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Global–in–time existence for liquid mixtures subject to a generalised incompressibility constraint

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Abstract

We consider a system of partial differential equations describing diffusive and convective mass transport in a fluid mixture of $N > 1$ chemical species. A weighted sum of the partial mass densities of the chemical species is assumed to be constant, which expresses the incompressibility of the fluid, while accounting for different reference sizes of the involved molecules. This condition is different from the usual assumption of a constant total mass density, and it leads in particular to a non-solenoidal velocity field in the Navier-Stokes equations. In turn, the pressure gradient occurs in the diffusion fluxes, so that the PDE–system of mass transport equations and momentum balance is fully coupled. Another striking feature of such incompressible *mixtures* is the algebraic formula connecting the pressure and the densities, which can be exploited to prove a pressure bound in L^1. In this paper, we consider incompressible initial states with bounded energy and show the global existence of weak solutions with defect measure.

1 Introduction

The paper is devoted to the weak solutions analysis of recent models for the diffusive mass transport in multicomponent *incompressible* fluids. Consider a molecular mixture, assumed in a fluid phase, of $N \in \mathbb{N}$ chemical substances $\mathrm{A}_1, \ldots, \mathrm{A}_\mathrm{N}$. The temperature $\theta > 0$ of the physical system is assumed constant. The mass densities of the chemical species *in the mixture* are denoted $\rho_1, \ldots, \rho_N$. Emphasis is to be put on the fact that a specific incompressibility constraint, generalising the assumption of a constant mass density considered in other analytical investigations (a. o. [CJ15], [MT15], [BP17]), shall be investigated here:

$$\sum_{i=1}^{N} \rho_i \, \bar{V}_i = 1 \, , \tag{1}$$

where $\bar{V}_1, \ldots, \bar{V}_N > 0$ are specific volumes of the molecules at reference temperature and pressure. Only in the very special case of constituents that at pressure p and temperature θ would occupy in equilibrium the same volume, it is possible to assume that (1) is equivalent with the total mass of the mixture being constant. In this paper we are therefore interested only in the case that at least two indexes exist such that $\bar{V}_{i_1} \neq \bar{V}_{i_2}$, or in vectorial notation, that for all $\lambda \in \mathbb{R}$

$$\bar{V} \neq \lambda \, 1^N \, , \tag{2}$$

where $\bar{V} = (\bar{V}_1, \, \bar{V}_2, \ldots, \bar{V}_N)$ and $1^N = (1, \, 1, \ldots, 1) \in \mathbb{R}^N$.

Though it remains linear, the relation (1) yields essential differences in the structure of the mathematical problem, and in the analysis of the equations of momentum balance, as we shall see. Let us note that the relation (1) is a correct description of volume effects for the wide class of mixtures that

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are approximately ideal, meaning that the mixing provokes no essential loss or gain of volume (see [Mil66, JHH96, BD15, DNB$^+$15] for more insights into the model).

The PDE model describes the mass transport by convection and diffusion, and the momentum balance. The barycentric velocity of the mixture is called v, the thermodynamic pressure is p, and these variables are assumed obeying

$$\partial_t\rho_i + \operatorname{div}(\rho_i\, v + J^i) = 0 \text{ for } i = 1, \ldots, N \tag{3}$$

$$\partial_t(\varrho\, v) + \operatorname{div}(\varrho\,(v\otimes v) - \mathbb{S}(\nabla v)) + \nabla p = 0\,. \tag{4}$$

The total mass is related to the partial mass densities via $\varrho := \sum_{i=1}^N \rho_i$. The diffusion fluxes $J^1, \ldots, J^N$ satisfy by definition the side condition $\sum_{i=1}^N J^i = 0$, which entails the conservation of the total mass ϱ: $\partial_t\varrho + \operatorname{div}(\varrho\, v) = 0$. A thermodynamic consistent Fickean closure respecting this constraint is assumed. This approach is described in great generality among others by [BD15], [DGM13], [Guh14] following older ideas by [MR59], [dM63]. The diffusions fluxes $J^1, \ldots, J^N$ then obey

$$J^i = -\sum_{j=1}^N M_{i,j}(\rho_1, \ldots, \rho_N)\, \nabla\mu_j \text{ for } i = 1, \ldots, N\,. \tag{5}$$

The *Onsager matrix* $M(\rho_1, \ldots, \rho_N)$ is a *symmetric*, positive semi-definite $N\times N$ matrix for every $(\rho_1, \ldots, \rho_N) \in \mathbb{R}^N_+$. Due to the vanishing of the net diffusion flux, the consistent Onsager closure is usually performed with a kinetic matrix M satisfying

$$\sum_{j=1}^N M_{i,j}(\rho_1, \ldots, \rho_N) = 0 \text{ for all } (\rho_1, \ldots, \rho_N) \in \mathbb{R}^N_+ \tag{6}$$

which guaranties that the necessary side-condition $\sum_{i=1}^N J^i = 0$ is satisfied. Note that the closure by means of the kinetic Maxwell–Stefan theory exhibits after inversion the same structure.[1]

The quantities $\mu_1, \ldots, \mu_N$ are the *chemical potentials* from which the thermodynamic driving forces for the diffusion phenomena are inferred. They are related to the mass densities $\rho_1, \ldots, \rho_N$ and to the pressure via

$$\mu_i = \bar{V}^i\, p + \partial_{\rho_i} k(\theta,\, \rho_1, \ldots, \rho_N)\,. \tag{7}$$

In this paper, the chemical contribution to the free energy k is specified to be:

$$k(\theta,\, \rho) = \sum_{i=1}^N \mu_i^{\text{ref}}\, \rho_i + k_B\,\theta \sum_{i=1}^N n_i\, \ln y_i \tag{8}$$

where $n_i := \rho_i/m_i$ are the number densities ($m_1, \ldots, m_N > 0$ are the molecular masses), $y_i = n_i/\sum_{j=1}^N n_j$ are the number fractions, and μ_i^{ref} are reference values of the chemical potentials. More general structures in (8) are of strong interest, but here we first want to overcome the technical problems with the PDE analysis, and will consider them in further publications.

In the case that the vector of the reference volumes $\bar{V}$ is not parallel to $1^N = (1,\, 1, \ldots, 1) \in \mathbb{R}^N$ (cf. (2)), which occurs for most of the real mixtures, then if we multiply the equations (3) with the constants $\bar{V}_i$ and sum up, we obtain for the local change of volume the equation

$$\operatorname{div} v = -\operatorname{div}(\sum_{i=1}^N \bar{V}_i\, J^i)\,. \tag{9}$$

[1] Private communication by D. Bothe

A net local change of volume is possible, not because of mechanical compression, but because other effects like diffusion induce a local change in the molecular composition.

The intrinsic relationship between (1), (7) and the pressure is contained in the isothermal Gibbs-Duhem equation: $dp = \sum_{i=1}^N \rho_i \, d\mu_i$. To anticipate on more systematic explanations in the paper [BDa], note that the relation (7) occurs in the limit case that the bulk free energy density of the system adopts the singular form

$$h^{\infty}(\theta,\, \rho) := \begin{cases} k(\theta,\, \rho) & \text{if } \sum_{i=1}^N \rho_i \, \bar{V}_i = 1 \\ +\infty & \text{otherwise} \end{cases} . \tag{10}$$

The relation (7) is the equivalent expression of $\mu \in \partial h^{\infty}(\theta,\, \rho)$ where ∂ denote the sub-differential of the convex function $h^{\infty}(\theta, \cdot)$, and the function p can be understood as a Kuhn–Tucker vector associated with the constraint. The reader interested in the justification of the model can also, in heuristic spirit, follow at first the intention in [BD15], Section 16 (see also [DGM13], [DGL14]) where (1) and (7) were derived passing to the formal limit $K \to +\infty$ ($K =$ a constant compression modulus of the mixture) in a compressible mixture theory.

At last, we note that for the sake of technical simplicity we shall restrict in (4) to a Newtonian form of the stress tensor $\mathbb{S}(\nabla v) := \eta \, (D(v) - \frac{2}{3} \, \mathrm{div}\, v \, \mathrm{Id}) + \lambda \, \mathrm{div}\, v \, \mathrm{Id}$, where $D(v) = \nabla v + (\nabla v)^T$ and $\eta,\, \lambda$ are positive constants.

Remark on boundary conditions. We investigate the equations (3), (4) in a cylindrical domain $Q_T := \Omega \times]0, T[$ where T is a finite time and $\Omega \subset \mathbb{R}^3$ a bounded domain. It is possible to treat also the case $\Omega \subset \mathbb{R}^d$ for general $d \geq 2$ with similar methods, but we restrict here to the physical dimension $d = 3$.

We consider initial conditions for the mass densities and the velocity:

$$\rho_i(x,\, 0) = \rho_i^0(x) \text{ for } x \in \Omega,\; i = 1, \ldots, N \tag{11}$$

$$v_j(x,\, 0) = v_j^0(x) \text{ for } x \in \Omega,\; j = 1, 2, 3\,. \tag{12}$$

For simplicity, we consider only linear boundary conditions on the lateral surface $S_T := \partial\Omega \times]0, T[$

$$v = 0 \qquad \text{on } S_T \tag{13}$$

$$\nu \cdot J^i = M^{\Gamma}_{i,j}(x,\, t) \, (\mu_j - \mu^{\Gamma}_j) \text{ on } S_T \text{ for } i = 1, \ldots, N\,. \tag{14}$$

In the latter condition, we make the assumption that M^{Γ} is a map from S_T into the positive semi-definite, symmetric $N \times N$ matrices of rank $N-1$ satisfying like in (6)

$$\sum_{i=1}^N M^{\Gamma}_{i,j}(x,\, t) = 0 \text{ for all } j = 1, \ldots, N, \quad (x,\, t) \in S_T\,.$$

The vector μ^{Γ} (surface chemical potential) is given; It reflects the state in adjacent matter. As to the conditions (14), they describe linear adsorption/desorption phenomena at participating boundaries, and turn out convenient for the analysis. At the price of increasing the technicality, we can also treat the case $M^{\Gamma} = 0$ in (14). The case of non-linear boundary effects, like for instance boundary reactions investigated in [DDGG16], leads to however to other problems that shall require further publications.

2 Analysis

State of the art. Multicomponent flow models under an incompressiblity constraint have up to few exceptions not been investigated in mathematical analysis. In [CJ15], the Navier-Stokes equations are considered in connection with multicomponent Maxwell-Stefan diffusion. In [MT15], the same setting is investigated but for a non-isothermal case (heat equation).

In these investigations, the constraint $\varrho =$ Const. (or equivalently $\bar{V} = \lambda_0 \, 1^N$ in (1)) was used as expression of the incompressibility. Summing up for $i = 1, \ldots, N$ in (3), one then obtains $\operatorname{div} v = 0$, and the Navier-Stokes equations simplify to their mono–molecular variant. Therein the pressure p is used as 'Lagrange multiplier' to satisfy the constraint. In fact, this should turn out to be very problematic, since the mathematical theory of the incompressible Navier-Stokes equations does not allow to introduce the pressure as function[2]. The analysis is possible, because the value of p is not necessary to resolve the equations (3)!

In order to understand this point, note that the original variables of the problem (P) are the mass densities $\rho_1, \rho_2, \ldots, \rho_N$, the pressure p and the velocity $v_1, \, v_2, \, v_3$. The algebraic relation (7) serves as definition for derivates, or secondary variables $\mu_1, \ldots, \mu_N$.

In terms of the main variables, the diffusion flux J^i has for $i = 1, \ldots, N$ now has the form

$$J^i = -\sum_{j=1}^N M_{i,j}(\rho) \, \nabla \mu_j = -\sum_{j=1}^N M_{i,j}(\rho) \, \nabla (\bar{V}_j \, p + \partial_{\rho_j} k(\rho)) \, , \tag{15}$$

where the direction $\bar{V} \in \mathbb{R}^N_+$ expresses different volumes of the molecules at equilibrium. If $\bar{V} = \lambda_0 \, 1^N$ then the relation (6) and the Onsager symmetry impose $\sum_{j=1}^N M_{i,j} \, \bar{V}_j = \lambda_0 \sum_{j=1}^N M_{i,j} = 0$, so that the pressure does not contribute to the diffusion flux.

In fact, in both [CJ15] and [MT15] the Navier-Stokes equations can be resolved independently. From this viewpoint, the investigation [HMPW17] devoted to local–in–time well posedness by mechanical equilibrium. ($v = 0$ and $p =$ Const.) also ranges in this line of investigation. In [BP17], the considered incompressibility constraint is $\varrho =$ Const. too, but mass transfer due to phase change is also considered. In [BFS14], [FS17], [CI18] (mixtures with charged carriers) the considered incompressibility constraint is not (1) and the diffusion law being diagonal, it is not compatible with (15).

From our viewpoint in this paper, the condition $\varrho =$ Const. is only a special case, which occurs when all molecules in the mixture have approximately the same volume near the reference temperature and pressure of the physical system. It can also be a valid approximation for a dilute mixture where the properties of the liquid are largely dominated by one substance (see [BS16]).

We remark that much more mathematical context is available concerning compressible mixtures, and to find a. o. in the references: [Gio99], [Bot11] (Modelling), [FPT08], [Zat15], [MPZ15] (Analysis). Let us also mention the recent investigations [FLM16] and [FLN18] devoted to the compressible Navier-Stokes system with a pressure law of 'hard-sphere' type. This means the pressure is infinite outside some bounded interval of $\mathbb{R}_+$. In the latter context, similar problems and techniques as in the present paper arise for estimating the pressure.

At last, we notice that the problem here under consideration is investigated in [BDb] from the point of view of its local–in–time well posedness in classes of strong solutions, with essentially positive answers.

Our approach. In the case that $\bar{V}$ is not parallel to 1^N, then (15) exhibits a coupling between the diffusion fluxes and the pressure. A corollary of this fact is that if we multiply the equations (3) with the constants $\bar{V}_i$ and sum up, we obtain for the local change of volume the equation (9). These are obviously major differences, leading also to a different structure of the fluid dynamical problem. Indeed, if the vectors $\bar{V}$ and 1^N are not parallel in $\mathbb{R}^N$:

[2] At least by Dirichlet boundary conditions, this is a well known problem

(a) The viscous stress tensor does not simplify to the velocity gradient;

(b) The total mass density is calculated from the continuity equation $\partial_t \varrho + \operatorname{div}(\varrho\, v) = 0$;

(c) The pressure remains partly connected to the other variables by algebraic formula.

As to the latter point, we shall show that the pressure can be written as algebraic expression of the total mass density ϱ and of $N-1$ 'diffusive variables' $q_1, \ldots, q_{N-2}, \zeta$. These variables are sufficiently under control due to the presence of diffusion gradients. In fact we obtain a representation $p = P(\varrho, q_1, \ldots, q_{N-2}) + \zeta$, where the non-linear part P possesses a logarithmic singularity:

$$|P(\varrho, q_1, \ldots, q_{N-2})| \approx C\, (1 + |q| + |\ln \min\{\varrho_{\max} - \varrho,\, \varrho - \varrho_{\min}\}|) \ . \tag{16}$$

Here, the constants $0 < \varrho_{\min} < \varrho_{\max} < +\infty$ are the thresholds of the total mass for states $\rho_1, \ldots, \rho_N$ that satisfy the constraint (1):

$$\varrho_{\min} := \inf_{\rho\,:\, \sum_{i=1}^N \rho_i \bar{V}_i = 1} \sum_{i=1}^N \rho_i = \frac{1}{\max \bar{V}}, \quad \varrho_{\max} := \sup_{\rho\,:\, \sum_{i=1}^N \rho_i \bar{V}_i = 1} \sum_{i=1}^N \rho_i = \frac{1}{\min \bar{V}}$$

The properties (a), (b), (c) are rather known for compressible Navier-Stokes equations. Here they apply to the incompressible model. And surprisingly, they help proving the global existence of certain weak solutions of bounded positive mass densities $\rho_1, \ldots, \rho_N$. The most striking part of our main result is certainly the possibility to introduce the pressure as a measure, and in fact

$$p = (P(\varrho, q_1, \ldots, q_{N-2}) + \zeta)\, dxdt + d\kappa \tag{17}$$

where the Lebesgue part in $L^1(Q_T)$ is the expected function, and the singular part $d\kappa$ is further decomposed in a jump part and a Cantor part. The jump part, defined as the absolutely continuous part with respect to the Hausdorff measure $d\mathcal{H}^3$ is concentrated in a countable union of hyperplanes $\bigcup_{j=1}^N \Omega \times \{t_j\}$ in which the singular values $\varrho = \varrho_{\min}$ and $\varrho = \varrho_{\max}$ might be attained on some subset of Ω of positive $\mathcal{H}^3-$ measure.

Our main method to approach the PDE problem is a switch of variables in the transport problem. Instead of the original variables $(\rho_1, \ldots, \rho_N)$ and $(p,\, v_1,\, v_2,\, v_3)$, we look for approximations taking the chemical potentials $(\mu_1, \ldots, \mu_N)$ and the velocity field as main variables. This idea has been already exploited for a relatively long time in the analysis of transport problems by mechanical equilibrium, for instance of semi-conductor equations (overview of methods a. o. in [J1̈5], [J1̈7]), and more directly related to the present context, in [CJ15]. For mixtures of phases, the Cahn-Hilliard models is another good example where the mathematical theory considers the chemical potential as the main variable. Let us emphasize that our approach in [DDGG16] based on methods of convex analysis, for being related to these ideas, is not completely reducible to them, because the mechanical contribution to the free energy model is not neglected. This was a main result of the models proposed in [BD15], [DGM13] and [DGM18]. Here we extend the 'compressible' method of [DDGG16] to the case of a singular free energy of the form (10) due to the incompressibility constraint. After the transformation we attain for the free variables $(\varrho,\, q_1, \ldots, q_{N-2},\, \zeta,\, v)$ instead of (3), (4) the equations

$$\begin{aligned}
\partial_t R_k(\varrho,\, q) + \operatorname{div}(R_k(\varrho,\, q)\, v - \widetilde{M}_{k,\ell}(\varrho,\, q)\, \nabla q_\ell - A_k(\varrho,\, q)\, \nabla \zeta) &= 0 \text{ for } k = 1, \ldots, N-2\\
\operatorname{div}(v - \, A(\varrho,\, q) \cdot \nabla q - b(\varrho,\, q)\, \nabla \zeta) &= 0\\
\partial_t \varrho + \operatorname{div}(\varrho\, v) &= 0\\
\partial_t(\varrho\, v) + \operatorname{div}(\varrho\, (v \otimes v) - \mathbb{S}(\nabla v)) + \nabla P(\varrho,\, q) + \nabla \zeta &= 0\,.
\end{aligned}$$

The non-linear fields $R,\ A$, the positive matrix $\widetilde{M}$, and the positive coefficient b will be constructed below. We are faced with a non-linear parabolic–elliptic–hyperbolic system. All variables are *unconstrained*, but for the restriction $\varrho_{\min} \leq \varrho \leq \varrho_{\max}$ on the total mass density, so that the latter problem is accessible to functional analytic methods.

In order to solve the highly non-linear equations we further need compactness properties. A main tool for obtaining compactness is the Lions–Feireisl method for Navier-Stokes, which provide the strong convergence of the the total mass density. Here we extend it to a special algebraic implicitly form of the pressure function. A major challenge comes from the fact that the L^1 bound on the pressure cannot be improved. These arguments go through, because (1) implies a L^∞ bound on the mass densities. In [FLM16] and [FLN18], the defect measure can be avoided assuming that the singularity of $P(\varrho)$ at the thresholds $\{\varrho_{\min},\ \varrho_{\max}\}$ is of inverse polynomial type[3]. For sufficiently large exponents, such pressure laws allow to prove the higher integrability of the pressure in L^q for a $q > 1$. As a matter of fact, we would also propose to modify the function k in (8) – and, by these means, to avoid the defect measure – if the logarithmic singularity would not be of major practical importance.

Notations. Vectors are denoted $\rho = (\rho_1, \rho_2, \ldots, \rho_N) \in \mathbb{R}^N$.

The vector $(1, 1, \ldots, 1)$ is called 1^N. The vector $\bar{V} = (\bar{V}_1,\ \bar{V}_2, \ldots, \bar{V}_N)$ refers to the reference specific volumes of the condition (1). We introduce

$$\mathbb{R}^N_+ := \{\rho \in \mathbb{R}^N\ :\ \rho_i > 0 \text{ for } i = 1, \ldots, N\}, \quad \mathbb{R}^N_{+,0} := \quad \{\rho \in \mathbb{R}^N\ :\ \rho_i \geq 0 \text{ for } i = 1, \ldots, N\}\,.$$

The orthogonal complement of $\operatorname{span}\{1^N\}$ is $\{1^N\}^\perp$. We call $\mathcal{P}$ the projection on $\{1^N\}^\perp$ in $\mathbb{R}^N$:

$$\mathcal{P}\xi := \xi - \frac{1}{N}\,\xi \cdot 1^N\, 1^N = (\mathrm{Id}_N - \frac{1}{N}\, 1^N \otimes 1^N)\,\xi \qquad \text{for } \xi \in \mathbb{R}^N\,.$$

We denote S_0 the relatively open, planar surface associated with the constraint (1):

$$S_0 := \Big\{\rho \in \mathbb{R}^N_+\ :\ \sum_{i=1}^N \bar{V}_i\, \rho_i = 1\Big\}\,. \tag{18}$$

For a bounded cylindrical domain $Q = Q_T := \Omega\times]0, T[\subset \mathbb{R}^4$ and $1 \leq p, q \leq \infty$, the spaces $L^{q,p}(Q_T)$ are well known (space index first!). We call $W^{1,0}_p(Q)$ the Sobolev space of functions in $L^p(Q)$ having $p-$integrable spatial derivatives and $W^{1,1}_{p,q}(Q)$ the space of functions of $L^{p,q}(Q_T)$ having a $L^{p,q}-$integrable time and space derivatives. We denote $V^{1,0}_2(Q)$ the Sobolev space of functions of $W^{1,0}_2(Q)$ having finite norm in $L^{2,\infty}(Q)$,

3 Main result

We consider a bounded domain $\Omega \subset \mathbb{R}^3$ and $T > 0$. We denote $Q = Q_T := \Omega\times]0, T[$ and $S = S_T := \partial\Omega\times]0, T[$.

We first formulate the concept of a weak solution to the problem (P) in its original variables: The mass densities $\rho_1, \rho_2, \ldots, \rho_N$, the pressure p and the velocity $(v_1,\ v_2,\ v_3)$. The algebraic relation (7) serves as definition for secondary variables $\mu_1, \ldots, \mu_N$ by means of a nonlinear mapping $\mu = \mu(\rho,\ p)$.

[3]We thank Prof. E. Feireisl for hinting us at these references

3.1 Weak solutions and their possible singularities

The concept of a weak solution is subject to necessary **constraints** that we now formulate more precisely. Recalling first the formula (15), we notice that none of our estimates below will allow to make sense of spatial derivatives of the function p, and even not of the densities ρ_i. In order to define weak solutions as vectors $(\rho,\, v,\, p)$ we must therefore add the condition that the nonlinear mapping $\mu = \mu(\rho,\, p)$ defined by (7) possesses spatial derivatives in some weak sense. But here is a second subtle point, due to the fact that the kinetic matrix M possesses the kernel $\mathrm{span}\{1^N\}$. Not the entire map $\mu(\rho,\, p)$ can be expected to possess spatial derivatives, but only $N-1$ coordinates $\mu(\rho,\, p)\cdot\eta^k$ ($k = 1,\ldots,N-1$). Here, $\eta^1,\ldots,\eta^{N-1}$ can be chosen to any basis of the $N-1$ dimensional linear space $\{1^N\}^{\perp}$. In other words, only the standard projection applied to μ, that is $\mathcal{P}\,\mu(\rho,\, p)$, must be (weakly) differentiable in x. At third, more obviously, we have the physical restriction $\rho_i \geq 0$ for $i = 1,\ldots,N$. But if we assume that the function k obeys (8), then the derivatives $\partial_{\rho_i}k$ blow up at $\partial\mathbb{R}^N_+$. Thus, we can make sense of the expressions $\mu(\rho,\, p)$ as differentiable functions only if the strict positivity $\rho_i > 0$ is valid (almost everywhere) for $i = 1,\ldots,N$. Adding the incompressibility constraint (1), the mass densities are subject to the restrictions $\rho \in S_0(\cap\mathbb{R}^N_+)$.

We first formulate the concept of weak solution expected from the standard estimates:

Definition 3.1. *We call weak solution to the problem (P) a vector $(\rho,\, v,\, p)$ satisfying*

$$\rho \in L^\infty(Q_T;\, S_0), \quad v \in V_2^{1,0}(Q_T;\, \mathbb{R}^3), \quad p \in L^1(Q_T)\,, \tag{19}$$

such that the mappings $\mu(\rho,\, p) := \bar{V}\,p + \partial_\rho k(\rho)$ satisfy

$$\mathcal{P}\,\mu(\rho,\, p) \in W_2^{1,0}(Q;\, \mathbb{R}^N)\,, \tag{20}$$

and such that the following integral relations are valid

$$\begin{aligned} -\int_Q \rho\cdot\partial_t\psi\,dxdt - \int_Q \rho\cdot\nabla\psi\cdot v\,dxdt + \int_Q M(\rho)\nabla\mu(\rho,\, p)\,\nabla\psi\,dxdt \\ + \int_Q M^\Gamma\,(\mu-\mu_\Gamma)\,\psi\,dSdt = \int_\Omega \rho^0(x)\cdot\psi(x,\, 0)\,dx \end{aligned} \tag{21}$$

$$\begin{aligned} -\int_Q \varrho\,v\cdot\partial_t\eta\,dxdt - \int_Q \varrho\,v\otimes v\,:\,\nabla\eta\,dxdt + \int_Q \mathbb{S}(\nabla v)\,:\,\nabla\eta\,dxdt - \int_Q p\,\operatorname{div}\eta\,dxdt \\ = \int_\Omega \varrho_0(x)\,v^0(x)\cdot\eta(x,\, 0)\,dx \end{aligned} \tag{22}$$

for all $\psi \in C^1(\overline{Q}_T;\, \mathbb{R}^N)$ such that $\psi(T) = 0$ and all $\eta \in C^1_c(\Omega\times[0,\, T[;\, \mathbb{R}^3)$.

Singularities at the density thresholds.

It does not seem possible to improve the pressure estimate in $L^1(Q_T)$, and this fact yields troubles in proving existence for the concept of Definition 3.1. Moreover, we can show that the class of solutions generated by this definition is not closed. This is related to the point (c): There is a connection between pressure and total mass density, so that $p = +\infty$ at the threshold value $\varrho = \varrho_{\max}$ and $p = -\infty$ at $\varrho = \varrho_{\min}$. This fact allows to construct simple examples where one must leave the expected class of weak solutions.

Example 3.2. *We consider a sequence of regular initial conditions $\{\rho^0_m\}_{m\in\mathbb{N}}$ admitting values in S_0. Assume $\rho^0_m \to \rho^0$, and the limit total mass $\varrho_0 := \sum_{i=1}^N \rho^0_i$ attains the critical value $\varrho_{\max}$ on an open* *set U of positive measure in Ω. At the same time, we consider initial conditions for the velocity v^0 satisfying $\operatorname{div} v^0 < 0$ in U. Suppose next that there are weak solutions $(\rho^m,\, v^m,\, p_m)$ bounded in the solution class, and the solution class is closed (weakly closed), so that they are converging (weakly) to a weak solution $(\rho,\, v,\, p)$ to the limit problem with initial data $\rho^0,\, v^0$.*

We fix any non-negative test function ϕ with support in U, and from the continuity equation we infer for $t > 0$ that

$$\int_\Omega \varrho(x,\, t)\, \phi(x)\, dx = \int_\Omega \varrho_0(x)\, \phi(x)\, dx + \int_0^t \int_\Omega \varrho\, v \cdot \nabla\phi\, dxds$$
$$= \varrho_{\max} \int_\Omega \phi(x)\, dx + \int_0^t \int_\Omega \varrho\, v \cdot \nabla\phi dxds\,.$$

Since $p \in L^1(Q_T)$, we must have $\varrho < \varrho_{\max}$ almost everywhere in Q (see the Prop. 5.3 below). Therefore it is possible to find a sequence $t \to 0$ such that $\int_\Omega (\varrho(x,\, t) - \varrho_{\max})\, \phi\, dx < 0$. This yields

$$\limsup_{t\to 0} \frac{1}{t} \int_0^t \int_\Omega \varrho\, v \cdot \nabla\phi dxds \leq 0\,.$$

Thus, if $t = 0$ would be a regular point of the function $s \mapsto \int_\Omega \varrho(x,\, s)\, v(x,\, s) \cdot \nabla\phi(x)\, dx$, it would follow that

$$0 \geq \int_\Omega \varrho_0(x)\, v^0(x) \cdot \nabla\phi(x)\, dx = \varrho_{\max} \int_\Omega v^0(x) \cdot \nabla\phi(x)\, dx$$
$$= -\, \varrho_{\max} \int_\Omega \operatorname{div} v^0(x)\, \phi(x)\, dx\,.$$

Since ϕ was arbitrary in $C_c(U)$, it follows that $\operatorname{div} v^0 \geq 0$, a contradiction to the choice of v^0. We conclude that $t = 0$ is no continuity point of $s \mapsto \int_\Omega \varrho(x,\, s)\, v(x,\, s) \cdot \nabla\phi(x)\, dx$ for at least one ϕ. But then we see that the jump of this function at zero is missing in the equation (22)*. Indeed, the latter relation implies that*

$$\int_\Omega (\varrho(x,\, t)\, v(x,\, t) - \varrho_0(x)\, v^0(x)) \cdot \nabla\phi\, dx = \int_0^t \int_\Omega L(x,\, s) \cdot D^2\phi\, dxds$$
$$L = \varrho\, v \otimes v - \mathbb{S}(\nabla v) + p\, I,$$

and by the absolute continuity of the integral, the function $s \mapsto \int_\Omega \varrho(x,\, s)\, v(x,\, s) \cdot \nabla\phi(x)\, dx$ must be continuous in zero.

3.2 The main theorem

For the data, we are going to require the following conditions.

We assume that $M\,:\, \mathbb{R}^N_{+,0} \to \mathbb{R}^{N\times N}$ is a continuous mapping into the positive semi-definite matrices of rank $N-1$ with constant kernel $\operatorname{span}\{1^N\}$, and that they are constants $0 < \underline{\lambda} \leq \overline{\lambda}$ such that for all $\rho \in \mathbb{R}^N_+$

$$\underline{\lambda}\, |\mathcal{P}\xi|^2 \leq M(\rho)\xi \cdot \xi \leq \overline{\lambda}\, (1 + |\rho|)\, |\mathcal{P}\xi|^2 \text{ for all } \xi \in \mathbb{R}^N\,. \tag{23}$$

We assume that $M^\Gamma\,:\, S_T \to \mathbb{R}^{N\times N}$ is a continuous, bounded mapping into the positive semi-definite matrices of rank $N-1$ with constant kernel $\operatorname{span}\{1^N\}$, and that there is a constant $0 < \underline{\lambda}^\Gamma$ such that for all $(x,\, t) \in S_T$

$$\underline{\lambda}^\Gamma\, |\mathcal{P}\xi|^2 \leq M^\Gamma(x,\, t)\xi \cdot \xi \text{ for all } \xi \in \mathbb{R}^N\,. \tag{24}$$

We moreover assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain of class $\mathcal{C}^2$. This assumption is inessential, but provides some relief in the technical discussions. Local–in–time weak solutions starting from regular (sufficiently smooth) initial data with $\rho^0 \in S_0$ are studied in the paper [BDb], that shows that the problem (P) is locally well-posed. For global weak solutions, we must amend the Definition 3.1, so as to allow for instance for jump singularities like constructed in Example 3.2. This can be done by introducing a defect measure. Since these measures occurs only if $\varrho \in \{\varrho_{\min},\, \varrho_{\max}\}$ which corresponds to $p = \pm\infty$, it is not clear that they possess a qualified meaning in the relevant applications. Our research is done in the spirit of helping understanding the singularities of the model, since the properties of the relevant measure are relatively well characterised:

Definition 3.3. *Consider a vector $(\rho,\, v,\, p,\, \kappa)$ where $(\rho,\, v,\, p)$ satisfy the conditions* (19) *and* (20)*, and $\kappa \in \mathcal{M}(\overline{Q})$ is a regular measure satisfying the following conditions:*

(a) $\kappa = \kappa_{\lfloor A}$ where $A \subset Q$ is a set with $\lambda_4(A) = 0$;

(b) If $(x,\, t) \in A$ is such that there exists the Lebesgue value $\lambda := \lim_{\delta\to 0} \fint_{B_\delta(x)} \varrho(y,\, t)\, dy$, then $\lambda \in \{\varrho_{\min},\, \varrho_{\max}\}$;

(c) The measure κ possesses the orthogonal decomposition $\kappa = \kappa^{(j)} + \kappa^{(C)}$, where $\kappa^{(j)}$ (the jump part) is absolutely continuous with respect to the $3-$dimensional Hausdorff measure $\mathcal{H}^3$, and $\kappa^{(C)}$ (the Cantor part) is orthogonal to both λ_4 and $\mathcal{H}^3$;

(d) There is a countable set of times $t_1,\, t_2, \ldots \in [0,\, T]$ and functions $\mathfrak{p}_1,\, \mathfrak{p}_2, \ldots \in W^{1,2}(\Omega)$ such that $\int_Q f\, d\kappa^{(j)} = \sum_{k=1}^{\infty} \int_\Omega \mathfrak{p}_k(x)\, f(x,\, t_k)\, dx$ for all $f \in C_c(Q)$.

If for all $\psi \in C^1(\overline{Q}_T;\, \mathbb{R}^N)$ such that $\psi(T) = 0$, the integral relation (21) *is valid, and instead of* (22) *we have for all $\eta \in C^1_c(\Omega \times [0,\, T[;\, \mathbb{R}^3)$*

$$
\begin{aligned}
-\int_Q \varrho\, v \cdot \partial_t \eta\, dxdt - \int_Q \varrho\, v \otimes v \,:\, \nabla \eta\, dxdt + \int_Q \mathbb{S}(\nabla v) \,:\, \nabla\eta\, dxdt - \int_Q p\, \operatorname{div} \eta\, dxdt \\
= \int_Q \operatorname{div} \psi\, d\kappa + \int_\Omega \varrho_0(x)\, v^0(x) \cdot \eta(x,\, 0)\, dx\,,
\end{aligned}
\tag{25}
$$

then we call $(\rho,\, v,\, p,\, \kappa)$ a weak solution with defect measure to (P).

Remark 3.4. *In order that a jump singularity occurs, it is necessary that the Lebesgue representant $\tilde{\varrho}(t_k)$ takes only value in $\{\varrho_{\min},\, \varrho_{\max}\}$ on a subset of three-dim. positive measure of Ω (singular initial data, infinite pressure).*

Theorem 3.5. *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain of class $\mathcal{C}^2$, and $T > 0$. Assume that M, M^Γ satisfy the assumptions* (23) *and* (24)*, and that $k \,:\, \mathbb{R}^N_+ \to \mathbb{R}$ is the function* (8)*. Suppose that the initial data $\rho_{0,1}, \ldots \rho_{0,N}$ are of class $L^\infty(\Omega;\, \overline{S_0})$, in particular are non-negative almost everywhere in Ω. Suppose further that the initial total mass density $\varrho_0 := \sum_{i=1}^N \rho_i^0$ satisfies*

$$
\varrho_{\min} < \fint_\Omega \varrho_0(x)\, dx < \varrho_{\max}\,. \tag{26}
$$

Then, the problem (P) possesses a global weak solution (as in Definition 3.3).

Method. We already noted in our study [DDGG16] on compressible systems, that the formulation in the original variables $(\rho,\, v,\, p)$ contains several constraints difficult to handle by functional analytic methods. Here, the incompressibility constraint: $\rho \in S_0$ even makes things worse, since the operator generated by the equations (3) is not tangential on this constraint. [Recall: the singular direction of M is not the vector $\bar{V}$ but 1^N!]. The non-linear condition $\mathcal{P}\, \mu(\rho,\, p)$ is another important obstacle to the application of Banach space methods.

In the next sections we will show that the problem admits a reformulation in terms of equivalent variables such that all up to one constraint vanish. This formulation will aid us to construct the appropriate weak solutions. Similar ideas are applied in [BDb] to the problem of local–in–time existence. We will massively exploit an equivalent reformulation of (7) based on the convex conjugate function for a singular free energy h^{∞} of the type (10). The convex conjugate is defined for $X \in \mathbb{R}^N$ via

$$f(X) := \sup_{\rho>0}\{X\cdot\rho - h^{\infty}(\rho)\} = \sup_{\rho\in S_0}\{X\cdot\rho - k(\rho)\}\,. \tag{27}$$

The function f is continuously differentiable. The image of the gradient mapping ∇f on $\mathbb{R}^N$ is contained in S_0, with in particular the consequence that f is globally Lipschitz. Further, the function f is affine with slope one in the direction of the vector $\bar{V}$:

$$f(\mu + s\,\bar{V}) = f(\mu) + s \text{ for all } \mu\in\mathbb{R}^N,\; s\in\mathbb{R}\,. \tag{28}$$

The Hessian $D^2_{\mu,\mu}f$ is positive semi-definite. Its kernel is associated with the singular direction $\bar{V}$. By definition (7) is valid if and only if μ, p and ρ are connected via

$$p = f(\mu), \quad \rho = \nabla_{\mu} f(\mu)\,. \tag{29}$$

All these facts are proved in the paper [BDa] for a very general setting. In this paper we provide an alternative proof by direct algebraic computations in the special case that k is given by the mixing entropy of formula (8). This is the object of the next section.

4 Preliminary: The singular free energy function and its conjugate

For the commodity of the next computations, we introduce for $i = 1,\ldots,N$ new constants $V_i := \bar{V}_i\, m_i$, where $\bar{V}$ are the constants occurring in the incompressibility constraint (1). By the definitions (8) of the function k and (7) of the chemical potentials

$$\mu_i = \mu_i^{\text{ref}} + \bar{V}_i\, p + \frac{k_B\,\theta}{m_i}\, \ln y_i \text{ for } i = 1,\ldots,N\,. \tag{30}$$

This implies that $p = f(\mu_1,\ldots,\mu_N)$, where the function f is implicitly defined by the equation

$$\sum_{i=1}^N e^{\frac{m_i\,(\mu_i-\mu_i^{\text{ref}}) - V_i\, f}{k_B\,\theta}} = 1\,. \tag{31}$$

Lemma 4.1. *For all $\mu \in \mathbb{R}^N$, the equation* (31) *possesses a unique solution $f = f(\mu)$. The function $\mu \mapsto f(\mu)$ belongs to $C^2(\mathbb{R}^N)$. Defining $c_0 := |\max_{i=1,\ldots,n}\{\bar{V}_i\, \text{sign}(f(\mu))\}|^{-1}$ and $c_1 := |\min_{i=1,\ldots,n}\{\bar{V}_i\, \text{sign}(f(\mu))\}|^{-1}$, the following inequalities are valid:*

$$c_0\,\big(\max_{i=1,\ldots,m} \mu_i - \|\mu^{\text{ref}}\|_{\infty}\big) \le f(\mu) \le c_1\,\big(\max_{i=1,\ldots,m} \mu_i + \|\mu^{\text{ref}}\|_{\infty} + \frac{k_B\,\theta}{\min_{i=1,\ldots,N} m_i}\, \ln N\big)\,.$$

Moreover, the identity $\sum_{j=1}^N \bar{V}_j\, \partial_{\mu_j} f = 1$ holds uniformly on $\mathbb{R}^N$. For all $\mu \in \mathbb{R}^N$, the Hessian $D^2 f(\mu)$ is positive semi-definite. Its kernel is the span of the vector $\bar{V}$.

Proof. For $\mu \in \mathbb{R}^N$ and $f \in \mathbb{R}$, denote $G(\mu,\, f) := \sum_{i=1}^N e^{\frac{m_i\,(\mu_i-\mu_i^{\text{ref}})-V_i\,f}{k_B\,\theta}}$. We directly verify that $\lim_{f\to\pm\infty} G(\mu,\, f) = \mp\infty$. Moreover $\partial_f G(\mu,\, f) < 0$. Therefore, there exists a unique solution $f(\mu)$ to the equation $G(\mu,\, f) = 1$.

Define $Y_i(\mu) := e^{\frac{m_i\,(\mu_i-\mu_i^{\text{ref}})-V_i\,f(\mu)}{k_B\,\theta}}$. Then, $0 < Y_i < 1$ for all $i \in \{1,\dots,N\}$, and $\sum_{i=1}^N Y_i = 1$. This definition also implies the identities

$$\mu_i = \mu_i^{\text{ref}} + \bar{V}_i\, f(\mu) + \tfrac{k_B\,\theta}{m_i}\, \ln Y_i(\mu) \text{ for } i = 1,\dots,N\,. \tag{32}$$

For $i = 1,\dots,N$, it follows that $\mu_i \leq \mu_i^{\text{ref}} + \bar{V}_i\, f(\mu)$, and that $\max_{i=1,\dots,m} \mu_i \leq \|\mu^{\text{ref}}\|_\infty + \max_{i=1,\dots,n}\{\bar{V}_i\, \operatorname{sign}(f(\mu))\}\, |f(\mu)| = \|\mu^{\text{ref}}\|_\infty + c_0\, f(\mu)$. On the other hand, for each index i_1 such that $Y_{i_1} = \max_{i=1,\dots,N} Y_i \geq N^{-1}$, one has

$$\max_{i=1,\dots,m} \mu_i \geq \mu_{i_1} \geq -\|\mu^{\text{ref}}\|_\infty + |f(\mu)| \min_{i=1,\dots,N}\{\bar{V}_i\, \operatorname{sign}(f(\mu))\} + \tfrac{k_B\,\theta}{m_{i_1}}\, \ln\tfrac{1}{N}\,.$$

By direct computations we next verify that

$$\partial_{\mu_j} f(\mu) = \frac{m_j\, e^{\frac{m_j\,(\mu_j-\mu_j^{\text{ref}})-V_j\,f(\mu)}{k_B\,\theta}}}{\sum_{i=1}^N V_i\, e^{\frac{m_i\,(\mu_i-\mu_i^{\text{ref}})-V_i\,f(\mu)}{k_B\,\theta}}}\,. \tag{33}$$

It follows that $0 < \partial_{\mu_j} f \leq \frac{1}{\bar{V}_j}$ for $j = 1,\dots,N$, and obviously that $\sum_{j=1}^N \bar{V}_j\, \partial_{\mu_j} f = 1$. Using (33), we further can compute that

$$\partial^2_{\mu_j,\mu_\ell} f = m_\ell\, \partial_{\mu_j} f\, \delta_j^\ell - \partial_{\mu_j} f\, \partial_{\mu_\ell} f\, (V_j + V_\ell) + \partial_{\mu_j} f\, \partial_{\mu_\ell} f\, \left(\sum_{i=1}^N \frac{V_i^2}{m_i}\, \partial_{\mu_i} f\right)\,. \tag{34}$$

For $\xi \in \mathbb{R}^N$ arbitrary, it follows that $D^2_{\mu_j,\mu_\ell} f\, \xi_j\, \xi_\ell = \sum_{i=1}^N \partial_{\mu_i} f\, (\xi_i\, \sqrt{m_i} - \frac{V_i}{\sqrt{m_i}}\, \xi\cdot\nabla f)^2$. We see that $D^2 f$ is positive semi-definite. Using that $\partial_\mu f$ is positive, we verify that $D^2_{\mu,\mu} f\, \xi\, \xi$ vanishes if and only if $\xi_i = \bar{V}_i\, \xi\cdot\nabla f$ for $i = 1,\dots,N$, that is, if ξ is parallel to $\bar{V}$. Therefore, $D^2 f$ has a one-dim. kernel associated with the vector $\bar{V}$. □

Next we prove the identities (29).

Lemma 4.2. *Assume that $\rho \in \mathbb{R}^N_+$ satisfies $\sum_{i=1}^N \bar{V}_i\, \rho_i = 1$. Assume that $\mu \in \mathbb{R}^N$, $\rho \in S_0$ and p are related via* (30)*. Then $p = f(\mu)$ and $\rho = \nabla_\mu f(\mu)$.*

Proof. The relation (30) implies that $p = f(\mu)$ by the definition of f. Comparing (30) and (32), we see that the fractions y are given as functions of μ via $y_i = Y_i(\mu)$.

Due to the incompressibility constraint $\sum_{i=1}^N \bar{V}_i\, \rho_i = 1$, the formula $\rho_i = \frac{m_i\, y_i}{\sum_{j=1}^N m_j\, \bar{V}_j\, y_j}$ are valid for $i = 1,\dots,N$. The mass densities satisfy (see (33)) $\rho_i = \frac{m_i\, Y_i(\mu)}{\sum_{j=1}^N m_j\, \bar{V}_j\, Y_j(\mu)} = \partial_{\mu_j} f(\mu)$. □

We will need several properties of the Hessian matrix $D^2 f$. We recall the definition

$$\varrho_{\min} := \min_{i=1,\dots,N} \frac{1}{\bar{V}_i}, \quad \varrho_{\max} := \max_{i=1,\dots,N} \frac{1}{\bar{V}_i}\,. \tag{35}$$

Lemma 4.3. *Let $\mu \in \mathbb{R}^N$ and $\rho = \nabla_\mu f(\mu)$. Let $1^N := (1, 1, \ldots, 1)$, and $\varrho := \rho \cdot 1^N$. For $i = 1, \ldots, N$, define $V_i := m_i \bar{V}_i$. We further define*

$$d_0 := \frac{1}{2} \inf_{\bar{V}_i \neq \bar{V}_j} \left|\frac{m_i}{V_i} - \frac{m_j}{V_j}\right| = \frac{1}{2} \inf_{\bar{V}_i \neq \bar{V}_j} \left|\frac{1}{\bar{V}_i} - \frac{1}{\bar{V}_j}\right| .$$

For $\varrho \in [\varrho_{\min}, \varrho_{\max}]$, we define

$$\begin{aligned} I_0 =& I_0(\varrho) := \{i_0 \in \{1, \ldots, N\} \,:\, |\varrho - \tfrac{m_{i_0}}{V_{i_0}}| = \min_{i=1,\ldots,N} |\varrho - \tfrac{m_i}{V_i}|\} \\ J_0 =& J_0(\varrho) := \{1, \ldots, N\} \setminus I_0 . \end{aligned}$$

The following properties are valid:

(a) The Hessian $D^2 f(\mu)$ depends only on ρ and it possesses the representation

$$\partial^2_{\mu_j,\mu_\ell} f(\mu) = m_\ell \, \rho_j \, \delta_j^\ell - \rho_j \, \rho_\ell \, (V_j + V_\ell) + \rho_j \, \rho_\ell \, (\sum_{i=1}^N \tfrac{V_i^2}{m_i} \, \rho_i) ;$$

(b) $D^2 f(\mu) 1^N \cdot 1^N = \sum_{i=1}^N \frac{\rho_i}{m_i} (m_i - \varrho \, V_i)^2$;

(c) $D^2 f(\mu) 1^N \cdot 1^N \geq \Big[\min_{i=1,\ldots,N} \frac{V_i^2}{m_i}\Big] \Big\{\varrho_{\min} [\min_{i=1,\ldots,N} |\varrho - \frac{m_i}{V_i}|]^2 + d_0^2 \sum_{i \in J_0(\varrho)} \rho_i\Big\}$;

(d) There is $C_0 > 0$ such that $D^2 f(\mu) 1^N \cdot 1^N \geq C_0 \, \min_{i=1,\ldots,N} |\varrho - \frac{m_i}{V_i}|$;

(e) There is a constant $C_1 > 0$ such that $|D^2 f(\mu) 1^N \cdot e^j| \leq C_1 \, D^2 f(\mu) 1^N \cdot 1^N$ for all $j = 1, \ldots, N$.

Proof. The representation (a) of $D^2 f(\mu)$ is identical with (34). The representation (b) of $D^2 f(\mu) 1^N \cdot 1^N$ is then its straightforward consequence.

We choose $i_0(\rho)$ as an index such that $|\varrho - \frac{m_{i_0}}{V_{i_0}}| = \min_{i=1,\ldots,N} \, |\varrho - \frac{m_i}{V_i}| =: \alpha$. Obviously we have $\alpha \leq \frac{1}{2} \, \inf_{\bar{V}_i \neq \bar{V}_j} |\frac{m_i}{V_i} - \frac{m_j}{V_j}| = d_0$. Then it must follow that $\inf_{i \in J_0} |\varrho - \frac{m_i}{V_i}| \geq d_0$. Therefore, in view of (b)

$$D^2 f(\mu) 1^N \cdot 1^N \geq \tfrac{\rho_{i_0}}{m_{i_0}} \, |V_{i_0} \, \varrho - m_{i_0}|^2 + \left[\min_{j=1,\ldots,N} \tfrac{V_j^2}{m_j}\right] \, d_0^2 \sum_{i \in J_0} \rho_i \, , \tag{36}$$

which implies the estimate (c). To prove (d), we make use of the identity $1 = \sum_{i=1}^N \bar{V}_i \, \rho_i$. Subtracting $\bar{V}_{i_0} \, \varrho$, and exploiting that $\bar{V}_i = \bar{V}_{i_0}$ for all $i \in I_0$, we see that $1 - \bar{V}_{i_0} \, \varrho = \sum_{i \in J_0} (\bar{V}_i - \bar{V}_{i_0}) \, \rho_i$. In turn, this entails $\sum_{i \in J_0} \rho_i \geq \frac{1 - \bar{V}_{i_0} \, \varrho}{\sup_{\bar{V}_i \neq \bar{V}_j} |\bar{V}_i - \bar{V}_j|}$, and (d) follows from (c). It remains to prove (e). We notice

$$D^2 f(\mu) 1^N \cdot e^j = \rho_j \, \left(m_j - \varrho \, V_j + \varrho \, (\sum_{i=1}^N \tfrac{V_i^2}{m_i} \, \rho_i) - \rho \cdot V \right) .$$

Consider $j \in J_0(\varrho)$. Then by means of (36)

$$|D^2 f(\mu) 1^N \cdot e^j| \leq c \, \rho_j \leq \tilde{c} \, D^2 f(\mu) 1^N \cdot 1^N . \tag{37}$$

On the other hand, for $j \in I_0(\varrho)$

$$D^2f(\mu)1^N \cdot e^j = \rho_j \left(m_j - \varrho\, V_j + \varrho\, (\sum_{i=1}^N \tfrac{V_i^2}{m_i}\, \rho_i) - \rho \cdot V \right)$$
$$= \rho_j \left(m_j - \varrho\, V_j + \varrho\, (\sum_{i\in I_0} \frac{V_i^2}{m_i}\, \rho_i) - \sum_{i\in I_0} \rho_i\, V_i + \varrho\, (\sum_{i\in J_0} \tfrac{V_i^2}{m_i}\, \rho_i) - \sum_{i\in J_0} \rho_i\, V_i \right) .$$

For all $i \in I_0$, recall that $\bar{V}_i = \bar{V}_{i_0}$. Thus

$$\varrho\, (\sum_{i\in I_0} \frac{V_i^2}{m_i}\, \rho_i) - \sum_{i\in I_0} \rho_i\, V_i = \bar{V}_{i_0} \sum_{i\in I_0} \rho_i\, (V_i\, \varrho - m_i) = \bar{V}_{i_0}\, (\varrho\, \bar{V}_{i_0} - 1) \sum_{i\in I_0} m_i\, \rho_i$$

implying that

$$D^2f(\mu)1^N \cdot e^j = \rho_j \left((1 - \varrho\, \bar{V}_{i_0})\, (m_j - \bar{V}_{i_0} \sum_{i\in I_0} m_i\, \rho_i) + \sum_{i\in J_0} \rho_i\, V_i\, (\bar{V}_i\, \varrho - 1) \right) .$$

It follows that $|D^2f(\mu)1^N \cdot e^j| \leq c_1\, |1 - \varrho\, \bar{V}_{i_0}| + c_2 \sum_{i\in J_0} \rho_i$. Combining (d) and (c), we obtain (e). ☐

We now possess the tools to introduce an algebraic variable transformation on the PDE system.

5 Change of variables

We propose a reformulation of the equations (3), (4) subject to the constitutive equations (5), (7) and (8) and to the volume constraint (1) in order to eliminate:

(a) Positivity constraints on ρ;

(b) The singular direction due to $M\, 1^N = 0$ (cp. (6));

(c) The singular direction due to the fact that the function f, interpreted as dual of the free energy, is affine in the direction of $\bar{V}$ ($D^2f\, \bar{V} = 0$, Lemma 4.1);

5.1 General ideas

We choose a basis of $\mathbb{R}^N$: $\{\xi^1, \ldots, \xi^{N-2}, \xi^{N-1}, \xi^N\}$ with $\xi^N = 1^N$ and $\xi^{N-1} = \bar{V}$. We then find $\eta^1, \ldots, \eta^N$ such that $\xi^i \cdot \eta^j = \delta^i_j$ for $i, j = 1, \ldots, N$ (inverse matrix). We define variables $q^1, \ldots, q^{N-2}$ and ζ via

$$q_\ell := \eta^\ell \cdot \mu := \sum_{i=1}^N \eta^\ell_i\, \mu_i \text{ for } \ell = 1, \ldots, N-2 \tag{38}$$

$$\zeta (= q_{N-1}) := \eta^{N-1} \cdot \mu = \sum_{i=1}^N \eta^{N-1}_i\, \mu_i\, . \tag{39}$$

For $\rho \in \mathbb{R}^N_+$ such that $\sum_{i=1}^N \rho_i \bar{V}_i = 1$, we want to invert the relation $\mu_i = \bar{V}_i p + \partial_{\rho_i} k(\rho)$ for $i = 1, \ldots, N$. In the particular case (8) here under study, we only exploit the result of Lemma 4.2 saying that (30) is equivalent to $\rho_i = \partial_{\mu_i} f(\mu_1, \ldots, \mu_N)$ for $i = 1, \ldots, N$. The vector μ is next decomposed into its projection onto $\{1^N\}^\perp$ expressed by the variables q and ζ, and its projection on the $\operatorname{span}\{1^N\}$ according to

$$\mu = \sum_{\ell=1}^{N-2} q^\ell \, \xi^\ell + \zeta \, \bar{V} + \mu \cdot \eta^N \, 1^N \, .$$

The last coordinate $\mu \cdot \eta^N$ is now eliminated using the equation

$$\varrho = \sum_{i=1}^N \rho_i = 1^N \cdot \nabla_\mu f(\mu_1, \ldots, \mu_N) = 1^N \cdot \nabla_\mu f \left(\sum_{\ell=1}^{N-2} q_\ell \, \xi^\ell + \zeta \, \bar{V} + (\mu \cdot \eta^N) \, 1^N \right) .$$

Now, the properties of f imply that the gradient $\nabla_\mu f$ is invariant in the direction $\bar{V}$ (cf. (28) and Lemma 4.1) and therefore, the variable ζ decouples from the latter equation, that now reads

$$\varrho - 1^N \cdot \nabla_\mu f \left(\sum_{\ell=1}^{N-2} q_\ell \, \xi^\ell + (\mu \cdot \eta^N) \, 1^N \right) = 0 \, . \tag{40}$$

This representation is an algebraic equation $F(\mu \cdot \eta^N, \, q_1, \ldots, q_{N-2}, \, \varrho) = 0$. In view of Lemma 4.3, note that $\partial_{\mu \cdot \eta^N} F(\mu \cdot \eta^N, \, q_1, \ldots, q_{N-2}, \, \varrho) = -D^2 f(\mu) 1^N \cdot 1^N < 0$, due the fact that 1^N is not parallel to $\bar{V}$! Thus, the last component $\mu \cdot \eta^N$ is defined implicitly as a differentiable function of ϱ and q. We call this function $\mathscr{M}$ and obtain the equivalent formulas

$$\begin{aligned} \mu = & \sum_{\ell=1}^{N-2} q_\ell \, \xi^\ell + \zeta \, \bar{V} + \mathscr{M}(\varrho, \, q_1, \ldots, q_{N-2}) \, 1^N \, , \\ \rho = & \nabla_\mu f \left(\sum_{\ell=1}^{N-2} q_\ell \, \xi^\ell + \mathscr{M}(\varrho, \, q_1, \ldots, q_{N-2}) \, 1^N \right) =: \mathscr{R}(\varrho, \, q) \, , \end{aligned}$$

where only the total mass density ϱ and the *relative chemical potentials* $q_1, \ldots, q_{N-2}$ and ζ occur as free variables. Note moreover that ζ and ρ decouple.

Similarly, we obtain a representation of the pressure

$$\begin{aligned} p = f(\mu) = & f \left(\sum_{\ell=1}^{N-2} q_\ell \, \xi^\ell + \zeta \, \bar{V} + \mathscr{M}(\varrho, \, q_1, \ldots, q_{N-2}) \, 1^N \right) \\ = & f \left(\sum_{\ell=1}^{N-2} q_\ell \, \xi^\ell + \mathscr{M}(\varrho, \, q_1, \ldots, q_{N-2}) \, 1^N \right) + \zeta \\ =: & P(\varrho, \, q) + \zeta \, . \end{aligned} \tag{41}$$

All this is summarised in the following Lemma.

Lemma 5.1. *Define* $\varrho_{\min} := \min_{i=1,\ldots,N} \frac{1}{\bar{V}_i}$ *and* $\varrho_{\max} := \max_{i=1,\ldots,N} \frac{1}{\bar{V}_i}$. *We call* I *the open interval* $]\varrho_{\min}, \, \varrho_{\max}[$. *There exists a function* $\mathscr{M} \in C^1(I \times \mathbb{R}^{N-2})$ *and a field* $\mathscr{R} \in C^1(I \times \mathbb{R}^{N-2}; \, S_0)$ *such that the equations* $\rho = \nabla_\mu f(\mu)$ *are valid if and only if there are* $\varrho \in I$, $q \in \mathbb{R}^{N-2}$ *and* $\zeta \in \mathbb{R}$ *such that*

$$\sum_{i=1}^N \rho_i = \varrho, \; \rho = \mathscr{R}(\varrho, \, q), \qquad \mu = \sum_{j=1}^{N-2} q_j \, \xi^j + \zeta \, \bar{V} + \mathscr{M}(\varrho, \, q) \, 1^N =: \mu(\varrho, \, q, \, \zeta) \, .$$

5.2 Relevant properties of the transformed coefficient functions

By means of the implicit representation (40) of $\mathscr{M} = \mu \cdot \eta^N$, we can obtain a representation of derivatives. Recall in this respect that the Hessian $D^2 f$ does not depend on the complete vector μ, but only on $\mu'(\varrho, q) := \sum_{j=1}^{N-2} q_j \, \xi^j + \mathscr{M}(\varrho, q) \, 1^N$.

Lemma 5.2. *At $(\varrho, q) \in I \times \mathbb{R}^{N-2}$, the derivatives of the function $\mathscr{M}$ satisfy*

$$\partial_\varrho \mathscr{M}(\varrho, q) = \frac{1}{D^2 f(\mu'(\varrho, q))1^N \cdot 1^N}, \quad \partial_{q_j} \mathscr{M}(\varrho, q) = \frac{D^2 f(\mu'(\varrho, q))1^N \cdot \xi^j}{D^2 f(\mu'(\varrho, q))1^N \cdot 1^N} \, .$$

The derivatives of the vector field $\mathscr{R}$ satisfy

$$\begin{aligned} \partial_\varrho \mathscr{R}_i(\varrho, q) =& \frac{D^2 f(\mu'(\varrho, q))1^N \cdot e^i}{D^2 f(\mu'(\varrho, q))1^N \cdot 1^N} \, , \\ \partial_{q_j} \mathscr{R}_i(\varrho, q) =& D^2 f(\mu'(\varrho, q))\xi^j \cdot e^i - \frac{D^2 f(\mu'(\varrho, q))1^N \cdot e^i \, D^2 f(\mu'(\varrho, q))1^N \cdot \xi^j}{D^2 f(\mu'(\varrho, q))1^N \cdot 1^N} \, . \end{aligned}$$

An essential ingredient for the analysis of the model is the **pressure function**.

Proposition 5.3. *Assumptions of Lemma 5.1. We define*

$$P(\varrho, q) := f\left(\sum_{i=1}^{N-2} q_i \, \xi^i + \mathscr{M}(\varrho, q) \, 1^N\right) \, .$$

(a) There are $c_1, \, C_1 > 0$ such that $\partial_\varrho P(\varrho, q) \geq c_1$ and $|\partial_q P(\varrho, q)| \leq C_1$ for all $(\varrho, q) \in I \times \mathbb{R}^{N-2}$.

(b) For each compact subset $K \subset I \times \mathbb{R}^{N-2}$, there is $c = c_K$ such that $|\partial_\varrho P(\varrho, q)| \leq c_K$ for all $(\varrho, q) \in K$.

(c) Suppose that $\bar{V}_{i_N} \leq \bar{V}_{i_{N-1}} \leq \ldots \leq \bar{V}_{i_1}$ is an ordering of the critical values, and denote $I_k =]V_{i_k}^{-1}, \, V_{i_{k+1}}^{-1}[$. Suppose that I' is a compact subset of I_k for a $k \in \{1, \ldots, N-1\}$, then there is a constant $c = c_{I'}$ such that $|\partial_\varrho P(\varrho, q)| \leq c_{I'}$ for all $\varrho \in I'$, $q \in \mathbb{R}^{N-2}$.

(d) There are constants $c_i > 0$ and $C_i > 0$ ($i = 1, 2, 3$) such that for all $(\varrho, q) \in I \times \mathbb{R}^{N-2}$

$$\begin{aligned} -c_1 - c_2 \, |q| + c_3 \, \ln \max &\left\{1, \frac{1}{\varrho_{\max} - \varrho}, \, \frac{1}{\varrho - \varrho_{\min}}\right\} \\ &\leq |P(\varrho, q)| \leq C_1 + C_2 \, |q| + C_3 \, \ln \max \left\{1, \frac{1}{\varrho_{\max} - \varrho}, \, \frac{1}{\varrho - \varrho_{\min}}\right\} \, . \end{aligned}$$

Proof. We employ the Lemma 5.2 to obtain the representation of the derivatives:

$$\begin{aligned} \partial_\varrho P(\varrho, q) =& \frac{\varrho}{D^2 f(\mu'(\varrho, q))1^N \cdot 1^N} \, , \\ \partial_{q_j} P(\varrho, q) =& \mathscr{R}(\varrho, q) \cdot \xi^j - \varrho \, \frac{D^2 f(\mu'(\varrho, q))1^N \cdot \xi^j}{D^2 f(\mu'(\varrho, q))1^N \cdot 1^N} \text{ for } j = 1, \ldots, N-2 \, . \end{aligned}$$

We obtain the constant c_1 in (a) using that $\varrho \geq \varrho_{\min}$ and $D^2 f 1^N \cdot 1^N \leq C$. We obtain the bound for $\sup |\partial_q P|$ using $\mathscr{R} \subset S_0$ and from Lemma 4.3, (e).

Suppose that $\bar{V}_{i_N} \leq \bar{V}_{i_{N-1}} \leq \ldots \leq \bar{V}_{i_1}$ is an ordering of the critical values, and denote $I_k =]V_{i_k}^{-1},\, V_{i_{k+1}}^{-1}[$. Suppose that for $k \in \{1, \ldots, N-1\}$, I' is a compact subset of I_k. If $\varrho \in I'$, then

$$\min_{i=1,\ldots,N} |\varrho - \bar{V}_i^{-1}| \geq \operatorname{dist}(I',\, \partial I_k) > 0\,,$$

and the inequality of Lemma 4.3, (d) together with the representation of $\partial_\varrho P$ imply that

$$\sup_{\varrho \in I'} \partial_\varrho P(\varrho,\, q) \leq C_0^{-1}\, \varrho_{\max}\, \frac{1}{\operatorname{dist}(I',\, \partial I_k)}\,.$$

This proves (c). In order to obtain further bounds, we recall that Lemma 5.1 implies for $i = 1, \ldots, N$ the identity $\mu_i = \sum_{k=1}^{N-2} q_k\, \xi_i^k + \zeta\, \bar{V}_i + \mathscr{M}(\varrho,\, q)$, while the Lemma 4.1, identity (32) (or also the ground relation (7) and the pressure representation (41)) yield $\mu_i = \bar{V}_i\, (P(\varrho,\, q) + \zeta) + \mu_i^{\text{ref}} + \frac{1}{m_i}\, \ln y_i$. Comparing both relations, we obtain that

$$\sum_{k=1}^{N-2} q_k\, \xi_i^k + \mathscr{M}(\varrho,\, q) = \bar{V}_i\, P(\varrho,\, q) + \frac{1}{m_i}\, \ln y_i + \mu_i^{\text{ref}}\,. \tag{42}$$

We define i_0 to be an index associated with the largest among the mass fractions $x_i := \rho_i/\varrho$. Since the fractions sum up to one, this in particular implies that $x_{i_0} \geq N^{-1}$. Thus, for the number fractions $y_i = n_i/n_{\text{tot}}$, we obtain that $y_{i_0} \geq r\, x_{i_0} > \frac{r}{N}$ with $r := \min_i m_i / \max_i m_i$. Subtracting now the line with index i_0 to the other lines in the (42) yields for all $j \neq i_0$

$$\frac{1}{m_j}\, \ln y_j - \frac{1}{m_{i_0}}\, \ln y_{i_0} = -P(\varrho,\, q)\, (\bar{V}_j - \bar{V}_{i_0}) + \sum_{k=1}^{N-2} q_k\, (\xi_j^k - \xi_{i_0}^k) - \mu_j^{\text{ref}} - \mu_{i_0}^{\text{ref}}\,. \tag{43}$$

For technical simplicity, we shall assume that the constants $\mu_j^{\text{ref}} - \mu_{i_0}^{\text{ref}}$, that actually play no role, are all zero. We first focus on the case $q = 0$ in $\mathbb{R}^{N-2}$. Then for all $j \neq i_0$ the identities (43) yield

$$\frac{1}{m_j}\, \ln y_j - \frac{1}{m_{i_0}}\, \ln y_{i_0} = -P(\varrho,\, 0)\, (\bar{V}_j - \bar{V}_{i_0})\,. \tag{44}$$

In particular, since y is a vector of fractions all smaller than one

$$-\frac{1}{m_{i_0}}\, \ln y_{i_0} \geq -P(\varrho,\, 0)\, (\bar{V}_{i_N} - \bar{V}_{i_0}), \quad -\frac{1}{m_{i_0}}\, \ln y_{i_0} \geq -P(\varrho,\, 0)\, (\bar{V}_{i_1} - \bar{V}_{i_0})\,.$$

If it is possible to choose i_0 such that $i_1 \neq i_0(\varrho) \neq i_N$, we obtain the obvious consequence that

$$|P(\varrho,\, 0)| \leq \tfrac{1}{2\, d_0\, \min_i m_i}\, |\ln \tfrac{r}{N}| =: c_2\,, \tag{45}$$

where d_0 is the constant defined in Lemma 4.3. Let us discuss the cases $i_0(\varrho) \in \{i_1,\, i_N\}$, which means that the largest mass fraction is x_{i_1} or x_{i_N}. In the case of $i_0 = i_N$ we exploit the identities $\sum_{i=1}^N x_i\, \bar{V}_i = \frac{1}{\varrho}$ and $\bar{V}_{i_N} = \frac{1}{\varrho_{\max}}$ to see that

$$\sum_{i \neq i_N} x_i\, (\bar{V}_i - \bar{V}_{i_N}) = \frac{1}{\varrho} - \frac{1}{\varrho_{\max}} = \frac{\varrho_{\max} - \varrho}{\varrho_{\max}\, \varrho}\,.$$

The latter identity possesses two consequences. At first, there always exists at least one $j'(\varrho) \neq i_N$ such that

$$x_{j'(\varrho)}\, \inf_{\bar{V}_k \neq i_N} (\bar{V}_k - \bar{V}_{i_N}) \leq \frac{\varrho_{\max} - \varrho}{\varrho_{\max}\, \varrho}\,. \tag{46}$$

At second, the largest mass fraction having $\bar{V}_j \neq \bar{V}_{i_N}$, denoted $x_{j(\varrho)}$ satisfies

$$(N-1) \sup_{\bar{V}_i \neq \bar{V}_{i_N}} \left(\bar{V}_i - \bar{V}_{i_N}\right) x_{j(\varrho)} \geq \frac{\varrho_{\max} - \varrho}{\varrho_{\max}\, \varrho} . \tag{47}$$

We implement (47) into (44) choosing $j = j(\varrho)$. Observe that the number fraction y_j satisfies $y_j \geq r\, x_j$ and therefore

$$y_j \geq \frac{r}{N-1} \frac{\varrho_{\min}}{\varrho_{\max}} \frac{\varrho_{\max} - \varrho}{\varrho_{\max} - \varrho_{\min}} =: \alpha_0 \left(\varrho_{\max} - \varrho\right) .$$

By the choice of $j(\varrho)$, we have $\bar{V}_j > \bar{V}_{i_N}$ and $\bar{V}_j - \frac{1}{\varrho_{\max}} \geq 2\, d_0$. Thus

$$\begin{aligned} P(\varrho,\, 0) =& \frac{1}{\bar{V}_j - \bar{V}_{i_N}} \left(\frac{1}{m_{i_0}} \ln y_{i_0} - \frac{1}{m_j} \ln y_j\right) \\ \leq & \frac{1}{m_j \, (\bar{V}_j - \bar{V}_{i_N})} \ln \frac{1}{\alpha_0 \, (\varrho_{\max} - \varrho)} . \end{aligned} \tag{48}$$

Implement now (46) into (44) choosing $j = j'(\varrho)$. We observe that $y_{j'} \leq \frac{x_{j'}}{r}$. Thus

$$y_{j'} \leq \frac{1}{r \, \inf_{\bar{V}_k \neq i_N} (\bar{V}_k - \bar{V}_{i_N})} \frac{\varrho_{\max} - \varrho}{\varrho_{\max}\, \varrho} =: \beta_0 \left(\varrho_{\max} - \varrho\right) .$$

It follows that

$$\begin{aligned} P(\varrho,\, 0) =& \frac{1}{\bar{V}_{j'} - \bar{V}_{i_N}} \left(\frac{1}{m_{i_0}} \ln y_{i_0} - \frac{1}{m_{j'}} \ln y_{j'}\right) \\ \geq & - \frac{1}{2\, d_0\, m_{i_0}} \, |\ln \frac{r}{N}| + \frac{1}{m_{j'} \, (\bar{V}_{j'} - \bar{V}_{i_N})} \ln \frac{1}{\beta_0 \, (\varrho_{\max} - \varrho)} . \end{aligned} \tag{49}$$

In the case that x_{i_N} is the largest mass fraction, we combine (48) and (49) to easily prove that

$$-c_3 + c_4 \ln \frac{1}{\varrho_{\max} - \varrho} \leq P(\varrho,\, 0) \leq C_3 + C_4 \frac{1}{\varrho_{\max} - \varrho} .$$

for certain fixed constant $c_3,\, c_4 > 0$ and $C_3,\, C_4 > 0$. We can discuss the case $i_0 = i_1$ similarly. In this case, we find the estimates

$$\tilde{c}_3 - \tilde{c}_4 \ln \frac{1}{\varrho - \varrho_{\min}} \geq P(\varrho,\, 0) \geq -\tilde{C}_3 - \tilde{C}_4 \frac{1}{\varrho - \varrho_{\min}} .$$

Recalling also (45), we have proved that

$$-c_5 + c_6 \, |\ln \min\{\varrho_{\max} - \varrho,\ \varrho - \varrho_{\min}\}| \leq |P(\varrho,\, 0)| \leq C_5 + C_6 \, |\ln \min\{\varrho_{\max} - \varrho,\ \varrho - \varrho_{\min}\}| .$$

It remains to estimate $|P(\varrho,\, q)| \leq |P(\varrho,\, q) - P(\varrho,\, 0)| + |P(\varrho,\, 0)|$ and to use the boundedness of P_q to obtain (d). At last, we reconsider (43), and get for all $j \neq i_0$

$$\ln y_j \geq \frac{m_j}{m_{i_0}} \ln y_{i_0} - C \left(|P(\varrho,\, q)| + |q| + \|\mu^{\text{ref}}\|_{\infty}\right) \geq -c \left(1 + |q| + |P(\varrho,\, 0)\right) .$$

Thus, if $(\varrho,\, q)$ is in a compact of $I \times \mathbb{R}^{N-2}$, all fractions y_j are bounded strictly from zero by a constant depending on $\varrho - \varrho_{\min}$, $\varrho_{\max} - \varrho$ and $\sup_{(\varrho,\, q) \in K} |q|$. Invoking the representation of $D^2 f 1^N \cdot 1^N$ in Lemma 4.3, (b), we then have $D^2 f 1^N \cdot 1^N \geq c_K \, \inf_{i=1,\ldots,N} \rho_i$ and $|\partial_\varrho P| \leq c_K$. This proves (b). □

5.3 Reformulation of the partial differential equations

We make use of the relation (6) and the equivalence of Lemma 5.1 to see that the diffusion fluxes possess the equivalent form

$$\begin{aligned} J^i &= -M_{i,j}(\rho_1,\ldots,\rho_N)\,\nabla\mu_j \\ &= -\sum_{\ell=1}^{N-2} M_{i,j}(\rho_1,\ldots,\rho_N)\,\xi^\ell_j\,\nabla q_\ell - M_{i,j}(\rho_1,\ldots,\rho_N)\,\bar V_j\,\nabla\zeta - M_{i,j}(\rho_1,\ldots,\rho_N)\,\nabla\mathscr{M}(\varrho,\,q) \\ &= -\sum_{\ell=1}^{N-2} M_{i,j}(\rho_1,\ldots,\rho_N)\,\xi^\ell_j\,\nabla q_\ell - M_{i,j}(\rho_1,\ldots,\rho_N)\,\bar V_j\,\nabla\zeta\,. \end{aligned}$$

If we introduce the rectangular projection matrix $\Pi_{j,\ell} = \xi^\ell_j$ for $\ell = 1,\ldots,N-2$ and $j = 1,\ldots,N$, then $J = -M\,\Pi\nabla q - M\,\bar V\,\nabla\zeta$. Thus, we consider equivalently

$$\begin{aligned} \partial_t\rho + \operatorname{div}(\rho\,v - M\,\Pi\,\nabla q - M\,\bar V\,\nabla\zeta) = 0 \\ \partial_t(\varrho\,v) + \operatorname{div}(\varrho\,(v\otimes v) - \mathbb{S}(\nabla v)) + \nabla P(\varrho,\,q) + \nabla\zeta = 0\,. \end{aligned}$$

In the latter system we have $\rho = \mathscr{R}(\varrho,\,q)$ and $(\varrho, q_1,\ldots,q_{N-2},\zeta,\,v_1,v_2,v_3)$ are the independent variables.

Next we define for $k = 1,\ldots,N-2$ the maps

$$\begin{aligned} R_k(\varrho,\,q) :=& \sum_{j=1}^N \xi^k_j\,\rho_j = \Pi^T\rho \\ =& \sum_{j=1}^N \xi^k_j\,f_{\mu_j}\left(\sum_{\ell=1}^{N-2} q_\ell\,\xi^\ell + \mathscr{M}(\varrho,\,q_1,\ldots,q_{N-2})\,1^N\right)\,. \end{aligned}$$

Multiplying the mass transfer equations with ξ^k_i, we obtain that

$$\partial_t R_k(\varrho,\,q) + \operatorname{div}\Big(R_k(\varrho,\,q)\,v - [\Pi^T\,M(\rho)\,\Pi]_{k,\ell}\,\nabla q_\ell - [\Pi^T\,M(\rho)\,\bar V]_k\,\nabla\zeta\Big) = 0\,.$$

It turns out that the matrix $\Pi^T\,M(\rho)\,\Pi \in \mathbb{R}^{(N-2)\times(N-2)}$ is symmetric and strictly positive definite on all states $\rho \in \mathbb{R}^N_+$ thanks to the assumption (23). We then multiply the mass balance equations with $\bar V_i$, and making use of the constraint yields

$$\operatorname{div}(v - \bar V\cdot M(\rho)\,\Pi\,\nabla q - \bar V\cdot M(\rho)\,\bar V\,\nabla\zeta) = 0\,.$$

Using once again the identity $\rho = \mathscr{R}(\varrho,\,q)$, we introduce for notational simplicity

$$\begin{aligned} \widetilde{M}(\varrho,\,q) :=& \Pi^T\,M(\mathscr{R}(\varrho,\,q))\,\Pi \in \mathbb{R}^{(N-2)\times(N-2)} \\ A(\varrho,\,q) :=& \Pi^T\,M(\mathscr{R}(\varrho,\,q))\,\bar V \in \mathbb{R}^{N-2} \\ b(\varrho,\,q) :=& \bar V\cdot M(\mathscr{R}(\varrho,\,q))\,\bar V\,. \end{aligned}$$

Overall we get for the variables $(\varrho,\,q_1,\ldots,q_{N-2},\,\zeta,\,v)$ instead of (3), (4) the equations

$$\partial_t R_k(\varrho,\,q) + \operatorname{div}(R_k(\varrho,\,q)\,v - \widetilde{M}_{k,\ell}(\varrho,\,q)\,\nabla q_\ell - A_k(\varrho,\,q)\,\nabla\zeta) = 0 \qquad (50)$$

$$\operatorname{div}(v - A(\varrho,\,q)\cdot\nabla q - b(\varrho,\,q)\,\nabla\zeta) = 0 \qquad (51)$$

$$\partial_t\varrho + \operatorname{div}(\varrho\,v) = 0 \qquad (52)$$

$$\partial_t(\varrho\,v) + \operatorname{div}(\varrho\,(v\otimes v) - \mathbb{S}(\nabla v)) + \nabla P(\varrho,\,q) + \nabla\zeta = 0\,. \qquad (53)$$

The problem (P') consisting of (50), (51), (52) and (53) for the variables $(\varrho,\, q,\, \zeta,\, v)$ exhibits more non-linearities than the original problem (P). However it has the overwhelming advantage that up to the restriction on the total mass density $\varrho_{\min} \leq \varrho \leq \varrho_{\max}$, it is completely free of constraints.

Further it is to note that the differential operator is linear in the variable ζ.

In order to reformulate the boundary conditions (14), we apply the same concepts:

$$M^{\Gamma}\,(\mu - \mu^{\Gamma}) = M^{\Gamma}\,\Pi\,(q - q^{\Gamma}) + M^{\Gamma}\,\bar{V}\,(\zeta - \zeta^{\Gamma})\,,$$

with the obvious definitions $q^{\Gamma}_{\ell} := \mu^{\Gamma}\cdot\eta^{\ell}$ for $\ell = 1, \ldots, N-2$ and $\zeta^{\Gamma} = \mu^{\Gamma}\cdot\eta^{N-1}$. We introduce

$$\begin{aligned}\widetilde{M}^{\Gamma}(x,\, t) :=& \Pi^{T}\, M^{\Gamma}(x,\, t)\,\Pi \in \mathbb{R}^{(N-2)\times(N-2)}\\ A^{\Gamma}(x,\, t) :=& \Pi^{T}\, M^{\Gamma}\,\bar{V} \in \mathbb{R}^{N-2}\\ b^{\Gamma}(x,\, t) :=& M^{\Gamma}\,\bar{V}\cdot\bar{V}\,.\end{aligned}$$

Thus, we obtain for the fluxes the boundary conditions

$$-\widetilde{M}_{k,\ell}(\varrho,\, q)\,\nabla_{\nu} q_{\ell} - A_{k}(\varrho,\, q)\,\nabla_{\nu}\zeta = \widetilde{M}^{\Gamma}_{\ell}\,(q_{\ell} - q^{\Gamma}_{\ell}) + A^{\Gamma}_{k}\,(\zeta - \zeta^{\Gamma}) \tag{54}$$

$$- A(\varrho,\, q)\cdot\nabla_{\nu} q - b(\varrho,\, q)\,\nabla_{\nu}\zeta = [A^{\Gamma}]^{T}\cdot(q - q^{\Gamma}) + b^{\Gamma}\,(\zeta - \zeta^{\Gamma})\,. \tag{55}$$

Initial conditions $q^0_1, \ldots, q^0_{N-2}$ for the variables $q_1, \ldots, q_{N-2}$ are derived from the equations

$$\rho^0(x) = \mathscr{R}(\varrho_0(x),\, q^0(x)) = \sum_{\ell=1}^{N-2} R_{\ell}(\varrho_0(x),\, q^0(x))\,\eta^{\ell} + \eta^{N-1} + \varrho_0(x)\,\eta^{N}\ .$$

Definition 5.4. *We call weak solution to the problem (P') a vector $(\varrho,\, q,\, \zeta,\, v)$ such that*

$$\varrho \in L^{\infty}(Q_T;\, [\varrho_{\min},\, \varrho_{\max}]),\ q \in W^{1,0}_2(Q_T;\, \mathbb{R}^{N-2}),\ \zeta \in W^{1,0}_2(Q_T),\ v \in V^{1,0}_2(Q_T;\, \mathbb{R}^3) \tag{56}$$

such that $P(\varrho,\, q) \in L^1(Q_T)$ and the following integral relations are valid:

$$\begin{aligned}- \int_Q R(\varrho,\, q)\cdot(\partial_t\psi + v\cdot\nabla\psi)\,dxdt + \int_Q \{\widetilde{M}(\varrho,\, q)\,\nabla q + A(\varrho,\, q)\,\nabla\zeta\}\cdot\nabla\psi\,dxdt&\\ + \int_S \{\widetilde{M}^{\Gamma}\,(q - q^{\Gamma}) + A^{\Gamma}\,(\zeta - \zeta^{\Gamma})\}\,\psi\,dSdt = \int_{\Omega} \Pi^T \rho^0(x)\cdot\psi(x,\, 0)\,dx&\end{aligned} \tag{57}$$

$$\begin{aligned}- \int_Q v\cdot\nabla\phi\,dxdt + \int_Q \{b(\varrho,\, q)\,\nabla\zeta + A(\varrho,\, q)\cdot\nabla q\}\cdot\nabla\phi\,dxdt&\\ + \int_S \{b^{\Gamma}\,(\zeta - \zeta^{\Gamma}) + A^{\Gamma}\cdot(q - q^{\Gamma})\}\,\phi\,dSdt = 0&\end{aligned} \tag{58}$$

$$- \int_Q \varrho\,(\partial_t\phi + v\cdot\nabla\phi)\,dxdt = \int_{\Omega} \varrho_0(x)\,\phi(x,\, 0)\,dx \tag{59}$$

$$\begin{aligned}- \int_Q \varrho\, v\cdot(\partial_t\eta + (v\cdot\nabla)\eta)\,dxdt + \int_Q \mathbb{S}(\nabla v)\ :\ \nabla\eta\,dxdt - \int_Q p\,\operatorname{div}\eta\,dxdt&\\ = \int_Q \operatorname{div}\eta\,d\kappa + \int_{\Omega} \varrho_0(x)\,v^0(x)\cdot\eta(x,\, 0)\,dx&\end{aligned} \tag{60}$$

for all $\psi \in C^1(\overline{Q}_T;\, \mathbb{R}^{N-2})$, all $\phi \in C^1(\overline{Q_T})$ such that $\psi(T) = 0$ and $\phi(T) = 0$, and for all $\eta \in C^1_c(\Omega\times[0,\, T[;\, \mathbb{R}^3)$. Here $\kappa \in \mathcal{M}(\overline{Q})$ is the measure satisfying the conditions of Definition 3.3.

We then have the following equivalence:

Proposition 5.5. *The problem* (P') *possesses a global weak solution* $(\varrho,\, q,\, \zeta,\, v)$ *as in Definition 5.4 if and only if the problem* (P) *possesses the weak solution* $(\mathscr{R}(\varrho,\, q),\, v,\, P(\varrho,\, q)+\zeta)$ *in the sense of Def. 3.3.*

From now we will concentrate on proving existence for (P'), which is directly accessible to functional analytic methods. The proof consists in the steps: Regularisation in Section 6, derivation of uniform bounds in Section 7, limit passage with weakly convergent subsequences in Section 8, compactness statements in Section 9 and representation of the limit pressure in Section 10 and Section 11.

6 Regularisation

We next want to prove global existence for the system (50), (51), (52) and (53) for the variables $(\varrho,\, q_1, \ldots, q_{N-2},\, \zeta,\, v)$. The boundary conditions are (54), (55). A result in this direction does not follow directly of standard Theorems, so we shall construct approximate problems that are easier to solve. There is of course much arbitrariness in the way to construct such an approximation scheme, but some restriction comes from the need of respecting the energy identity, which reflects the thermodynamic consistency. We stabilise the kinetic matrix M. For $\sigma > 0$, we define

$$M^{\sigma}(\rho) := M(\rho) + \sigma\, 1^N \otimes 1^N \text{ for } \rho \in \mathbb{R}^N_+ \,. \tag{61}$$

Due to the assumptions on M, we can verify that M^{σ} has full rank, and the condition (23) yields

$$\underline{\lambda}\, |\mathcal{P}\xi|^2 + \sigma\, |1^N\cdot \xi|^2 \leq M^{\sigma}(\rho) \leq \overline{\lambda}\, (1+|\rho|)\, |\mathcal{P}\xi|^2 + \sigma\, |1^N\cdot\xi|^2 \text{ for all } \xi \in \mathbb{R}^N \,. \tag{62}$$

If the regularised kinetic matrix has full rank, it is not necessary to pass to the coordinates $(\varrho,\, q)$. In the first level of our approximation scheme, we look for a vector $(\mu_1, \ldots, \mu_N,\, v_1,\, v_2,\, v_3)$ solution to

$$\partial_t f_{\mu_i}(\mu) + \operatorname{div}\Big(f_{\mu_i}(\mu)\, v - M^{\sigma}(f_{\mu_i}(\mu))\, \nabla\mu\Big) = 0 \tag{63}$$

$$\partial_t(\varrho\, v) + \operatorname{div}(\varrho\, (v\otimes v) - \mathbb{S}(\nabla v)) + \nabla f(\mu) = \operatorname{div}(\sum_{i=1}^N J^i\, v) \tag{64}$$

$$\varrho := 1^N \cdot \nabla_{\mu} f(\mu) = \sum_{i=1}^N f_{\mu_i}(\mu)\,.$$

This is supplemented by the boundary conditions (13) and

$$-M^{\sigma}(f_{\mu}(\mu))\, \nabla_{\nu}\mu = M^{\Gamma,\sigma}\, (\mu - \mu^{\Gamma}) \text{ on } S_T \tag{65}$$

Here we have introduced $M^{\Gamma,\sigma} := M^{\Gamma} + \sigma\, 1^N \otimes 1^N$. We call (P_{σ}) the problem of finding $\mu : Q_T \to \mathbb{R}^N$ and $v :\, Q_T \to \mathbb{R}^3$ subject to (63), (64), (65) and (13), together with the initial conditions $f_{\mu_i}(\mu(x,\, 0)) = \rho_i^0(x)$ and $v(x,\, 0) = v^0(x)$ for $x \in \Omega$.

It turns out that the proof of global existence for the regularised problem (P_{σ}) with $\sigma > 0$ is not direct. In fact the problem (63) for μ is a doubly non-linear *degenerated* parabolic system, because the coefficient $D^2 f$ in front of the time derivative μ_t has rank $(N-1)$ (the kernel is the span of $\bar{V}$, Lemma 4.1), while the positive eigenvalues might tend to zero for $|\mu| \to \infty$. In order to simplify the proof of existence, we have to apply another stabilisation step. This will be discussed separately in the Appendix, Section A. The reason is that we regard the limit passage $\sigma \to 0$ as the main technical step of the paper, to be discussed with priority. For (P_{σ}), we thus obtain the following existence statement.

Proposition 6.1. *Adopt the assumptions* (23), (24)*. Assume that* $\rho^0 \in L^\infty(\Omega; \overline{S_0})$ *satisfies the condition* (26)*, that* $v^0 \in L^2(\Omega; \mathbb{R}^3)$ *and that* $\mu^\Gamma \in L^2(S_T; \mathbb{R}^N)$*. Then, there exists a pair* (μ, v) *with* $\mu \in W^{1,0}_2(Q_T; \mathbb{R}^N)$ *and* $v \in V^{1,0}_2(Q_T; \mathbb{R}^3)$ *such that*

$$p := f(\mu) \in W^{1,0}_2(Q_T)\,,$$
$$\rho := \nabla_\mu f(\mu) \in L^\infty(Q_T; S_0) \cap W^{1,0}_2(Q; \mathbb{R}^N)\,,$$

such that $\rho > 0$ *almost everywhere in* Q_T*, and for all* $\psi \in C^1(\overline{Q}_T; \mathbb{R}^N)$ *such that* $\psi(T) = 0$ *and* $\eta \in C^1_c(\Omega \times [0, T[; \mathbb{R}^3)$ *the following integral relations are valid*

$$- \int_Q \rho \cdot \partial_t \psi \, dxdt - \int_Q \rho \cdot \nabla\psi \cdot v \, dxdt + \int_Q M^\sigma(\rho)\, \nabla\mu\, \nabla\psi \, dxdt$$
$$+ \int_S M^{\Gamma,\sigma}\, (\mu - \mu^\Gamma)\, \psi dSdt = \int_\Omega \rho^0(x) \cdot \psi(x, 0)\, dx \tag{66}$$
$$- \int_Q \varrho\, v \cdot \partial_t\eta \, dxdt - \int_Q \varrho\, v \otimes v \,:\, \nabla\eta \, dxdt + \int_Q \mathbb{S}(\nabla v)\,:\, \nabla\eta\, dxdt - \int_Q p \,\operatorname{div}\eta \, dxdt$$
$$= \int_\Omega \varrho_0(x)\, v^0(x) \cdot \eta(x, 0)\, dx - \int_Q \left(\sum_{i=1}^N J^i \cdot \nabla\right) \eta \cdot v \, dxdt \tag{67}$$

in which $J = -M^\sigma(\rho)\, \nabla\mu$ *and* $\varrho := \sum_{i=1}^N \rho_i$*. Moreover* $\rho \in C([0,T]; L^2(\Omega; \mathbb{R}^N))$*,* $\varrho v \in C([0,T]\, L^2(\Omega; \mathbb{R}^3))$*, and for all* $t \in]0,T[$ *the following dissipation inequality is valid:*

$$\int_\Omega [k(\rho(t)) + \tfrac{1}{2}\, \varrho(t)\, |v(t)|^2]\, dx + \int_{Q_t} \{M^\sigma(\rho)\nabla\mu \cdot \nabla\mu + \mathbb{S}(\nabla v) \cdot \nabla v\}\, dxds$$
$$+ \int_{S_t} M^{\Gamma,\sigma}\, (\mu - \mu^\Gamma) \cdot \mu\, dSds \leq \int_\Omega [k(\rho^0) + \tfrac{1}{2}\, \varrho_0\, |v^0|^2]\, dx\,.$$

Remark 6.2. *If in* (66) *we choose* $\psi = 1^N\, \tilde\psi$ *with scalar* $\tilde\psi \in C^1(\overline{Q})$ *such that* $\tilde\psi(T) = 0$*, we obtain the perturbed continuity equation*

$$- \int_Q \varrho \cdot \partial_t\tilde\psi \, dxdt - \int_Q \varrho\, v \cdot \nabla\tilde\psi \, dxdt = \int_\Omega \varrho_0(x) \cdot \tilde\psi(x, 0)\, dx$$
$$+ \int_Q \sum_{i=1}^N J^i \cdot \nabla\tilde\psi \, dxdt + \int_S 1^N \cdot \jmath^\Gamma\, \tilde\psi \, dSdt \tag{68}$$

with $J = -M^\sigma(\rho)\, \nabla\mu$ *and* $\jmath^\Gamma = -M^{\Gamma,\sigma}\, (\mu - \mu^\Gamma)$*.*

7 Uniform bounds

Now we want to explore the passage to the limit $\sigma \to 0$ for the family $\{(\mu^\sigma, v^\sigma)\}_{\sigma>0}$ constructed in Proposition 6.1. First we will derive uniform estimates exploiting the dissipation inequality. We previously recall the definitions of new variables which are simply linear combinations of elements of the vector μ^σ (cf. Lemma 5.1)

$$q^\sigma := (\mu^\sigma \cdot \eta^1, \ldots, \mu^\sigma \cdot \eta^{N-2}) \in W^{1,0}_2(Q_T; \mathbb{R}^{N-2}), \quad \zeta_\sigma := \mu^\sigma \cdot \eta^{N-1} \in W^{1,0}_2(Q_T) \tag{69}$$

We moreover use the abbreviations $J^\sigma := -M^\sigma(\rho)\nabla\mu^\sigma$ and $\jmath^{\Gamma,\sigma} := -M^{\Gamma,\sigma}(\mu^\sigma - \mu^\Gamma)$.

Proposition 7.1. *For the family $\{(\mu^\sigma,\, v^\sigma)\}_{\sigma>0}$ constructed in Proposition 6.1, the following uniform bounds are valid:*

$$\begin{aligned}
&\|\rho^\sigma\|_{L^\infty(Q;\,\mathbb{R}^N)} + \|v^\sigma\|_{L^{2,\infty}(Q;\,\mathbb{R}^3)} \leq C_0\\
&\|v^\sigma\|_{W^{1,0}_2(Q;\,\mathbb{R}^3)} + \|q^\sigma\|_{W^{1,0}_2(Q;\,\mathbb{R}^{N-2})} + \|\zeta^\sigma\|_{W^{1,0}_2(Q)} \leq C_0\\
&\|1^N\cdot J^\sigma\|_{L^2(Q;\,\mathbb{R}^3)} + \|1^N\cdot \jmath^{\Gamma,\sigma}\|_{L^2(S)} \leq C_0\,\sqrt{\sigma},\, .
\end{aligned}$$

Proof. Owing to the statement of Prop. 6.1, we know that $\rho^\sigma(x,\,t) = \partial_\mu f(\mu^\sigma(x,\,t))$ belongs to S_0 for almost all $(x,\,t)\in Q$. This yields in particular that $\varrho_{\min}\leq\varrho_\sigma\leq\varrho_{\max}$ almost everywhere in Q proving the L^∞ bound for $\{\rho^\sigma\}$. From the dissipation inequality, we directly obtain that $\|\sqrt{\varrho_\sigma}\,v^\sigma\|_{L^{2,\infty}(Q;\,\mathbb{R}^3)}\leq C_0$, and since $\varrho_{\min}\leq\varrho_\sigma$ we obtain that $\|v^\sigma\|_{L^{2,\infty}(Q;\,\mathbb{R}^3)}\leq C_0$. The bound on ∇v^σ in $L^2(Q;\,\mathbb{R}^9)$ follows from the fact that $\int_Q \mathbb{S}(\nabla v^\sigma)\,:\,\nabla v^\sigma\,dxdt\leq C_0$.

Due to the fact that $\int_Q M^\sigma_{i,j}\,\mu^\sigma_{i,x}\,\mu^\sigma_{j,x}\,dxdt\leq C_0$, the condition (23) guaranties that

$$\underline{\lambda}\int_Q |\mathcal{P}\,\nabla\mu^\sigma|^2\,dxdt \leq \int_Q M^\sigma_{i,j}\,\nabla\mu^\sigma_i\cdot\nabla\mu^\sigma_j\,dxdt\leq C_0 \tag{70}$$

$$\sigma\int_Q |\nabla(1^N\cdot\mu^\sigma)|^2\,dxdt \leq \int_Q M^\sigma_{i,j}\,\nabla\mu^\sigma_i\cdot\nabla\mu^\sigma_j\,dxdt\leq C_0\,. \tag{71}$$

From the first bound (70), we obtain that $\underline{\lambda}\,(\|q^\sigma_x\|_{L^2(Q;\,\mathbb{R}^{N-2})} + \|\zeta^\sigma_x\|_{L^2(Q)}) \leq C_0$. Moreover, the dissipation inequality directly yields $\int_S M^{\Gamma,\sigma}\,\mu^\sigma\cdot\mu^\sigma\,dSdt\leq C_0$ which implies that

$$\underline{\lambda}^\Gamma\int_S |\mathcal{P}\,\mu^\sigma|^2\,dSdt\leq\int_Q M^{\Gamma,\sigma}\,\mu^\sigma\cdot\mu^\sigma\,dSdt\leq C_0\,,$$

resulting into $\underline{\lambda}^\Gamma\,(\|q^\sigma\|_{L^2(S;\,\mathbb{R}^{N-2})} + \|\zeta^\sigma\|_{L^2(S)})\leq C_0$. From the control of the spatial gradients and the L^2-norm on the surface, we deduce a control on $\|q^\sigma\|_{W^{1,0}_2(Q;\,\mathbb{R}^{N-2})} + \|\zeta^\sigma\|_{W^{1,0}_2(Q)}$.

Finally, observe that $\sum_{i=1}^N J^{\sigma,i} = -N\,\sigma\,\nabla(1^N\cdot\mu^\sigma)$, and therefore, making use of (71), we infer that

$$\|\sum_{i=1}^N J^{\sigma,i}\|_{L^2(Q;\,\mathbb{R}^3)}\leq N\,\sqrt{\sigma}\,\left(\sigma\int_Q|\nabla(1^N\cdot\mu^\sigma)|^2\,dxdt\right)^{\frac{1}{2}}\leq C_0\,\sqrt{\sigma}\,.$$

The same arguments apply to estimating $\sum_{i=1}^N \jmath^{\Gamma,\sigma}_i$. □

Remark 7.2. *Thanks to the embedding $V^{1,0}_2(Q)\subset L^{\frac{10}{3}}(Q)$ and Hölder's inequality we obtain additional bounds*

$$\begin{aligned}
\|v^\sigma\,v^\sigma\|_{L^{\frac{5}{3}}(Q;\,\mathbb{R}^{3\times3})} &\leq c\,\|v^\sigma\|^2_{V^{1,0}_2}\leq C_0\,,\\
\|\sum_{i=1}^N J^{\sigma,i}\,v^\sigma\|_{L^{\frac{5}{4}}(Q;\,\mathbb{R}^{3\times3})} &\leq c\,\|v^\sigma\|_{V^{1,0}_2}\,\|\sum_{i=1}^N J^{\sigma,i}\|_{L^2(Q)}\leq C_0\,\sqrt{\sigma}\,.
\end{aligned}$$

Lemma 7.3. *Tor the sequence $\{(\mu^\sigma,\, v^\sigma)\}_{\sigma>0}$ constructed in Proposition 6.1 we define $m_\sigma(t) := \fint_\Omega \varrho_\sigma(x,\,t)\,dx$, and $M_0 := \fint_\Omega\varrho_0(x)\,dx$. Then $|m_\sigma(t)-M_0|\leq C_1\,\sqrt{\sigma}$.*

Proof. Due to Remark 6.2 with $\tilde{\psi} = 1$, the identity $m'_\sigma(t) = \frac{N}{\lambda_3(\Omega)} \, \sigma \, \int_{\partial\Omega} 1^N \cdot (\mu^\sigma - \mu^\Gamma) \, dS$ is valid. Thus

$$|m_\sigma(t) - M_0| \leq \sigma \, \frac{N^{\frac{3}{2}}}{\lambda_3(\Omega)} \int_{S_t} |\mu^\sigma - \mu^\Gamma| \, dSds \leq \sqrt{\sigma} \, C_0 \, \sqrt{\sigma} \, \|\mu^\sigma - \mu^\Gamma\|_{L^2(S)} \leq C_1 \, \sqrt{\sigma} \, .$$

□

Now we obtain the pressure bound which can be regarded as one of the most important steps in the paper.

Proposition 7.4. *Consider the family $\{(\mu^\sigma, \, v^\sigma)\}_{\sigma>0}$ constructed in Proposition 6.1 and $p_\sigma := f(\mu^\sigma)$. Then $p_\sigma = P(\varrho_\sigma, \, q^\sigma) + \zeta_\sigma$, with the function P of Proposition 5.3, and the norms $\|P(\varrho_\sigma, \, q^\sigma)\|_{L^1(Q)}$ is are uniformly bounded.*

Proof. The pressure decomposition is a direct consequence of the algebraic reduction in Lemma 5.1 (cf. (41)). In order to prove the bound we employ the properties of the so-called Bogovski operator. There exists a solutions $Y \in W^{1,2}_0(\Omega; \, \mathbb{R}^3)$ to the equations $\operatorname{div} Y = \varrho_\sigma(t) - m_\sigma(t)$ satisfying moreover for all $1 < r < +\infty$ the estimate $\|\nabla Y(t)\|_{L^r(\Omega)} \leq c_r \, \|\varrho_\sigma(t) - m_\sigma(t)\|_{L^r(\Omega)}$. Due to Remark 6.2 the derivative $\partial_t(\varrho_\sigma(t) - m_\sigma(t))$ is well defined in the sense of distributions and

$$\begin{aligned}
&\partial_t(\varrho_\sigma(t) - m_\sigma(t)) = f(t)\\
&f(t)(\tilde{\psi}) := \int_\Omega (\varrho_\sigma \, v^\sigma + \sum_{i=1}^N J^{\sigma,i})(x, \, t) \cdot \nabla\tilde{\psi}(x) \, dx\\
&\qquad + \int_{\partial\Omega} \sum_{i=1}^N \jmath_i^{\Gamma,\sigma}(x, \, t) \, \tilde{\psi}(x) \, dS(x) - m'_\sigma(t) \, \int_\Omega \tilde{\psi}(x) \, dx\\
&= \int_\Omega (\varrho_\sigma \, v^\sigma + \sum_{i=1}^N J^{\sigma,i})(x, \, t) \cdot \nabla\tilde{\psi}(x) \, dx + \int_{\partial\Omega} \sum_{i=1}^N \jmath_i^{\Gamma,\sigma}(x, \, t) \, (\tilde{\psi}(x) - \fint_\Omega \tilde{\psi}) \, dS(x) \, .
\end{aligned}$$

For $F(\tilde{\psi}) := \int_{\partial\Omega} \sum_{i=1}^N \jmath_i^{\Gamma,\sigma}(t) \, (\tilde{\psi} - \fint_\Omega \psi \, dx) \, dS$, we can verify that $F(1) = 0$ and that

$$\|F\|_{[W^{1,2}(\Omega)]^*} \leq c \, \|1^N \cdot \jmath^{\Gamma,\sigma}(t)\|_{L^2(\partial\Omega)} \, .$$

Consequently, we can solve the weak Neumann problem and find $u_\sigma = u_\sigma(t) \in W^{1,2}(\Omega)$ such that $\int_\Omega \nabla u_\sigma \cdot \nabla\phi = F(\phi)$ for all $\phi \in W^{1,2}(\Omega)$ and $\|\nabla u_\sigma(t)\|_{L^2(\Omega)} \leq c \, \|F\|_{[W^{1,2}(\Omega)]^*}$. Thus, the distributional time derivative $f(t)$ also possesses the representation

$$f(t)(\tilde{\psi}) = \int_\Omega (\varrho_\sigma \, v^\sigma + \sum_{i=1}^N J^{\sigma,i} + \nabla u_\sigma)(x, \, t) \cdot \nabla\tilde{\psi}(x) \, dx \, .$$

Employing the properties of the Bogovski operator recalled in the Appendix, we then obtain for $Y = \mathcal{B}(\varrho_\sigma - m_\sigma)$ the estimate

$$\begin{aligned}
\|\partial_t Y(t)\|_{L^2(\Omega; \, \mathbb{R}^3)} \leq & c \, \|\partial_t(\varrho_\sigma(t) - m_\sigma(t))\|_{[W^{1,2}(\Omega)]^*}\\
\leq & c \, (\|\varrho_\sigma(t) \, v^\sigma(t)\|_{L^2(\Omega)} + \|\sum_{i=1}^N J^\sigma(t)\|_{L^2(\Omega)} + c \, \|\sum_{i=1}^N \jmath_i^{\Gamma,\sigma}(t)\|_{L^2(\partial\Omega)}) \, .
\end{aligned}$$

Since $\| \sum_{i=1}^N J^{\sigma}\|_{L^2(Q)} + \| \sum_{i=1}^N \jmath_i^{\Gamma,\sigma}\|_{L^2(S)} \leq C_0\, \sqrt{\sigma}$, it follows also that

$$\|\partial_t Y\|_{L^2(Q;\,\mathbb{R}^3)} \leq c\, (\|\varrho_\sigma\, v^\sigma(t)\|_{L^2(Q)} + C_0\, \sqrt{\sigma})\,.$$

Therefore, in view of the estimates of Proposition 7.1, we obtain a uniform bound

$$\|\nabla Y\|_{L^r(Q)} + \|\partial_t Y\|_{L^2(Q)} \leq C_0(r) \text{ for } 1 \leq r < +\infty \text{ arbitrary }\,.$$

We can modify the relation (67) to

$$\begin{aligned} & - \int_Q \varrho_\sigma\, v^\sigma \cdot \partial_t \eta + \int_\Omega \varrho_\sigma(x,\,T)\, v^\sigma(x,\,T) \cdot \eta(x,\,T)\, dx - \int_Q \{\varrho_\sigma\, v^\sigma \otimes v^\sigma - \mathbb{S}(\nabla v^\sigma)\} \,:\, \nabla \eta \\ & = \int_Q p_\sigma\, \operatorname{div} \eta + \int_\Omega \varrho_0(x)\, v^0(x) \cdot \eta(x,\,0)\, dx - \int_Q (\sum_{i=1}^N J^{\sigma,i} \cdot \nabla) \eta \cdot v^\sigma \end{aligned} \tag{72}$$

valid first for all $\eta \in C^1_c(\Omega \times [0,\,T];\, \mathbb{R}^3)$. We can verify that this relation in fact makes sense for all $\eta \in C_c(\Omega \times [0,\,T];\, \mathbb{R}^3)$ having $\|\eta_x\|_{L^5(Q)} + \|\eta_t\|_{L^2(Q)}$ finite, and even that (72) is valid for such vector fields. The proof being fairly standard, we shall spare it. We thus employ $\eta = Y$ as test function in (72) which yields a bound

$$\begin{aligned} & \left|\int_Q p_\sigma\, (\varrho_\sigma(t) - m_\sigma(t))\right| \leq c\, (\|\varrho_\sigma\, v^\sigma\|_{L^2(Q)} + \|\varrho_\sigma\, |v^\sigma|^2\|_{L^{\frac{5}{3}}(Q)} + \|v^\sigma_x\|_{L^2(Q)} + \|J^\sigma\, v^\sigma\|_{L^{\frac{5}{4}}(Q)}) \times \\ & \times (\|Y_t\|_{L^2(Q)} + \|Y_x\|_{L^5(Q)}) + 2\, \|\varrho_\sigma\|_{L^\infty(Q)}\, \|v^\sigma\|_{L^{2,\infty}(Q)}\, \|Y\|_{L^{2,\infty}(Q)} \leq C_0\,. \end{aligned}$$

We express $p_\sigma = P(\varrho_\sigma,\, q^\sigma) + \zeta_\sigma$. Since ζ_σ is bounded in $L^2(Q)$ and ϱ_σ even in $L^\infty(Q)$, it therefore next follows that $\left|\int_Q P(\varrho_\sigma,\, q^\sigma)\, (\varrho_\sigma(t) - m_\sigma(t))\right| \leq C_0$.

Due to Lemma 7.3, we know that $|m_\sigma(t) - M_0| \leq C_1\, \sqrt{\sigma}$. Making use of the properties of P (Prop. 5.3), we obtain that

$$\begin{aligned} & \int_Q |P(\varrho_\sigma,\, q^\sigma) - P(m_\sigma,\, q^\sigma)|\, |\varrho_\sigma(t) - m_\sigma(t)| \leq C_0 + \int_Q |P(m_\sigma,\, q^\sigma)|\, |\varrho_\sigma(t) - m_\sigma(t)| \\ & \leq C_0 + |\varrho_{\max} - \varrho_{\min}|\, \int_Q \left[C_1 + C_2\, |q^\sigma| + C_3\, \ln \max\left\{1,\, \frac{1}{m_\sigma(t) - \varrho_{\min}},\, \frac{1}{\varrho_{\max} - m_\sigma(t)}\right\}\right] \end{aligned} \tag{73}$$

which is uniformly bounded for all $\sqrt{\sigma} \leq \frac{1}{2C_1}\, \operatorname{dist}(M_0,\, \{\varrho_{\min},\, \varrho_{\max}\})$.

Next we distinguish two cases: If $|\varrho_\sigma(x,\,t) - m_\sigma| \geq \frac{1}{2}\, \operatorname{dist}(M_0,\, \{\varrho_{\min},\, \varrho_{\max}\}) =: a_0$, it follows that that

$$|P(\varrho_\sigma,\, q^\sigma) - P(m_\sigma,\, q^\sigma)|\, |\varrho_\sigma(t) - m_\sigma(t)| \geq a_0\, [|P(\varrho_\sigma,\, q^\sigma)| - |P(m_\sigma,\, q^\sigma)|]\,. \tag{74}$$

If otherwise $|\varrho_\sigma(x,\,t) - m_\sigma| < a_0$, then $\varrho_\sigma(x,\,t) \in [m_\sigma - a_0,\, m_\sigma + a_0]$, and due to the choice of M_0 and to Lemma 7.3, this interval is contained in a compact subset of $]\varrho_{\min},\, \varrho_{\max}[$ for σ small. We then express

$$\begin{aligned} |P(\varrho_\sigma,\, q^\sigma) - P(m_\sigma,\, q^\sigma)| \leq & |P(\varrho_\sigma,\, q^\sigma) - P(\varrho_\sigma,\, 0)| + |P(m_\sigma,\, 0) - P(m_\sigma,\, q^\sigma)| \\ & + |P(\varrho_\sigma,\, 0) - P(m_\sigma,\, 0)|\,. \end{aligned}$$

Since $|\partial_q P| \leq C_1$, the first terms are bounded by $\bar{C}\,|q^\sigma|$ which is bounded in $L^2(Q)$. In order to estimate the remaining difference, we invoke Prop. 5.3 proving the $\partial_\varrho P$ is bounded on compact subsets of $I \times \mathbb{R}^{N-2}$. Thus

$$|P(\varrho_\sigma,\, q^\sigma) - P(m_\sigma,\, q^\sigma)| \leq \max_{r\in[M_0-a_0-c\,\sqrt{\sigma},\, M_0+a_0+c\,\sqrt{\sigma}]} |P_\varrho(r,\, q^\sigma)|\,|\varrho_\sigma - m_\sigma| + \bar{C}\,|q^\sigma|\,. \tag{75}$$

For almost all $(x,\, t) \in Q$, we combine the two estimates (74) and (75) and get

$$\begin{aligned}|P(\varrho_\sigma,\, q^\sigma)| \leq & a_0^{-1}\,|P(\varrho_\sigma,\, q^\sigma) - P(m_\sigma,\, q^\sigma)|\,|\varrho_\sigma(t) - m_\sigma(t)| \\ & + \bar{C}\,(|P(m_\sigma,\, q^\sigma)| + |q^\sigma| + C(\mathrm{dist}(M_0,\, \{\varrho_{\min},\, \varrho_{\max}\})))\,,\end{aligned}$$

yielding with the help of (73) the L^1-bound. □

8 Convergence analysis

In this section, we extract (weakly) convergent subsequences and construct their appropriate limits which are the candidate weak solutions.

Proposition 8.1. *From the family* $\{(\mu^\sigma,\, v^\sigma)\}_{\sigma>0}$ *constructed in Proposition 6.1 it is possible to extract a subsequence* $\{\sigma_n\}_{n\in\mathbb{N}}$ *and to find limit elements*

$$\rho \in L^\infty(Q;\, \overline{S_0}), \quad q \in W^{1,0}_2(Q;\, \mathbb{R}^{N-2}), \quad \zeta \in W^{1,0}_2(Q), \quad v \in V^{1,0}_2(Q;\, \mathbb{R}^3)$$

such that

$\rho^{\sigma_n} \to \rho$ *weakly in* $L^2(Q;\, \mathbb{R}^N)$

$q^{\sigma_n} \to q$ *weakly in* $W^{1,0}_2(Q;\, \mathbb{R}^{N-2})$

$\zeta_{\sigma_n} \to \zeta$ *weakly in* $W^{1,0}_2(Q)$

$v^{\sigma_n} \to v$ *weakly in* $W^{1,0}_2(Q;\, \mathbb{R}^3)$ *and strongly in* $L^r(Q;\, \mathbb{R}^3)$ *for all* $1 \leq r < \frac{10}{3}$

$\rho_{\sigma_n}\, v^{\sigma_n} \to \rho\, v$ *weakly in* $L^2(Q;\, \mathbb{R}^{N\times 3})$

$\varrho_{\sigma_n}\, v^{\sigma_n} \otimes v^{\sigma_n} \to \varrho\, v \otimes v$ *weakly in* $L^{\frac{5}{3}}(Q;\, \mathbb{R}^{3\times 3})$

$\sum_{i=1}^N J^{\sigma_n,i}\, v^{\sigma_n} \to 0$ *strongly in* $L^{\frac{5}{4}}(Q;\, \mathbb{R}^{3\times 3})\,.$

and such that for almost all $t \in]0, T[$

$$\rho^{\sigma_n}(t) \to \rho(t) \text{ weakly in } L^2(\Omega;\, \mathbb{R}^N), \quad \varrho_{\sigma_n}(t)\, v^\sigma(t) \to \varrho(t)\, v(t) \text{ weakly in } L^2(\Omega;\, \mathbb{R}^3)\,.$$

Proof. First exploiting the reflexivity of L^p spaces, the bounds in Proposition 7.1 and the Remark 7.2, we can extract a subsequence such that

$$\begin{aligned}
&\rho^{\sigma_n} \to \rho \text{ weakly in } L^2(Q;\, \overline{S_0})\\
&q^{\sigma_n} \to q \text{ weakly in } W^{1,0}_2(Q;\, \mathbb{R}^{N-2}), \quad \zeta_{\sigma_n} \to \zeta \text{ weakly in } W^{1,0}_2(Q)\\
&v^{\sigma_n} \to v \text{ weakly in } W^{1,0}_2(Q;\, \mathbb{R}^3)\\
&\rho_{\sigma_n}\, v^{\sigma_n} \to \xi \text{ weakly in } L^2(Q;\, \mathbb{R}^{N\times 3}), \quad \varrho_{\sigma_n}\, v^{\sigma_n} \otimes v^{\sigma_n} \to \chi \text{ weakly in } L^{\frac{5}{3}}(Q;\, \mathbb{R}^{3\times 3})\\
&\sum_{i=1}^N J^{\sigma_n,i}\, v^{\sigma_n} \to 0 \text{ strongly in } L^{\frac{5}{4}}(Q;\, \mathbb{R}^{3\times 3})\,.
\end{aligned} \tag{76}$$

Moreover, we can assume that $J^{\sigma_n} = -M(\rho^{\sigma_n})\,\nabla\mu^{\sigma_n}$ and $\jmath^{\Gamma,\sigma_n} = -M^{\Gamma,\sigma_n}\,(\mu^{\sigma_n} - \mu^{\Gamma})$ satisfy, for appropriate subsequences and limits J and $\jmath^{\Gamma}$,

$$J^{\sigma_n} \to J \text{ weakly in } L^2(Q;\, \mathbb{R}^{N\times 3}), \quad \jmath^{\Gamma,\sigma_n} \to \jmath^{\Gamma} \text{ weakly in } L^2(S;\, \mathbb{R}^N)\,.$$

In the relation (66), we choose a test function of the form $\psi(x,\, t) = u(t)\,\phi(x)$ with $\phi \in C^1(\overline{\Omega};\, \mathbb{R}^N)$ and $u \in C^1_c(]0,\, T[)$ yielding

$$\begin{aligned} -\int_0^T u'(s)\,\left(\int_\Omega \rho^\sigma(x,\, s)\cdot\phi(x)\,dx\right)\,ds &= \int_0^T u(s)\,a_\sigma(s;\,\phi)\,ds\\ a_\sigma(s;\,\phi) := \int_\Omega (\rho^\sigma\,v^\sigma + J^\sigma)(x,\, s)\,:\,\nabla\phi(x)\,dx &- \int_{\partial\Omega} \jmath^\Gamma_\sigma(x,\, s)\cdot\phi\,dS(x)\,. \end{aligned}$$

This allows to interpret $a_\sigma(s;\,\phi)$ as the weak derivative of the function $\int_\Omega \rho^\sigma(x,\, s)\cdot\phi(x)\,dx$. For all $t\in[0,\, T]$ the Leibniz formula yields

$$\begin{aligned} \int_\Omega \rho^\sigma(x,\, t)\cdot\phi(x)\,dx &= \int_\Omega \rho^\sigma(x,\, 0)\cdot\phi(x)\,dx + \int_0^t a_\sigma(s;\,\phi)\,ds\\ &= \int_\Omega \rho^0(x)\cdot\phi(x)\,dx + \int_0^t a_\sigma(s;\,\phi)\,ds\,. \end{aligned}$$

Exploiting the weak convergence properties (76) associated with the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$, we can verify for all $t\in[0,\, T]$ that $\int_0^t a_{\sigma_n}(s;\,\phi)\,ds \to \int_{Q_t}(\rho\,v + J)(x,\, s)\,:\,\nabla\phi(x)\,dxds - \int_{S_t}\jmath^\Gamma(x,\, s)\cdot\phi(x)\,dS(x)ds$. Consequently, there exists $\lim_{n\to\infty}\int_\Omega \rho^{\sigma_n}(x,\, t)\cdot\phi(x)\,dx$ for all $t\in[0,\, T]$ and all $\phi\in C^1(\overline{\Omega};\,\mathbb{R}^N)$. This shows that $\{\rho^{\sigma_n}(t)\}$ converges as distribution in Ω for all t. But due to the fact that $\{\rho^{\sigma_n}(t)\}$ is uniformly bounded in $L^\infty(\Omega;\,\mathbb{R}^N)$ we obtain that $\{\rho^{\sigma_n}(t)\}$ converges weakly in $L^2(\Omega)$ for all t. Then it is clear that the limit must be identical for almost all t with $\rho(t)$ showing that

$$\rho^{\sigma_n}(t) \to \rho(t) \text{ weakly in } L^2(\Omega;\,\mathbb{R}^N) \text{ for almost all } t\in]0,\, T[\,.$$

Thanks to the Remark B.1, this allows to identify $\xi = \rho\,v$.

We choose in the relation (67) a test function of the form $\eta(x,\, t) = u(t)\,\phi(x)$ with $\phi\in C^1_c(\Omega;\,\mathbb{R}^3)$ and $u\in C^1_c(]0,\, T[)$ yielding

$$\begin{aligned} -\int_0^T u'(s)\,\left(\int_\Omega \varrho_\sigma\,v^\sigma(x,\, s)\cdot\phi(x)\,dx\right)\,ds &= \int_0^T u(s)\,(a_\sigma(s;\,\phi) + b_\sigma(s;\,\phi))\,ds\\ a_\sigma(s;\,\phi) &:= \int_\Omega \{\varrho_\sigma\,v^\sigma\otimes v^\sigma - \mathbb{S}(\nabla v^\sigma) - \sum_{i=1}^N J^{\sigma,i}\otimes v^\sigma\}\,:\,\nabla\phi\\ b_\sigma(s;\,\phi) &:= \int_\Omega p_\sigma(s)\,\operatorname{div}\phi\,. \end{aligned}$$

For all t the Leibniz formula yields

$$\int_\Omega \varrho_\sigma\,v^\sigma(x,\, t)\cdot\phi(x)\,dx = \int_\Omega \varrho_0(x)\,v^0(x)\cdot\phi(x)\,dx + \int_0^t a_\sigma(s;\,\phi)\,ds + \int_0^t b_\sigma(s;\,\phi)\,ds\,.$$

We easily show that the the weak convergence (76) associated with the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ implies the existence of a limit for $\int_0^t a_{\sigma_n}(s;\,\phi)\,ds$ for all $t\in[0,\, T]$.

In order to prove the convergence of $\int_0^t b_\sigma(s;\,\phi)\,ds = \int_0^t \int_\Omega p_\sigma(s,\,x)\,\operatorname{div}\phi(x)\,dxds$ we apply the Lemma 8.2 hereafter. Specialising to $\phi \in C^3_c(\Omega;\,\mathbb{R}^3)$, we see that $\lim_{n\to\infty} \int_0^t b_{\sigma_n}(s;\,\phi)\,ds$ exists for almost all $t \in]0,\,T[$. Overall there exists $\lim_{n\to\infty} \int_\Omega \varrho_{\sigma_n} v^{\sigma_n}(x,\,t)\cdot\phi(x)\,dx$ for almost all $t \in [0,\,T]$ and all $\phi \in C^3_c(\Omega;\,\mathbb{R}^3)$. This shows that $\{\varrho_{\sigma_n}\, v^{\sigma_n}(t)\}$ converges as distributions in Ω for almost all t. But due to the fact that $\{\varrho_{\sigma_n} v^{\sigma_n}(t)\}$ is uniformly bounded in $L^2(\Omega;\,\mathbb{R}^N)$ we obtain that the sequence converges weakly in $L^2(\Omega)$ for almost all $t \in [0,\,T]$. Then, the limit must be identical for almost all t with $\varrho(t)\,v(t)$ showing that

$$\varrho_{\sigma_n}(t)\, v^{\sigma_n}(t) \to \varrho(t)\, v(t) \text{ weakly in } L^2(\Omega;\,\mathbb{R}^N) \text{ for almost all } t \in]0,\,T[\,.$$

Thanks to the Remark B.1, this allows to next identify $\chi = \rho\, v\otimes v$.

Applying the Corollary B.2, we obtain moreover that $v^{\sigma_n} \to v$ strongly in $L^2(Q;\,\mathbb{R}^3)$. But since $\|v^{\sigma_n}\|_{L^{\frac{10}{3}}(Q)} \leq C_0$ (the embedding $V^{1,0}_2 \subset L^{\frac{10}{3}}$ is continuous), we obtain for an appropriate subsequence the strong convergence in every $L^r(Q)$ for $1 \leq r < \frac{10}{3}$. The proof is complete. □

Lemma 8.2. *From the family $\{(\mu^\sigma,\,v^\sigma)\}_{\sigma>0}$ constructed in Proposition 6.1 it is possible to extract a subsequence $\{\sigma_n\}_{n\in\mathbb{N}}$, such that for almost all $t \in [0,\,T]$ and all $\phi \in C^2_c(\Omega)$, there exists $\lim_{n\to\infty} \int_0^t \int_\Omega p_\sigma(x,\,s)\,\phi(x)\,dxds$.*

Proof. We introduce the functions $\pi_\sigma(x,\,t) := \int_0^t p_\sigma(x,\,s)\,ds$. Clearly, $\{\pi_\sigma\}$ satisfies a uniform bound in $W^{0,1}_1(Q) = W^1_1(0,\,T;\,L^1(\Omega))$ (Prop. 7.4). For each $t \in [0,\,T]$, we now consider a solution $u_\sigma(t)$ to the problem $-\triangle u_\sigma = \pi_\sigma(\cdot,\,t)$ in Ω with $u_\sigma(t) = 0$ on $\partial\Omega$. For this problem, existence of a unique solution $u_\sigma(t)$ is known in the class $W^{1,r}_0(\Omega)$ for every $1 \leq r < \frac{3}{2}$. Moreover, we can differentiate in time the equation to obtain that

$$\|u_\sigma\|_{L^\infty(0,T;\,W^{1,r}_0(\Omega))} + \|\partial_t u_\sigma\|_{L^1(0,T;\,W^{1,r}_0(\Omega))} \leq c_r\,\|p_\sigma\|_{L^1(Q)}\,.$$

With the Lemma 4 and the Theorem 4 of [Sim86], we conclude from the latter estimate that $\{u_\sigma\}_{\sigma>0}$ is relatively compact in $L^z(0,T;\,L^{r^*}(\Omega))$ for all $1 \leq z < \infty$ and all $1 \leq r^* < 3$. In particular, we find $u \in L^\infty(0,T;\,W^{1,r}_0(\Omega))$ and a subsequence such that $u_{\sigma_n} \to u$ strongly in $L^1(Q)$. The latter means further that $f_n(t) := \int_\Omega |u_{\sigma_n}(x,\,t) - u(x,\,t)|\,dx \to 0$ strongly in $L^1(0,T)$. Thus, we can pass to a subsequence and obtain that for almost all $t \in]0,\,T[$, we have $u_{\sigma_n}(t) \to u(t)$ strongly in $L^1(\Omega)$.

Consider arbitrary $\phi \in C^2_c(\Omega)$ and t in the set where $u_{\sigma_n}(t) \to u(t)$ strongly in $L^1(\Omega)$. Then $\int_0^t \int_\Omega p_{\sigma_n}(x,\,s)\,\phi(x)\,dxds = -\int_\Omega u_\sigma(x,\,t)\,\triangle\phi(x)\,dx \to \int_\Omega u(x,\,t)\,\triangle\phi(x)\,dx$. □

For the family $\{p_\sigma\}$, we remark that the L^1-bound of Prop. 7.4 allows to obtain weak convergence as measures or distributions. In order to introduce the limit pressure and show its connection to the limits ρ, q and ζ, we first want to obtain the strong convergence of the variables ϱ_σ and q^σ. To this aim we first derive in known manner a strong convergence result for the acceleration terms in the momentum balance equations. For pairs $r \geq r_0 := \frac{5}{2}$ and $q \geq q_0 := \frac{10}{7}$ we introduce the Banach-space

$$\mathcal{Y}(r,\,q) := \{Y \in L^1(Q;\,\mathbb{R}^3)\,:\,\partial_t Y \in L^q(Q),\,\partial_x Y \in L^r(Q)\}\,. \tag{77}$$

We remark that the convergence properties stated in Proposition 8.1 allow to show that the limits $\varrho := \sum_{i=1}^N \rho_i$ and v satisfy

$$-\int_Q \varrho\,(\partial_t\psi + v\cdot\nabla\psi)\,dxdt = \int_\Omega \varrho_0(x)\,\psi(x,\,0)\,dx \text{ for all } \psi \in W^1_2(Q)\,:\,\psi(T) = 0\,. \tag{78}$$

Lemma 8.3. *From the family* $\{(\mu^\sigma,\, v^\sigma)\}_{\sigma>0}$ *constructed in Proposition 6.1 it is possible to extract a subsequence* $\{\sigma_n\}_{n\in\mathbb{N}}$ *such that for all* $r > r_0$ *and* $q > q_0$ *the functionals*

$$\mathcal{M}_n(Y) := -\int_Q \varrho_{\sigma_n}\, v^{\sigma_n} \cdot (\partial_t Y + (v^{\sigma_n}\cdot\nabla)Y)\, dxdt$$

converge strongly in $[\mathcal{Y}(r,\, q)]^*$ *to* $\mathcal{M}(Y) := -\int_Q \varrho\, v\cdot(\partial_t Y + (v\cdot\nabla)Y)\, dxdt$ *with the limits* ϱ *and* v *constructed in Proposition 8.1.*

Proof. Suppose that $\{Y^n\}_{n\in\mathbb{N}}$ is a weakly converging sequence in $\mathcal{Y}(r,\, q)$ with weak limit Y. We consider the sequence $\beta_n := -\varrho_{\sigma_n}\,\partial_t Y^n - \varrho_{\sigma_n}\, v^{\sigma_n}\cdot\nabla\, Y^n$. Then, defining $s = \max\{q,\, \frac{10r}{3r+10}\} > \frac{10}{7}$, we obtain the bound

$$\|\beta_n\|_{L^s(Q;\,\mathbb{R}^3)} \leq c(Q)\, \|\varrho_{\sigma_n}\|_{L^\infty(Q)}\, (\|\partial_t Y_n\|_{L^q(Q)} + \|v^{\sigma_n}\|_{L^{\frac{10}{3}}(Q)}\, \|\nabla Y_n\|_{L^r(Q)}) \leq C_0\,.$$

Consider any subsequence $\{n_k\}_{k\in\mathbb{N}}$. Then, there is a subsequence $\{\beta_{n_{k_j}}\}_{j\in\mathbb{N}}$ such that for a $\beta \in L^s(Q;\,\mathbb{R}^3)$ we have $\beta_{n_{k_j}} \to \beta$ weakly in $L^s(Q;\,\mathbb{R}^3)$. For $\eta \in C^1_c(Q;\,\mathbb{R}^3)$ arbitrary, the Gauss divergence theorem implies that

$$\begin{aligned}\int_Q \beta_{n_{k_j}}\cdot\eta =& -\int_Q [\varrho_{\sigma_{n_{k_j}}}\,\partial_t(Y^{n_{k_j}}\;\eta) + \varrho_{\sigma_{n_{k_j}}}\, v^{\sigma_{n_{k_j}}}\cdot\nabla(Y^{n_{k_j}}\;\eta)]\\ &+ \int_Q Y^{n_{k_j}}\,\varrho_{\sigma_{n_{k_j}}}\,[\partial_t\eta + v^{\sigma_{n_{k_j}}}\cdot\nabla\eta]\,.\end{aligned}$$

Choosing on the other hand $\psi = Y^{n_{k_j}}\;\phi\, 1^N$ with $\phi\in C^1(\overline{Q})$ in the equations (66) yields

$$\begin{aligned}&-\int_Q [\varrho_{\sigma_{n_{k_j}}}\,\partial_t(Y^{n_{k_j}}\;\phi) + \varrho_{\sigma_{n_{k_j}}}\, v^{\sigma_{n_{k_j}}}\cdot\nabla(Y^{n_{k_j}}\;\phi)]\\ &\quad= -\int_Q 1^N\cdot J^{\sigma_{n_{k_j}}}\,\nabla(Y^{n_{k_j}}\;\phi) + \int_S 1^N\cdot j^{\Gamma,\sigma_{n_{k_j}}}\, Y^{n_{k_j}}\;\phi\,.\end{aligned}$$

We infer that

$$\begin{aligned}&\left|\int_Q [\varrho_{\sigma_{n_{k_j}}}\,\partial_t(Y^{n_{k_j}}\;\phi) + \varrho_{\sigma_{n_{k_j}}}\, v^{\sigma_{n_{k_j}}}\cdot\nabla(Y^{n_{k_j}}\;\phi)]\right|\\ &\quad\leq \|\phi\|_{C^1(\overline{Q})}\,(\|1^N\cdot J^{\sigma_{n_{k_j}}}\|_{L^2(Q)} + \|1^N\cdot j^{\Gamma,\sigma_{n_{k_j}}}\|_{L^2(S)})\,\|Y^{n_{k_j}}\|_{W^{1,0}_2(Q)} \to 0\,.\end{aligned}$$

Exploiting now that $\{Y^{n_{k_j}}\}$ is bounded in $\mathcal{Y}(r,\, q)$, passing possibly to a further subsequence yields $Y^{n_{k_j}} \to Y$ strongly in $L^2(Q;\,\mathbb{R}^3)$. Thus, invoking Prop. 8.1, we have $Y^{n_{k_j}}\;\varrho_{\sigma_{n_{k_j}}} \to Y\,\varrho$ weakly in $L^2(Q)$ and $Y^{n_{k_j}}\;\varrho_{\sigma_{n_{k_j}}}\; v^{n_{k_j}} \to Y\,\varrho\, v$ weakly in $L^1(Q)$. Thus

$$\begin{aligned}\lim_{j\to\infty}\int_Q \beta_{n_{k_j}}\cdot\eta =& \lim_{j\to\infty}\int_Q Y^{n_{k_j}}\,\varrho_{\sigma_{n_{k_j}}}\,[\partial_t\eta + v^{\sigma_{n_{k_j}}}\cdot\nabla\eta]\\ =& \int_Q Y\,\varrho\,[\partial_t\eta + v\cdot\nabla\eta]\\ =& \int_Q \varrho\,[\partial_t(\eta\, Y) + v\cdot\nabla(\eta\, Y)] - \int_Q \varrho\,(\partial_t Y + v\cdot\nabla\, Y)\,\phi\,.\end{aligned}$$

This identity together with (78), in which we choose $\psi = \eta\, Y_i$, shows that $\beta = \varrho\,(\partial_t Y + v \cdot \nabla\, Y)$. Since this is valid for any subsequence $\{n_k\}_{k\in\mathbb{N}}$ we see that the entire sequence must converge, that is, $\varrho_{\sigma_n}\,(\partial_t Y^n + v^{\sigma_n} \cdot \nabla\, Y^n) \to \varrho\,(\partial_t Y + v \cdot \nabla\, Y)$ weakly in $L^s(Q;\,\mathbb{R}^3)$. Since $s > 10/7$ and $v^{\sigma_n} \to v$ strongly in $L^p(Q)$ for all $1 \leq p < 10/3$, the products satisfy

$$v^{\sigma_n}\,[\varrho_{\sigma_n}\,(\partial_t Y^n + v^{\sigma_n} \cdot \nabla\, Y^n)] \to v\,[\varrho\,(\partial_t Y + v \cdot \nabla\, Y)] \text{ weakly in } L^1(Q)\,.$$

□

9 The question of compactness

9.1 Compactness of the total mass densities

The first main technical problem is to obtain the strong convergence of the sequence $\{\varrho_{\sigma_n}\}$. We apply a derivate of the method invented by P.-L. Lions in the context of compressible Navier-Stokes equations. Even if we do not essentially enrich this method, we have to show that we can extend it to our case, in which the pressures satisfy a bound only in $L^1(Q)$. Several technical steps are to be performed to verify this claim.

For this section we introduce the positive constant $\lambda' := \lambda + \frac{4}{3}\,\eta$, where $\lambda,\, \eta$ are the viscosity constants.

Lemma 9.1. *For $r > r_1 := 5$ and $q > q_0 := \frac{10}{7}$, we define Banach-spaces*

$$\begin{aligned} \mathcal{X}(r,\,q) :=& \{f \in L^r(Q;\,\mathbb{R}^3)\,:\, \partial_t f \in L^q(]0,\,T[;\,[W^{1,q'}(\Omega)]^*))\} \\ \overset{0}{\mathcal{X}}(r,\,q) :=& \{f \in \mathcal{X}(r,\,q)\,:\, f(T) = 0 \text{ in } [W^{1,q'}(\Omega)]^*\}\,. \end{aligned} \tag{79}$$

and define $\mathcal{X}_0 := \mathcal{X}(r_1,\,q_0)$. From the family $\{(\mu^\sigma,\,v^\sigma)\}_{\sigma>0}$ constructed in Proposition 6.1 it is possible to extract a subsequence and finding $F \in [\mathcal{X}_0]^*$ such that

$$\begin{aligned} & p_\sigma \to F \text{ weakly star in } \mathcal{X}_0 \\ & p_\sigma - \lambda'\,\operatorname{div} v^\sigma \to F - \lambda'\,\operatorname{div} v \text{ strongly in } [\overset{0}{\mathcal{X}}(r,\,q)]^*\,. \end{aligned}$$

Proof. For $f \in \mathcal{X}_0 := \mathcal{X}(r_1,\,q_0)$, we first define functionals

$$F_\sigma(f) := \int_Q p_\sigma\,(f - \fint_\Omega f)\,dxdt\,.$$

We further define $\mathcal{Y}(r,\,q)$ as in (77) and $\mathcal{Y}_0 = \mathcal{Y}(r_1,\,q_0)$. For $Y \in \mathcal{Y}_0$ we define functionals

$$\mathcal{M}_\sigma(Y) := -\int_Q \varrho_\sigma\, v^\sigma \cdot (\partial_t Y + (v^\sigma \cdot \nabla) Y)\,dxdt, \quad \mathcal{S}_\sigma(Y) := \int_Q \mathbb{S}(\nabla v^\sigma)\,:\,\nabla Y\,dxdt$$

$$\mathcal{E}_\sigma(Y) := \int_Q (\sum_{i=1}^N J^{\sigma_n,i} \cdot \nabla) Y \cdot v^{\sigma_n}\,dxdt\,.$$

Due to Hölder's inequality, Prop. 7.1 and Remark 7.2, we obviously see that $\mathcal{M}_\sigma$, $\mathcal{S}_\sigma$ and $\mathcal{E}_\sigma$ are well defined elements of $[\mathcal{Y}_0]^*$, and are bounded sequences in this space.

The identity (67) now implies that $F_\sigma(\operatorname{div} Y) = (\mathcal{M}_\sigma + \mathcal{S}_\sigma + \mathcal{E}_\sigma)(Y) + \mathcal{A}_0(Y)$ for all $Y \in \mathcal{Y}_0$ such that $Y = 0$ on S_T and $Y(T) = 0$. Here $\mathcal{A}_0(Y) := -\int_\Omega \varrho_0(x)\, v^0(x) \cdot Y(x, 0)\, dx$. The identity (72) implies that $F_\sigma(\operatorname{div} Y) = (\mathcal{M}_\sigma + \mathcal{S}_\sigma + \mathcal{E}_\sigma + \mathcal{A}_0(Y) + \mathcal{A}_{\sigma,T})(Y)$, with $\mathcal{A}_{\sigma,T}(Y) := -\int_\Omega \varrho_\sigma(x, T)\, v^\sigma(x, T) \cdot Y(x, T)\, dx$.

For $f \in \mathcal{X}(r, q)$, we denote $\mathcal{B}(f) := (\operatorname{div})^{-1}(f - \text{mean}_\Omega(f))$, where $(\operatorname{div})^{-1}$ is the so-called Bogovski operator. Exploiting its properties (Appendix B.3), we show that $\mathcal{B} \in \mathscr{L}(\mathcal{X}(r, q), \mathcal{Y}(r, q))$. Thus, we can also show that $F_\sigma(f) = [(\mathcal{A}_0 + \mathcal{A}_{\sigma,T} + \mathcal{M}_\sigma + \mathcal{S}_\sigma + \mathcal{E}_\sigma) \circ \mathcal{B}](f)$. It follows that

$$|F_\sigma(f)| \leq c_{\mathcal{B}}\, C_0\, \|f\|_{\mathcal{X}_0}\,. \tag{80}$$

Thus $\{F_\sigma\}_{\sigma>0}$ is uniformly bounded in $\mathcal{X}_0^*$. Since $\mathcal{X}_0$ is separable, we find first a limit element $F_0 \in \mathcal{X}_0^*$ such that for a subsequence: $F_{\sigma_n} \to F_0$ weakly star in $\mathcal{X}_0$.

Note that for $f \in \overset{0}{\mathcal{X}}_0$, we by definition have $\mathcal{B}(f)(T) = 0$, so that

$$F_\sigma(f) = [(\mathcal{A}_0 + \mathcal{M}_\sigma + \mathcal{S}_\sigma + \mathcal{E}_\sigma) \circ \mathcal{B}](f) \text{ for } f \in \overset{0}{\mathcal{X}}_0\,. \tag{81}$$

Exploiting the Lemma 8.3, we find on the other hand that

$$\mathcal{M}_{\sigma_n} \to \mathcal{M} \text{ strongly in } [\mathcal{Y}(r, q)]^*\,, \quad \mathcal{M}(Y) := -\int_Q \varrho\, v \cdot (\partial_t Y + (v \cdot \nabla)Y)\, dxdt\,.$$

Clearly $\mathcal{S}_{\sigma_n} \to \mathcal{S}$ weakly star in $\mathcal{Y}_0$ with $\mathcal{S}(Y) := \int_Q \mathbb{S}(\nabla v) : \nabla Y\, dxdt$. This allows to easily establish the identities

$$F_0(\operatorname{div} Y) =(\mathcal{M} + \mathcal{S} + \mathcal{A}_0)(Y) \qquad \text{for all } Y \in \mathcal{Y}_0, \quad Y = 0 \text{ on } S_T \cup \Omega \times \{T\}\,, \tag{82}$$

$$F_0(f) =[(\mathcal{M} + \mathcal{S} + \mathcal{A}_0) \circ \mathcal{B}](f) \text{ for all } f \in \overset{0}{\mathcal{X}}_0\,. \tag{83}$$

Consider now $\{f_n\}_{n>0}$ a bounded sequence in $\overset{0}{\mathcal{X}}(r, q)$ and $f \in \overset{0}{\mathcal{X}}(r, q)$ such that $f_n \to f$ weakly in $\mathcal{X}(r, q)$. For $t \in]0, T[$, we introduce functions $\{\psi_n\}_{n>0}$ solutions to

$$-\triangle\psi_n(t) = f_n(t) - \text{Mean}_\Omega(f_n(t)) \text{ in } \Omega, \quad -\nabla_\nu\psi_n(t) = 0 \text{ on } \partial\Omega\,. \tag{84}$$

We claim that $Z^n := -\nabla\psi_n$ is a bounded sequence in $\mathcal{Y}(r, q)$, so that for a subsequence

$$\begin{aligned} Z^n \to Z :=& -\nabla\psi \text{ weakly in } \mathcal{Y}(r, q) \text{ and strongly in } C([0, T];\, L^2(\Omega;\, \mathbb{R}^3))\,, \\ -\triangle\psi(t) =& f(t) - \text{Mean}_\Omega(f(t)) \text{ in } \Omega\,, \quad -\nabla_\nu\psi(t) = 0 \text{ on } \partial\Omega\,. \end{aligned}$$

To show this, we employ standard theory that implies for the solutions to (84) that $\|\psi_n(t)\|_{W^{2,r}(\Omega)} \leq c\,\|f_n(t)\|_{L^r(\Omega)}$, that $\|\partial_t\psi_n(t)\|_{W^{1,q}(\Omega)} \leq c\,\|f_n'(t)\|_{[W^{1,q'}(\Omega)]^*}$. Thus

$$\|\nabla\psi_n\|_{\mathcal{Y}(r,q)} \leq c\,\|f_n\|_{\mathcal{X}(r,q)}\,. \tag{85}$$

Moreover, $\nabla\psi_n(T) = 0$ due to the considered restriction $f_n \in \overset{0}{\mathcal{X}}(r, q)$.

In particular, $Z^n = -\nabla\psi_n$ is bounded in $L^\infty(0,T;\, L^q(\Omega;\, \mathbb{R}^3))$, in $L^r(0,T;\, W^{1,r}(\Omega;\, \mathbb{R}^3))$ and $\partial_t Z_n$ is bounded in $L^q(Q;\, \mathbb{R}^3)$. The compactness in $C([0,T];\, L^2(\Omega))$ follows from the Theorem 3 of [Sim86]. Be now $\phi \in C_c^\infty(\Omega)$ fixed. In (67), we use the testfunction $Y = Z^n\,\phi$. Using well known identities of vector analysis, we show the identity

$$\mathcal{S}_{\sigma_n}(Z^n\,\phi) =\lambda' \int_Q \operatorname{div} v^{\sigma_n}\,(-\triangle\psi_n\,\phi - \nabla\psi_n \cdot \nabla\phi) + \eta \int_Q \operatorname{curl} v^{\sigma_n} \cdot (\nabla\psi_n \times \nabla\phi)\,. \tag{86}$$

Therefore, there exists $\lim_{n\to\infty}\{\mathcal{S}_{\sigma_n}(Z^n\,\phi) - \lambda' \int_Q \operatorname{div} v^{\sigma_n}\,[-\triangle\psi_n]\,\phi\}$ and

$$\lim_{n\to\infty}\left\{\mathcal{S}_{\sigma_n}(Z^n\,\phi) - \lambda' \int_Q \operatorname{div} v^{\sigma_n}\,[-\triangle\psi_n]\,\phi\right\}$$
$$= \lambda' \int_Q \operatorname{div} v\,(-\nabla\psi\cdot\nabla\phi) + \eta \int_Q \operatorname{curl} v\cdot(\nabla\psi\times\nabla\phi)\,. \tag{87}$$

By the properties of the sequence $\mathcal{M}_{\sigma_n}$ of Lemma 8.3, we directly obtain that $\mathcal{M}_{\sigma_n}(Z^n\,\phi) \to \mathcal{M}(-\nabla\psi\,\phi)$. Thus, employing (87), and in the limit momentum equations (82) the same transformation (86) of the stress tensor yield

$$\begin{aligned}
&F_{\sigma_n}(\operatorname{div}(Z^n\,\phi)) - \lambda' \int_Q \operatorname{div} v^{\sigma_n}\,[-\triangle\psi_n]\,\phi\\
&= \mathcal{M}_{\sigma_n}(Z^n\,\phi) + \left\{\mathcal{S}_{\sigma_n}(Z^n\,\phi) - \lambda' \int_Q \operatorname{div} v^{\sigma_n}\,[-\triangle\psi_n]\,\phi\right\} + \mathcal{A}_0(Z^n\,\phi) + \mathcal{E}_{\sigma_n}(Z^n\,\phi)\\
&\to \mathcal{M}(-\nabla\psi\,\phi) + \lambda' \int_Q \operatorname{div} v\,(-\nabla\psi\cdot\nabla\phi) + \eta\int_Q \operatorname{curl} v\cdot(\nabla\psi\times\nabla\phi) + \mathcal{A}_0(Z\,\phi)\\
&= \mathcal{M}(-\nabla\psi\,\phi) + \left\{\mathcal{S}(-\nabla\psi\,\phi) - \lambda'\int_Q \operatorname{div} v\,[-\triangle\psi]\,\phi\right\} + \mathcal{A}_0(Z\,\phi)\\
&= F_0(\operatorname{div}(-\nabla\psi\,\phi)) - \lambda'\int_Q \operatorname{div} v\,[-\triangle\psi]\,\phi\,.
\end{aligned}$$

Here we made use of $|\mathcal{E}_{\sigma_n}(Z^n\,\phi)| \leq \|1^N\cdot J^{\sigma_n}\,v^{\sigma_n}\|_{L^{\frac{5}{4}}(Q)}\,\|Z^n\,\phi\|_{\mathcal{Y}_0} \to 0$. Moreover $\mathcal{A}_0(Z^n\,\phi) \to \mathcal{A}_0(Z\,\phi)$, since $Z^n \to Z$ in $C([0,T];\,L^2(Q))$. Next we want to 'eliminate' the cut-off ϕ from the latter formula by constructing a sequence $\{\phi_m\}_{m\in\mathbb{N}}$ of non-negative, sufficiently smooth functions converging to one. We choose ϕ_m of the form $\phi_m(x) = h_m(d(x))$ where $d(x)$ is the distance of x to the boundary $\partial\Omega$; Moreover, $h_m \in C^2(\mathbb{R}_+)$ is a non-decreasing function satisfying $h_m(d) = 1$ for $d > 2/m$, $h_m(d) = 0$ for $d < 1/m$ and $|h_m'| \leq c\,m$. Then, the support of ϕ_m is the complement in Ω of the strip $\Omega_m := \{x\in\Omega\,:\,d(x)\leq 2/m\}$.

For arbitrary $u\in W^{2,r}(\Omega)$, the Hölder inequality yield

$$\int_\Omega |\triangle u|^5\,|1-\phi_m|^5 \leq \|u\|^5_{W^{2,r}(\Omega_m)}\,|\Omega_m|^{1-\frac{5}{r}}\,. \tag{88}$$

With elementary covering techniques, we can finitely cover Ω_m with cubes $Q_{r_i}(x^i)$ such that the intersection $Q_{r_i}(x^i)\cap\Omega_m$ is up to a rotation and a translation contained in $\{\bar{x}\in\mathbb{R}^3\,:\,\max_{i=1,2}|\bar{x}_i| < 1,\,0\leq\bar{x}_3 < 2/m\}$. If u satisfies $\nabla_\nu u = 0$ on $\partial\Omega$, we can invoke the Poincaré inequality to see that

$$\begin{aligned}
\int_\Omega |\nabla u\cdot\nabla\phi_m|^5 \leq = \int_{\Omega_m} |h_m'(d)|^5\,|\nabla u\cdot\nabla d|^5 \leq c\,m^5 \int_{\Omega_m}|\nabla u\cdot\nabla d|^5\\
\leq c\,m^5\sum_{i=1}^I \int_{Q_{r_i}(x^i)\cap\Omega_m} |\nabla u\cdot\nabla d|^5\\
\leq C_0\sum_{i=1}^I\int_{Q_{r_i}(x^i)\cap\Omega_m}|\nabla(\nabla u\cdot\nabla d)|^5 \leq C\,\|u\|^5_{W^{2,r}(\Omega_m)}\,|\Omega_m|^{1-\frac{5}{r}}\,.
\end{aligned} \tag{89}$$

The inequalities (88) and (89) yield $\| \operatorname{div}(-\nabla u\, \phi_m) + \triangle u\|_{L^5(\Omega)}^{\frac{5}{2}} \leq C\, \|u\|_{W^{2,r}(\Omega_m)}^5\, |\Omega_m|^{1-\frac{5}{r}}$. Further, consider $b \in W^{1,\frac{10}{3}}(\Omega)$. Then $\int_\Omega (\partial_t \operatorname{div}(-\nabla u\, \phi_m),\, b) = \int_\Omega \nabla u_t \cdot \nabla b\, \phi_m$, and therefore

$$\|\partial_t[\operatorname{div}(-\nabla u\, \phi_m) + \triangle u]\|_{[W^{1,\frac{10}{3}}(\Omega)]^*} \leq \|\nabla u_t\, (\phi_m - 1)\|_{L^{\frac{10}{7}}(\Omega)} \leq \|\nabla u_t\|_{L^q(\Omega_m)}\, |\Omega_m|^{\frac{7}{10}-\frac{1}{q}}\, .$$

With similar ideas, and setting $\alpha := \min\{\frac{7}{10} - \frac{1}{q},\, \frac{1}{5} - \frac{1}{r}\}$

$$\| \operatorname{div}(-\nabla u\, \phi_m) + \triangle u\|_{\mathcal{X}_0} \leq C\, |\Omega_m|^\alpha\, \|\nabla u\|_{\mathcal{Y}(r,\,q)}\, .$$

Thus, recalling (85), we can show that

$$|F_{\sigma_n}(\operatorname{div}(Z^n\, \phi_m)) - F_{\sigma_n}(-\triangle \psi_n)| \leq C\, \| \operatorname{div}(-\nabla \psi_n\, \phi_m) + \triangle \psi_n\|_{\mathcal{X}_0} \leq C\, |\Omega_m|^\alpha\, \|f_n\|_{\mathcal{X}(r,\,q)}$$

so that $F_{\sigma_n}(f_n) - \lambda' \int_Q \operatorname{div} v^{\sigma_n}\, f_n \rightarrow F_0(f) - \lambda' \int_Q \operatorname{div} v\, f$.

It remains to construct limits that have no mean-value restriction. To this aim, we note that for $f \in \mathcal{X}(r,\, q)$, we have $\partial_t\text{Mean}_\Omega f(t) = \langle \partial_t f,\, 1\rangle$, so that $\|\partial_t\text{Mean}_\Omega f(t)\|_{L^q(0,T)} \leq \|1\|_{W^{1,q'}(\Omega)}\, \|f\|_{\mathcal{X}(r,\,q)}$. Thus, $\text{Mean}_\Omega f(t) \in W^{1,q}(0,T) \subset C([0,T])$ with compact embedding. We note that

$$\int_Q p_{\sigma_n}\, f\, dxdt = F_{\sigma_n}(f) + \int_0^T \text{Mean}_\Omega f(t)\, \text{Mean}_\Omega(p_{\sigma_n})(t)\, dt\, .$$

Since $\|\text{Mean}_\Omega(p_{\sigma_n})\|_{L^1(0,T)}$, we can extract a subsequence and find a regular measure $\bar{p} \in \mathcal{M}([0,T])$ such that $\text{Mean}_\Omega(p_{\sigma_n})\, d\lambda_1 \rightarrow d\bar{p}$ weakly as measures in $[0,T]$. Thus, for $f \in \mathcal{X}_0$ arbitrary,

$$\int_Q p_{\sigma_n}\, f\, dxdt \rightarrow F_0(f) + \int_0^T \text{Mean}_\Omega f(t)\, d\bar{p} =: F(f)\, .$$

Moreover, if $f_n \rightarrow f$ weakly in $\overset{0}{\mathcal{X}}(r,\, q)$, then as claimed

$$\int_Q (p_{\sigma_n} - \lambda' \operatorname{div} v^{\sigma_n})\, f_n = F_{\sigma_n}(f_n) - \lambda' \int_Q \operatorname{div} v^{\sigma_n}\, f_n + \int_0^T \text{Mean}_\Omega f_n(t)\, \text{Mean}_\Omega(p_{\sigma_n})(t)\, dt$$
$$\rightarrow F_0(f) - \lambda' \int_Q \operatorname{div} v\, f + \int_0^T \text{Mean}_\Omega f(t)\, d\bar{p} = F(f) - \lambda' \int_Q \operatorname{div} v\, f\, .$$

□

This was the most technical step. In order to establish the strong convergence of the mass densities, we need a second technical preliminary:

Lemma 9.2. *Consider arbitrary* $u \in C^1(\overline{Q}_T;\,]\varrho_{\min},\, \varrho_{\max}[)$ *with range in a compact of* $]\varrho_{\min},\, \varrho_{\max}[$ *and satisfying* $⨍_\Omega u(t)\, dx = M_0$ *(recall* $M_0 := ⨍_\Omega \varrho_0$*). Consider arbitrary* $\phi \in C^1(\overline{Q})$*. Then, there are: A function* $\beta = \beta(u,\, \phi) \in L^2(Q_T)$*, a constant* $c > 0$ *and a function* $G \in L^2(Q_T)$ *both independent on* u *and* ϕ *such that*

$$|\beta(x,\, t)| \leq G(x,\, t) + c\, (|\ln(\varrho_{\max} - u(x,\, t))| + |\ln(u(x,\, t) - \varrho_{\min})| \quad \textit{for almost all } (x,\, t) \in Q$$

and $\limsup_{n\to\infty} \int_Q [P(u,\, q^{\sigma_n}) + \zeta_{\sigma_n}]\, (\varrho_{\sigma_n} - u)\, \phi\, dxdt = \int_Q \beta\, (\varrho - u)\, \phi\, dxdt.$

Proof. Since $\{\zeta_{\sigma_n}\}$ converges weakly in $W^{1,0}_2(Q)$ and $\{\varrho_{\sigma_n}\}$ is bounded in $\mathcal{X}_0$, we can show by the product Lemma in Appendix B.1 that $\zeta_{\sigma_n}\,(\varrho_{\sigma_n} - u) \to \zeta\,(\varrho - u)$ weakly in $L^2(Q)$. Thus, this term converges elementarily. For a further subsequence, we can find $\tilde{G} \in L^2(Q)$ such that $|q^{\sigma_n}| + |\zeta_{\sigma_n}| \to \tilde{G}$ weakly in $L^2(Q)$. We assume that it is the case for the original sequence. Consider arbitrary $\phi \in C^1(\overline{Q})$. We extract a subsequence $\{n_j\}_{j\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \int_Q P(u,\, q^{\sigma_n})\,(\varrho_{\sigma_n} - u)\,\phi\,dxdt = \lim_{j\to\infty} \int_Q P(u,\, q^{\sigma_{n_j}})\,(\varrho_{\sigma_{n_j}} - u)\,\phi\,dxdt\,.$$

Define functions $\beta_j = \beta_j(u) := P(u,\, q^{\sigma_{n_j}}) + \zeta_{\sigma_{n_j}}$. Due to the Proposition 5.3, we obtain

$$|\beta_j| \leq C\,(1 + |q^{\sigma_{n_j}}|) + c\,(|\ln(\varrho_{\max} - u)| + |\ln(u - \varrho_{\min})| + |\zeta_{\sigma_{n_j}}|\,,$$

and we see that $\{\beta_j\}_{j\in\mathbb{N}}$ is bounded in $L^2(Q)$. We find $\beta \in L^2(Q)$ and extract a subsequence $\beta_j \to \beta$ weakly in $L^2(Q)$. The subsequence extractions do not need being indicated with separate labels. We then see that for $G = C\,(1 + \tilde{G})$ independent on u and ϕ, we have

$$|\beta(x,\,t)| \leq G(x,\,t) + c\,(|\ln(\varrho_{\max} - u(x,\,t))| + |\ln(u(x,\,t) - \varrho_{\min})| \text{ for almost all } (x,\,t) \in Q\,. \tag{90}$$

We next investigate the convergence of the product $\beta_j\,(\varrho_{\sigma_{n_j}} - u)$.

We fix $R \in \mathbb{N}$ and define $[q^{\sigma_{n_j}}]^{(R)}$ to be the componentwise cut-off of $q^{\sigma_{n_j}}$ at levels $\pm R$. We denote $\beta_j^R := P(u,\, [q^{\sigma_{n_j}}]^{(R)})$. Then, we find weak limits $\{\beta^R\}_{R\in\mathbb{N}} \subset L^2(Q)$ and a diagonal subsequence of n_j such that $\beta_j^R \to \beta^R$ weakly in $L^2(Q)$ for all R, such that (90) is valid for all β^R. In addition, due to Proposition 5.3, we have $|\beta_j - \beta_j^R| \leq |P_q|_\infty\,|q^{\sigma_{n_j}} - [q^{\sigma_{n_j}}]^{(R)}|$, from which follows that $\|\beta^R - \beta\|_{L^2(Q)} \leq C_0\,R^{-1}$. We now compute

$$\nabla\beta_j^R = P_\varrho(u,\; [q^{\sigma_{n_j}}]^{(R)})\,\nabla u + P_q(u,\; [q^{\sigma_{n_j}}]^{(R)})\,\nabla[q^{\sigma_{n_j}}]^{(R)}\,.$$

Due to Proposition 5.3, P_ϱ is uniformly bounded on compact sets of $I \times \mathbb{R}^{N-2}$, while P_q is uniformly bounded, and therefore

$$\|\nabla\beta_j^R\|_{L^2(Q)} \leq (c(u,\,R)\,\|\nabla u\|_{L^2(Q)} + \|\nabla q^{\sigma_{n_j}}\|_{L^2(Q)}) \leq C(u,\,R)\,.$$

This shows that $\{\beta_j^R\}_{j\in\mathbb{N}}$ is uniformly bounded in $W^{1,0}_2(Q)$. Thus, we can extract a weakly convergent sequence in this space with limit β^R.

Denote ψ_n a solution to $-\triangle\psi_n = (\varrho_{\sigma_n} - u) - \text{Mean}_\Omega(\varrho_{\sigma_n} - u)$ in Ω with $\nabla_\nu\psi_n = 0$ on $\partial\Omega$. Then, $\nabla\psi_n \to \nabla\psi \in L^p(Q_T;\,\mathbb{R}^3)$ for all $1 \leq p < +\infty$. This allows to show that

$$\int_Q \beta_j^R\,(\varrho_{\sigma_{n_j}} - u)\,\phi = \int_Q \nabla(\beta_j^R\,\phi)\cdot\nabla\psi_{n_j} + \int_0^T \text{Mean}_\Omega(\varrho_{\sigma_{n_j}} - u)\,\int_\Omega \beta_j^R\,\phi\,.$$

Since $\text{Mean}_\Omega(\varrho_{\sigma_{n_j}} - u) \to 0$ strongly in $L^2(0,T)$ (Lemma 7.3 and choice of $\text{Mean}_\Omega u$), it follows that

$$\lim_{j\to\infty} \int_Q \beta_j^R\,(\varrho_{\sigma_{n_j}} - u)\,\phi = \int_Q \beta^R\,(\varrho - u)\,\phi\,.$$

Thus, recalling that $\|\beta_j^R - \beta_j\|_{L^2} \leq C_0\,R^{-1}$, it follows that

$$\left|\lim_{j\to\infty} \int_Q \beta_j\,(\varrho_{\sigma_{n_j}} - u)\,\phi - \int_Q \beta\,(\varrho - u)\,\phi\right| \leq C\,R^{-1}\,.$$

and letting R tend to infinity, we obtain that the left hand is zero. Thus $\limsup_{n\to\infty}\int_Q P(u,\,q^{\sigma_n})\,(\varrho_{\sigma_n} - u)\,\phi = \int_Q \beta\,(\varrho - u)\,\phi$ by the definition of the sequence $\{n_j\}$. □

Lemma 9.3. *For a subsequence $\{\sigma_n\}_{n\in\mathbb{N}}$ with the properties in Proposition 8.1 and Lemma 9.1, we have $\varrho_{\sigma_n} \to \varrho$ strongly in $L^2(Q)$ and even $\varrho_{\sigma_n}(t) \to \varrho(t)$ strongly in $L^2(\Omega)$ for almost all $t \in]0,T[$.*

Proof. For $\sigma > 0$, recall that the total mass density ϱ_σ satisfies in the sense of weak solutions (cf. Remark 6.2) $\partial_t \varrho_\sigma + \operatorname{div}(\varrho_\sigma\, v^\sigma) - N\,\sigma\,\triangle\bar{\mu}_\sigma$, where $\bar{\mu}_\sigma = \sum_{i=1}^N \mu_i^\sigma = 1^N \cdot \mu^\sigma$.

Since ϱ_σ is bounded from below and above, and moreover belongs to $W^{1,0}_2(Q) \cap C([0,T];\, L^2(\Omega))$, we can multiply with the testfunction $1 + \ln \varrho_\sigma$ and obtain for all $t \in]0,T[$ the identity

$$\frac{d}{dt}\int_\Omega \varrho_\sigma(t)\, \ln\varrho_\sigma(t)\, dx + \int_\Omega \varrho_\sigma\, \operatorname{div} v^\sigma + N\,\sigma \int_\Omega \nabla\bar{\mu}_\sigma \cdot \nabla \ln \varrho_\sigma = \int_{\partial\Omega} 1^N \cdot \jmath^{\Gamma,\sigma}\,(1 + \ln\varrho_\sigma)\,.$$

Recall now that $\varrho_\sigma = 1^N \cdot \nabla_\mu f(\mu^\sigma)$. Then

$$\begin{aligned}\nabla\bar{\mu}_\sigma \cdot \nabla\ln\varrho_\sigma &= \frac{1}{\varrho_\sigma}\, 1^N \cdot D^2 f(\mu^\sigma) e^j\, \nabla\bar{\mu}_\sigma \cdot \nabla\mu_j^\sigma\\ &= \frac{1}{N\varrho_\sigma}\, 1^N \cdot D^2 f(\mu^\sigma) 1^N\, |\nabla\bar{\mu}_\sigma|^2 + \frac{1}{\varrho_\sigma}\, 1^N \cdot D^2 f(\mu^\sigma) e^j\, \nabla\bar{\mu}_\sigma \cdot \nabla(\mu_j^\sigma - \frac{1}{N}\,\bar{\mu}_\sigma)\,.\end{aligned}$$

Since $|D^2 f|_\infty \leq C$, we see that $\nabla\bar{\mu}_\sigma \cdot \nabla\ln\varrho_\sigma \geq -c_0\, |\nabla\mu^\sigma|\, |\nabla\mathcal{P}(\mu^\sigma)|$, where $\mathcal{P}$ is the orthogonal projection on $\{1^N\}^\perp$. By means of the estimates in Prop. 7.1, it thus follows that

$$\begin{aligned}N\,\sigma \int_Q \nabla\bar{\mu}_\sigma \cdot \nabla\ln\varrho_\sigma &\geq -\, c_0\,\sqrt{\sigma}\, \|\nabla\mathcal{P}(\mu^\sigma)\|_{L^2(Q)}\, (\sqrt{\sigma}\, \|\nabla\mu^\sigma\|_{L^2(Q)})\\ &\geq -\, c_1\,\sqrt{\sigma}\, (\|\nabla q^\sigma\|_{L^2(Q)} + \|\nabla\zeta_\sigma\|_{L^2(Q)})\, \sqrt{\sigma}\, \|\nabla\mu^\sigma\|_{L^2(Q)} \geq -C_0\,\sqrt{\sigma}\,.\end{aligned}$$

This yields for all $t \in [0,T]$ the consequence

$$\int_\Omega \varrho_\sigma(t)\, \ln\varrho_\sigma(t)\, dx + \int_0^t \int_\Omega \varrho_\sigma\, \operatorname{div} v^\sigma \leq \int_\Omega \varrho_0\, \ln\varrho_0\, dx + C_0\,\sqrt{\sigma}\,. \tag{91}$$

We extend the latter inequality via

$$\int_0^t \int_\Omega \varrho_\sigma\, \operatorname{div} v^\sigma = \frac{1}{\lambda'} \int_{Q_t} \varrho_\sigma\, (\lambda'\, \operatorname{div} v^\sigma - p_\sigma) + \frac{1}{\lambda'} \int_{Q_t} p_\sigma\, \varrho_\sigma\,.$$

To proceed, we shall exploit the structure $p_\sigma = P(\varrho_\sigma,\, q^\sigma) + \zeta_\sigma$ (cf. Proposition 7.4).

For any function $u\,:\; Q_T \to]\varrho_{\min},\, \varrho_{\max}[$, the function $s \mapsto P(s,\, q)$ being non-decreasing for every $q \in \mathbb{R}^{N-2}$ yields (cp. Proposition 5.3)

$$\begin{aligned}c_1\,(\varrho_\sigma - u)^2 \leq &(P(\varrho_\sigma,\, q^\sigma) - P(u,\, q^\sigma))\,(\varrho_\sigma - u)\\ =&[P(\varrho_\sigma,\, q^\sigma) + \zeta_\sigma - (P(u,\, q^\sigma) + \zeta_\sigma)]\,(\varrho_\sigma - u)\\ =&p_\sigma\,\varrho_\sigma - p_\sigma\, u - (P(u,\, q^\sigma) + \zeta_\sigma)\,(\varrho_\sigma - u)\,.\end{aligned} \tag{92}$$

Consider arbitrary $u \in C^1(\overline{Q}_T;\,]\varrho_{\min},\, \varrho_{\max}[)$ having range in a compact of $]\varrho_{\min},\, \varrho_{\max}[$ and satisfying $\fint_\Omega u(t)\, dx = M_0$. Consider further arbitrary $\phi \in C^1(\overline{Q})$. Due to Lemma 9.2, there is a function $\beta = \beta(u,\phi) \in L^2(Q_T)$ such that $\limsup_{n\to\infty} \int_Q (P(u,\, q^{\sigma_n}) + \zeta_{\sigma_n})\,(\varrho_{\sigma_n} - u)\,\phi = \int_Q \beta(u)\,(\varrho - u)\,\phi$. Due to Lemma 9.1, we know that $\int_Q p_\sigma\, u\,\phi \to F(u\,\phi)$. Therefore, the relation (92) implies that

$$\limsup_{\sigma\to 0} \int_Q p_\sigma\,\varrho_\sigma\,\phi \geq c_1\, \liminf_{\sigma\to 0} \int_Q |\varrho_\sigma - u|^2\,\phi + F(u\,\phi) + \int_Q \beta(u)\,(\varrho - u)\,\phi\,.$$

In the latter identity, we now choose $u = u_m$ for a sequence $\{u_m\}_{m\in\mathbb{N}} \subset C^1(\overline{Q};\ [\varrho_{\min}+\delta,\ \varrho_{\max}-\delta])$ such that

$$u_m \to \varrho^{(\delta)} \text{ strongly in } \mathcal{X}_0\,, \quad \varrho^{(\delta)} := \begin{cases} \varrho_{\min}+\delta & \text{if } \varrho \leq \varrho_{\min}+\delta \\ \varrho & \text{if } \varrho_{\min}+\delta < \varrho < \varrho_{\max}-\delta \\ \varrho_{\max}-\delta & \text{otherwise} \end{cases}.$$

By these means we obtain the identity

$$\limsup_{\sigma\to 0} \int_Q p_\sigma\, \varrho_\sigma\, \phi \geq c_1 \liminf_{\sigma\to 0} \int_Q |\varrho_\sigma - \varrho^{(\delta)}|^2\, \phi + F(\varrho^{(\delta)}\, \phi) + \int_Q \beta^\delta\, (\varrho - \varrho^{(\delta)})\, \phi\,, \tag{93}$$

where β^δ is a weak limit in L^2 for $\beta(u_m)$ satisfying for almost all $(x,\, t) \in Q$

$$|\beta^\delta(x,\, t)| \leq G(x,\, t) + c\,(|\ln(\varrho_{\max} - \varrho^{(\delta)}(x,\, t))| + |\ln(\varrho^{(\delta)}(x,\, t) - \varrho_{\min})|\,.$$

The latter implies also that

$$\begin{aligned} |\beta^\delta|\, |\varrho - \varrho^{(\delta)}| \leq & |G|\, |\varrho - \varrho^{(\delta)}| + c\,(|\ln(\varrho_{\max} - \varrho^{(\delta)})| + |\ln(\varrho^{(\delta)} - \varrho_{\min})|\,(\varrho - \varrho^{(\delta)}) \\ \leq & \delta\,(|G| + \tilde{c}\,|\ln\delta|)\,. \end{aligned}$$

so that $\beta^\delta\,(\varrho - \varrho^{(\delta)})$ strongly converges to zero in $L^2(Q)$ for $\delta \to 0$. Letting δ tend to zero in (93), we now see that $\limsup_{\sigma\to 0} \int_Q p_\sigma\, \varrho_\sigma\, \phi \geq c_1 \liminf_{\sigma\to 0} \int_Q |\varrho_\sigma - \varrho|^2\, \phi + F(\varrho\, \phi)$, in which $\phi \in C^1(\overline{Q})$ is arbitrary.

Consider next the integrals $\int_Q \varrho_\sigma\, (\lambda'\, \operatorname{div} v^\sigma - p_\sigma)$. Obviously, we can assume that $\varrho_\sigma \to \varrho$ in $L^r(Q)$ for all $1 \leq r < +\infty$. Moreover, we have

$$\int_Q \partial_t \varrho_\sigma \cdot \psi = \int_Q (\varrho_\sigma v^\sigma + \sum_{i=1}^N J^{\sigma,i}) \cdot \nabla\psi + \int_S \sum_{i=1}^N \jmath_i^{\Gamma,\sigma} \cdot \psi\,.$$

Owing to $\varrho_\sigma v^\sigma \to \varrho\, v$ weakly in $L^2(Q)$, to $\sum_{i=1}^N J^{\sigma,i} \to 0$ strongly in $L^2(Q)$ and to $\sum_{i=1}^N \jmath_i^{\Gamma,\sigma} \to 0$ strongly in $L^2(S)$ we obtain that $\partial_t \varrho_\sigma \to \partial_t \varrho$ weakly in $L^2(0, T;\ [W^{1,2}(\Omega)]^*)$. Thus, $\varrho_\sigma \to \varrho$ weakly in $[\mathcal{X}(r,\, q)]^*$ for all $1 \leq r < +\infty$ and $q \leq 2$. Considering a fixed cut-off in time $\phi_0 \in C_c^1([0,\, T[)$, then it follows that $\varrho_\sigma\, \phi_0 \to \varrho\, \phi_0$ weakly in $[\overset{0}{\mathcal{X}}(r,\, q)]^*$. We make use of the Lemma 9.1, in order to obtain that

$$\lim_{\sigma\to 0} \int_Q \varrho_\sigma\, \phi_0\, (\lambda'\, \operatorname{div} v^\sigma - p_\sigma) = \lambda' \int_Q \varrho\, \phi_0\, \operatorname{div} v - F(\varrho\, \phi_0)\,.$$

We define $k := \liminf_{\sigma\to 0} \int_Q \varrho_\sigma\, (1 - \phi_0)\, \operatorname{div} v^\sigma$, and obtain that $|k| \leq C_0\, \|1 - \phi_0\|_{L^2(0,T)}$ It follows that

$$\begin{aligned} &\limsup_{\sigma\to 0} \int_Q \varrho_\sigma\, \operatorname{div} v^\sigma \geq \limsup_{\sigma\to 0} \int_Q \varrho_\sigma\, \phi_0\, \operatorname{div} v^\sigma + k \\ &\geq \frac{1}{\lambda'} \lim_{\sigma\to 0} \int_Q \varrho_\sigma\, \phi_0\, (\lambda'\, \operatorname{div} v^\sigma - p_\sigma) + \frac{1}{\lambda'} \limsup_{\sigma\to 0} \int_Q p_\sigma\, \varrho_\sigma\, \phi_0 + k \\ &\geq \int_Q \varrho\, \phi_0\, \operatorname{div} v - \frac{1}{\lambda'} F(\varrho\, \phi_0) + \frac{c_1}{\lambda'} \liminf_{\sigma\to 0} \int_Q |\varrho_\sigma - \varrho|^2\, \phi_0 + \frac{1}{\lambda'} F(\varrho\, \phi_0) + k \\ &= \int_Q \varrho\, \phi_0\, \operatorname{div} v + \frac{c_1}{\lambda'} \liminf_{\sigma\to 0} \int_Q |\varrho_\sigma - \varrho|^2\, \phi_0 + k \\ &= \int_Q \varrho\, \operatorname{div} v + \frac{c_1}{\lambda'} \liminf_{\sigma\to 0} \int_Q |\varrho_\sigma - \varrho|^2 + \tilde{k}\,. \end{aligned}$$

Clearly, $|\tilde{k}| \leq |k| + c\,\|1-\phi_0\|_{L^2(0,T)}$. We let ϕ_0 tend to one in $L^2(0,T)$ and get

$$\limsup_{\sigma\to 0} \int_Q \varrho_\sigma \, \operatorname{div} v^\sigma \geq \int_Q \varrho \, \operatorname{div} v + \frac{c_1}{\lambda'} \liminf_{\sigma\to 0} \int_Q |\varrho_\sigma - \varrho|^2 \, .$$

Thus, recalling also (91) for $t = T$

$$\liminf_{\sigma\to 0} \int_\Omega \varrho_\sigma(T) \, \ln \varrho_\sigma(T) \, dx + \int_0^T \int_\Omega \varrho \, \operatorname{div} v + \frac{c_1}{\lambda'} \liminf_{\sigma\to 0} \int_{Q_T} |\varrho_\sigma - \varrho|^2 \leq \int_\Omega \varrho_0 \, \ln \varrho_0 \, dx \, .$$

It remains to show that the weak solution ϱ to the continuity equation is a renormalised solution. This is standard for $v \in W^{1,0}_2(Q;\, \mathbb{R}^3)$ and $\varrho \in L^\infty(Q)$. By standard means we then can show that

$$\begin{aligned}\int_\Omega \varrho_0 \, \ln \varrho_0 \, dx &= \int_\Omega \varrho(T) \, \ln \varrho(T) \, dx + \int_0^T \int_\Omega \varrho \, \operatorname{div} v \\ &\leq \liminf_{\sigma\to 0} \int_\Omega \varrho_\sigma(T) \, \ln \varrho_\sigma(T) \, dx + \int_0^T \int_\Omega \varrho \, \operatorname{div} v\end{aligned}$$

Thus, we obtain at first that $\liminf_{\sigma\to 0} \int_{Q_T} |\varrho_\sigma - \varrho|^2 = 0$. This allows to extract a subsequence such that $\varrho_\sigma \to \varrho$ strongly in $L^2(Q)$. Then, we can pass to the limit directly in (91) to obtain the strong convergence in $L^1(\Omega)$ for all t. □

9.2 Compactness of the other components

Having proved that the total mass densities converge strongly, we obtain the convergence of the partial mass densities.

Lemma 9.4. *For a subsequence $\{\sigma_n\}_{n\in\mathbb{N}}$ with the properties in Proposition 8.1 and Lemma 9.1*

$$\rho^{\sigma_n} \to \rho = \mathscr{R}(\varrho,\, q) \text{ in } L^2(Q;\, \mathbb{R}^N)\, , \quad q^{\sigma_n} \to q \text{ in } L^2(Q;\, \mathbb{R}^{N-2})\, .$$

Proof. Denote $u^{n,m} = \mathscr{R}(\phi_m \star \varrho_{\sigma_n},\, q^{\sigma_n})$, where ϕ_m is an averaging kernel in space. Then,

$$|u^{n,m}_i - \rho^n_i| \leq |\mathscr{R}_\varrho|_\infty \, |\phi_m \star \varrho_{\sigma_n} - \varrho_{\sigma_n}|\, ,$$

and owing to the compactness of $\{\varrho_{\sigma_n}\}$ in $L^2(Q)$ it follows that

$$\sup_{n\in\mathbb{N}} \|u^{n,m}_i - \rho^n_i\|_{L^2(Q)} \to 0 \text{ for } m \to \infty\, . \tag{94}$$

Now, for each fixed m, the sequence $\{u^{n,m}_i\}_{n\in\mathbb{N}}$ is bounded in $W^{1,0}_2(Q)$ since

$$\|\nabla u^{n,m}_i\|_{L^2(Q)} \leq |\mathscr{R}_\varrho|_\infty \, \|\nabla(\phi_m \star \varrho_{\sigma_n})\|_{L^2} + |\mathscr{R}_q|_\infty \, \|\nabla q^{\sigma_n}\|_{L^2} \leq C(m)\, .$$

Thus, by the means of Remark B.1, $\rho^{\sigma_n}_i \, u^{n,m}_i \to \rho_i \, u^m_i$ weakly in $L^2(Q)$, where u^m_i is a weak limit of $\{u^{n,m}_i\}_{n\in\mathbb{N}}$. But then, (94) shows that $(\rho^{\sigma_n}_i)^2 \to \rho_i^2$ weakly in $L^2(Q)$, so that $\rho^{\sigma_n} \to \rho_i$ strongly in $L^2(Q)$. Passing to a subsequence, we obtain that $\rho^{\sigma_n}(x,\, t) \to \rho_i(x,\, t)$ for almost all $(x,\, t) \in Q$.

In order to show that $\{q^{\sigma_n}\}$ strongly converges, we invoke the properties of the function f (cp. Lemma 4.1). By definition, we indeed have (cp. Lemma 4.1, relation (32))

$$\begin{aligned}\mu^{\sigma_n}_i =& \mu^{\text{ref}}_i + \bar{V}_i \, f(\mu^{\sigma_n}) + k_B \, \theta \, \frac{1}{m_i} \, \ln y^{\sigma_n}_i \\ y^{\sigma_n}_i =& Y_i(\mu^{\sigma_n}) = \frac{1}{m_i} \, \frac{\rho^{\sigma_n}_i}{\sum_j \frac{1}{m_j} \, \rho^{\sigma_n}_j}, \quad \rho^{\sigma_n} := \nabla_\mu f(\mu^{\sigma_n})\, .\end{aligned} \tag{95}$$

On the other hand $\mu^{\sigma_n} = \Pi\, q^{\sigma_n} + \zeta_{\sigma_n}\, \bar{V} + \mathscr{M}(\varrho_{\sigma_n},\, q^{\sigma_n})\, 1^N$ by the properties of the change of variables. Together with (95), this implies that

$$k_B\, \theta\, \frac{1}{m}\, \ln y^{\sigma_n} = \Pi\, q^{\sigma_n} + \mathscr{M}(\varrho_{\sigma_n},\, q^{\sigma_n})\, 1^N - P(\varrho_{\sigma_n},\, q^{\sigma_n})\, \bar{V} - \mu^{\text{ref}}\,. \tag{96}$$

We know that $P(\varrho_{\sigma_n},\, q^{\sigma_n})$ is uniformly bounded in $L^1(Q)$ (Lemma 7.4). Moreover, the properties of the functions P and $\mathscr{M}$ imply that $\{\mathscr{M}(\varrho_{\sigma_n},\, q^{\sigma_n})\}$ is also uniformly bounded in $L^1(Q)$ (Lemma 4.1). By means of (96), we thus see that $\ln y_i^{\sigma_n}$ is bounded in $L^1(Q)$. But $\ln y_i^{\sigma_n}$ converges pointwise, because $\{\rho^n\}$ converges pointwise. Consequently, $\ln y_i^{\sigma_n} \to \ln y_i$ pointwise almost everywhere in Q, and the limits $y_i = \frac{1}{m_i}\, \frac{\rho_i}{\sum_j \frac{1}{m_j}\, \rho_j}$ are almost everywhere strictly positive. We multiply (95) with any of the vectors η^k, $k < N-1$, and recalling that $1^N \cdot \eta^k = 0 = \bar{V} \cdot \eta^k$, we obtain that

$$q_k^{\sigma_n} = \mu^{\sigma_n} \cdot \eta^k = \mu^{\text{ref}} \cdot \eta^k + k_B\, \theta \sum_{i=1}^N \frac{\eta_i^k}{m_i}\, \ln y_i^{\sigma_n} \to \mu^{\text{ref}} \cdot \eta^k + k_B\, \theta \sum_{i=1}^N \frac{\eta_i^k}{m_i}\, \ln y_i$$

showing the pointwise convergence. Since $\{q^{\sigma_n}\}$ converges almost everywhere to its weak limit q, this gives the strong convergence in $L^2(Q_T;\, \mathbb{R}^{N-2})$. □

10 Convergence of the pressure

We have shown the convergence $p_\sigma \to F$ weakly in the space $\mathcal{X}_0^*$. Note also that our compactness statements 9.4 allow to show $P(\varrho_\sigma,\, q^\sigma) \to P(\varrho,\, q)$ almost everywhere in Q. Since we have only a L^1-bound, we do not obtain the desired representation of F. However, by Fatou's Lemma, the candidate function $p := P(\varrho,\, q) + \zeta$ at least belongs to $L^1(Q_T)$. In this section we want to use the weak convergence of $\{p_\sigma\}$ as measures, and to show that: 1. The absolutely continuous part of the limit measure is given by $p\, d\lambda_4$, and 2. that the singular part of the limit is concentrated in an appropriate small set. We commence our considerations with the following decomposition of the pressure:

$$\begin{aligned} p_\sigma =& P(\varrho_\sigma,\, q^\sigma) + \zeta_\sigma \\ =& P(\varrho_\sigma,\, q^\sigma) - P(\varrho_\sigma,\, 0) + \zeta_\sigma + P(\varrho_\sigma,\, 0) =: a_\sigma + \zeta_\sigma + P(\varrho_\sigma,\, 0)\,. \end{aligned} \tag{97}$$

The functions a_σ satisfy $|a_\sigma| \leq |\partial_q P|_\infty\, |q^\sigma|$ and therefore they enjoy a uniform bound in $L^2(Q)$. Moreover $a_\sigma \to P(\varrho,\, q) - P(\varrho,\, 0)$ pointwise. Therefore, the Theorem of Vitali implies that $a_\sigma \to P(\varrho,\, q) - P(\varrho,\, 0)$ strongly in $L^2(Q)$. Moreover ζ_σ converges weakly in $W^{1,0}_2(Q)$. It remains therefore to investigate the family $\{P(\varrho_\sigma,\, 0)\}_{\sigma>0}$.

10.1 First observations

Using the weak form of the Navier-Stokes equations (67), we can make first statements on converging quantities.

Lemma 10.1. *There is $C_0 > 0$ depending on $\sup_\sigma \|F_\sigma\|_{\mathcal{X}_0^*}$ and $\sup_\sigma \|a_\sigma + \zeta_\sigma\|_{L^{\frac{5}{3}}(Q)}$ such that*

$$\int_Q |P(\varrho_\sigma,\, 0)|\, |\varrho_\sigma - \varrho|\, dxdt \leq C_0\, \|\varrho_\sigma - \varrho\|_{\mathcal{X}_0} + 2 \int_Q |P(\varrho,\, 0)|\, |\varrho_\sigma - \varrho|\, dxdt\,.$$

For all $f \in W^{1,\infty}(]\varrho_{\min},\, \varrho_{\max}[)$ we have $\limsup_{\sigma\to 0} \int_Q |P(\varrho_\sigma,\, 0)|\, |f(\varrho_\sigma) - f(\varrho)| = 0.$

Proof. Recall that $\varrho_\sigma \to \varrho$ strongly in $L^r(Q)$ for all $1 \leq r < +\infty$. Multiplying (66) with $\psi = \phi\, 1^N$, $\phi \in C^1_c([0,T[\times\Omega)$ arbitrary, we easily show by means of Prop. 8.1 that ϱ is a renormalised solution of the continuity equation. Employing (66), we see that

$$-\int_Q (\varrho_\sigma - \varrho)\, \partial_t \phi \, dxdt = \int_Q (\varrho_\sigma\, v^\sigma - \varrho\, v) \cdot \nabla\phi \, dxdt + \int_Q \sum_{i=1}^N J^{i,\sigma} \cdot \nabla\psi \, dxdt \, . \tag{98}$$

By standard techniques $\|\partial_t(\varrho_\sigma - \varrho)\|_{L^2(0,T;\,[W^{1,2}(\Omega)]^*)} \leq \|\varrho_\sigma\, v^\sigma - \varrho\, v\|_{L^2(Q)} + C_0\, \sqrt{\sigma} \to 0$. This proves that $\varrho_\sigma - \varrho \to 0$ in $\mathcal{X}(r,\, 2) \subset \mathcal{X}_0$, and so $\int_Q p_\sigma\, (\varrho_\sigma - \varrho)\, dxdt = F_\sigma(\varrho_\sigma - \varrho) \to 0$. Now, since P is monotone in the first argument

$$\begin{aligned}
&|P(\varrho_\sigma,\, 0)\, (\varrho_\sigma - \varrho)| \leq |P(\varrho_\sigma,\, 0) - P(\varrho,\, 0)|\, |\varrho_\sigma - \varrho| + |P(\varrho,\, 0)\, (\varrho_\sigma - \varrho)| \\
=&P(\varrho_\sigma,\, 0)\, (\varrho_\sigma - \varrho) + (|P(\varrho,\, 0)\, (\varrho_\sigma - \varrho)| - P(\varrho,\, 0)\, (\varrho_\sigma - \varrho)) \\
=&p_\sigma\, (\varrho_\sigma - \varrho) - (a_\sigma + \zeta_\sigma)\, (\varrho_\sigma - \varrho) + (|P(\varrho,\, 0)\, (\varrho_\sigma - \varrho)| - P(\varrho,\, 0)\, (\varrho_\sigma - \varrho)) \, .
\end{aligned}$$

Integrating over Q, it follows that

$$\begin{aligned}
\int_Q |P(\varrho_\sigma,\, 0)\, (\varrho_\sigma - \varrho)|\, dxdt \leq& \sup_\sigma \|F_\sigma\|_{\mathcal{X}_0^*}\, \|\varrho_\sigma - \varrho\|_{\mathcal{X}_0} \\
&+ \|a_\sigma + \zeta_\sigma\|_{L^{\frac{5}{3}}(Q)}\, \|\varrho_\sigma - \varrho\|_{L^{\frac{5}{2}}(Q)} + 2\int_Q |P(\varrho,\, 0)|\, |\varrho_\sigma - \varrho)|\, dxdt \\
\leq& C_0\, \|\varrho_\sigma - \varrho\|_{\mathcal{X}_0} + 2\int_Q |P(\varrho,\, 0)|\, |\varrho_\sigma - \varrho)|\, dxdt \, .
\end{aligned}$$

This implies that $\int_Q |P(\varrho_\sigma,\, 0)|\, |\varrho_\sigma - \varrho| \to 0$, and the claim for general $f \in W^{1,\infty}(]\varrho_{\min},\, \varrho_{\max}[)$ follows easily. ☐

Lemma 10.2. *Suppose that $h \in W^{1,\infty}_0(]\varrho_{\min},\, \varrho_{\max}[)$. Then $P(\varrho_\sigma,\, 0)\, h(\varrho) \to P(\varrho,\, 0)\, h(\varrho)$ strongly in $L^1(Q)$.*

Proof. Due to the choice of h, and the fact that $P(\varrho,\, 0)$ has logarithmic singularity, the sequence $\{P(\varrho_\sigma,\, 0)\, h(\varrho_\sigma)\}_{\sigma>0}$ is readily shown bounded in $L^\infty(Q)$. Thus, it converges to $P(\varrho,\, 0)\, h(\varrho)$ strongly in $L^r(Q)$ for all $r < +\infty$. Due to the Lemma 10.1, $P(\varrho_\sigma,\, 0)\, (h(\varrho_\sigma) - h(\varrho))$ converges strongly to zero in $L^1(Q)$. The claim follows. ☐

The main statement used in our considerations for the pressure convergence is given next.

Lemma 10.3. *There is $C_0 > 0$ such that for all $f \in W^{1,\infty}(]\varrho_{\min},\, \varrho_{\max}[)$, for all $\sigma > 0$, and for all $u \in \mathcal{X}_0 \cap C(\overline{Q})$ satisfying $P(u,\, 0) \in L^1(Q)$*

$$\begin{aligned}
\int_Q |P(\varrho_\sigma,\, 0)|\, |f(\varrho) - f(u)|\, dxdt \leq& \|f'\|_{L^\infty(]\varrho_{\min},\, \varrho_{\max}[)} \Big[\quad C_0\, (\|\varrho_\sigma - u\|_{\mathcal{X}_0} + \|\varrho_\sigma - \varrho\|_{\mathcal{X}_0}) \\
&+2\int_Q (|P(u,\, 0)| + |P(\varrho,\, 0)|)\, |\varrho_\sigma - u|\, dxdt \Big] \, .
\end{aligned}$$

Proof. We employ the abbreviation $P_\sigma = P(\varrho_\sigma,\, 0)$. We estimate $|P_\sigma|\, |\varrho - u| \leq |P_\sigma|\, (|\varrho - \varrho_\sigma| + |\varrho_\sigma - u|)$. Then, we express

$$\begin{aligned}
|P_\sigma|\, |\varrho_\sigma - u| \leq& |P(\varrho_\sigma,\, 0) - P(u,\, 0)|\, |\varrho_\sigma - u| + |P(u,\, 0)|\, |\varrho_\sigma - u| \\
=&(P(\varrho_\sigma,\, 0) - P(u,\, 0))\, (\varrho_\sigma - u) + |P(u,\, 0)|\, |\varrho_\sigma - u| \\
\leq& P(\varrho_\sigma,\, 0)\, (\varrho_\sigma - u) + 2\, |P(u,\, 0)|\, |\varrho_\sigma - u| \, .
\end{aligned}$$

Thus, $|P_\sigma|\,|\varrho-u| \leq P(\varrho_\sigma,\,0)\,(\varrho_\sigma-u)+2\,|P(u,\,0)|\,|\varrho_\sigma-u|+|P_\sigma|\,|\varrho-\varrho_\sigma|$. We integrate the inequality over Q. We obtain that

$$\int_Q |P_\sigma|\,|\varrho-u|\,dxdt \leq F_\sigma(\varrho_\sigma-u)-\int_Q (a_\sigma+\zeta_\sigma)\,(\varrho_\sigma-u)\,dxdt \\ + 2\int_Q |P(u,\,0)|\,|\varrho_\sigma-u|\,dxdt+\int_Q |P_\sigma|\,|\varrho-\varrho_\sigma|\,dxdt\,.$$

Invoking at last Lemma 10.1, the claim follows for $f=$ id. The claim for general $f\in W^{1,\infty}(]\varrho_{\min},\,\varrho_{\max}[)$ is then obvious. □

10.2 The weak convergence as measures

We denote $P_\sigma^+ := \max\{P(\varrho_\sigma,\,0),\,0\}$ and $P_\sigma^- := P(\varrho_\sigma,\,0)-P_\sigma^+$. Then, there are positive measures $k_1,\,k_2$ such that for some subsequence $P_\sigma^+\to k_1$ and $-P_\sigma^-\to k_2$ weakly as measures in Q. Thus, $P_\sigma := P(\varrho_\sigma,\,0)$ converge weakly as measures to the regular signed measure $k:=k_1-k_2$, while $|P_\sigma|$ converges to $|k|=k_1+k_2$. We define

$$dk^s := dk - P(\varrho,\,0)\,d\lambda_4, \quad d|k^s| := d|k|-|P(\varrho,\,0)|\,d\lambda_4\,. \tag{99}$$

Our next task will be to show that the measure k^s concentrates on a *closed* set of measure zero. We shall employ the following technical statement.

Lemma 10.4. *For $\epsilon>0$ arbitrary, there is an open set $U_\epsilon\subseteq Q$ enjoying the properties*

$$\lambda_4(Q\setminus U_\epsilon)\leq\epsilon, \quad |k^s|(U_\epsilon)\leq\epsilon\,.$$

Proof. We consider the sequence of functions $\{h_m\}_{m\in\mathbb{N}}$ defined on the interval $]\varrho_{\min},\,\varrho_{\max}[$ via

$$h_m(r) := \begin{cases} 0 & \text{for } r>\varrho_{\max}-\frac{1}{m} \\ m\,(\varrho_{\max}-\frac{1}{m}-r) & \text{for } \varrho_{\max}-\frac{1}{m}\geq r>\varrho_{\max}-\frac{2}{m} \\ 1 & \text{for } \varrho_{\max}-\frac{2}{m}\geq r>\varrho_{\min}+\frac{2}{m} \\ m\,(r-\varrho_{\min}-\frac{1}{m}) & \text{for } \varrho_{\min}+\frac{2}{m}\geq r>\varrho_{\min}+\frac{1}{m} \\ 0 & \text{for } \varrho_{\min}+\frac{1}{m}\geq r \end{cases}\,.$$

It is then readily established that $h_m(\varrho)\nearrow 1$ pointwise almost everywhere in Q_T. Since $\{h_m(\varrho)\}_{m\in\mathbb{N}}$ is bounded in $L^\infty(Q_T)$, it follows that $h_m(\varrho)\to 1$ in $L^r(Q_T)$ for all $1\leq r<+\infty$. Due to the Lemma 10.3, we have for every $u\in\mathcal{X}_0\cap C(\overline{Q})$ such that $P(u,\,0)\in L^1(Q)$, every $\sigma>0$ and every $m\in\mathbb{N}$, the inequality

$$\int_Q |P(\varrho_\sigma,\,0)|\,|h_m(\varrho)-h_m(u)|\,dxdt \leq m\,\Big[\quad C_0\,(\|\varrho_\sigma-u\|_{\mathcal{X}_0}+\|\varrho_\sigma-\varrho\|_{\mathcal{X}_0}) \\ +2\int_Q (|P(u,\,0)|+|P(\varrho,\,0)|)\,|\varrho_\sigma-u|\,dxdt\Big]\,.$$

Note also that $\int_Q |P_\sigma|\, h_m(u)\, dxdt \leq \int_Q |P_\sigma|\, h_m(\varrho)\, dxdt + \int_Q |P_\sigma|\, |h_m(\varrho) - h_m(u)|\, dxdt$. Making use of the latter inequality and of the Lemma 10.2, we obtain that

$$\int_Q |P(\varrho,\, 0)|\, h_m(u)\, dxdt + \int_Q h_m(u)\, d|k^s| = \int_Q h_m(u)\, d|k|$$
$$\leq \liminf_{\sigma\to 0} \int_Q |P_\sigma|\, h_m(u)\, dxdt$$
$$\leq \int_Q |P(\varrho,\, 0)|\, h_m(\varrho)\, dxdt + m\, \left[C_0\, \|\varrho - u\|_{\mathcal{X}_0} + 2\int_Q (|P(u,\, 0)| + |P(\varrho,\, 0)|)\, |\varrho - u|\, dxdt\right] .$$

Since also $\int_Q |P(\varrho,\, 0)|\, |h_m(u) - h_m(\varrho)|\, dxdt \leq m\, \int_Q |P(\varrho,\, 0)|\, |u - \varrho|\, dxdt$ it next follows that

$$\int_Q h_m(u)\, d|k^s| \leq m\, \left[C_0\, \|\varrho - u\|_{\mathcal{X}_0} + \int_Q (2\,|P(u,\, 0)| + 3\,|P(\varrho,\, 0)|)\, |\varrho - u|\, dxdt\right] .$$

We introduce the set $U_m := \{(x,\, t) \,:\, \varrho_{\min} + \frac{2}{m} < u(x,\, t) < \varrho_{\max} - \frac{2}{m}\}$. This set is open whenever u is continuous. Since the characteristic function χ_{U_m} is majorated by $h_m(u)$, we next see that

$$|k^s|(U_m) \leq m\, \left[C_0\, \|\varrho - u\|_{\mathcal{X}_0} + \int_Q (2\,|P(u,\, 0)| + 3\,|P(\varrho,\, 0)|)\, |\varrho - u|\, dxdt\right] . \tag{100}$$

Now, for $\delta > 0$ arbitrary, we choose $u = u_\delta(x,\, t) = \phi_\delta \star_{\mathbb{R}^3} (\chi_\Omega \varrho(\cdot,\, t))$. Here ϕ_δ is a smooth averaging kernel in space. Recall that the function $s \mapsto |P(s,\, 0)|$ is up to constant equivalent to the function $\ln \max\{\frac{1}{\varrho_{\max}-s},\, \frac{1}{s-\varrho_{\min}}\}$ (Prop. 5.3). Thanks to the Jensen inequality applied to the convex functions $g(s) := \ln \frac{1}{\varrho_{\max}-s}$, we obtain that

$$g(u_\delta(t)) \leq \phi_\delta \star_{\mathbb{R}^3} (\chi_\Omega\, g(\varrho(\cdot,\, t)) \text{ in } Q\, ,$$

and we see that $\|P^+(u_\delta,\, 0)\|_{L^1(Q)} \leq C_0\, \|P^+(\varrho,\, 0)\|_{L^1(Q)} + C_1 \leq C_2$. We can employ the convexity of $\ln \frac{1}{s-\varrho_{\min}}$ as well to show that $\|P^-(u_\delta,\, 0)\|_{L^1(Q)} \leq C_0\, \|P^-(\varrho,\, 0)\|_{L^1(Q)} + C_1 \leq C_2$. Overall, we obtain independently on δ that

$$\|P(u_\delta,\, 0)\|_{L^1(Q)} \leq C_2\, . \tag{101}$$

Recall next that $u_\delta(x,\, t) \to \varrho$ in $\mathcal{X}_0$ for $\delta \to 0$. Therefore, to any $m \in \mathbb{N}$, we find $\delta = \delta(m) > 0$, such that

$$C_0\, \|\varrho - u_\delta\|_{\mathcal{X}_0} + \int_Q (2\,|P(u_\delta,\, 0)| + 3\,|P(\varrho,\, 0)|)\, |\varrho - u_\delta|\, dxdt \leq m^{-2}\, .$$

Recalling in addition the inequality (100) applied to $U_m := \{(x,\, t) \,:\, \varrho_{\min} + \frac{2}{m} < u_{\delta(m)}(x,t) < \varrho_{\max} - \frac{2}{m}\}$ we get

$$|k^s|(U_m) \leq m^{-1}\, . \tag{102}$$

We next decompose $Q \setminus U_m = V_m^+ \cup V_m^-$ according to

$$V_m^+ := \{(x,\, t) \,:\, u_{\delta(m)} \geq \varrho_{\max} - \frac{2}{m}\}, \quad V_m^- := \{(x,\, t) \,:\, u_{\delta(m)} \leq \varrho_{\min} + \frac{2}{m}\}\, .$$

Since $\{P^+(u_{\delta(m)},\, 0)\}$ is uniformly bounded in $L^1(Q)$ (cf. (101)), we directly obtain that $\ln \frac{m}{2}\, \lambda_4(V_m^+) \leq \int_{V_m^+} P^+(u_{\delta(m)},\, 0)\, dxdt \leq C_0$. Applying the same inequality to the negative part, we just get

$$\lambda_4(Q \setminus U_m) \leq C\, \frac{1}{\ln m}\, . \tag{103}$$

We combine (102) and (103) to obtain the claim for $m = e^{C/\epsilon}$. □

Now we are able to deduce the main property needed for characterising the absolutely continuous part of the measure.

Corollary 10.5. *(1) The measure $|k^s|$ is concentrated in a set $A \subset Q$ with $\lambda_4(A) = 0$;*

(2) For all $(x,\, t) \in A$ at which the Lebesgue value $\tilde{\varrho}(x,\, t) := \lim_{\delta\to 0} \fint_{\Omega_\delta(x)} \varrho(y,\, t)\, dy$ exists, the latter attains one of the critical numbers $\{\varrho_{\min},\, \varrho_{\max}\}$.

Proof. Consider $\epsilon > 0$ arbitrary but fixed. For $i \in \mathbb{N}$, we find with the help of Lemma 10.4 an open set $U_i \subset Q$ such that

$$|k^s|(U_i) \leq 2^{-i}\,\epsilon, \quad \lambda_4(Q \setminus U_i) \leq 2^{-i}\,\epsilon\,.$$

We define $U_\epsilon := \bigcup_{i=1}^\infty U_i$ open. We easily show that $|k^s|(U_\epsilon) \leq \frac{3}{2}\,\epsilon$ while $\lambda_4(Q \setminus U_\epsilon) = 0$. Since ϵ was arbitrary, the measure k^s concentrates on $A := \bigcap_{\epsilon>0} U_\epsilon \setminus Q$ which has Lebesgue-measure zero.

Suppose now that $(x,\, t) \in A$. Then for all $i \in \mathbb{N}$, $(x,\, t) \notin U_i$ meaning for $m := e^{-C\frac{2^i}{\epsilon}}$ and $\delta = \delta(m)$

$$\varrho_{\min} + \frac{2}{m} \geq \phi_\delta \star (\chi_\Omega\, \varrho(\cdot,\, t))(x) \text{ or } \varrho_{\max} - \frac{2}{m} \leq \phi_\delta \star (\chi_\Omega\, \varrho(\cdot, t))(x)\,.$$

If the Lebesgue value λ of $\varrho(\cdot,\, t)$ exists in x, then $\phi_{\delta(m)} \star (\chi_\Omega\, \varrho(\cdot,\, t))(x) \to \lambda$ as $m \to \infty$ and thus $\lambda \in \{\varrho_{\min},\, \varrho_{\max}\}$. □

Remark 10.6. *Corollary 10.5 shows that $|k^s| \perp \lambda_4$. Thus, the relation* (99) *defines the unique orthogonal decomposition of the measure k (Lebesgue's decomposition Theorem, to be found for instance in [GMS98] I.1.1.4 Th. 1), and we have also $P(\varrho_\sigma,\, 0) \overset{ACP}{\rightharpoonup} P(\varrho,\, 0)$ (See [GMS98] I.1.2.7 Def. 2).*

11 The global characterisation of the limit pressure

We at last show some structural properties of the limit of $\{p_\sigma\}$. As shown previously the functions $P_\sigma := P(\varrho_\sigma,\, 0)$ converge weakly as measures to the regular signed measure $k := k_1 - k_2$, and moreover k admits the orthogonal decomposition

$$dk = P(\varrho,\, 0)\, d\lambda_4 + d(k_1^s - k_2^s)\,. \tag{104}$$

It is possible to precise the structure of the defect measure. We have shown the convergence $F_\sigma \to F$ weakly in the space $\mathcal{X}_0^*$ (Lemma 9.1). Combining this information with the weak convergence as measures, we obtain for every function $f \in \mathcal{X}_0 \cap C(\overline{Q})$ that $\int_Q p\, f\, dxdt + \int_Q f\, dk^s = F(f)$. We next consider the restriction of F on the Sobolev space $W^{0,1}_z(Q)$, which is continuously embedded into $\mathcal{X}_0$ for $z \geq \frac{5}{2}$. For a functional on $W^{0,1}_z(Q)$ there exist $a,\, b \in L^z(Q)$ such that $F(f) = \int_Q (a\, f + b\, \partial_t f)\, dxdt$ for all $f \in W^{0,1}_z(Q)$. It follows that

$$\int_Q p\, f dxdt + \int_Q f\, dk^s = \int_Q (a\, f + b\, \partial_t f)\, dxdt \text{ for all } f \in W^{0,1}_z(Q) \cap C(\overline{Q})\,.$$

We now see that for all $f \in C^1_c(Q)$ we must have the identity $-\int_Q b\, \partial_t f\, dxdt = \int_Q (a-p)\, f\, dxdt - \int_Q f\, dk^s$. Thus, the function b possesses a measure-valued partial time derivative $\partial_t b = (a -$

$p)\,d\lambda_4 + dk^s$. Another consequence of the representation $F(f) = \int_Q (a\,f + b\,\partial_t f)\,dxdt$ being that $b \in L^\infty W^{1,q'}$, we now see that $b \in BV(Q)$. This allows to apply the celebrated BV structure theorem showing that

$$(a-p)\,d\lambda_4 + dk^s = \partial_t b = (\partial_t b)^{(a)} + (\partial_t b)^{(j)} + (\partial_t b)^{(C)} \tag{105}$$

with $(\partial_t b)^{(a)}$ being absolutely continuous with respect to λ_4, the jump part $(\partial_t b)^{(j)}$ absolutely continuous with respect to $d\mathcal{H}^3$, and the Cantor part $(\partial_t b)^{(C)}$ of the gradient measure $\partial_t b$. The decompositions are orthogonal on both sides of (105). Thus $(\partial_t b)^{(a)} = (a-p)\,d\lambda_4$ and $dk^s = (\partial_t b)^{(j)} + (\partial_t b)^{(C)}$. Thus, the measure $\kappa := k^s$ vanishes on all sets of Hausdorff dimension less than three. We denote $\kappa^{(j)}$ the absolutely continuous part of κ with respect to $\mathcal{H}^3$, and $\kappa^{(C)} \perp \mathcal{H}^3$ the Cantor part. The behaviour of the jump part is readily characterised.

Corollary 11.1. *The measure $\kappa^{(j)}$ is concentrated in a countable union of hypersurface $\Omega \times \{t_k\}$. If $|\kappa^{(j)}|(\Omega\times\{t\}) > 0$, then the Lebesgue value $\tilde{\varrho}(x,\,t)$ of $\varrho(\cdot,t)$ exists for $|\kappa^{(j)}|$ almost all $x \in \Omega\times\{t\}$, and it attains only the critical values $\{\varrho_{\min},\,\varrho_{\max}\}$.*

Proof. From the formula (105), we see that $\kappa^{(j)}$ is the jump part of the measure $D_t b$, which is concentrated in a countable union $\bigcup_{k=1}^\infty \mathcal{M}_k$ of three-dim. manifolds of class $\mathcal{C}^1$. Since the spatial gradients $D_x b$ exist in the sense of weak derivatives, we formally directly see that the jump direction (Chapter 4.1.4, Theorem 2, (ii) of [GMS98]) can be given only by $(0,\,0,\,0,\,1)$, so that every $\mathcal{M}_k$ can be chosen as a subset of some $\Omega \times \{t_k\}$. There are other more elementary proof of this fact.

The measure $\kappa^{(j)}$ is concentrated in the set A constructed in Corollary 10.5. Since $\varrho(\cdot,t) \in L^1(\Omega)$ for all t, the Lebesgue value of $\varrho(\cdot,\,t_k)$ exists for λ_3 almost all $x \in \Omega$ and all singular times $t_1,\,t_2,\ldots$. Again according to Corollary 10.5, this value is almost everywhere $\varrho_{\min}$ or $\varrho_{\max}$ on A. □

The last statement is a representation of the jump measure.

Lemma 11.2. *There are $\mathfrak{p}_1, \mathfrak{p}_2, \ldots \in W^{1,2}(\Omega)$ such that $\sup_{k\in\mathbb{N}} \|\mathfrak{p}_k\|_{W^{1,2}(\Omega)} < C$ and the measure $\kappa^{(j)}$ possesses the representation $\kappa^{(j)}(\operatorname{div} Y) := \sum_{k=1}^\infty \int_\Omega \nabla\mathfrak{p}_k(x)\cdot Y(x,\,t_k)\,dx$ for all $Y \in C^1_c([0,T[\times\Omega;\,\mathbb{R}^3)$.*

Proof. According to the preceding results, there are at most countably many $t_1,\,t_2,\ldots \in [0,\,T]$ such that $\kappa^{(j)}$ is identical with its restriction to $\bigcup_{k=1}^\infty \Omega\times\{t_k\}$, and there absolutely continuous with respect to the three-dim Hausdorff measure.

In the limit relation (25), we consider a testfunction $\eta(x,\,t) = \phi_m(t)\,Y(x)$, where the sequence $\{\phi_m\}$ approximates from below the characteristic function of some interval $I =]a,\,b[\subset]0,\,T[$. For almost all $0 < a < b < T$, we obtain after few elementary steps:

$$-\int_\Omega [(\varrho\,v)(x,\,b) - (\varrho\,v)(x,\,a)]\cdot Y = \int_a^b \int_\Omega L(x,\,t) : \nabla Y(x)\,dx + \kappa(\chi_{]a,\,b[}\operatorname{div} Y)$$
$$L := \varrho\,v\otimes v - \mathbb{S}(\nabla v) + p\,\mathsf{Id}\,.$$

Suppose now that $a = a_\ell$ and $b = b_\ell$ ($\ell \in \mathbb{N}$) are sequences of numbers such that $\bigcap_{\ell=1}^\infty]a_\ell,\,b_\ell[= t_k$ for some t_k in the singular set. The Functions $d_\ell := [(\varrho\,v)(x,\,b_\ell) - (\varrho\,v)^-(x,\,a_\ell)]$ are uniformly bounded in $L^2(\Omega)$, and, passing to a subsequence if necessary, we can therefore assume that they converge weakly in $L^2(\Omega)$ to an element $u_k = u_{t_k}$. Thus

$$\int_\Omega u_k\cdot Y\,dx = \int_{\Omega\times\{t_k\}} \operatorname{div} Y\,d\kappa^{(j)} = \int_{\Omega\times\{t_k\}} \operatorname{div} Y\,\frac{d\kappa^{(j)}}{dx}\,dx\,.$$

It follows that the density $\mathfrak{p}_k := \frac{d\kappa^{(j)}}{dx}$ on $\Omega \times \{t_k\}$ belongs to $W^{1,2}(\Omega)$ and that the vector field u_k is its gradient. Therefore all $\mathfrak{p}_k$ satisfy a uniform estimate in $W^{1,2}(\Omega)$. □

12 Performing the limit passage $\sigma \to 0$

We make use of the Proposition 6.1 and of the convergence properties established for the densities and the relative chemical potentials in the Lemma 9.4, and for the pressure in the Sections 10 and 11 to show the existence of weak solutions.

The main tool is the passage to the equivalent formulation (P') (see (57), (58), (59), (60)). In the relation (66), we choose the test function ψ of the special form $\psi := \sum_{k=1}^{N-2} \hat{\psi}_k\, \xi^k + \phi\, \bar{V} + \hat{\phi}\, 1^N$, with $\hat{\psi}_k,\, \phi,\, \hat{\phi} \in C^1_c([0,T[\times\overline{\Omega})$. Then

$$\begin{aligned}
f_\mu(\mu^\sigma)\cdot\partial_t\psi =& \sum_{k=1}^{N-2} \eta^k \cdot f_\mu(\mu^\sigma)\,\partial_t\hat{\psi}_k + \partial_t\phi + 1^N\cdot f_\mu(\mu^\sigma)\,\partial_t\hat{\phi}\\
=& \sum_{k=1}^{N-2} \eta^k\cdot f_\mu\left(\sum_{j=1}^{N-2} q^\sigma_j\,\xi^j + \mathscr{M}(\varrho_\sigma,\, q^\sigma)\,1^N\right)\,\partial_t\hat{\psi}_k + \partial_t\phi + \varrho_\sigma\,\partial_t\hat{\phi}\\
=& R(\varrho_\sigma,\, q^\sigma)\cdot\partial_t\hat{\psi} + \partial_t\phi + \varrho_\sigma\,\partial_t\hat{\phi}\,.
\end{aligned}$$

Employing similar arguments, we derive from (66) the identities

$$\begin{aligned}
&-\int_Q R(\varrho_\sigma,\, q^\sigma)\cdot\partial_t\hat{\psi} - \int_Q R(\varrho_\sigma,\, q^\sigma)\cdot\nabla\hat{\psi}\cdot v^\sigma + \int_Q \{\widetilde{M}(\varrho_\sigma,\, q^\sigma)\,\nabla q + A(\varrho_\sigma,\, q^\sigma)\,\nabla\zeta_\sigma\}\,\nabla\hat{\psi}\\
&\qquad + \int_S \{\widetilde{M}^\Gamma\,(q^\sigma - q^\Gamma) + A^\Gamma\,(\zeta_\sigma - \zeta^\Gamma)\}\,\hat{\psi} = \int_\Omega \Pi^T\rho^0(x)\cdot\hat{\psi}(x,\,0)\,dx + e^1_\sigma(\hat{\psi}) \qquad (106)\\
&-\int_Q v^\sigma\cdot\nabla\phi + \int_Q \{b(\varrho_\sigma,\, q^\sigma)\,\nabla\zeta_\sigma + A(\varrho_\sigma,\, q^\sigma)\cdot\nabla q^\sigma\}\cdot\nabla\phi\\
&\qquad + \int_S \{b^\Gamma\,(\zeta_\sigma - \zeta^\Gamma) + A^\Gamma\cdot(q^\sigma - q^\Gamma)\}\,\phi = e^2_\sigma(\phi) \qquad (107)\\
&-\int_Q \varrho_\sigma\,\partial_t\hat{\phi} - \int_Q \varrho_\sigma\, v^\sigma\cdot\nabla\hat{\phi} = \int_\Omega \varrho_0(x)\,\hat{\phi}(x,\,0)\,dx + e^3_\sigma(\hat{\phi})\,. \qquad (108)
\end{aligned}$$

Here the error terms are given by

$$\begin{aligned}
e^1_\sigma(\hat{\psi}) :=& -\sigma\,\left(\int_Q \Pi^T 1^N\,\nabla\hat{\psi}\cdot\nabla(1^N\cdot\mu^\sigma)\,dxdt + \int_S (\Pi^T 1^N\cdot\hat{\psi})\,1^N\cdot(\mu^\sigma - \mu^\Gamma)\,dSdt\right)\\
e^2_\sigma(\phi) =& -\sigma\,\bar{V}\cdot 1^N\,\left(\int_Q \nabla\phi\cdot\nabla(1^N\cdot\mu^\sigma)\,dxdt + \int_S \phi\,1^N\cdot(\mu^\sigma - \mu^\Gamma)\,dSdt\right)\\
e^3_\sigma(\hat{\phi}) =& -\sigma\,|1^N|^2\,\left(\int_Q \nabla\hat{\phi}\cdot\nabla(1^N\cdot\mu^\sigma)\,dxdt + \int_S \hat{\phi}\,1^N\cdot(\mu^\sigma - \mu^\Gamma)\,dSdt\right)\,.
\end{aligned}$$

Owing to the bound $\sqrt{\sigma}\,\|\mu^\sigma\|_{W^{1,0}_2(Q)} \leq C_0$, we have $|e^i_\sigma| \leq c\,\sqrt{\sigma}$, so that these perturbations clearly tend to zero. Thus, the passage to the limit in the integral relations (106), (107), (108) is readily performed using Lemma 9.4. To see that the strong limits ϱ and q and the weak limit ζ satisfy (57), (58), (59). It is of course essential that the differential operator is linear in ζ, so that we only need the weak convergence of this variable.

It remains to pass to the limit in the momentum equation (67). Here we clearly obtain for $p = P(\varrho,\, q) + \zeta$ in general that

$$
\begin{aligned}
- \int_Q \varrho\, v \cdot \partial_t \eta - \int_Q \varrho\, v \otimes v \,:\, \nabla \eta + \int_Q \mathbb{S}(\nabla v) \,:\, \nabla \eta - \int_Q p \,\operatorname{div} \eta \\
= \int_Q \operatorname{div} \eta \, d\kappa + \int_\Omega \varrho_0(x)\, v^0(x) \cdot \eta(x,\, 0)\, dx
\end{aligned}
$$

and the solution might be affected by a defect measure. This achieves to prove the Theorem 5.5, and thus the equivalent Th. 3.5.

A Galerkin approximation

In this section we show how to approximate (P_σ) and prove the statement of Prop. 6.1. One possible idea would be to regularise the function f in order that its Hessian possesses full rank. This would correspond to approximating with 'compressible' models. In this paragraph we rather construct an approximation scheme that exploits the essential features of the incompressible system (50), (51), (52) and (53).

Some further stabilisation steps. We choose a function $\omega \in C^2(\mathbb{R})$ which is convex and increasing on all $\mathbb{R}$, and for which we impose the growth conditions

$$
\begin{aligned}
c_0\, (\sqrt{|s^-|} + |s^+|^2) \leq \omega'(s)\, s - \omega(s) \leq c_1\, (\sqrt{|s^-|} + |s^+|^2 + 1) \\
\omega'(s) \leq c_2\, (1 + \omega'(s)\, s - \omega(s))^{\frac{1}{2}}\, .
\end{aligned} \tag{109}
$$

In these conditions, c_i is constant for $i = 0, 1, 2$, $s^+ := \max\{s,\, 0\}$ and $s^+ + s^- = s$. For example, we propose the function

$$
\omega(s) := \begin{cases} -2\, \sqrt{|s|} & \text{for } s \leq -1 \\ \frac{1}{4}\, s^2 + \frac{3}{2}\, s - \frac{3}{4} & \text{for } -1 < s \end{cases} \tag{110}
$$

which satisfies these assumptions with $c_0 = 1 = c_1$ and $c_2 = 3$. For $\mu \in \mathbb{N}$, denote $q := \Pi\mu \in \mathbb{R}^{N-2}$ the vector $(\mu \cdot \eta^1, \ldots, \mu \cdot \eta^{N-2})$ according to (38), and $\mu' := \sum_{i=1}^{N-2} q_i\, \xi^i + \mu \cdot \eta^N\, 1^N$. For $\mu \in \mathbb{R}^N$ and $n \in \mathbb{N}$, we define

$$
f^n(\mu) := f(\mu) + \frac{1}{n}\, \left[\sum_{k=1}^{N-2} \omega(\mu \cdot \eta^k) + \omega(\mu \cdot \eta^N)\right] . \tag{111}
$$

For $n \in \mathbb{N}$, we also introduce

$$
T^n(s) := \frac{s}{(1 + \frac{s^2}{n^2})^{\frac{1}{2}}} \text{ for } s \in \mathbb{R}\, , \tag{112}
$$

$$
\hat{P}^n(\mu) := f\left(\sum_{k=1}^{N-2} T^n(\mu \cdot \eta^k)\, \xi^k + T^n(\mu \cdot \eta^N)\, 1^N\right) + \mu \cdot \eta^{N-1} \text{ for } \mu \in \mathbb{R}^N\, . \tag{113}
$$

In this section, the gradients of a function f of the variable μ on $\mathbb{R}^N$ are denoted f_μ.

Remark A.1. *Note the following properties: For all $\mu \in \mathbb{R}^N$*

(1) $f^n_\mu(\mu) = f^n_\mu(\mu')$*;*

(2) $f^n_\mu(\mu) \cdot 1^N = f_\mu(\mu) \cdot 1^N + \frac{1}{n}\,\omega'(\mu \cdot \eta^N)$*;*

(3) $f^n_\mu(\mu) \cdot \bar{V} = 1$*;*

(4) $\hat{P}^n_\mu(\mu) \cdot \bar{V} = 1$*;*

Proof. The first property is obvious since $f_\mu(\mu) = f_\mu(\mu')$, and the stabilisation in (111) does not depend on the coordinate $\mu \cdot \eta^{N-1}$. For the same reasons, $f^n_\mu \cdot \bar{V} = f_\mu \cdot \bar{V} \equiv 1$. Similarly, denoting $B^n(\mu) := \sum_{k \neq N-1} T^n(\mu \cdot \eta^k)\, \xi^k$ we compute

$$
\begin{aligned}
\hat{P}^n_{\mu_i}(\mu) &= \sum_{j=1}^N f_{\mu_j}(B^n(\mu))\, \partial_{\mu_i} B^n_j(\mu) + \eta_i^{N-1}\\
&= \sum_{j=1}^N f_{\mu_j}(B^n(\mu)) \left(\sum_{k=1}^{N-2} \xi^k_j\, (T^n)'(\mu \cdot \eta^k)\, \eta^k_i + (T^n)'(\mu \cdot \eta^N)\, \eta^N_i\right) + \eta_i^{N-1}\,. \qquad (114)
\end{aligned}
$$

Recall that $\bar{V} \cdot \eta^k = \delta^k_{N-1}$, an the claim (4) also follows. □

The approximation scheme $(P_{n,\sigma})$. For variables $(\mu_1, \ldots, \mu_N,\, v_1,\, v_2,\, v_3) : Q_T \to \mathbb{R}$, we consider the system:

$$\partial_t f^n_{\mu_i}(\mu) + \operatorname{div}(\hat{P}^n_{\mu_i}(\mu)\, v - M^\sigma_{i,j}(f_\mu(\mu))\, \nabla \mu_j) = 0 \text{ for } i = 1, \ldots, N \qquad (115)$$

$$[1^N \cdot f^n_\mu(\mu)]\, \partial_t v + [1^N \cdot \hat{P}^n_\mu(\mu)](v \cdot \nabla) v - \operatorname{div} \mathbb{S}(\nabla v) + \nabla \hat{P}^n(\mu) = -(\sum_{i=1}^N J^i \cdot \nabla) v\,. \qquad (116)$$

In the latter relation we use the abbreviations $J^i := -M^\sigma_{i,j}(f_\mu(\mu))\, \nabla \mu_j$. This is supplemented by the boundary conditions (13) and

$$-M^\sigma(f_\mu(\mu))\, \nabla_\nu \mu = M^{\Gamma,\sigma}\, (\mu - \mu^\Gamma) \text{ on } S_T\,. \qquad (117)$$

Moreover, we impose initial conditions $\mu(x,\, 0) = \mu^0(x)$ and $v(x,\, 0) = v^0(x)$ in Ω. We prove existence for the problem (P_σ) at $\sigma > 0$ using the problems $(P_{n,\sigma})$ defined for $n > 1$ and σ positive by the relations (115), (116) and (117) as approximation scheme.

Construction of approximate solutions for $(P_{n,\sigma})$. We do not solve $(P_{n,\sigma})$ exactly, but use a Galerkin approximation in the spatial variable. We recall the notation $\{\bar{V}\}^\perp := \{\mu \in \mathbb{R}^N \,:\, \mu \cdot \bar{V} = 0\}$. We then choose

(1) A countable linearly independent system $\phi^1,\, \phi^2, \ldots \in W^{1,\infty}_0(\Omega;\, \mathbb{R}^3)$ dense in $W^{1,2}_0(\Omega;\, \mathbb{R}^3)$ in order to approximate the variable v;

(2) In order to approximate the variables μ we need a countable system $\psi^1,\, \psi^2, \ldots$ of lin. independent elements of the space $W^{1,\infty}(\Omega;\, \{\bar{V}\}^\perp)$ dense in $W^{1,2}(\Omega;\, \{\bar{V}\}^\perp)$, and a countable system of lin. independent $w^1,\, w^2, \ldots$ of the space $W^{1,\infty}(\Omega;\, \operatorname{span}\{\bar{V}\})$, dense in $W^{1,2}(\Omega;\, \operatorname{span}\{\bar{V}\})$. [The latter means equivalently that there are $u_1,\, u_2, \ldots \in W^{1,\infty}(\Omega)$ dense in $W^{1,2}(\Omega)$ such that $w^i = u_i\, \bar{V}$ for $i \in \mathbb{N}$.]

For technical reasons, we have to require additional properties of the latter sets. For $n \in \mathbb{N}$, and $i, j \in \{1, \ldots, n\}$ such that $i \leq j$, we introduce the functions $\tilde{\phi}^{i,j} = \phi^i \cdot \phi^j$ with $\phi^1, \ldots, \phi^n$ from (1). By means of an obvious renumbering, we denote these functions $\tilde{\phi}^s$ for $s = 1, \ldots, n \, \frac{n+1}{2}$. For all $n \in \mathbb{N}$, we assume that there are natural numbers $p = p(n)$ and $m = m(n)$, such that additionally to (1) and (2) we have:

$$\begin{cases} 1^N \in \operatorname{span}\{\psi^1, \ldots, \psi^p, \, w^1, \ldots, w^m\} \\ \tilde{\phi}^s \, 1^N \in \operatorname{span}\{\psi^1, \ldots, \psi^p, \, w^1, \ldots, w^m\} \quad \text{for all } s = 1, \ldots, n \, \frac{n+1}{2} \, . \end{cases} \tag{118}$$

It is then possible to in particular verify that

$$v \in \operatorname{span}\{\phi^1, \ldots, \phi^n\} \Longrightarrow |v|^2 \, 1^N \in \operatorname{span}\{\psi^1, \ldots, \psi^{p(n)}, \, w^1, \ldots, w^{m(n)}\} \, . \tag{119}$$

For $n \in \mathbb{N}$, we are looking for approximate solutions of the form

$$\mu^n \in C^1([0,T]; \operatorname{span}\{\psi^1, \ldots, \psi^{p(n)}, \, w^1, \ldots, w^{m(n)}\}) \quad v^n \in C^1([0,T]; \operatorname{span}\{\phi^1, \ldots, \phi^n\})$$

$$\mu^n(x, \, t) = \sum_{\ell=1}^{p(n)} a_\ell(t) \, \psi^\ell(x) + \sum_{\ell=1}^{m(n)} \bar{a}_\ell(t) \, w^\ell(x) \text{ and } v^n(x, \, t) = \sum_{\ell=1}^{n} b_\ell(t) \, \phi^\ell(x) \, . \tag{120}$$

We moreover need defining a projection of μ on $\{\bar{V}\}^\perp$ via

$$\mu^{n,\prime} := \sum_{\ell=1}^{p} a_\ell \, \psi^\ell = \mu^n - \sum_{\ell=1}^{m} \bar{a}_\ell \, w^\ell \, . \tag{121}$$

In order to approximate the equations (115), we first consider for $s = 1, \ldots, p(n)$ their projection

$$\int_\Omega \partial_t f^n_\mu(\mu^n) \cdot \psi^s \, dx = \int_\Omega \Big((\hat{P}^n_\mu(\mu^n) \, v^n + J^n) \cdot \nabla \psi^s \Big) \, dx + \int_{\partial\Omega} \jmath^{\Gamma,n} \cdot \psi^s \, dS \, . \tag{122}$$

Here J^n is an abbreviation for the fluxes $-M^\sigma_{i,j}(f_\mu(\mu^n)) \, \nabla \mu^n_j$, while $\jmath^{\Gamma,n}$ stands for $-M^{\Gamma,\sigma} \, (\mu^n - \mu^\Gamma)$. For $\mu \in C([0,T]; \, L^1(\Omega; \mathbb{R}^p))$, we define $A^1(\mu) = \{a^1_{i,j}(\mu)\}_{i,j=1,\ldots,p(n)}$ via

$$a^1_{\ell,s}(\mu(t)) := \int_\Omega f^n_{\mu_i,\mu_j}(\mu(x, \, t)) \, \psi^\ell_i(x) \, \psi^s_j(x) \, dx \, . \tag{123}$$

Owing to the convexity of f^n, we see that $A^1(\mu(t))$ is symmetric and positive semi-definite. Due to the restriction of $\psi^1, \ldots, \psi^p$ mapping into $\bar{V}^\perp$, we can moreover verify that $A^1(\mu(t))$ is strictly positive on $\mathbb{R}^{p \times p}$ for μ finite. Recalling the Remark A.1, we know that $f_\mu(\mu^n) = f_\mu(\mu^{n,\prime})$ so that $A^1(\mu^n) = A^1(\mu^{n,\prime})$ depends actually only on the coefficient a. We can now express (122) in the equivalent form

$$\begin{aligned} A^1(a(t)) \, a'(t) &= F^1(t, \, a(t), \, \bar{a}(t), \, b(t)), \\ F^1_s &:= \int_\Omega (\hat{P}^n_\mu(\mu^n) \, v^n + J^n) \cdot \nabla \psi^s \, dx + \int_{\partial\Omega} \jmath^{\Gamma,n} \cdot \psi^s \, dS \, . \end{aligned}$$

We next observe for $r = 1, \ldots, m(n)$ that $\partial_t f^n_\mu(\mu^n) \cdot w^r = D^2 f^n(\mu^n) \, \partial_t \mu^n \cdot w^r = 0$ because all w^r's are parallel to $\bar{V}$. We multiply (115) with w^r and obtain after integrating over Ω

$$0 = \int_\Omega \Big((\hat{P}^n_\mu(\mu^n) \, v^n + J^n) \cdot \nabla w^r \Big) \, dx + \int_{\partial\Omega} \jmath^{\Gamma,n} \cdot w^r \, dS \, . \tag{124}$$

We recall that all w^r possess the form $w^r = u_r\,\bar{V}$ for some scalar function $u_r \in W^{1,\infty}(\Omega)$. The Remark A.1 then implies that $\int_\Omega \hat{P}^n_\mu(\mu^n)\, v^n \cdot \nabla w^r = \int_\Omega v^n \cdot \nabla u_r$. We can re-express

$$\int_\Omega J^n \cdot \nabla w^r = -\int_\Omega M^\sigma_{i,j}(f_\mu(\mu^n))\, \nabla \mu^{n,\prime}_j \cdot \nabla w^r_i - \sum_{\ell=1}^m \bar{a}_\ell \int_\Omega M^\sigma_{i,j}(f_\mu(\mu^n))\, \nabla w^\ell_j \cdot \nabla w^r_i\,,$$

$$\int_{\partial\Omega} M^{\Gamma,\sigma}\, \mu^n \cdot w^r = \int_{\partial\Omega} M^{\Gamma,\sigma}\, \mu^{n,\prime} \cdot w^r + \sum_{\ell=1}^m \bar{a}_\ell \int_{\partial\Omega} M^{\Gamma,\sigma}\, w^\ell \cdot w^r\,.$$

We introduce the matrix $\bar{A}^1(\mu) = \{\bar{a}^1_{i,j}(\mu)\}_{i,j=1,\ldots,m(n)}$ via

$$\bar{a}^1_{\ell,r}(\mu(t)) := \int_\Omega M^\sigma_{i,j}(f_\mu(\mu^n(x,\,t))\, \nabla w^\ell_j(x) \cdot \nabla w^r_i(x)\, dx + \int_{\partial\Omega} M^{\Gamma,\sigma}\, w^\ell(x) \cdot w^r(x)\, dS(x)\,,$$

which is symmetric and positive in $\mathbb{R}^{m\times m}$. Moreover, we note again that $\bar{A}^1(\mu^n) = \bar{A}^1(\mu^{n,\prime})$ depends only on a. We can therefore reformulate (124) via

$$\bar{A}^1(a(t))\,\bar{a}(t) = \bar{F}^1(t,\,a(t),\,b(t))\,, \quad \bar{F}^1_r := \int_\Omega (v^n \cdot \nabla u_r + J^{n,\prime} \cdot \nabla w^r)\; dx + \int_{\partial\Omega} \jmath^{\Gamma,n,\prime} \cdot w^r\, dS$$

in which $J^{n,\prime}$ is an abbreviation for the fluxes $-M^\sigma_{i,j}(f_\mu(\mu^n))\, \nabla \mu^{n,\prime}_j$, and $\jmath^{\Gamma,n,\prime} = -M^{\Gamma,\sigma}\,(\mu^{n,\prime} - \mu^\Gamma)$. Thus the components $\bar{a}$ can be eliminated by means of the formula

$$\bar{a}(t) = [\bar{A}^1(a(t))]^{-1}\, \bar{F}^1(t,\,a(t),\,b(t)) =: G(t,\,a(t),\,b(t))\,. \tag{125}$$

In order to approximate the equations (4), we consider for $s \in \{1,\ldots,n\}$

$$\int_\Omega [1^N \cdot f^n_\mu(\mu^n)]\, \partial_t v^n \cdot \phi^s\, dx = -\int_\Omega [1^N \cdot \hat{P}^n_\mu(\mu^n)]\, (v^n \cdot \nabla) v^n \cdot \phi^s\, dx + \int_\Omega \hat{P}^n(\mu^n)\; \operatorname{div} \phi^s\, dx$$
$$- \int_\Omega \mathbb{S}(\nabla v^n) \cdot \nabla \phi^s\, dx - \int_\Omega (\sum_{i=1}^N J^{n,i} \cdot \nabla) v^n \cdot \phi^s\, dx\,. \tag{126}$$

We define $A^2(\mu) = \{a^{(2)}_{i,j}(\mu)\}_{i,j=1,\ldots,n}$ via $a^{(2)}_{i,j}(\mu(t)) := \int_\Omega f^n_\mu(\mu(x,\,t)) \cdot 1^N\, \eta^i(x) \cdot \eta^j(x)\, dx$. Owing to the properties of f, we see that $1^N \cdot f_\mu \geq \varrho_{\min}$. Thus $1^N \cdot f^n_\mu = 1^N \cdot f_\mu + \frac{1}{n}\,\omega'(\mu \cdot \eta^N) > \varrho_{\min}$, and we see that $A^2(\mu)$ is symmetric and positive definite. Moreover $A^2(\mu) = A^2(\mu')$. We can express (126) in the equivalent form

$$A^2(a(t))\, b'(t) = F^2(a(t),\,\bar{a}(t),\,b(t))$$
$$F^2_s := \int_\Omega \left\{ (-[1^N \cdot \hat{P}^n_\mu(\mu^n)]\, v^n - \sum_{i=1}^N J^{n,i} \cdot \nabla) v^n \cdot \phi^s + (\hat{P}^n(\mu^n)\, \mathsf{I} - \mathbb{S}(\nabla v^n))\; :\; \nabla \phi^s \right\} dx\,.$$

Overall, making use of the algebraic elimination (125), the Galerkin approximation (122), (124), (126) has the reduced form

$$\begin{pmatrix} A^1(a(t)) & 0 \\ 0 & A^2(a(t)) \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \tilde{F}^1(t,\,a(t),\,b(t)) \\ \tilde{F}^2(t,\,a(t),\,b(t)) \end{pmatrix} \tag{127}$$

with $\tilde{F}^i(t,\,a(t),\,b(t)) := F^i\big(t,\,a(t),\,G(t,\,a(t),\,b(t)),\,b(t)\big)$ for $i = 1, 2$. We consider the initial conditions

$$a(0) = a^{0,n} \in \mathbb{R}^p, \quad b(0) = b^{0,n} \in \mathbb{R}^n\,. \tag{128}$$

Here we require for the reason of consistency that $\mu^{0,n,\prime} := \sum_{\ell=1}^{p(n)} a^{0,n}_\ell\, \psi^\ell$ and $v^{0,n} = \sum_{\ell=1}^n b^{0,n}_\ell\, \phi^\ell$ satisfy

$$\rho^{0,n} := f_\mu(\mu^{0,n,\prime}) \to \rho^0 \text{ in } L^2(\Omega;\,\mathbb{R}^N), \quad v^{0,n} \to v^0 \text{ in } L^2(\Omega;\,\mathbb{R}^3) \text{ for } n \to \infty\,. \tag{129}$$

Existence for $(P_{n,\sigma})$**.** The eigenvalues of $A^1(\mu)$ are strictly positive only for μ finite. Therefore we must solve the problem (127), (128) employing successive local solutions. The local existence at least is obvious, since the block diagonal matrix associated with (127) is invertible in $\mu = \mu^{0,n}$, and since it can be easily verified that: 1. The right-hand sides $\tilde{F}^i$ are continuous in time for $\mu^\Gamma \in C([0,T];\ L^2(\partial\Omega))$; 2. The map $(a,\, b) \mapsto \tilde{F}^i(t,\, a,\, b)$ is globally Lipschitz on $\mathbb{R}^p\times\mathbb{R}^n$. To verify these properties it is essential that we have employed a cut-off in the argument of the pressure function (see (113)).

Lemma A.2. *There is* $T^* = T^*(n,\, \|(a^{0,n},\, b^{0,n})\|_\infty) > 0$ *such that the problem* (127)*,* (128) *possesses a unique solution* $(a,\, b) \in C^1([0,T^*];\ \mathbb{R}^p \times \mathbb{R}^n)$*.*

We spare the proof of this obvious statement. To show that the solution can be extended to the entire interval, we need *a priori* estimates.

Proposition A.3. *Assume that the approximate system* (127)*,* (128) *possesses a solution* $(a,\, b) \in C^1([0,T^*];\ \mathbb{R}^p \times \mathbb{R}^n)$ *for a* $T^* > 0$*. Then the vector fields* μ^n *and* v^n *satisfy on* $]0,\, T^*[$*, the identity*

$$\frac{d}{dt}\int_\Omega \left[k(f_\mu(\mu^n)) + \frac{1}{n}\sum_{k\neq N-1} \Phi_\omega(\mu^n\cdot\eta^k) + \frac{1}{2}\,\varrho_n\,|v^n|^2\right]\,dx \\ + \int_\Omega \{M^\sigma\nabla\mu^n\cdot\nabla\mu_n + \mathbb{S}(\nabla v^n)\cdot\nabla v^n\}\,dx + \int_{\partial\Omega} M^{\Gamma,\sigma}\,(\mu^n-\mu^\Gamma)\cdot\mu^n\,dS = 0\,,$$

in which $\Phi_\omega(s) := \omega'(s)\,s - \omega(s)$ *for* $s\in\mathbb{R}$*.*

Proof. Recall that (127) is equivalent to (122), (124) and (126). We can multiply (122) with a, (124) with $\bar{a}$ to obtain for μ^n that

$$\int_\Omega \partial_t f^n_\mu(\mu^n)\cdot\mu^n\,dx = \int_\Omega \Big((\hat{P}^n_\mu(\mu^n)\,v^n + J^n)\cdot\nabla\mu^n\Big)\,dx - \int_{\partial\Omega} M^{\Gamma,\sigma}\,(\mu^n-\mu^\Gamma)\cdot\mu^n\,dS \\ = \int_\Omega (v^n\cdot\nabla\hat{P}^n(\mu^n) + J^n\cdot\nabla\mu^n)\,dx - \int_{\partial\Omega} M^{\Gamma,\sigma}\,(\mu^n-\mu^\Gamma)\cdot\mu^n\,dS \tag{130}$$

We notice that

$$\partial_t f^n_\mu(\mu^n)\cdot\mu^n = \partial_t(f^n_\mu(\mu^n)\cdot\mu^n - f^n(\mu^n)) \\ = \partial_t\left[f_\mu(\mu^n)\cdot\mu^n - f(\mu^n) + \frac{1}{n}\sum_{k\neq N-1}\{\omega'(\mu^n\cdot\eta^k)\,\mu^n\cdot\eta^k - \omega(\mu^n\cdot\eta^k)\}\right] \\ = \partial_t\left[k(f_\mu(\mu_n)) + \frac{1}{n}\sum_{k\neq N-1}\Phi_\omega(\mu^n\cdot\eta^k)\right]\,.$$

Second, we multiply (126) with v^n. This yields

$$\int_\Omega \frac{1}{2}\,[1^N\cdot f^n_\mu(\mu^n)]\,\partial_t|v^n|^2\,dx = -\int_\Omega \frac{1}{2}\,[1^N\cdot\hat{P}^n_\mu(\mu^n)]\,(v^n\cdot\nabla)|v^n|^2\,dx + \int_\Omega \hat{P}^n(\mu^n)\,\operatorname{div} v^n\,dx \\ - \int_\Omega \mathbb{S}(\nabla v^n)\cdot\nabla v^n\,dx - \int_\Omega \frac{1}{2}\,(\sum_{i=1}^N J^{n,i}\cdot\nabla)|v^n|^2\,dx\,. \tag{131}$$

Due to the additional property (118) and to (119), we can also choose $|v^n|^2\, 1^N$ as a test function in (122). Recall also that $v^n = 0$ on $\partial\Omega$ to obtain that

$$\int_\Omega \partial_t[1^N \cdot f^n_\mu(\mu^n)]\, |v^n|^2\, dx = \int_\Omega ([1^N \cdot \hat{P}^n_\mu(\mu^n)]\, v^n + 1^N \cdot J^n) \cdot \nabla |v^n|^2\, dx\,.$$

Thus, making use of the latter identity and of (131) yields

$$\frac{1}{2}\int_\Omega \partial_t([1^N \cdot f^n_\mu(\mu^n)]\, |v^n|^2)\, dx + \int_\Omega \mathbb{S}(\nabla v^n) \cdot \nabla v^n\, dx = \int_\Omega \hat{P}^n(\mu^n)\, \operatorname{div} v^n\, dx\,. \tag{132}$$

We add (132) to (130). Observe that $\int_\Omega (v^n \cdot \nabla \hat{P}^n(\mu^n) + \hat{P}^n(\mu^n)\, \operatorname{div} v^n)\, dx = 0$, and the claim follows. □

We integrate the inequality of Proposition A.3 on $[0,\, t]$ for $t \leq T^*$ arbitrary. Using that $f_\mu(\mu_n) \in S_0$ is bounded, we see that $|k(f_\mu(\mu_n(t))|$ is uniformly bounded. Thus, the quantities $\frac{1}{n}\, \|\Phi_\omega(\mu^n \cdot \eta^k)\|_{L^{1,\infty}(Q_{T^*})}$ for $k \neq N-1$, and $\|v^n\|_{L^{2,\infty}(Q_{T^*})}$ are uniformly bounded by functions of $\frac{1}{n}\, \|\Phi_\omega(\mu^{0,n} \cdot \eta^k)\|_{L^1(\Omega)}$, of $\|\sqrt{f^n_\mu(\mu^{0,n}) \cdot 1^N}\, v^{0,n}\|_{L^2(\Omega)}$, and of $\|\mu^\Gamma\|_{L^2(S_{T^*})}$.

By means of the conditions (109), we can verify that $\|\Phi_\omega(\mu^n \cdot \eta^k)\|_{L^{1,\infty}(Q_{T^*})} < +\infty$ bounded implies that $\sqrt{1 + |\mu^n \cdot \eta^k|}$ is bounded in $L^{1,\infty}(Q_{T^*})$. Since μ^n and v^n live on a discrete spaces, we obtain from these informations uniform bounds (depending only on n) for $(a,\, b)$ in $L^\infty(]0, T^*];\, \mathbb{R}^p \times \mathbb{R}^n)$. Thus, the solution can be extended beyond T^*. The size of the extension interval depends only on n. In this way, we obtain a global solution after finitely many steps. We next state the global existence result, referring for instance to [DDGG16] for a working out of the argument in more details.

Corollary A.4. *The problem* (127)*,* (128) *possesses a unique global solution* $(a,\, b) \in C^1([0, T];\, \mathbb{R}^p \times \mathbb{R}^n)$.

Uniform estimates. We define $p_n := \hat{P}^n(\mu^n)$ and

$$\rho^n := f_\mu(\mu^n), \quad \tilde{\rho}^n := \hat{P}^n_\mu(\mu^n), \quad r^n := f_\mu(\mu^n) + \frac{1}{n} \sum_{k \neq N-1} \omega'(\mu^n \cdot \eta^k)\, \eta^k\,. \tag{133}$$

Thanks to the dissipation identity of Proposition A.3, we then prove *a priori* bounds.

Proposition A.5. *There is a number* $C_0 > 0$ *such that the main variables* μ^n*,* v^n *and the auxiliary variables* ρ^n*,* $\tilde{\rho}^n$*,* r^n *and* p^n *defined in* (133) *satisfy*

$$\begin{aligned}
&\|\rho_n\|_{L^\infty(Q)} + \|\tilde{\rho}^n\|_{L^\infty(Q)} \leq C_0\,,\\
&\|r^n\|_{L^{2,\infty}(Q)} + \frac{1}{n}\, \|\Phi_\omega(\mu^n)\|_{L^{1,\infty}(Q)} + \|v^n\|_{L^{2,\infty}(Q)} \leq C_0\,,\\
&\|v^n\|_{W^{1,0}_2(Q)} + \|\mu^n\|_{W^{1,0}_2(Q)} + \|p_n\|_{W^{1,0}_2(Q)} \leq C_0\,,\\
&\|n^{-1}\, \omega'(\mu^n \cdot \eta^k)\|_{L^{2,\infty}(Q;\, \mathbb{R}^N)} \leq \frac{C_0}{\sqrt{n}}\,.
\end{aligned}$$

Proof. Since $\rho^n \in S_0$ by construction, we have $0 \leq \rho^n_i \leq \varrho_{\max}$ for $i = 1, \ldots, N$. Using (114), we see that also $\|\tilde{\rho}^n\|_{L^\infty(Q)}$ is uniformly bounded. We integrate the identity of Proposition A.3 in order to obtain bounds for $\frac{1}{n}\, \|\Phi_\omega(\mu^n)\|_{L^{1,\infty}(Q)}$, $\|\sqrt{1^N \cdot f^n_\mu}\, v^n\|_{L^{2,\infty}(Q)}$, $\|v^n\|_{W^{1,0}_2(Q)}$ and $\|\mu^n\|_{W^{1,0}_2(Q)}$. Since

$1^N \cdot f_\mu^n > \varrho_{\min}$, we also obtain the bound for $\|v^n\|_{L^{2,\infty}(Q)}$. Further using (114), we can verify that $\mu \mapsto \hat{P}^n(\mu)$ is globally Lipschitz continuous. Thus $\|p_n\|_{W_2^{1,0}(Q)} \leq C_0$.

Due to the conditions (109), we verify that $|\omega'| \leq c_2\,(1+\Phi_\omega)^{\frac{1}{2}}$ and this directly yields

$$\|n^{-1}\,\omega'(\mu^n\cdot\eta^k)\|^2_{L^{2,\infty}(Q)} \leq \frac{c_2^2}{n}\,(\|n^{-1}\,\Phi_\omega(\mu)\|_{L^{\infty,1}(Q)} + \frac{1}{n}\,|\Omega|) \leq \frac{C_0}{n}\,.$$

Therefore $\|r^n\|_{L^{2,\infty}(Q)} \leq C_0$. □

Remark A.6. *Thanks to Hölder's inequality and the Sobolev embedding for $V_2^{1,0} = W_2^{1,0} \cap L^{2,\infty}$ we further derive uniform bounds for $\|r^n\,v^n\|_{L^{\frac{3}{2},2}(Q;\,\mathbb{R}^{N\times 3})}$ and $\|r^n\,v^n\|_{L^{\frac{4}{3},\infty}(Q;\,\mathbb{R}^{N\times 3})}$, as well as for $\|v^n\otimes v^n\|_{L^{\frac{5}{3}}(Q;\,\mathbb{R}^{3\times 3})}$ and for $\|\sum_{i=1}^N J^{n,i}\,v^n\|_{L^{\frac{5}{4}}(Q;\,\mathbb{R}^{3\times 3})}$.*

Convergence. To pursue, we extract convergent subsequences.

Lemma A.7. *There are $\mu \in W_2^{1,0}(Q;\,\mathbb{R}^N)$ and $v \in W_2^{1,0}(Q;\,\mathbb{R}^3)$ and a subsequence such that as $n\to\infty$:*

$\mu^n \to \mu$ weakly in $W_2^{1,0}(Q;\,\mathbb{R}^N)$ and $\mu^{n,\prime} \to \mu'$ strongly in $L^2(Q;\,\mathbb{R}^N)$,
$v^n \to v$ weakly in $W_2^{1,0}(Q;\,\mathbb{R}^3)$ and strongly in $L^2(Q;\,\mathbb{R}^3)$,
$\rho^n,\,r^n,\,\tilde{\rho}^n \to \rho = f_\mu(\mu)$ strongly in $L^2(Q;\,\mathbb{R}^N)$,
$p_n \to p := f(\mu)$ weakly in $L^2(Q)$.

Proof. At first, using the bounds obtained in Proposition A.5, we extract a subsequence such that

$$\mu^n \to \mu \text{ weakly in } W_2^{1,0}(Q;\,\mathbb{R}^N),\quad v^n \to v \text{ weakly in } W_2^{1,0}(Q;\,\mathbb{R}^3)\,.$$

In the remainder of the proof, we will pass several times to subsequences but omit for convenience indicating this step by means of additional labels.

On the basis of Prop. A.5 and Remark A.6, there are $J \in L^2(Q;\,\mathbb{R}^{N\times 3})$, $\xi \in L^{\frac{3}{2},2}(Q;\,\mathbb{R}^{N\times 3})$, $\zeta \in L^{\frac{5}{3}}(Q;\,\mathbb{R}^{3\times 3})$ and $\chi \in L^{\frac{5}{4}}(Q;\,\mathbb{R}^3)$ such that

$$\begin{aligned}
&J^n \to J \text{ weakly in } L^2(Q;\,\mathbb{R}^{N\times 3})\\
&\jmath^{\Gamma,n} \to \jmath^\Gamma := -M^{\Gamma,\sigma}\,(\mu-\mu^\Gamma) \text{ weakly in } L^2(S_T;\,\mathbb{R}^{N\times 3})\\
&r^n\,v^n \to \xi \text{ weakly in } L^{\frac{3}{2},2}(Q;\,\mathbb{R}^{N\times 3})\\
&p_n \to p \text{ weakly in } W_2^{1,0}(Q;\,\mathbb{R}^N)\\
&[1^N\cdot\tilde{\rho}^n]\,v^n\otimes v^n \to \zeta \text{ weakly in } L^{\frac{5}{3}}(Q;\,\mathbb{R}^{3\times 3})\\
&\sum_{i=1}^N J^{n,i}\,v^n \to \chi \text{ weakly in } L^{\frac{5}{4}}(Q;\,\mathbb{R}^{3\times 3})\,.
\end{aligned} \tag{134}$$

We identify the limits J, ξ, ζ and χ only later, but need them to prove first the strong convergence of $\{\rho^n\}$ and $\{r^n\}$. Consider the decomposition $r^n = f_\mu(\mu^n) + (n)^{-1}\sum_{k\neq N-1}\omega'(\mu^n\cdot\eta^k)\,\eta^k =: \rho^n + b^n$. Due to Proposition A.5, we have $b^n \to 0$ strongly in $L^{2,\infty}(Q)$. Integrating in time the equations (122), we obtain for all $s = 1,\ldots,p(n_0)$ and $n \geq n_0$, and all $t\in[0,\,T]$

$$\int_\Omega \rho^n(t)\cdot\psi^s\,dx = \int_0^t\int_\Omega(\tilde{\rho}^n\,v^n + J^n)\cdot\nabla\psi^s\,dxdt + \int_0^t\int_{\partial\Omega}\jmath^{\Gamma,n}\cdot\psi^s\,dSdt$$
$$+\int_\Omega(\rho^{0,n} - b^n(t))\cdot\psi_s\,dx$$

Thus, we see that for the chosen subsequence, there exists $\lim_{n\to\infty}\int_\Omega \rho^n(t)\cdot\psi^s\,dx$ for $s = 1,\ldots,p(n_0)$. Moreover, for $r = 1,\ldots,m(n_0)$, we have $\rho^n(t)\cdot w^r = f_\mu(\mu^n)\cdot\bar{V}\,u_r = u_r$ is independent of n and t. Since ρ^n is bounded in $L^\infty(Q_T;\,\mathbb{R}^N)$ and $\{\psi^1,\ldots,\psi^{p(n)},w^1,\ldots,w^{m(n)}\}$ is dense in $W^{1,2}(\Omega;\,\mathbb{R}^N)$ for $n\to\infty$, we easily show that $\{\rho^n(t)\}_{k\in\mathbb{N}}$ converges as distributions for all $t\in[0,\,T]$.

Since $D^2 f$ is uniformly bounded (cf. Lemma 4.3, (a)), we moreover obtain a bound

$$\|\nabla\rho^n\|_{L^2(Q;\mathbb{R}^N)} \leq |D^2 f|_\infty\,\|\nabla\mu^n\|_{L^2(Q;\mathbb{R}^N)} \leq C_0\,.$$

Classical arguments allow to show that $\{\rho^n\}$ is a Cauchy sequence in $L^2(Q;\,\mathbb{R}^N)$ (see Lemma B.1). Thus there is a limit $\rho\in L^2(Q;\,\mathbb{R}^N)$ such that $\rho^n\to\rho$ strongly in $L^2(Q;\,\mathbb{R}^N)$, and even $r^n\to\rho$ in $L^2(Q;\,\mathbb{R}^N)$. Since $\rho^n = f_\mu(\mu^n)$ are all elements of the image of f_μ, that is the surface S_0, we directly obtain that $\rho\in L^2(Q;\,S_0)$. Thus, we also can identify $\xi = \rho\,v$ and $J = -M_\sigma(\rho)\,\nabla\mu$.

The next point is to obtain the strong convergence of $\mu^{n,\prime}$. We use the properties of the function f which is constructed as to guaranty the identity (recall (95))

$$\mu^n = \mu^{\text{ref}} + \bar{V}\,f(\mu^n) + k_B\,\theta\,\frac{1}{m_i}\,\ln y_i^n$$
$$y_i^n = Y_i(\mu^n) = \frac{1}{m_i}\,\frac{\rho_i^n}{\sum_j \frac{1}{m_j}\,\rho_j^n},\quad \rho^n := f_\mu(\mu^n)\,.$$

Using that f is globally Lipschitz, we see that $\{f(\mu^n)\}$ is bounded in $W^{1,0}_2(Q)$. Thus, $\ln y_i^n$ is bounded in $W^{1,0}_2$. But $\ln y_i^n$ converges poinwise, because ρ^n converges pointwise. Consequently, $\ln y_i^n\to\ln y_i$ strongly in $L^2(Q)$, in which $y_i = \frac{1}{m_i}\,\frac{\rho_i}{\sum_j\frac{1}{m_j}\,\rho_j}$. Multiplying with a vector η^k, $k\neq N-1$, we obtain that $\mu^n\cdot\eta^k\to\mu^{\text{ref}}\cdot\eta^k + k_B\,\theta\,\sum_{i=1}^n\frac{\eta_i^k}{m_i}\,\ln y_i$ pointwise. This yields the strong convergence of $\mu^{n,\prime}$ in $L^2(Q)$. Now, it is not difficult to recognise that $p_n = \hat{P}^n(\mu^n) = f(B^n(\mu^n)) + \mu^n\cdot\eta^{N-1}$ which is bounded in $W^{1,0}_2(Q)$ converges weakly to $f(\mu') + \mu\cdot\eta^{N-1} = f(\mu)$. Moreover, invoking (114) again, we find $\tilde{\rho}^n\to\rho$ strongly in $L^2(Q)$. □

Passage to the limit $n\to\infty$. For all $\psi\in\operatorname{span}\{\psi^1,\ldots,\psi^p,\,w^1,\ldots,w^m\}$, our approximations μ^n, v^n satisfy (cf. (115), (122), (124))

$$\int_\Omega \partial_t r^n\cdot\psi - \int_\Omega(\tilde{\rho}_i^n\,v^n - M^\sigma_{i,j}(\rho^n)\,\nabla\mu_j^n)\cdot\nabla\psi_i + \int_{\partial\Omega} M^{\Gamma,\sigma}\,(\mu^n-\mu^\Gamma)\cdot\psi = 0\,. \tag{135}$$

In particular, we can insert $\psi := \phi^s\cdot\phi^\ell\,1^N$ for $s,\ell = 1,\ldots,n$ (cp. (118), (119)) to obtain that

$$\int_\Omega \partial_t[1^N\cdot r^n]\,\phi^s\cdot\phi^\ell - \int_\Omega([1^N\cdot\tilde{\rho}^n]\,v^n + \sum_{i=1}^N J^{i,n})\cdot\nabla(\phi^s\cdot\phi^\ell) = 0\,.$$

Recall indeed that $\phi^s = 0$ on $\partial\Omega$. Thus, since $v^n\in\operatorname{span}\{\phi^1,\ldots,\phi^s\}$, it follows that

$$\int_\Omega \partial_t[1^N\cdot r^n]\,\phi^s\cdot v^n - \int_\Omega([1^N\cdot\tilde{\rho}^n]\,v^n + \sum_{i=1}^N J^{i,n})\cdot\nabla(\phi^s\cdot v^n) = 0\,. \tag{136}$$

Moreover, for all ϕ^s, the relations (116), (126) yield

$$\int_\Omega[1^N\cdot r^n]\,\partial_t v^n\cdot\phi^s + \int_\Omega[1^N\cdot\tilde{\rho}^n]\,(v^n\cdot\nabla)v^n\cdot\phi^s + \int_\Omega \mathbb{S}(\nabla v^n))\;:\;\nabla\phi^s$$
$$= \int_\Omega p_n\,\operatorname{div}\phi^s - \int_\Omega(\sum_{i=1}^N J^{i,n}\nabla)v^n\cdot\phi^s\,.$$

By means of adding (136) to the latter, we also obtain that

$$\int_\Omega \partial_t([1^N \cdot r^n]\, v^n) \cdot \phi^s - \int_\Omega [1^N \cdot \tilde{\rho}^n]\, (v^n \otimes v^n) \,:\, \nabla\phi^s - \int_\Omega \mathbb{S}(\nabla v^n) \,:\, \nabla\phi_s$$
$$= \int_\Omega p_n \,\operatorname{div}\phi^s + \int_\Omega \sum_{i=1}^N J^{i,n} \otimes v^n \,:\, \nabla\phi^s\,. \tag{137}$$

We multiply in (135) with $\zeta \in C^1_c(0,T)$, integrate over time to obtain for $\psi(x,\,t) := \psi(x)\,\zeta(t)$

$$-\int_Q r^n \cdot \partial_t\psi - \int_Q (\tilde{\rho}^n\, v^n - M^\sigma(\rho^n)\,\nabla\mu^n)\cdot\nabla\psi + \int_S M^{\Gamma,\sigma}\,(\mu^n - \mu^\Gamma)\cdot\psi = \int_\Omega r^n(0)\cdot\psi(0)\,,$$

and doing the same in (137), we obtain for $Y(x,\,t) = \eta^s(x)\,\zeta(t)$

$$-\int_Q ([1^N \cdot r^n]\, v^n)\cdot\partial_t Y - \int_Q \{[1^N \cdot \tilde{\rho}^n]\,(v^n \otimes v^n) - \mathbb{S}(\nabla v^n))\} \,:\, \nabla Y$$
$$= \int_\Omega ([1^N \cdot r^n(0)]\, v^{0,n})\cdot Y(0) + \int_Q p_n\,\operatorname{div} Y + \int_\Omega \sum_{i=1}^N J^{i,n} \otimes v^n \,:\, \nabla Y\,.$$

We use Proposition A.7 to easily identify the limits.

A noticeable peculiarity: It is not necessary that the initial conditions $\mu^{0,n}$ be uniformly bounded. Indeed the entire argument remains valid if only $\frac{1}{n}\,\|\mu^{0,n}\|_{L^2(\Omega)} \to 0$. We conclude with the proof of Prop. 6.1.

Lemma A.8. *The pair of limits $\mu \in W^{1,0}_2(Q;\,\mathbb{R}^N)$ and $v \in W^{1,0}_2(Q;\,\mathbb{R}^3)$ obtain in Proposition A.7 is a weak solution to (P_σ).*

B Technical points

The following two points are readily established.

Remark B.1.
- *Let $1 \le p \le +\infty$. Let $\mathcal{K} :\; L^p(\Omega) \to W^{1,p}(\Omega)$ be a linear, bounded, compact operator. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset L^p(Q)$ is a sequence such that $u_n(t) \to u(t)$ weakly in $L^p(\Omega)$ for almost all $t \in]0,T[$. Then $\mathcal{K}(u_n(t)) \to \mathcal{K}(u(t))$ strongly in $W^{1,p}(\Omega)$ for almost all $t \in]0,T[$.*
- *If $v_n \to v$ weakly in $W^{1,0}_2(Q)$ and $u_n(t) \to u(t)$ strongly in $[W^{1,2}(\Omega)]^*$ for almost all $t \in]0,T[$, then $u_n\, v_n \to u\, v$ weakly in $L^1(Q)$.*

The strong convergence of the velocity field is in principle known (see [Lio98], page 9) in the context of Navier-Stokes equations.

Corollary B.2. *Assumptions of Proposition 8.1. Then, there is a subsequence such that $v^{\sigma_n} \to v$ strongly in $L^2(Q;\,\mathbb{R}^3)$ and pointwise almost everywhere in Q.*

We also need a suitable solution operator to the equation $\operatorname{div} X = f$.

Remark B.3. *Assume that Ω is a bounded domain of class $\mathcal{C}^1$. There is a solution operator to the linear problem*

$$\operatorname{div} X = f \text{ in } \Omega, \quad X = 0 \text{ on } \partial\Omega\,, \tag{138}$$

for all f having mean value zero over Ω. For all $1 < q < +\infty$ the continuity estimates

$$\|X\|_{W^{1,q}_0(\Omega;\mathbb{R}^3)} \leq c_q\,\|f\|_{L^q(\Omega)}, \quad \|X\|_{L^q(\Omega)} \leq c_q\,\|f\|_{[W^{1,q'}(\Omega)]^*} \tag{139}$$

are valid. For more details about the solution operator, see among others [FNP01], Section 3.1.

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