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Longtime behavior of a branching process controlled by branching catalysts

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Abstract

The model under consideration is a catalytic branching model constructed in [DF96], where the catalysts themselves suffer a spatial branching mechanism. Main attention is paid to dimension $d = 3$. The key result is a convergence theorem towards a limit with full intensity (persistence), which in a sense is comparable with the situation for the "classical" continuous super-Brownian motion. As by-products, strong laws of large numbers are derived for the Brownian collision local time controlling the branching of reactants, and for the catalytic occupation time process. Also, the occupation measures are shown to be absolutely continuous.

Contents

1 Introduction and main results

Consider two types of "particles" situated in R^d, one of them we call the *catalysts*, the others the *reactants.* The catalysts perform a continuous super-Brownian motion (SBM) ϱ with constant branching rate $\gamma > 0$. The reactants are also super-Brownian, however given ϱ, their branching rate at time t in the volume element $\mathrm{d}b$ of R^d is just given by $\varrho_t(\mathrm{d}b)$. In other words, first ϱ is realized, and then a continuous SBM $X = X^\varrho = (X^\varrho, P^\varrho_{s,\mu})$ evolves with *varying* branching rates $\varrho_t(\mathrm{d}b)$ (*quenched* approach). More precisely, the rate of branching of a reactant with (Brownian) path W is controlled (in the sense of Dynkin's additive functional approach to superprocesses) by the *Brownian collision local time (BCLT)* $L_{[W,\varrho]}$ of ϱ, formally described by

$$L_{[W,\varrho]}(\mathrm{d}r) := \mathrm{d}r \int \varrho_r(\mathrm{d}b)\, \delta_b(W_r), \tag{1}$$

which exists non-trivially for dimensions $d \le 3$ (cf. Barlow et al. [BEP91]). In higher dimensions instead, W and ϱ do not collide (see Barlow and Perkins [BP94, Proposition 1.3]), and therefore branching should not occur, which means that X^ϱ degenerates to the heat flow. That *catalytic SBM* X^ϱ in R^d, $d \le 3$, was constructed as a continuous process in detail in [DF96].

It might be useful at this point to recall the longtime behavior of SBM with *constant* branching rate, starting with a Lebesgue measure ℓ ([Daw77]). In dimension one, it suffers local extinction almost surely, in dimension two stochastically, whereas in $d \ge 3$ it converges in law to a non-trivial steady state with expectation ℓ (persistence).

The study of the longtime behavior of the catalytic SBM X^ϱ was initiated also in [DF96], but restricted only to dimension $d = 1$. In this case, X^ϱ behaves quite different from the usual spatial branching models in law dimensions. In fact, if both the catalyst process ϱ and the catalytic SBM X^ϱ start off with the Lebesgue measure ℓ, then, for almost all catalyst process realizations, X^ϱ_t converges stochastically to the starting Lebesgue measure ℓ (persistence).

Here we continue the study of this model X^ϱ in the time-space catalytic medium ϱ. In dimension $d = 2$, we get only some partial results, namely some *self-similarity* properties (Proposition 12) and a *random ergodic limit* (Theorem 14). The question whether or not persistence occurs in this "delicate" dimension is an *open* problem (see also Remark 13).

But our *main result* concerns dimension $d = 3$. Here we alternatively allow ϱ to start off also with the ergodic steady state (of the catalyst process) leading to a time-stationary (in law) medium. Then the random (with respect to ϱ) distribution of X_t^ϱ converges in law to some (possibly random) distribution of a random measure of full intensity (*convergence and persistence Theorem* 17 (b)). From this point of view, the time averaged process should obey a *strong law of large numbers* (Theorem 9). Both can be considered as a random medium analog of properties of the classical SBM in higher dimensions. But note that it can be expected that the limit is dependent on the medium ϱ, hence is different from the classical one. To complete the picture, we also establish a strong law of large numbers for the BCLT $L_{[W,\varrho]}$ (Theorem 5).

We also show that the (weighted) *occupation time process* $Y_t := \int_0^t \mathrm{d}r\ X_r$ has *absolutely continuous states.*

The log-Laplace functional $v_t = v_t^\varrho$ of the catalytic SBM X_t^ϱ at time t satisfies (formally) the following *reaction-diffusion equation*

$$-\frac{\partial}{\partial s} v_t(s,a) = \frac{1}{2}\Delta v_t(s,a) - \varrho_s(\mathrm{d}a)\, v_t^2(s,a), \qquad s \leq t, \quad a \in \mathsf{R}^d, \qquad (2)$$

with a terminal condition $v_t(s,\cdot)|_{s=t-} = f \geq 0$. (The backward setting reflects the fact that, for ϱ fixed, the deterministic process v^ϱ is "dual" to the stochastic process X^ϱ, realized by the log-Laplace functional.) Via this connection, our results can also be understood as a probabilistic contribution to the study of that equation with a (random) *singular* reaction coefficient $\varrho_s(\mathrm{d}a)$, describing a spatially *heterogeneous* catalytic reaction. Actually, our results give information on the longtime behavior of the L^1–norm $\int \ell(\mathrm{d}a)\, v(s,a)$ of the solution to (2) as $s \to -\infty$ if it "starts" at time t with a finite mass $\int \ell(\mathrm{d}a)\, f(a)$. In fact, we proved in [DF96] that in the one-dimensional case one has convergence to the starting mass $\int \ell(\mathrm{d}a)\, f(a)$ (persistence). Dimension two is *open*. But the main result of the present paper establishes in dimension three a.s. convergence to a non-zero limit (possibly depending on the medium ϱ).

Note that the one-dimensional case resembles a bit a *reaction-diffusion process of electrically charged species* studied by Glitzky et al. [GGH95]. They got convergence to an equilibrium with exponential velocity. But our three-dimensional model behaves different in that we do not get an equilibrium at the equation level.

2 Brownian collision local time

In this section we introduce the Brownian collision local time $L = L_{[W,\varrho]}$, and state in dimension $d = 3$ a strong law of large numbers (Theorem 5 at p.7).

2.1 Preliminaries

Fix a constant $p > d$ with $d \geq 1$ the dimension of space, and introduce the *reference function*

$$\phi_p(b) := (1 + |b|^2)^{-p/2}, \qquad b \in \mathsf{R}^d. \tag{3}$$

Let $\mathcal{B}^p$ denote the space of all measurable functions f defined on R^d such that $|f| \leq c_f \phi_p$ for some constant c_f. Write $\mathcal{C}^{p;\ell}$ for the subset of all continuous functions f in $\mathcal{B}^p$ such that $f(b)/\phi_p(b)$ has a finite limit as $|b| \to \infty$. Equipped with the norm $\|f\| := \|f/\phi_p\|_\infty$, the Banach space $\mathcal{C}^{p;\ell}$ is *separable.*

Set $I := [0, T]$, $T \geq 0$. Write $\mathcal{C}^{p,I}$ for the set of all continuous functions ψ defined on $I \times \mathsf{R}^d$ such that $|\psi_s| \leq c_\psi \phi_p$, $s \in I$, for some constant c_ψ.

Let $\mathcal{M}_p$ refer to the cone of all (non-negative) measures μ defined on R^d such that

$$\|\mu\|_p := \langle \mu, \phi_p \rangle := \int \mu(\mathrm{d}b)\, \phi_p(b) < +\infty. \tag{4}$$

$\mathcal{M}_p$ is endowed with the coarsest topology such that the maps $\mu \to \langle \mu, f \rangle$ are continuous where $f = \phi_p$ or $f \in \mathcal{C}_+^{\mathrm{comp}}$. Here $\mathcal{C}^{\mathrm{comp}}$ denotes the space of continuous functions on R^d with compact support (and the index $+$ indicates the subset of all non-negative members). Recall that each Lebesgue measure ℓ belongs to $\mathcal{M}_p$. Write i_ℓ for the unit volume in R^d measured with respect to ℓ.

Let $W = (W, \Pi_{s,a})$ denote the *Brownian motion* in R^d on canonical path space of continuous functions, with "generator" $\frac{1}{2}\Delta$. Let $\mathrm{p}_t(a, b) = \mathrm{p}_t(b - a)$ refer to its continuous transition density function, and $S = \{S_t : t \geq 0\}$ to its semigroup. Set $\Pi_{s,\mu} := \int \mu(\mathrm{d}a)\, \Pi_{s,a}$. We also introduce the (time-inhomogeneous) *Brownian potential kernel*

$$\mathrm{q}_{s,t}(a, b) = \mathrm{q}_{s,t}(b - a) := \int_s^t \mathrm{d}r\; \mathrm{p}_r(a, b), \qquad 0 \leq s \leq t, \quad a, b \in \mathsf{R}^d. \tag{5}$$

2.2 Catalyst process ϱ

For convenience, we expose the following definition of the *catalyst process* ϱ.

Definition 1 (catalyst process ϱ) Write $\varrho = (\varrho, \mathbb{P}_{s,\mu})$ for the continuous SBM in R^d with constant branching rate $\gamma > 0$. Consequently, for fixed $t \geq 0$, the *log-Laplace functional* of ϱ is given by

$$-\log \mathbb{P}_{s,\mu} \exp \langle \varrho_t \,, -f \rangle = \langle \mu, -v_t(s,\cdot) \rangle, \qquad s \leq t, \ \mu \in \mathcal{M}_p, \ f \in \mathcal{B}^p, \tag{6}$$

where v_t is the unique non-negative solution to (2) with $\varrho_s(\mathrm{d}a)$ replaced by the constant γ, and with terminal condition $v_t(s,\cdot)|_{s=t-} = f$. Here we always work with a mild solution, that is with a solution to the equation in the integrated and Dynkin's probabilistic form

$$v_t(s,a) \ = \ \Pi_{s,a}\Big[f(W_t) - \int_s^t \gamma \mathrm{d}r \ v_t^2(r, W_r)\Big], \qquad s \leq t, \quad a \in \mathsf{R}^d. \tag{7}$$

ϱ is called the *catalyst process*. ◇

Recall the *expectation* formula

$$\mathbb{P}_{s,\mu} \, \langle \varrho_t \,, f \rangle = \langle \mu \,, S_{t-s} f \rangle, \qquad s \leq t, \quad \mu \in \mathcal{M}_p, \quad f \in \mathcal{B}^p. \tag{8}$$

Recall also, that in dimensions $d \geq 3$, with respect to $\mathbb{P}_{0,\ell}$, the catalyst process ϱ_t has a non-trivial limit ϱ_∞ in law as $t \uparrow \infty$ which has full intensity measure ℓ (see, e.g. [DP91, Proposition 6.1]). Hence, here we can form the *time-stationary* continuous $\mathcal{M}_p$–valued process $\varrho = \{\varrho_t : t \in \mathsf{R}\}$ which one-dimensional laws $\mathcal{L}(\varrho_t)$ coincide with $\mathcal{L}(\varrho_\infty)$. In this case we write $\mathbb{P}$ and sometimes $\mathbb{P}_{-\infty,\ell}$ for the law of ϱ.

The following mixing property is taken from a general result in [Fle82b], which was formulated in a time-discrete setting. In the present particular situation, a simplified proof will be given.

Lemma 2 (time-space mixing) *In dimensions $d \geq 3$, the catalyst process ϱ with time-space-shift invariant law $\mathbb{P}$ is time-space mixing of all orders. In particular, for $f, g \in C_+^{\mathrm{comp}}$, the vector*

$$\Big[\langle \varrho_{t_1} \,, f \rangle \,, \langle \varrho_{t_2} \,, g \rangle\Big] \quad \textit{is asymptotically independent as} \quad |t_1 - t_2| \to \infty.$$

Proof First of all, recall the following *covariance formula* for ϱ :

$$\mathbb{C}\mathrm{ov}_{s,\mu}\Big[\langle\varrho_{t_1}\,,f\rangle\,,\langle\varrho_{t_2}\,,g\rangle\Big] \;=\; 2\,\Pi_{s,\mu}\int_s^{t_1\wedge t_2}\gamma\mathrm{d}r\;S_{t_1-r}f\,(W_r)\,S_{t_2-r}g\,(W_r), \qquad (9)$$

$s\le t_1,t_2$, $\mu\in\mathcal{M}_p$, and $f,g\in C_+^{\mathrm{comp}}$; see e.g. [DF96, Proposition 11 (b)]. Hence, the *covariance density* function of ϱ at $[[t_1\,,b_1],[t_2\,,b_2]]$ with respect to $\mathbb{P}_{s,\ell}$ is given by

$$2\int_s^{t_1\wedge t_2}\gamma\mathrm{d}r\;\;\mathrm{p}_{t_1+t_2-2r}(b_1\,,b_2).$$

Letting $s\downarrow-\infty$, we arrive at the covariance density function at $[t_1\,,b_1]$ and $[t_2\,,b_2]$ of the catalyst process ϱ with respect to $\mathbb{P}$. Since $\mathbb{P}$ is invariant with respect to the time-space shift and infinitely divisible, it suffices to show that this covariance density function converges to 0 as $\|[t_1\,,b_1]-[t_2\,,b_2]\|\to\infty$ on the sets $\{|b_1-b_2|\ge\varepsilon\}$, $\varepsilon>0$; see the remark after Theorem 2.0.2 in [Fle82a]. Here we may set $[t_2\,,b_2]=0$ without loss of generality. Thus it is sufficient to demonstrate that

$$\int_{-\infty}^0\mathrm{d}r\;\;\mathrm{p}_{t-r}(b)\longrightarrow 0\quad\text{as}\quad[t,|b|]\to+\infty\quad\text{on}\quad\mathsf{R}_+\times\{|b|\ge\varepsilon\},\quad\varepsilon>0.$$

But the latter integral equals $\int_t^\infty\mathrm{d}r\;\;\mathrm{p}_r(b)$ and can be estimated from above by $\le\mathrm{const}\,[|b|^{2-d}\wedge t^{-1/2}]$ with a constant *const* depending on ε. This finishes the proof. ■

2.3 Brownian collision local time $L_{[W,\varrho]}$

Assumption 3 From now on we restrict our attention to dimensions $d\le 3$, and assume, if not otherwise indicated, that the catalyst process ϱ is distributed according to $\mathbb{P}_{0,\ell}$ or to the stationary $\mathbb{P}$, the latter of course only if $d=3$. ◇

For $\varepsilon>0$ and given ϱ, consider the following continuous additive functional $L^\varepsilon_{[W,\varrho]}$ of Brownian motion W :

$$L^\varepsilon(\mathrm{d}r)\;:=\;L^\varepsilon_{[W,\varrho]}(\mathrm{d}r)\;:=\;\mathrm{d}r\int\varrho_r(\mathrm{d}b)\;\mathrm{p}_\varepsilon(W_r\,,b)\,, \qquad (10)$$

describing the collision local time of the measure-valued path ϱ with the "ε–vicinity" of the Brownian path W.

Lemma 4 (Brownian collision local time $L_{[W,\varrho]}$) *Suppose Assumption 3, and fix a constant $\xi \in (0, \frac{1}{4})$. Then for almost all paths ϱ of the catalyst process, there exists a continuous additive functional $L = L_{[W,\varrho]}$ of the Brownian motion W, called the* Brownian collision local time (BCLT) *of ϱ, with the following properties.*

(a) (convergence) *If ψ is a (strictly) positive function in $C^{p,I}$, $I = [0,T]$, $T > 0$, then*

$$\sup_{s\in I,\, a\in \mathsf{R}^d} \Pi_{s,a} \sup_{s\le t\le T} \left| \int_s^t L^\varepsilon(\mathrm{d}r)\, \psi_r(W_r) - \int_s^t L(\mathrm{d}r)\, \psi_r(W_r) \right|^2 \underset{\varepsilon\downarrow 0}{\longrightarrow} 0.$$

(b) (moments) *For measurable $\psi : \mathsf{R}_+ \times \mathsf{R}^d \mapsto \mathsf{R}_+$, and $s \le t$, $a \in \mathsf{R}^d$,*

$$\Pi_{s,a} \int_s^t L_{[W,\varrho]}(\mathrm{d}r)\, \psi_r(W_r) = \int_s^t \mathrm{d}r \int \varrho_r(\mathrm{d}b)\, \mathrm{p}_{r-s}(a,b)\, \psi_r(b),$$

$$\Pi_{s,a}\Big[\int_s^t L_{[W,\varrho]}(\mathrm{d}r)\, \psi_r(W_r) \Big]^2$$
$$= 2 \int_s^t \mathrm{d}r \int_r^t \mathrm{d}r' \int \varrho_r(\mathrm{d}b) \int \varrho_{r'}(\mathrm{d}b')\, \mathrm{p}_{r-s}(a,b)\, \mathrm{p}_{r'-r}(b,b')\, \psi_r(b)\, \psi_{r'}(b').$$

Proof This follows from Proposition 38 and Theorem 42 in [DF96]. ■

2.4 A strong law of large numbers for $L_{[W,\varrho]}$ in $d = 3$

In this subsection we assume that Brownian motion W is distributed according to $\Pi_{0,0}$. First we recall that in dimension $d = 1$ the *total* BCLT $L_{[W,\varrho]}(\mathsf{R}_+)$ of ϱ is *finite,* for almost all $[W, \varrho]$ ([DF96, Proposition 4.8]). Next we mention that in $d = 2$ we have a self-similarity property for $L = L_{[W,\varrho]}$, see Corollary 11 below. But in dimension $d = 3$, a strong law of large numbers holds:

Theorem 5 (strong LLN for the BCLT) *If $d = 3$, then*

$$T^{-1} L_{[W,\varrho]}[0,T] \underset{T\uparrow\infty}{\longrightarrow} i_\ell, \qquad \Pi_{0,0} \times \mathsf{P}_{0,\ell}\text{–}a.s. \text{ and } \Pi_{0,0} \times \mathsf{P}\text{–}a.s.$$

Proof *Step* 1° First of all, for $s \le 0$, by the expectation formula in Lemma 4 (b),

$$\Pi_{0,0} \times \mathbb{P}_{s,\ell} T^{-1} L[0,T] \;=\; \mathbb{P}_{s,\ell} T^{-1} \int_0^T \mathrm{d}r \;\langle \varrho_r \,,\, \mathrm{p}_r \rangle \;\equiv\; i_\ell \tag{11}$$

since by the expectation formula (8)

$$\mathbb{P}_{s,\ell}\varrho_r \;\equiv\; \ell \tag{12}$$

(independent of the dimension d). Hence, the claimed a.s. convergence follows if we show that the $\Pi_{0,0} \times \mathbb{P}_{s,\ell}$–variance of $T^{-1}L[0,T]$ is of order $O(T^{-1/2})$ as $T \uparrow \infty$, uniformly in $s \le 0$ (covering the cases $s = 0$ and $s = -\infty$ corresponding to $\mathbb{P}_{0,\ell}$ and $\mathbb{P}$, respectively, we are interested in).

Step 2° In view of the second moment formula in Lemma 4 (b),

$$\Pi_{0,0}(L[0,T])^2 \;=\; 2 \int_0^T \mathrm{d}r \int_r^T \mathrm{d}r' \int \varrho_r(\mathrm{d}b) \int \varrho_{r'}(\mathrm{d}b')\, \mathrm{p}_r(b)\, \mathrm{p}_{r'-r}(b,b').$$

Therefore, by (12), by

$$\int \ell(\mathrm{d}b) \int \ell(\mathrm{d}b')\, \mathrm{p}_r(b)\, \mathrm{p}_{r'-r}(b,b') \;=\; i_\ell^2,$$

and by step 1°,

$$\left.\begin{aligned} &\Pi_{0,0} \times \mathbb{P}_{s,\ell} \Big| T^{-1}L[0,T] - i_\ell \Big|^2 \\ &= 2\,T^{-2} \int_0^T \mathrm{d}r \int_r^T \mathrm{d}r'\; \mathbb{C}\mathrm{ov}_{s,\ell}\,[\varrho_r(\mathrm{d}b), \varrho_{r'}(\mathrm{d}b')]\mathrm{p}_r(b)\, \mathrm{p}_{r'-r}(b,b'). \end{aligned}\right\} \tag{13}$$

But by (9), the latter covariance expression equals

$$2\,\Pi_{s,\ell} \int_s^r \gamma \mathrm{d}t \int \mathrm{d}b \;\; \mathrm{p}_{r-t}(W_t\,,b)\, \mathrm{p}_r(b)\, \mathrm{p}_{2r'-r-t}(W_t\,,b).$$

Therefore we may continue (13) with

$$=\; 4\,i_\ell T^{-2} \int_0^T \mathrm{d}r \int_r^T \mathrm{d}r' \int_s^r \gamma \mathrm{d}t \;\; \mathrm{p}_{2r'-2t}(0).$$

However, the internal integral is of order $O((r'-r)^{-1/2})$, uniformly in $s \le 0$. Hence, altogether we get an order $O(T^{-1/2})$ uniformly in $s \le 0$, finishing the proof. ■

3 Occupation times

Here we introduce the catalytic SBM X^ϱ, verify that its occupation time Y^ϱ has absolutely continuous states, and satisfies a strong law of large numbers, the latter in the case $d = 3$.

3.1 Catalytic SBM

Since the BCLT $L = L_{[W,\varrho]}$ of Lemma 4 is a locally admissible additive functional with "small" increments one can conclude for the existence of the *catalytic SBM* X^ϱ *in the catalytic medium* ϱ :

Lemma 6 (catalytic SBM ϱ) *Under Assumption* 3*, for almost all realizations* ϱ *of the catalyst process, the following statements hold:*

(a) (existence) *There exists the continuous SBM* $X = X^\varrho = (X^\varrho, P^\varrho_{s,\mu})$ *with branching functional given by the BCLT* $L = L_{[W,\varrho]}$.

(b) (log-Laplace functional) *The log-Laplace functional of* X *is given by*

$$-\log P^\varrho_{s,\mu} \exp \langle X_t, -f\rangle = \langle \mu, -v_t(s,\cdot)\rangle, \tag{14}$$

$s \le t$, $\mu \in \mathcal{M}_p$, $f \in \mathcal{B}^p$, *where* v_t *is the unique non-negative solution to Dynkin's log-Laplace equation*

$$v_t(s,a) = \Pi_{s,a}\Big[f(W_t) - \int_s^t L(\mathrm{d}r)\, v_t^2(r, W_r)\Big], \tag{15}$$

$s \le t, \quad a \in \mathsf{R}^d$.

(c) (moments) *Expectation and covariance of* X^ϱ *are given by*

$$P^\varrho_{s,\mu} \langle X_t, f\rangle = \langle \mu, S_{t-s} f\rangle, \qquad s \le t, \quad \mu \in \mathcal{M}_p, \quad f \in \mathcal{B}^p,$$

and

$$\mathrm{Cov}^\varrho_{s,\mu}\big[\langle X_{t_1}, f\rangle, \langle X_{t_2}, g\rangle\big] = 2\Pi_{s,\mu} \int_s^{t_1\wedge t_2} L(\mathrm{d}r)\, S_{t_1-r} f(W_r)\, S_{t_2-r} g(W_r),$$

$s \le t_1, t_2$, $\mu \in \mathcal{M}_p$, *and* $f, g \in C_+^{\mathrm{comp}}$.

Proof See [DF96], Definition 44 which is based on Theorem 18 (b), and formula (95). ∎

Note that (15) is the precise meaning of the reaction diffusion equation (2) with reaction rate $\varrho_s(da)$.

3.2 Absolutely continuous occupation time states

Since X^ϱ is pathwise continuous, we may introduce the *(weighted) occupation time process* $Y = Y^\varrho = \{Y_t^\varrho : t \geq 0\}$ related to $X = X^\varrho$, defined by $Y_t := \int_0^t \mathrm{d}r \, X_r$.

Recall that for the "classical" continuous SBM, say X^ℓ, in dimensions $d \leq 3$, the related occupation measures Y_t^ℓ are absolutely continuous, i.e. density functions y_t^ℓ exist (see e.g. [Fle88]). This property is shared also by the SBM X^ϱ in the catalytic medium ϱ :

Theorem 7 (densities of Y_t) *Under Assumption* 3, *for almost all catalyst process realizations* ϱ, *and fixed* $T > 0$ *and* $z \in \mathsf{R}^d$, *the following statements hold:*

(a) (L^2–densities) *The* $L^2(P_{0,\ell}^\varrho)$*–limit of* $\langle Y_T^\varrho, \mathrm{p}_\varepsilon(z,\cdot)\rangle$ *as* $\varepsilon \downarrow 0$ *exists and is denoted by* $y_T^\varrho(z)$.

(b) (absolutely continuous states) *With respect to* $P_{0,\ell}^\varrho$, *the random measure* Y_T^ϱ *is absolutely continuous with density function* y_T^ϱ :

$$P_{0,\ell}^\varrho\Big(Y_T(\mathrm{d}b) = y_T^\varrho(b)\,\mathrm{d}b\Big) \;=\; 1.$$

(c) (moments) *The following expectation and variance formulas hold:*

$$\begin{aligned} P_{0,\ell}^\varrho y_T^\varrho(z) &\equiv i_\ell, \\ \mathrm{Var}_{0,\ell}^\varrho y_T^\varrho(z) &= 2\, i_\ell \int_0^T \mathrm{d}r \int \varrho_r(\mathrm{d}b)\, \mathrm{q}_{0,T-r}^2(b,z) \end{aligned}$$

(*recall definition* (5) *of the Brownian potential kernel* q).

Proof According to [DF96, Proposition 24] it suffices to show that for almost all ϱ

$$\Pi_{0,\ell} \int_0^T L_{[W,\varrho]}(\mathrm{d}r)\; \mathrm{q}_{r',\varepsilon+r'}^2(W_r\,, z) \underset{\varepsilon\downarrow 0}{\longrightarrow} 0 \qquad \text{for} \quad r' = 0 \quad \text{and} \quad r = T - r. \tag{16}$$

By the expectation formula in Lemma 4 (b)

$$\Pi_{s,\ell} \int_s^T L_{[W,\varrho]}(\mathrm{d}r)\, \psi_r(W_r) \;=\; i_\ell \int_s^T \mathrm{d}r \int \varrho_r(\mathrm{d}b)\, \psi_r(b). \tag{17}$$

Hence, the l.h.s. in (16) equals

$$i_\ell \int_0^T \mathrm{d}r \int \varrho_r(\mathrm{d}b)\ \mathrm{q}^2_{r',\varepsilon+r'}(b,z).$$

Since this is monotone in ε, the limit as $\varepsilon \downarrow 0$ exists (for each ϱ). Thus it is sufficient to show that even the expectation over ϱ converges to 0 as $\varepsilon \downarrow 0$. But by (12), the latter $\mathbb{P}_{s,\ell}$–expectation, $s \le 0$, is independent of s and equals

$$i_\ell \int_0^T \mathrm{d}r \int \ell(\mathrm{d}b)\ \mathrm{q}^2_{r',\varepsilon+r'}(b,z) \;=\; i_\ell^2 \int_0^T \mathrm{d}r \int_{r'}^{\varepsilon+r'} \mathrm{d}t \int_{r'}^{\varepsilon+r'} \mathrm{d}t'\ \mathrm{p}_{t+t'}(0).$$

Because the integrand is monotone decreasing in t and t', we may replace r' by 0, and since T is fixed we continue with

$$\le\ \text{const} \int_0^\varepsilon \mathrm{d}t \int_0^\varepsilon \mathrm{d}t'\ (t+t')^{-d/2} \;\le\; \text{const}\ \varepsilon^{1/2} \underset{\varepsilon\downarrow 0.}{\longrightarrow} 0 \quad \text{since} \quad d \le 3.$$

This finishes the proof. ■

Remark 8 (occupation density field) It can be expected that for the catalytic occupation time process Y^ϱ in all dimensions ($d \le 3$) a jointly continuous occupation density field y^ϱ exists, as it is for the "classical" continuous SBM, established by Sugitani [Sug89] (and reproved in [DF96, Lemma 25]). ◇

3.3 A strong law of large numbers for Y^ϱ in $d = 3$

First we recall that in dimension $d = 1$, the catalytic SBM X_t^ϱ converges $P^\varrho_{0,\ell}$–stochastically to ℓ, for $\mathbb{P}_{0,\ell}$–almost all ϱ, see [DF96, Theorem 51]. This of course implies a law of large numbers for the related occupation time process Y_t^ϱ. In $d = 2$, a "random self-similarity" holds instead, see Proposition 12 (b) below. Here now we restrict our attention to dimension $d = 3$. For $s \in [-\infty, 0]$, set

$$\mathbb{P}_{s,\ell} \otimes P^{\cdot}_{0,\ell}\,(\mathrm{d}[\varrho, X^\varrho]) := \mathbb{P}_{s,\ell}(\mathrm{d}\varrho) P^\varrho_{0,\ell}(\mathrm{d}X)$$

for the law of the (coupled) pair $[\varrho, X^\varrho]$.

Theorem 9 (LLN for Y) *If $d = 3$, then*

$$T^{-1}Y_T^{\varrho} \underset{T\uparrow\infty}{\longrightarrow} \ell \qquad \mathbb{P}_{0,\ell} \otimes P_{0,\ell}^{\cdot}-a.s. \ \ and \ \ \mathbb{P} \otimes P_{0,\ell}^{\cdot}-a.s.$$

Consequently, in dimension $d = 3$, the time-averaged X^{ϱ}–process behaves for almost all ϱ just as the "classical" SBM. That is, here the *averaging principle* holds: Finally only the expectation ℓ of the medium ϱ_t is "effective".

Proof Since $P_{0,\ell}^{\varrho} T^{-1}Y_T^{\varrho} \equiv \ell$ by the expectation formula in Lemma 6 (c), it suffices to show that for fixed $f \geq 0$ in the separable Banach space $\mathcal{C}^{p;\ell}$,

$$\mathbb{P}_{s,\ell} \otimes P_{0,\ell}^{\cdot} \Big| T^{-1} \langle Y_T, f \rangle - \langle \ell, f \rangle \Big|^2 \leq \text{ const } T^{-1/2}, \tag{18}$$

uniformly in $s \leq 0$. But

$$\text{Var}_{0,\ell}^{\varrho} \langle Y_T, f \rangle = 2\,\Pi_{0,\ell} \int_0^T L(\mathrm{d}r) \Big[\int_r^T \mathrm{d}t \ \Pi_{r,W_r} f(W_t) \Big]^2, \tag{19}$$

see [DF96, formula (46)]. Hence, the l.h.s. in (18) equals

$$2\,T^{-2}\,\mathbb{P}_{s,\ell}\Pi_{0,\ell} \int_0^T L(\mathrm{d}r) \Big[\int_r^T \mathrm{d}t \ \Pi_{r,W_r} f(W_t) \Big]^2.$$

Using the expectation formulas (17) and (12), we may continue with

$$= 2\,i_\ell T^{-2} \int_0^T \mathrm{d}r \int \ell(\mathrm{d}b) \Big[\int_r^T \mathrm{d}t \int \mathrm{d}z \ \ \mathrm{p}_{t-r}(b,z)\, f(z) \Big]^2.$$

Interchanging the order of integrations, and calculating the $\ell(\mathrm{d}b)$–integral, this can be estimated from above by

$$\leq 2 \,\langle \ell, f \rangle^2\, T^{-2} \int_0^T \mathrm{d}r \int_r^T \mathrm{d}t \int_r^T \mathrm{d}t' \ \ \mathrm{p}_{t+t'-2r}(0).$$

As in the end of proof of Theorem 5, the internal integral can be estimated by $\leq$ const $(t-r)^{-1/2}$, and the claim follows. ■

4 Random self-similarity in dimension $d = 2$

Recall that in dimension $d = 2$ the "classical" SBM ϱ with law $\mathbb{P}_{0,\ell}$ is *self-similar:* For $K > 0$,

$$\{K^{-1}\varrho_{Kt}(K^{1/2}\cdot) : t \geq 0\} \stackrel{\mathcal{L}}{=} \{\varrho_t : t \geq 0\}. \tag{20}$$

This has some consequences for the catalytic SBM X^{ϱ}.

4.1 A scaling property of $L_{[W,\varrho]}$

We start with a scaling property of the Brownian collision local time:

Lemma 10 (scaling of the BCLT) *Fix* $d = 2$, $s \geq 0$, $a \in \mathsf{R}^2$, *and* $K > 0$. *Then for* $\Pi_{s,a} \times \mathbb{P}_{0,\ell}$*-almost all* $[W, \varrho]$, *and measurable* $g : \mathsf{R}_+ \mapsto \mathsf{R}_+$,

$$\int L_{[W,\varrho]}(\mathrm{d}r)\, K^{-1} g(K^{-1}r) \;=\; \int L_{[K^{-1/2}W_{K\cdot},\, K^{-1}\varrho_{K\cdot}(K^{1/2}\cdot)]}(\mathrm{d}r)\, g(r). \qquad (21)$$

Proof Recalling the definition (10) of L^ε, by definition of the BCLT $L = L_{[W,\varrho]}$ it suffices to verify the claim with L replaced by $L^{K\varepsilon}$ and L^ε, respectively. Then by (10),

$$\int L^{K\varepsilon}_{[W,\varrho]}(\mathrm{d}r)\, K^{-1} g(K^{-1}r) \;=\; \int \mathrm{d}r \int \varrho_r(\mathrm{d}b)\, \mathrm{p}_{K\varepsilon}(W_r, b)\, K^{-1} g(K^{-1}r).$$

By a change of variables, and using the self-similarity of the Brownian transition density p, this can be written as

$$= \int \mathrm{d}r \int K^{-1} \varrho_{Kr}(K^{1/2}\mathrm{d}b)\, \mathrm{p}_\varepsilon(K^{-1/2}W_{Kr}, b)\, g(r).$$

Again by (10), we arrive at the r.h.s. of (21) with L replaced by L^ε, finishing the proof. ■

Combining Lemma 10 with the self-similarity of Brownian motion W and ϱ (recall (20)) we get the following result.

Corollary 11 (self-similarity of the BCLT) *For* $d = 2$ *and* $K > 0$, *with respect to* $\Pi_{0,0} \times \mathbb{P}_{0,\ell}$,

$$K^{-1} L_{[W,\varrho]}(K\cdot) \stackrel{\mathcal{L}}{=} L_{[W,\varrho]}.$$

4.2 Random self-similarity of $[X^\varrho, Y^\varrho]$

Instead of the well-known self-similarity of the "classical" SBM in $d = 2$ (as in (20)), for the catalytic SBM we have a "randomized" version:

Proposition 12 (properties of $[X, Y]$) *Let* $d = 2$ *and* $K > 0$.

(a) (scaling) *For $\mathbb{P}_{0,\ell}$-almost all ϱ the following holds. If $[X,Y]$ is distributed according to $P^{\varrho}_{0,\ell}$ then the pair*

$$\Big[K^{-1}X_{K\cdot}(K^{1/2}\cdot),\ K^{-2}Y_{K\cdot}(K^{1/2}\cdot)\Big]$$

of processes has the same law as $[X,Y]$ with respect to $P^{K^{-1}\varrho_{K\cdot}(K^{1/2}\cdot)}_{0,\ell}$.

(b) (random self-similarity) *With respect to the* random *law $P^{\varrho}_{0,\ell}$,*

$$\Big[K^{-1}X_{K\cdot}(K^{1/2}\cdot),\ K^{-2}Y_{K\cdot}(K^{1/2}\cdot)\Big] \ \stackrel{\mathcal{L}}{=}\ [X,Y].$$

Proof By [DF96, Hypothesis 13 and notation (47)], for $T \geq 0$ and f,g in $\mathcal{B}^p_+$,

$$-\log P^{\varrho}_{0,\ell} \exp\Big[\langle X_T\,, -f\rangle + \langle Y_T\,, -g\rangle\Big] \;=\; \langle \ell,\, v_T(0,\cdot)\rangle$$

with

$$v_T(s,a) \;=\; \Pi_{s,a}\Big[f(W_T) + \int_s^T \mathrm{d}r\; g(W_r) - \int_s^T L_{[W,\varrho]}(\mathrm{d}r)\, v_T^2(r,W_r)\Big],$$

$0 \leq s \leq T,\ a \in \mathsf{R}^2$. Hence

$$\begin{aligned}
&-\log P^{\varrho}_{0,\ell} \exp\Big[\Big\langle X_{KT}\,, -K^{-1}f(K^{-1/2}\cdot)\Big\rangle + \Big\langle Y_{KT}\,, -K^{-2}g(K^{-1/2}\cdot)\Big\rangle\Big] \\
&= \langle \ell,\, v_{KT}(0,\cdot)\rangle \;=\; \Big\langle \ell,\, Kv_{KT}(0,K^{1/2}\cdot)\Big\rangle
\end{aligned} \tag{22}$$

with

$$\begin{aligned}
Kv_{KT}(Ks,K^{1/2}a) = \Pi_{Ks,K^{1/2}a}\Big[&f(K^{-1/2}W_{KT}) + \int_{Ks}^{KT} \mathrm{d}r\; K^{-1}g(K^{-1/2}W_r) \\
&- K^{-1}\int_{Ks}^{KT} L_{[W,\varrho]}(\mathrm{d}r)\, K^2 v_{KT}^2(r,W_r)\Big].
\end{aligned}$$

Setting $u_T(s,a) := Kv_{KT}(Ks,K^{1/2}a)$ (for the fixed K), by a change of variables and using the scaling Lemma 10 (with $[s,a]$ replaced by $(Ks,K^{1/2}a)$), the latter equation can be written as

$$\begin{aligned}
u_T(s,a) \;=\; \Pi_{Ks,K^{1/2}a}\Big[&f(K^{-1/2}W_{KT}) + \int_s^T \mathrm{d}r\; g(K^{-1/2}W_{Kr}) \\
&- \int_s^T L_{[K^{-1/2}W_{K\cdot},\,K^{-1}\varrho_{K\cdot}(K^{1/2}\cdot)]}(\mathrm{d}r)\, u_T^2(r,K^{-1/2}W_{Kr})\Big].
\end{aligned}$$

But W distributed according to $\Pi_{Ks,K^{1/2}a}$ implies by scaling that the process $t \to K^{-1/2}W_{Kt}$ has the law $\Pi_{s,a}$. Therefore we may continue with

$$= \Pi_{s,a}\Big[f(W_T) + \int_s^T \mathrm{d}r\; g(W_r) - \int_s^T L_{[W,K^{-1}\varrho_K\cdot(K^{1/2}\cdot)]}(\mathrm{d}r)\, u_T^2(r, W_r)\Big].$$

Hence, by uniqueness of the solution to the log-Laplace equation, we conclude that (22) equals

$$= -\log P_{0,\ell}^{K^{-1}\varrho_K\cdot(K^{1/2}\cdot)} \exp\big[\langle X_T, -f\rangle + \langle Y_T, -g\rangle\big].$$

This proves the first claim. But the second one then immediately follows from the self-similarity (20) of ϱ, finishing the proof. ■

Remark 13 (open problems) By the random self-similarity of X^ϱ, the random law of X_T^ϱ coincides with the random law of $TX_1^\varrho(T^{-1/2}\cdot)$. Passing formally to $T \uparrow \infty$, we arrive at X_∞^ϱ and $x_1^\varrho(0)$. This relates the questions of existence of a non-trivial limit X_∞^ϱ and of absolute continuity of X_1^ϱ in the critical dimension $d = 2$. But whether or not a non-trivial limit X_∞^ϱ exists remains *open*. ◇

4.3 A random ergodic limit

Recall that for the continuous SBM X^ℓ in R^2 with constant branching rate and with law $P_{0,\ell}^\ell$ we have the following "random" ergodic limit:

$$T^{-1}Y_T^\ell \xrightarrow[T\uparrow\infty]{} y_1^\ell(0)\,\ell \quad \text{in law,}$$

where $y_1^\ell(0)$ is the random density of the occupation measure Y_1^ℓ at time 1 at the origin 0; see e.g. [Fle88]. The two-dimensional catalytic SBM X^ϱ satisfies a "randomized" version of this, expressed in convergence in law of random probability distributions:

Theorem 14 (random ergodic limit) *Let $d = 2$ and consider the catalytic SBM X^ϱ with $\mathsf{P}_{0,\ell}$-random law $P_{0,\ell}^\varrho$. Then the $\mathsf{P}_{0,\ell}$-random law of $T^{-1}Y_T^\varrho$ converges in $\mathsf{P}_{0,\ell}$-law as $T \uparrow \infty$ towards the $\mathsf{P}_{0,\ell}$-random law of the multiple $y_1^\varrho(0)\ell$ of Lebesgue measure ℓ, where $y_1^\varrho(0)$, given ϱ, is the $L^2(P_{0,\ell}^\varrho)$-density at 0 of the occupation measure Y_1^ϱ at time 1, according to Theorem 7 (a).*

Proof Using the random self-similarity in Proposition 12 (b), the $\mathbb{P}_{0,\ell}$–random law of $T^{-1}Y_T^{\varrho}$ coincides with that of $T\,Y_1(T^{-1/2}\,\cdot\,)$. But by Theorem 7 (a), for $f \in \mathcal{C}_+^{p;\ell}$ and $\mathbb{P}_{0,\ell}$–almost all ϱ

$$\left\langle T\,Y_1^{\varrho}(T^{-1/2}\,\cdot\,), f\right\rangle \underset{T\uparrow\infty}{\longrightarrow} y_1^{\varrho}(0)\;\langle \ell, f\rangle \quad \text{in } L^2(P_{0,\ell}^{\varrho}),$$

implying the claim. ■

Note that opposed to dimensions 1 and 3, here the limit after the averaging procedure remains random, even in a double sense (by the random medium).

5 Persistence in dimension $d = 3$

In this final section we pay attention to the following situation.

Assumption 15 (time-space-shift invariance) Let $d = 3$ and assume that the catalyst process ϱ is distributed according to the time-space-shift invariant $\mathbb{P}$ (introduced in § 2.2). Write $\mathcal{P}$ for the *annealed* law $\mathbb{P} P_{0,\ell}^{\varrho}$. ◇

Remark 16 (approximation) Working with the non-stationary catalyst process ϱ distributed according to $\mathbb{P}_{0,\ell}$ would require some additional approximation. ◇

5.1 Main result

Now we are in a position to formulate our *main result*:

Theorem 17 (convergence and persistence) *Impose Assumption* 15.

(a) (annealed convergence) *With respect to the annealed $\mathcal{P}$, the catalytic SBM X_T converges in law as $T \uparrow \infty$ to some limit X_∞ with full intensity ℓ (persistence).*

(b) (random convergence) *The $\mathbb{P}$–random distribution of*

$$P_{0,\ell}^{\varrho}(X_T \in \cdot) =: Q_T^{\varrho}$$

converges in $\mathbb{P}$*–law as* $T \uparrow \infty$ *to some* $\mathbb{P}$*–random distribution* Q_∞^ϱ *with full intensity (persistence):*

$$\int Q_\infty^\varrho(\mathrm{d}\nu)\,\nu = \ell, \quad \mathbb{P}\text{–}a.s.$$

Consequently, at the first sight, our catalytic SBM X^ϱ behaves similar to the classical continuous SBM X^ℓ. However, the main difference should be that a new limit occurs. For instance, the limiting random measure X_∞ of (a) should be different from the classical steady state X_∞^ℓ. Such statements will be investigated in a forthcoming paper.

5.2 Proof of the main theorem

The key of proof will be some *backward technique*: By the time-stationarity of the random medium ϱ we may start X^ϱ at time $-T$ with ℓ, and observe the state at time 0. Then we may continue for *fixed* realization ϱ, sending $-T$ to $-\infty$, by exploiting some backward monotonicities.

1° (*convergence*) First of all, for $\mathbb{P}$–almost all ϱ, the law Q_T^ϱ coincides with the law $P_{-T,\ell}^{\varrho_{T+\cdot}}(X_0 \in \cdot\,)$. Here $\varrho_{T+\cdot}$ is the catalyst process shifted by T. Hence, by the time-shift invariance of the catalyst process ϱ, the distribution of the random law Q_T^ϱ coincides with that of the random law $P_{-T,\ell}^\varrho(X_0 \in \cdot\,)$.

Given ϱ, we turn to the log-Laplace functional according to Lemma 6 (b): For $f \in \mathcal{B}_+^p$, writing $\langle \ell, f\rangle =: \|f\|_1$,

$$-\log P_{-T,\ell}^\varrho \exp\langle X_0, -f\rangle \;=\; \|v_0(-T,\cdot)\|_1$$

where by the log-Laplace equation (15),

$$\|v_0(-T,\cdot)\|_1 \;=\; \Pi_{-T,\ell}\Big[f(W_0) - \int_{-T}^{0} L_{[W,\varrho]}(\mathrm{d}r)\, v_0^2(r, W_r)\Big].$$

Using the expectation formula in Lemma 4 (b), we may continue with

$$=\; \|f\|_1 - i_\ell \int_{-T}^{0} \mathrm{d}r \int \varrho_r(\mathrm{d}b)\, v_0^2(r,b).$$

But this non-negative expression is non-increasing[1)] in T. Hence, the limit of $\|v_0(-T,\cdot)\|_1$ exists (for the fixed ϱ) and determines (note that the family $\{Q_T^\varrho : T \ge 0\}$ is tight, see (23) below) a log-Laplace functional of a random measure, its law denoted by Q_∞^ϱ. This gives the convergence claim in (b), and by averaging over ϱ, also the convergence claim of (a) follows.

2° (*expectations*) For almost all ϱ, from the expectation formula in Lemma 6 (c),

$$\int Q_T^\varrho(\mathrm{d}\mu)\,\mu \;=\; P_{0,\ell}^\varrho X_T \;\equiv\; \ell, \tag{23}$$

which implies that

$$\int Q_\infty^\varrho(\mathrm{d}\mu)\,\mu \;\le\; \ell, \quad \text{hence} \quad \int \mathbb{P} Q_\infty^\varrho(\mathrm{d}\mu)\,\mu \;\le\; \ell. \tag{24}$$

Consequently, the limiting intensity measures in (b) and (a) are bounded by ℓ.

3°(*variances*) Let again $f \in \mathcal{B}_+^p$. Given ϱ, by the variance formula in Lemma 6 (c),

$$\mathrm{Var}_{-T,\ell}^\varrho \langle X_0, f\rangle \;=\; 2\,\Pi_{-T,\ell} \int_{-T}^0 L_{[W,\varrho]}(\mathrm{d}r)\,[S_{-r}f(W_r)]^2 .$$

In view of the expectation formula (17), we continue with

$$=\; 2\,i_\ell \int_{-T}^0 \mathrm{d}r \int \varrho_r(\mathrm{d}b)\,[S_{-r}f(b)]^2$$

which monotonously converges to

$$2\,i_\ell \int_{-\infty}^0 \mathrm{d}r \int \varrho_r(\mathrm{d}b)\,[S_{-r}f(b)]^2 \quad \text{as} \quad T \uparrow \infty. \tag{25}$$

Integrating additionally ϱ with $\mathbb{P}$, by the expectation formula (12) we get the monotone convergence

$$\mathbb{P}\,\mathrm{Var}_{0,\ell}^\varrho \langle X_T, f\rangle \underset{T\uparrow\infty}{\nearrow} 2\int \ell(\mathrm{d}x)\,f(x) \int \ell(\mathrm{d}y)\,f(y) \int_0^\infty \mathrm{d}r\; \mathrm{p}_{2r}(x,y). \tag{26}$$

[1)] Note that this monotonicity would be violated if we started ϱ at time $-T$ with $\varrho_{-T} = \ell$. That is, the present method only works for the time-stationary process ϱ on the whole time axis R.

Note that by (12), the l.h.s. is the variance of $\langle X_T, f\rangle$ with respect to the annealed law $\mathcal{P}$. On the other hand, the r.h.s. is the variance expression related to the classical steady state X_∞^ℓ (see, e.g. [Daw77]), hence is finite. Therefore also the limit (25) is finite $\mathbb{P}$–a.s. But this implies that in (24) even equalities must hold. In other words, we get persistence in both cases (a) and (b). This finishes the proof. ∎

References

[BEP91] M.T. Barlow, S.N. Evans, and E.A. Perkins. Collision local times and measure-valued processes. *Can. J. Math.*, 43(5):897–938, 1991.

[BP94] M.T. Barlow and E.A. Perkins. On the filtration of historical Brownian motion. *Ann. Probab.*, 22:1273–1294, 1994.

[Daw77] D.A. Dawson. The critical measure diffusion process. *Z. Wahrsch. verw. Gebiete*, 40:125–145, 1977.

[DF96] D.A. Dawson and K. Fleischmann. A continuous super-Brownian motion in a super-Brownian medium. WIAS Berlin, Preprint Nr. 165, 1995, *Journ. Theoret. Probab.*, to appear 1996.

[DP91] D.A. Dawson and E.A. Perkins. Historical processes. *Mem. Amer. Math. Soc.*, 454, 1991.

[Fle82a] K. Fleischmann. Mixing properties of infinitely divisible random measures and an application in branching theory. *Carleton Mathematical Lecture Notes*, 43, 1982.

[Fle82b] K. Fleischmann. Space-time mixing in a branching model. In *Advances in Filtering and Optimal Control*, volume 42 of *Lecture Notes in Control and Information Sciences*, pages 125–130, Cocoyoc, Mexico, 1982.

[Fle88] K. Fleischmann. Critical behavior of some measure-valued processes. *Math. Nachr.*, 135:131–147, 1988.

[GGH95] A. Glitzky, K. Gröger, and R. Hünlich. Free energy and dissipation rate for reaction diffusion processes of electrically charged species. WIAS Berlin, Preprint Nr. 207, 1995.

[Sug89] S. Sugitani. Some properties for the measure-valued branching diffusion process. *J. Math. Soc. Japan*, 41(3):437–462, 1989.

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