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Robert Lasarzik, Elisabetta Rocca, Giulio Schimperna

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Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: robert.lasarzik@wias-berlin.de

Universita' degli Studi di Pavia
Dipartimento di Matematica
Via Ferrata 5
27100, Pavia, Italy
E-Mail: elisabetta.rocca@unipv.it

Universita' degli Studi di Pavia
Dipartimento di Matematica
Via Ferrata 5
27100, Pavia, Italy
E-Mail: giusch04@unipv.it

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

In this paper we prove the existence of weak solutions for a thermodynamically consistent phase-field model introduced in [26] in two and three dimensions of space. We use a notion of solution inspired by [18], where the pointwise internal energy balance is replaced by the total energy inequality complemented with a weak form of the entropy inequality. Moreover, we prove existence of local-in-time strong solutions and, finally, we show weak-strong uniqueness of solutions, meaning that every weak solution coincides with a local strong solution emanating from the same initial data, as long as the latter exists.

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1 Introduction

This paper is concerned with the analysis of the initial boundary-value problem for the following PDE system:

$$\theta_t + \theta\varphi_t - \kappa\Delta\theta = |\varphi_t|^2 \qquad \text{in } \Omega \times (0,T)\,, \tag{1a}$$

$$\varphi_t - \Delta\varphi + F'(\varphi) = \theta \qquad \text{in } \Omega \times (0,T)\,, \tag{1b}$$

which describes phase transition phenomena occurring in a bounded connected container $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, with sufficiently smooth boundary and fixing a reference time interval $[0,T]$. The state variables are the *absolute* temperature θ and the order parameter (or phase-field) φ describing the locally attained phase. We have

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denoted by F the interaction potential entering the free energy functional and by $\kappa > 0$ the heat conductivity, assumed to be constant. This system is equipped with homogeneous NEUMANN boundary conditions, *i.e.,*

$$\boldsymbol{n} \cdot \nabla\theta = 0 = \boldsymbol{n} \cdot \nabla\varphi \qquad \text{on } \partial\Omega \times (0,T) \tag{1c}$$

and initial conditions

$$\theta(0) = \theta_0 \,, \quad \varphi(0) = \varphi_0 \qquad \text{in } \Omega \,. \tag{1d}$$

System (1a)-(1b) can be seen as one of the simplest diffuse-interface models describing non-isothermal phase transition processes in a thermodynamically consistent setting in the case when no external heat source is present. More precisely, thermodynamic consistency holds for a wide range of temperature values and not only in proximity of the equilibrium temperature. Nonetheless, the global-in-time well posedness of the model is still open both in 2 and in 3 space dimensions.

There are various ways to derive equations (1a)-(1b) from the laws of Thermodynamicas. We sketch here an approach that follows the lines of the so-called FRÉMOND theory of phase transitions (*cf.* [26, p. 5]) for a particular choice of the free-energy functional and of the pseudo-potential of dissipation. An alternative physical derivation is provided in the paper [1].

We start from the following expression for the volumetric free energy:

$$\Psi(\theta,\varphi,\nabla\varphi) = c_V\theta(1-\log\theta) - \frac{\lambda}{\theta_c}(\theta-\theta_c)\varphi + F(\varphi) + \frac{\nu}{2}|\nabla\varphi|^2, \tag{2}$$

where the constants c_V, θ_c, and $\nu > 0$ represent, respectively, the specific heat, the equilibrium temperature, and the interfacial energy coefficient, while λ stands for the latent heat of the system. The term $F(\varphi) + (\nu/2)|\nabla\varphi|^2$ accounts for a mixture or interaction free-energy. Hereafter, for simplicity, we shall set $c_V = \nu = \lambda/\theta_c = 1$ and we incorporate the term $\theta_c\varphi$ into $F(\varphi)$. Indeed, since in the following the potential F will be assumed to be a λ-convex function, we may suppose without loss of generality that $\theta_c\varphi$ contributes to the non-convex part. A typical example of a potential F that we can include in our analysis is the so-called "regular double-well potential" $F(r) = (r^2-1)^2$. Dissipation effects are described by means of a pseudo-potential of dissipation Φ depending on the dissipative variables $\nabla\theta$ and φ_t:

$$\Phi(\nabla\theta,\varphi_t) = \frac{1}{2}|\varphi_t|^2 + \frac{h(\theta)|\nabla\theta|^2}{2\theta}\,, \tag{3}$$

where h stands for a positive function representing the heat conductivity of the process and, for the sake of simplicity, the other physical parameters have been set equal to 1.

The evolution of the phase variable φ is ruled by an equation derived from a generalization of the principle of virtual power (*cf.* [26, Sec. 2]):

$$B - \operatorname{div}\mathbf{H} = 0 \tag{4}$$

in case the volume amount of mechanical energy provided to the domain by the external actions (which do not involve macroscopic motions) is zero. Here B (a density or energy function) and $\mathbf{H}$ (an energy flux vector) represent the internal microscopic forces responsible for the mechanically induced heat sources:

$$B = \frac{\partial\Psi}{\partial\varphi} + \frac{\partial\Phi}{\partial\varphi_t} = -\theta + F'(\varphi) + \varphi_t\,, \quad \mathbf{H} = \frac{\partial\Psi}{\partial\nabla\varphi} = \nabla\varphi. \tag{5}$$

With trivial computations, from (4)–(5) we derive exactly (1b). Moreover, if the surface amount of mechanical energy provided by the external local surface actions (not involving macroscopic motions) is zero as well, then the natural boundary condition for this equation of motion is exactly the second one in (1c).

Finally, the energy balance equation reads

$$e_t + \operatorname{div}\mathbf{q} = B\varphi_t + \mathbf{H}\cdot\nabla\varphi_t\,, \tag{6}$$

where e, the (specific) internal energy, is linked to the free energy Ψ by the standard HELMHOLTZ relation

$$e = \Psi + \theta s, \quad s = -\frac{\partial \Psi}{\partial \theta}, \tag{7}$$

in which we have denoted by s the specific entropy of the system. Following FRÉMOND's perspective (*cf.* [26]), on the right-hand side of (6) there appears the mechanically induced heat sources, related to microscopic stresses, while the heat flux $\mathbf{q}$ is defined by the following constitutive relation (*cf.* (3)):

$$\mathbf{q} = -\theta \frac{\partial \Phi}{\partial \nabla \theta} = -h(\theta)\nabla\theta. \tag{8}$$

Using the no-flux boundary condition and the FOURIER heat flux law (*i.e.* setting $h(\theta) \equiv \kappa$), we get the first boundary condition in (1c) and equation (1a).

This model turns out to be thermodynamically consistent in the sense that it complies with the Second Principle of Thermodynamics: indeed, the CLAUSIUS–DUHEM inequality

$$s_t + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) \geq 0 \tag{9}$$

holds true. To check (9), it is sufficient to note that the internal energy balance (6) can be expressed in terms of the entropy s in this way:

$$\theta\left(s_t + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right)\right) = |\varphi_t|^2 - \frac{\mathbf{q}}{\theta}\cdot\nabla\theta = |\varphi_t|^2 + h(\theta)\frac{|\nabla\theta|^2}{\theta}, \tag{10}$$

where we used (8) and the formal identity $e_t - B\varphi_t - \mathbf{H}\cdot\nabla\varphi_t = \theta s_t - |\varphi_t|^2$, which follows from (5) and (7). Therefore, (9) ensues from the positivity of θ, a fact that we shall prove in the sequel.

Coming to our results, in this paper we shall first prove existence of weak solutions (θ, φ). These will comply with equation (1b) satisfied almost everywhere in the space-time domain $\Omega \times (0, T)$ and supplemented with homogeneous NEUMANN boundary conditions and initial conditions. Moreover, weak solutions will satisfy the *total energy inequality*:

$$E(t) \leq E(0), \quad \text{for a.e. } t \in (0,T), \quad \text{where } E \equiv \int_\Omega \left(\theta + F(\varphi) + \frac{1}{2}|\nabla\varphi|^2\right) \mathrm{d}\boldsymbol{x} \tag{11}$$

together with the following *entropy inequality*:

$$\begin{aligned}
&-\int_\Omega \vartheta(t)(\log\theta(t) + \varphi(t))\,\mathrm{d}\boldsymbol{x} + \int_\Omega \vartheta(0)(\log\theta_0 + \varphi_0)\,\mathrm{d}\boldsymbol{x} + \int_0^t\int_\Omega \vartheta\left(\kappa|\nabla\log\theta|^2 + \frac{|\varphi_t|^2}{\theta}\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\quad \leq \int_0^t\int_\Omega \left(\kappa\nabla\log\theta\cdot\nabla\vartheta - \vartheta_t(\log\theta + \varphi)\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,
\end{aligned} \tag{12}$$

for a.e. $t \in (0, T)$ and for every sufficiently regular nonnegative function ϑ. Inequality (12) implies that the entropy is controlled by the dissipation of the system. It is worth noticing that the weak formulation corresponds to the underlying physical laws, the two Thermodynamic Principles, *i.e.,* energy conservation and entropy production in the case of (11) and (12), respectively.

From the mathematical viewpoint, initial-boundary value problems for equations (1a)-(1b) or variations of them have been addressed in a number of apers. Starting from the pioneering work [7], there is a comprehensive literature on the models of phase change with microscopic movements proposed by FRÉMOND (we may refer to the PhD thesis [45] and the references therein).

However, system (1) needs to be carefully handled, mainly because of the the presence of the terms $\varphi_t\,\theta$ and $|\varphi_t|^2$ in (1a). Due to the difficulties arising from these two higher order nonlinearities, there has not been any global-in-time well-posedness result for the initial-boundary value problem related to system (1) in the two or three-dimensional case. A global existence result for system (1) has only been proved in the one-dimensional setting in [34, 35], while in [22] (*cf.* also [21]) a global-in-time well-posedness result has been obtained in the 2 and 3D cases only for power-like type growth of the heat flux law ($h(\theta) \sim \theta^\eta$, for $\eta \geq 3$ for θ large).

Other approaches to phase transition models based on various forms of the *entropy balance* and possibly including mechanical effects are available in the literature. A possibility (*cf.* [3, 4, 5, 6]) consists in coupling an *entropy equation* (instead of the standard internal energy balance equation (1a)) with a microscopic motion equation. Within this approach the resulting PDE system couples an equation for φ of the type (1b) with an entropy balance equation, which can be written as $s_t + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) = R$ and which is obtained rewriting the internal energy balance in terms of s by means of the standard HELMHOLTZ relation (7) and assuming the right-hand side R, which now has the meaning of an *entropy source*, to be known. Note that in [2] the entropy source is allowed to depend (somehow singularly) on θ. A second approach has been used in [38, 39, 40, 41]: the main novelty of these contributions lies in the fact that the equations for θ and φ (analogous to our (1a) and (1b)) are coupled to a hyperbolic stress-strain relation for the displacement variable $\mathbf{u}$. In the 3D case in [38] a *local-in-time* well-posedness result is obtained, whereas in [41] *entropic solutions* have been proved to exist, but only in the case of a power-like type heat flux law ($h(\theta) \sim \theta^\kappa$, with $\kappa > 1$ for θ large). Finally, in [39] the *global* existence and the *long-time behavior of solutions* are investigated in the 1D case.

Since weak solutions of system (1) are not known to be unique, and the source of non-uniqueness stems from insufficient regularity properties holding in the setting of weak solutions, a natural concept generalizing uniqueness is the so-called weak-strong uniqueness. It is fulfilled whenever every weak solution coincides with a local strong solution emanating from the same initial data, as long as the latter exists. In this way, the property also guarantees that the generalized solution concept is indeed a generalization of strong solutions.

There are prominent examples in the context of fluid dynamics for these kinds of results, such as SERRIN's uniqueness result [44] for LERAY's weak solutions [32] to the incompressible NAVIER–STOKES equation in three space dimensions, or the weak-strong uniqueness for suitable weak-solutions to the incompressible NAVIER–STOKES system [19] or to the full NAVIER–STOKES–FOURIER system [20].

A recurrent tool to prove such a weak-strong uniqueness result is the formulation of a relative energy. For thermodynamical systems this idea goes back to DAFERMOS [10]. In the context of fluid dynamics, the relative energy approach has also been used to show the stability of a stationary solution [17], the convergence to a singular limit [23], or to derive *a posteriori* estimates for simplified models [25].

In the article at hand, this approach is adapted to a λ-convex energy functional. Actually, there are very few articles dealing with the relative energy approach for nonconvex energies, and all of them seem to pertain to the context of liquid crystals. Actually, the paper [24] deals with a Q-tensor model equipped with a λ-convex energy and the authors are able to show weak-strong uniqueness for dissipative solutions. The weak-strong uniqueness for weak solutions to the penalized ERICKSEN–LESLIE model in three space dimensions has been proved in [16]. In [31], weak-strong uniqueness of weak solutions (measure-valued [30] and dissipative [29]) has been shown for the ERICKSEN–LESLIE model equipped with the OSEEN–FRANK energy, which is an energy with nonconvex leading order term.

Our proofs of existence and of weak-strong uniqueness for system (1a)-(1b) combine the use of more or less established methods in the mathematical theory of phase transition models with two new ideas, which constitute the key points of our argument:

- a proper notion of weak solutions to (1), which is based on a new *a-priori* estimate holding for *polynomial* potentials F: actually, the estimates following from the total energy inequality (11) and (12) are not sufficient in order to pass to the limit in a suitable regularized problem;
- a concept of relative energy which combines the natural contribution of the "physical" energy with an additional L^1-term. Indeed, the latter is crucial in order to overcome the nonconvex character of the physical energy functional and to obtain an effective estimate.

The plan of the paper is as follows: in the next Section 2 we give our precise assumptions on data and state our main results. The remainder of the paper is devoted to proofs: in Section 3, we show our global existence result for weak solutions; in Section 4, we deal with weak-strong uniqueness; finally, in Section 5, we prove local-in-time existence for strong solutions.

2 Assumptions and main results

We start this section by presenting our basic hypotheses on the nonlinear function F. These assumptions are collected, together with a number of notable consequences of them, in the following statement. It is worth noting that, despite the length of what follows, it is very easy to check that the usual double-well potentials of polynomial growth (as the commonly used quartic potential $F(r) = (r^2 - 1)^2$) satisfy all the assumptions (A)-(D) listed below.

Hypothesis 2.1. (A) We let $F \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}) \cap \mathcal{C}^{2,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$.

(B) We assume F to be λ-*convex*, *i.e.*, convex up to a quadratic perturbation. Namely there exists a constant $\lambda > 0$ such that $F''(y) \geq -\lambda$ for all $y \in \mathbb{R}$. We can then define a convex modification of F, subsequently named G, as

$$G(y) = F(y) + \lambda y^2 \quad y \in \mathbb{R}\,. \tag{13}$$

By construction, G is “strongly convex”, *i.e.*, $G''(y) \geq \lambda > 0$ for all $y \in \mathbb{R}$. Moreover, it is not restrictive to assume G to be nonnegative and so normalized that $G'(0) = 0$.

(C) Next, we assume a minimal coercivity assumption at ∞, namely

$$\liminf_{|y| \to \infty} F'(y) \operatorname{sign} y > 0. \tag{14}$$

As a consequence of (14), we can first observe that $F(y) \geq -c$ for some constant $c > 0$ and every $y \in \mathbb{R}$. Moreover, it is easy to verify that the physical energy controls the H^1-norm of φ from above. Namely, there exist $\gamma > 0$ and $c \geq 0$ such that

$$\frac{1}{2}\|\nabla\varphi\|^2_{L^2(\Omega)} + \int_\Omega F(\varphi)\, \mathrm{d}\boldsymbol{x} \geq \gamma\|\varphi\|_{H^1(\Omega)} - c, \tag{15}$$

for every $\varphi \in H^1(\Omega)$ such that $F(\varphi) \in L^1(\Omega)$.

(D) Finally, a growth condition is assumed to hold, *i.e.*, there exists a constant $c > 0$ such that

$$|F'(y)| \log(e + |F'(y)|) \leq c(1 + |F(y)|) \quad \text{for all } y \in \mathbb{R}\,. \tag{16}$$

Possibly modifying the value of c one can see that the analogue of (16) holds also for the convex modification G, *i.e.*, we have

$$|G'(y)| \log(e + |G'(y)|) \leq c(1 + G(y)) \quad \text{for all } y \in \mathbb{R}\,. \tag{17}$$

To check that (16) implies (17), a number of straightforward but somehow technical computations would be required. We leave them to the reader because no real difficulty is involved.

We can now define weak solutions to our system in a rigorous way:

Definition 2.2. A couple (θ, φ) is called a weak solution to (1) over the time interval $(0, T)$ if the following conditions are satisfied. First, there hold the regularity properties

$$\theta \in L^\infty(0,T;L^1(\Omega)) \quad \text{with } \theta(\boldsymbol{x},t) > 0 \text{ a.e. in } \Omega \times (0,T)\,, \tag{18a}$$
$$\theta \log\theta \in L^1(\Omega \times (0,T))\,, \tag{18b}$$
$$\log\theta \in L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{18c}$$
$$\partial_t \log\theta \in (\mathcal{M}(\overline{\Omega}\times[0,T]) + L^2(0,T;(W^{1,2}(\Omega)^*))\,, \tag{18d}$$
$$\varphi \in L^\infty(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^1(\Omega))\,, \tag{18e}$$
$$\Delta\varphi \in L^1(0,T;L^1(\Omega))\,, \tag{18f}$$
$$F(\varphi) \in L^\infty(0,T;L^1(\Omega))\,, \tag{18g}$$
$$\theta^{-1/2}\partial_t\varphi \in L^2(0,T;L^2(\Omega))\,. \tag{18h}$$

Next, the entropy inequality holds in the integral form

$$\begin{aligned} - \int_\Omega \vartheta(t)(\log\theta(t) + \varphi(t))\,\mathsf{d}\boldsymbol{x} + \int_\Omega \vartheta(0)(\log\theta_0 + \varphi_0)\,\mathsf{d}\boldsymbol{x} \\ + \int_0^t \int_\Omega \vartheta\left(\kappa|\nabla\log\theta|^2 + |\theta^{-1/2}\partial_t\varphi|^2\right)\mathsf{d}\boldsymbol{x}\,\mathsf{d}t \\ \le \int_0^t \int_\Omega (\kappa\nabla\log\theta\cdot\nabla\vartheta - \partial_t\vartheta(\log\theta + \varphi))\,\mathsf{d}\boldsymbol{x}\,\mathsf{d}t\,, \end{aligned} \tag{19}$$

for a.e. $t \in (0,T)$ and for every $\vartheta \in C^0([0,T]\times\overline{\Omega}) \cap H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ such that $\vartheta(t,\boldsymbol{x}) \ge 0$ for every $t \in [0,T]$ and $\boldsymbol{x} \in \overline{\Omega}$. Moreover, the phase field equation

$$\partial_t\varphi - \Delta\varphi + F'(\varphi) = \theta \tag{20}$$

holds a.e. in $\Omega \times (0,T)$ with the initial and boundary conditions

$$\varphi(0) = \varphi_0, \qquad \boldsymbol{n}\cdot\nabla\varphi = 0 \tag{21}$$

in the sense of traces respectively in Ω and on $\partial\Omega \times (0,T)$. Finally, we require validity of the energy inequality for a.e. $t \in (0,T)$:

$$\int_\Omega \left(\frac{1}{2}|\nabla\varphi(t)|^2 + F(\varphi(t)) + \theta(t)\right)\mathsf{d}\boldsymbol{x} \le \int_\Omega \left(\frac{1}{2}|\nabla\varphi_0|^2 + F(\varphi_0) + \theta_0\right)\mathsf{d}\boldsymbol{x}\,. \tag{22}$$

Remark 2.1. Note that the traces are well-defined a.e. in Ω and on $\partial\Omega\times(0,T)$, respectively. Indeed, from (18e) we observe that $\varphi \in \mathcal{C}_{\mathsf{w}}([0,T];H^1(\Omega))$. Moreover, the normal-trace operator is well-defined as a mapping from $W^{2,1}(\Omega)$ to $L^1(\partial\Omega)$, see for instance [9, Thm. 2.7.4, (2.7.10) with (2.7.4)], [13, Prop. 3.80], or [14].

The first result of this paper is devoted to proving the global in time existence of weak solutions in the sense of Definition 2.2. As noted in the introduction, this seems to be the first rigorous existence result for system (1a)-(1b) in absence of regularizing power-like terms in the heat equation (*cf.* [21]).

Theorem 2.3. Let Ω be sufficiently smooth and let Hypothesis 2.1 be fulfilled. Let us also assume

$$\theta_0 \in L^1(\Omega), \quad \theta_0 > 0 \text{ a.e. in } \Omega, \quad \log\theta_0 \in L^1(\Omega)\,, \tag{23a}$$
$$\varphi_0 \in H^1(\Omega), \quad F(\varphi_0) \in L^1(\Omega), \quad \varphi_0\log\theta_0 \in L^1(\Omega)\,, \tag{23b}$$
$$\varphi_0(\boldsymbol{x}) \ge -K > -\infty \quad \text{for some } K \ge 0 \text{ and a.e. } \boldsymbol{x} \in \Omega\,. \tag{23c}$$

Then, there exists at least one weak solution in the sense of Definition 2.2. Moreover, if we have in addition

$$\operatorname{ess\,inf}_{\boldsymbol{x}\in\Omega}\theta_0(\boldsymbol{x}) > 0\,, \tag{24}$$

then there follows the minimum principle property

$$\operatorname{ess\,inf}_{\Omega\times(0,T)} \theta(\boldsymbol{x},t) > 0\,, \tag{25}$$

providing also the additional regularity

$$\log\theta \in \mathsf{BV}(0,T;(W^{1,p}(\Omega))^*) \cap L^\infty(0,T;L^q(\Omega)) \quad \text{for } p>3 \text{ and } q\in(1,\infty)\,, \tag{26}$$

so that in particular the entropy inequality (19) holds for every (and not just a.e.) $t \in [0,T]$.

Observe that (26) yields that there exists a countable set $D \subset [0,T]$, such that $\log\theta \in C^0([0,T]\setminus D;(W^{1,p}(\Omega))^*)$.

The second result of this paper states the local-in-time existence of strong solutions. Here and below we note as $H^2_{\boldsymbol{n}}(\Omega)$ the space of the functions in $H^2(\Omega)$ having zero normal derivative on the boundary. We also need an additional assumption on F:

Hypothesis 2.4. Let F be three times continuously differentiable, *i.e.,* $F \in \mathcal{C}^3(\mathbb{R};\mathbb{R})$.

Theorem 2.5. Let Hypotheses 2.1 and 2.4 hold true. Moreover, let us assume the following additional conditions on the initial data:

$$\theta_0 \in H^1(\Omega), \quad \operatorname{ess\,inf}_{\boldsymbol{x}\in\Omega} \theta_0(\boldsymbol{x}) > 0\,, \tag{27a}$$

$$\varphi_0 \in H^2_{\boldsymbol{n}}(\Omega), \qquad \varphi_1 := \Delta\varphi_0 - F'(\varphi_0) + \theta_0 \in H^1(\Omega) \tag{27b}$$

(notice that the latter is equivalent to just assuming $\varphi_0 \in H^3(\Omega)\cap H^2_{\boldsymbol{n}}(\Omega)$). Then, there exists a local-in-time solution to the system (1), namely, a couple (θ,φ) satisfying the regularity properties

$$\theta \in \mathcal{C}^0([0,T^*);H^1(\Omega)) \cap L^2(0,T^*;H^2(\Omega)) \cap W^{1,2}(0,T^*;L^2(\Omega))\,, \tag{28a}$$

$$\varphi \in \mathcal{C}^0([0,T^*);H^2(\Omega)) \cap C^1([0,T^*);H^1(\Omega)) \cap W^{1,2}(0,T^*;H^2(\Omega))\,, \tag{28b}$$

for a certain $T^*>0$. Moreover, the analogue of (25) holds on the interval $[0,T^*)$ and the strong solution fulfills a supplementary initial condition in the sense that $\|\varphi_t(t)-\varphi_1\|_{H^1} \to 0$ as $t \searrow 0$. Finally, the strong solution satisfies the energy equality (22) with equality sign for any $t\in[0,T^*)$.

Actually, the regularity proved above for strong solution is not sufficient for our purposes. However, it is easy to prove additional results. For instance we have the following

Corollary 2.6. Let the assumptions of Theorem 2.5 hold and let, in addition,

$$\theta_0 \in H^2_{\boldsymbol{n}}(\Omega). \tag{29}$$

Then, the local strong solution provided by Theorem 2.5 satisfies the additional regularity condition

$$\theta \in \mathcal{C}^0([0,T^*);H^2(\Omega)) \cap W^{1,2}(0,T^*;H^1(\Omega)). \tag{30}$$

Proof. It is sufficient to observe that, thanks to (28a)-(28b), (1a) can be rewritten as

$$\theta_t - \kappa\Delta\theta = (\partial_t\varphi)^2 - \theta\partial_t\varphi \in L^2(0,T^*;H^1(\Omega)). \tag{31}$$

Hence, the assertion follows from the standard theory of linear parabolic equations by also using the regularity (29) of the initial datum (the a-priori estimate used here corresponds to testing (31) by $-\Delta\theta_t$ and using the regularity of the right-hand side resulting from (28a)–(28b)). □

Remark 2.2. Via standard bootstrap arguments, one can even deduce more regularity for the local-strong solution, *e.g.* that all terms appearing in (1) are even continuous in the case that $\theta_0, \varphi_0 \in \mathcal{C}^2(\overline{\Omega})$. Indeed, from $\varphi, \theta \in \mathcal{C}^0([0,T^*); H^2(\Omega))$ we infer by the SOBOLEV-embedding theorem that $\varphi, \theta \in \mathcal{C}^0(\overline{\Omega} \times [0,T^*))$. The continuity of the function F' implies that

$$\partial_t \varphi - \Delta \varphi = \theta - F'(\varphi) \in C^0(\overline{\Omega} \times [0,T^*)) ,$$

which grants together with the regularity of the heat equation (see [46, Prop. 1.1A]) that there exists some $0 < \tilde{T} < T^*$ such that $\varphi \in \mathcal{C}^1([0,\tilde{T}]; \mathcal{C}^0(\overline{\Omega})) \cap \mathcal{C}^0([0,\tilde{T}]; \mathcal{C}^2(\overline{\Omega}))$, *i.e.*, $\partial_t\varphi$, $\Delta\varphi \in \mathcal{C}^0(\overline{\Omega} \times [0,\tilde{T}])$. This in turn implies that the right-hand side of (31) is even in $C^0(\overline{\Omega} \times [0,\tilde{T}])$ such that, we may infer the regularity $\theta \in \mathcal{C}^1([0,\tilde{T}]; \mathcal{C}^0(\overline{\Omega})) \cap \mathcal{C}^0([0,\tilde{T}]; \mathcal{C}^2(\overline{\Omega}))$, (see [46, Prop. 1.1A]), *i.e.* $\partial_t\theta$, $\Delta\theta \in C^0(\overline{\Omega} \times [0,\tilde{T}])$. This regularity is then sufficient to give a sense to all terms appearing in the weak-strong uniqueness proof (see Sec. 4) and especially to bound the right-hand side of the inequality (89). It is worth noticing that this regularity is exactly what is needed to fulfill the system (1) in a classical sense, *i.e.*, pointwise.

We can finally state our second main result providing weak-strong uniqueness for system (1a)-(1b). We may note that the proof of this result neither requires Hypothesis 2.4, nor any additional assumption on F and on the initial data leading to the validity of (1a)-(1b) in L^∞ according to the above remark. On the other hand, such additional conditions are needed in the construction of local strong solutions with the desired properties (see Theorem 2.5 above).

Theorem 2.7. Let Hypothesis 2.1 hold true. Let (θ, φ) be a weak solution according to Definition 2.2 and $(\tilde{\theta}, \tilde{\varphi})$ a strong solution according to Theorem 2.5 and satisfying (1a)–(1b) in the $\mathcal{C}^0$-sense. Let us also assume that (θ, φ) and $(\tilde{\theta}, \tilde{\varphi})$ start from the same initial data assumed to fulfill (27a)-(27b) and (29) and that they are defined over the same reference interval $(0, T)$. Then the two solutions coincide, *i.e.*, $\theta \equiv \tilde{\theta}$ and $\varphi \equiv \tilde{\varphi}$ a.e. in $\Omega \times (0, T)$.

It is worth observing that, in view of (29), the strong solution additionally satisfies (30); moreover, thanks to the last condition in (27a), both the weak and the strong solution comply with (25).

Remark 2.3. The two main theorems of this article, *i.e.,* Theorem 2.3 and Theorem 2.7, are not restricted to the space dimensions two and three. Actually, the proofs of these results do not use any dimension-depending inequality. Only Theorem 2.5 makes explicit use of the restriction onto at most three space dimensions, since several dimension-dependent embeddings are crucial. Nevertheless, we expect that the local existence might be generalized to higher space dimensions up to technical variants in the proof.

The rest of the paper is devoted to the proof of the above results, along the scheme outlined at the end of the introduction.

3 Existence of weak solutions

In this section, we prove the existence of weak solutions according to Definition 2.2.

3.1 Regularized system

As a regularized system to approximate (1), we consider

$$\partial_t \theta - \kappa \Delta \theta + \varepsilon \theta^p + \theta \partial_t \varphi = |\partial_t \varphi|^2 , \tag{32a}$$

$$\partial_t \varphi - \Delta \varphi + F'(\varphi) = \theta , \tag{32b}$$

where $\varepsilon \in (0, 1)$ is intended to pass to 0 in the limit and $p \in (3, \infty)$ is arbitrary but fixed. The system is equipped with the boundary conditions

$$\boldsymbol{n} \cdot \nabla \varphi = 0 = \boldsymbol{n} \cdot \nabla \theta .$$

In order to prove existence of weak solutions according to Definition 2.2, we follow the standard path of first proving existence of solutions to the regularized system (32) (see Subsection 3.2) (complemented with regularized initial data), then deriving *a priori* estimates (see Subsection 3.3) uniform with respect to ε, and showing convergence in the end (see Subsection 3.4).

3.2 Existence of solutions to the regularized system

We consider the following regularized decoupled system

$$\partial_t\theta - \kappa\Delta\theta + \varepsilon(\theta)^p + \theta\partial_t\varphi = |\partial_t\varphi|^2\,, \tag{33a}$$

$$\partial_t\varphi - \Delta\varphi + F'(\varphi) = \overline{\theta}\,, \tag{33b}$$

equipped with the boundary conditions (1c) and initial data $\theta_0^\varepsilon \in \mathcal{C}^2(\overline{\Omega})$ with $\theta_0^\varepsilon(\boldsymbol{x}) \geq c_\varepsilon > 0$ and $\varphi_0^\varepsilon \in \mathcal{C}^2(\overline{\Omega})$ for every $\varepsilon > 0$ such that θ_0^ε and φ_0^ε strongly converge to θ_0 and φ_0 in the spaces indicated in (23a) and (23b), respectively. Additionally, we assume that $\varepsilon\|\varphi_0\|_{L^p(\Omega)}^p \leq c$ and that the product $\varphi_0^\varepsilon \log\theta_0^\varepsilon$ remains bounded in $L^1(\Omega)$. Here and below, for $r \in \mathbb{R}$ and $\sigma > 1$, we use the notation $(r)^\sigma = |r|^{\sigma-1}r$. This assures that the associated operator is monotone. This is necessary in this step, because we have not established yet that θ is positive.

We prove the existence of local solutions by means of the SCHAUDER fixed point argument. To do this, we define the set

$$\mathcal{M}_R := \big\{\theta \in L^p(\Omega\times(0,T_\varepsilon))\big|\|\theta\|_{L^p(\Omega\times(0,T_\varepsilon))} \leq R\big\},$$

where the radius $R > 0$, possibly depending on ε, and the final time $T_\varepsilon > 0$ will be specified in the course of the procedure.

The operator $\mathcal{T} : \mathcal{M}_R \to \mathcal{M}_R$ is defined by the following procedure. For given $\overline{\theta} \in \mathcal{M}_R$, solve (33b) with the boundary condition (1c) and the initial value φ_0^ε so obtaining φ in a proper regularity class.

Then, for given $\partial_t\varphi$, solve (33a) with the boundary condition (1c) and the initial value θ_0^ε obtaining θ. Correspondingly, we set $\mathcal{T}(\overline{\theta}) = \theta$. Then, to be able to apply the SCHAUDER fixed point theorem, we need to prove several claims.

Claim 1: The operator $\mathcal{T}$ is well defined and maps $\mathcal{M}_R$ into itself. Since showing the existence of weak solutions to the two equations separately is a standard procedure (one may use ROTHE's method or a suitable GALERKIN discretization like in [42, chapter 8]), we rather focus on the relevant estimates.

First of all, recalling (13) and testing (33b) by $(\varphi)^{p-1}/p$, we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}\|\varphi\|_{L^p(\Omega)}^p + \frac{4(p-1)}{p^3}\|\nabla(\varphi)^{p/2}\|_{L^2(\Omega)}^2 + \frac{1}{p}\int_\Omega G'(\varphi)(\varphi)^{p-1}\,\mathrm{d}\boldsymbol{x}&\\ \leq \frac{2\lambda}{p}\|\varphi\|_{L^p(\Omega)}^p + \frac{1}{p^2}\|\bar\theta\|_{L^p(\Omega)}^p + \frac{p-1}{p^2}\|\varphi\|_{L^p(\Omega)}^p\,.&\end{aligned}$$

Due to the monotonicity of G' and to the assumption $G'(0) = 0$, the GRONWALL inequality shows that

$$\|\varphi\|_{L^\infty(0,T_\varepsilon;L^p(\Omega))} \leq \|\varphi_0^\varepsilon\|_{L^p(\Omega)} + c\|\bar\theta\|_{L^p(0,T_\varepsilon;L^p(\Omega))} \leq \|\varphi_0^\varepsilon\|_{L^p(\Omega)} + cR\,. \tag{34}$$

Here and below, the constant c, whose value may vary on occurrence, is allowed to depend both on ε and, *in a monotonic way*, on the final time T_ε to be chosen later.

Let us now set

$$G_p(s) := \int_0^s (G'(r))^{p-1}\,\mathrm{d}\,r \tag{35}$$

and note that, by construction, G_p is convex and coercive at infinity. Then, testing (33b) by $(G'(\varphi))^{p-1}$, we observe by YOUNG's inequality that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G_p(\varphi)\,\mathrm{d}x + (p-1)\int_\Omega |\nabla\varphi|^2 G''(\varphi)|G'(\varphi)|^{p-2}\,\mathrm{d}\boldsymbol{x} + \|G'(\varphi)\|^p_{L^p(\Omega)}\\ \le \frac{1}{2}\|G'(\varphi)\|^p_{L^p(\Omega)} + c\left(\|\bar{\theta}\|^p_{L^p(\Omega)} + \|\varphi\|^p_{L^p(\Omega)}\right).\end{aligned}$$

We then obtain

$$\|G'(\varphi)\|_{L^p(\Omega\times(0,T_\varepsilon))} \le \int_\Omega G_p(\varphi_0^\varepsilon)\,\mathrm{d}\boldsymbol{x} + c\left(\|\bar{\theta}\|_{L^p(\Omega\times(0,T_\varepsilon))} + \|\varphi\|_{L^p(\Omega\times(0,T_\varepsilon))} + 1\right).$$

Comparing terms in (33b) and using the maximal L^p-regularity of the operator associated to the heat equation (see [15]), we conclude that there exists a constant $c > 0$ such that

$$\begin{aligned}&\|\partial_t\varphi\|_{L^p(\Omega\times(0,T_\varepsilon))} + \|\Delta\varphi\|_{L^p(\Omega\times(0,T_\varepsilon))}\\ &\qquad \le c(\|\varphi_0^\varepsilon\|_{\mathcal{C}^0(\overline{\Omega})} + \|\varphi_0^\varepsilon\|_{W^{2-2/p,p}(\Omega)} + \|\bar{\theta}\|_{L^p(\Omega\times(0,T_\varepsilon))}) \le c(1+R).\end{aligned} \quad (36)$$

In the following, we may consider equation (33a) for $\partial_t\varphi \in L^p(\Omega\times(0,T))$. We define $q = p^2/2$ and test equation (33a) by $\theta^{(q-p)}$ to find

$$\begin{aligned}\frac{1}{(q-p+1)}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^{q-p+1}_{L^{(q-p+1)}(\Omega)} + \frac{4\kappa(q-p)}{(q-p+1)^2}\|\nabla(\theta^{(q-p+1)/2})\|^2_{L^2(\Omega)} + \varepsilon\|\theta\|^q_{L^q(\Omega)}\\ = \int_\Omega |\partial_t\varphi|^2\theta^{(q-p)} - \partial_t\varphi\theta^{(q-p+1)}\,\mathrm{d}\boldsymbol{x}\\ \le \frac{\varepsilon}{2}\|\theta\|^q_{L^q(\Omega)} + c_\varepsilon\left(\|\partial_t\varphi\|^p_{L^p(\Omega)} + \|\partial_t\varphi\|^{q/(p-1)}_{L^{q/(p-1)}(\Omega)}\right).\end{aligned}$$

Note that $p \ge 2$ such that $q/(p-1) \le p$. Integrating in time, we conclude that

$$\|\theta\|^{p^2/2}_{L^{p^2/2}(\Omega\times(0,T_\varepsilon))} \le c_\varepsilon\left(\|\theta_0^\varepsilon\|^{(p^2-2p+2)/2}_{L^{(p^2-2p+2)/2}(\Omega)} + \|\partial_t\varphi\|^p_{L^p(\Omega\times(0,T_\varepsilon))}\right). \quad (37)$$

The previous estimates and maximal $L^{p/2}$-regularity results for the operator associated to the heat equation allow us to infer that

$$\|\partial_t\theta\|_{L^{p/2}(\Omega\times(0,T_\varepsilon))} + \|\Delta\theta\|_{L^{p/2}(\Omega\times(0,T_\varepsilon))} \le c. \quad (38)$$

Using interpolation, it is now not difficult to verify (we leave details to the reader) that there exists an $s > 0$ such that

$$\|\theta\|_{W^{s,2p}(0,T_\varepsilon;L^p(\Omega))} \le Q(\|\varphi_0^\varepsilon\|_{\mathcal{C}^2(\overline{\Omega})}, \|\theta_0^\varepsilon\|_{\mathcal{C}^2(\overline{\Omega})}, R),$$

where Q is a computable nonnegative-valued and continuous function, which is monotone in R. As the previous constants c, also the expression of Q may depend on T_ε. However, this dependence is monotonic, hence taking a small final time T_ε does not alter the expression of Q. Further on, we can find a $\beta > 0$ such that

$$\|\theta\|_{L^p(\Omega\times(0,T_\varepsilon))} \le cT_\varepsilon^\beta\|\theta\|_{W^{s,2p}(0,T_\varepsilon;L^p(\Omega))}.$$

Hence, we may conclude that for R big enough, we can take T_ε so small that

$$\|\theta\|_{L^p(\Omega\times(0,T_\varepsilon))} \le cT_\varepsilon^\beta\|\theta\|_{W^{s,2p}(0,T_\varepsilon;L^p(\Omega))} \le T_\varepsilon^\beta Q(\|\varphi_0^\varepsilon\|_{\mathcal{C}^2(\overline{\Omega})}, \|\theta_0^\varepsilon\|_{\mathcal{C}^2(\overline{\Omega})}, R) \le R.$$

Claim 2: The image of $\mathcal{T}$ is compact in $\mathcal{M}_R$. The above estimate (38) allows us to apply the LIONS–AUBIN compactness lemma (*cf.* [33, p. 58]) granting compactness in $L^{p/2}(\Omega\times(0,T))$. Combining this with the boundedness (37), we obtain compactness in $L^p(\Omega\times(0,T))$. Note that we used $p > 2$ such that $p^2 > 2p$.

Claim 3: The operator $\mathcal{T}$ is continuous. Assume that φ_1 and φ_2 are the solutions to (33b) corresponding to two different right-hand sides $\overline{\theta}_1$ and $\overline{\theta}_2$, respectively. Subtracting the equations from each other and testing the difference by $\varphi_1 - \varphi_2$, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\|\varphi_1 - \varphi_2\|^2_{L^2(\Omega)} + \|\nabla\varphi_1 - \nabla\varphi_2\|^2_{L^2(\Omega)} + \int_\Omega \left(G'(\varphi_1) - G'(\varphi_2)\right)(\varphi_1 - \varphi_2)\,\mathrm{d}\boldsymbol{x} \leq \frac{1}{2}\|\overline{\theta}_1 - \overline{\theta}_2\|^2_{L^2(\Omega)} + \left(\frac{1}{2} + 2\lambda\right)\|\varphi_1 - \varphi_2\|^2_{L^2(\Omega)}\,,$$

which grants the continuity of the solution operator to the equation (33b) from $L^2(\Omega\times(0,T))$ to $L^\infty(0,T;L^2(\Omega))$.

This fact, together with the previous estimates and the standard theory of the heat equation, guarantees that the solution operator to (33b) maps $\overline{\theta}$ to $\partial_t\varphi$ continuously from $L^2(\Omega\times(0,T))$ to $L^2(\Omega\times(0,T))$. Since we already established the boundedness in a better space (see (36)), we even find that the solution operator to (33b) maps $\overline{\theta}$ to $\partial_t\varphi$ continuously from $L^p(\Omega\times(0,T))$ to $L^q(\Omega\times(0,T))$ for some $q<p$, for instance for $q = p/(p-2)$.

Assume now that θ_1 and θ_2 are the solutions to (33a) corresponding to two different inputs $\partial_t\varphi_1$ and $\partial_t\varphi_2$. Subtracting the equations from each other and testing the difference by $\theta_1-\theta_2$, we then find

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\|\theta_1 - \theta_2\|^2_{L^2(\Omega)} + \kappa\|\nabla\theta_1 - \theta_2\|^2_{L^2(\Omega)} + \varepsilon\int_\Omega((\theta_1)^p - (\theta_2)^p)(\theta_1-\theta_2)\,\mathrm{d}\boldsymbol{x}\\
&= \int_\Omega(|\partial_t\varphi_1|^2 - |\partial_t\varphi_2|^2 - \partial_t\varphi_1\theta_1 + \partial_t\varphi_2\theta_2)(\theta_1-\theta_2)\,\mathrm{d}\boldsymbol{x}\\
&\leq \|\partial_t\varphi_1 - \partial_t\varphi_2\|_{L^{p/(p-2)}(\Omega)}\left(\|\partial_t\varphi_1\|_{L^p(\Omega)} + \|\partial_t\varphi_2\|_{L^p(\Omega)}\right)\|\theta_1-\theta_2\|_{L^p(\Omega)}\\
&\quad + \|\partial_t\varphi_1\|_{L^p(\Omega)}\|\theta_1-\theta_2\|^2_{L^{2p/(p-1)}(\Omega)} + \|\theta_2\|_{L^p(\Omega)}\|\partial_t\varphi_1 - \partial_t\varphi_2\|_{L^{p/(p-2)}(\Omega)}\|\theta_1-\theta_2\|_{L^p(\Omega)}\,.
\end{aligned} \tag{39}$$

Since the function $s\mapsto(s)^p$ is strongly monotone, there exists a $c_p>0$ such that

$$((\theta_1)^p - (\theta_2)^p)(\theta_1-\theta_2) \geq c_p|\theta_1-\theta_2|^{p+1}\,.$$

This allows us to absorb the terms $\|\theta_1-\theta_2\|_{L^p(\Omega)}$ on the right-hand side of (39) via YOUNG's inequality into the corresponding term on the left-hand side. The estimation of the other terms in (39) is simpler and hence it is left to the reader. Note that $p>3$, so that $2p/(p-1)>p$. Then, a number of straightforward checks permits us to conclude that the solution operator to (33a) is continuous as a mapping from $L^{p/(p-2)}(0,T;L^{p/(p-2)}(\Omega))$ to $L^\infty(0,T;L^2(\Omega))\cap L^{p+1}(0,T;L^{p+1}(\Omega))\cap L^2(0,T;H^1(\Omega))$. Eventually, we have obtained that $\mathcal{T}$ is a continuous mapping from $\mathcal{M}_R$ to $\mathcal{M}_R$, as desired.

Claim 4: The solution θ is positive. We observe that from (33a) there follows

$$\begin{aligned}
\partial_t\theta - \kappa\Delta\theta &= -\varepsilon(\theta)^p + |\partial_t\varphi|^2 - \partial_t\varphi\theta\\
&\geq -\varepsilon(\theta)^p + \frac{1}{2}|\partial_t\varphi|^2 - \frac{1}{2}\theta^2 \geq -(\theta)^p - \frac{1}{2}\theta^2.
\end{aligned}$$

Then, one may check that, for every $\varepsilon\in(0,1)$, the solution to the ODE

$$\partial_t h + (h)^p + \frac{1}{2}h^2 = 0\,,\quad h(0) := \operatorname{ess\,inf}_{\boldsymbol{x}\in\Omega}\theta_0^\varepsilon(\boldsymbol{x})\,,$$

is a positive subsolution to (33a) on $\Omega\times(0,T)$. Consequently, $\theta(\boldsymbol{x},t)\geq h(t)>0$ for all $\boldsymbol{x}\in\Omega$ and $t\in(0,T)$. Moreover, it is worth observing that the above bound is, in the case of the additional assumption (24), independent of ε. This fact is not essential here, but it will be in the sequel.

As an outcome of the proved Claims 1–4, we may now apply SCHAUDER's fixed point theorem to the map $\mathcal{T}$. Hence, for every $\varepsilon>0$ there exists a local-in-time solution $(\theta_\varepsilon,\varphi_\varepsilon)$ to the regularized system (32), where the local existence interval $(0,T_\varepsilon)$ may actually depend on ε.

Remark 3.1. We have only established local-in-time existence of solutions to the approximate system. On the other hand, all the *a priori* estimates we shall perform in the next subsection will be independent of T_ε. Hence, as a consequence of standard extension arguments (details are left to the reader) the weak solution we obtain in the limit will be in fact defined on the whole reference interval $(0,T)$ given in the beginning.

3.3 *A priori* estimates

In order to find *a priori* bounds independent of ε, we derive a number of estimates. Firstly, testing the equation (32a) and (32b) with 1 and φ_t, respectively, and adding the results, we obtain the energy equality

$$\frac{1}{2}\|\nabla\varphi_\varepsilon(t)\|^2_{L^2(\Omega)} + \int_\Omega F(\varphi_\varepsilon(t)) + \theta_\varepsilon(t)\,\mathrm{d}\boldsymbol{x} + \varepsilon\int_0^t\int_\Omega \theta_\varepsilon^p\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s = \frac{1}{2}\|\nabla\varphi_0^\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega F(\varphi_0^\varepsilon) + \theta_0^\varepsilon\,\mathrm{d}\boldsymbol{x}\,, \tag{40}$$

where θ_0^ε and φ_0^ε are the regularized initial data for system (33).

Similarly, testing (32a) by $-\vartheta/\theta_\varepsilon$, we find the entropy production rate

$$-\int_\Omega(\log\theta_\varepsilon(t)+\varphi_\varepsilon(t))\vartheta(t)\,\mathrm{d}\boldsymbol{x} + \int_0^t\int_\Omega \kappa|\nabla\log\theta_\varepsilon|^2\vartheta + \left|\frac{\partial_t\varphi_\varepsilon}{\sqrt{\theta_\varepsilon}}\right|^2\vartheta - \varepsilon\theta_\varepsilon^{p-1}\vartheta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s = -\int_\Omega(\log\theta_0^\varepsilon+\varphi_0^\varepsilon)\vartheta(0)\,\mathrm{d}\boldsymbol{x} + \int_0^t\int_\Omega\kappa\nabla\log\theta_\varepsilon\cdot\nabla\vartheta - (\log\theta_\varepsilon+\varphi_\varepsilon)\partial_t\vartheta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\,, \tag{41}$$

holding true for every $\vartheta\in C^0([0,T]\times\overline{\Omega})\cap H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$ such that $\vartheta(t,\boldsymbol{x})\geq 0$ for every $t\in[0,T]$ and $\boldsymbol{x}\in\overline{\Omega}$.

Using the boundedness of F from below (see Hypothesis 2.1), we conclude from the energy equality (40) that

$$\|\nabla\varphi_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|F(\varphi_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} + \|\theta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \varepsilon\|\theta_\varepsilon\|^p_{L^p(0,T;L^p(\Omega))} \leq c\,. \tag{42}$$

Then, on account of (15), we have more precisely

$$\|\varphi_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}\leq c\,. \tag{43}$$

From (42) and Hypothesis 2.1, we observe that

$$\|F'(\varphi_\varepsilon)\log(e+|F'(\varphi_\varepsilon)|)\|_{L^\infty(0,T;L^1(\Omega))}\leq c\,. \tag{44}$$

Choosing now $\vartheta=1$ in (41) and using (42), we obtain the bounds

$$\|\log\theta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla\log\theta_\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \left\|\frac{\partial_t\varphi_\varepsilon}{\sqrt{\theta_\varepsilon}}\right\|_{L^2(0,T;L^2(\Omega))}\leq c\,. \tag{45}$$

As observed in the introduction, the above information (which is the direct outcome of the basic laws of Thermodynamics) is not sufficient to provide a control of the temperature uniformly with respect to ε in a space better than L^1 with respect to the space variables. This is a critical issue and it is the reason why existence of weak solutions to system (1a)-(1b) has been proved so far only in space dimension 1, or in the case when the heat flux law has a power-like expression [22], *i.e.*, the diffusion term in (1a) has the form $-\kappa\Delta\theta^b$, for b large enough. In what follows we actually deduce a new *a-priori* estimate that will allow us to show that

$$\int_0^T\int_\Omega\theta_\varepsilon\log\theta_\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\leq c\,. \tag{46}$$

Such a uniform integrability property, which is a somehow unexpected fact because it does not seem to be linked to any physical principle, will be the key for showing that (a subsequence of) θ_ε converges strongly in L^1 to a limit temperature. In turn, this will imply solvability of the limit system in the sense of weak solutions with no occurrence of defect measures. It is worth noting that, in order for this argument to work, assuming a polynomial growth of the potential (16) is a crucial point.

That said, we first observe that the solution to

$$\partial_t \underline{\varphi} - \Delta \underline{\varphi} + F'(\underline{\varphi}) = \partial_t \underline{\varphi} - \Delta \underline{\varphi} + G'(\underline{\varphi}) - 2\lambda \underline{\varphi} = 0 \quad \text{with } \underline{\varphi}(0) = -K \tag{47}$$

is a subsolution to (32b), because we have proved that $\theta \geq 0$. Note that we have chosen the initial value $\underline{\varphi}(0)$ to be constant in Ω, where the constant K is given in (23c).

Then, assuming for simplicity but without loss of generality that $G'(0) = 0$, for the solution of (47), we observe by testing with $\underline{\varphi}$ that

$$\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}|\underline{\varphi}|^2 - \Delta|\underline{\varphi}|^2\right) = -|\nabla\underline{\varphi}|^2 - G'(\underline{\varphi})\underline{\varphi} + 2\lambda|\underline{\varphi}|^2 \leq 2\lambda|\underline{\varphi}|^2 \quad \text{with } |\underline{\varphi}(0)|^2 = K^2\,,$$

which permits us to conclude that the solution to the ODE $\partial_t h = 4\lambda h$ with $h(0) = K^2$ is a supersolution to $|\underline{\varphi}|^2$ and deduce that $|\underline{\varphi}(\boldsymbol{x},t)|^2 \leq K^2 e^{4\lambda T}$ for all $(\boldsymbol{x},t) \in \Omega \times (0,T)$. This in turn gives a lower bound on the solution of (32b), *i.e.*, there exists a constant $K' > 0$ such that

$$\varphi_\varepsilon(\boldsymbol{x},t) \geq -K' \quad \text{for all } (\boldsymbol{x},t) \in \Omega \times (0,T)\,. \tag{48}$$

Multiplying now (32a) by $1/\theta_\varepsilon$, we find the entropy production rate for the approximate system:

$$\partial_t \log\theta_\varepsilon + \partial_t\varphi_\varepsilon - \kappa\Delta\log\theta_\varepsilon - \kappa|\nabla\log\theta_\varepsilon|^2 + \varepsilon\theta_\varepsilon^{p-1} = \frac{|\partial_t\varphi_\varepsilon|^2}{\theta_\varepsilon}\,.$$

Testing this equation by $\varphi_\varepsilon + K'$, we observe

$$\begin{aligned}\int_\Omega (\varphi_\varepsilon + K')\partial_t\log\theta_\varepsilon \,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\|\varphi_\varepsilon + K'\|^2_{L^2(\Omega)} \\ + \int_\Omega \kappa\nabla\log\theta_\varepsilon\cdot\nabla\varphi_\varepsilon - \kappa(\varphi_\varepsilon + K')|\nabla\log\theta_\varepsilon|^2 - (\varphi_\varepsilon + K')\frac{|\partial_t\varphi_\varepsilon|^2}{\theta_\varepsilon} + \varepsilon\theta_\varepsilon^{p-1}(\varphi_\varepsilon + K')\,\mathrm{d}\boldsymbol{x} = 0\,.\end{aligned} \tag{49}$$

Now, we test (32b) by $\log\theta_\varepsilon$, which implies

$$\int_\Omega \log\theta_\varepsilon\partial_t\varphi_\varepsilon + \nabla\varphi_\varepsilon\cdot\nabla\log\theta_\varepsilon + F'(\varphi_\varepsilon)\log\theta_\varepsilon\,\mathrm{d}\boldsymbol{x} = \int_\Omega \theta_\varepsilon\log\theta_\varepsilon\,\mathrm{d}\boldsymbol{x}\,. \tag{50}$$

Adding (49) and (50) and integrating in time, we may observe

$$\begin{aligned}\int_0^T\int_\Omega \theta_\varepsilon\log\theta_\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,t + \int_0^T\int_\Omega(\varphi_\varepsilon + K')\left(\kappa|\nabla\log\theta_\varepsilon|^2 + \frac{|\partial_t\varphi_\varepsilon|^2}{\theta_\varepsilon}\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,t \\ = \frac{1}{2}\|\varphi_\varepsilon + K'\|^2_{L^2(\Omega)}\Big|_0^T + \int_\Omega \log\theta_\varepsilon(\varphi_\varepsilon + K')\,\mathrm{d}\boldsymbol{x}\Big|_0^T \\ + \int_0^T\int_\Omega(\kappa+1)\nabla\log\theta_\varepsilon\cdot\nabla\varphi_\varepsilon + F'(\varphi_\varepsilon)\log\theta_\varepsilon + \varepsilon\theta_\varepsilon^{p-1}(\varphi_\varepsilon + K')\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,t\,.\end{aligned} \tag{51}$$

Now, we provide a control of the various terms on the right-hand side of the above relation. First of all, the lower bound on $\{\varphi_\varepsilon\}$ (see (48)) allows us to observe that

$$\int_0^t\int_\Omega(\varphi_\varepsilon + K')\left(\kappa|\nabla\log\theta_\varepsilon|^2 + \frac{|\partial_t\varphi_\varepsilon|^2}{\theta_\varepsilon}\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s \geq 0\,.$$

Moreover, the term $\|\varphi_\varepsilon(t)\|^2_{L^2(\Omega)}$ in (51) is controlled due to the $L^\infty(0,T;H^1(\Omega))$-bound of $\{\varphi_\varepsilon\}$ established in (43). Next, using Hölder's inequality, we have

$$\int_0^t \int_\Omega (\kappa+1)\nabla\varphi_\varepsilon \cdot \nabla \log\theta_\varepsilon \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le c\|\nabla\varphi_\varepsilon\|_{L^2(0,T;L^2(\Omega))}\|\nabla\log\theta_\varepsilon\|_{L^2(0,T;L^2(\Omega))}\,,$$

where the right-hand side is bounded due to (42) and (45). The regularizing term can be estimated as follows:

$$\varepsilon\int_0^t \int_\Omega \theta_\varepsilon^{p-1}\varphi_\varepsilon \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le \left(\varepsilon^{1/p}\|\theta_\varepsilon\|_{L^p(0,T;L^p(\Omega))}\right)^{p-1}\varepsilon^{1/p}\|\varphi_\varepsilon\|_{L^p(0,T;L^p(\Omega))} \le c\,,$$

where we also used (42) and (34).

Considering the second term on the right-hand side of (51), we distinguish the cases for the different signs of $\log\theta_\varepsilon$:

$$\int_\Omega \log\theta_\varepsilon(\varphi_\varepsilon+K')\,\mathrm{d}\boldsymbol{x} = \int_{\{\theta_\varepsilon\le 1\}}\log\theta_\varepsilon(\varphi_\varepsilon+K')\,\mathrm{d}\boldsymbol{x} + \int_{\{\theta_\varepsilon>1\}}\log\theta_\varepsilon(\varphi_\varepsilon+K')\,\mathrm{d}\boldsymbol{x}\,, \tag{52}$$

where all terms are evaluated at the time t (the counterparts at $t=0$ are controlled thanks to the assumptions (23b) and the assumptions on the regularized initial data below formula (33)). Thanks to the lower bound (48) on $\{\varphi_\varepsilon\}$, we find

$$\int_{\{\theta_\varepsilon\le 1\}}\log\theta_\varepsilon(\varphi_\varepsilon+K')\,\mathrm{d}\boldsymbol{x} \le 0\,.$$

Additionally, we may estimate the last term on the right-hand side of (52) using the LEGENDRE–FENCHEL–YOUNG inequality for the convex function $\psi(r)=e^r$ and its convex conjugate $\psi^*(s)=s\log s - s$ (where we intend that $\psi^*(0)=0$ and $\psi^*(s)\equiv+\infty$ for $s<0$):

$$\begin{aligned}\int_{\{\theta_\varepsilon>1\}}\log\theta_\varepsilon(\varphi_\varepsilon+K')\,\mathrm{d}\boldsymbol{x} &\le \int_{\{\theta_\varepsilon>1\}}\psi(\log\theta_\varepsilon)+\psi^*(\varphi_\varepsilon+K')\,\mathrm{d}\boldsymbol{x}\\ &= \int_{\{\theta_\varepsilon>1\}}\theta_\varepsilon+(\varphi_\varepsilon+K')\log(\varphi_\varepsilon+K')-(\varphi_\varepsilon+K')\,\mathrm{d}\boldsymbol{x}\le c\,,\end{aligned}$$

the last inequality following from (42) and (43).

We now focus on the term depending on F' in (51). Actually, by (48) and (14), it is clear that $F'(\varphi_\varepsilon)\ge -c$ a.e. in $\Omega\times(0,T)$ and for all $\varepsilon>0$. Hence, we can decompose that term as

$$\begin{aligned}\int_0^t\int_\Omega F'(\varphi_\varepsilon)\log\theta_\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s &= \iint_{\{\theta_\varepsilon\le1\}}F'(\varphi_\varepsilon)\log\theta_\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \iint_{\{\theta_\varepsilon>1\}}F'(\varphi_\varepsilon)\log\theta_\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\ &\le c\iint_{\{\theta_\varepsilon\le1\}}|\log\theta_\varepsilon|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \iint_{\{\theta_\varepsilon>1\}}|F'(\varphi_\varepsilon)|\log\theta_\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\,.\end{aligned}$$

The first term on the right-hand side is estimated thanks to (45), whereas the second term can be controlled once more by using the LEGENDRE–FENCHEL–YOUNG inequality with the same ψ as before. Namely, we have

$$\begin{aligned}&\iint_{\{\theta_\varepsilon>1\}}|F'(\varphi_\varepsilon)|\log\theta_\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \le \iint_{\{\theta_\varepsilon>1\}}\psi^*(|F'(\varphi_\varepsilon)|)+\psi(\log\theta_\varepsilon)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\ &\quad\le \iint_{\{\theta_\varepsilon>1\}}|F'(\varphi_\varepsilon)|(\log|F'(\varphi_\varepsilon)|-1)+\theta_\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\,,\end{aligned}$$

and the right-hand side is controlled thanks to the growth assumption (16) of Hypothesis 2.1 and the bound (42). Inserting everything back into (51), we eventually obtain

$$\int_0^t\int_\Omega \theta_\varepsilon\log\theta_\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \le c\,,$$

which proves the estimate (46).

In order to improve our bound for $\partial_t\varphi_\varepsilon$, we consider a new convex function $\psi : \mathbb{R} \to [1, +\infty]$ defined as $\psi(r) = (1/4)(r^2(2\log r - 1) + 1)$, where it is intended that $\psi(1) = 0$ and $\psi(r) \equiv +\infty$ as $r < 1$. Determining the precise expression of the conjugate function $\psi^*(s)$ is difficult, but we can at least estimate it appropriately. We recall that

$$\psi^*(s) = \max_{r\in\mathbb{R}} \big(sr - \psi(r)\big)$$

and a simple computation shows that the maximum is attained at $s = r\log r$. Hence, if r is the maximizer, using first that $s = r\log r$ and then that $s + 1 \le r^2$ (which holds as $r \ge 1$), we have

$$\begin{aligned}\psi^*(s) &= r^2\log r - \psi(r) = \frac{1}{2}r^2\log r + \frac{1}{4}r^2 - \frac{1}{4} = \frac{s^2}{\log r^2} + \frac{s^2}{\log^2 r^2} - \frac{1}{4}\\ &\le \frac{s^2}{\log(s+1)} + \frac{s^2}{\log^2(s+1)} - \frac{1}{4}\,.\end{aligned}$$

Additionally, we observe for ψ^* that for any $y \in [2,\infty)$ it holds

$$\psi^*(y\log^{1/2} y) \le \frac{y^2\log y}{\log\Big(1 + y\log^{1/2} y\Big)} + \frac{y^2\log y}{\log^2\Big(1 + y\log^{1/2} y\Big)} - \frac{1}{4} \le c(y^2+1)\,, \tag{53}$$

since the function

$$y \mapsto \frac{\log y}{\log\Big(1 + y\log^{1/2} y\Big)} + \frac{\log y}{\log^2\Big(1 + y\log^{1/2} y\Big)}$$

is bounded for $y \in [2,\infty)$, which is obvious for any compact subset in $[2,\infty)$ and also holds as $y \nearrow \infty$ as an easy check shows.

Now, setting for simplicity $u := \sqrt{\theta_\varepsilon} + 1$ and $v := |\partial_t\varphi_\varepsilon| + 2u$, we have

$$\begin{aligned}v\log^{1/2} v &= \frac{v}{u}\left[\log\Big(\frac{v}{u}\Big) + \frac{1}{2}\log(u^2)\right]^{1/2} u\\ &\le \frac{v}{u}\left[\log^{1/2}\Big(\frac{v}{u}\Big) + \frac{1}{\sqrt{2}}\log^{1/2}(u^2)\right] u\\ &\le \frac{v}{u}\log^{1/2}\Big(\frac{v}{u}\Big)u + \frac{1}{\sqrt{2}}\frac{v}{u}u\log^{1/2}(u^2)\\ &\le \psi^*\left(\frac{v}{u}\log^{1/2}\Big(\frac{v}{u}\Big)\right) + \psi(u) + \frac{1}{\sqrt{8}}\left(\frac{v^2}{u^2} + u^2\log(u^2)\right)\\ &\le c\left(\frac{v^2}{u^2} + 1\right) + \frac{1}{4}u^2(2\log(u^2) - 1) + \frac{1}{4} + \frac{1}{\sqrt{8}}\left(\frac{v^2}{u^2} + u^2\log(u^2)\right)\\ &\le c\left(\frac{v^2}{u^2} + u^2\log(u^2) + 1\right)\,,\end{aligned} \tag{54}$$

where we used calculation rules for the logarithm, properties of the square root under the additional observation that $\log(v/u) \ge 0$, the LEGENDRE–FENCHEL–YOUNG inequality as well as the standard YOUNG's inequality, and (53) as well as the definition of ψ. Finally, integrating (54) over $\Omega \times (0,T)$, we observe that the right-hand side is bounded due to (45) and (46). From the left-hand side, we deduce with the bound (46) that

$$\int_0^T\int_\Omega |\partial_t\varphi_\varepsilon|\log^{1/2}\big(1 + |\partial_t\varphi_\varepsilon|\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le c. \tag{55}$$

Collecting the above estimates and comparing terms in the equality (41), we also obtain an estimate for the time derivatives of $\log\theta_\varepsilon$, *i.e.*,

$$\|\partial_t\log\theta_\varepsilon\|_{L^1(0,T;(W^{1,p}(\Omega))^*)} \le c \quad \text{for } p > 3\,.$$

3.4 Convergence of the approximate solutions

The *a priori* estimates deduced above allow us to infer that there exists a (nonrelabelled) subsequence of $\varepsilon \searrow 0$ such that the following convergence relations hold:

$$\varphi_\varepsilon \overset{*}{\rightharpoonup} \varphi \quad \text{in } L^\infty(0,T;H^1(\Omega))\,, \tag{56}$$

$$\log\theta_\varepsilon \rightharpoonup \ell \quad \text{in } L^2(0,T;H^1(\Omega))\,, \tag{57}$$

$$\partial_t \log\theta_\varepsilon \overset{*}{\rightharpoonup} \partial_t\ell \quad \text{in } \mathcal{M}([0,T];(W^{1,p}(\Omega))^*) \text{ for } p>3\,, \tag{58}$$

where (57) comes from combining the estimates of the first two quantities in (45).

By estimate (55), the family $\{\partial_t\varphi_\varepsilon\}$ is uniformly integrable. Hence, by the DUNFORD-PETTIS theorem (*cf.* [11, Thms. 21, 22]) we have

$$\partial_t\varphi_\varepsilon \rightharpoonup \partial_t\varphi \quad \text{in } L^1(0,T;L^1(\Omega))\,. \tag{59}$$

Then, the generalized LIONS-AUBIN Lemma (*cf.* [42, Cor. 7.9] or [27, Thm. 3.19]) allows us to extract strongly converging subsequences:

$$\varphi_\varepsilon \to \varphi \quad \text{in } L^2(0,T;L^2(\Omega))\,, \tag{60}$$

$$\log\theta_\varepsilon \to \ell \quad \text{in } L^2(0,T;L^2(\Omega))\,. \tag{61}$$

In particular, up to extracting further subsequences, $\{\varphi_\varepsilon\}$ and $\{\log\theta_\varepsilon\}$ converge to φ and ℓ pointwise a.e. in $\Omega\times(0,T)$. Using estimates (44) and (46), we obtain that the families $\{F'(\varphi_\varepsilon)\}$ and $\{\theta_\varepsilon\}$ are also uniformly integrable. Hence, noting that $F'(\varphi_\varepsilon)\to F'(\varphi)$ pointwise a.e. in $\Omega\times(0,T)$ by continuity of F' and that $\theta_\varepsilon\to e^\ell =: \theta$ pointwise a.e. in $\Omega\times(0,T)$, and applying VITALI's theorem (*cf.* [47]), we obtain

$$\theta_\varepsilon \to \theta \quad \text{in } L^1(0,T;L^1(\Omega))\,, \tag{62}$$

$$F'(\varphi_\varepsilon) \to F'(\varphi) \quad \text{in } L^1(0,T;L^1(\Omega)) \tag{63}$$

and in particular we identify $\ell = \log\theta$ in (57)-(58).

Comparing terms in (32b), we obtain that also $\{\Delta\varphi_\varepsilon\}$ is uniformly integrable. Hence,

$$\Delta\varphi_\varepsilon \rightharpoonup \Delta\varphi \quad \text{in } L^1(0,T;L^1(\Omega))\,. \tag{64}$$

We can thus pass to the limit in all terms in (32b) and attain the equation (20).

In the next step, we want to pass to the limit in the energy equality (40). First, we observe that $\varepsilon\theta^p\geq 0$. Hence, removing that term we get an inequality at the approximate level.

From (56) and (59) we deduce that

$$\varphi_\varepsilon \to \varphi \quad \text{in } \mathcal{C}_{\mathsf{w}}([0,T];H^1(\Omega))\,, \tag{65}$$

i.e., the sequence is converging pointwise in $[0,T]$ with respect to the weak topology in $L^2(\Omega)$. The weakly-lower semi-continuity of the $L^2(\Omega)$-norm grants that, for every $t\in[0,T]$,

$$\|\nabla\varphi(t)\|^2_{L^2(\Omega)} \leq \liminf_{\varepsilon>0} \|\nabla\varphi_\varepsilon(t)\|^2_{L^2(\Omega)}\,.$$

Finally, using pointwise (a.e.) convergence of $\{\log\theta_\varepsilon\}$ and $\{\varphi_\varepsilon\}$ as well as FATOU's lemma, we find that

$$\int_\Omega F(\varphi(t)) + \theta(t)\,\mathrm{d}\boldsymbol{x} \leq \liminf_{\varepsilon\to 0}\int_\Omega F(\varphi_\varepsilon(t)) + \theta_\varepsilon(t)\,\mathrm{d}\boldsymbol{x} \quad \text{a.e. in } (0,T)\,.$$

Therefore, the limit fulfills the energy inequality (22).

It remains to pass to the limit in the entropy inequality (41) for $\vartheta \geq 0$. To remove the regularizing term, we observe that

$$\int_0^t \int_\Omega \varepsilon \theta_\varepsilon^{p-1} \vartheta \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t \leq \varepsilon^{1/p} \left(\varepsilon^{1/p} \|\theta_\varepsilon\|_{L^p(\Omega\times(0,T))} \right)^{p-1} \|\vartheta\|_{L^p(\Omega\times(0,T))} \to 0$$

as $\varepsilon \to 0$, where again the bound (42) is used. Additionally, combining estimate (45) with the pointwise convergence resulting from (62) and (59), we infer

$$\frac{\partial_t \varphi_\varepsilon}{\sqrt{\theta_\varepsilon}} \rightharpoonup \frac{\partial_t \varphi}{\sqrt{\theta}} \quad \text{in } L^2(0,T;L^2(\Omega))\,.$$

Considering now the equality (41), we find from the two previous convergences, (57) and the weak lower semicontinuity of convex functions (*cf.* [23, Thm. 10.20] or [28]) that

$$\liminf_{\varepsilon\to 0} \int_0^t \int_\Omega \kappa |\nabla \log \theta_\varepsilon|^2 \vartheta + \left| \frac{\partial_t \varphi_\varepsilon}{\sqrt{\theta_\varepsilon}} \right|^2 \vartheta - \varepsilon \theta_\varepsilon^{p-1} \vartheta \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}s \geq \int_0^t \int_\Omega \kappa |\nabla \log \theta|^2 \vartheta + \left| \frac{\partial_t \varphi}{\sqrt{\theta}} \right|^2 \vartheta \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}s\,, \tag{66}$$

where, from now on, we consider $\vartheta \in C^0([0,T]\times\overline{\Omega}) \cap H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ such that $\vartheta(t,\boldsymbol{x}) \geq 0$ for every $t \in [0,T]$ and $\boldsymbol{x} \in \overline{\Omega}$. Then, the convergence (57) allows us to pass to the limit in the second term on the right-hand side of (41):

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_0^t \int_\Omega & \kappa \nabla \log \theta_\varepsilon \cdot \nabla \vartheta - (\log \theta_\varepsilon + \varphi_\varepsilon) \partial_t \vartheta \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}s \\ &= \int_0^t \int_\Omega \kappa \nabla \log \theta \cdot \nabla \vartheta - (\log \theta + \varphi) \partial_t \vartheta \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}s \quad \text{for all } t \in [0,T]\,. \end{aligned} \tag{67}$$

Due to the choice of the approximation of the initial values (below formula (33)) together with (23a) as well as (23b), also the first term on the right-hand side of (41) converges appropriately.

From (65), we find that

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi_\varepsilon(t) \vartheta(t) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \varphi(t) \vartheta(t) \,\mathrm{d}\boldsymbol{x} \quad \text{for all } t \in (0,T)\,. \tag{68}$$

Finally, we consider the first term in (41). Decomposing the logarithm in its positive and negative part,

$$\log \theta_\varepsilon = \log_+ \theta_\varepsilon - \log_- \theta_\varepsilon\,,$$

we may observe that the positive part is bounded in $L^p(\Omega)$ for any $p \in (0,\infty)$. Indeed, by the estimate $y^q \leq c e^y$ for any $q \geq 0$, we find

$$\int_\Omega |\log_+ \theta_\varepsilon(t)|^q \,\mathrm{d}\boldsymbol{x} \leq \int_\Omega c e^{\log_+ \theta_\varepsilon(t)} \,\mathrm{d}\boldsymbol{x} \leq c \int_\Omega \theta_\varepsilon(t) \,\mathrm{d}\boldsymbol{x} \tag{69}$$

for $q \in (0,\infty)$ and a.e. $t \in (0,T)$, where the right-hand side is bounded due to (42). This implies that the sequence $\{\log_+ \theta_\varepsilon(\boldsymbol{x},t)\}$ is weakly compact in $L^1(\Omega)$ for a.e. $t \in (0,T)$ (see for instance [43, Thm. 1.4.5] for different characterizations of weak compactness in L^1) . The Theorem by VITALI (*cf.* [47]) implies by the pointwise (a.e.) strong convergence of $\{\log \theta_\varepsilon\}$ to $\log \theta$ that

$$\log_+ \theta_\varepsilon(t) \to \log_+ \theta(t) \quad \text{in } L^q(\Omega) \text{ for } q \in [1,\infty) \text{ and a.e. } t \in (0,T)\,. \tag{70}$$

From the a.e. pointwise convergence of $\{\log \theta_\varepsilon\}$ to $\log \theta$ and FATOU's Lemma, we deduce from the positivity of $\log_-$ and of ϑ that

$$\int_\Omega \log_- \theta(t) \vartheta(t) \,\mathrm{d}\boldsymbol{x} \leq \liminf_{\varepsilon \searrow 0} \left(\int_\Omega \log_- \theta_\varepsilon(t) \vartheta(t) \,\mathrm{d}\boldsymbol{x} \right) \quad \text{a.e. in } (0,T)\,. \tag{71}$$

Finally, going to the limit in the entropy inequality (41) for $\vartheta \geq 0$ using (66)–(71), we attain relation (19).

Note that the pointwise lower bound on the approximate solutions of **Claim 4** in Sec. 3.2 does not depend on ε in the case of the additional assumption (24). The strong convergence (62) preserves this pointwise lower bound, which results in (25). In this case, the sequence $\{\log\theta_\varepsilon\}$ is bounded in $L^\infty(0,T;L^q(\Omega))$ due to the lower bound and (69). This allows us to infer from convergence (58) and [27, Lemma 2.17] that

$$\log\theta_\varepsilon \overset{*}{\rightharpoonup} \log\theta \quad \text{in } \mathsf{BV}(0,T;(W^{1,p}(\Omega))^*) \cap L^\infty(0,T;L^q(\Omega)) \text{ for } p>3 \text{ and } q \in (1,\infty)\,,$$

which proves regularity (26). From [41, Thm. A.5] with the choices $Y = W^{1,p}(\Omega)$ and $V = L^q(\Omega)$ we also deduce that

$$\log\theta_\varepsilon(t) \rightharpoonup \log\theta(t) \quad \text{in } L^q(\Omega) \quad \text{for } q\in(1,\infty) \text{ and a.e. } t\in(0,T)\,.$$

Moreover, using a generalization of HELLY's theorem (*cf.*, *e.g.*, [36, Thm. 3.1] or [12, Lemma 7.2]) we also obtain

$$\log\theta_\varepsilon(t) \overset{*}{\rightharpoonup} \bar\ell(t) \quad \text{in } (W^{1,p}(\Omega))^* \quad \text{for all } t\in(0,T).$$

It is easy then to check that $\bar\ell$ coincides with $\log\theta$ almost everywhere. Hence, up to changing the representative of $\log\theta$, we may assume that $\bar\ell = \log\theta$ everywhere on $[0,T]$.

4 Weak-strong uniqueness

In this section we prove Theorem 2.7. In order to do that we first introduce a proper *relative energy functional* $\mathcal{E}$ (*cf.* (72)), which plays the role of a *distance* between a solution (θ,φ) and a generic couple $(\tilde\theta,\tilde\varphi)$. Then we estimate in a suitable way the various terms appearing in the expression of $\mathcal{E}$, assuming that $(\tilde\theta,\tilde\varphi)$ is a strong solution of our problem, with the aim of finding a suitable *relative energy inequality* (*cf.* (82)). With proper manipulations, we will then see that the relative energy inequality can be interpreted as a differential inequality to which a Gronwall-type argument (*cf.* (89)) can be applied. As a consequence of that, we will obtain that any weak solution necessarily coincides with a strong solution originating from the same initial data on the existence interval of the latter.

4.1 Relative energy inequality

We use the relative energy approach (see FEIREISL, JIN and NOVOTNÝ [19]) to prove weak-strong uniqueness. In the case of a convex energy functional, this idea goes back to DAFERMOS [10] in the context of thermodynamical systems.

We define the relative energy for system (1) as

$$\mathcal{E}(\theta,\varphi|\tilde\theta,\tilde\varphi) := \frac12\|\nabla\varphi-\nabla\tilde\varphi\|^2_{L^2(\Omega)} + M\|\varphi-\tilde\varphi\|^2_{L^1(\Omega)} - \lambda\|\varphi-\tilde\varphi\|^2_{L^2(\Omega)} \tag{72a}$$

$$+\int_\Omega G(\varphi)-G(\tilde\varphi)-G'(\tilde\varphi)(\varphi-\tilde\varphi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \Lambda(\theta|\tilde\theta)\,\mathrm{d}\boldsymbol{x}\,, \tag{72b}$$

which plays the role of a *distance* between a solution (θ,φ) and a generic couple $(\tilde\theta,\tilde\varphi)$. In (72) we defined for convenience

$$\Lambda(\theta|\tilde\theta) := \theta-\tilde\theta-\tilde\theta(\log\theta-\log\tilde\theta)\,. \tag{73}$$

Moreover, the constant $M>0$ is taken big enough so that, from the GAGLIARDO-NIRENBERG inequality (*cf.* [37, p. 125]) $\|\phi\|_{L^2(\Omega)} \leq c(\|\nabla\phi\|^\alpha_{L^2(\Omega)}\|\phi\|^{(1-\alpha)}_{L^1(\Omega)} + \|\phi\|_{L^1(\Omega)})$ for an $\alpha\in(0,1)$ and the YOUNG inequality, line (72a) can be estimated from below by

$$\mathcal{E}(\theta,\varphi|\tilde\theta,\tilde\varphi) \geq \frac14\|\nabla\varphi-\nabla\tilde\varphi\|^2_{L^2(\Omega)} + \|\varphi-\tilde\varphi\|^2_{L^1(\Omega)}\,. \tag{74}$$

Moreover, due to convexity of G, we may conclude that the first term in line (72b) is nonnegative.

We can also observe that, thanks to definition (13),

$$\begin{aligned} G(\varphi) - G(\tilde\varphi) - G'(\tilde\varphi)(\varphi-\tilde\varphi) &= F(\varphi) - F(\tilde\varphi) - F'(\tilde\varphi)(\varphi-\tilde\varphi) \\ &\quad + \lambda(|\varphi|^2 - |\tilde\varphi|^2 - 2\tilde\varphi(\varphi-\tilde\varphi)) \\ &= F(\varphi) - F(\tilde\varphi) - F'(\tilde\varphi)(\varphi-\tilde\varphi) + \lambda|\varphi-\tilde\varphi|^2 \,. \end{aligned}$$

Finally, we note that also the second term in (72b) (*cf.* (73)) is nonnegative thanks to convexity of the exponential.

Hence, regrouping of some terms in (72) gives

$$\begin{aligned} \mathcal{E}(\theta,\varphi|\tilde\theta,\tilde\varphi) = {}& \int_\Omega \frac12|\nabla\varphi|^2 + F(\varphi) + \theta \,\mathrm{d}\boldsymbol{x} - \int_\Omega \frac12 |\nabla\tilde\varphi|^2 + F(\tilde\varphi) + \tilde\theta\,\mathrm{d}\boldsymbol{x} \\ &+ \int_\Omega \nabla\tilde\varphi\cdot(\nabla\tilde\varphi - \nabla\varphi) + F'(\tilde\varphi)(\tilde\varphi-\varphi)\,\mathrm{d}\boldsymbol{x} \qquad (75a) \\ &- \int_\Omega \tilde\theta(\log\theta+\varphi) - \tilde\theta(\log\tilde\theta+\tilde\varphi)\,\mathrm{d}\boldsymbol{x} \\ &+ \int_\Omega \tilde\theta(\varphi-\tilde\varphi)\,\mathrm{d}\boldsymbol{x} + M\|\varphi-\tilde\varphi\|^2_{L^1(\Omega)}\,. \qquad (75b) \end{aligned}$$

Then, we assume that $(\tilde\theta,\tilde\varphi)$ is a strong solution of our problem with the aim of finding a suitable *relative energy inequality*. First, we observe by the energy inequality (22) for the weak solution and the energy equality ((22) with equality) for the strong solution that

$$\begin{aligned} &\int_\Omega \frac12|\nabla\varphi(t)|^2 + F(\varphi(t)) + \theta(t)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \frac12|\nabla\tilde\varphi(t)|^2 + F(\tilde\varphi(t)) + \tilde\theta(t)\,\mathrm{d}\boldsymbol{x} \\ &\qquad \le \int_\Omega \frac12|\nabla\varphi_0|^2 + F(\varphi_0) + \theta_0\,\mathrm{d}\boldsymbol{x} - \int_\Omega \frac12|\nabla\tilde\varphi_0|^2 + F(\tilde\varphi_0) + \tilde\theta_0\,\mathrm{d}\boldsymbol{x}\,. \qquad (76) \end{aligned}$$

Note that the energy equality is valid for the strong solution since its regularity suffices to test the equations (1a) and (1b) by 1 and $\tilde\varphi_t$, respectively. For brevity, we denote the time derivative by the subscript t.

Next, we choose $\vartheta = \tilde\theta$ in (19). Note that this is possible on account of Corollary 2.6 and Remark 2.2.

We then get

$$\begin{aligned} &- \int_\Omega \tilde\theta(\log\theta+\varphi)\,\mathrm{d}\boldsymbol{x}\Big|_0^t + \int_0^t\int_\Omega \tilde\theta\left(\kappa|\nabla\log\theta|^2 + \frac{|\varphi_t|^2}{\theta}\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &\qquad \le \int_0^t\int_\Omega \kappa\nabla\log\theta\cdot\nabla\tilde\theta - \tilde\theta_t(\log\theta+\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,. \qquad (77) \end{aligned}$$

Testing now equation (1a) for the strong solution $(\tilde\theta,\tilde\varphi)$ by 1 and observing that $\tilde\theta_t + \tilde\varphi_t\tilde\theta = \partial_t(\tilde\theta(\log\tilde\theta+\tilde\varphi)) - \tilde\theta_t(\log\tilde\theta+\tilde\varphi)$, we find

$$\int_\Omega \tilde\theta(\log\tilde\theta+\tilde\varphi)\,\mathrm{d}\boldsymbol{x}\Big|_0^t - \int_0^t\int_\Omega |\tilde\varphi_t|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^t\int_\Omega \tilde\theta_t(\log\tilde\theta+\tilde\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,.$$

Note that the diffusive term of (1a) vanished due to the homogeneous NEUMANN boundary conditions (see (1c)). Similarly, we find by testing equation (1a) for the strong solution $(\tilde\theta,\tilde\varphi)$ by $(\theta-\tilde\theta)/\tilde\theta$ that

$$\int_0^t\int_\Omega (\theta-\tilde\theta)\left(\kappa|\nabla\log\tilde\theta|^2 + \kappa\Delta\log\tilde\theta + \frac{|\tilde\varphi_t|^2}{\tilde\theta}\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^t\int_\Omega (\theta-\tilde\theta)\partial_t(\log\tilde\theta+\tilde\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,.$$

Adding the last two equations leads to

$$\int_\Omega \tilde\theta(\log\tilde\theta+\tilde\varphi)\,\mathrm{d}\boldsymbol{x}\Big|_0^t+\int_0^t\int_\Omega(\theta-\tilde\theta)\left(\kappa|\nabla\log\tilde\theta|^2+\kappa\Delta\log\tilde\theta+\frac{|\tilde\varphi_t|^2}{\tilde\theta}\right)-|\tilde\varphi_t|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,t$$
$$=\int_0^t\int_\Omega\tilde\theta_t(\log\tilde\theta+\tilde\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,t+\int_0^t\int_\Omega(\theta-\tilde\theta)\partial_t(\log\tilde\theta+\tilde\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,t\,. \tag{78}$$

Applying the fundamental theorem of calculus, we may infer for the terms in line (75a) that

$$\begin{aligned}&\int_\Omega\big(\nabla\tilde\varphi\cdot(\nabla\tilde\varphi-\nabla\varphi)+F'(\tilde\varphi)(\tilde\varphi-\varphi)\big)\,\mathrm{d}\boldsymbol{x}\Big|_0^t\\&=\int_0^t\int_\Omega\big(\nabla\tilde\varphi_t\cdot(\nabla\tilde\varphi-\nabla\varphi)+F''(\tilde\varphi)\tilde\varphi_t(\tilde\varphi-\varphi)\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s\\&\quad-\int_0^t\int_\Omega(\tilde\varphi_t-\varphi_t)(\Delta\tilde\varphi-F'(\tilde\varphi))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s\,.\end{aligned} \tag{79}$$

Note that the procedure leading to equality (79) is formal, but it could be easily made rigorous by density arguments.

Next, writing (1b) both for the strong solution $(\tilde\theta,\tilde\varphi)$ and for the weak solution (θ,φ) (indeed, the expression (20) in the definition of weak solutions is equivalent), taking the difference, and testing it by $\partial_t\tilde\varphi$, we deduce

$$\int_0^t\int_\Omega\nabla\tilde\varphi_t\cdot(\nabla\tilde\varphi-\nabla\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s$$
$$=\int_0^t\int_\Omega(\tilde\theta-\theta)\tilde\varphi_t-\tilde\varphi_t(F'(\tilde\varphi)-F'(\varphi))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s-\int_0^t\int_\Omega(\tilde\varphi_t-\varphi_t)\tilde\varphi_t\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s\,.$$

Similarly, we find by testing equation (1b) for the strong solution $(\tilde\theta,\tilde\varphi)$ with $(\tilde\varphi_t-\varphi_t)$ that

$$-\int_0^t\int_\Omega(\Delta\tilde\varphi-F'(\tilde\varphi))(\tilde\varphi_t-\varphi_t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s=\int_0^t\int_\Omega\tilde\theta(\tilde\varphi_t-\varphi_t)-\tilde\varphi_t(\tilde\varphi_t-\varphi_t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s\,.$$

Inserting the latter two relations into (79), we may conclude that

$$\begin{aligned}&\int_\Omega\big(\nabla\tilde\varphi\cdot(\nabla\tilde\varphi-\nabla\varphi)+F'(\tilde\varphi)(\tilde\varphi-\varphi)\big)\,\mathrm{d}\boldsymbol{x}\Big|_0^t\\&=\int_0^t\int_\Omega\tilde\varphi_t\big(F''(\tilde\varphi)(\tilde\varphi-\varphi)-F'(\tilde\varphi)+F'(\varphi)\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s\\&\quad+\int_0^t\int_\Omega(\tilde\theta-\theta)\tilde\varphi_t\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s-\int_0^t2\int_\Omega(\tilde\varphi_t-\varphi_t)\tilde\varphi_t-(\tilde\varphi_t-\varphi_t)\tilde\theta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s\,.\end{aligned} \tag{80}$$

To handle the first term in line (75b), we apply again the fundamental theorem of calculus, leading to

$$\int_\Omega\tilde\theta(\varphi-\tilde\varphi)\,\mathrm{d}\boldsymbol{x}\Big|_0^t=\int_0^t\int_\Omega\tilde\theta_t(\varphi-\tilde\varphi)+(\varphi_t-\tilde\varphi_t)\tilde\theta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s\,. \tag{81}$$

Now, we insert (76), (77), (78), (80), and (81) back into (75), which leads to the estimate

$$
\begin{aligned}
&\mathcal{E}(\theta(t),\varphi(t)|\tilde\theta(t),\tilde\varphi(t)) + \int_0^t\int_\Omega \tilde\theta\frac{|\varphi_t|^2}{\theta} + (\theta-\tilde\theta)\frac{|\tilde\varphi_t|^2}{\tilde\theta} - |\tilde\varphi_t|^2 + 2(\tilde\varphi_t-\varphi_t)\tilde\varphi_t \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&+\kappa\int_0^t\int_\Omega \tilde\theta|\nabla\log\theta|^2 - \nabla\log\theta\cdot\nabla\tilde\theta + (\theta-\tilde\theta)|\nabla\log\tilde\theta|^2 + \Delta\log\tilde\theta(\theta-\tilde\theta)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\qquad\le \mathcal{E}(\theta_0,\varphi_0|\tilde\theta_0,\tilde\varphi_0) + \int_0^t\int_\Omega \tilde\varphi_t\left(F''(\tilde\varphi)(\tilde\varphi-\varphi) - F'(\tilde\varphi)+F'(\varphi)\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\qquad\quad - \int_0^t\int_\Omega \tilde\theta_t(\log\theta+\varphi) - \tilde\theta_t(\log\tilde\theta+\tilde\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\qquad\quad + \int_0^t\int_\Omega (\theta-\tilde\theta)\partial_t(\log\tilde\theta+\tilde\varphi) + (\tilde\theta-\theta)\tilde\varphi_t\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\qquad\quad + \int_0^t\int_\Omega (\tilde\varphi_t-\varphi_t)\tilde\theta + \tilde\theta_t(\varphi-\tilde\varphi) + (\varphi_t-\tilde\varphi_t)\tilde\theta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + M\|\varphi(t)-\tilde\varphi(t)\|^2_{L^1(\Omega)}\,.
\end{aligned}
\tag{82}
$$

For the last three lines of the forgoing estimate, we observe due to several cancellations that

$$
\begin{aligned}
&-\int_\Omega \left(\tilde\theta_t(\log\theta+\varphi) - \tilde\theta_t(\log\tilde\theta+\tilde\varphi) - (\theta-\tilde\theta)\partial_t(\log\tilde\theta+\tilde\varphi) - (\tilde\theta-\theta)\tilde\varphi_t\right)\mathrm{d}\boldsymbol{x} \\
&\qquad\qquad + \int_\Omega(\tilde\varphi_t-\varphi_t)\tilde\theta\,\mathrm{d}\boldsymbol{x} + \int_\Omega \tilde\theta_t(\varphi-\tilde\varphi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega(\varphi_t-\tilde\varphi_t)\tilde\theta\,\mathrm{d}\boldsymbol{x} \\
&\qquad = -\int_\Omega \tilde\theta_t(\log\theta+\varphi) - \tilde\theta_t(\log\tilde\theta+\tilde\varphi) - \tilde\theta_t(\varphi-\tilde\varphi)\,\mathrm{d}\boldsymbol{x} \\
&\qquad\quad + \int_\Omega (\theta-\tilde\theta)\partial_t(\log\tilde\theta+\tilde\varphi) + (\tilde\theta-\theta)\tilde\varphi_t\,\mathrm{d}\boldsymbol{x} \\
&\qquad = \int_\Omega \partial_t(\log\tilde\theta)(\theta-\tilde\theta-\tilde\theta(\log\theta-\log\tilde\theta))\,\mathrm{d}\boldsymbol{x} \\
&\qquad = \int_\Omega (\partial_t\log\tilde\theta)\Lambda(\theta|\tilde\theta)\,\mathrm{d}\boldsymbol{x}\,.
\end{aligned}
$$

Here and below, we are actually using notation (73).

4.2 Estimates for the dissipative terms and the nonconvex part

For the dissipative terms due to heat conduction, *i.e.*, the terms in the second line of (82), we observe with some algebraic transformations that

$$
\begin{aligned}
&\tilde\theta|\nabla\log\theta|^2 - \nabla\log\theta\cdot\nabla\tilde\theta + (\theta-\tilde\theta)|\nabla\log\tilde\theta|^2 + \Delta\log\tilde\theta(\theta-\tilde\theta) \\
&\quad = \left(\tilde\theta\nabla\log\theta\cdot(\nabla\log\theta-\nabla\log\tilde\theta) + (\theta-\tilde\theta)|\nabla\log\tilde\theta|^2 + \Delta\log\tilde\theta(\theta-\tilde\theta)\right) \\
&\quad = \tilde\theta\left(|\nabla\log\theta-\nabla\log\tilde\theta|^2 + \nabla\log\tilde\theta\cdot(\nabla\log\theta-\nabla\log\tilde\theta)\right) \\
&\qquad + \left(\theta-\tilde\theta-\tilde\theta(\log\theta-\log\tilde\theta)|\nabla\log\tilde\theta|^2 + \tilde\theta(\log\theta-\log\tilde\theta)|\nabla\log\tilde\theta|^2\right) \\
&\qquad + \Delta\log\tilde\theta(\theta-\tilde\theta-\tilde\theta(\log\theta-\log\tilde\theta)) + \Delta\log\tilde\theta(\tilde\theta(\log\theta-\log\tilde\theta))\,.
\end{aligned}
$$

From an integration-by-parts on the last term, using the fact that $\nabla\log\tilde\theta\cdot\boldsymbol{n} = (\nabla\tilde\theta\cdot\boldsymbol{n})/\tilde\theta = 0$ on the boundary (see (1c)), and the product rule, we may infer

$$
\begin{aligned}
&\int_\Omega \tilde\theta\nabla\log\tilde\theta\cdot(\nabla\log\theta-\nabla\log\tilde\theta) + \tilde\theta(\log\theta-\log\tilde\theta)|\nabla\log\tilde\theta|^2\,\mathrm{d}\boldsymbol{x} \\
&\qquad\qquad - \int_\Omega \nabla\log\tilde\theta\cdot\nabla(\tilde\theta(\log\theta-\log\tilde\theta))\,\mathrm{d}\boldsymbol{x} = 0\,.
\end{aligned}
$$

We may conclude that

$$\int_\Omega \tilde{\theta}|\nabla\log\theta|^2 - \nabla\log\theta\cdot\nabla\tilde{\theta} + (\theta-\tilde{\theta})|\nabla\log\tilde{\theta}|^2 + \Delta\log\tilde{\theta}(\theta-\tilde{\theta})\,\mathrm{d}\boldsymbol{x}$$
$$= \int_\Omega \tilde{\theta}|\nabla\log\theta - \nabla\log\tilde{\theta}|^2\,\mathrm{d}\boldsymbol{x} + \int_\Omega \Lambda(\theta|\tilde{\theta})(|\nabla\log\tilde{\theta}|^2 + \Delta\log\tilde{\theta})\,\mathrm{d}\boldsymbol{x}\,.$$

Thanks to the above manipulations, the estimate (82) may be written as

$$\begin{aligned}
&\mathcal{E}(\theta(t),\varphi(t)|\tilde{\theta}(t),\tilde{\varphi}(t)) + \kappa\int_0^t\int_\Omega \tilde{\theta}|\nabla\log\theta - \nabla\log\tilde{\theta}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&+\int_0^t\int_\Omega \tilde{\theta}\frac{|\varphi_t|^2}{\theta} + (\theta-\tilde{\theta})\frac{|\tilde{\varphi}_t|^2}{\tilde{\theta}} - |\tilde{\varphi}_t|^2 + 2(\tilde{\varphi}_t-\varphi_t)\tilde{\varphi}_t\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&\le \mathcal{E}(\theta_0,\varphi_0|\tilde{\theta}_0,\tilde{\varphi}_0) + \int_0^t\int_\Omega \tilde{\varphi}_t\left(F''(\tilde{\varphi})(\tilde{\varphi}-\varphi) - F'(\tilde{\varphi}) + F'(\varphi)\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&+\int_0^t\int_\Omega (\partial_t(\log\tilde{\theta}) - \kappa|\nabla\log\tilde{\theta}|^2 - \kappa\Delta\log\tilde{\theta})\Lambda(\theta|\tilde{\theta})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&+ M\|\varphi(t)-\tilde{\varphi}(t)\|_{L^1(\Omega)}^2\,.
\end{aligned} \tag{83}$$

For the terms in the second line of the right-hand side of the previous estimate, we find with equation (1a) for the strong solution $(\tilde{\theta},\tilde{\varphi})$

$$(\log\tilde{\theta})_t - \kappa|\nabla\log\tilde{\theta}|^2 - \kappa\Delta\log\tilde{\theta} = \frac{\tilde{\theta}_t - \kappa\Delta\tilde{\theta}}{\tilde{\theta}} = \frac{|\tilde{\varphi}_t|^2 - \tilde{\varphi}_t\tilde{\theta}}{\tilde{\theta}} = \frac{|\tilde{\varphi}_t|^2}{\tilde{\theta}} - \tilde{\varphi}_t\,.$$

Moreover, rearranging the terms in the second line of (83), we obtain

$$\begin{aligned}
&\int_0^t\int_\Omega \tilde{\theta}\frac{|\varphi_t|^2}{\theta} + (\theta-\tilde{\theta})\frac{|\tilde{\varphi}_t|^2}{\tilde{\theta}} - |\tilde{\varphi}_t|^2 + 2(\tilde{\varphi}_t-\varphi_t)\tilde{\varphi}_t\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&= \int_0^t\int_\Omega \tilde{\theta}\frac{|\varphi_t|^2}{\theta} + \theta\frac{|\tilde{\varphi}_t|^2}{\tilde{\theta}} - 2\varphi_t\tilde{\varphi}_t\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\
&= \int_0^t\int_\Omega \left|\sqrt{\frac{\tilde{\theta}}{\theta}}\varphi_t - \sqrt{\frac{\theta}{\tilde{\theta}}}\tilde{\varphi}_t\right|^2\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\,.
\end{aligned} \tag{84}$$

Proposition 4.1. Let (θ,φ) be a weak solution according to Definition 2.2 and let $(\tilde{\theta},\tilde{\varphi})$ be a strong solution according to Theorem 2.5. Then it holds

$$\|\varphi-\tilde{\varphi}\|_{L^1(\Omega)}^2\Big|_0^t \le (4\lambda+2)\int_0^t \|\varphi-\tilde{\varphi}\|_{L^1(\Omega)}^2\,\mathrm{d}s + \int_0^t \|\Lambda(\theta|\tilde{\theta})\|_{L^1(\Omega)}^2\,\mathrm{d}s$$
$$+\int_0^t\left(\int_\Omega \tilde{\theta}|\log\theta - \log\tilde{\theta}|\,\mathrm{d}\boldsymbol{x}\right)^2\mathrm{d}s\,,$$

for all $t\in[0,T]$.

Proof. The idea of the proof is to use $\operatorname{sign}(\varphi-\tilde{\varphi})$ as a test function in equation (20) and subtract equation (1b) for the strong solution equally tested with $\operatorname{sign}(\varphi-\tilde{\varphi})$. Since all terms in (20) are elements of $L^1(\Omega\times(0,T))$, this procedure is allowed. Setting $\hat{\varphi}=\varphi-\tilde{\varphi}$, we then find

$$\int_\Omega \partial_t\hat{\varphi}\operatorname{sign}(\hat{\varphi}) - \Delta\hat{\varphi}\operatorname{sign}(\hat{\varphi}) + (F'(\varphi)-F'(\tilde{\varphi}))\operatorname{sign}(\hat{\varphi})\,\mathrm{d}\boldsymbol{x} = \int_\Omega (\theta-\tilde{\theta})\operatorname{sign}(\hat{\varphi})\,\mathrm{d}\boldsymbol{x}\,.$$

Moreover, it is clear that

$$\partial_t\hat{\varphi}\operatorname{sign}(\hat{\varphi}) = \partial_t|\hat{\varphi}|$$

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and the λ-convexity of F guarantees that

$$(F'(\varphi) - F'(\tilde\varphi))\,\mathrm{sign}(\hat\varphi) = (G'(\varphi) - G'(\tilde\varphi))(\mathrm{sign}\,\hat\varphi) - 2\lambda|\hat\varphi| \geq -2\lambda|\hat\varphi|\,.$$

Additionally, proceeding by approximation it is not difficult to show that

$$-\int_\Omega \Delta\hat\varphi\,\mathrm{sign}(\hat\varphi)\,\mathrm{d}\boldsymbol{x} \geq 0\,,$$

where we point out that the boundary conditions (1c) of φ are crucial for this argument.

Collecting the above relations, observing additionally that $|\,\mathrm{sign}(\hat\varphi)| \leq 1$, we find

$$\begin{aligned}\partial_t \int_\Omega |\hat\varphi|\,\mathrm{d}\boldsymbol{x} &\leq 2\lambda \int_\Omega |\hat\varphi|\,\mathrm{d}\boldsymbol{x} + \int_\Omega |\theta - \tilde\theta|\,\mathrm{d}\boldsymbol{x}\\ &\leq 2\lambda \int_\Omega |\hat\varphi|\,\mathrm{d}\boldsymbol{x} + \int_\Omega \Lambda(\theta|\tilde\theta)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \tilde\theta|\log\theta - \log\tilde\theta|\,\mathrm{d}\boldsymbol{x}\,.\end{aligned}$$

Multiplying the above relation by $\|\hat\varphi\|_{L^1(\Omega)}$ and applying YOUNG's inequality, we may conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}\|\hat\varphi\|^2_{L^1(\Omega)} \leq 2(2\lambda+1)\|\hat\varphi\|^2_{L^1(\Omega)} + \left(\int_\Omega \Lambda(\theta|\tilde\theta)\,\mathrm{d}\boldsymbol{x}\right)^2\\ + \left(\int_\Omega \tilde\theta|\log\theta - \log\tilde\theta|\,\mathrm{d}\boldsymbol{x}\right)^2\end{aligned}$$

for a.e. $t \in (0,T)$. Integrating in time provides the assertion. □

Proposition 4.2. Let θ be a weak solution provided by Theorem (2.3) and $\tilde\theta$ a strong solution according to Theorem 2.5, both originating from the same initial data satisfying (27a)-(27b) and (29) and defined over the same time interval $(0,T)$. Then there exists a constant $c > 0$ such that

$$\|\log\theta - \log\tilde\theta\|^2_{L^1(\Omega)} \leq c\int_\Omega \Lambda(\theta|\tilde\theta)\,\mathrm{d}\boldsymbol{x}\,, \tag{85}$$

where c only depends on the given data of the system.

Proof. In view of the second condition (27a), (25) is satisfied both by θ and by $\tilde\theta$. Setting $\eta = \log\theta$ and $\tilde\eta = \log\tilde\theta$, using Taylor's expansion, we can directly compute

$$\Lambda(\theta|\tilde\theta) = e^\eta - e^{\tilde\eta} - e^{\tilde\eta}(\eta - \tilde\eta) = \frac{1}{2}e^\xi(\eta-\tilde\eta)^2, \tag{86}$$

where the above formula holds at a.e. point $(\boldsymbol{x},t) \in \Omega \times (0,T)$ and $\xi = \xi(\boldsymbol{x},t)$ lies between $\eta(\boldsymbol{x},t)$ and $\tilde\eta(\boldsymbol{x},t)$.

In view of (25), there exists $\delta > 0$ depending only on the initial data and on T such that $\theta(\boldsymbol{x},t) \geq \delta$ and $\tilde\theta(\boldsymbol{x},t) \geq \delta$. As a consequence, we also have $\xi \geq \log\delta$. Thus, rewriting (86) in terms of θ and $\tilde\theta$ and integrating in space, we readily obtain (85) (which is stated in term of the L^1-rather than L^2-norm just for later convenience). □

4.3 Estimate of the convex modification

It remains to control the term resulting from the nonlinear potential F, *i.e.*, the second term on the right-hand side of (83):

$$\begin{aligned}&\int_0^t\int_\Omega \tilde\varphi_t(F''(\tilde\varphi)(\tilde\varphi - \varphi) - F'(\tilde\varphi) + F'(\varphi))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s\\ &= \int_0^t\int_\Omega \tilde\varphi_t(G''(\tilde\varphi)(\tilde\varphi - \varphi) - G'(\tilde\varphi) + G'(\varphi))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s,\end{aligned}$$

the equality holding as a consequence of (13). We first observe that, by (28b) and standard Sobolev embeddings, there exists $\overline{\varphi} > 0$ depending only on the data of the problem such that $|\tilde{\varphi}(x,t)| \le \overline{\varphi}$ a.e. in $\Omega \times (0,T)$. For simplicity, we set $\mathcal{R}(\varphi,\tilde{\varphi}) := G''(\tilde{\varphi})(\tilde{\varphi}-\varphi) - G'(\tilde{\varphi}) + G'(\varphi)$ and, for $\overline{M} > \overline{\varphi}$ to be chosen below, we also define

$$\Omega_-(t) := \big\{x \in \Omega : \ |\varphi(t,\boldsymbol{x})| \le \overline{M}\big\}$$

and, correspondingly, $\Omega_+(t) := \Omega \setminus \Omega_-(t)$. It is then clear that for a.e. $t \in (0,T)$ and $\boldsymbol{x} \in \Omega_-(t)$, there holds

$$\begin{aligned}
\big|\mathcal{R}(\varphi,\tilde{\varphi})\big| &= \left| G''(\tilde{\varphi})(\tilde{\varphi}-\varphi) - \int_0^1 G''(\tilde{\varphi} + (1-s)(\varphi-\tilde{\varphi}))\,\mathrm{d}\,s(\tilde{\varphi}-\varphi)\right| \\
&\le \int_0^1 |G''(\tilde{\varphi}) - G''(\tilde{\varphi}+(1-s)(\varphi-\tilde{\varphi}))|\,\mathrm{d}\,s|\tilde{\varphi}-\varphi| \\
&\le \int_0^1 c(\|F\|_{\mathcal{C}^{2,1}}, \overline{\varphi}, \overline{M})|\tilde{\varphi} - \tilde{\varphi} - (1-s)(\varphi-\tilde{\varphi})|\,\mathrm{d}\,s|\tilde{\varphi}-\varphi| \\
&\le \frac{1}{2} c(\|F\|_{\mathcal{C}^{2,1}}, \overline{\varphi}, \overline{M})|\varphi-\tilde{\varphi}|^2 \\
&=: c|\tilde{\varphi}-\varphi|^2,
\end{aligned}$$

the last constant c depending on $\overline{M}$, $\overline{\varphi}$ and the problem data. On the other hand, for $\boldsymbol{x} \in \Omega_+(t)$, thanks to the growth condition (17) of Hypothesis 2.1 we have

$$\begin{aligned}
|\mathcal{R}(\varphi,\tilde{\varphi})| &\le |G''(\tilde{\varphi})||\tilde{\varphi}-\varphi| + |G'(\varphi)| + |G'(\tilde{\varphi})| \\
&\le c(1 + |\varphi-\tilde{\varphi}|^2) + |G'(\varphi)| \\
&\le c(1 + |\varphi-\tilde{\varphi}|^2 + G(\varphi))\,.
\end{aligned}$$

As a consequence of the above argument, we deduce

$$\begin{aligned}
&\int_0^t \int_\Omega |\tilde{\varphi}_t|(G''(\tilde{\varphi})(\tilde{\varphi}-\varphi) - G'(\tilde{\varphi}) + G'(\varphi))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\,s \qquad (87)\\
&\le c\int_0^t \|\tilde{\varphi}_t\|_{L^\infty(\Omega)} \left(\|\tilde{\varphi}-\varphi\|^2_{L^2(\Omega)} + \int_{\Omega_+(s)} G(\varphi)\,\mathrm{d}\boldsymbol{x} + |\Omega_+(s)| \right) \mathrm{d}\,s\,,
\end{aligned}$$

where, recalling (74) and using the GAGLIARDO–NIRENBERG (*cf.* [37, p. 125]) and YOUNG inequalities, the difference in the L^2-norm can be estimated by

$$\|\varphi-\tilde{\varphi}\|^2_{L^2(\Omega)} \le c\left(\|\nabla\varphi - \nabla\tilde{\varphi}\|^2_{L^2(\Omega)} + \|\varphi-\tilde{\varphi}\|^2_{L^1(\Omega)}\right) \le c\mathcal{E}(\theta,\varphi|\tilde{\theta},\tilde{\varphi})\,.$$

Let us also notice that, by Hypothesis 2.1 (see in particular (14)), we can choose $\overline{M}$ so large that $G(r) \ge \lambda r^2/2$ for every $|r| \ge \overline{M}$. As a consequence, for $\boldsymbol{x} \in \Omega_+(t)$ there holds

$$\begin{aligned}
\frac{1}{2}G(\varphi) - G(\tilde{\varphi}) - G'(\tilde{\varphi})(\varphi-\tilde{\varphi}) &\ge \frac{\lambda}{4}\varphi^2 - G(\tilde{\varphi}) + G'(\tilde{\varphi})\tilde{\varphi} - \frac{\lambda}{8}\varphi^2 - \frac{2}{\lambda}G'(\tilde{\varphi})^2 \\
&\ge \frac{\lambda}{8}\varphi^2 - c(\overline{\varphi}) \ge \frac{\lambda}{8}\overline{M}^2 - c(\overline{\varphi}) \ge K(\overline{M},\overline{\varphi},\lambda),
\end{aligned}$$

where the last inequality follows for suitable $K > 0$ by possibly taking a larger value of $\overline{M}$ (still in a way that only depends on $\overline{\varphi}$, hence on the fixed data of the problem). We eventually conclude that

$$\begin{aligned}
\mathcal{E}(\theta,\varphi|\tilde{\theta},\tilde{\varphi})(s) &\ge \int_{\Omega_+(s)} \big(G(\varphi) - G(\tilde{\varphi}) - G'(\tilde{\varphi})(\varphi-\tilde{\varphi})\big)\,\mathrm{d}\boldsymbol{x} \\
&\ge \frac{1}{2}\int_{\Omega_+(s)} G(\varphi)\,\mathrm{d}\boldsymbol{x} + K|\Omega_+(s)|\,,
\end{aligned}$$

so that (87) gives

$$\int_0^t \int_\Omega |\tilde\varphi_t| (G''(\tilde\varphi)(\tilde\varphi - \varphi) - G'(\tilde\varphi) + G'(\varphi)) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}s \le c \int_0^t \|\tilde\varphi_t\|_{L^\infty(\Omega)} \mathcal{E}(\theta, \varphi | \tilde\theta, \tilde\varphi) \, \mathrm{d}s \,. \tag{88}$$

Putting everything together and going back to relation (83), we may observe

$$\begin{aligned}
&\mathcal{E}(\theta(t), \varphi(t) | \tilde\theta(t), \tilde\varphi(t)) + \kappa \int_0^t \int_\Omega \tilde\theta |\nabla \log\theta - \nabla \log\tilde\theta|^2 + \left| \sqrt{\frac{\tilde\theta}{\theta}} \varphi_t - \sqrt{\frac{\theta}{\tilde\theta}} \tilde\varphi_t \right|^2 \mathrm{d}\boldsymbol{x} \, \mathrm{d}s \\
&\le \mathcal{E}(\theta_0, \varphi_0 | \tilde\theta_0, \tilde\varphi_0) + c \int_0^t \|\tilde\varphi_t\|_{L^\infty(\Omega)} \mathcal{E}(\theta, \varphi | \tilde\theta, \tilde\varphi) + \left(\left\| \frac{|\tilde\varphi_t|^2}{\tilde\theta} \right\|_{L^\infty(\Omega)} + \|\tilde\varphi_t\|_{L^\infty(\Omega)} \right) \Lambda(\theta | \tilde\theta) \, \mathrm{d}s \\
&\quad + M \int_0^t (4\lambda + 2) \|\varphi - \tilde\varphi\|^2_{L^1(\Omega)} + \|\Lambda(\theta|\tilde\theta)\|^2_{L^1(\Omega)} + \|\tilde\theta\|_{L^\infty(\Omega)} \|\log\theta - \log\tilde\theta\|^2_{L^1(\Omega)} \, \mathrm{d}s \,.
\end{aligned}$$

We point out that $\mathcal{E}(\theta, \varphi | \tilde\theta, \tilde\varphi)$ is bounded in $L^\infty(0,T)$ by a constant only depending on the given data of the system. Moreover, by (30), $\|\tilde\theta\|_{L^\infty(\Omega)}$ is controlled uniformly in time. This in turn also holds for $\|\Lambda(\theta|\tilde\theta)\|_{L^1(\Omega)}$. Hence we may arrive with Proposition 4.2 at

$$\mathcal{E}(\theta(t), \varphi(t) | \tilde\theta(t), \tilde\varphi(t)) + \int_0^t \mathcal{W}(\theta, \varphi | \tilde\theta, \tilde\varphi) \, \mathrm{d}s \le \mathcal{E}(\theta_0, \varphi_0 | \tilde\theta_0, \tilde\varphi_0) + c \int_0^t \mathcal{K}(\tilde\theta, \tilde\varphi) \mathcal{E}(\theta, \varphi | \tilde\theta, \tilde\varphi) \, \mathrm{d}s \,,$$

where we have defined

$$\mathcal{W}(\theta, \varphi | \tilde\theta, \tilde\varphi) = \int_\Omega \left(\kappa\tilde\theta |\nabla \log\theta - \nabla \log\tilde\theta|^2 + \left| \sqrt{\frac{\tilde\theta}{\theta}} \varphi_t - \sqrt{\frac{\theta}{\tilde\theta}} \tilde\varphi_t \right|^2 \right) \mathrm{d}\boldsymbol{x}$$

and

$$\mathcal{K}(\tilde\theta, \tilde\varphi) = c \left(\|\tilde\varphi_t(s)\|_{L^\infty(\Omega)} + \left\| \frac{|\tilde\varphi_t(s)|^2}{\tilde\theta(s)} \right\|_{L^\infty(\Omega)} + 1 \right) .$$

Applying GRONWALLâĂŹs inequality, we conclude that

$$\begin{aligned}
&\mathcal{E}(\theta(t), \varphi(t) | \tilde\theta(t), \tilde\varphi(t)) + \int_0^t \mathcal{W}(\theta(s), \varphi(s) | \tilde\theta(s), \tilde\varphi(s)) e^{\int_s^t \mathcal{K}(\tilde\theta(\tau), \tilde\varphi(\tau)) \, \mathrm{d}\tau} \, \mathrm{d}s \\
&\qquad\qquad \le \mathcal{E}(\theta_0, \varphi_0 | \tilde\theta_0, \tilde\varphi_0) e^{\int_0^t \mathcal{K}(\tilde\theta(s), \tilde\varphi(s)) \, \mathrm{d}s} \,.
\end{aligned} \tag{89}$$

The above estimate concludes the proof of Theorem 2.7.

Remark 4.1. It is worth noting that the previous estimate (89) could also be adapted to provide a result on the continuous dependence of the weak solution on the initial datum, holding as long as a strong solution exists.

Remark 4.2. In the case when the polynomial growth condition (16) fails but F has at most exponential growth (in such a way that $|F'|$ is somehow controlled by $|F|$, so excluding the so-called “singular” potentials), it may be still possible to prove existence of some notion of weak solution. However one expects the occurrence of defect measures in the phase field equation because the uniform integrability estimates for θ, φ_t and $F'(\varphi)$ are now expected to fail. On the other hand, strong positivity (25) of θ is still likely holding under the additional assumption (24), but it should be reinterpreted in the sense of measures. Note, however, that in such a setting, the interpretation of the term φ_t^2/θ in the (weak analogue of the) entropy inequality (19) may be troublesome because both φ_t and θ are now measure-valued objects. Extending weak-strong uniqueness to this weakened regularity framework seems also a nontrivial issue; indeed, with the occurrence of defect measures, several terms we could treat by taking advantage of L^1-regularity may become difficult to be managed.

5 Local strong solutions

In what follows, we focus on the proof of Theorem 2.5.

To prove local solvability, we derive some local-in-time estimates. In order to avoid technical complications, we work directly on system (1a)-(1b). It is however easy to check that the argument could be reproduced and made fully rigorous by working on the regularized system (32). In the sequel, for notational simplicity, we will write, for instance, $\|\cdot\|_{L^2}$ in place of $\|\cdot\|_{L^2(\Omega)}$.

Differentiating (1b) in time leads to

$$\varphi_{tt} - \Delta\varphi_t + F''(\varphi)\varphi_t = \theta_t \,. \tag{90}$$

Testing (90) by $-\Delta\varphi_t + \varphi_t$ provides

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\varphi_t\|_{H^1}^2 + \|\Delta\varphi_t\|_{L^2}^2 + \|\nabla\varphi_t\|_{L^2}^2 &= \int_\Omega F''(\varphi)\varphi_t(\Delta\varphi_t - \varphi_t)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \theta_t(\Delta\varphi_t - \varphi_t)\,\mathrm{d}\boldsymbol{x}\\
&= -\int_\Omega F''(\varphi)\left(|\nabla\varphi_t|^2 + |\varphi_t|^2\right) + F'''(\varphi)\varphi_t\nabla\varphi\cdot\nabla\varphi_t\,\mathrm{d}\boldsymbol{x} - \int_\Omega \theta_t(\Delta\varphi_t - \varphi_t)\,\mathrm{d}\boldsymbol{x}\\
&\le \lambda\|\varphi_t\|_{H^1}^2 + \int_\Omega F'''(\varphi)\varphi_t\nabla\varphi\cdot\nabla\varphi_t\,\mathrm{d}\boldsymbol{x} + \frac{9}{16}\|\theta_t\|_{L^2}^2 + \frac{1}{2}\|\Delta\varphi_t\|_{L^2}^2 + 4\|\varphi_t\|_{L^2}^2 \,.
\end{aligned} \tag{91}$$

In the above formula we used the λ-convexity of F (see Hypothesis 2.1) together with HÖLDER's and YOUNG's inequalities.

Testing now equation (1a) with θ_t, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\kappa}{2}\|\nabla\theta\|_{L^2}^2 + \|\theta_t\|_{L^2}^2 &= \int_\Omega |\varphi_t|^2\theta_t - \theta\theta_t\varphi_t\,\mathrm{d}\boldsymbol{x}\\
&\le \|\varphi_t\|_{L^4}^2\|\theta_t\|_{L^2} + \|\theta_t\|_{L^2}\|\theta\|_{L^4}\|\varphi_t\|_{L^4}\\
&\le \frac{1}{8}\|\theta_t\|_{L^2}^2 + 6\|\varphi_t\|_{L^4}^4 + 2\|\theta\|_{L^4}^4 \,,
\end{aligned} \tag{92}$$

where we used again HÖLDER's and YOUNG's inequalities. With the chain rule, HÖLDER's and YOUNG's inequalities, we observe the estimates

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\Delta\varphi\|_{L^2}^2 &= \int_\Omega \Delta\varphi_t\Delta\varphi\,\mathrm{d}\boldsymbol{x} \le \frac{1}{4}\|\Delta\varphi_t\|_{L^2}^2 + \|\Delta\varphi\|_{L^2}^2 \,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\varphi\|_{L^2}^2 &= \int_\Omega \varphi_t\varphi\,\mathrm{d}\boldsymbol{x} \le \frac{1}{2}\|\varphi_t\|_{L^2}^2 + \frac{1}{2}\|\varphi\|_{L^2}^2 \,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\kappa}{2}\|\theta\|_{L^2}^2 &= \kappa\int_\Omega \theta_t\theta\,\mathrm{d}\boldsymbol{x} \le \frac{1}{16}\|\theta_t\|_{L^2}^2 + 4\kappa^2\|\theta\|_{L^2}^2 \,.
\end{aligned} \tag{93}$$

Adding (91), (92), and (93) and using on $H^2_{\boldsymbol{n}}$ the norm $\|\cdot\|_{\tilde{H}^2}^2 := \|\cdot\|_{L^2}^2 + \|\Delta\cdot\|_{L^2}^2$, which is equivalent to the standard H^2-norm as far as functions in $H^2_{\boldsymbol{n}}$ are considered, we obtain the inequality

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left(\|\varphi_t\|_{H^1}^2 + \kappa\|\theta\|_{H^1}^2 + \|\varphi\|_{\tilde{H}^2}^2\right) + \frac{1}{4}\left(\|\Delta\varphi_t\|_{L^2}^2 + \|\theta_t\|_{L^2}^2\right)\\
&\quad\le c\left(\|\varphi_t\|_{H^1}^2 + \|\varphi\|_{\tilde{H}^2}^2 + \|\theta\|_{L^2}^2\right) + \int_\Omega F'''(\varphi)\varphi_t\nabla\varphi\cdot\nabla\varphi_t\,\mathrm{d}\boldsymbol{x} + 6\|\varphi_t\|_{L^4}^4 + 2\|\theta\|_{L^4}^4 \,.
\end{aligned} \tag{94}$$

For the term including the nonconvex potential, we observe that

$$\begin{aligned}
\int_\Omega F'''(\varphi)\varphi_t\nabla\varphi\cdot\nabla\varphi_t\,\mathrm{d}\boldsymbol{x} &\le \|\nabla\varphi_t\|_{L^2}\|\nabla\varphi\|_{L^3}\|\varphi_t\|_{L^6}\|F'''(\varphi)\|_{L^\infty}\\
&\le c\|\varphi_t\|_{H^1}^2\|\nabla\varphi\|_{L^3}\left(\max_{s\in[-\|\varphi\|_{L^\infty},\|\varphi\|_{L^\infty}]}|F'''(s)|\right)\\
&\le c\|\varphi_t\|_{H^1}^2 Q(\|\varphi\|_{\tilde{H}^2}).
\end{aligned}$$

Here and below, $Q : [0, \infty) \to [0, \infty)$ denotes a computable, continuous and increasingly monotone function whose expression may vary on occurrence. Here, in the specific, we used the condition that F''' is continuous (see Hypothesis 2.4) and the continuity of the embedding of H^2 into L^∞.

Defining now $\xi(t) = \frac{1}{2}\big(\|\varphi_t\|^2_{H^1} + \kappa\|\theta\|^2_{H^1} + \|\varphi\|^2_{H^2}\big)$, we find with the continuous embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ that the inequality

$$\frac{\mathsf{d}}{\mathsf{d}\,t}\xi(t) \leq c\,(1 + Q(\xi(t))) \tag{95}$$

holds for $Q : [0, \infty) \to [0, \infty)$ with the properties specified above. Then, a simple application of the comparison principle for ODE's guarantees the existence of $T^* > 0$ and $C_0 > 0$ such that

$$\|\xi\|_{L^\infty(0,T^*)} \leq C_0. \tag{96}$$

We used here conditions (27a)-(27b) on the initial data. Indeed, it is not difficult to verify that the finiteness of the initial value $\xi|_{t=0}$ corresponds exactly to the regularity of θ_0, φ_0 and $\varphi_1 = \varphi_t(0)$ specified in (27a)-(27b). Then, using (96) and subsequently integrating (94) over $(0, T^*)$, we deduce the properties (28a)-(28b), with the exception of the regularity condition $\theta \in L^2(0, T^*; H^2(\Omega))$. The latter, however, can be inferred *a posteriori* by comparing terms in (1a) and applying standard elliptic regularity results. Note, finally, that the energy equality (22) is valid for the strong solution with equality sign because its regularity suffices to test the equations (1a) and (1b) by 1 and φ_t, respectively. This concludes the proof of Theorem 2.5.

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