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Well-posedness and regularity for a fractional tumor growth model

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Abstract

In this paper, we study a system of three evolutionary operator equations involving fractional powers of selfadjoint, monotone, unbounded, linear operators having compact resolvents. This system constitutes a generalization of a phase field system of Cahn–Hilliard type modelling tumor growth that has been proposed in Hawkins-Daarud et al. (*Int. J. Numer. Math. Biomed. Eng.* **28** *(2012), 3–24*) and investigated in recent papers co-authored by the present authors and E. Rocca. The model consists of a Cahn–Hilliard equation for the tumor cell fraction φ, coupled to a reaction-diffusion equation for a function S representing the nutrient-rich extracellular water volume fraction. Effects due to fluid motion are neglected. The generalization investigated in this paper is motivated by the possibility that the diffusional regimes governing the evolution of the different constituents of the model may be of different (e.g., fractional) type. Under rather general assumptions, well-posedness and regularity results are shown. In particular, by writing the equation governing the evolution of the chemical potential in the form of a general variational inequality, also singular or nonsmooth constributions of logarithmic or of double obstacle type to the energy density can be admitted.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ denote an open, bounded, and connected set with smooth boundary Γ and unit outward normal $\mathbf{n}$, let $T > 0$ be given, and set $Q_t := \Omega \times (0,t)$ for $t \in (0,T)$ and $Q := \Omega \times (0,T)$, as well as $\Sigma := \Gamma \times (0,t)$. We investigate in this paper the evolutionary system

$$\alpha\,\partial_t\mu + \partial_t\varphi + A^{2\rho}\mu = P(\varphi)(S-\mu) \qquad \text{in } Q, \tag{1.1}$$

$$\mu = \beta\,\partial_t\varphi + B^{2\sigma}\varphi + f(\varphi) \qquad \text{in } Q, \tag{1.2}$$

$$\partial_t S + C^{2\tau}S = -P(\varphi)(S-\mu) \qquad \text{in } Q, \tag{1.3}$$

$$\mu(0) = \mu_0, \quad \varphi(0) = \varphi_0, \quad S(0) = S_0, \qquad \text{in } \Omega. \tag{1.4}$$

In the above system, $\alpha > 0$ and $\beta > 0$, and $A^{2\rho}$, $B^{2\sigma}$, $C^{2\tau}$, with $r,\,\sigma,\,\tau > 0$, denote fractional powers of the selfadjoint, monotone, and unbounded linear operators A, B, and C, respectively, which are supposed to be densely defined in $H := L^2(\Omega)$ and to have compact resolvents. Moreover, f denotes the derivative of a double-well potential F. Typical and physically significant examples of F are the so-called *classical regular potential*, the *logarithmic potential*, and the *double obstacle*

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potential, which are given, in this order, by

$$F_{reg}(r) := \frac{1}{4}\,(r^2-1)^2\,, \quad r\in\mathbb{R}, \tag{1.5}$$

$$F_{log}(r) := \begin{cases} \big((1+r)\ln(1+r)+(1-r)\ln(1-r)\big)-c_1 r^2\,, & r\in(-1,1) \\ 2\log(2)-c_1\,, & r\in\{-1,1\} \\ +\infty\,, & r\not\in[-1,1] \end{cases}\,, \tag{1.6}$$

$$F_{2obs}(r) := c_2\left(1-r^2\right) \quad \text{if } |r|\le 1 \quad \text{and} \quad F_{2obs}(r) := +\infty \quad \text{if } |r|>1. \tag{1.7}$$

Here, the constants c_i in (1.6) and (1.7) satisfy $c_1 > 1$ and $c_2 > 0$, so that the corresponding functions are nonconvex. In cases like (1.7), one has to split F into a nondifferentiable convex part F_1 (the indicator function of $[-1,1]$, in the present example) and a smooth perturbation F_2. Accordingly, in the term $f(\varphi)$ appearing in (1.2), one has to replace the derivative F_1' of the convex part F_1 by the subdifferential $f_1 := \partial F_1$ and interpret (1.2) as a differential inclusion or as a variation inequality involving F_1 rather than f_1. Furthermore, the function P occurring in (1.1) and (1.3) is nonnegative and smooth. Finally, the terms on the right-hand sides in (1.4) are prescribed initial data.

The above system is a generalization of a phase field system of Cahn–Hilliard type [5] modelling tumor growth. The original model was proposed in [39], then extended in [9, 40], and investigated in [11, 12, 27] from the viewpoint of well-posedness, regularity, and asymptotic analyses; instead, the papers [6, 13, 46–48] were concerned with various optimal control problems that have been set for this class of models. In the mentioned contributions, the three operators $A^{2\rho}$, $B^{2\sigma}$, $C^{2\tau}$ are nothing but the operator $-\Delta$, with homogeneous Neumann boundary conditions. Concerning the meanings of the variables of the system (1.1)–(1.4), φ represents an order parameter accounting for the tumor fraction, and S stands for a nutrient concentration, while the third unknown μ is the related chemical potential, specified by (1.2) as for the viscous Cahn–Hilliard equation. Some interest of (1.1)–(1.3) becomes immediately evident and relies to the fact that we can admit different fractional operators in the description of the evolution of a tumor growth.

Modelling the dynamics of tumor growth has recently become an important issue in applied mathematics (see, e.g., [19, 53]). Indeed, a noteworthy interest arose among mathematicians and applied scientists on the dynamics of tumor cells inside parts of the human body. Thus, a significant number of models have been introduced and discussed, with numerical simulations as well, in connection and comparison with the behavior of other special materials: one may see [3,18,19,23,25,26,38,45,53,54].

As diffuse interface models are concerned, we note that these models mostly use the Cahn–Hilliard framework, which is related to the theory of phase transitions, and which is used extensively in materials science and multiphase fluid flow. Actually, one can distinguish between two main classes of models. The first one considers the tumor and healthy cells as inertialess fluids including effects generated by fluid flow development, postulating a Darcy or a Brinkman law. To this concern, we refer to [20, 22, 29, 30, 33, 34, 36, 41, 43, 49, 52] (see also [4, 17, 21, 24, 37, 50, 51] for local or nonlocal Cahn–Hilliard systems with Darcy or Brinkman law), and we point out that further mechanisms such as chemotaxis and active transport can be taken into account. The other class, to which the model leading to (1.1)–(1.4) belongs, neglects the velocity and admits as variables concentrations and chemical potential. Let us quote a group of contributions inside this class, namely [6, 7, 10, 28, 31, 32, 35, 44]. To our knowledge, up to now fractional operators have not yet been dealt with in either of these two groups of models, although one may also wonder about nonlocal operators.

All in all, fractional operators represent nowadays a challenging subject for mathematicians: they have been used in a number of situations, and there is already a wide literature about equations and systems with fractional terms. In particular, different variants of fractional operators have been considered and employed. For a review of some related work, let us refer the interested reader to our recent papers [14, 15] and [8], which offer a recapitulation of various contributions. In our approach here, which follows closely the setting used in [8, 14–16], we deal with fractional operators defined via spectral theory. Then we can easily consider powers of a second-order elliptic operator with either Dirichlet or Neumann or Robin homogeneous boundary conditions, as well as other operators like, e.g., fourth-order ones or systems involving the Stokes operator. The precise framework for our fractional operators $A^{2\rho}$, $B^{2\sigma}$, $C^{2\tau}$, is given in the first part of Section 2.

The remainder of the paper is organized as follows. In the next section, we list our assumptions and notations and state our results. The uniqueness of the solution is proved in Section 3, while its existence is established in Section 4. The proof is prepared by the study of approximating discrete problems, which are introduced and solved in the subsections of the same section. Finally, the last section is devoted to the regularity of the solution.

2 Statement of the problem and results

In this section, we state precise assumptions and notations and present our results. As mentioned above, the set $\Omega \subset \mathbb{R}^3$ is bounded, connected, and smooth, with volume $|\Omega|$ and outward unit normal vector field $\mathbf{n}$ on $\Gamma := \partial\Omega$. Moreover, $\partial_{\mathbf{n}}$ stands for the corresponding normal derivative. We set

$$H := L^2(\Omega) \tag{2.1}$$

and denote by $\|\cdot\|$ and $(\,\cdot\,,\,\cdot\,)$ the standard norm and inner product of H. Now, we start introducing our assumptions. As for the operators, we first postulate that

$$A : D(A) \subset H \to H, \quad B : D(B) \subset H \to H \quad \text{and} \quad C : D(C) \subset H \to H \quad \text{are}$$
$$\text{unbounded, monotone, selfadjoint, linear operators with compact resolvents.} \tag{2.2}$$

Therefore, there are sequences $\{\lambda_j\}$, $\{\lambda'_j\}$, $\{\lambda''_j\}$ and $\{e_j\}$, $\{e'_j\}$, $\{e''_j\}$ of eigenvalues and of corresponding eigenfunctions satisfying

$$Ae_j = \lambda_j e_j, \quad Be'_j = \lambda'_j e'_j \quad \text{and} \quad Ce''_j = \lambda''_j e''_j$$
$$\text{with} \quad (e_i, e_j) = (e'_i, e'_j) = (e''_i, e''_j) = \delta_{ij} \quad \text{for } i, j = 1, 2, \dots \tag{2.3}$$

$$0 \le \lambda_1 \le \lambda_2 \le \dots, \quad 0 \le \lambda'_1 \le \lambda'_2 \le \dots \quad \text{and} \quad 0 \le \lambda''_1 \le \lambda''_2 \le \dots$$
$$\text{with} \quad \lim_{j\to\infty} \lambda_j = \lim_{j\to\infty} \lambda'_j = \lim_{j\to\infty} \lambda''_j = +\infty, \tag{2.4}$$

$$\{e_j\}, \{e'_j\} \text{ and } \{e''_j\} \text{ are complete systems in } H. \tag{2.5}$$

As a consequence, we can define the powers of these operators with arbitrary positive real exponents as done below. As far as the first operator is concerned, we have for $r > 0$

$$V_A^\rho := D(A^\rho) = \Big\{ v \in H : \sum_{j=1}^{\infty} |\lambda_j^\rho (v, e_j)|^2 < +\infty \Big\} \quad \text{and} \tag{2.6}$$

$$A^\rho v = \sum_{j=1}^{\infty} \lambda_j^\rho (v, e_j) e_j \quad \text{for } v \in V_A^\rho, \tag{2.7}$$

the series being convergent in the strong topology of H, due to the properties (2.6) of the coefficients. We endow V_A^ρ with the graph norm, i.e., we set

$$(v,w)_{A,\rho} := (v,w) + (A^\rho v, A^\rho w) \quad \text{and} \quad \|v\|_{A,\rho} := (v,v)_{A,\rho}^{1/2} \quad \text{for } v,w \in V_A^\rho, \tag{2.8}$$

and obtain a Hilbert space. In the same way, we can define the power B^σ and C^τ for every $\sigma > 0$ and $\tau > 0$, starting from (2.2)–(2.5) for B and C. We therefore set

$$\begin{aligned} & V_B^\sigma := D(B^\sigma) \quad \text{and} \quad V_C^\tau := D(C^\tau) \quad \text{with the norms } \|\cdot\|_{B,\sigma} \text{ and } \|\cdot\|_{C,\tau} \\ & \quad \text{associated with the inner products} \\ & (v,w)_{B,\sigma} := (v,w) + (B^\sigma v, B^\sigma w) \quad \text{and} \quad (v,w)_{C,\tau} := (v,w) + (C^\tau v, C^\tau w), \\ & \quad \text{for } v,w \in V_B^\sigma \text{ and } v,w \in V_C^\tau, \text{ respectively.} \end{aligned} \tag{2.9}$$

Since $\lambda_j \geq 0$ for every j, one immediately deduces from the definition of A^ρ that

$$\begin{aligned} & A^\rho : V_A^\rho \subset H \to H \quad \text{is maximal monotone and} \\ & \varepsilon I + A^\rho : V_A^\rho \to H \quad \text{is a topological isomorphism for every } \varepsilon > 0, \end{aligned} \tag{2.10}$$

where $I : H \to H$ is the identity operator. Similar results hold for B^σ and C^τ. It is clear that, for every $\rho_1,\ \rho_2 > 0$, we have that

$$(A^{\rho_1+\rho_2} v, w) = (A^{\rho_1} v, A^{\rho_2} w) \quad \text{for every } v \in V_A^{\rho_1+\rho_2} \text{ and } w \in V_A^{\rho_2}, \tag{2.11}$$

and that similar relations holds for the other two types of fractional operators. Due to these properties, we can define proper extensions of the operators that allow values in dual spaces. In particular, we can write variational formulations of the equation (1.1)–(1.3). It is convenient to use the notations

$$V_A^{-\rho} := (V_A^\rho)^*, \quad V_B^{-\sigma} := (V_B^\sigma)^*, \quad \text{and} \quad V_C^{-\tau} := (V_C^\tau)^*, \quad \text{for } \rho,\ \sigma,\ \tau > 0. \tag{2.12}$$

Thus, we have that

$$A^{2\rho} \in \mathcal{L}(V_A^\rho; V_A^{-\rho}), \quad B^{2\sigma} \in \mathcal{L}(V_B^\sigma; V_B^{-\sigma}), \quad \text{and} \quad C^{2\tau} \in \mathcal{L}(V_C^\tau; V_C^{-\tau}), \tag{2.13}$$

as well as

$$A^\rho \in \mathcal{L}(H; V_A^{-\rho}), \quad B^\sigma \in \mathcal{L}(H; V_B^{-\sigma}), \quad \text{and} \quad C^\tau \in \mathcal{L}(H; V_C^{-\tau}). \tag{2.14}$$

The symbol $\langle\,\cdot\,,\,\cdot\,\rangle_{A,\rho}$ will be used for the duality pairing between $V_A^{-\rho}$ and V_A^ρ. Moreover, we identify H with a subspace of $V_A^{-\rho}$ in the usual way, i.e., such that

$$\langle v, w\rangle_{A,\rho} = (v,w) \quad \text{for every } v \in H \text{ and } w \in V_A^\rho. \tag{2.15}$$

Analogously, we have that $H \subset V_B^{-\sigma}$ and $H \subset V_C^{-\tau}$ and use similar notations.

From now on, we assume that

$$\alpha,\ \beta,\ \rho,\ \sigma \text{ and } \tau \text{ are fixed positive real numbers.} \tag{2.16}$$

Moreover, for some of our results we have to require the following continuous embeddings of Sobolev type:

$$V_A^\rho \subset L^4(\Omega) \quad \text{and} \quad V_C^\tau \subset L^4(\Omega)\,. \tag{2.17}$$

Under these assumptions, we can choose some $M \geq 1$ such that

$$\|v\|_4 \leq M\,\|v\|_{A,\rho} \quad \text{and} \quad \|v\|_4 \leq M\|v\|_{C,\tau} \tag{2.18}$$

for every $v \in V_A^\rho$ and $v \in V_C^\tau$, respectively.

Remark 2.1. For instance, the first embedding (2.17) is satisfied if $A = -\Delta$, the (negative) Laplace operator, with domain $H^2(\Omega) \cap H^1_0(\Omega)$ (thus, with homogeneous Dirichlet conditions, but similarly for the Neumann boundary conditions). Indeed, $V_A^{1/2} = H^1_0(\Omega)$ in this case. Clearly, the same embedding holds true if ρ is sufficiently close to $1/2$.

For the nonlinear functions entering the equations (1.1)–(1.3) of our system, we postulate the properties listed below. The notation F_1^λ stands for the Moreau–Yosida regularization of F_1 at the level $\lambda > 0$ (see, e.g., [2, p. 39]).

$$F := F_1 + F_2, \quad \text{where:} \tag{2.19}$$

$$F_1 : \mathbb{R} \to [0, +\infty] \quad \text{is convex, proper, and l.s.c., with} \quad F_1(0) = 0; \tag{2.20}$$

$$F_2 : \mathbb{R} \to \mathbb{R} \quad \text{is of class } C^1 \text{ with a Lipschitz continuous first derivative;} \tag{2.21}$$

$$F_1^\lambda(s) + F_2(s) \geq -C_0 \quad \text{for some constant } C_0 \text{ and every } s \in \mathbb{R}; \tag{2.22}$$

$$P : \mathbb{R} \to [0, +\infty) \quad \text{is bounded and Lipschitz continuous.} \tag{2.23}$$

Remark 2.2. The assumption (2.22) can be supposed to hold just for sufficiently small $\lambda > 0$. A sufficient condition for this (see [16, formula (3.1)] for some explanation) is that F satisfies an inequality of type

$$F(s) \geq c_1 s^2 - c_2\,, \quad \text{for some constants } c_i > 0 \text{ and every } s \in \mathbb{R}. \tag{2.24}$$

Hence, (2.20)–(2.22) are fulfilled by all of the important potentials (1.5)–(1.7).

We set, for convenience,

$$f_1 := \partial F_1 \quad \text{and} \quad f_2 := F_2'\,. \tag{2.25}$$

Moreover, we term $D(F_1)$ and $D(f_1)$ the effective domains of F_1 and f_1, respectively. We notice that f_1 is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and use the same symbol f_1 for the maximal monotone operators induced in L^2 spaces. Observe that $D(F_1) = D(f_1) = \mathbb{R}$ for $F = F_{reg}$, while $D(F_1) = [-1, 1]$ and $D(f_1) = (-1, 1)$ for $F = F_{log}$. Finally, we have that $D(F_1) = D(f_1) = [-1, 1]$ if $F = F_{2obs}$.

On account of (2.11) and its analogues for B and C, we give a weak formulation of the equations (1.1)–(1.3). Moreover, we present (1.2) as a variational inequality. For the data, we make the following assumptions:

$$\mu_0 \in H, \quad \varphi_0 \in V_B^\sigma \quad \text{with} \quad F_1(\varphi_0) \in L^1(\Omega), \quad \text{and} \quad S_0 \in H. \tag{2.26}$$

We then look for a triplet (μ, φ, S) satisfying

$$\mu \in H^1(0,T; V_A^{-\rho}) \cap L^\infty(0,T;H) \cap L^2(0,T;V_A^\rho), \tag{2.27}$$

$$\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V_B^\sigma), \tag{2.28}$$

$$S \in H^1(0,T;V_C^{-\tau}) \cap L^\infty(0,T;H) \cap L^2(0,T;V_C^\tau), \tag{2.29}$$

$$F_1(\varphi) \in L^1(Q), \tag{2.30}$$

and solving the system

$$\alpha\langle\partial_t\mu(t), v\rangle_{A,\,\rho} + \big(\partial_t\varphi(t), v\big) + (A^\rho\mu(t), A^\rho v) = \big(P(\varphi(t))(S(t) - \mu(t)), v\big)$$
$$\text{for every } v \in V_A^\rho \text{ and for a.a. } t \in (0,T), \tag{2.31}$$

$$\beta\big(\partial_t\varphi(t),\varphi(t)-v\big)+\big(B^\sigma\varphi(t),B^\sigma(\varphi(t)-v)\big)$$
$$+\int_\Omega F_1(\varphi(t))+\big(f_2(\varphi(t)),\varphi(t)-v\big)\leq\big(\mu(t),\varphi(t)-v\big)+\int_\Omega F_1(v)$$
$$\text{for every } v\in V_B^\sigma \text{ and for a.a. } t\in(0,T), \tag{2.32}$$

$$\langle\partial_t S(t),v\rangle_{C,\tau}+(C^\tau S(t),C^\tau v)=-\big(P(\varphi(t))(S(t)-\mu(t)),v\big)$$
$$\text{for every } v\in V_C^\tau \text{ and for a.a. } t\in(0,T), \tag{2.33}$$

$$\mu(0)=\mu_0\,,\quad \varphi(0)=\varphi_0\,,\quad \text{and}\quad S(0)=S_0\,. \tag{2.34}$$

Here, it is understood that $\int_\Omega F_1(v)=+\infty$ whenever $F_1(v)\not\in L^1(\Omega)$.

We notice at once that (2.32) is equivalent to its time-integrated variant, that is,

$$\beta\int_0^T\big(\partial_t\varphi(t),\varphi(t)-v(t)\big)\,dt+\int_0^T\big(B^\sigma\varphi(t),B^\sigma(\varphi(t)-v(t))\big)\,dt$$
$$+\int_Q F_1(\varphi)+\int_0^T\big(f_2(\varphi(t)),\varphi(t)-v(t)\big)\,dt$$
$$\leq\int_0^T\big(\mu(t),\varphi(t)-v(t)\big)\,dt+\int_Q F_1(v)\quad\text{for every } v\in L^2(0,T;V_B^\sigma). \tag{2.35}$$

Here is our well-posedness and continuous dependence result.

Theorem 2.3. *Let the assumptions* (2.2)*,* (2.16)*,* (2.19)*–*(2.23) *and* (2.25) *on the structure of the system, and* (2.26) *on the data be fulfilled. Then there exists at least one triplet* (μ,φ,S) *satisfying* (2.27)*–*(2.30) *and solving problem* (2.31)*–*(2.34)*. Moreover, for this solution we have the estimates*

$$\alpha^{1/2}\|\mu\|_{L^\infty(0,T;H)}+\|A^\rho\mu\|_{L^2(0,T;H)}$$
$$+\beta^{1/2}\|\partial_t\varphi\|_{L^2(0,T;H)}+\|B^\sigma\varphi\|_{L^\infty(0,T;H)}+\|F(\varphi)\|_{L^\infty(0,T;L^1(\Omega))}$$
$$+\|S\|_{L^\infty(0,T;H)}+\|C^\tau S\|_{L^2(0,T;H)}+\|P^{1/2}(\varphi)(S-\mu)\|_{L^2(0,T;H)}$$
$$\leq C_1\big(\alpha^{1/2}\|\mu_0\|+\|B^\sigma\varphi_0\|+\|F(\varphi_0)\|_{L^1(\Omega)}+\|S_0\|+1\big), \tag{2.36}$$

$$\|\partial_t(\alpha\mu+\varphi)\|_{L^2(0,T;V_A^{-\rho})}+\|\partial_t S\|_{L^2(0,T;V_C^{-\tau})}$$
$$\leq C_2\big(\alpha^{1/2}\|\mu_0\|+\|B^\sigma\varphi_0\|+\|F(\varphi_0)\|_{L^1(\Omega)}+\|S_0\|+1\big), \tag{2.37}$$

with a constant C_1 *that depends only on* Ω *and the constant* C_0 *from* (2.22)*, and a constant* C_2 *that also depends on* P*. If, in addition,* (2.24) *is satisfied, then we also have*

$$\|\varphi\|_{L^\infty(0,T;V_B^\sigma)}\leq C_3\big(\alpha^{1/2}\|\mu_0\|+\|B^\sigma\varphi_0\|+\|F(\varphi_0)\|_{L^1(\Omega)}+\|S_0\|+1\big), \tag{2.38}$$

where C_3 *depends on* Ω *and the constants* C_0*,* c_1 *and* c_2 *from* (2.22) *and* (2.24)*. Finally, the solution* (μ,φ,S) *is unique if the spaces* V_A^ρ *and* V_C^τ *satisfy* (2.17)*.*

Remark 2.4. More generally, we could add known forcing terms u_μ, u_φ and u_S to the right-hand sides of equations (1.1), (1.2) and (1.3), respectively, and accordingly modify the definition of solution. If we assume that

$$u_\mu,\ u_\varphi\,,u_S\in L^2(0,T;H)\,, \tag{2.39}$$

then we have a similar well-posedness result. In estimate (2.36), one has to modify the right-hand side by adding the norms corresponding to (2.39) (possibly multiplied by negative powers of α and β). This remark is useful if one has in mind to perform a control theory on the above system with distributed controls.

Our next aim is to prove further properties of the solution. Indeed, from one side, one wishes to improve the regularity requirements (2.27)–(2.30) under suitable new assumptions on the data. On the other hand, one wishes to have an equation (or at least a differential inclusion) in place of the variational inequality (2.32). The next results deal with these problems independently from each other, in principle. The first one requires just something more on the initial data, indeed. Namely, we assume that

$$\mu_0 \in V_A^\rho\,, \quad \varphi_0 \in V_B^{2\sigma} \quad \text{with} \quad f_1^\circ(\varphi_0) \in H, \quad \text{and} \quad S_0 \in V_C^\tau\,, \tag{2.40}$$

where for $s \in D(f_1)$, the symbol $f_1^\circ(s)$ stands for the element of $f_1(s)$ having minimum modulus. We notice that (2.40) implies (2.26) since $F_1(s) \leq F_1(0) + s f_1^\circ(s)$ for every $s \in D(f_1)$ by convexity and $F_1(0) = 0$. Hence the existence of a solution is still ensured.

For the other problem, one cannot expect anything that is similar to (1.2), since no estimate for $f_1(\varphi)$ is available in the general case. However, if the assumptions on the structure are reinforced, then one can recover (1.2) at least as a differential inclusion. The crucial condition is the following:

$$\psi(v) \in H \quad \text{and} \quad \big(B^{2\sigma}v, \psi(v)\big) \geq 0, \quad \text{for every } v \in V_B^{2\sigma} \text{ and every monotone and Lipschitz continuous function } \psi : \mathbb{R} \to \mathbb{R} \text{ vanishing at the origin.} \tag{2.41}$$

We notice that this assumption is fulfilled if $B^{2\sigma} = -\Delta$ with zero Neumann boundary conditions. Indeed, in this case it results that $V_B^{2\sigma} = \{v \in H^2(\Omega) :\ \partial_{\mathbf{n}} v = 0\}$, and, for every ψ as in (2.41) and $v \in V_B^{2\sigma}$, we have that $\psi(v) \in H^1(\Omega)$ (since $v \in H^1(\Omega)$) as well as

$$\big(B^{2\sigma}v, \psi(v)\big) = \int_\Omega (-\Delta v)\, \psi(v) = \int_\Omega \nabla v \cdot \nabla \psi(v) = \int_\Omega \psi'(v) |\nabla v|^2 \geq 0.$$

More generally, in place of the Laplace operator we can take the principal part of an elliptic operator in divergence form with Lipschitz continuous coefficients, provided that the normal derivative is replaced by the conormal derivative. In any case, we can take the (zero) Dirichlet boundary conditions instead of the Neumann boundary conditions, since the functions ψ for which (2.41) is required has to satisfy $\psi(0) = 0$.

Theorem 2.5. *Let the assumptions* (2.2), (2.16)–(2.23), *and* (2.25) *on the structure of the system be fulfilled. Moreover, let the data satisfy* (2.40). *Then the unique solution* (μ, φ, S) *to problem* (2.31)–(2.34) *enjoys the further regularity*

$$\mu \in H^1(0,T;H) \cap L^\infty(0,T;V_A^\rho) \cap L^2(0,T;V_A^{2\rho}), \tag{2.42}$$

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V_B^\sigma), \tag{2.43}$$

$$S \in H^1(0,T;H) \cap L^\infty(0,T;V_C^\tau) \cap L^2(0,T;V_C^{2\tau}). \tag{2.44}$$

Theorem 2.6. *Besides the assumptions* (2.2), (2.16), (2.19)–(2.23), *and* (2.25) *on the structure of the system, let* (2.41) *be fulfilled. Moreover, let the data satisfy* (2.26). *Then there exist a solution* (μ, φ, S) *to problem* (2.31)–(2.34) *and some* ξ *such that*

$$\varphi \in L^2(0,T;V_B^{2\sigma}) \quad \textit{and} \quad \xi \in L^2(0,T;H)\,, \tag{2.45}$$

$$\beta\, \partial_t \varphi + B^{2\sigma}\varphi + \xi + f_2(\varphi) = \mu \quad \textit{and} \quad \xi \in f_1(\varphi) \quad \textit{a.e. in } Q\,. \tag{2.46}$$

Furthermore, even ξ *is unique under the further assumption* (2.17).

Corollary 2.7. *Assume* (2.2)*,* (2.16)*–*(2.23)*,* (2.25)*, and* (2.41) *for the structure of the system, and* (2.40) *for the data. Then the unique solution* (μ, φ, S) *to problem* (2.31)*–*(2.34) *and the corresponding* ξ *satisfy* (2.42)*–*(2.46) *as well as*

$$\varphi \in L^\infty(0,T;V_B^{2\sigma}) \quad \textit{and} \quad \xi \in L^\infty(0,T;H). \tag{2.47}$$

In the following, we make use of the elementary identity and of the Young inequality

$$a(a-b) = \frac{1}{2}\,a^2 + \frac{1}{2}\,(a-b)^2 - \frac{1}{2}\,b^2 \quad \text{for every } a,b \in \mathbb{R}, \tag{2.48}$$

$$ab \le \delta a^2 + \frac{1}{4\delta}\,b^2 \quad \text{for every } a,b \in \mathbb{R} \text{ and } \delta > 0. \tag{2.49}$$

Moreover, if V is a Banach space and v is any function in $L^2(0,T;V)$, then we define $1 * v \in H^1(0,T;V)$ by setting

$$(1*v)(t) := \int_0^t v(s)\,ds \quad \text{for } t \in [0,T]. \tag{2.50}$$

Notice that, for every $t \in [0,T]$, we have

$$\|(1*v)(t)\|_V^2 \le T \int_0^t \|v(s)\|_V^2\,ds\,, \tag{2.51}$$

$$\bigl(1*(uv)\bigr)(t) = u(t)(1*v)(t) - \int_0^t u'(s)(1*v)(s)\,ds \quad \text{if } u \in H^1(0,T). \tag{2.52}$$

As for the notation concerning norms, we use the symbol $\|\cdot\|_V$ for the norm in the generic Banach space V (as done in (2.51)) with the following exceptions: the simpler symbol $\|\cdot\|$ denotes the norms in H, as already said; for the norms in the spaces V_A^ρ, V_B^σ and V_C^τ we use the notations introduced above; if $1 \le q \le \infty$, the norm in any L^q space is denoted by $\|\cdot\|_q$.

Finally, we state a general rule that we follow throughout the paper as far as the constants are concerned. We use a small-case italic c without subscripts for different constants that may only depend on the final time T, the operators A^ρ, B^σ and C^τ, the shape of the nonlinearities F and P, and the properties of the data involved in the statements at hand. The values of such constants might change from line to line and even within the same formula or chain of inequalities. The symbol c_δ stand for (possibly different) constants that depend on the parameter δ in addition. It is clear that c and c_δ do not depend on the regularization parameter λ and the time step h we introduce in the next sections. With the aim of performing some asymptotic analyses as the parameters α and/or β tend to zero, we clearly specify that the values of c or c_δ do not depend on α or β. Constants (possibly different from each other) that depend, e.g., on both α and β are denoted by $c_{\alpha,\beta}$. In contrast, we use different symbols (like M in (2.18) or C_0 in (2.22)) for precise values of constants that we want to refer to.

3 Uniqueness

In this section, we give the proof of the uniqueness part of Theorem 2.3. We pick two solutions (μ_i, φ_i, S_i), $i = 1,2$, and set for convenience $\mu := \mu_1 - \mu_2$, $\varphi := \varphi_1 - \varphi_2$ and $S := S_1 - S_2$. Now, we write equation (2.31) for these solutions and integrate the difference with respect to time. Then, we test the equality thus obtained by μ. At the same time, we test the difference of (2.33) written for the two solutions by S. Then, we sum up, integrate over $(0,t)$ with an arbitrary $t \in (0,T]$, and rearrange. The resulting left-hand side contains the term $\frac{1}{2}\|A^\rho(1*\mu)(t)\|^2$. Thus, we add the same quantity

$$\frac{1}{2}\,\|(1*\mu)(t)\|^2 = \int_0^t \bigl((1*\mu)(s),\mu(s)\bigr)\,ds$$

to both sides, by choosing the former expression for the left-hand side and the latter for the right-hand side, and use the definition (2.8) with $v = (1*\mu)(t)$. Similarly, we add the same quantity to both sides in order to reconstruct the full norm $\|S(\,\cdot\,)\|_{C,\tau}$ in the corresponding integral. We then obtain the identity

$$\begin{aligned}
&\alpha\int_0^t \|\mu(s)\|^2\,ds + \int_0^t \bigl(\varphi(s),\mu(s)\bigr)\,ds + \frac{1}{2}\,\|(1*\mu)(t)\|^2_{A,\rho} \\
&\quad + \frac{1}{2}\,\|S(t)\|^2 + \int_0^t \|S(s)\|^2_{C,\tau}\,ds \\
&= \int_0^t \Bigl(\bigl(1*[P(\varphi_1)(S_1-\mu_1) - P(\varphi_2)(S_2-\mu_2)]\bigr)(s),\mu(s)\Bigr)\,ds \\
&\quad - \int_0^t \Bigl(\bigl(P(\varphi_1)(S_1-\mu_1) - P(\varphi_2)(S_2-\mu_2)\bigr)(s),S(s)\Bigr)\,ds \\
&\quad + \int_0^t \bigl((1*\mu)(s),\mu(s)\bigr)\,ds + \int_0^t \|S(s)\|^2\,ds\,.
\end{aligned} \tag{3.1}$$

Now, we treat the first term of the right-hand side. In the sequel, δ is a positive parameter. By an integration by parts, we get that

$$\begin{aligned}
&\int_0^t \Bigl(\bigl(1*[P(\varphi_1)(S_1-\mu_1) - P(\varphi_2)(S_2-\mu_2)]\bigr)(s),\mu(s)\Bigr)\,ds \\
&= -\int_0^t \Bigl(\bigl(P(\varphi_1)(S_1-\mu_1) - P(\varphi_2)(S_2-\mu_2)\bigr)(s),(1*\mu)(s)\Bigr)\,ds \\
&\quad + \Bigl(\bigl(1*[P(\varphi_1)(S_1-\mu_1) - P(\varphi_2)(S_2-\mu_2)]\bigr)(t),(1*\mu)(t)\Bigr)\,,
\end{aligned}$$

and we denote by Y_1 and Y_2, in this order, the summands on the right-hand side. To handle these terms, we owe to the Hölder and Young inequalities, the boundedness and the Lipschitz continuity of P, and the embeddings (2.17). As for Y_1, we have

$$\begin{aligned}
Y_1 &= -\int_0^t \bigl([P(\varphi_1)(S-\mu)](s),(1*\mu)(s)\bigr)\,ds \\
&\quad - \int_0^t \bigl([(P(\varphi_1)-P(\varphi_2))(S_2-\mu_2)](s),(1*\mu)(s)\bigr)\,ds \\
&\le \delta\int_0^t \bigl(\|S(s)\|^2 + \|\mu(s)\|^2\bigr)\,ds + c_\delta\int_0^t \|(1*\mu)(s)\|^2\,ds \\
&\quad + c\int_0^t \|\varphi(s)\|_2\bigl(\|S_2(s)\|_4 + \|\mu_2(s)\|_4\bigr)\|(1*\mu)(s)\|_4\,ds \\
&\le \delta\int_0^t \bigl(\|S(s)\|^2 + \|\mu(s)\|^2\bigr)\,ds + c_\delta\int_0^t \|(1*\mu)(s)\|^2\,ds \\
&\quad + \delta\int_0^t \|\varphi(s)\|^2_{B,\sigma}\,ds + c_\delta\int_0^t \bigl(\|S_2(s)\|^2_{C,\tau} + \|\mu_2(s)\|^2_{A,\rho}\bigr)\|(1*\mu)(s)\|^2_{A,\rho}\,ds\,,
\end{aligned} \tag{3.2}$$

and we notice that the function $s \mapsto \|S_2(s)\|_{C,\tau}^2 + \|\mu_2(s)\|_{A,\rho}^2$ belongs to $L^1(0,T)$, thanks to the regularity assumed for the solution (μ_2, φ_2, S_2). In order to deal with Y_2, we prepare an estimate of a delicate term with the help of (2.52). Since P is nonnegative, one of the resulting terms turns out to be nonpositive. Thus, on account of Hölder's inequality, (2.51), and (2.17), we have

$$\begin{aligned}
&-\big((1 * [P(\varphi_1)\mu])(t), (1*\mu)(t)\big) \\
&= -\big(P(\varphi_1(t))(1*\mu)(t), (1*\mu)(t)\big) + \Big(\textstyle\int_0^t P'(\varphi_1(s))\partial_t\varphi_1(s)(1*\mu)(s)\,ds, (1*\mu)(t)\Big) \\
&\leq \delta\, \|(1*\mu)(t)\|_4^2 + c_\delta \Big\| \textstyle\int_0^t P'(\varphi_1(s))\partial_t\varphi_1(s)(1*\mu)(s)\,ds \Big\|_{4/3}^2 \\
&\leq \delta\, \|(1*\mu)(t)\|_4^2 + c_\delta \int_0^t \|\partial_t\varphi_1(s)\|_2^2\, \|(1*\mu)(s)\|_4^2\,ds \\
&\leq \delta\, M\, \|(1*\mu)(t)\|_{A,\rho}^2 + c_\delta \int_0^t \|\partial_t\varphi_1(s)\|^2\, \|(1*\mu)(s)\|_{A,\rho}^2\,ds\,,
\end{aligned}$$

where we observe that the function $s \mapsto \|\partial_t\varphi_1(s)\|^2$ belongs to $L^1(0,T)$. At this point, we can estimate the term Y_2 by using (2.51), this inequality, and (2.18) with $M \geq 1$. We have

$$\begin{aligned}
Y_2 &= \big((1 * [P(\varphi_1)(S-\mu) - (P(\varphi_1) - P(\varphi_2))(S_2-\mu_2)])(t), (1*\mu)(t)\big) \\
&\leq \delta\, \|(1*\mu)(t)\|^2 + c_\delta\, \|(1*[P(\varphi_1)S])(t)\|^2 \\
&\quad - \big((1*[P(\varphi_1)\mu])(t), (1*\mu)(t)\big) \\
&\quad + \delta\, \|(1*\mu)(t)\|_4^2 + c_\delta \|(1*[(P(\varphi_1) - P(\varphi_2))(S_2-\mu_2)])(t)\|_{4/3}^2 \\
&\leq \delta\, \|(1*\mu)(t)\|^2 + c_\delta \int_0^t \|(P(\varphi_1)S)(s)\|^2\,ds \\
&\quad + \delta\, M\, \|(1*\mu)(t)\|_{A,\rho}^2 + c_\delta \int_0^t \|\partial_t\varphi_1(s)\|^2\, \|(1*\mu)(s)\|_{A,\rho}^2\,ds \\
&\quad + \delta\, \|(1*\mu)(t)\|_4^2 + c_\delta \int_0^t \|[(P(\varphi_1) - P(\varphi_2))(S_2-\mu_2)](s)\|_{4/3}^2\,ds \\
&\leq 3\,\delta\, M\, \|(1*\mu)(t)\|_{A,\rho}^2 + c_\delta \int_0^t \|S(s)\|^2\,ds + c_\delta \int_0^t \|\partial_t\varphi_1(s)\|^2\, \|(1*\mu)(s)\|_{A,\rho}^2\,ds \\
&\quad + c_\delta \int_0^t \big(\|S_2(s)\|_4^2 + \|\mu_2(s)\|_4^2\big) \|\varphi(s)\|^2\,ds\,,
\end{aligned} \tag{3.3}$$

where the function $s \mapsto \|S_2(s)\|_4^2 + \|\mu_2(s)\|_4^2$ is known to belong to $L^1(0,T)$. Indeed, we have $S_2 \in L^2(0,T;V_C^\tau) \subset L^2(0,T;L^4(\Omega))$ and $\mu_2 \in L^2(0,T;V_A^\rho) \subset L^2(0,T;L^4(\Omega))$.

Now, we come back to (3.1) and estimate the second term on the right-hand side, which we call Y_3 for simplicity. We have

$$Y_3 = -\int_0^t \Big(\big(P(\varphi_1)(S-\mu) + (P(\varphi_1) - P(\varphi_2))(S_2-\mu_2)\big)(s), S(s)\Big)\,ds$$

$$
\begin{aligned}
&\leq \delta \int_0^t \|\mu(s)\|^2 \, ds + c_\delta \int_0^t \|S(s)\|^2 \, ds \\
&\quad + \delta \int_0^t \|S(s)\|_4^2 \, ds + c_\delta \int_0^t \big(\|S_2(s)\|_4^2 + \|\mu_2(s)\|_4^2 \big) \|\varphi(s)\|^2 \, ds \\
&\leq \delta \int_0^t \|\mu(s)\|^2 \, ds + c_\delta \int_0^t \|S(s)\|^2 \, ds \\
&\quad + \delta M \int_0^t \|S(s)\|_{C,\tau}^2 \, ds + c_\delta \int_0^t \big(\|S_2(s)\|_{C,\tau}^2 + \|\mu_2(s)\|_{A,\rho}^2 \big) \|\varphi(s)\|^2 \, ds \,. \qquad (3.4)
\end{aligned}
$$

At this point, we recall (3.1)–(3.4) and use the Schwarz and Young inequalities to estimate the first term of the last line of (3.1), in order to get the estimate

$$
\begin{aligned}
&\alpha \int_0^t \|\mu(s)\|^2 \, ds + \int_0^t \big(\varphi(s), \mu(s)\big) \, ds + \frac{1}{2} \|(1 * \mu)(t)\|_{A,\rho}^2 \\
&\quad + \frac{1}{2} \|S(t)\|^2 + \int_0^t \|S(s)\|_{C,\tau}^2 \, ds \\
&\leq \delta \int_0^t \big(\|S(s)\|^2 + \|\mu(s)\|^2 \big) \, ds + c_\delta \int_0^t \|(1 * \mu)(s)\|^2 \, ds \\
&\quad + \delta \int_0^t \|\varphi(s)\|_{B,\sigma}^2 \, ds + c_\delta \int_0^t \big(\|S_2(s)\|_{C,\tau}^2 + \|\mu_2(s)\|_{A,\rho}^2 \big) \|(1 * \mu)(s)\|_{A,\rho}^2 \, ds \\
&\quad + 3\,\delta\, M \, \|(1 * \mu)(t)\|_{A,\rho}^2 + c_\delta \int_0^t \|S(s)\|^2 \, ds + c_\delta \int_0^t \|\partial_t \varphi_1(s)\|^2 \, \|(1 * \mu)(s)\|_{A,\rho}^2 \, ds \\
&\quad + c_\delta \int_0^t \big(\|S_2(s)\|_4^2 + \|\mu_2(s)\|_4^2 \big) \|\varphi(s)\|^2 \, ds + \delta \int_0^t \|\mu(s)\|^2 \, ds + c_\delta \int_0^t \|S(s)\|^2 \, ds \\
&\quad + \delta M \int_0^t \|S(s)\|_{C,\tau}^2 \, ds + c_\delta \int_0^t \big(\|S_2(s)\|_{C,\tau}^2 + \|\mu_2(s)\|_{A,\rho}^2 \big) \|\varphi(s)\|^2 \, ds \\
&\quad + \delta \int_0^t \|\mu(s)\|^2 \, ds + c_\delta \int_0^t \|(1 * \mu)(s)\|^2 \, ds + \int_0^t \|S(s)\|^2 \, ds \,. \qquad (3.5)
\end{aligned}
$$

Next, we use the variational inequality (2.32), writing it for the two solutions, and testing the resulting inequalities by φ_2 and φ_1, respectively. Now, we sum up and notice that the contributions involving F_1 cancel out. By integrating over $(0, t)$, using the Lipschitz continuity of f_2, and adding the same term to both sides in order to recover the full V_B^σ-norm on the left-hand side, we then obtain that

$$
\frac{\beta}{2} \|\varphi(t)\|^2 + \int_0^t \|\varphi(s)\|_{B,\sigma}^2 \, ds \;\leq\; c \int_0^t \|\varphi(s)\|^2 \, ds + \int_0^t \big(\mu(s), \varphi(s)\big) \, ds. \qquad (3.6)
$$

Finally, we add (3.5) to (3.6), choose $\delta > 0$ sufficiently small, and apply Gronwall's lemma. We then conclude that $(\mu, \varphi, S) = (0, 0, 0)$, and the proof is complete.

Remark 3.1. In connection with Remark 2.4, we could consider the equations obtained by adding the forcing terms, say, controls, to the right-hand sides of the equations. It is clear that no change is necessary in the above proof in order to obtain uniqueness also in this more general situation. Furthermore, just minor modifications lead to a continuous dependence result. More precisely, if $u_{\mu,i}$, $u_{\varphi,i}$ and $u_{S,i}$, $i = 1, 2$, are two choices of the controls and u_μ, u_φ and u_S denote their differences, then we obtain, with the notation used in the proof,

$$
\begin{aligned}
&\|\mu\|_{L^2(0,T;H)} + \|1 * \mu\|_{L^\infty(0,T;V_A^\rho)} + \|\varphi\|_{L^\infty(0,T;H) \cap L^2(0,T;V_B^\sigma)} + \|S\|_{L^\infty(0,T;H) \cap L^2(0,T;V_C^\tau)} \\
&\leq C_4 \left(\|u_\mu\|_{L^2(0,T;H)} + \|u_\varphi\|_{L^2(0,T;H)} + \|u_S\|_{L^2(0,T;H)} \right), \qquad (3.7)
\end{aligned}
$$

where $C_4 > 0$ depends only on the structure, i.e., the linear operators, the shape of the nonlinearities, the parameters α and β, and the final time T.

4 Existence

In this section, we prove the existence of a solution to problem (2.31)–(2.34) as stated in Theorem 2.3. To help the reader, we start with a formal estimate that gives a flavor of the regularity to be expected and, at the same time, indicates the direction one can take for a rigorous proof. Then, in the next subsections, we introduce the approximating problem and its discretization, solve the discrete problem, perform rigorous estimates, and solve first the regularized problem and then problem (2.31)–(2.34).

4.1 Preliminaries

Here is the formal estimate just mentioned. We multiply (1.1), (1.2), and (1.3), by μ, $-\partial_t\varphi$, and S, respectively, in the scalar product of H. Then we sum up and integrate over $(0,t)$, where $t \in (0,T)$ is arbitrary, noting that the terms involving the product $\mu\,\partial_t\varphi$ cancel each other. By accounting for (2.11) and its analogues for the other two types of operators, we obtain the identity

$$
\begin{aligned}
&\frac{\alpha}{2}\,\|\mu(t)\|^2 + \int_0^t \|A^\rho\mu(s)\|^2\,ds \\
&\quad + \beta\int_{Q_t} |\partial_t\varphi|^2 + \frac{1}{2}\,\|B^\sigma\varphi(t)\|^2 + \int_\Omega F(\varphi(t)) \\
&\quad + \frac{1}{2}\,\|S(t)\|^2 + \int_0^t \|C^\tau S(s)\|^2\,ds + \int_{Q_t} P(\varphi)(S-\mu)^2 \\
&= \frac{\alpha}{2}\,\|\mu_0\|^2 + \frac{1}{2}\,\|B^\sigma\varphi_0\|^2 + \int_\Omega F(\varphi_0) + \frac{1}{2}\,\|S_0\|^2 .
\end{aligned}
$$

By reading F_1 instead of F_1^λ in (2.22), and adding $|\Omega|C_0$ to both sides of the above equality, we conclude that

$$
\begin{aligned}
&\alpha^{1/2}\,\|\mu\|_{L^\infty(0,T;H)} + \|A^\rho\mu\|_{L^2(0,T;H)} \\
&\quad + \beta^{1/2}\,\|\partial_t\varphi\|_{L^2(0,T;H)} + \|B^\sigma\varphi\|_{L^\infty(0,T;H)} + \|F(\varphi)+C_0\|_{L^\infty(0,T;L^1(\Omega))} \\
&\quad + \|S\|_{L^\infty(0,T;H)} + \|C^\tau S\|_{L^2(0,T;H)} + \|P^{1/2}(\varphi)(S-\mu)\|_{L^2(0,T;H)} \\
&\leq C\left(\alpha^{1/2}\,\|\mu_0\| + \|B^\sigma\varphi_0\| + \|F(\varphi_0)+C_0\|_1 + \|S_0\|\right), \qquad (4.1)
\end{aligned}
$$

where $C > 0$ is a universal constant. Eliminating C_0 in the norms by means of the triangle inequality, we obtain an estimate that is nothing but (2.36).

In the next subsections, after introducing and solving the discrete problem, we implement the above argument to derive a rigorous a priori estimate for the discrete solution. Then, we use it for the necessary limiting procedures and solve the original problem. For this purpose, it is convenient to introduce some notations at once.

Notation 4.1. Let N be a positive integer, and let Z be one of the spaces H, V_A^ρ, V_B^σ, V_C^τ. We set $h := T/N$ and $I_n := ((n-1)h, nh)$ for $n = 1,\dots,N$. Given $z = (z_0, z_1, \dots, z_N) \in Z^{N+1}$, we define the piecewise constant and piecewise linear interpolants

$$\overline{z}_h \in L^\infty(0,T;Z), \quad \underline{z}_h \in L^\infty(0,T;Z), \quad \text{and} \quad \widehat{z}_h \in W^{1,\infty}(0,T;Z),$$

by setting

$$\overline{z}_h(t) = z^n \quad \text{and} \quad \underline{z}_h(t) = z^{n-1} \quad \text{for a.a. } t \in I_n,\ n = 1,\dots,N, \tag{4.2}$$

$$\widehat{z}_h(0) = z_0 \quad \text{and} \quad \partial_t \widehat{z}_h(t) = \frac{z^{n+1} - z^n}{h} \quad \text{for a.a. } t \in I_n,\ n = 1,\dots,N. \tag{4.3}$$

For the reader's convenience, we summarize the relations between the finite set of values and the interpolants in the following proposition, whose proof follows from straightforward computations.

Proposition 4.2. *With Notation 4.1, we have that*

$$\|\overline{z}_h\|_{L^\infty(0,T;Z)} = \max_{n=1,\dots,N} \|z^n\|_Z\,, \quad \|\underline{z}_h\|_{L^\infty(0,T;Z)} = \max_{n=0,\dots,N-1} \|z^n\|_Z\,, \tag{4.4}$$

$$\|\partial_t \widehat{z}_h\|_{L^\infty(0,T;Z)} = \max_{0\le n\le N-1} \|(z^{n+1} - z^n)/h\|_Z\,, \tag{4.5}$$

$$\|\overline{z}_h\|^2_{L^2(0,T;Z)} = h \sum_{n=1}^{N} \|z^n\|^2_Z\,, \quad \|\underline{z}_h\|^2_{L^2(0,T;Z)} = h \sum_{n=0}^{N-1} \|z^n\|^2_Z\,, \tag{4.6}$$

$$\|\partial_t \widehat{z}_h\|^2_{L^2(0,T;Z)} = h \sum_{n=0}^{N-1} \|(z^{n+1} - z^n)/h\|^2_Z\,, \tag{4.7}$$

$$\|\widehat{z}_h\|_{L^\infty(0,T;Z)} = \max_{n=1,\dots,N} \max\left\{\|z^{n-1}\|_Z, \|z^n\|_Z\right\} = \max\left\{\|z_0\|_Z, \|\overline{z}_h\|_{L^\infty(0,T;Z)}\right\}, \tag{4.8}$$

$$\|\widehat{z}_h\|^2_{L^2(0,T;Z)} \le h \sum_{n=1}^{N} \big(\|z^{n-1}\|^2_Z + \|z^n\|^2_Z\big) \le h\,\|z_0\|^2_Z \;+\; 2\,\|\overline{z}_h\|^2_{L^2(0,T;Z)}\,. \tag{4.9}$$

Moreover, it holds that

$$\|\widehat{z}_h(t) - \overline{z}_h(t)\|_Z \le \|\overline{z}_h(t) - \underline{z}_h(t)\|_Z, \quad \|\widehat{z}_h(t) - \underline{z}_h(t)\|_Z \le \|\overline{z}_h(t) - \underline{z}_h(t)\|_Z \quad \text{for a.a. } t \in (0,T)\,, \tag{4.10}$$

$$\|\overline{z}_h - \widehat{z}_h\|_{L^\infty(0,T;Z)} = \max_{n=0,\dots,N-1} \|z^{n+1} - z^n\|_Z = h\,\|\partial_t \widehat{z}_h\|_{L^\infty(0,T;Z)}\,, \tag{4.11}$$

$$\|\overline{z}_h - \widehat{z}_h\|^2_{L^\infty(0,T;Z)} \le h \sum_{n=0}^{N-1} h \left\|\frac{z^{n+1} - z^n}{h}\right\|^2_Z = h\,\|\partial_t \widehat{z}_h\|^2_{L^2(0,T;Z)}\,, \tag{4.12}$$

$$\|\overline{z}_h - \widehat{z}_h\|^2_{L^2(0,T;Z)} = \frac{h}{3} \sum_{n=0}^{N-1} \|z^{n+1} - z^n\|^2_Z = \frac{h^2}{3}\,\|\partial_t \widehat{z}_h\|^2_{L^2(0,T;Z)}\,, \tag{4.13}$$

and similar identities for the difference $\underline{z}_h - \widehat{z}_h$. As a consequence, we have the inequalities

$$\|\overline{z}_h - \underline{z}_h\|_{L^\infty(0,T;Z)} \le 2h\,\|\partial_t \widehat{z}_h\|_{L^\infty(0,T;Z)}\,, \tag{4.14}$$

$$\|\overline{z}_h - \underline{z}_h\|^2_{L^2(0,T;Z)} \le \frac{2h^2}{3}\,\|\partial_t \widehat{z}_h\|^2_{L^2(0,T;Z)}\,. \tag{4.15}$$

Finally, we have that

$$h \sum_{n=0}^{N-1} \|(z^{n+1} - z^n)/h\|^2_Z \;\le\; \|\partial_t z\|^2_{L^2(0,T;Z)}\,, \quad \text{if } z \in H^1(0,T;Z) \quad \text{and} \quad z^n = z(nh) \text{ for } n = 0,\dots,N. \tag{4.16}$$

4.2 Approximation and discretization

In this subsection, we introduce an approximation of problem (2.31)–(2.34) and its time discretization. Then, we solve the discrete problem. We first introduce the Moreau–Yosida regularizations F_1^λ and f_1^λ of F_1 of f_1 at the level $\lambda > 0$ (see, e.g., [2, p. 28 and p. 39]). We set, for convenience,

$$F^\lambda := F_1^\lambda + F_2 \quad \text{and} \quad f^\lambda := f_1^\lambda + f_2\,. \tag{4.17}$$

By accounting for well-known properties of this regularization and the assumptions (2.20)–(2.22), we have

$$F_1^\lambda(s) = \int_0^s f_1^\lambda(s')\,ds'\,, \quad 0 \le F_1^\lambda(s) \le F_1(s)\,, \quad \text{and} \quad F^\lambda(s) \ge -C_0\,, \tag{4.18}$$

for every $s \in \mathbb{R}$, as well as

$$|f_1^\lambda(s)| \le |f_1^\circ(s)| \quad \text{for every } s \in D(f_1)\,, \tag{4.19}$$

where $f_1^\circ(s)$ is the element of $f_1(s)$ having minimum modulus. By replacing F_1 in (2.32) by F_1^λ, we obtain the following system:

$$\alpha\,\langle \partial_t \mu^\lambda(t), v\rangle_{A,\rho} + \bigl(\partial_t\varphi^\lambda(t), v\bigr) + (A^\rho\mu^\lambda(t), A^\rho v) = \bigl(P(\varphi^\lambda(t))(S^\lambda(t) - \mu^\lambda(t)), v\bigr)$$
$$\text{for every } v \in V_A^\rho \text{ and for a.a. } t \in (0,T), \tag{4.20}$$

$$\beta\,\bigl(\partial_t\varphi^\lambda(t), \varphi^\lambda(t) - v\bigr) + \bigl(B^\sigma\varphi^\lambda(t), B^\sigma(\varphi^\lambda(t) - v)\bigr)$$
$$+ \int_\Omega F_1^\lambda(\varphi^\lambda(t)) + \bigl(f_2(\varphi^\lambda(t)), \varphi^\lambda(t) - v\bigr)$$
$$\le \bigl(\mu^\lambda(t), \varphi^\lambda(t) - v\bigr) + \int_\Omega F_1^\lambda(v) \quad \text{for every } v \in V_B^\sigma \text{ and for a.a. } t \in (0,T), \tag{4.21}$$

$$\langle \partial_t S^\lambda(t), v\rangle_{C,\tau} + (C^\tau S^\lambda(t), C^\tau v) = -\bigl(P(\varphi^\lambda(t))(S^\lambda(t) - \mu^\lambda(t)), v\bigr)$$
$$\text{for every } v \in V_C^\tau \text{ and for a.a. } t \in (0,T), \tag{4.22}$$

$$\mu^\lambda(0) = \mu_0\,, \quad \varphi^\lambda(0) = \varphi_0\,, \quad \text{and} \quad S^\lambda(0) = S_0\,. \tag{4.23}$$

We stress that (4.21) is equivalent to both the time-integrated variational inequality

$$\beta \int_0^T \bigl(\partial_t\varphi^\lambda(t), \varphi^\lambda(t) - v(t)\bigr)\,dt + \int_0^T \bigl(B^\sigma\varphi^\lambda(t), B^\sigma(\varphi^\lambda(t) - v(t))\bigr)\,dt$$
$$+ \int_Q F_1^\lambda(\varphi^\lambda) + \int_0^T \bigl(f_2(\varphi^\lambda(t)), \varphi^\lambda(t) - v(t)\bigr)\,dt$$
$$\le \int_0^T \bigl(\mu^\lambda(t), \varphi^\lambda(t) - v(t)\bigr)\,dt + \int_Q F_1^\lambda(v) \quad \text{for every } v \in L^2(0,T;V_B^\sigma), \tag{4.24}$$

and the pointwise variational equation (since F_1^λ is differentiable and f_1^λ is its derivative)

$$\beta\bigl(\partial_t\varphi^\lambda(t), v\bigr) + \bigl(B^\sigma\varphi^\lambda(t), B^\sigma v\bigr) + \bigl(f^\lambda(\varphi^\lambda(t)), v\bigr) = \bigl(\mu^\lambda(t), v\bigr)$$
$$\text{for every } v \in V_B^\sigma \text{ and for a.a. } t \in (0,T). \tag{4.25}$$

Theorem 4.3. *Under the same assumptions as in Theorem 2.3, the problem* (4.20)–(4.23) *has at least a solution satisfying the analogues of* (2.27)–(2.29).

Remark 4.4. The above statement does not ensure uniqueness. On the other hand, no uniqueness for the solution to the approximating problem is necessary for our purpose. However, uniqueness is guaranteed if the spaces V_A^ρ and V_C^τ satisfy (2.17). Indeed, in this case, what we have proved in Section 3 can be applied since F^λ satisfies all the properties we have postulated for F.

The major part of the present section is devoted to the proof of Theorem 4.3, which is based on the discretization procedure. Thus, we introduce and solve the discrete problem and then take the limits of the interpolants as the time step size tends to zero.

The discrete problem. We fix an integer $N > 1$ and set $h := T/N$. Moreover, we fix a constant L satisfying

$$L > \operatorname{Lip} f_2 \,, \tag{4.26}$$

where $\operatorname{Lip} f_2$ is the Lipschitz constant of f_2. Then, the discrete problem consists in finding three $(N+1)$-tuples $(\mu^0, \dots, \mu^N)$, $(\varphi^0, \dots, \varphi^N)$, and $(S^0, \dots, S^N)$, satisfying

$$\mu^0 = \mu_0 \,, \quad \varphi^0 = \varphi_0 \,, \quad \text{and} \quad S^0 = S_0 \,, \tag{4.27}$$

$$\begin{aligned} &(\mu^1, \dots, \mu^N) \in (V_A^{2\rho})^N \,, \quad (\varphi^1, \dots, \varphi^N) \in (V_B^{2\sigma})^N \,, \\ &\quad \text{and} \quad (S^1, \dots, S^N) \in (V_C^{2\tau})^N \,, \end{aligned} \tag{4.28}$$

and solving

$$\alpha \, \frac{\mu^{n+1} - \mu^n}{h} + \frac{\varphi^{n+1} - \varphi^n}{h} + A^{2\rho} \mu^{n+1} + P(\varphi^n) \mu^{n+1} = P(\varphi^n) S^{n+1} \,, \tag{4.29}$$

$$\beta \, \frac{\varphi^{n+1} - \varphi^n}{h} + B^{2\sigma} \varphi^{n+1} + (f^\lambda + L\, I)(\varphi^{n+1}) = L \varphi^n + \mu^{n+1} \,, \tag{4.30}$$

$$\frac{S^{n+1} - S^n}{h} + C^{2\tau} S^{n+1} + P(\varphi^n) S^{n+1} = P(\varphi^n) \mu^{n+1} \tag{4.31}$$

a.e. in Ω, for $n = 0, 1, \dots, N-1$. This problem can be solved inductively for $n = 0, \dots, N-1$. Namely, for a given $(\mu^n, \varphi^n, S^n) \in H \times H \times H$, we show that there exists a unique triplet $(\mu^{n+1}, \varphi^{n+1}, S^{n+1}) \in V_A^{2\rho} \times V_B^{2\sigma} \times V_C^{2\tau}$ satisfying a problem equivalent to (4.29)–(4.31). Here is the construction of the new problem. We first observe that the linear operator

$$A_h v := \frac{\alpha}{h} \, v + A^{2\rho} v + P(\varphi^n) v, \quad v \in V_A^{2\rho}, \tag{4.32}$$

is an isomorphism from $V_A^{2\rho}$ to H. To see this, it suffices to apply (2.10) to $A^{2\rho}$ and to notice that the linear operator given by the last contribution $v \mapsto P(\varphi^n) v$ is monotone and continuous from H into itself, since P is bounded and nonnegative. By the way, one also sees that $A_h^{-1} \in \mathcal{L}(H; H)$ is monotone and that its norm is bounded by h/α. Hence, (4.29) can be solved for μ^{n+1}, and we can write

$$\mu^{n+1} = A_h^{-1} \Big(\frac{\alpha}{h} \, \mu^n - \frac{\varphi^{n+1} - \varphi^n}{h} + P(\varphi^n) S^{n+1} \Big). \tag{4.33}$$

So, we replace (4.29) by (4.33), and (4.30) by the equation obtained by inserting in (4.30) the expression for μ^{n+1} given by (4.33) in place of μ^{n+1}. Thus, the new second equation reads

$$\begin{aligned} &\beta \, \frac{\varphi^{n+1} - \varphi^n}{h} + B^{2\sigma} \varphi^{n+1} + (f^\lambda + L\, I)(\varphi^{n+1}) \\ &= \ L \varphi^n + A_h^{-1} \Big(\frac{\alpha}{h} \mu^n - \frac{\varphi^{n+1} - \varphi^n}{h} + P(\varphi^n) S^{n+1} \Big) \end{aligned}$$

or, even better,

$$\beta\,\frac{\varphi^{n+1}-\varphi^n}{h}+B^{2\sigma}\varphi^{n+1}+(f^\lambda+L\,I)(\varphi^{n+1})+\frac{1}{h}\,A_h^{-1}\varphi^{n+1}$$
$$=\;L\varphi^n+A_h^{-1}\Big(\frac{\alpha}{h}\mu^n+\frac{1}{h}\,\varphi^n+P(\varphi^n)S^{n+1}\Big)\,. \tag{4.34}$$

We rewrite (4.31) here, for convenience,

$$\frac{S^{n+1}-S^n}{h}+C^{2\tau}S^{n+1}+P(\varphi^n)S^{n+1}=P(\varphi^n)\mu^{n+1}\,, \tag{4.35}$$

and the new problem is given by (4.33)–(4.35). We now show that it can be solved by a fixed point argument, provided that $h>0$ is small enough. To this end, we construct some mappings. In doing this, for simplicity, we use the symbols μ^{n+1}, φ^{n+1}, and S^{n+1}, as if they were independent variables. The subscripts we choose remind the order of appearance of the equations. Here are the mappings:

$$\begin{aligned}
\Phi_3 : H \to V_C^{2\tau}\subset H; &\qquad \mu^{n+1}\mapsto S^{n+1} \quad \text{by solving (4.35) for } S^{n+1},\\
\Phi_2 : H \to V_B^{2\sigma}\subset H; &\qquad S^{n+1}\mapsto \varphi^{n+1} \quad \text{by solving (4.34) for } \varphi^{n+1},\\
\Phi_1 : H\times H \to V_A^{2\rho}\subset H; &\qquad (\varphi^{n+1},S^{n+1})\mapsto \mu^{n+1} \quad \text{by just applying (4.33)},\\
\Phi : H \to H; &\qquad \overline{\mu}\mapsto \Phi_1\Big(\Phi_2\big(\Phi_3(\overline{\mu})\big),\Phi_3(\overline{\mu})\Big).
\end{aligned}$$

Once we prove that these mappings are well defined and that Φ has a unique fixed point μ^*, it is clear that the unique solution $(\mu^{n+1},\varphi^{n+1},S^{n+1})$ we are looking for is given by $(\mu^*,\Phi_2(\Phi_3(\mu^*)),\Phi_3(\mu^*))$. Let us start. As for Φ_3, one adopts the same argument used to define A_h^{-1}. Concerning Φ_2, the proof is similar, if one notes that the monotonicity of A_h^{-1} follows from the one of A_h and that even f^λ and f_2+LI are everywhere defined monotone operators, the last due to (4.26). Thus, all of the mappings are well defined. Now, we consider Φ_3 and take any $\overline{\mu}_1$, $\overline{\mu}_2\in H$. By writing (4.35) with $\overline{S}_i$ and $\overline{\mu}_i$, $i=1,2$, in place of S^{n+1} and μ^{n+1}, respectively, and multiplying the difference by $\overline{S}_1-\overline{S}_2$, we immediately find that

$$\frac{1}{h}\,\|\overline{S}_1-\overline{S}_2\|\le\sup_{s\in\mathbb{R}}P(s)\,\|\overline{\mu}_1-\overline{\mu}_2\|\,.$$

This implies that

$$\|\Phi_3(\overline{\mu}_1)-\Phi_3(\overline{\mu}_2)\|\le K_3h\,\|\overline{\mu}_1-\overline{\mu}_2\| \quad \text{for every } \overline{\mu}_1,\overline{\mu}_2\in H\,,$$

where K_3 is the supremum of P. Similarly, one shows that

$$\|\Phi_2(\overline{S}_1)-\Phi_2(\overline{S}_2)\|\le K_2\,h\,\|\overline{S}_1-\overline{S}_2\|\,,$$
$$\|\Phi_1(\overline{\varphi}_1,\overline{S}_1)-\Phi_1(\overline{\varphi}_2,\overline{S}_2)\|\le K_1\big(\|\overline{\varphi}_1-\overline{\varphi}_2\|+\|\overline{S}_1-\overline{S}_2\|\big)\,,$$

for every $\overline{S}_i\in H$ and $\overline{\varphi}_i\in H$, $i=1,2$, and some constants K_2 and K_1. Hence, there is a constant K such that

$$\|\Phi(\overline{\mu}_1)-\Phi(\overline{\mu}_2)\|\le Kh\,\|\overline{\mu}_1-\overline{\mu}_2\| \quad \text{for every } \overline{\mu}_1,\overline{\mu}_2\in H.$$

Therefore, if $Kh<1$, Φ is a contraction in H and thus has a unique fixed point. We conclude that the discrete problem is uniquely solvable by assuming that $0<h<K^{-1}$.

4.3 Solution of the approximating problem

As announced in the Introduction, we prove the existence of a solution to the approximating problem (4.20)–(4.23) by taking the limit of the interpolants of the solution to the discrete problem as the time step size h tends to zero. According to Notation 4.1, we remark at once that the regularity required for discrete solution implies that

$$\widehat{\mu}_h \in L^\infty(0,T;V_A^\rho), \quad \underline{\mu}_h \in L^\infty(0,T;H), \quad \text{and} \quad \overline{\mu}_h \in L^\infty(0,T;V_A^{2\rho}), \tag{4.36}$$

$$\widehat{\varphi}_h \in W^{1,\infty}(0,T;V_B^\sigma), \quad \underline{\varphi}_h \in L^\infty(0,T;V_B^\sigma), \quad \text{and} \quad \overline{\varphi}_h \in L^\infty(0,T;V_B^{2\sigma}), \tag{4.37}$$

$$\widehat{S}_h \in L^\infty(0,T;V_C^\tau), \quad \underline{S}_h \in L^\infty(0,T;H), \quad \text{and} \quad \overline{S}_h \in L^\infty(0,T;V_C^{2\tau}), \tag{4.38}$$

and that the discrete problem also reads

$$\alpha\, \partial_t\widehat{\mu}_h + \partial_t\widehat{\varphi}_h + A^{2\rho}\overline{\mu}_h + P(\underline{\varphi}_h)\overline{\mu}_h = P(\underline{\varphi}_h)\overline{S}_h \quad \text{a.e. in } Q, \tag{4.39}$$

$$\beta\, \partial_t\widehat{\varphi}_h + B^{2\sigma}\overline{\varphi}_h + (f^\lambda + L\,I)(\overline{\varphi}_h) = L\underline{\varphi}_h + \overline{\mu}_h \quad \text{a.e. in } Q, \tag{4.40}$$

$$\partial_t\widehat{S}_h + C^{2\tau}\overline{S}_h + P(\underline{\varphi}_h)\overline{S}_h = P(\underline{\varphi}_h)\overline{\mu}_h \quad \text{a.e. in } Q, \tag{4.41}$$

$$\widehat{\mu}_h(0) = \mu_0\,, \quad \widehat{\varphi}_h(0) = \varphi_0\,, \quad \widehat{S}_h(0) = S_0 \quad \text{a.e. in } \Omega. \tag{4.42}$$

We point out that the equations (4.39)–(4.41) have been written a.e. in Q, and in this case all of the terms, including $A^{2\rho}\overline{\mu}_h$, $B^{2\sigma}\overline{\varphi}_h$, and $C^{2\tau}\overline{S}_h$, are interpreted as functions of space and time; another way of reading (4.39)–(4.41) could be in H, a.e. in $(0,T)$, as the single terms make sense in the space H as well.

So, our aim is to let h tend to zero in (4.39)–(4.42) (or in some equivalent formulation). Hence, we start estimating. We do this on the solution to the discrete problem (4.27)–(4.31), by adapting the procedure that led to the formal estimate of Section 4.1. Then, we express the bounds we find in terms of the interpolants. According to the general rule stated at the end of Section 2, the (possibly different) values of the constants termed c are independent of the parameters h, λ, α, and β.

Basic a priori estimate. We test (4.29), (4.30) and (4.31) (by taking the scalar product in H) by μ^{n+1}, $(\varphi^{n+1} - \varphi^n)/h$ and S^{n+1}, respectively, and add the resulting identities to each other. Noting an obvious cancellation, we obtain the equality

$$\begin{aligned}
&\frac{\alpha}{h}\,(\mu^{n+1}, \mu^{n+1} - \mu^n) + \|A^\rho\mu^{n+1}\|^2 + \int_\Omega P(\varphi^n)(\mu^{n+1} - S^{n+1})^2 \\
&\quad + \beta\left\|\frac{\varphi^{n+1} - \varphi^n}{h}\right\|^2 + \frac{1}{h}\big(B^\sigma\varphi^{n+1}, B^\sigma(\varphi^{n+1} - \varphi^n)\big) \\
&\quad + \frac{1}{h}\big((f^\lambda + L)(\varphi^{n+1}), \varphi^{n+1} - \varphi^n\big) \\
&\quad + \frac{1}{h}\,(S^{n+1}, S^{n+1} - S^n) + \|C^\tau S^{n+1}\|^2 \\
&= \frac{L}{h}\,(\varphi^n, \varphi^{n+1} - \varphi^n).
\end{aligned}$$

Now, we observe that the function $s \mapsto F^\lambda(s) + \frac{L}{2}\,r^2 = F_1^\lambda(s) + F_2(s) + \frac{L}{2}\,r^2$ is convex on $\mathbb{R}$, since F_1^λ is convex and L satisfies (4.26). Thus, we have that

$$\begin{aligned}&\bigl((f^\lambda + L\,I)(\varphi^{n+1}), \varphi^{n+1} - \varphi^n\bigr)\\&\geq \int_\Omega F^\lambda(\varphi^{n+1}) + \frac{L}{2}\,\|\varphi^{n+1}\|^2 - \int_\Omega F^\lambda(\varphi^n) - \frac{L}{2}\,\|\varphi^n\|^2.\end{aligned}$$

Therefore, by using this inequality and applying the identity (2.48) to some terms of the previous equality, we deduce that

$$\begin{aligned}&\frac{\alpha}{2h}\,\|\mu^{n+1}\|^2 + \frac{\alpha}{2h}\,\|\mu^{n+1}-\mu^n\|^2 - \frac{\alpha}{2h}\,\|\mu^n\|^2 + \|A^\rho \mu^{n+1}\|^2\\&\quad + \int_\Omega P(\varphi^n)(\mu^{n+1} - S^{n+1})^2 + \beta\,\left\|\frac{\varphi^{n+1}-\varphi^n}{h}\right\|^2\\&\quad + \frac{1}{2h}\,\|B^\sigma\varphi^{n+1}\|^2 + \frac{1}{2h}\,\|B^\sigma(\varphi^{n+1}-\varphi^n)\|^2 - \frac{1}{2h}\,\|B^\sigma\varphi^n\|^2\\&\quad + \frac{1}{h}\int_\Omega F^\lambda(\varphi^{n+1}) + \frac{L}{2h}\,\|\varphi^{n+1}\|^2 - \frac{1}{h}\int_\Omega F^\lambda(\varphi^n) - \frac{L}{2h}\,\|\varphi^n\|^2\\&\quad + \frac{1}{2h}\,\|S^{n+1}\|^2 + \frac{1}{2h}\,\|S^{n+1}-S^n\|^2 - \frac{1}{2h}\,\|S^n\|^2 + \|C^\tau S^{n+1}\|^2\\&\leq \frac{L}{2h}\,\|\varphi^{n+1}\|^2 - \frac{L}{2h}\,\|\varphi^n\|^2 - \frac{L}{2h}\,\|\varphi^{n+1}-\varphi^n\|^2.\end{aligned}$$

At this point, we first note two cancellations; then, we multiply by h and sum up with respect to $n = 0, \dots, m-1$ for $m = 1, \dots, N$. We obtain

$$\begin{aligned}&\frac{\alpha}{2}\,\|\mu^m\|^2 - \frac{\alpha}{2}\,\|\mu_0\|^2 + \frac{\alpha}{2}\sum_{n=0}^{m-1}\|\mu^{n+1}-\mu^n\|^2 + \sum_{n=0}^{m-1} h\,\|A^\rho\mu^{n+1}\|^2\\&\quad + \sum_{n=0}^{m-1} h\int_\Omega P(\varphi^n)(\mu^{n+1}-S^{n+1})^2 + \beta\sum_{n=0}^{m-1} h\,\left\|\frac{\varphi^{n+1}-\varphi^n}{h}\right\|^2\\&\quad + \frac{1}{2}\,\|B^\sigma\varphi^m\|^2 - \frac{1}{2}\,\|B^\sigma\varphi_0\|^2 + \frac{1}{2}\sum_{n=0}^{m-1}\|B^\sigma(\varphi^{n+1}-\varphi^n)\|^2 + \int_\Omega F^\lambda(\varphi^m) - \int_\Omega F^\lambda(\varphi_0)\\&\quad + \frac{1}{2}\,\|S^m\|^2 - \frac{1}{2}\,\|S_0\|^2 + \frac{1}{2}\sum_{n=0}^{m-1}\|S^{n+1}-S^n\|^2 + \sum_{n=0}^{m-1} h\,\|C^\tau S^{n+1}\|^2\\&\leq -\frac{L}{2}\sum_{n=0}^{m-1}\|\varphi^{n+1}-\varphi^n\|^2.\end{aligned}$$

Clearly, this inequality also holds for $m = 0$ if it is understood that all the sums vanish since the set of the indices is empty. Therefore, by rearranging, accounting for (4.18), adding $|\Omega|C_0$ to both sides and owing to the assumption (2.26) on the initial data, we obtain an estimate (the analogue of (4.1)) that in terms of the interpolants reads

$$\begin{aligned}&\alpha\,\|\overline{\mu}_h\|^2_{L^\infty(0,T;H)} + \frac{\alpha}{h}\,\|\overline{\mu}_h - \underline{\mu}_h\|^2_{L^2(0,T;H)} + \|A^\rho\overline{\mu}_h\|^2_{L^2(0,T;H)}\\&\quad + \|(P(\underline{\varphi}_h))^{1/2}(\overline{\mu}_h - \overline{S}_h)\|^2_{L^2(0,T;H)} + \beta\,\|\partial_t\widehat{\varphi}_h\|^2_{L^2(0,T;H)} + \frac{L}{h}\,\|\overline{\varphi}_h - \underline{\varphi}_h\|^2_{L^2(0,T;H)}\\&\quad + \|B^\sigma\overline{\varphi}_h\|^2_{L^\infty(0,T;H)} + \frac{1}{h}\,\|B^\sigma(\overline{\varphi}_h - \underline{\varphi}_h)\|^2_{L^2(0,T;H)} + \|F^\lambda(\overline{\varphi}_h)\|_{L^\infty(0,T;L^1(\Omega))}\end{aligned}$$

$$
\begin{aligned}
&+ \|\overline{S}_h\|^2_{L^\infty(0,T;H)} + \frac{1}{h}\,\|\overline{S}_h - \underline{S}_h\|^2_{L^2(0,T;H)} + \|C^\tau \overline{S}_h\|^2_{L^2(0,T;H)} \\
&\le C_0' \bigl(\alpha\,\|\mu_0\|^2 + \|B^\sigma \varphi_0\|^2 + \|F(\varphi_0)\|_1 + \|S_0\|^2 + 1\bigr)\,, \qquad (4.43)
\end{aligned}
$$

where C_0' depends only on Ω and the constant C_0.

First consequences. We observe that (see also (4.12))

$$
\begin{aligned}
&\|\overline{\varphi}_h(t)\| \le \|\widehat{\varphi}_h(t)\| + \|\overline{\varphi}_h(t) - \widehat{\varphi}_h(t)\| \\
&\le \|\varphi_0\| + T^{1/2}\|\partial_t\widehat{\varphi}_h\|_{L^2(0,T;H)} + h^{1/2}\|\partial_t\widehat{\varphi}_h\|_{L^2(0,T;H)}
\end{aligned}
$$

for a.a. $t \in (0,T)$. Moreover, the inequality $P(s) \le c\,(P(s))^{1/2}$ holds true for every $s \in \mathbb{R}$ due to the boundedness of P. Hence, we infer from (4.43) that

$$
\begin{aligned}
&\|\overline{\mu}_h\|_{L^\infty(0,T;H)\cap L^2(0,T;V_A^\rho)} + \|\overline{\varphi}_h\|_{L^\infty(0,T;V_B^\sigma)} + \|\overline{S}_h\|_{L^\infty(0,T;H)\cap L^2(0,T;V_C^\tau)} \\
&\quad + \|\widehat{\varphi}_h\|_{H^1(0,T;H)} + \|F^\lambda(\overline{\varphi}_h)\|_{L^\infty(0,T;L^1(\Omega))} + \|P(\underline{\varphi}_h)(\overline{\mu}_h - \overline{S}_h)\|_{L^2(0,T;H)} \\
&\le c_{\alpha,\beta}\,, \qquad (4.44)
\end{aligned}
$$

as well as (due to (4.10))

$$
\begin{aligned}
&\|\overline{\mu}_h - \widehat{\mu}_h\|_{L^2(0,T;H)} + \|\overline{\varphi}_h - \underline{\varphi}_h\|_{L^2(0,T;V_B^\sigma)} + \|\overline{\varphi}_h - \widehat{\varphi}_h\|_{L^2(0,T;H)} \\
&\quad + \|\overline{S}_h - \widehat{S}_h\|_{L^2(0,T;H)} \le c_\alpha\, h^{1/2}. \qquad (4.45)
\end{aligned}
$$

By combining with (4.44), we deduce that

$$
\|\underline{\varphi}_h\|_{L^2(0,T;V_B^\sigma)} \le c_{\alpha,\beta}\,. \qquad (4.46)
$$

We also derive an estimate that we will use later on. Since F_2 grows at most quadratically due to (2.21), the inequality (4.44) yields an estimate for $F_2(\overline{\varphi}_h)$ in $L^\infty(0,T;L^1(\Omega))$. Therefore, owing to the estimate of $F^\lambda(\overline{\varphi}_h)$ given by (4.43), we deduce that

$$
\|F_1^\lambda(\overline{\varphi}_h)\|_{L^\infty(0,T;L^1(\Omega))} \le c_{\alpha,\beta}\,. \qquad (4.47)
$$

Second a priori estimate. By direct computation, for $n = 0,\dots,N-1$ and for a.e. $t \in (nh, (n+1)h)$, we have that

$$
\begin{aligned}
&\|B^\sigma(\widehat{\varphi}_h(t) - \underline{\varphi}_h(t))\| = \|B^\sigma(\varphi^n + \tfrac{t-nh}{h}\,(\varphi^{n+1} - \varphi^n) - \varphi^n)\| \\
&= \tfrac{t-nh}{h}\,\|B^\sigma(\varphi^{n+1} - \varphi^n)\| = \tfrac{t-nh}{h}\,\|B^\sigma(\overline{\varphi}_h(t) - \underline{\varphi}_h(t))\| \le \|B^\sigma(\overline{\varphi}_h(t) - \underline{\varphi}_h(t))\|\,,
\end{aligned}
$$

whence

$$
\|B^\sigma(\widehat{\varphi}_h - \underline{\varphi}_h)\|_{L^2(0,T;H)} \le \|B^\sigma(\overline{\varphi}_h - \underline{\varphi}_h)\|_{L^2(0,T;H)}\,.
$$

By also accounting for (4.45), we deduce that

$$
\|\widehat{\varphi}_h - \underline{\varphi}_h\|_{L^2(0,T;V_B^\sigma)} \le c_\alpha\, h^{1/2}\,,
$$

and (4.46) yields that

$$
\|\widehat{\varphi}_h\|_{L^2(0,T;V_B^\sigma)} \le c_{\alpha,\beta}\,. \qquad (4.48)
$$

Third a priori estimate. By equation (4.39) and assumption (2.23), we have

$$\begin{aligned}&\alpha\,\|\partial_t\widehat{\mu}_h\|_{L^2(0,T;V_A^{-\rho})}\\&\le c\left(\|\partial_t\widehat{\varphi}_h\|_{L^2(0,T;H)}+\|A^{2\rho}\overline{\mu}_h\|_{L^2(0,T;V_A^{-\rho})}+\|\overline{\mu}_h\|_{L^2(0,T;H)}+\|\overline{S}_h\|_{L^2(0,T;H)}\right).\end{aligned}$$

Then, we account for (4.44) and the first of (2.13) to obtain an estimate for the time derivative $\partial_t\widehat{\mu}_h$. By proceeding analogously with equation (4.41), we conclude that

$$\|\partial_t\widehat{\mu}_h\|_{L^2(0,T;V_A^{-\rho})}+\|\partial_t\widehat{S}_h\|_{L^2(0,T;V_C^{-\tau})}\le c_{\alpha,\beta}\,. \tag{4.49}$$

Convergence. By recalling (4.44)–(4.49), we see that there exist a triplet $(\mu^\lambda,\varphi^\lambda,S^\lambda)$ such that

$$\overline{\mu}_h\to\mu^\lambda \quad \text{weakly star in } L^\infty(0,T;H)\cap L^2(0,T;V_A^\rho)\,, \tag{4.50}$$
$$\widehat{\mu}_h\to\mu^\lambda \quad \text{weakly star in } H^1(0,T;V_A^{-\rho})\cap L^\infty(0,T;H)\,, \tag{4.51}$$
$$\overline{\varphi}_h\to\varphi^\lambda \quad \text{weakly star in } L^\infty(0,T;V_B^\sigma)\,, \tag{4.52}$$
$$\underline{\varphi}_h\to\varphi^\lambda \quad \text{weakly in } L^2(0,T;V_B^\sigma)\,, \tag{4.53}$$
$$\widehat{\varphi}_h\to\varphi^\lambda \quad \text{weakly in } H^1(0,T;H)\cap L^2(0,T;V_B^\sigma)\,, \tag{4.54}$$
$$\overline{S}_h\to S^\lambda \quad \text{weakly star in } L^\infty(0,T;H)\cap L^2(0,T;V_C^\tau)\,, \tag{4.55}$$
$$\widehat{S}_h\to S^\lambda \quad \text{weakly star in } H^1(0,T;V_C^{-\tau})\cap L^\infty(0,T;H)\,, \tag{4.56}$$

at least for some sequence $h_k\searrow 0$. From (4.51), (4.54), (4.56), and (4.42), we deduce that the initial conditions (4.23) are satisfied by the limiting triplet. Next, we prove that (4.20)–(4.22) are fulfilled as well. By first applying the Aubin–Lions lemma (see, e.g., [42, Thm. 5.1, p. 58]) to $\widehat{\varphi}_h$ on account of (4.54), and then owing to (4.45), we deduce that

$$\widehat{\varphi}_h\to\varphi^\lambda,\quad \overline{\varphi}_h\to\varphi^\lambda,\quad\text{and}\quad \underline{\varphi}_h\to\varphi^\lambda,\quad \text{strongly in } L^2(0,T;H). \tag{4.57}$$

In particular note that the limit φ^λ is in $H^1(0,T;H)\cap L^\infty(0,T;V_B^\sigma)$, thanks to (4.52) and (4.54). Next, by recalling that f_2 and P are Lipschitz continuous (see (2.21), (2.23), and (2.25)), and that the same holds for f_1^λ due to the general properties of the Yosida approximation, we infer that

$$f^\lambda(\overline{\varphi}_h)\to f^\lambda(\varphi^\lambda)\quad\text{and}\quad P(\underline{\varphi}_h)\to P(\varphi^\lambda)\quad \text{strongly in } L^2(0,T;H).$$

The latter, (4.55), and (4.50) imply that

$$P(\underline{\varphi}_h)(\overline{S}_h-\overline{\mu}_h)\to P(\varphi^\lambda)(S^\lambda-\mu^\lambda)\quad \text{weakly in } L^1(Q).$$

On the other hand, $P(\underline{\varphi}_h)(\overline{S}_h-\overline{\mu}_h)$ is bounded in $L^2(0,T;H)$ by (4.44). Therefore, we conclude that

$$P(\underline{\varphi}_h)(\overline{S}_h-\overline{\mu}_h)\to P(\varphi^\lambda)(S^\lambda-\mu^\lambda)\quad \text{weakly in } L^2(0,T;H).$$

In view of (4.57), we have that, possibly taking another subsequence of h,

$$\overline{\varphi}_h(t)\to\varphi^\lambda(t)\quad \text{strongly in } H,\quad \text{for a.a. } t\in(0,T).$$

Hence, by lower semicontinuity it turns out that

$$\int_\Omega F_1^\lambda(\varphi^\lambda(t))\le\liminf_{h\searrow 0}\int_\Omega F_1^\lambda(\overline{\varphi}_h(t))\le c_{\alpha,\beta}\quad \text{for a.a. } t\in(0,T). \tag{4.58}$$

At this point, we write (4.39)–(4.41) in the equivalent form

$$\int_0^T \Big(\alpha \, \langle \partial_t \widehat{\mu}_h(s), v(s) \rangle_{A,\rho} + (\partial_t \widehat{\varphi}_h(s), v(s)) + (A^\rho \overline{\mu}_h(s), A^\rho v(s))\Big)\, ds$$

$$= \int_0^T \big(P(\underline{\varphi}_h(s))(\overline{S}_h(s) - \overline{\mu}_h(s)), v(s)\big)\, ds \qquad \text{for every } v \in L^2(0,T;V_A^\rho)\,,$$

$$\int_0^T \Big(\big(\beta\, \partial_t \widehat{\varphi}_h(s), v(s)\big) + \big(B^\sigma \overline{\varphi}_h(s), B^\sigma v(s)\big) + \big((f_1^\lambda + f_2 + L\, I)(\overline{\varphi}_h(s)), v(s)\big)\Big)\, ds$$

$$= \int_0^T \big(L \underline{\varphi}_h(s) + \overline{\mu}_h(s), v(s)\big)\, ds \qquad \text{for every } v \in L^2(0,T;V_B^\sigma)\,,$$

$$\int_0^T \Big(\big(\partial_t \widehat{S}_h(s), v(s)\big) + \big(C^\tau \overline{S}_h(s), C^\tau v(s)\big)\Big)\, ds$$

$$= - \int_0^T \big(P(\underline{\varphi}_h(s))(\overline{S}_h(s) - \overline{\mu}_h(s)), v(s)\big)\, ds \qquad \text{for every } v \in L^2(0,T;V_C^\tau)\,,$$

and let h tend to zero on account of the convergence properties we have established. We obtain the integrated versions of (4.20), (4.22), and (4.25). Now, starting from (4.25), we can perform the formal procedure that led to the estimate (4.1), by observing that the argument used there is now correct. One obtains the estimate

$$\begin{aligned} &\alpha^{1/2}\, \|\mu^\lambda\|_{L^\infty(0,T;H)} + \|A^\rho \mu^\lambda\|_{L^2(0,T;H)} \\ &\quad + \beta^{1/2}\, \|\partial_t \varphi^\lambda\|_{L^2(0,T;H)} + \|B^\sigma \varphi^\lambda\|_{L^\infty(0,T;H)} + \|F^\lambda(\varphi^\lambda) + C_0\|_{L^\infty(0,T;L^1(\Omega))} \\ &\quad + \|S^\lambda\|_{L^\infty(0,T;H)} + \|C^\tau S^\lambda\|_{L^2(0,T;H)} + \|P^{1/2}(\varphi^\lambda)(S^\lambda - \mu^\lambda)\|_{L^2(0,T;H)} \\ &\leq C\big(\alpha^{1/2}\|\mu_0\| + \|B^\sigma \varphi_0\| + \|F^\lambda(\varphi_0) + C_0\|_1 + \|S_0\|\big), \end{aligned} \tag{4.59}$$

where C_0 is given by (4.18) and C is a universal constant. Just something on the regularity is missing, namely, the requirements for the time derivatives $\partial_t \mu^\lambda$ and $\partial_t S^\lambda$. But these regularities immediately follow from (4.49), which also yields that

$$\|\partial_t \mu^\lambda\|_{L^2(0,T;V_A^{-\rho})} + \|\partial_t S^\lambda\|_{L^2(0,T;V_C^{-\tau})} \leq c_{\alpha,\beta}\,. \tag{4.60}$$

This concludes the proof of Theorem 4.3.

4.4 Solution to the original problem

In this section, we conclude the proof of Theorem 2.3. Namely, we costruct a solution (μ, φ, S) by letting λ tend to zero in the approximating problem. From (4.59)–(4.60) and the boundedness of P (which implies $P \leq c\, P^{1/2}$), we derive the following estimate:

$$\begin{aligned} &\|\mu^\lambda\|_{H^1(0,T;V_A^{-\rho}) \cap L^\infty(0,T;H) \cap L^2(0,T;V_A^\rho)} + \|\varphi^\lambda\|_{H^1(0,T;H) \cap L^\infty(0,T;V_B^\sigma)} \\ &+ \|S^\lambda\|_{H^1(0,T;V_C^{-\tau}) \cap L^\infty(0,T;H) \cap L^2(0,T;V_C^\tau)} + \|P(\varphi^\lambda)(\mu^\lambda - S^\lambda)\|_{L^2(0,T;H)} \;\leq\; c_{\alpha,\beta}\,. \end{aligned} \tag{4.61}$$

Therefore, by using the same arguments of the previous subsection and the generalized Ascoli theorem, we deduce that (for some sequence $\lambda_k \searrow 0$)

$\mu^\lambda \to \mu$ weakly in $H^1(0,T;V_A^{-\rho}) \cap L^2(0,T;V_A^\rho)$ and strongly in $L^2(0,T;H)$,
$\varphi^\lambda \to \varphi$ weakly star in $H^1(0,T;H) \cap L^\infty(0,T;V_B^\sigma)$ and strongly in $C^0([0,T];H)$,
$S^\lambda \to S$ weakly in $H^1(0,T;V_C^{-\tau}) \cap L^2(0,T;V_C^\tau)$ and strongly in $L^2(0,T;H)$.

Similarly as before, we obtain the initial conditions, and we also have that

$$f_2(\varphi^\lambda) \to f_2(\varphi) \quad \text{and} \quad P(\varphi^\lambda) \to P(\varphi) \quad \text{strongly in } L^2(0,T;H),$$
$$P(\varphi^\lambda)(S^\lambda - \mu^\lambda) \to P(\varphi)(S-\mu) \quad \text{weakly in } L^2(0,T;H) \text{ and strongly in } L^1(Q).$$

In particular, we can pass to the limit in (4.20) and (4.22) to obtain (2.31) and (2.33), respectively. On the contrary, some more work has to be done for the equation for φ, in particular to argue on the $\liminf$ in the left-hand side of the inequality (4.21) or, equivalently, (4.24). To this concern, we can show that

$$\int_\Omega F_1(\varphi(t)) \le \liminf_{\lambda\searrow 0} \int_\Omega F_1^\lambda(\varphi^\lambda(t)) \quad \text{for all } t\in[0,T]. \tag{4.62}$$

Indeed, let us recall the definitions of the resolvent J_λ of $f_1 = \partial F_1$ and the Moreau-Yosida approximation F_1^λ of F_1, which are given by

$$J_\lambda := (I+\lambda f_1)^{-1}, \quad F_1^\lambda(r) := \min_{s\in\mathbb{R}} \bigl\{ \tfrac{1}{2\lambda} |s-r|^2 + F_1(s) \bigr\},$$

in order to point out the property (see, e.g., [2, Prop. 2.11, p. 39])

$$F_1^\lambda(r) = F_1(J_\lambda(r)) + \frac{1}{2\lambda}|J_\lambda(r) - r|^2 \quad \text{for all } r\in\mathbb{R}. \tag{4.63}$$

Now, we know that $\varphi^\lambda(t)$ converges to $\varphi(t)$ in H as $\lambda\searrow 0$ and that $\int_\Omega F_1^\lambda(\varphi^\lambda(t))$ is nonnegative and bounded independently of λ, by virtue of (4.59) and (2.22). Then, using the representation (4.63), it immediately follows that $\int_\Omega F_1(J_\lambda\varphi^\lambda(t))$ is bounded independently of λ and that also $J_\lambda\varphi^\lambda(t)$ converges to $\varphi(t)$ in H as $\lambda\searrow 0$. Hence, (4.62) follows from the lower semicontinuity of the convex functional $v\mapsto \int_\Omega F_1(v)$ in H. In addition, this argument also entails that $F_1(\varphi)\in L^\infty(0,T;L^1(\Omega))$ and

$$0 \le \int_Q F_1(\varphi) \le \liminf_{\lambda\searrow 0} \int_Q F_1^\lambda(\varphi^\lambda), \tag{4.64}$$

which ensures (2.30). At this point, since

$$\int_0^T \bigl(B^\sigma\varphi(t), B^\sigma\varphi(t)\bigr)\, dt \le \liminf_{\lambda\searrow 0} \int_0^T \bigl(B^\sigma\varphi^\lambda(t), B^\sigma\varphi^\lambda(t)\bigr)\, dt$$

by the weak lower semicontinuity of the norm in $L^2(0,T;H)$, it is sufficient to let λ tend to zero in (4.24) in order to obtain (2.35). This concludes the proof of the existence of a solution in the sense of Theorem 2.3; it remains to complete the proof of the estimates (2.36)–(2.38) for the solution we have constructed.

In view of (4.59) and (2.22), we claim that

$$\begin{aligned} 0 \le \int_\Omega (F(\varphi(t)) + C_0) &\le \liminf_{\lambda\searrow 0} \int_\Omega \bigl(F_1^\lambda(\varphi^\lambda(t)) + F_2(\varphi^\lambda(t)) + C_0\bigr) \\ &\le C\bigl(\alpha^{1/2}\|\mu_0\| + \|B^\sigma\varphi_0\| + \|F(\varphi_0)+C_0\|_1 + \|S_0\|\bigr) \end{aligned} \tag{4.65}$$

for all $t\in[0,T]$. Indeed, the first inequality in (4.65) is a consequence of (2.22) when taking the limit as $\lambda\searrow 0$. Moreover, as, for all $t\in[0,T]$, $\varphi^\lambda(t)$ converges to $\varphi(t)$ in H and $F_2\in C^1(\mathbb{R})$ has a Lipschitz continuous derivative (i.e., f_2), using the Taylor formula it is not difficult to verify that

$$F_2(\varphi^\lambda(t)) \to F_2(\varphi(t)) \quad \text{strongly in } L^1(\Omega).$$

Furthermore, the last term in (4.65) comes from the right-hand side of (4.59) as a consequence of $0 \leq \int_\Omega F_1^\lambda(\varphi_0) \leq \int_\Omega F_1(\varphi_0)$, and $\int_\Omega F_1(\varphi_0)$ if finite because of (2.26). Then, (4.65) follows easily from (4.62).

Now, by (4.59), (4.65), and the weak or weak star lower semicontinuity of norms, we easily obtain (2.36). As for (2.37), we find the right bound for the time derivative $\partial_t(\alpha\mu + \varphi)$. We observe that (4.20) yields for every $v \in L^2(0,T;V_A^\rho)$ that

$$
\begin{aligned}
&\int_0^T \langle \partial_t(\alpha\mu^\lambda + \varphi^\lambda)(t), v(t)\rangle_{A,\rho}\, dt \\
&= -\int_0^T \big(A^\rho \mu^\lambda(t), A^\rho v(t)\big)\, dt + \int_0^T \big(P(\varphi^\lambda(t))(S^\lambda(t) - \mu^\lambda(t)), v(t)\big)\, dt \\
&\leq \|A^\rho\mu^\lambda\|_{L^2(0,T;H)}\, \|A^\rho v\|_{L^2(0,T;H)} + \big(\sup P^{1/2}\big)\, \|P^{1/2}(\varphi^\lambda)(S^\lambda - \mu^\lambda)\|_{L^2(0,T;H)}\, \|v\|_{L^2(0,T;H)} \\
&\leq \big\{\|A^\rho\mu^\lambda\|_{L^2(0,T;H)} + \big(\sup P^{1/2}\big)\, \|P^{1/2}(\varphi^\lambda)(S^\lambda - \mu^\lambda)\|_{L^2(0,T;H)}\big\}\|v\|_{L^2(0,T;V_A^\rho)}\,.
\end{aligned}
$$

This, along with (4.59), provides the analogue of the desired estimate for $\partial_t(\alpha\mu^\lambda + \varphi^\lambda)$, and the estimate for $\partial_t(\alpha\mu + \varphi)$ follows immediately. Since the treatment of $\partial_t S$ is quite similar, (2.37) is completely proved. Finally, to obtain (2.38), it suffices to remark that the further assumption (2.24) we make implies that

$$
\begin{aligned}
&\|\varphi\|^2_{L^\infty(0,T;V_B^\sigma)} = \|\varphi\|^2_{L^\infty(0,T;H)} + \|B^\sigma\varphi\|^2_{L^\infty(0,T;H)} \\
&\leq \frac{1}{c_1}\big(\|F(\varphi)\|_{L^\infty(0,T;L^1(\Omega))} + c_2\big) + \|B^\sigma\varphi\|^2_{L^\infty(0,T;H)}\,,
\end{aligned}
$$

so that (2.36) plainly leads to the correct estimate (2.38). Then, Theorem 2.3 turns out to be completely proved.

5 Regularity

This section is devoted to establish further properties of the solution to problem (2.31)–(2.34). Namely, we prove Theorems 2.5 and 2.6 as well as Corollary 2.7. We start with the first of these results.

Proof of Theorem 2.5. The rigorous proof is based on a priori estimates for the solution to the discrete problem obtained by first performing the discrete differentiation of (4.30) and then suitably testing the resulting equality as well as (4.29) and (4.31), and finally summing up. Since the details are rather heavy, we prefer to deal with the approximating problem (4.20)–(4.23), directly, by taking into account that the use of the regularity assumption (2.40) on the initial data would be essentially the same for the rigorous procedure and the formal one.

We differentiate (4.21) with respect to time and test the resulting equality by $\partial_t\varphi^\lambda$. At the same time, we test (4.20) and (4.22) by $\partial_t\mu^\lambda$ and $\partial_t S^\lambda$, respectively. Then, we sum up and integrate over $(0,t)$.

The terms involving the product $\partial_t\varphi^\lambda\,\partial_t\mu^\lambda$ cancel each other, and we obtain

$$\begin{aligned}
&\alpha\int_{Q_t}|\partial_t\mu^\lambda|^2+\frac12\,\|A^\rho\mu^\lambda(t)\|^2\\
&\quad+\frac{\beta}{2}\,\|\partial_t\varphi^\lambda(t)\|^2+\int_0^t\|B^\sigma\partial_t\varphi^\lambda(t)\|^2+\int_{Q_t}(f_1^\lambda)'(\varphi^\lambda)|\partial_t\varphi^\lambda|^2\\
&\quad+\int_{Q_t}|\partial_t S^\lambda|^2+\frac12\,\|C^\tau S^\lambda(t)\|^2\\
&=-\int_{Q_t}f_2'(\varphi^\lambda)|\partial_t\varphi^\lambda|^2+\frac12\,\|A^\rho\mu_0\|^2+\frac{\beta}{2}\,\|\partial_t\varphi^\lambda(0)\|^2+\frac12\,\|C^\tau S_0\|^2\\
&\quad+\int_{Q_t}P(\varphi^\lambda)\bigl[(S^\lambda-\mu^\lambda)\partial_t\mu^\lambda-(S^\lambda-\mu^\lambda)\partial_t S^\lambda\bigr]\,. \qquad (5.1)
\end{aligned}$$

All of the terms on the left-hand side are nonnegative, and the first one on the right-hand side is estimated by a constant proportional to $1/\beta$, due to (4.59). Moreover, the last integral, which we denote by I for brevity, can be dealt with by using the Young inequality and (4.59):

$$\begin{aligned}
I\ &\le\ c\int_{Q_t}|(S^\lambda-\mu^\lambda)\partial_t\mu^\lambda|+c\int_{Q_t}|(S^\lambda-\mu^\lambda)\partial_t S^\lambda|\\
&\le\ \frac{\alpha}{2}\int_{Q_t}|\partial_t\mu^\lambda|^2+\frac12\int_{Q_t}|\partial_t S^\lambda|^2+c\left(\frac{1}{\alpha}+1\right)\int_0^t\bigl(\|S^\lambda(s)\|^2+\|\mu^\lambda(s)\|^2\bigr)\,ds\,,\\
&\le\ \frac{\alpha}{2}\int_{Q_t}|\partial_t\mu^\lambda|^2+\frac12\int_{Q_t}|\partial_t S^\lambda|^2+c_\alpha. \qquad (5.2)
\end{aligned}$$

It remains to deal with the H-norm of $\partial_t\varphi(0)$. To this end, we observe that (4.25) yields

$$\beta\,\partial_t\varphi^\lambda(0)=\mu_0-B^{2\sigma}\varphi_0-f_1^\lambda(\varphi_0)-f_2(\varphi_0)\,,$$

whence (see (2.40) and (4.19))

$$\|\partial_t\varphi^\lambda(0)\|\le\frac{1}{\beta}\bigl(\|\mu_0\|+\|\varphi_0\|_{B,2\sigma}+\|f_1^\circ(\varphi_0)\|+c(\|\varphi_0\|+1)\bigr)\le c_\beta\,.$$

Therefore, if we come back to (5.1) and account for (5.2) and the estimate (4.61) of the previous section, we see that we have proved that

$$\|\mu^\lambda\|_{H^1(0,T;H)\cap L^\infty(0,T;V_A^\rho)}+\|\varphi^\lambda\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V_B^\sigma)}+\|S\|_{H^1(0,T;H)\cap L^\infty(0,T;V_C^\tau)}\le c_{\alpha,\beta}\,.$$

Since $(\mu^\lambda,\varphi^\lambda,S^\lambda)$ converges to (μ,φ,S) as shown in the previous section, the above estimate implies many of the regularity properties stated in (2.42)–(2.44). Indeed, by accounting for what we have already shown in the first part, we see that just the conditions $\mu\in L^2(0,T;V_A^{2\rho})$ and $S\in L^2(0,T;V_C^{2\tau})$ are missing. But these properties immediately follow by comparison in the equations (2.31) and (2.33). This concludes the proof.

Proof of Theorem 2.6. In contrast to the proof of Theorem 2.5, we here use a completely rigorous argument since the details are not complicated. However, a remark is necessary. We recall that the assumption (2.17) on the spaces V_A^ρ and V_C^τ is not required in the statement, so that uniqueness is neither ensured for problem (2.31)–(2.34) nor for the approximating problem. Hence, we have to be precise. Namely, we fix any solution (μ, φ, S) that can be obtained by the procedure adopted in Section 4 and prove both its further regularity and the existence of some ξ satisfying the conditions of the statement if (2.41) is fulfilled. Thus, the interpolants of the discrete solution converge as h tend to zero (along a suitable subsequence) to some solution to the approximating problem, which converges as λ tends to zero (along a subsequence) to the solution we have chosen. So, coming back to the discrete problem (4.27)–(4.31), we multiply (4.30) by $f_1^\lambda(\varphi^{n+1})$. We notice that $f_1^\lambda(\varphi^{n+1}) \in H$, due to (2.41) with $v = \varphi^{n+1}$ and $\psi = f_1^\lambda$, since $\varphi^{n+1} \in V_B^{2\sigma}$, and because f_1^λ is monotone and Lipschitz continuous and vanishes at the origin. We obtain

$$\begin{aligned} &\frac{\beta}{h} \left(\varphi^{n+1} - \varphi^n, f_1^\lambda(\varphi^{n+1})\right) + \left(B^{2\sigma}\varphi^{n+1}, f_1^\lambda(\varphi^{n+1})\right) + \|f_1^\lambda(\varphi^{n+1})\|^2 \\ &= \left(\mu^{n+1} - f_2(\varphi^{n+1}) + L(\varphi^n - \varphi^{n+1}), f_1^\lambda(\varphi^{n+1})\right). \end{aligned} \tag{5.3}$$

For the first term on the left-hand side, we use the convexity in this way:

$$\left(\varphi^{n+1} - \varphi^n, f_1^\lambda(\varphi^{n+1})\right) \geq \int_\Omega F_1^\lambda(\varphi^{n+1}) - \int_\Omega F_1^\lambda(\varphi^n).$$

The second term of (5.3) is nonnegative by assumption (2.41). Finally, the right-hand side is estimated by owing to the Young inequality and to the linear growth of f_2 given by its Lipschitz continuity. Namely, we have that

$$\begin{aligned} &\left(\mu^{n+1} - f_2(\varphi^{n+1}) + L(\varphi^n - \varphi^{n+1}), f_1^\lambda(\varphi^{n+1})\right) \\ &\leq \frac{1}{2} \|f_1^\lambda(\varphi^{n+1})\|^2 + c\,(\|\mu^{n+1}\|^2 + \|\varphi^{n+1}\|^2 + \|\varphi^n - \varphi^{n+1}\|^2 + 1). \end{aligned}$$

Therefore, combining with (5.3), rearranging, multiplying by h, and summing up with respect to $n = 0, \dots, N-1$, we deduce that

$$\begin{aligned} &\beta \int_\Omega F_1^\lambda(\varphi^N) + \frac{1}{2} \sum_{n=0}^{N-1} h\, \|f_1^\lambda(\varphi^{n+1})\|^2 \\ &\leq \beta \int_\Omega F_1^\lambda(\varphi_0) + c\Big(\sum_{n=0}^{N-1} h\, \|\mu^{n+1}\|^2 + \sum_{n=0}^{N-1} h\, \|\varphi^{n+1}\|^2 + \sum_{n=0}^{N-1} h\, \|\varphi^n - \varphi^{n+1}\|^2\Big). \end{aligned}$$

Since the first term on the left-hand side is nonnegative, the above inequality and the second inequality in (4.18) imply the following estimate for the interpolants:

$$\|f_1^\lambda(\overline{\varphi}_h)\|^2_{L^2(0,T;H)} \leq \beta \int_\Omega F_1(\varphi_0) + c\left(\|\overline{\mu}_h\|^2_{L^2(0,T;H)} + \|\overline{\varphi}_h\|^2_{L^2(0,T;H)} + \|\overline{\varphi}_h - \underline{\varphi}_h\|^2_{L^2(0,T;H)}\right).$$

By recalling (2.26) for φ_0 and the estimates (4.43)–(4.44), we conclude that

$$\|f_1^\lambda(\overline{\varphi}_h)\|_{L^2(0,T;H)} \leq c_{\alpha,\beta}.$$

Since $\overline{\varphi}_h \to \varphi^\lambda$ strongly in $L^2(0,T;H)$ (see (4.57)) and f_1^λ is Lipschitz continuous, we infer that

$$\|f_1^\lambda(\varphi^\lambda)\|_{L^2(0,T;H)} \leq c_{\alpha,\beta}.$$

Moreover, φ^λ converges to φ strongly in $L^2(0,T;H)$. Therefore, by using weak compactness and applying, e.g., [1, Lemma 2.3, p. 38], we conclude that

$$f_1^\lambda(\varphi^\lambda) \to \xi \quad \text{weakly in } L^2(0,T;H), \quad \text{for some } \xi \text{ with } \quad \xi \in f_1(\varphi) \quad \text{a.e. in } Q.$$

At this point, we can let λ tend to zero in the integrated version (4.25) and deduce that

$$\beta \int_0^T \big(\partial_t\varphi(t), v(t)\big)\, dt + \int_0^T \big(B^\sigma\varphi(t), B^\sigma v(t)\big)\, dt + \int_0^T \big(\xi(t) + f_2(\varphi(t), v(t)\big)\, dt$$
$$= \int_0^T \big(\mu^\lambda(t), v(t)\big)\, dt \quad \text{for every } v \in L^2(0,T;V_B^\sigma).$$

This variational equation is equivalent to

$$\beta\, \partial_t\varphi + B^{2\sigma}\varphi + \xi + f_2(\varphi) = \mu \quad \text{a.e. in } (0,T) \text{ in the sense of } V_B^{-\sigma},$$

and this implies both (2.45) and (2.46). In order to prove the last sentence, it suffices to recall that the embedding properties (2.17) ensure uniqueness for the solution (μ,φ,S). Hence the uniqueness of ξ simply follows by comparison in (2.46).

Proof of Corollary 2.7. The assumptions of the statement guarantee that the solution (μ,φ,S) is unique and that there exists a unique ξ satisfying the properties stated in Theorem 2.6. In particular, by the above proofs, the (unique) solution $(\mu^\lambda,\varphi^\lambda,S^\lambda)$ to the approximating problem and the corresponding $f_1^\lambda(\varphi^\lambda)$ converge to (μ,φ,S) and to ξ, respectively, in the proper topologies. Moreover, as f_1^λ satisfies the same assumptions as those we have postulated for f_1, we can apply Theorem 2.6 to the approximating problem. Hence, φ^λ belongs to $L^2(0,T;V_B^{2\sigma})$, and (4.21) can be replaced by the equation

$$B^{2\sigma}\varphi^\lambda(t) + f_1^\lambda(\varphi^\lambda(t)) = \mu^\lambda(t) - \beta\,\partial_t\varphi^\lambda(t) - f_2(\varphi^\lambda(t)) \quad \text{for a.a. } t \in (0,T). \tag{5.4}$$

For a while, we argue for a fixed t (a.e. in $(0,T)$). We multiply (5.4) by $f_1^\lambda(\varphi(t)) \in H$. Since $\varphi^\lambda(t) \in V_B^{2\sigma}$ and f_1^λ is monotone, Lipschitz continuous, and vanishes at the origin, we can apply (2.41) and have that

$$\big(B^{2\sigma}\varphi^\lambda(t), f_1^\lambda(\varphi^\lambda(t))\big) \geq 0\,.$$

Therefore, we obtain the inequality

$$\|f_1^\lambda(\varphi^\lambda(t))\| \leq \|\mu^\lambda(t) - \beta\,\partial_t\varphi^\lambda(t) - f_2(\varphi^\lambda(t))\|\,.$$

Since $\mu^\lambda - \beta\,\partial_t\varphi^\lambda - f_2(\varphi^\lambda)$ is bounded in $L^\infty(0,T;H)$ by (4.59), the same is true for $f_1^\lambda(\varphi^\lambda)$. By comparison in (5.4), we deduce that $B^{2\sigma}\varphi^\lambda$ is bounded as well. Hence, it immediately follows that $B^{2\sigma}\varphi \in L^\infty(0,T;H)$, whence also $\xi \in L^\infty(0,T;H)$ by comparison in (2.46).

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