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Optimal Neumann boundary control of a vibrating string with uncertain initial data and probabilistic terminal constraints

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Abstract

In optimal control problems, often initial data are required that are not known exactly in practice. In order to take into account this uncertainty, we consider optimal control problems for a system with an uncertain initial state. A finite terminal time is given. On account of the uncertainty of the initial state, it is not possible to prescribe an exact terminal state. Instead, we are looking for controls that steer the system into a given neighborhood of the desired terminal state with sufficiently high probability. This neighborhood is described in terms of an inequality for the terminal energy. The probabilistic constraint in the considered optimal control problem leads to optimal controls that are robust against the inevitable uncertainties of the initial state. We show the existence of such optimal controls. Numerical examples with optimal Neumann control of the wave equation are presented.

1 Introduction

Many applications in engineering sciences are modeled by initial boundary value problems with a hyperbolic system, see for example [3, 2, 6]. In the applications in engineering, often some data are uncertain. In order to provide analytical insights for such a situation, in this paper we consider a system that is governed by a wave equation with uncertain initial data that are modeled by random Fourier series. The corresponding Cauchy problems have been analyzed in [4].

For the situation with uncertain initial data, we consider an optimal control problem with conditions on the terminal state. For a given control, the terminal state is also uncertain. Optimal control problems with terminal constraints have been studied before in the classical deterministic setting where the desired terminal state is prescribed exactly, see for example [22, 26, 15]. This is possible since the system is exactly controllable, that is for a known initial state and a sufficiently large control time there exists a control such that the desired terminal state is reached exactly (see [23]). This exact terminal condition is equivalent to a sequence of moment equations for the terminal state, see for example [12]. Control to a position of rest can also be characterized by the requirement that the terminal energy is equal to zero.

In our probabilistic setting, since the initial state is uncertain for a fixed control function it is impossible to predict the terminal state exactly. Therefore, instead of a terminal constraint we require that the terminal energy of the system state is less than or equal to a given upper bound ε at least with a given probability. So similar as in [14], we present a relaxation of the exact terminal constraint that increases the robustness of the optimal controls.

In this way, we obtain optimal controls for which the probability that the terminal position is contained in a certain ε-neighborhood of the desired terminal position is greater than or equal to a prescribed parameter $p \in (0, 1)$. So starting from the uncertain initial state, the optimal controls generate a

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terminal state that satisfies a *probabilistic constraint*. The solution of this optimal control problem can be approximated by a suitable numerical method. By truncating the infinite series that represents the terminal energy we obtain a sequence of auxiliary problems whose solutions converge to the solution of the optimal control problem with the probabilistic contraint on the terminal energy. In this way, we can compute approximations for a control that satisfies the probabilistic constraints and minimizes the objective function of the optimal control problem. Here we consider the L^2–norm of the control as the objective function.

The problem we consider falls into the class of PDE-constrained optimization subject to risk-averse decisions. This topic has gained much interest recently. One direction of approaching such problems is the consideration of the conditional value-at-risk in the objective or constraints (e.g., [17, 21]). Another perspective, the one taken in this paper, is the use of probabilistic constraints which are very popular in engineering problems but traditionally applied in finite dimension. Their application in the environment of PDE-constrained optimization is still in its infancy (e.g., [9, 25]).

The paper has the following structure. In Section 2 we discuss the deterministic initial-boundary value problem and the corresponding problem with uncertain initial data. In Section 3 we discuss the corresponding problem of norm-minimal Neumann control for the system with uncertain initial state and a probabilistic constraint for the terminal energy. Also an approximation of the energy with finite sums is introduced.

In Section 4 it is shown that the probabilistic constraint in the optimal control problem with approximated energy is convex. Based upon this fact, methods for the computation of the probabilities and the corresponding gradients with respect to the control are presented in Section 5. The existence of optimal controls for the original problem and the problems with the approximated energy constraint is shown in Section 6. In Section 7 numerical examples are illustrated and discussed.

2 System states for Neumann boundary of the wave equation control with uncertain initial data

We consider a vibrating string of length $L > 0$ with homogeneous Dirichlet boundary conditions at one end and Neumann–boundary control at the other end. To study this problem we use the Sobolev space $H^1(0, L) = \{f \in L^2(0, L)$: The derivative f' in the sense of distributions is in $L^2(0, L)\}$. Let $y_0 \in H^1(0, L)$ and $y_1 \in L^2(0, L)$ be given. Let $T > 0$ denote a given terminal time and let $c > 0$ denote the wave speed. For a given control $u \in L^2(0, T)$ the deterministic initial boundary value problem

$$(NIBVP) \begin{cases} y(0,x) = y_0(x), & x \in (0,L) \\ y_t(0,x) = y_1(x), & x \in (0,L) \\ y_{tt}(t,x) = c^2\, y_{xx}(t,x), & (t,x) \in (0,T) \times (0,L) \\ y(t,0) = 0, & t \in (0,T) \\ y_x(t,L) = u(t), & t \in (0,T). \end{cases}$$

has been analyzed for example in [13], Theorem 2.3. In engineering applications, often the initial data are uncertain. As a model for the uncertain initial data, we use random Fourier series as studied in [19]. For $n \in \{0, 1, 2, ...\}$ define the complete orthonormal series

$$\varphi_n(x) := \frac{\sqrt{2}}{\sqrt{L}}\, \sin\left(\left(\frac{\pi}{2} + n\pi\right)\frac{x}{L}\right)$$

and the corresponding coefficients $\alpha_n^0, \alpha_n^1 \in \mathbb{R}$ with

$$\alpha_n^0 := \int_0^L y_0(x)\,\varphi_n(x)\,dx, \;\; \alpha_n^1 := \int_0^L y_1(x)\,\varphi_n(x)\,dx.$$

Then we have the series representations

$$y_0(x) = \sum_{n=0}^{\infty} \alpha_n^0\,\varphi_n(x), \;\; y_1(x) = \sum_{n=0}^{\infty} \alpha_n^1\,\varphi_n(x). \tag{1}$$

Unless otherwise stated, we shall make the following standing assumption for our model of uncertain initial data:

(A) Assume that for all $n \in \{0, 1, 2, ...\}$, identically distributed random variables a_n, b_n are given on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$.

Consider the random initial data

$$y_0^\omega(x) = \sum_{n=0}^{\infty} a_n^\omega\,\alpha_n^0\,\varphi_n(x), \;\; y_1^\omega(x) = \sum_{n=0}^{\infty} b_n^\omega\,\alpha_n^1\,\varphi_n(x), \tag{2}$$

where the superscript ω indicates the evaluation of the random variable at this outcome $\omega \in \Omega$. Then, our assumption (**A**) along with an argument related to the Paley–Zygmund Theorem implies that for these random series almost surely we have $y_0^\omega \in H^1(0, L)$, $y_1^\omega \in L^2(0, L)$ (see [18, Lemma 4.3],[4, 7]). The corresponding initial boundary value problem with uncertain initial data is given by

$$(NIBVPU) \left\{ \begin{array}{lcll} y^\omega(0, x) &=& y_0^\omega(x), & x \in (0, L) \\ y_t^\omega(0, x) &=& y_1^\omega(x), & x \in (0, L) \\ y_{tt}^\omega(t, x) &=& c^2\, y_{xx}^\omega(t, x), & (t, x) \in (0, T) \times (0, L) \\ y^\omega(t, 0) &=& 0, & t \in (0, T) \\ y_x^\omega(t, L) &=& u(t), & t \in (0, T), \end{array} \right.$$

where $\omega \in \Omega$. The solution of $(NIBVPU)$ is almost surely at least as well-behaved as the solution of $(NIBVP)$. In fact, we have again a series representation that is presented in the following theorem.

Theorem 1 *Assume that $y_0 \in H^1(0, L)$ with $y_0(0) = 0$, $y_1 \in L^2(0, L)$ and $u \in L^2(0, T)$. For $n \in \{0, 1, 2, ...\}$ define*

$$\lambda_n := \frac{1}{L^2} \left(\frac{\pi}{2} + n\pi\right)^2$$

and the random variables

$$\begin{array}{rcl} \alpha_n^\omega(t) &:=& a_n^\omega\,\alpha_n^0\,\cos\left(\sqrt{\lambda_n}\,c\,t\right) + b_n^\omega\,\alpha_n^1\,\frac{1}{\sqrt{\lambda_n}\,c}\sin\left(\sqrt{\lambda_n}\,c\,t\right) \\ &+& c^2\,\varphi_n(L)\,\frac{1}{\sqrt{\lambda_n}\,c} \displaystyle\int_0^t u(s)\sin\left(\sqrt{\lambda_n}\,c\,(t-s)\right)\,ds. \end{array}$$

Then

$$y^\omega(t, x) = \sum_{n=0}^{\infty} \alpha_n^\omega(t)\,\varphi_n(x) \tag{3}$$

is almost surely the unique solution of $(NIBVPU)$. *For all* $t \in (0,T)$ *we have almost surely* $y^\omega(t,\cdot) \in L^2(0,L)$ *and*

$$\int_0^L y^\omega(t,x)^2\,dx = \sum_{n=0}^\infty \left(\alpha_n^\omega(t)\right)^2.$$

Moreover we have almost surely $y^\omega \in C((0,T),\ L^2(0,L))$.

Proof: The functions φ_n are the solutions of the eigenvalue problem

$$\varphi_{xx}(x) = -\lambda\,\varphi(x),\ x\in[0,L],\ \varphi(0)=0,\ \varphi_x(L)=0$$

with the eigenvalues λ_n and the normalization

$$\int_0^L \varphi_n(x)^2\,dx = 1,\ n\in\{0,1,2,...\}.$$

The sequence of functions $(\varphi_n)_{n=0}^\infty$ is a complete orthonormal system in the Hilbert space $L^2(0,L)$. The definition of the functions $\alpha_n^\omega(t)$ implies that almost surely $\alpha_n^\omega(0) = a_n^\omega\,\alpha_n^0$ and almost surely $(\alpha_n^\omega)'(0) = b_n^\omega\,\alpha_n^1$. Hence we have almost surely $\sum_{n=0}^\infty \alpha_n^\omega(0)\,\varphi_n(x) = y_0^\omega(x)$ and $\sum_{n=0}^\infty (\alpha_n^\omega)'(0)\,\varphi_n(x) = y_1^\omega(x)$ almost everywhere on $(0,\ L)$. Thus we have almost surely $y^\omega(0,\ x) = y_0^\omega(x)$ and $y_t^\omega(0,\ x) = y_1^\omega(x)$ almost everywhere on $(0,\ L)$, that is the initial conditions of $(NIBVPU)$ are satisfied. Now we consider the boundary traces. Almost everywhere on $(0,\ T)$ we have almost surely $y^\omega(t,\ 0) = \sum_{n=0}^\infty \alpha_n^\omega(t)\,\varphi_n(0) = 0$.

The definition of $\alpha_n^\omega(t)$ implies that for all $n\in\{0,1,2,...\}$ we have almost surely

$$(\alpha_n^\omega)''(t) = -\frac{c^2}{L^2}\left(\frac{\pi}{2}+n\pi\right)^2 \alpha_n^\omega(t) + (-1)^n\,c^2\,\frac{\sqrt{2}}{\sqrt{L}}\,u(t).$$

For a test function $\varphi\in\mathcal{D}([0,\ T]\times[0,\ L])$ we have

$$\varphi(t,\ x) = \sum_{n=0}^\infty \int_0^L \varphi(t,\ s)\,\varphi_n(s)\,ds\,\varphi_n(x).$$

Hence $0 = \varphi(t,\ L) = \sum_{n=0}^\infty \int_0^L \varphi(t,\ s)\,\varphi_n(s)\,ds\,\varphi_n(L)$. Thus the definition of the distributional derivative implies almost surely

$$\begin{aligned}
&\int_0^T\int_0^L \left[y_{tt}^\omega(t,x) - c^2\,y_{xx}^\omega(t,x)\right]\,\varphi(t,\ x)\,dx\,dt\\
=\ &\int_0^T\int_0^L y^\omega(t,x)\,\varphi_{tt}(t,\ x) - c^2\,y^\omega(t,x)\,\varphi_{xx}(t,\ x)\,dx\,dt\\
=\ &\int_0^T\int_0^L \sum_{n=0}^\infty \alpha_n^\omega(t)\,\varphi_n(x)\,\varphi_{tt}(t,\ x) - c^2\,\alpha_n^\omega(t)\,\varphi_n(x)\,\varphi_{xx}(t,\ x)\,dx\,dt\\
=\ &\sum_{n=0}^\infty\int_0^T\int_0^L \varphi_n(x)\,(\alpha_n^\omega)''^2\ \alpha_n^\omega(t)\,\varphi_n''(x)\,\varphi(t,\ x)\,dx\,dt\\
=\ &\sum_{n=0}^\infty\int_0^T\int_0^L \varphi_n(x)\,\alpha_n^\omega(t)\left[-c^2\,\lambda_n + c^2\lambda_n\right]\,\varphi(t,\ x) + \varphi_n(x)\,c^2\varphi_n(L)\,u(t)\,\varphi(t,\ x)\,dx\,dt\\
=\ &\int_0^T c^2\,u(t)\sum_{n=0}^\infty\int_0^L \varphi(t,\ s)\,\varphi_n(s)\,ds\,\varphi_n(L)\,dt\\
=\ &\int_0^T c^2\,u(t)\,\varphi(t,\ L)\ dt = 0.
\end{aligned}$$

Thus almost surely y^ω satisfies the wave equation in the sense of distributions.

Now we choose a test function $\varphi(x) \in C^2([0,\, L])$ such that $\varphi(0) = 0$ and $\varphi_x(L) = 0$. Then we have $\varphi(L) = \sum_{n=0}^\infty \int_0^L \varphi(x)\, \varphi_n(x)\, dx\, \varphi_n(L)$. Hence we have almost surely for $t \in (0,\, T)$ almost everywhere

$$\begin{aligned}
& \int_0^L \left[\tfrac{1}{c^2}\, y_{tt}^\omega(t,\, x) - y_{xx}^\omega(t,\, x)\right]\, \varphi(x)\, dx \\
= & \int_0^L \tfrac{1}{c^2}\, y_{tt}^\omega(t,\, x)\, \varphi(x) - y^\omega(t,\, x)\, \varphi_{xx}(x)\, dx\, - y_x^\omega(t,\, x)\, \varphi(x)|_{x=0}^L + y^\omega(t,\, x)\, \varphi_x(x)|_{x=0}^L \\
= & \sum_{n=0}^\infty \left(\tfrac{1}{c^2}\, \alpha_n''(t) + \lambda_n \alpha_n(t)\right) \int_0^L \varphi(x)\, \varphi_n(x)\, dx - y_x^\omega(t,\, L)\, \varphi(L) \\
= & \sum_{n=0}^\infty \varphi_n(L)\, u(t) \int_0^L \varphi(x)\, \varphi_n(x)\, dx - y_x^\omega(t,\, L)\, \varphi(L) \\
= & \; u(t)\, \varphi(L) - y_x^\omega(t,\, L)\, \varphi(L) \\
= & \; [u(t) - y_x^\omega(t,\, L)]\, \varphi(L).
\end{aligned}$$

In this weak sense y^ω satisfies the Neumann boundary condition at $x = L$.

Hence the series (3) almost surely solves $(NIBVPU)$. Consider the classical energy

$$E^\omega(u,t) := \int_0^L y_x^\omega(t,x)^2 + \frac{1}{c^2}\, \left(y_t^\omega(t,x)\right)^2\, dx.$$

Since the energy decays almost surely for $u(t) = 0$, the uniqueness of the solution follows. The continuity of the solution with respect to time follows from the regularity of the $\alpha_n(t)$. Thus Theorem 1 is proved.□

Remark 2.1 *Note that the functions* $(\frac{1}{\sqrt{\lambda_n}}\, \varphi_n')_{n=0}^\infty$ *also form a complete orthonormal system in* $L^2(0, L)$.

3 Optimal Neumann control

In this section we look at the problem of optimal control from an uncertain initial state to a desired terminal position in a probabilistic sense in a given finite time for a system that is governed by the initial boundary value problem $(NIBVPU)$. We assume for simplicity that $L = 1$. Let $T \geq 2$ be given. Let an expected initial position $y_0 \in H^1(0,1)$ with $y_0(0) = 0$ and an expected initial velocity $y_1 \in L^2(0,1)$ be given. For the convenience of the reader we first restate the deterministic optimal control problem $(\mathbf{NEC})$ for the case of initial states without uncertainty with an exact terminal condition that has been studied in [12]:

$$(\mathbf{NEC}) \begin{cases} \min_{u\in L^2(0,T)}\, \|u\|_{L^2(0,T)}^2 \text{ subject to} \\ y(0,x) = y_0(x),\; y_t(0,x) = y_1(x),\; x \in (0,1) \\ y(t,0) = 0,\; y_x(t,1) = u(t), t \in (0,T) \\ y_{tt}(t,x) = c^2\, y_{xx}(t,x),\; (t,x) \in (0,T) \times (0,1) \\ y(T,x) = 0,\; y_t(T,x) = 0,\; x \in (0,1) \end{cases}$$

The terminal constraints can also be replaced by the requirement that the terminal energy is equal to zero. For the convenience of the reader we include the representation of the optimal control for the deterministic case $(\mathbf{NEC})$ with wave speed $c = 1$.

Theorem 2 (Representation of the optimal Neumann control, see [12]) *Let $T \geq 2, k := \max\{n \in \mathbb{N} : 2n \leq T\}$ and $\Delta := T - 2k$.... For $t \in [0, 2)$, let*

$$d(t) := \begin{cases} k+1, & t \in (0, \Delta], \\ k, & t \in (\Delta, 2). \end{cases} \tag{4}$$

*Then the optimal control u_0 that solves **(NEC)** is 4–periodic, with*

$$u_0(t) = \begin{cases} \frac{1}{2d(t)} \left[y_0'(1-t) - y_1(1-t) \right], & t \in (0,1), \\ \\ \frac{1}{2d(t)} \left[y_0'(t-1) + y_1(t-1) \right], & t \in (1,2). \end{cases}$$

For $t \in (0, 2)$, $l \in \{0, 1, ..., k\}$ with $t + 2l \leq T$ we have $u_0(t + 2l) = (-1)^l \, u_0(t)$.

Now we present a problem that is suitable for uncertain initial data. Let an accuracy parameter $\varepsilon > 0$ and a preset probability threshold $p \in (0, 1)$ be given. We consider the problem of optimal exact control and uncertain initial data

$$(\mathbf{NECU})(\varepsilon, p) \begin{cases} \min_{u \in L^2(0,T)} \|u\|^2_{L^2(0,T)} \text{ subject to} \\ \\ y^\omega(0,x) = y_0^\omega(x), \; y_t^\omega(0,x) = y_1^\omega(x), \; x \in (0,1) \\ \\ y^\omega(t,0) = 0, \; y_x^\omega(t,1) = u(t), \, t \in (0,T) \\ \\ y_{tt}^\omega(t,x) = c^2 \, y_{xx}^\omega(t,x), \; (t,x) \in (0,T) \times (0,1) \\ \\ \mathbb{P}\,(\; E^\omega(u,T) \leq \varepsilon \;) \geq p. \end{cases}$$

The parameter ε is an upper bound for the terminal energy that is valid at least with the given probability p. For the terminal energy we have

$$E^\omega(u,T) = \int_0^L y_x^\omega(T,x)^2 + \frac{1}{c^2} \, y_t^\omega(T,x)^2 dx$$

$$= \sum_{n=0}^{\infty} \left| \int_0^L y_x^\omega(T,x) \, \tfrac{1}{\sqrt{\lambda_n}} \, \varphi_n'(x) \, dx \right|^2 + \frac{1}{c^2} \left| \int_0^L y_t^\omega(T,x) \, \varphi_n(x) \, dx \right|^2 dx$$

$$= \sum_{n=0}^{\infty} \lambda_n \, \left(\alpha_n^\omega(T)\right)^2 + \frac{1}{c^2} \, \left((\alpha_n^\omega)'(T)\right)^2.$$

For $n \in \{1, 2, 3, ...\}$ define

$$\sigma_n^\omega(t) := a_n^\omega \, \alpha_n^0 \, \cos\left(\sqrt{\lambda_n} \, c \, t\right) + b_n^\omega \, \alpha_n^1 \, \tfrac{1}{\sqrt{\lambda_n} \, c} \sin\left(\sqrt{\lambda_n} \, c \, t\right)$$

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and

$$c_n^{(1)}(u) := c^2\, \varphi_n(1)\, \tfrac{1}{\sqrt{\lambda_n}\, c} \int_0^T u(s) \sin\left(\sqrt{\lambda_n}\, c\, (T-s)\right)\, ds.$$

With the explicit representation of $\alpha_n^\omega(T)$ from Theorem 1 we obtain

$$\alpha_n^\omega(T) = c_n^{(1)}(u) + \sigma_n^\omega(T).$$

We have

$$(\sigma_n^\omega)'(t) = -a_n^\omega\, \alpha_n^0\, \sqrt{\lambda_n}\, c\, \sin\left(\sqrt{\lambda_n}\, c\, t\right) + b_n^\omega\, \alpha_n^1\, \cos\left(\sqrt{\lambda_n}\, c\, t\right).$$

Define

$$c_n^{(2)}(u) := c^2\, \varphi_n(1) \int_0^T u(s) \cos\left(\sqrt{\lambda_n}\, c\, (T-s)\right)\, ds.$$

Then

$$(\alpha_n^\omega)'(T) = c_n^{(2)}(u) + (\sigma_n^\omega)'(T).$$

For the terminal energy this yields

$$E^\omega(u,\, T) = \sum_{n=0}^{\infty} \lambda_n\, \left(c_n^{(1)}(u) + \sigma_n^\omega(T)\right)^2 + \frac{1}{c^2}\, \left(c_n^{(2)}(u) + (\sigma_n^\omega)'(T)\right)^2.$$

Thus the inequality constraint in $(\mathbf{NECU})(\varepsilon,\, p)$ has the form

$$\mathbb{P}\left(\sum_{n=0}^{\infty} \lambda_n\, \left(c_n^{(1)}(u) + \sigma_n^\omega(T)\right)^2 + \frac{1}{c^2}\, \left(c_n^{(2)}(u) + (\sigma_n^\omega)'(T)\right)^2 \leq \varepsilon\right) \geq p. \tag{5}$$

In this way, the probabilistic constraint in the definition of $(\mathbf{NECU})(\varepsilon,\, p)$ demands that the probability that the control is successful in a reduction of the energy such that $E^\omega(u,\, T) \leq \varepsilon$ is greater than or equal to p.

In order to make the energy accessible to numerical computations, the infinite series that defines $E^\omega(u,\, T)$ is approximated by the finite sum of the first N terms with $N \in \{1,\, 2,\, 3, ...\}$. Define

$$E_N^\omega(u,\, T) := \sum_{n=0}^{N} \lambda_n\, \left(c_n^{(1)}(u) + \sigma_n^\omega(T)\right)^2 + \frac{1}{c^2}\, \left(c_n^{(2)}(u) + (\sigma_n^\omega)'(T)\right)^2. \tag{6}$$

We obtain the problem

$$(\mathbf{NECU})(\varepsilon,\, p,\, N) \begin{cases} \min_{u\in L^2(0,T)}\ \|u\|^2_{L^2(0,T)} \ \text{ subject to} \\ \\ \mathbb{P}\left(\ E_N^\omega(u,\, T) \leq \varepsilon\ \right) \geq p. \end{cases}$$

4 Convexity of the probabilistic constraint in $(\mathbf{NECU})(\varepsilon,\, p,\, N)$

In this and the following section, we shall be dealing with finite-dimensional random vectors in the context of problem $(\mathbf{NECU})(\varepsilon,\, p,\, N)$ with cut-off N-term energy 6. Therefore we will not have to impose our standing assumption (**A**) on identical distributions of all of its components. This assumption will become critical again, when showing in Section 6 the existence of solutions to the original problem $(\mathbf{NECU})(\varepsilon,\, p)$ where the whole sequence of random variables comes into play.

In this section, we show that the probabilistic constraint in $(\mathbf{NECU})(\varepsilon,\, p,\, N)$ defines a convex set of admissible controls u. To this aim, let ξ be a m-dimensional random vector, U a vector space and $\tilde{g} : U \times \mathbb{R}^m \to \mathbb{R}^k$ a given mapping. The following Theorem is due to Prékopa (in the original formulation, U was supposed to be finite-dimensional, but this restrictive property is not exploited in the proof of the result, see also [9, Prop. 4]):

Theorem 3 ([24]) *Let ξ have a log-concave density, i.e., a density whose log is (possibly extended-valued) concave. If all components $\tilde{g}_i$ of $\tilde{g}$, (i=1,...,k), are quasi-convex, then the probability function $\tilde{\phi} : U \to \mathbb{R}$ defined by*

$$\tilde{\phi}(u) := \mathbb{P}\left(\tilde{g}_i(u,\xi) \leq 0 \quad (i = 1, \ldots, k)\right) \tag{7}$$

is log-concave.

We introduce the probability functions $\phi_N : L^2(0,T) \to \mathbb{R}$ for $N \in \mathbb{N}$ associated with the inequality constraints in $(\mathbf{NECU})(\varepsilon,\, p,\, N)$ as

$$\phi_N(u) := \mathbb{P}\left(\, E_N^{\omega}(u,T) \leq \varepsilon \,\right),$$

or more explicitly (see (6)):

$$\phi_N(u) := \mathbb{P}\left(\sum_{n=0}^{N} \lambda_n \, \left(c_n^{(1)}(u) + \sigma_n^{\omega}(T)\right)^2 + \frac{1}{c^2} \, \left(c_n^{(2)}(u) + (\sigma_n^{\omega})'(T)\right)^2 \leq \varepsilon \right). \tag{8}$$

Proposition 1 *Let the random vector $\xi := ((a_n, b_n))_{n=1}^{N}$, have a log-concave density. Then, ϕ_N, defined in (8), is log-concave too.*

Proof 1 *By definition, ϕ_N can be written as $\phi_N(u) = \mathbb{P}\left(g\left(u,\xi\right) \leq 0\right)$, where for $n = 1, \ldots, N$:*

$$\begin{aligned}
g\left(u,z\right) &:= \sum_{n=1}^{N} g_n\left(u,z\right) - \varepsilon \\
g_n\left(u,z\right) &:= \lambda_n \left(c_n^{(1)}(u) + \langle A_n, z\rangle\right)^2 + \frac{1}{c^2}\left(c_n^{(2)}(u) + \langle B_n, z\rangle\right)^2 \\
A_n &:= \Big((0,0), \ldots, (0,0), \\
&\qquad \left(\alpha_n^0 \cos\left(\sqrt{\lambda_n} cT\right), \frac{\alpha_n^1}{\sqrt{\lambda_n} c} \sin\left(\sqrt{\lambda_n} cT\right)\right), (0,0), \ldots, (0,0)\Big) \\
B_n &:= \Big((0,0), \ldots, (0,0), \\
&\qquad \left(-\alpha_n^0 \sqrt{\lambda_n} c \sin\left(\sqrt{\lambda_n} cT\right), \alpha_n^1 \cos\left(\sqrt{\lambda_n} cT\right)\right), (0,0), \ldots, (0,0)\Big) \\
z &:= \left(\left(z_n^a, z_n^b\right)\right)_{n=1}^{N},
\end{aligned}$$

where the nonzero expressions in A_n, B_n appear at position n. Since the functions $c_n^{(1)}(u)$ and $c_n^{(2)}(u)$ are linear in u, the functions g_n are convex as squares of linear functions in (u,z) jointly. Hence g is convex too. Now, the result follows from Theorem 3 upon putting $\tilde{\phi} := \phi_N$ and $\tilde{g} := g$.

Corollary 1 *In the setting of Proposition 1, the set of feasible controls u defined by the probabilistic constraint in problem $(\mathbf{NECU})(\varepsilon,\, p,\, N)$ is convex and can be equivalently represented by a convex inequality $h_N(u) \leq 0$, where*

$$h_N(u) := -\log \phi_N(u) + \log p \tag{9}$$

and ϕ_N is defined in (8).

Remark 4.1 *We observe that many prominent multivariate distributions share the property of having log-concave densities, e.g., multivariate Gaussian, Dirichlet, Wishart, Gamma or the uniform distribution on convex compact sets, see [24]*

5 Algorithmic approach to the solution of problem $(\mathbf{NECU})(\varepsilon,\, p,\, N)$

For the algorithmic solution of the optimization problem $(\mathbf{NECU})(\varepsilon,\, p,\, N)$ the numerical evaluation/approximation of the probability function ϕ_N in (8) along with its sensitivity with respect to the control variable u is necessary. We will describe next an approach via the so-called *spheric-radial decomposition* of Gaussian random vectors (see, e.g., [8, p. 105], [10, p.30]). Although the same idea applies to a whole class of distributions (e.g., Gaussian-like such as log-normal, truncated Gaussian or elliptically symmetric distributions such as *Student*), we will content ourselves here with the probably most prominent case of purely Gaussian distributions. The following result is well-known (see [8, eq. (8)] or in its present form [11, Th. 3.1.]):

Lemma 1 *Let $\xi \sim \mathcal{N}(\mu, \Sigma)$ be an m-dimensional Gaussian random vector having expectation μ and (nondegenerate) covariance matrix Σ. Then, for any Borel-measurable subset $M \subseteq \mathbb{R}^m$ its probability with respect to this distribution can be represented as*

$$\mathbb{P}(\xi \in M) = \int_{\mathbb{S}^{m-1}} \nu_\chi \{r \geq 0 | \mu + rLv \in M\}\, d\nu_\eta(v).$$

Here, $\mathbb{S}^{m-1}$ is the $(m-1)$-dimensional unit sphere in $\mathbb{R}^m$, ν_η is the uniform distribution on $\mathbb{S}^{m-1}$, ν_χ denotes the one-dimensional χ-distribution with m degrees of freedom and L is such that

$$\Sigma = LL^T \tag{10}$$

(e.g., Cholesky decomposition).

Next, we apply the previous lemma to the probability function $\tilde{\phi}(u)$ defined in Theorem 3 for $k = 1$ and consider

$$\tilde{\phi}(u) := \mathbb{P}(\tilde{g}(u, \xi) \leq 0), \tag{11}$$

where $\tilde{g} : U \times \mathbb{R}^m \to \mathbb{R}$ for some Banach space U and ξ is a Gaussian random vector as in Lemma 1.

Corollary 2 *Fix any $u \in U$ with the following properties:*

1 $\tilde{\phi}(u) > 0.5$

2 $\tilde{g}(u,\cdot)$ is convex.

3 The convex inequality $\tilde{g}(u,\cdot) \leq 0$ admits a Slater point, i.e., there is some $\bar{z}$ with $\tilde{g}(u,\bar{z}) < 0$.

Then,

$$\tilde{\phi}(u) = \int_{\mathbb{S}^{m-1}} e(u,v)\, d\nu_\eta(v), \tag{12}$$

where $e(u,v) = F_{\nu_\chi}(\rho(u,v))$ (with F_{ν_χ} referring to the distribution function of ν_χ) if the equation

$$\tilde{g}(u, \mu + rLv) = 0$$

admits a (unique) nonnegative solution $r = \rho(u,v) \geq 0$ and $e(u,v) = 1$ else (i.e., $\tilde{g}(u,\mu + rLv) < 0$ for all $r \geq 0$).

Proof 2 *Lemma 1 yields that*

$$\tilde{\phi}(u) = \int_{\mathbb{S}^{m-1}} \nu_\chi \{r \geq 0 | \tilde{g}(u, \mu + rLv) \leq 0\}\, d\nu_\eta(v). \tag{13}$$

For fixed $u \in U$, define the set $M := \{z \in \mathbb{R}^m \mid \tilde{g}(u,z) \leq 0\}$. Clearly, M is convex and nonempty by the assumption 3. From our assumptions, it follows that the mean vector μ of ξ satisfies the strict inequality $\tilde{g}(u,\mu) < 0$ (see [1, Prop. 3.11]). Now, by the convexity of M, one has either that $\tilde{g}(u,\mu + rLv) < 0$ for all $r \geq 0$ or that $\tilde{g}(u,\mu + rLv) = 0$ for exactly one $r = \rho(u,v) \geq 0$. In the first case,

$$\nu_\chi \{r \geq 0 | \tilde{g}(u, \mu + rLv) \leq 0\} = \nu_\chi(\mathbb{R}_+) = 1$$

since the support of the χ-distribution is the nonnegative reals. In the second case,

$$\begin{aligned}\nu_\chi \{r \geq 0 | \tilde{g}(u, \mu + rLv) \leq 0\} &= \nu_\chi([0, \rho(u,v)]) = F_{\nu_\chi}(\rho(u,v)) - F_{\nu_\chi}(0) \\ &= F_{\nu_\chi}(\rho(u,v)).\end{aligned}$$

Now, the assertion follows from (13).

A few words on the assumptions of Corollary 2 are in order: The convexity assumption 2. will hold true in the case of our problem $(\mathbf{NECU})(\varepsilon, p, N)$ because the function g defined in the proof of Proposition 1 and playing the role of $\tilde{g}$ in Corollary 2 was shown to be convex (actually in both arguments simultaneously) in the mentioned proof. Requiring the value of the probability function to be larger than 0.5 as in assumption 1. is no practical restriction because in probabilistic constraints one is typically dealing with probabilities close to one. Finally, it is well known that, without the Slater point assumption 3., the probability function $\tilde{\phi}$ may even fail to be continuous.

For numerical purposes, the spheric integral in Corollary 2 will be approximated by a finite sum based on a sample $v^1, \ldots, v^K \in \mathbb{S}^{m-1}$ from the uniform distribution on the sphere. One possibility to do so efficiently, consists in generating a Quasi-Monte Carlo sample $w^1, \ldots, w^K$ of the m-dimensional standard Gaussian distribution in $\mathbb{R}^m$ and to normalize it, so that $v^k := w^k / \left\| w^k \right\|$ for $k = 1, \ldots, K$. Then, the desired value of the probability function in (11) can be approximated as

$$\tilde{\phi}(u) \approx K^{-1} \sum_{k=1}^{K} e\left(u, v^k\right). \tag{14}$$

Of course, the larger K, the better this approximation. It remains to clarify, how the function $e\,(u,v)$ in Corollary 2 can be evaluated in the context of our optimization problem $(\mathbf{NECU})(\varepsilon,\,p,\,N)$. To this end, the general function $\tilde{g}$ defined in (11) will be specified now as the function g introduced in the proof of Proposition 1. Hence, we have

$$\tilde{g}\,(u,\mu+rLv) := g\,(u,\mu+rLv) =$$
$$\sum_{n=1}^{N}\left(\lambda_n\left(c_n^{(1)}(u)+\langle A_n,\mu+rLv\rangle\right)^2+\frac{1}{c^2}\left(c_n^{(2)}(u)+\langle B_n,\mu+rLv\rangle\right)^2\right)-\varepsilon.$$

Now, to solve the equation $\tilde{g}\,(u,\mu+rLv)=0$ for given u and v in $r\geq 0$ as mentioned in Corollary 2, we can regroup and rewrite it as a quadratic equation in r:

$$\alpha\,(u,v)\,r^2+\beta\,(u,v)\,r+\gamma\,(u,v)=0, \tag{15}$$

where

$$\begin{aligned}
\alpha\,(u,v) &:= \sum_{n=1}^{N}\lambda_n\,\langle A_n,Lv\rangle^2+\frac{1}{c^2}\,\langle B_n,Lv\rangle^2\\
\beta\,(u,v) &:= \sum_{n=1}^{N}2\lambda_n\,\langle A_n,Lv\rangle\left(\langle A_n,\mu\rangle+c_n^{(1)}(u)\right)+\frac{2}{c^2}\,\langle B_n,Lv\rangle\left(\langle B_n,\mu\rangle+c_n^{(2)}(u)\right)\\
\gamma\,(u,v) &:= \sum_{n=1}^{N}\lambda_n\left(c_n^{(1)}(u)+\langle A_n,\mu\rangle\right)^2+\frac{1}{c^2}\left(c_n^{(2)}(u)+\langle B_n,\mu\rangle\right)^2-\varepsilon.
\end{aligned}$$

Under the assumptions of Corollary 2 it follows that $\tilde{g}\,(u,\mu)<0$ (see proof there), which means that $\gamma\,(u,v)<0$. Since, moreover, $\alpha\,(u,v)\geq 0$, it follows that exactly one of the following cases may occur:

(a) If $\alpha\,(u,v)=0$ and $\beta\,(u,v)\leq 0$, then (15) has no solution $r\geq 0$ at all and, hence, $e\,(u,v)=1$ (see Corollary 2).

(b) If $\alpha\,(u,v)=0$ and $\beta\,(u,v)>0$, then the unique solution of (15) is

$$r=\rho\,(u,v):=-\frac{\gamma\,(u,v)}{\beta\,(u,v)}>0.$$

(c) If $\alpha\,(u,v)>0$, then (15) has exactly one solution $r\geq 0$ given by

$$r=\rho\,(u,v):=\frac{-\beta\,(u,v)+\sqrt{\beta^2\,(u,v)-4\gamma\,(u,v)\,\alpha\,(u,v)}}{2\alpha\,(u,v)}.$$

We note, that the one-dimensional distribution function F_{ν_χ} of the χ-distribution is a direct built-in function of many mathematical software packages or can be easily derived from the corresponding distribution functions of the χ^2- or Γ-distributions (for instance: $F_{\nu_\chi}\,(t)=F_{\nu_{\chi^2}}\,(t^2)$).

After having described, how to evaluate fairly precise values for the probability function $\tilde{\phi}(u)$ via (14) (or in the context of optimization problem $(\mathbf{NECU})(\varepsilon,\,p,\,N)$ the probability function $\phi_N(u)$ defined in (8)), we address now the question, how to obtain its Derivative $D\tilde{\phi}(u)$ in order to set up a gradient-based optimization procedure for solving problem $(\mathbf{NECU})(\varepsilon,\,p,\,N)$. For the following result we refer to [16, Th. 6, Cor. 3, Rem.1]:

Theorem 4 *Let $\tilde{g}$ in (11) be continuously differentiable. Additionally to the assumptions of Corollary 2, suppose that the following growth condition is satisfied around $u \in U$:*

$$\exists l > 0 : \|D_u \tilde{g}\,(w, z)\| \leq l e^{\|z\|} \quad \forall w : \|w - u\| \leq 1/l \; \forall z : \|z\| \geq l.$$

Then, the probability function $\tilde{\phi}$ from (11) is (strictly) differentiable at u and the derivative $D\tilde{\phi}(u)$ is given as

$$D\tilde{\phi}(u)(h) = -\int\limits_{v \in F(u)} \frac{\chi\left(\rho\left(u, v\right)\right) \cdot D_u \tilde{g}\left(u, \rho\left(u, v\right) L v\right)(h)}{\left\langle \nabla_z \tilde{g}\left(u, \rho\left(u, v\right) L v\right), L v\right\rangle} d\nu_\eta(v) \quad \forall h \in U. \tag{16}$$

Here,

$$\chi(t) := \frac{t^{m-1} e^{-t^2/2}}{2^{m/2-1}\Gamma\left(k/2\right)} \quad (t \geq 0)$$

is the density of the one-dimensional χ-distribution with m degrees of freedom. Moreover, $F(u)$ refers to the set of directions $v \in \mathbb{S}^{m-1}$ such that the equation

$$\tilde{g}\left(u, \mu + r L v\right) = 0$$

admits a (unique) nonnegative solution $r = \rho\left(u, v\right) \geq 0$ (see Cor. 2).

The preceding Theorem can now be applied to the probability function of our problem $(\mathbf{NECU})(\varepsilon,\, p,\, N)$:

Theorem 5 *Let the random vector $\xi := (a_n, b_n)_{n=1}^N$ with coefficients introduced in (A) have a $2N$-dimensional Gaussian distribution $\xi \sim \mathcal{N}\left(\mu, \Sigma\right)$. Assume that $p \geq 0.5$ in $(\mathbf{NECU})(\varepsilon,\, p,\, N)$. Consider any $u \in L^2(0, T)$ such that $E_N^\omega(u) < \varepsilon$ for at least one scenario ω. Then, the probability function $\phi_N(u) := \mathbb{P}\left(\, E_N^\omega(u) \leq \varepsilon \,\right)$ is (strictly) differentiable at u and its gradient is given by*

$$\nabla \phi_N(u) = -\int\limits_{v \in F(u)} \frac{\chi\left(\rho\left(u, v\right)\right) \cdot \nabla_u g\left(u, \rho\left(u, v\right) L v\right)}{\left\langle \nabla_z g\left(u, \rho\left(u, v\right) L v\right), L v\right\rangle} d\nu_\eta(v), \tag{17}$$

where, with the defintions of A_n, B_n in the proof of Proposition 1,

$$\nabla_u g(u, z) = 2 \sum_{n=1}^{N} \left\{ \lambda_n \left(c_n^{(1)}(u) + \langle A_n, z \rangle \right) \nabla c_n^{(1)}(u) + \frac{c_n^{(2)}(u) + \langle B_n, z \rangle}{c^2} \nabla c_n^{(2)}(u) \right\} \tag{18}$$

$$\nabla_z g(u, z) = 2 \sum_{n=1}^{N} \left\{ \lambda_n \left(c_n^{(1)}(u) + \langle A_n, z \rangle \right) A_n + \frac{c_n^{(2)}(u) + \langle B_n, z \rangle}{c^2} B_n \right\}. \tag{19}$$

Proof 3 *Clearly, the function g introduced in Proposition 1 is continuously differentiable with partial gradients as in (18), (19). As already stated in the proof of Proposition 1, g is convex. Furthermore, $E_N^\omega(u) < \varepsilon$ for some scenario ω means the existence of z such that $g\left(u, z\right) < 0$. Finally, we have that*

$$\begin{aligned} \|\nabla_u g(v, z)\| \leq 2 \sum_{n=1}^{N} \{ & \lambda_n \left\| \nabla c_n^{(1)}(v) \right\| \left(\left| c_n^{(1)}(v) \right| + \|A_n\| \, \|z\| \right) \\ & + \frac{1}{c^2} \left\| \nabla c_n^{(2)}(v) \right\| \left(\left| c_n^{(2)}(v) \right| + \|B_n\| \, \|z\| \right) \}. \end{aligned}$$

By linearity of $c_n^{(1)}$, $c_n^{(2)}$*, the norms* $\left\|\nabla c_n^{(1)}(v)\right\|$, $\left\|\nabla c_n^{(2)}(v)\right\|$ *are constants and, moreover, there exists some* $\varkappa, \delta > 0$ *such that*

$$\max\{\left|c_n^{(1)}(v)\right|, \left|c_n^{(2)}(v)\right|\} \leq \varkappa \quad \forall v : \|v - u\| \leq \delta.$$

From here, one easily deduces the existence of some $l > 0$ *such that*

$$\|\nabla_u g(w, z)\| \leq l e^{\|z\|} \quad \forall w : \|w - u\| \leq 1/l \; \forall z.$$

Putting, $\tilde{g} := g$ *and* $\tilde{\phi} := \varphi_N$ *with the latter from (8), we observe that all assumptions of Theorem 4 (including those of Corollary 2) are fulfilled. Accordingly, (16) yields by the Fubini Theorem that for all* $h \in L^2(0, T)$*,*

$$\begin{aligned}
\langle \nabla\phi_N(u), h\rangle &= D\tilde{\phi}(u)(h) \\
&= -\int\limits_{v\in F(u)} \frac{\chi(\rho(u,v)) \cdot \langle \nabla_u g(u, \rho(u,v)Lv), h\rangle}{\langle \nabla_z g(u, \rho(u,v)Lv), Lv\rangle} d\nu_\eta(v) \\
&= -\int\limits_{v\in F(u)} \int\limits_0^T \frac{\chi(\rho(u,v)) \cdot \nabla_u g(u, \rho(u,v)Lv)(t) \cdot h(t)}{\langle \nabla_z g(u, \rho(u,v)Lv), Lv\rangle} dt\, d\nu_\eta(v) \\
&= -\int\limits_0^T \int\limits_{v\in F(u)} \frac{\chi(\rho(u,v)) \cdot \nabla_u g(u, \rho(u,v)Lv)(t) \cdot h(t)}{\langle \nabla_z g(u, \rho(u,v)Lv), Lv\rangle} d\nu_\eta(v) dt \\
&= \left\langle -\int\limits_{v\in F(u)} \frac{\chi(\rho(u,v)) \cdot \nabla_u g(u, \rho(u,v)Lv)}{\langle \nabla_z g(u, \rho(u,v)Lv), Lv\rangle} d\nu_\eta(v), h \right\rangle.
\end{aligned}$$

This yields the asserted formula of our Theorem.

We recall that the functions $c_n^{(1)}, c_n^{(2)}$ occuring in the formulae (18), (19) are defined below problem $(\mathbf{NECU})(\varepsilon, p)$. In particular, one calculates their gradients occuring in (18) as

$$\nabla c_n^{(1)}(u)(s) = \frac{c}{\sqrt{\lambda_n}} \varphi_n(1) \sin\left(\sqrt{\lambda_n}\, c\,(T-s)\right) \quad (s \in (0,T)) \tag{20}$$

$$\nabla c_n^{(2)}(u)(s) = c^2 \varphi_n(1) \cos\left(\sqrt{\lambda_n}\, c\,(T-s)\right) \quad (s \in (0,T)) \tag{21}$$

The results presented in this section suggest an algorithmic scheme for determining approximations of the probability function $\phi_N(u) = \mathbb{P}\left(\, E_N^\omega(u) \leq \varepsilon \,\right)$ and its gradients $\nabla\phi_N(u)$ in the iterative solution of our optimization problem $(\mathbf{NECU})(\varepsilon, p, N)$ at a given iterate $u \in L^2(0,T)$. The idea is to approximate the gradient $\nabla\phi_N(u)$ by its values on a finite subset $\{t_1, \ldots, t_M\} \subseteq (0,T)$. In this way, a gradient-based solution algorithm for $(\mathbf{NECU})(\varepsilon, p, N)$ (e.g., projected gradients) is easily set up. Observe that, thanks to Corollary 2 and Theorem 5, both the value $\phi_N(u)$ and its gradient $\nabla\phi_N(u)$ are represented as spherical integrals (in the latter case one reduces the sphere to its subset $v \in F(u)$ by a simple check of $\rho(u,v) < \infty$). Hence, the same sample $v \in \mathbb{S}^{m-1}$ can be used in order to update both $\phi_N(u)$ and $\nabla\phi_N(u)$. Here, one takes advantage of the fact, that the value $\rho(u,v)$ has only to be determined once. The following algorithm assigns to a given iterate $u \in L^2(0,T)$ approximations for $\phi_N(u)$ and for $\nabla\phi_N(u)$ on a given grid $\{t_1, \ldots, t_M\}$ under the given Gaussian distribution $(a_n, b_n)_{n=1}^N \sim \mathcal{N}(\mu, \Sigma)$ of the coefficients in (2):

Algorithm 5.1 *1 Generate a (Quasi Monte-Carlo) sample $v^1,\dots,v^K$ of the m-dimensional standard Gaussian distribution $\mathcal{N}(0,I)$.*

2 Find a Cholesky decomposition $\Sigma = LL^T$ for the covariance matrix of the given distribution of coefficients.

3 Initialise the desired approximation s_v for $\phi_N(u)$ and s_g for $\nabla\phi_N(u)$ on the grid $\{t_1,\dots,t_M\}$ by $s_v := 0$ and $s_g(t_i) := 0$ for $i = 1,\dots,M$. Initialise the iteration counter for the sample in Step 1. as $k := 1$.

4 For the given iterate u and the sampled direction $v := v^k \in \mathbb{S}$, check for and identify solution of (15). Compute values $\alpha\left(u,v^k\right)$, $\beta\left(u,v^k\right)$, $\gamma\left(u,v^k\right)$ as indicated below (15). Update the contribution of sample v^k to the discretized versions of the spherical integrals (12) and (17) according to the case distinction made above:

(a) If $\alpha\left(u,v^k\right) = 0$ and $\beta\left(u,v^k\right) \le 0$ (i.e., (15) has no solution, whence $e\left(u,v^k\right) = 1$ and $v^k \notin F(u)$), then update $s_v := s_v + 1$.

(b) If $\alpha\left(u,v^k\right) = 0$ and $\beta\left(u,v^k\right) > 0$, then put

$$\rho\left(u,v^k\right) := -\frac{\gamma\left(u,v^k\right)}{\beta\left(u,v^k\right)}.$$

(c) If $\alpha\left(u,v^k\right) > 0$, then put

$$\rho\left(u,v^k\right) := \frac{-\beta\left(u,v^k\right) + \sqrt{\beta^2\left(u,v^k\right) - 4\gamma\left(u,v^k\right)\alpha\left(u,v^k\right)}}{2\alpha\left(u,v^k\right)}.$$

In both cases (b) and (c), $v^k \in F(u)$. Update $s_v := s_v + F_{\nu_\chi}\left(\rho\left(u,v^k\right)\right)$ and (see (17)):

$$s_g(t_i) := s_g(t_i) + \frac{\chi\left(\rho\left(u,v^k\right)\right)\cdot\nabla_u g\left(u,\rho\left(u,v^k\right)Lv^k\right)(t_i)}{\left\langle\nabla_z g\left(u,\rho\left(u,v^k\right)Lv^k\right),Lv^k\right\rangle}\quad (i = 1,\dots,M).$$

In this last formula, use the representations (18), (19). Referring to (20) and (21), we obtain, for instance, the following fully explicit representation for $\nabla_u g$, required above:

$$\begin{gathered}\nabla_u g\left(u,\rho\left(u,v^k\right)Lv^k\right)(t_i) = \\ 2c\sum_{n=1}^{N}\left(\tfrac{c\varphi_n(1)}{\sqrt{\lambda_n}}\int_0^T u(s)\sin\left(\sqrt{\lambda_n}\,c\,(T-s)\right)\,ds + \left\langle A_n,\rho\left(u,v^k\right)Lv^k\right\rangle\right)\cdot \\ \sqrt{\lambda_n}\,\varphi_n(1)\,\sin\left(\sqrt{\lambda_n}\,c\,(T-t_i)\right) + \\ 2\sum_{n=1}^{N}\left((c^2\,\varphi_n(1)\int_0^T u(s)\cos\left(\sqrt{\lambda_n}\,c\,(T-s)\right)\,ds + \left\langle B_n,\rho\left(u,v^k\right)Lv^k\right\rangle\right)\cdot \\ \varphi_n(1)\,\cos\left(\sqrt{\lambda_n}\,c\,(T-t_i)\right)\end{gathered}\quad .$$

5 If $k < K$, then $k := k+1$ and goto 4.

6 STOP with $\phi_N(u) \approx K^{-1}s_v$ and $\nabla\phi_N(u)(t_i) \approx K^{-1}s_g(t_i)$ for $i = 1,\dots,M$.

6 Existence of Solutions

In this section first we show that for all natural numbers $N \in \{1, 2, 3, ...\}$ and $\varepsilon > 0$ a solution of $(\mathbf{NECU})(\varepsilon,\, p,\, N)$ exists if p is sufficiently small. For a given $\varepsilon > 0$, we set

$$p_{\mathsf{sup}}(\varepsilon, N) := \sup \left\{ p \in [0,1] \mid \exists u \in L^2(0,T) \text{ s.t. } \mathbb{P}\left(\, E_N^{\omega}(u,\, T) \leq \varepsilon \,\right) \geq p \right\},$$

which is decreasing with respect to N. Hence, for all $p < p_{\mathsf{sup}}(\varepsilon,\, N)$, there exists $u \in L^2(0,\, T)$ such that $\mathbb{P}\left(\, E_N^{\omega}(u,\, T) \leq \varepsilon \,\right) \geq p$, which means that the feasible set of the optimal control problem $(\mathbf{NECU})(\varepsilon,\, p,\, N)$ is nonempty. This fact is used to prove the following lemma, that ensures the existence of a unique solution to $(\mathbf{NECU})(\varepsilon,\, p,\, N)$.

Lemma 2 *Let $N \in \{1, 2, 3, ...\}$ be given. Assume that $p \in [0,\, p_{sup}(\varepsilon,\, N))$. Then $(\mathbf{NECU})(\varepsilon,\, p,\, N)$ has a unique solution $u \in L^2(0,\, T)$.*

Proof 4 *With the convex function h_N as defined in (9) we can state problem $(\mathbf{NECU})(\varepsilon,\, p,\, N)$ in the form*

$$\min_{u \in L^2(0,T)} \|u\|^2_{L^2(0,T)} \text{ subject to } h_N(u) \leq 0.$$

Then the Direct Method of the Calculus of Variations yields the solution as the weak limit point of a minimizing sequence. The strong convexity of the objective function implies the uniqueness.

In preparation of the following lemma, let $\nu(\varepsilon,\, p,\, N)$ and $\nu(\varepsilon,\, p)$ denote the optimal value of $(\mathbf{NECU})(\varepsilon,\, p,\, N)$ and $(\mathbf{NECU})(\varepsilon,\, p)$, respectively.

Lemma 3 *Assume that $p \in [0,\, \inf_N p_{sup}(\varepsilon,\, N))$ and $\nu(\varepsilon,\, p) < \infty$. Then the sequence of solutions $u^*(\varepsilon,\, p,\, N)$ of $(\mathbf{NECU})(\varepsilon,\, p,\, N)$ ($N \in \{1, 2, 3, ...\}$) contains a subsequence that converges strongly in $L^2(0,\, T)$.*

Proof 5 *Since $E_N^{\omega}(u,\, T) \leq E_{N+1}^{\omega}(u,\, T) \leq \ldots \leq E^{\omega}(u,\, T)$ we have*

$$\nu(\varepsilon,\, p,\, N) \leq \nu(\varepsilon,\, p) \text{ for all } N \in \{1, 2, 3, ...\}.$$

Moreover, the sequence $(\nu(\varepsilon,\, p,\, N))_{N \in \mathbb{N}}$ is increasing. Let $\tilde{L} := \lim_{N \to \infty} \nu(\varepsilon,\, p,\, N)$. The assumption $\nu(\varepsilon,\, p) < \infty$ implies that the sequence $(u^*(\varepsilon,\, p,\, N))_{N \in \mathbb{N}}$ is bounded. Hence there exists a weakly convergent subsequence with a weak limit $\tilde{u}$. We have $\tilde{L} \geq \|\tilde{u}\|^2_{L^2(0,\, L)}$. For all $N \in \{1, 2, 3, ...\}$, we have for all $M \geq N$,

$$h_N(u^*(\varepsilon,\, p,\, M)) \leq 0.$$

Since h_N is sequentially weakly lower semi-continuous, see [9, Proposition 1], we obtain for all $N \in \{1, 2, 3, ...\}$

$$h_N(\tilde{u}) \leq \liminf_{M \to \infty} h_N(u^*(\varepsilon,\, p,\, M)) \leq 0.$$

This implies that $\tilde{u}$ is feasible for $(\mathbf{NECU})(\varepsilon,\, p,\, N)$ for all $N \in \{1, 2, 3, ...\}$. But, $\nu(\varepsilon,\, p,\, N) \leq \|\tilde{u}\|^2_{L^2(0,\, T)}$ and hence $\tilde{L} \leq \|\tilde{u}\|^2_{L^2(0,\, T)}$. Thus we have $\tilde{L} = \|\tilde{u}\|^2_{L^2(0,\, T)}$. This implies in turn the strong convergence of the subsequence in $L^2(0,\, T)$ to $\tilde{u}$.

In the following Theorem we show that each strong limit point $\tilde{u}$ of the sequence of solutions $(u^*(\varepsilon,\, p,\, N))_{N\in\mathbb{N}}$ of the problems $(\mathbf{NECU})(\varepsilon,\, p, N)$ is feasible for $(\mathbf{NECU})(\varepsilon,\, p)$. In particular, this implies that a solution of $(\mathbf{NECU})(\varepsilon,\, p)$ exists.

Theorem 6 *Under the assumptions of Lemma 3 and under our standing assumption (**A**), $(\mathbf{NECU})(\varepsilon,\, p)$ has a solution $\tilde{u}$ and*

$$\lim_{N\to\infty} \|u^*(\varepsilon,\, p,\, N) - \tilde{u}\|_{L^2(0,T)} = 0,$$

where $(u^*(\varepsilon,\, p,\, N))_{N\in\mathbb{N}}$ is given by Lemma 3. Moreover we have

$$\lim_{N\to\infty} \nu(\varepsilon,\, p,\, N) = \nu(\varepsilon,\, p). \tag{22}$$

Proof 6 *Let $\tilde{u}$ be as in the proof of Lemma 3. Note that we have $\|\tilde{u}\|^2_{L^2(0,T)} \le \nu(\varepsilon,\, p)$. Since $\tilde{u}$ is feasible for $(\mathbf{NECU})(\varepsilon,\, p,\, N)$ for all $N \in \{1, 2, 3, ...\}$, we have for all $N \in \{1, 2, 3, ...\}$:*

$$\mathbb{P}\left(E^\omega_N(\tilde{u},\, T) \le \varepsilon\,\right) \ge p. \tag{23}$$

*Define the random variable $\delta^\omega_N := E^\omega(\tilde{u},\, T) - E^\omega_N(\tilde{u},\, T) \le 0$. We obtain that $\delta^\omega_{N+1} \le \delta^\omega_N$. Moreover, as a consequence of our standing assumption (**A**), Lemma 4.3 from [18] implies that almost surely we have $\sum_{n=1}^\infty \lambda_n \left(c^{(1)}_n(u) + \sigma^\omega_n(T)\right)^2 < \infty$ and $\sum_{n=1}^\infty \left(c^{(2)}_n(u) + (\sigma^\omega_n)'(T)\right)^2 < \infty$. Hence almost surely $\lim\limits_{N\to\infty} \delta^\omega_N = 0$. Moreover, we have*

$$\mathbb{P}(E^\omega_N(\tilde{u},\, T) \le \varepsilon) = \mathbb{P}(E^\omega(\tilde{u}) \le \varepsilon + \delta^\omega_N).$$

Consider the sets

$$S_N := \{\omega \in \Omega \mid E^\omega(\tilde{u},\, T) \le \varepsilon + \delta^\omega_N\} \text{ and } \cap_{N=1}^\infty S_N = \{\omega \in \Omega \mid E^\omega(\tilde{u}) \le \varepsilon\}.$$

Then we have $S_{N+1} \subset S_N$ and due to (23) we have $\mathbb{P}(S_N) \ge p$. Define the set

$$S := \{\omega \in \Omega \mid E^\omega(\tilde{u},\, T) \le \varepsilon\} \cup \{\omega \in \Omega \mid \lim_{N\to\infty} \delta^\omega_N \ne 0\},$$

where $\mathbb{P}(\{\omega \in \Omega \mid \lim\limits_{N\to\infty} \delta^\omega_N \ne 0\}) = 0$. Then $\mathbb{P}(S) = \lim\limits_{N\to\infty} \mathbb{P}(S_N)$. Thus we have shown

$$\mathbb{P}(E^\omega(\tilde{u}) \le \varepsilon) \ge p.$$

Hence $\tilde{u}$ is feasible for $(\mathbf{NECU})(\varepsilon,\, p)$, and hence also a solution of $(\mathbf{NECU})(\varepsilon,\, p)$. Since the arguments in the proof of Lemma 3 and Lemma 6 can be applied to any weak limit point of the sequence $(u^*(\varepsilon,\, p,\, N))_{N\in\mathbb{N}}$, this implies that the solution of $(\mathbf{NECU})(\varepsilon,\, p)$ is the strong limit of the sequence $(u^*(\varepsilon,\, p,\, N))_{N\in\mathbb{N}}$. Moreover we have (22).

7 Numerical solution of two examples

In this section, we discuss the numerical solution of two examples with different expected initial conditions for problem $(\mathbf{NECU})(\varepsilon,\, p)$. Referring to the explicit description of that problem as well as to (2) and (1), we consider the following problem data:

$$\begin{aligned} c &:= 1;\ \varepsilon := 0.1;\ T := 4;\ L := 1;\ p \in \{0.10, 0.15, 0.20\ldots, p^{\max}\} \\ a_n &\sim \mathcal{N}(1, 0.2)\ (n\in\mathbb{N});\ a_n \text{ pairwise uncorrelated; } b_n := 0\ (n\in\mathbb{N}); \\ y_0(x) &:= x \text{ (example 1); } y_0(x) := \pi^{-1}\sin(\pi x) \text{ (example 2); } y_1(x) := 0 \quad (x\in(0,1)) \end{aligned}$$

The coefficients α_n^0 and α_n^1 in (2) and (1), respectively are obtained as the Fourier coefficients of the chosen functions $y_0(x), y_1(x)$. In particular, $\alpha_n^1 = 0$ for all n. The latter implies that the (formal) multiplicative random coefficients b_n for perturbing $y_1(x)$ can be chosen arbitrarily without any effect. As for the coefficients a_n, they all follow an identical Gaussian distribution (with mean 1 and standard deviation 0.2) in order to satisfy our standing assumption (**A**). This allows to apply all the existence and convergence results of Section 6 to our examples. However, as pointed out earlier, this assumption is not necessary for the numerical solution of the approximating problem $(\mathbf{NECU})(\varepsilon, p)$. Moreover, assuming all coefficients to be pairwise uncorrelated is of absolutely no importance (recall Algorithm 5.1 allowing for correlated components of the Gaussian random vector) neither for the theory nor for the numerical solution and is just due to a lack of significant information about correlations here. With the a_n having expectation 1 it is ensured that the expected initial value coincides with the nominal one, i.e., $\mathbb{E}y_0^\omega(x) = y_0(x)$ for all $x \in (0,1)$.

In order to deal with $(\mathbf{NECU})(\varepsilon, p)$ numerically, one has to pass to finite-dimensional approximations on two sides simultaneously: first, the series expansion for the terminal energy has to be cut after N terms which leads us to the consideration of problem $(\mathbf{NECU})(\varepsilon, p, N)$. Second, we compute approximations for the optimal controls $u \in L^2(0, T)$ in the space of piecewise constant functions. Let a grid $0 = t_0 < t_1 < t_2 < \ldots < t_M = T$ be given. For $i \in \{1, \ldots, M\}$ let

$$v_j(t) := \begin{cases} 1 & \text{if } t \in [t_{j-1}, t_j), \\ 0 & \text{elsewhere,} \end{cases}$$

and, define the finite dimensional space $X_M(T)$ by

$$X_M(T) := \operatorname{span}\{v_j(\cdot) : j = 1, \ldots, M\}.$$

For any $u \in X_M(T)$ we use the representation

$$u(t) = \sum_{j=1}^{M} u(t_{j-1}) v_j(t) \qquad t \in [0, T).$$

Hence, we are finally led to solve the problem

$$(\mathbf{NECU})(\varepsilon, p, N, M) \begin{cases} \min_{u \in X_M(T)} \sum_{j=1}^{M} (t_j - t_{j-1})\, u(t_{j-1})^2 \text{ subject to} \\ \phi_N(u) \geq p. \end{cases}$$

The solution will be based on a projected gradient method using values and gradients of ϕ_N computed by means of Algorithm 5.1. Apart from explicit expressions occuring in these computations, we have to specify (see (17),(18),(19)) the integrals defining the functions $c_n^1(u), c_n^2(u)$ introduced below problem $(\mathbf{NECU})(\varepsilon, p)$. By elementary calculus, one obtains that for $u \in X_M(T)$ and for all $n \in \mathbb{N}$

$$\int_0^T u(s) \sin\left(\sqrt{\lambda_n}(T-s)\right)\, ds =$$
$$\frac{1}{\sqrt{\lambda_n}} \left[-\cos\left(\sqrt{\lambda_n} T\right) u(t_0) + \sum_{j=1}^{M-1} (u(t_{j-1}) - u(t_j)) \cos\left(\sqrt{\lambda_n}(T - t_j)\right) + u(t_M) \right],$$
$$\int_0^T u(s) \cos\left(\sqrt{\lambda_n}(T-s)\right)\, ds =$$
$$\frac{1}{\sqrt{\lambda_n}} \left[\sin\left(\sqrt{\lambda_n} T\right) u(t_0) - \sum_{j=1}^{M-1} (u(t_{j-1}) - u(t_j)) \sin\left(\sqrt{\lambda_n}(T - t_j)\right) \right].$$

7.1 First example

Here, we assume that the nominal (unperturbed, expected) initial state is given by $y_0(x) = x$. With the concrete problem data as specified above, it follows from Theorem 2 that the optimal deterministic control in problem $(\mathbf{NEC})$ is the bang-bang control

$$u^*(t) = 1/4 \quad \forall t \in (0,2); \quad u^*(t) = -1/4 \quad \forall t \in (2,4). \tag{24}$$

This solution takes a nominal deterministic initial state $y_0(x) = x$ and $y_1(x) = 0$ to a position of rest, i.e. a terminal state with zero energy, within the time $T = 4$. In contrast to the deterministic case, for uncertain initial values this same optimal control will no longer take the string to rest (apart from the unlikely event that the uncertain initial value coincides with the nominal or expected initial value). Instead, we search for a control which takes the initial state with a certain sufficiently high probability to a terminal state with an energy level below the chosen value of $\varepsilon = 0.1$. In order to approximate the theoretical solution of problem $(\mathbf{NECU})(\varepsilon, p)$, we solved problem $(\mathbf{NECU})(\varepsilon, p, N, M)$ for $N = 100$ and $M = 256$ with the probability level p varying from $p = 0.1$ to the maximum possible value $p = 0.9078$ in steps of 0.05. The corresponding optimal controls are illustrated in Fig. 7.1.

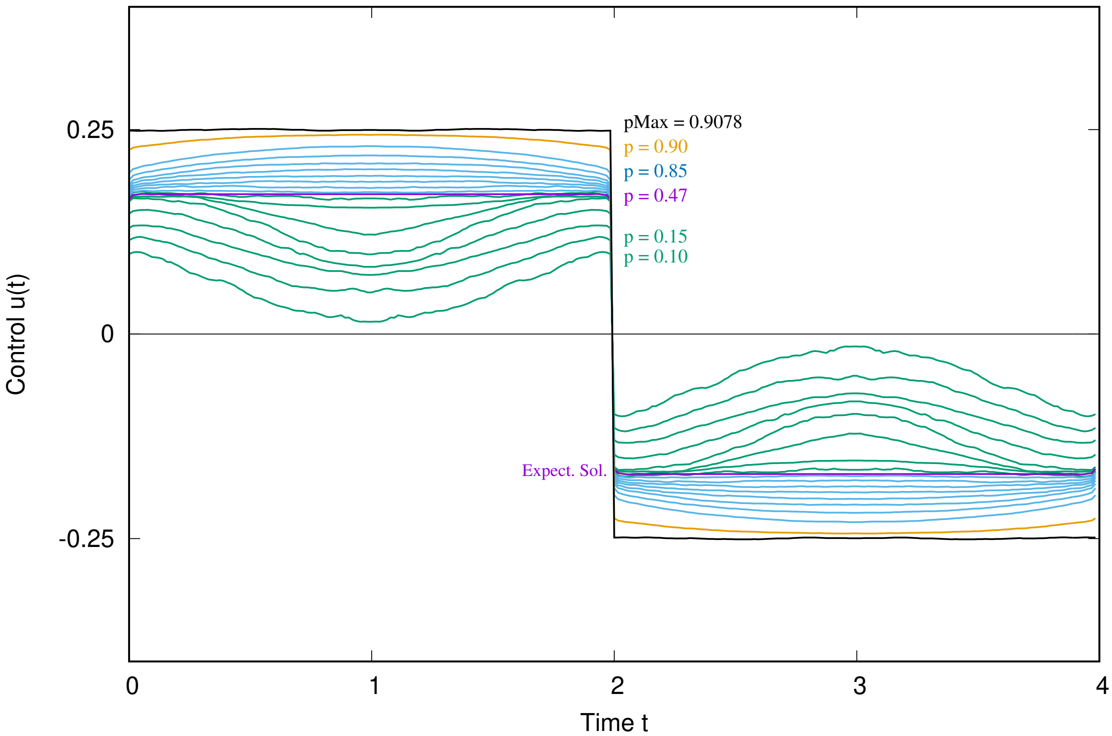

Figure 1: The figure shows the optimal controls under probabilistic terminal energy constraint as solutions of problem $(\mathbf{NECU})(\varepsilon, p, N, M)$ for a tolerance of $\varepsilon = 0.1$ and for different probability levels p. The two bang-bang controls in the figure refer to the optimal solution of the deterministic problem $(\mathbf{NEC})$ (zero terminal energy) (larger amplitude) and to the solution of the expected value counterpart of problem $(\mathbf{NECU})(\varepsilon, p, N, M)$ (lower amplitude, see text).

Not surprisingly, unlike the piecewise constant deterministic solution (24) of problem $(\mathbf{NEC})$ (largest control in Fig. 7.1) these controls are quite nonlinear due to the presence of the probabilistic constraint and they are increasing in amplitude for increasing probability level p. Moreover, they exhibit the same symmetry patterns as the deterministic controls. More surprisingly, the profiles change their shape from bimodal to concave when passing a certain medium probability level $p \approx 0.47$. At that probability, the optimal control is bang-bang again but with lower amplitude than the solution of $(\mathbf{NEC})$. It turns out that this is the solution of problem $(\mathbf{NECU})(\varepsilon,\, p,\, N,\, M)$ when replacing the probabilistic constraint $\phi_N(u) \geq p$ by the constraint

$$\int_0^L y_x(T,x)^2 + \frac{1}{c^2}\, y_t(T,x)^2 dx \leq \varepsilon, \tag{25}$$

which bounds the terminal energy of the nominal (expected) initial state by ε (whence the control can afford a lower amplitude than that of $(\mathbf{NEC})$ which would correspond to the stricter bound $\varepsilon = 0$).

Even less evident is the fact that the largest possible probability $p = 0.9078$ is achieved again by a bang-bang solution which is exactly the one of the deterministic problem $(\mathbf{NEC})$ imposing zero terminal energy while starting with the nominal initial state. Beyond that maximum probability, the feasible set of $(\mathbf{NECU})(\varepsilon,\, p,\, N,\, M)$ becomes empty and costs jump to infinity.

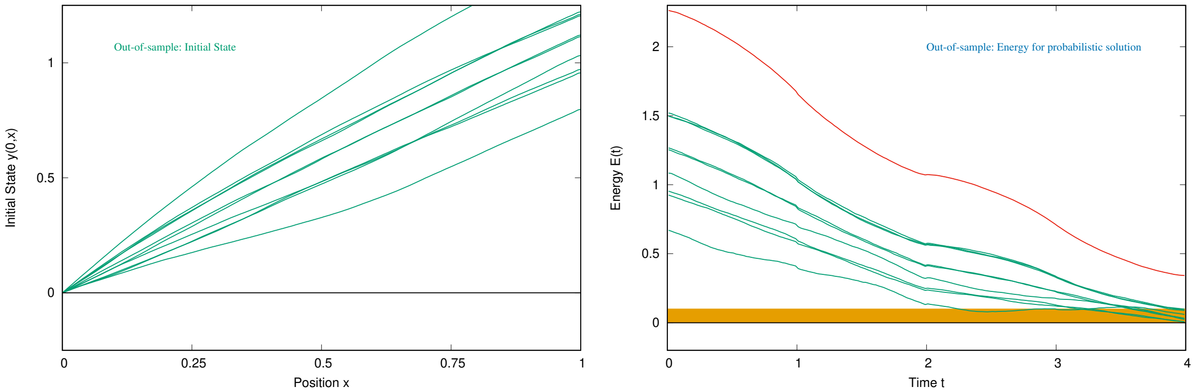

Figure 2: Illustration of ten simulated scenarios for the initial state (left diagram) and corresponding evolution of energy over time.

In order to illustrate the effect of the calculated solutions, we simulate a sample of 10 random scenarios for the initial state around the nominal (expected) initial state $y_0(x) = x$ according to the chosen Gaussian distribution of the multiplicative perturbations a_n of the nominal Fourier coefficients (see problem data above). They are illustrated in the left diagram of Fig. 2. Taking the optimal control for the (still feasible) probability level $p = 0.9$ (see Fig. 7.1) and applying it to these ten scenarios for the initial state yields a time-dependent development of the corresponding energy as illustrated in the right diagram of Fig. 2. At terminal time $T = 4$, nine out of these ten scenarios reach a terminal energy within the ε band around zero. This is in expected correspondence with the chosen probability 0.9 (of course, slight deviations could occur when repeating the simulation). Note however, that this is just an out-of-sample test (posterior test) and that the computation of optimal controls has not been based on simulated scenarios but on the parameters of the underlying continuous multivariate distribution.

7.2 Second example

We repeat the numerical experiment with the nominal initial state $y_0(x) = \pi^{-1} \sin(\pi x)$. The according optimal controls are illustrated in Fig. 3. Except monotonicity of profiles with respect to the probability level, we detect similar effects as in the previous example: again, the family of profiles passes, when increasing the probability level, through the solution of problem $(\mathbf{NECU})(\varepsilon,\, p,\, N,\, M)$ when replacing the probabilistic constraint $\phi_N(u) \geq p$ by the constraint (25) (terminal energy of expected

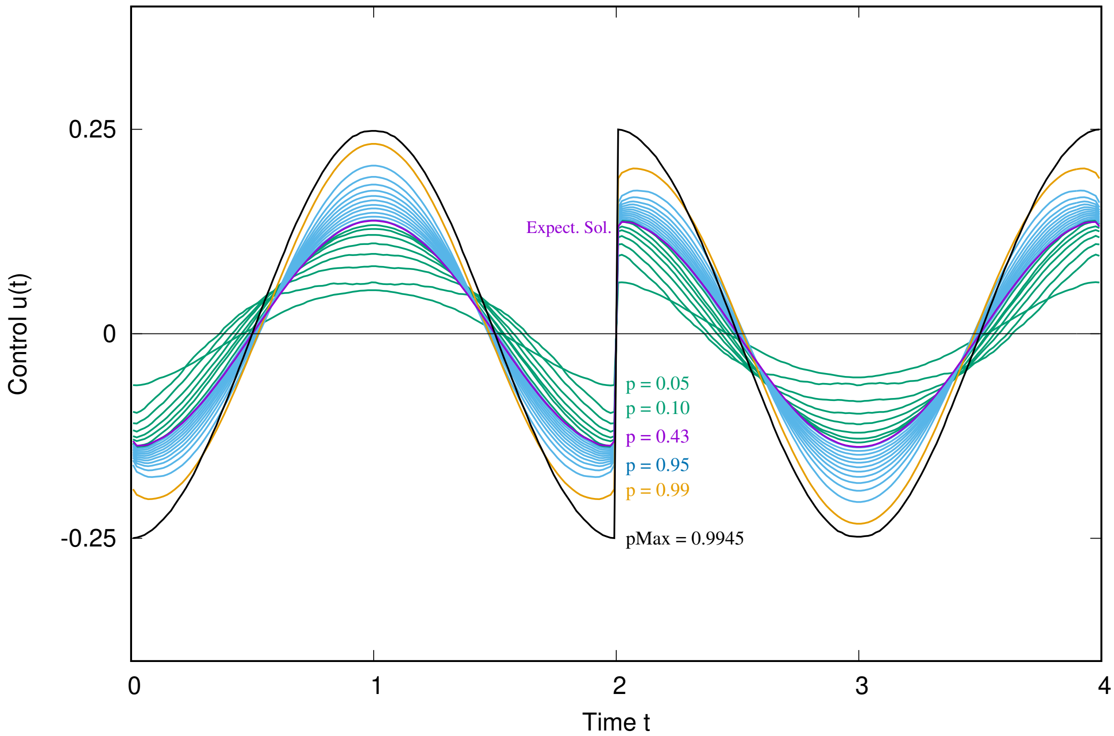

Figure 3: The meaning of the Figure is analogous to that in the previous example but now for a problem with different initial state

initial state smaller than ε) at $p = 0.43$ and reaches its maximum probability at $p = 0.9945$ when being identical with the deterministic solution of problem $(\mathbf{NEC})$. As this relation between the probabilistic and the two deterministic solutions is repeatedly observed in examples, we strongly believe, without having a proof yet, that it is generally true.

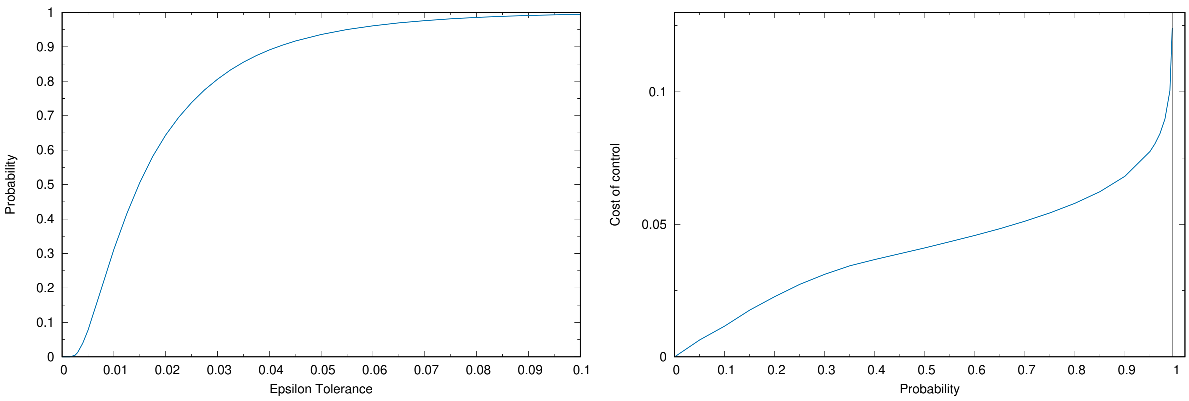

Figure 4: Plot of the maximum achievable probability as a function of the energy tolerance ε (left diagram) and of the cost for the control as a function of the chosen probability level (right diagram)

The left diagram of Fig. 7.2 illustrates the dependence of the maximum achievable probability on the chosen tolerance ε for the terminal energy. Of course, the larger this tolerance, the higher the maximum probability that can be achieved. It turns out that a slight increase of the tolerance from zero to 0.05 already ensures a sufficiently high maximum probability of around 0.9. The right diagram of Fig. 7.2 illustrates the dependence of the cost for the control on the chosen probability level. It can be seen, that the level can be quickly increased up to around 0.8-0.9 at a very moderate increase of costs. However, when approaching the maximum possible probability level, the additional costs are considerable.

8 Conclusions

We have studied optimal control problems with systems governed by the wave equation where the initial state is uncertain. In order to take into account the uncertainty, we have prescribed a probabilistic terminal constraint for the energy of the system. In the probabilistic constraint, an upper bound ε for the energy and a desired probability level p appear as parameters. We have shown that for reasonable choices of these parameters, optimal controls exist that solve the optimal control problems with a probabilistic terminal constraint. Examples illustrate that the optimal controls can also be approximated numerically.

Since the uncertainty of the initial states occurs for many optimal control problems, it is also interesting to study this type of problem for more complex nonlinear dynamics (see for example [5]). Also the analysis of uncertain non-linearities that occur as disturbance of the pde similar as in [20] is of interest. This will be a topic in future research.

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