Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Existence and uniqueness of solution for multidimensional parabolic PDAEs arising in semiconductor modeling

Giuseppe Alì[1], Nella Rotundo[2]

submitted: July 12, 2019

[1] Università della Calabria
Dipartimento di Fisica
Ponte P. Bucci 30B I-87036
Arcavacata di Rende (CS)
Italy
E-Mail: giuseppe.ali@unical.it

[2] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: nella.rotundo@wias-berlin.de

No. 2607
Berlin 2019

2010 *Mathematics Subject Classification.* 35K40, 35K55, 35K51, 82D37, 94C05.

Key words and phrases. Existence and uniqueness of solution, multidimensional parabolic PDAEs, network and semiconductor coupling model.

N. R. acknowledges support by DFG via SFB 787 Semiconductor Nanophotonics: Materials, Models, Devices, project B4 "Multi-dimensional Modeling and Simulation of electrically pumped semiconductor-based Emitters".

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Existence and uniqueness of solution for multidimensional parabolic PDAEs arising in semiconductor modeling

Giuseppe Alì, Nella Rotundo

Abstract

This paper concerns with a compact network model combined with distributed models for semiconductor devices. For linear RLC networks containing distributed semiconductor devices, we construct a mathematical model that joins the differential-algebraic initial value problem for the electric circuit with multi-dimensional parabolic-elliptic boundary value problems for the devices. We prove an existence and uniqueness result, and the asymptotic behavior of this mixed initial boundary value problem of partial differential-algebraic equations.

1 Introduction

The main ingredient used in circuit simulation is the lumped network equations for the simulation of the network designs. The application of the Modified Nodal Analysis (MNA) formalism yields a system of differential-algebraic equations (DAEs), that can be classified using the index concept [1]. The index determines the number of inherent derivatives that are needed to derive the ordinary differential equation. Different index cases can only be distinguished by structural means for the classical MNA equations. Using for instance the tractability index [2], one decomposes the set of variables accordingly and projects parts of the equations.

The transition from microelectronics to nano-electronics requires a more systematic study of the coupling effects. These effects are particularly relevant in integrated circuit modeling, and their relevance increases with the decreasing of the scales. Relevant examples are, e.g., electrothermal coupling [3–5], electromagnetic coupling [6, 7], or electric network-device coupling [8–10].

In the modeling of the coupling effects there is a basic set of “lumped” differential-algebraic equations, generally the electric network equations, and a set of “distributed” partial differential equations, which model phenomena which arise at a different, finer scale. The coupling involves an interplay between integrated quantities coming from the distributed model, which enter the lumped equations, and lumped variables which are related to boundary data for the distributed equations. This leads directly to coupled systems of differential-algebraic equations (DAEs) for the electric network and partial differential equations (PDEs) for the semiconductor devices. The coupling has two parts. On the one hand, an additional source term occurs in the current balance of the electric network. On the other hand, the boundary conditions of the device equations depend on the time-dependent node potentials, which are genuine unknowns of the electric network.

In this paper, we focus on the electric network-device coupling. We consider an electric network which contains semiconductor devices, modeled by multi-dimensional, parabolic, drift-diffusion equations.

The coupling of the drift-diffusion model with the electric network equations have been extensively studied. The well-posedness of the resulting coupled system has been investigated for the steady-state (elliptic) one-dimensional case in [3, 8] and in the index-1 and index-2 multidimensional case

DOI 10.20347/WIAS.PREPRINT.2607

respectively in [11] and [12]. The uniqueness for data close to equilibrium in the steady-state (elliptic) one-dimensional case is proved in [13]. In [14] is given a systematical approach for the decomposition in case of index grater than 2.

Other fundamental works in the direction of the study of coupled model within the framework of abstract differential-algebraic equations are [2, 15, 16]. The more recent publication [17] summarizes the state of the art of such coupled models in the simulation of electric circuits.

The present paper is strongly related to [9], where the authors prove existence and uniqueness result for the time-dependent (parabolic–elliptic) one-dimensional case. We establish existence and uniqueness of solution for an index-1 parabolic partial differential-algebraic equation.

The work is organized as follows. Section 2 covers the modeling of the coupled system; both subsystems are described in detail and the coupling terms are defined. In Section 3 we state the main result. First, we prove the local existence and uniqueness of solution and then using some a priori estimates we extend the solution globally. Section 4. concerns with the asymptotic behaviour of the solution when time tends to infinity.

2 Coupled circuit-device model

We consider electric networks which include some components described by distributed equations. The specific application we have in mind is a model for an integrated circuit with semiconductor devices. Nevertheless, this model is susceptible of different generalizations and extensions.

In this section we present the general coupled model, postponing to a later section the clarification of the needed mathematical assumptions.

An electric network is described by the electrical potentials at the nodes and by the currents through the branches. Using the approach of Modified Nodal Analysis (MNA) [18, 19], the electric network equations can be obtained by the Kirchhoff current law, replacing the constitutive equations for the currents through branches with capacitors and resistances, and by the constitutive equations for the remaining components.

2.1 Network models for electric circuits.

We consider a linear RLC network, that is, a network which connects n_C linear capacitors, n_L inductors and n_R resistors, and n_I independent voltage and n_V current sources. We assume that the network connects also semiconductor devices connected with circuit by Ohmic contacts. We assume that the network has m nodes plus the ground node, where the potential is zero. We denote by $\boldsymbol{u}(t) \in \mathbb{R}^m$ the node potentials, by $\boldsymbol{i}_L(t) \in \mathbb{R}^{n_L}$ the currents through inductors, by $\boldsymbol{i}_V(t) \in \mathbb{R}^{n_V}$ the currents through voltage sources, by $\boldsymbol{v}_V(t) \in \mathbb{R}^{n_V}$ the independent voltage sources, and by $\boldsymbol{i}_I(t) \in \mathbb{R}^{n_\mathrm{i}}$ the independent current sources. The MNA system of equations can be written as

$$\boldsymbol{A}_C \boldsymbol{C} \boldsymbol{A}_C^\top \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} + \boldsymbol{A}_R \boldsymbol{G} \boldsymbol{A}_R^\top \boldsymbol{u} + \boldsymbol{A}_L \boldsymbol{i}_L + \boldsymbol{A}_V \boldsymbol{i}_V + \boldsymbol{\ell}_D(\boldsymbol{x}) = \boldsymbol{A}_I \boldsymbol{i}_I(t), \tag{1a}$$

$$\boldsymbol{L}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{i}_L - \boldsymbol{A}_L^\top \boldsymbol{u} = \boldsymbol{O}, \tag{1b}$$

$$-\boldsymbol{A}_V^\top \boldsymbol{u} = \boldsymbol{v}_V(t), \tag{1c}$$

or in a compact form

$$\boldsymbol{M}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}+\boldsymbol{N}\boldsymbol{x}+\boldsymbol{s}_D(\boldsymbol{x})=\boldsymbol{s}(t). \tag{2}$$

which is a differential-algebraic equation for the unknown

$$\boldsymbol{x}=\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{i}_L\\ \boldsymbol{i}_V\end{bmatrix}\in\mathbb{R}^{n_{\mathsf{o}}},\quad n_{\mathsf{o}}=m+n_L+n_V.$$

The matrices in (2) are given by

$$\boldsymbol{M}=\begin{bmatrix}\boldsymbol{A}_C\boldsymbol{C}\boldsymbol{A}_C^\top & \boldsymbol{O} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{L} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O}\end{bmatrix},\quad \boldsymbol{N}=\begin{bmatrix}\boldsymbol{A}_R\boldsymbol{G}\boldsymbol{A}_R^\top & \boldsymbol{A}_L & \boldsymbol{A}_V\\ -\boldsymbol{A}_L^\top & \boldsymbol{O} & \boldsymbol{O}\\ -\boldsymbol{A}_V^\top & \boldsymbol{O} & \boldsymbol{O}\end{bmatrix},\quad \boldsymbol{s}(t)=\begin{bmatrix}\boldsymbol{A}_I\boldsymbol{i}_I(t)\\ \boldsymbol{O}\\ \boldsymbol{v}_V(t)\end{bmatrix}.$$

Here, $\boldsymbol{A}_C\in\mathbb{R}^{m\times n_C}$, $\boldsymbol{A}_R\in\mathbb{R}^{m\times n_R}$, $\boldsymbol{A}_L\in\mathbb{R}^{m\times n_L}$, $\boldsymbol{A}_V\in\mathbb{R}^{m\times n_V}$, $\boldsymbol{A}_I\in\mathbb{R}^{m\times n_I}$, are incidence matrices, which describe the topology of the network. Moreover $\boldsymbol{C}\in\mathbb{R}^{n_C\times n_C}$, $\boldsymbol{G}\in\mathbb{R}^{n_G\times n_G}$ and $\boldsymbol{L}\in\mathbb{R}^{n_L\times n_L}$ denote the capacitance matrix, the conductance matrix and the inductance matrix, respectively, which are regular.

The term $\boldsymbol{s}_D(\boldsymbol{x})$ in (2) represents the coupling with semiconductor devices and it will be described in details later on. It is related to the equation (1a) through the relation

$$\boldsymbol{s}_D(\boldsymbol{x})=\boldsymbol{\pi}^\top\boldsymbol{\ell}_D(\boldsymbol{\pi}\boldsymbol{x}),$$

where $\boldsymbol{\pi}=\begin{bmatrix}\boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O}\end{bmatrix}\in\mathbb{R}^{m\times n_{\mathsf{o}}}$ is a projection matrix which selects the first block of the circuit unknown $\boldsymbol{x}$, so that $\boldsymbol{u}=\boldsymbol{\pi}\boldsymbol{x}$.

We supplement equation (2) with consistent initial data

$$\boldsymbol{x}(t_0)=\boldsymbol{x}_0, \tag{3}$$

this consistency will be discussed in the next section.

2.2 The d-dimensional diode model.

For simplicity we consider a network containing only one semiconductor device. The case of circuits with many devices can be dealt using the same arguments, but the notation would be much heavier. We consider a semiconductor device with m_D+1 terminals. The device can be modeled by means of a domain $\Omega\subset\mathbb{R}^d$, characterized by a doping profile $C(\mathbf{x})$, with $\mathbf{x}\in\Omega$. For the derivation of the model we refer to the classical literature such as [20] or more recent [21]. We neglect all thermal effects, and assume that two carriers are responsible for the diodeÕs output current, that is, electrons with negative charge $-q$, and holes with positive charge q. The behavior of the device is described in terms of number densities of electrons and holes, denoted by $n(\mathbf{x},t), p(\mathbf{x},t)$, current densities for electrons and holes, denoted by $\boldsymbol{J}_n(\mathbf{x},t), \boldsymbol{J}_p(\mathbf{x},t)$, and electrostatic potential, denoted by $\psi(\mathbf{x},t)$, $t\in[0,T]$. These variables satisfy the following drift-diffusion system

$$-\nabla\cdot(\epsilon_s\nabla\psi)=q\left(p-n+C\right), \tag{4a}$$

$$-q\partial_t n+\nabla\cdot\boldsymbol{J}_n=qR, \tag{4b}$$

$$q\partial_t p+\nabla\cdot\boldsymbol{J}_p=-qR, \tag{4c}$$

for $\mathbf{x} \in \Omega$ and $t \in [0,T]$. In (4) the dielectric permittivity $\epsilon_s = \epsilon_0 \epsilon_r$ is given as the product of the vacuum dielectric permittivity ϵ_0 and the relative permittivity of the semiconductor material ϵ_r. The current densities in (4b) and (4c) are given by the usual expressions

$$\boldsymbol{J}_n = q(D_n \nabla n - \mu_n n \nabla \psi) \qquad \text{and} \qquad \boldsymbol{J}_p = -q(D_p \nabla p + \mu_p p \nabla \psi). \tag{4d}$$

In (4d), D_n and D_p are the diffusion coefficients related to the mobilities μ_n and μ_p of electrons and holes by the Einstein's relations

$$D_n = U_T \mu_n, \qquad D_p = U_T \mu_p, \tag{5}$$

where U_T is the thermal voltage given by $U_T = k_\mathsf{B}\theta/q$, in which k_B indicates the Boltzmann constant and θ is the globally constant temperature for both carrier species and the crystal lattice. The net recombination rate $R(n,p)$ on the right-hand side of equations (4b) and (4c) describes the generation or recombination of carriers due to various scattering effects. Considering Boltzmann statistics, the recombination rate has the form

$$R(n,p) = r(n,p)(np - n_\mathsf{i}^2), \tag{6}$$

where n_i is the constant intrinsic concentration. We assume that $r(n,p)$ is a regular function, with $r(n,p) > 0$ for all n, p, and such that $R(n,p)$ is Lipschitz continuous, which means for different electron and hole densities n_1, n_2 and p_1, p_2 respectively, we have

$$|R(n_1,p_1) - R(n_2,p_2)| \le L_R(|n_1 - n_2| + |p_1 - p_2|), \tag{7}$$

with Lipschitz constant L_R. The Shockley-Read-Hall recombination mechanism for example satisfies this property.

We supplement the system (4) with the initial conditions at time $t = 0$

$$n(\mathbf{x},0) = n_0(\mathbf{x}), \qquad p(\mathbf{x},0) = p_0(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega. \tag{8}$$

Concerning the boundary conditions, since we consider a semiconductor device containing $m_D + 1$ terminals we have thatthe boundary $\partial\Omega$ of the domain Ω contains m_D+1 open, disconnected subsets $\Gamma_{D,i}$, $i = 0, 1, \ldots, m_D$, representing the terminals of the device

$$\partial\Omega = \Gamma_D \cap \Gamma_N, \qquad \Gamma_D = \bigcup_{i=0}^{m_D} \Gamma_{D,i}, \qquad \Gamma_N = \partial\Omega \setminus \Gamma_D. \tag{9}$$

On Γ_D we assume Dirichlet boundary conditions, that is

$$n(\mathbf{x},t) = n_D(\mathbf{x}), \qquad p(\mathbf{x},t) = p_D(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_D \tag{10a}$$

$$\psi(\mathbf{x},t) = \psi_\mathsf{bi}(\mathbf{x}) + u_{D,i}(t), \qquad \mathbf{x} \in \Gamma_{D,i}, \quad i = 0, 1, \ldots m_D \tag{10b}$$

where

$$n_D = \frac{C}{2} + \sqrt{\left(\frac{C}{2}\right)^2 + n_\mathsf{i}^2}, \qquad p_D = -\frac{C}{2} + \sqrt{\left(\frac{C}{2}\right)^2 + n_\mathsf{i}^2}$$

and

$$\psi_\mathsf{bi} = U_T \ln \frac{n_D}{n_\mathsf{i}}$$

The external potentials $u_{D,i}, i = 0, \ldots m_D$ coincide with the electric potentials at the nodes of the network which correspond to the terminals. They represent the coupling with the circuit, and will be described later. On the remaining boundaries one typically imposes homogeneous Neumann boundary conditions, namely

$$\nabla\psi(\mathbf{x},t) \cdot \boldsymbol{\nu} = \boldsymbol{J}_n(\mathbf{x},t) \cdot \boldsymbol{\nu} = \boldsymbol{J}_p(\mathbf{x},t) \cdot \boldsymbol{\nu} = 0, \qquad \mathbf{x} \in \Gamma_N, \quad t \in [0,T]. \tag{10c}$$

In (10c), $\boldsymbol{\nu}$ denotes the outer normal vector on the boundary.

2.3 Coupling conditions

In this section we explain the coupling conditions between the network and the device. We describe in details the term $\boldsymbol{s}_D(\boldsymbol{x})$ in (2) and its relation with the external potentials $u_{D,i}, i = 0, \dots m_D$ in (10b).

We start defining the total electric current through the Dirichlet terminals. Taking the time derivative of the Poisson equation (4a) and using the continuity equations (4b) and (4c), we obtain

$$\nabla \cdot (-\epsilon_s \partial_t \nabla \psi + \boldsymbol{J}_n + \boldsymbol{J}_p) = 0. \tag{11}$$

Integrating the above identity on Ω and using Gauss's divergence theorem, recalling the Neumann's boundary conditions (10c), we get

$$\sum_{i=0}^{m_D} \int_{\Gamma_{D,i}} (-\epsilon_s \partial_t \nabla \psi + \boldsymbol{J}_n + \boldsymbol{J}_p) \cdot \boldsymbol{\nu}_i \, \mathrm{d}\sigma = 0. \tag{12}$$

We identify the total current through $\Gamma_{D,i}$ with

$$J_{D,i} := -\int_{\Gamma_{D,i}} (-\epsilon_s \partial_t \nabla \psi + \boldsymbol{J}_n + \boldsymbol{J}_p) \cdot \boldsymbol{\nu}_i \, \mathrm{d}\sigma, \qquad i = 0, 1, \dots, m_D, \tag{13}$$

which is the sum of the so-called displacement current – that is, the time derivative of the electric field – with the currents due to carriers. We note that, by construction, the currents satisfy the identity

$$\sum_{i=0}^{m_D} J_{D,i} = 0. \tag{14}$$

Next, we need to define in a proper way the electric currents flowing through the Ohmic contacts, which will be used for the coupling to the circuit. At this aim, we introduce the auxiliary functions w_i, $i = 0, 1, \dots, m_D$, defined by the following elliptic boundary value problem:

$$\begin{cases} -\nabla \cdot (\epsilon_s \nabla w_i) &= 0, & \text{in } \Omega, \\ w_i &= \delta_{ij}, & \text{on } \Gamma_{D,j}, \quad j = 0, 1, \dots, m_D, \\ \dfrac{\partial w_i}{\partial \nu} &= 0, & \text{on } \Gamma_N, \end{cases} \tag{15}$$

where δ_{ij} is Kronecker's delta ($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$).

Then, it is convenient to define the electric current $J_{D,i}$ flowing through the i-th Ohmic contact $\Gamma_{D,i}$ by the volume integral

$$J_{D,i}(t) = -\int_{\Omega} \nabla w_i \cdot (-\epsilon_s \partial_t \nabla \psi + \boldsymbol{J}_n + \boldsymbol{J}_p) \, \mathrm{d}\mathbf{x}. \tag{16}$$

For later use, we decompose the electric potential by means of the auxiliary functions w_i. We introduce the stationary part ψ^* of the electric potential, defined as the solution of the problem

$$\begin{cases} -\nabla \cdot (\epsilon_s \nabla \psi^*) = qC, & \text{in } \Omega, \\ \psi^* = \psi_{\mathrm{bi}}, & \text{on } \Gamma_D, \\ \boldsymbol{\nu} \cdot \nabla \psi^* = 0, & \text{on } \Gamma_N. \end{cases} \tag{17}$$

We define the linear functional $\mathcal{L}(\rho)$, which to the function ρ associates the function $\varphi \equiv \mathcal{L}(\rho)$, solution of the problem:

$$\begin{cases} -\nabla \cdot (\epsilon_s \nabla \varphi) = \rho, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \Gamma_D, \\ \boldsymbol{\nu} \cdot \nabla \varphi = 0, & \text{on } \Gamma_N. \end{cases}$$

Then we can write the electrostatic potential as

$$\psi = \psi^* + \sum_{i=0}^{m_D} w_i u_{D,i} + \mathcal{L}(qp - qn).$$

Using this decomposition, the electric current $J_{D,i}$ becomes

$$J_{D,i} = \sum_{j=0}^{m_D} C_{D,ij} \frac{\mathrm{d}u_{D,j}}{\mathrm{d}t} + \mathcal{J}_{D,i}(\boldsymbol{J}_n + \boldsymbol{J}_p), \qquad i = 0, 1, \ldots, m_D, \tag{18}$$

where

$$C_{D,ij} = \int_\Omega \epsilon_s \nabla w_i \cdot \nabla w_j \mathrm{d}\mathbf{x}, \qquad i, j = 0, 1, \ldots, m_D, \tag{19}$$

and we have introduced the functional $\mathcal{J}_{D,i}$ which to any vector function $\boldsymbol{J}(\mathbf{x})$ associates

$$\mathcal{J}_{D,i}(\boldsymbol{J}) = -\int_\Omega \nabla w_i \cdot (\epsilon_s \nabla \mathcal{L}(\nabla \cdot \boldsymbol{J}) + \boldsymbol{J}) \ \mathrm{d}\mathbf{x} \quad i = 0, 1, \ldots, m_D. \tag{20}$$

To write (18) in compact form, we introduce the vectors

$$\boldsymbol{J}_D = \begin{bmatrix} J_{D,0} \\ J_{D,1} \\ \vdots \\ J_{D,m_D} \end{bmatrix}, \quad \boldsymbol{u}_D = \begin{bmatrix} u_{D,0} \\ u_{D,1} \\ \vdots \\ u_{D,m_D} \end{bmatrix}, \quad \boldsymbol{\mathcal{J}}_D = \begin{bmatrix} \mathcal{J}_{D,0} \\ \mathcal{J}_{D,1} \\ \vdots \\ \mathcal{J}_{D,m_D} \end{bmatrix},$$

and the matrix $\widehat{\boldsymbol{C}}_D = (C_{D,ij})_{i,j=0,1,\ldots,m_D} \in \mathbb{R}^{(m_D+1)\times(m_D+1)}$. Then, we can write

$$\boldsymbol{J}_D = \widehat{\boldsymbol{C}}_D \frac{\mathrm{d}\boldsymbol{u}_D}{\mathrm{d}t} + \boldsymbol{\mathcal{J}}_D(\boldsymbol{J}_n + \boldsymbol{J}_p). \tag{21}$$

The currents through the interfaces are not independent, due to (14), which implies

$$J_{D,0} = -\sum_{i=1}^{m_D} J_{D,i}.$$

Moreover, it is possible to prove that

$$\sum_{j=0}^{m_D} C_{D,ij} = 0, \qquad i = 0, 1, \ldots, m_D. \tag{22}$$

It is convenient to introduce the vectors

$$\boldsymbol{I}_D = \begin{bmatrix} J_{D,1} \\ \vdots \\ J_{D,m_D} \end{bmatrix}, \qquad \boldsymbol{v}_D = \begin{bmatrix} u_{D,1} - u_{D,0} \\ \vdots \\ u_{D,m_D} - u_{D,0} \end{bmatrix}, \qquad \boldsymbol{\mathcal{I}}_D = \begin{bmatrix} \mathcal{J}_{D,1} \\ \vdots \\ \mathcal{J}_{D,m_D} \end{bmatrix}, \tag{23}$$

and the matrices $\boldsymbol{C}_D = (C_{D,ij})_{i,j=1,\dots,m_D} \in \mathbb{R}^{m_D \times m_D}$ and

$$\widehat{\boldsymbol{A}}_D = \begin{bmatrix} -1 & -1 & \cdots & -1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Then we can write

$$\boldsymbol{J}_D = \widehat{\boldsymbol{A}}_D \boldsymbol{I}_D, \quad \boldsymbol{v}_D = \widehat{\boldsymbol{A}}_D^\top \boldsymbol{u}_D, \tag{24}$$

and (21) can be replaced by

$$\boldsymbol{I}_D = \boldsymbol{C}_D \frac{\mathrm{d}\boldsymbol{v}_D}{\mathrm{d}t} + \boldsymbol{\mathcal{I}}_D(\boldsymbol{J}_n + \boldsymbol{J}_p). \tag{25}$$

In particular, combining (24) and (25), and comparing with (21), we have the identity $\widehat{\boldsymbol{C}}_D = \widehat{\boldsymbol{A}}_D \boldsymbol{C}_D \widehat{\boldsymbol{A}}_D^\top$.

Circuit-to-device coupling conditions

To relate the potentials $\boldsymbol{u}_D$, applied at the $m_D + 1$ contacts of the device, with the network potentials $\boldsymbol{u}$, we need to introduce a contact-to-node incidence matrix, which relates the device's contacts to the network nodes, selecting the node which corresponds to each contact. For this reason, we will call this matrix *selection matrix*, and denote it by $\boldsymbol{S}_D = (s_{D,ij}) \in \mathbb{R}^{m \times (m_D+1)}$. The sifting matrix is defined by:

$$s_{D,ij} = \begin{cases} 1, & \text{if the contact } j \text{ is connected to the node } i, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

By virtue of this definition, we can write

$$\boldsymbol{u}_D = \boldsymbol{S}_D^\top \boldsymbol{u}. \tag{27}$$

The components $\boldsymbol{u}$ are the first block of the circuit unknown $\boldsymbol{x}$ so that using the projection matrix $\boldsymbol{\pi}$ defined in Sec.2.1, we can also write

$$\boldsymbol{u}_D = \boldsymbol{S}_D^\top \boldsymbol{\pi} \boldsymbol{x}. \tag{28}$$

We refer to relation (28) as the *circuit-to-device coupling condition*, that can be also written using the voltage drop $\boldsymbol{v}_D$ defined in (23):

$$\boldsymbol{v}_D = \boldsymbol{A}_D^\top \boldsymbol{\pi} \boldsymbol{x}, \tag{29}$$

with $\boldsymbol{A}_D = \boldsymbol{S}_D \widehat{\boldsymbol{A}}_D$.

Device-to-circuit coupling conditions

Using the selection matrix $\boldsymbol{S}_D$ and the projection matrix $\boldsymbol{\pi}$, we can express the term $\boldsymbol{s}_D$ appearing in (2) as:

$$\boldsymbol{s}_D = \boldsymbol{\pi}^\top \boldsymbol{S}_D \boldsymbol{J}_D = \boldsymbol{\pi}^\top \boldsymbol{A}_D \boldsymbol{I}_D, \tag{30}$$

which also means $\boldsymbol{\ell}_D = \boldsymbol{S}_D \boldsymbol{J}_D = \boldsymbol{A}_D \boldsymbol{I}_D$. Using the representation (25), we get

$$\boldsymbol{s}_D = \boldsymbol{\pi}^\top \boldsymbol{A}_D \boldsymbol{C}_D \boldsymbol{A}_D^\top \boldsymbol{\pi} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \boldsymbol{\pi}^\top \boldsymbol{A}_D \boldsymbol{\mathcal{I}}_D(\boldsymbol{J}_n + \boldsymbol{J}_p). \tag{31}$$

We refer to this relation as the *device-to-circuit coupling condition*. The term $\boldsymbol{\mathcal{I}}_D(\boldsymbol{J}_n + \boldsymbol{J}_p)$ depends only on the voltage drops $\boldsymbol{v}_D = \boldsymbol{A}_D^\top \boldsymbol{\pi} \boldsymbol{x}$, so we will write $\boldsymbol{\mathcal{I}}_D(\boldsymbol{A}_D^\top \boldsymbol{\pi} \boldsymbol{x})$ to make this dependence explicit.

2.4 Circuit unknowns decomposition and topological conditions

The circuit equations (2) can be written in the form

$$\boldsymbol{E}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{x} + \mathcal{F}_D(\boldsymbol{x}) + \boldsymbol{s}(t), \tag{32}$$

where

$$\begin{aligned} &\boldsymbol{E} = \boldsymbol{M} + \boldsymbol{\pi}^\top \boldsymbol{A}_D \boldsymbol{C}_D \boldsymbol{A}_D^\top \boldsymbol{\pi}, \\ &\boldsymbol{A} = -\boldsymbol{N}, \\ &\mathcal{F}_D(\boldsymbol{x}) = -\boldsymbol{\pi}^\top \boldsymbol{A}_D \boldsymbol{\mathcal{I}}_D(\boldsymbol{A}_D^\top \boldsymbol{\pi}\boldsymbol{x}). \end{aligned} \tag{33}$$

Using a well-established procedure [22], we decompose the circuit's unknowns into a differential component and an algebraic component. Due to the special structure of the system, it is sufficient to impose index-1 topological conditions. To do so, first we write the system in the equivalent form:

$$\boldsymbol{E}_1\left(\boldsymbol{P}\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} + \boldsymbol{Q}\boldsymbol{z}\right) = \boldsymbol{A}_1\boldsymbol{y} + \mathcal{F}_D(\boldsymbol{P}\boldsymbol{y} + \boldsymbol{Q}\boldsymbol{z}) + \boldsymbol{s}(t), \tag{34}$$

where $\boldsymbol{y} = \boldsymbol{P}\boldsymbol{x}$, $\boldsymbol{z} = \boldsymbol{Q}\boldsymbol{x}$, with $\boldsymbol{Q}$ projector onto $\ker(\boldsymbol{E})$, $\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{Q}$ the complementary projector of $\boldsymbol{Q}$, and

$$\boldsymbol{E}_1 = \boldsymbol{E} - \boldsymbol{A}\boldsymbol{Q}, \quad \boldsymbol{A}_1 = \boldsymbol{A}\boldsymbol{P}. \tag{35}$$

We have explicitly

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_{CD} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I} \end{bmatrix}, \quad \boldsymbol{Q}_{CD} \text{ projector onto } \ker\begin{bmatrix}\boldsymbol{A}_C & \boldsymbol{A}_D\end{bmatrix}^\top. \tag{36}$$

Then the index-1 conditions should ensure that $\boldsymbol{E}_1$ is invertible and $\mathcal{F}_D$ depends only on $\boldsymbol{y}$. If this is the case, we can decompose (34) as follows:

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \boldsymbol{P}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y} + \mathcal{F}_D(\boldsymbol{P}\boldsymbol{y}) + \boldsymbol{s}(t)\right), \tag{37}$$

$$\boldsymbol{z} = \boldsymbol{Q}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y} + \mathcal{F}_D(\boldsymbol{P}\boldsymbol{y}) + \boldsymbol{s}(t)\right). \tag{38}$$

We can see that the first equation is decoupled from the second equation and can be solved for $\boldsymbol{y}$.

For our system, the index-1 conditions are [22, 23]:

$$\ker(\boldsymbol{A}_D, \boldsymbol{A}_C, \boldsymbol{A}_R, \boldsymbol{A}_V)^\top = \{0\}, \tag{39}$$

$$\ker \boldsymbol{Q}_{CD}^\top \boldsymbol{A}_V = \{0\}. \tag{40}$$

These conditions ensure also that $\mathcal{F}_D$ depends only on $\boldsymbol{y}$, since we have

$$\mathcal{F}_D(\boldsymbol{P}\boldsymbol{y} + \boldsymbol{Q}\boldsymbol{z}) = -\boldsymbol{\pi}^\top \boldsymbol{A}_D \boldsymbol{\mathcal{I}}_D(\boldsymbol{A}_D^\top \boldsymbol{\pi}(\boldsymbol{P}\boldsymbol{y} + \boldsymbol{Q}\boldsymbol{z})),$$

and $\boldsymbol{A}_D^\top \boldsymbol{\pi}\boldsymbol{Q} = \begin{bmatrix}\boldsymbol{A}_D^\top \boldsymbol{Q}_{CD} & \boldsymbol{O} & \boldsymbol{O}\end{bmatrix} = \boldsymbol{O}$ by construction.

We also have

$$\boldsymbol{Q}\boldsymbol{E}_1^{-1} = \bar{\boldsymbol{E}}_1^{-1}\boldsymbol{Q}^\top, \quad \bar{\boldsymbol{E}}_1 = \boldsymbol{E} - \boldsymbol{Q}^\top \boldsymbol{A}\boldsymbol{Q}, \tag{41}$$

which implies $\boldsymbol{Q}\boldsymbol{E}_1^{-1}\boldsymbol{\pi}^\top \boldsymbol{A}_D = 0$, and thus

$$\boldsymbol{Q}\boldsymbol{E}_1^{-1}\mathcal{F}_D(\boldsymbol{P}\boldsymbol{y}) = 0. \tag{42}$$

In turns, this yields a simplification of the decomposition (38), that is,

$$\boldsymbol{z} = \boldsymbol{Q}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y} + \boldsymbol{s}(t)\right). \tag{43}$$

Finally, we discuss the consistency of the initial data (3). From the above discussion, we see that we can only supplement the (32) with the initial data

$$\boldsymbol{y}(t_0) = \boldsymbol{y}_0 = \boldsymbol{P}\boldsymbol{x}_0. \tag{44}$$

Then the algebraic part $\boldsymbol{z}_0 = \boldsymbol{Q}\boldsymbol{x}_0$ of the initial data (3) must satisfy the consistency condition

$$\boldsymbol{z}_0 = \boldsymbol{Q}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{P}\boldsymbol{y}_0 + \boldsymbol{s}(t_0)\right).$$

3 Global existence and uniqueness results

The main results of this paper is the existence and uniqueness of the solution of the problem (2)-(4) with boundary and initial conditions (10) and (8) and coupling conditions (28) and (31) and additional topological index-1 conditions (39)-(40). We summarize here the equations for the reader's convenience:

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \boldsymbol{P}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y} + \mathcal{F}_D(\boldsymbol{P}\boldsymbol{y}) + \boldsymbol{s}(t)\right), \qquad \text{in } (0,T], \tag{1a}$$

$$\boldsymbol{z} = \boldsymbol{Q}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y} + \boldsymbol{s}(t)\right), \tag{1b}$$

$$\boldsymbol{y}(t_0) = \boldsymbol{y}_0 = \boldsymbol{P}\boldsymbol{x}_0, \tag{1c}$$

$$-\nabla\cdot(\epsilon_s\nabla\psi) = q\left(p - n + C\right), \tag{1d}$$

$$-q\partial_t n + \nabla\cdot\boldsymbol{J}_n = qR, \qquad \boldsymbol{J}_n = q(D_n\nabla n - \mu_n n\nabla\psi), \qquad \text{in } (0,T]\times\Omega, \tag{1e}$$

$$q\partial_t p + \nabla\cdot\boldsymbol{J}_p = -qR, \quad \boldsymbol{J}_p = -q(D_p\nabla p + \mu_p p\nabla\psi), \tag{1f}$$

$$n = n_D, \qquad p = p_D, \qquad \text{in } (0,T]\times\Gamma_D, \tag{1g}$$

$$\psi = \psi_{\text{bi}} + u_{D,i}, \qquad \text{in } (0,T]\times\Gamma_{D,i}, \quad i = 0,1,\dots m_D, \tag{1h}$$

$$\nabla\psi\cdot\boldsymbol{\nu} = \boldsymbol{J}_n\cdot\boldsymbol{\nu} = \boldsymbol{J}_p\cdot\boldsymbol{\nu} = 0 \qquad \text{in } (0,T]\times\Gamma_N, \tag{1i}$$

$$n(\mathbf{x},0) = n_0(\mathbf{x}), \quad p(\mathbf{x},0) = p_0(\mathbf{x}) \quad \text{in } \Omega, \tag{1j}$$

$$\boldsymbol{u}_D = \boldsymbol{S}_D^\top\boldsymbol{\pi}\boldsymbol{x}, \tag{1k}$$

$$\mathcal{F}_D(\boldsymbol{P}\boldsymbol{y}) = -\boldsymbol{\pi}^\top\boldsymbol{A}_D\boldsymbol{\mathcal{I}}_D(\boldsymbol{J}_n + \boldsymbol{J}_p). \tag{1l}$$

First, we show that a unique solution exists for small time interval, then we prove *a priori* estimates and, finally using these estimates we prove that this solution can be extended for all time intervals.

Let us start introducing the notation on functional spaces. We denote by $L^r = L^r(\Omega)$ and $W^{k,r} = W^{k,r}(\Omega)$ the usual spaces of functions, with norms $\|\cdot\|_r$ and $\|\cdot\|_{k,r}$, respectively, and we denote $H^k = W^{k,2}$, [24]. We also use the space L^2_+ of all functions in L^2 which are nonnegative almost everywhere. Let $[0,T]$ be a bounded time interval. For any Banach space V, we denote by $C([0,T];V)$, $L^r([0,T];V)$, and $H^k([0,T];V)$ the usual spaces of functions defined on $[0,T]$ with values in V.

For the purposes of this paper, we introduce the following Banach spaces:

$$
\begin{aligned}
&W_0^{2,r} = \{u \in W^{2,r} \mid u|_{\Gamma_D} = 0,\ \boldsymbol{\nu} \cdot \nabla u|_{\Gamma_N} = 0\},\\
&X = \{u \in H^1 \mid u|_{\Gamma_D} = 0\},\\
&Y = C([0,T];L^2) \cap L^2([0,T];X) \cap H^1([0,T];X^*),\\
&W = Y \times Y \times \{C([0,T];W_0^{2,2}) \cap L^2([0,T];W_0^{2,r}) \cap H^1([0,T];X)\},\\
&\mathcal{C}_L = C([0,T];\mathbb{R}^{m+n_L}),\\
&\mathcal{C}_V = C([0,T];\mathbb{R}^{m+n_V}),
\end{aligned}
$$

where X^* is the dual space of X.

We introduce a special steady-state solution $(n^{\mathsf{e}}, p^{\mathsf{e}}, \psi^{\mathsf{e}})$ of (1d)-(1f) which corresponds to zero external electric voltage source $\boldsymbol{v}_V$ and electric current source $\boldsymbol{i}_I$, where the potentials applied to the diode vanish. Furthermore, this solution shall satisfy the conditions

$$
n^{\mathsf{e}} = n_{\mathsf{i}} \exp(\psi^{\mathsf{e}}/U_T), \qquad p^{\mathsf{e}} = n_{\mathsf{i}} \exp(-\psi^{\mathsf{e}}/U_T). \tag{2}
$$

A steady-state solution corresponding to these constraints represents a physical state in total thermodynamic equilibrium. The equilibrium voltage ψ^{e} is uniquely determined by the following nonlinear elliptic problem:

$$
\begin{aligned}
&-\nabla \cdot (\epsilon_s \nabla \psi) = qC - qn_{\mathsf{i}} \left(\exp(\psi/U_T) - \exp(-\psi/U_T)\right),\\
&\psi(\mathbf{x},t) = \psi_{\mathsf{bi}}(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_{D,i}, \quad i = 0,1,\dots m_D,
\end{aligned} \tag{3}
$$

where

$$
\psi_{\mathsf{bi}} = U_T \ln \frac{n_D}{n_{\mathsf{i}}}, \qquad n_D = \frac{C}{2} + \sqrt{\left(\frac{C}{2}\right)^2 + n_{\mathsf{i}}^2}.
$$

Definition 1 (Solution to the coupled system) *A solution of the system of equation* (1a),(1b),(1d)-(1f) *with boundary conditions* (1g)-(1i) *and initial conditions* (1c) *and* (1j)*, coupling conditions* (1k) *and* (1l) *and additional topological index-1 conditions* (39)-(40) *is a tuple* $(n, p, \psi, \boldsymbol{y}, \boldsymbol{z})$ *such that:*

i) the unknown $\boldsymbol{y}$ *belongs to* $\mathcal{C}_L$ *and satisfies*

$$
\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \boldsymbol{P}\boldsymbol{E}_1^{-1} \left(\boldsymbol{A}_1 \boldsymbol{y} + \mathcal{F}_D(\boldsymbol{P}\boldsymbol{y}) + \boldsymbol{s}(t)\right);
$$

where the functional $\mathcal{F}_D$ *is defined in* (33)*;*

ii) $\boldsymbol{z}$ *belongs to* $\mathcal{C}_V$ *and is given by*

$$
\boldsymbol{z} = \boldsymbol{Q}\boldsymbol{E}_1^{-1} \left(\boldsymbol{A}_1 \boldsymbol{y} + \boldsymbol{s}(t)\right);
$$

iii) $(n - n^e, p - p^e, \psi - \psi^e) \in Y$*,* $n, p \in C([0,T]; L^2_+)$ *satisfy the initial conditions* (1j)*;*

iv) the triple (n, p, ψ) *satisfies the Poisson equation* (1d) *for all* $t \geq 0$*;*

v) for all test functions $\xi_n, \xi_p \in Y$*,* n *and* p *satisfy the weak formulation*

$$
\int_0^T \left[(\partial_t n, \xi_n) + (D_n \nabla n - \mu_n n \nabla \psi, \nabla \xi_n) + (R, \xi_n)\right] \mathrm{d}t = 0, \tag{4a}
$$

$$
\int_0^T \left[(\partial_t p, \xi_p) + (D_p \nabla p + \mu_p p \nabla \psi, \nabla \xi_p) + (R, \xi_p)\right] \mathrm{d}t = 0, \tag{4b}
$$

where $(\cdot,\cdot)$ *is the usual pairing between* X^* *and* X*.*

3.1 Local existence and uniqueness

We start by proving the following result.

Theorem 1 (Local existence and uniqueness) *Let the source functions $\boldsymbol{i}_I(t)$ and $\boldsymbol{v}_V(t)$ be continuous, let the network matrices be symmetric, positive definite and the topological conditions* (1k) *and* (1l) *be fulfilled, and let diffusivities and mobilities be constant. Then problem* (1) *admits a unique solution, provided $T > 0$ is sufficiently small.*

Proof. We introduce the Banach space

$$Z = (n^{\mathsf{e}}, p^{\mathsf{e}}) + C([0,T]; L^2 \times L^2) \cap L^2([0,T]; X^2)$$

and fix the pair of functions $(\hat{n}, \hat{p}) \in Z$ and $\hat{\boldsymbol{y}} \in \mathcal{C}_L$, with $\hat{n}(\cdot, 0) = n_0(\cdot)$, $\hat{p}(\cdot, 0) = p_0(\cdot)$, and $\hat{\boldsymbol{y}}(0) = \boldsymbol{y}_0$. We consider the following linearized problem for (n, p):

$$\int_0^T \big[(\partial_t n, \xi_n) + (D_n \nabla n + \mu_n \hat{n}^+ \hat{\mathbf{E}}, \nabla \xi_n) + (\hat{R}^+, \xi_n)\big] \mathrm{d}t = 0, \tag{5a}$$

$$\int_0^T \big[(\partial_t p, \xi_p) + (D_p \nabla p - \mu_p \hat{p}^+ \hat{\mathbf{E}}, \nabla \xi_p) + (\hat{R}^+, \xi_p)\big] \mathrm{d}t = 0, \tag{5b}$$

for all test functions ξ_n, $\xi_p \in Y$. We have indicated with $\hat{R}^+ {=} R(\hat{n}^+, \hat{p}^+)$, in which we have used the notation the notation $g^+{:=}\max(g, 0)$, for all function g and $\hat{\mathbf{E}} = -\nabla \hat{\psi}$ where $\hat{\psi}$ is the solution of the Poisson equation with source term $q(\hat{p}^+ - \hat{n}^+ + C)$ and boundary conditions

$$\psi(\mathbf{x}, t) = \psi_{\mathsf{bi}}(\mathbf{x}) + \hat{u}_{D,i}(t), \qquad \mathbf{x} \in \Gamma_{D,i}, \;\; i = 0, 1, \dots m_D,$$
$$\hat{\boldsymbol{u}}_D = \boldsymbol{S}_D^\top \boldsymbol{\pi} \boldsymbol{P} \hat{\boldsymbol{y}}.$$

We consider also the problem for $\boldsymbol{y}$

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \boldsymbol{P}\boldsymbol{E}_1^{-1} \left(\boldsymbol{A}_1 \hat{\boldsymbol{y}} + \mathcal{F}_D(\boldsymbol{P}\hat{\boldsymbol{y}}) + \boldsymbol{s}(t)\right) \tag{5c}$$

with

$$\mathcal{F}_D(\boldsymbol{P}\hat{\boldsymbol{y}}) = -\boldsymbol{\pi}^\top \boldsymbol{A}_D \boldsymbol{\mathcal{I}}_D(q(D_n \nabla \hat{n} - D_p \nabla \hat{p} + (\mu_n \hat{n} + \mu_p \hat{p})\hat{\mathbf{E}})).$$

The system of decoupled equations (5) admits a unique solution $(n, p, \boldsymbol{y})$ which satisfies $(n, p) \in (n^{\mathsf{e}}, p^{\mathsf{e}}) + Y^2, \boldsymbol{y} \in \mathcal{C}_L$, and have the initial data $n(\cdot, 0) = n_0(\cdot), p(\cdot, 0) = p_0(\cdot), y(0) = y_0$, the same initial data (1c) and (1j) of the original problem. In fact, the existence of a unique solution $\boldsymbol{y}$ to (5c) follows immediately by time integration of the equation over $[0, t]$. Because of the continuous embedding of H^1 into L^4 and the Poisson equation [25], we have the following estimates

$$\|\mu_n \hat{n}^+ \hat{\mathbf{E}}\|_2 \le c\|\hat{n}\|_4 \|\hat{\mathbf{E}}\|_4 \le c\|\hat{n}\|_{1,2} \|\hat{\mathbf{E}}\|_{1,2} \le c\|\hat{n}\|_{1,2}(1 + |\hat{\boldsymbol{y}}| + \|\hat{n}\|_2 + \|\hat{p}\|_2).$$

A similar estimates follows for the hole distribution. This means the terms $\mu_n \hat{n}^+ \hat{\mathbf{E}}$ and $\mu_p \hat{p}^+ \hat{\mathbf{E}}$ are in L^2 so we can apply the standard results, see for example [24], for linear parabolic equations with discontinuous coefficients to get the existence of a unique solution $(n, p, \boldsymbol{y})$ to (5a)–(5b). We define an operator $\mathcal{Q}$ from $Z \times \mathcal{C}_L$ to $Z \times \mathcal{C}_L$ which, to any $(\hat{n}, \hat{p}, \hat{\boldsymbol{y}}) \in Z \times \mathcal{C}_L$ associates the solution $(n, p, \boldsymbol{y})$ of the problem (5)

$$(\hat{n}, \hat{p}, \hat{\boldsymbol{y}}) \to (n, p, \boldsymbol{y}) =: \mathcal{Q}(\hat{n}, \hat{p}, \hat{\boldsymbol{y}}).$$

For all $(n, p, \boldsymbol{y}) \in Z \times \mathcal{C}_L$, we introduce the norm

$$|||(n,p,\boldsymbol{y})|||^2 = \max_{0\le t\le T}\left(\|n(\cdot,t)\|_2^2 + \|p(\cdot,t)\|_2^2 + |\boldsymbol{y}(t)|^2\right) + \int_0^T \left(\|\nabla n(\cdot,t)\|_2^2 + \|\nabla p(\cdot,t)\|_2^2\right) \mathsf{d}\tau.$$

We prove that $\mathcal{Q}$ is is strictly contractive with respect to this norm, for T small enough, in the set

$$S_a = \left\{(n,p,\boldsymbol{y}) \in Z \times \mathcal{C}_L : n(\cdot,t) = n_0(\cdot), p(\cdot,t) = p_0(\cdot), \boldsymbol{y}(0) = \boldsymbol{y}_0, |||(n,p,\boldsymbol{y})|||^2 \le a\right\}$$

where the constant a is such that $a > \|n_0\|_2^2 + \|p_0\|_2^2 + |\boldsymbol{y}_0|^2$. We consider two triples $(\hat{n}_\ell, \hat{p}_\ell, \hat{\boldsymbol{y}}_\ell) \in S_a$ for $\ell = 1, 2$ and we set

$$(n_\ell, p_\ell, \boldsymbol{y}_\ell) = \mathcal{Q}(\hat{n}_\ell, \hat{p}_\ell, \hat{\boldsymbol{y}}_\ell), \qquad \hat{\mathbf{E}}_\ell = \hat{\mathbf{E}}(\hat{n}_\ell, \hat{p}_\ell, \hat{\boldsymbol{y}}_\ell), \quad \text{for } \ell = 1,2$$

where $\hat{\mathbf{E}}_\ell = -\nabla\hat{\psi}_\ell$ depends on the solution $\hat{\psi}_\ell$ of the Poisson equation with source term $q(\hat{p}_\ell^+ - \hat{n}_\ell^+ + C)$ and the boundary term is related to $\hat{\boldsymbol{y}}_\ell$. We write (5a), with $n = n_1$ and $n = n_2$, both with test function $\xi_n = n_1 - n_2$. Subtracting the two equations and integrating over the time interval $[0,t]$, we obtain

$$\begin{aligned}
&\frac{1}{2}\|n_1 - n_2\|_2^2 + D_n \int_0^t \|\nabla(n_1 - n_2)\|_2^2 \mathsf{d}\tau \\
&= -\mu_n \int_0^t \left(\hat{n}_1^+\hat{\mathbf{E}}_1 - \hat{n}_2^+\hat{\mathbf{E}}_2, \nabla(n_1 - n_2)\right) \mathsf{d}\tau - \int_0^t \left(\hat{R}_1^+ - \hat{R}_2^+, n_1 - n_2\right) \mathsf{d}\tau \qquad (6)\\
&\le \mu_n \int_0^t \|\hat{n}_1^+\hat{\mathbf{E}}_1 - \hat{n}_2^+\hat{\mathbf{E}}_2\|_2 \|\nabla(n_1 - n_2)\|_2 \,\mathsf{d}\tau + \int_0^t \|\hat{R}_1^+ - \hat{R}_2^+\|_2 \|n_1 - n_2\|_2 \,\mathsf{d}\tau
\end{aligned}$$

in which $\hat{R}_\ell^+ = R(\hat{n}_\ell^+, \hat{p}_\ell^+)$, $\ell = 1, 2$.

To estimate the first term on the right-hand side of (6), we first observe that, using the Gagliardo-Nierenberg inequality and the weighted Young inequality, for any function $u \in H^1$ we have

$$\|u\|_r \le c\|u\|_2^{1-\eta}\|u\|_{1,2}^\eta + c\|u\|_2 \le c(\delta)\|u\|_2 + \delta\|u\|_{1,2}, \qquad (7)$$

where $\delta > 0$ can be chosen arbitrarily small, and $\eta = d\left(\frac{1}{2} - \frac{1}{r}\right) \in (0,1)$, if we choose $2 < r < 6$. Moreover, for $r > d$, we can write

$$\|\hat{\mathbf{E}}_1 - \hat{\mathbf{E}}_2\|_\infty \le c\|\hat{\mathbf{E}}_1 - \hat{\mathbf{E}}_2\|_{1,r} \le c(|\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2| + \|\hat{n}_1 - \hat{n}_2\|_r + \|\hat{p}_1 - \hat{p}_2\|_r).$$

and

$$\|\hat{\mathbf{E}}_2\|_6 \le c\|\hat{\mathbf{E}}_2\|_{1,2} \le c(1 + |\hat{\boldsymbol{y}}_2| + \|\hat{n}_2\|_2 + \|\hat{p}_2\|_2) \le c(a).$$

Then we have

$$\begin{aligned}
\|\hat{n}_1^+\hat{\mathbf{E}}_1 - \hat{n}_2^+\hat{\mathbf{E}}_2\|_2 &\le \|\hat{n}_1\|_2\|\hat{\mathbf{E}}_1 - \hat{\mathbf{E}}_2\|_\infty + \|\hat{\mathbf{E}}_2\|_6\|\hat{n}_1 - \hat{n}_2\|_3 \\
&\le c(a)\big[c|\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2| + c(\delta)(\|\hat{n}_1 - \hat{n}_2\|_2 + \|\hat{p}_1 - \hat{p}_2\|_2) \qquad (8)\\
&\quad + \delta(\|\hat{n}_1 - \hat{n}_2\|_{1,2} + \|\hat{p}_1 - \hat{p}_2\|_{1,2})\big].
\end{aligned}$$

So, we can estimate

$$\begin{aligned}
&\mu_n \int_0^t \|\hat{n}_1^+\hat{\mathbf{E}}_1 - \hat{n}_2^+\hat{\mathbf{E}}_2\|_2 \|\nabla(n_1 - n_2)\|_2 \,\mathrm{d}\tau \\
&\quad\le c(a,\delta)\int_0^t (|\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2|^2 + \|\hat{n}_1 - \hat{n}_2\|_2^2 + \|\hat{p}_1 - \hat{p}_2\|_2^2) \,\mathrm{d}\tau \\
&\qquad + \delta\int_0^t (\|\hat{n}_1 - \hat{n}_2\|_{1,2}^2 + \|\hat{p}_1 - \hat{p}_2\|_{1,2}^2) \,\mathrm{d}\tau + \delta\int_0^t \|\nabla(n_1 - n_2)\|_2^2 \,\mathrm{d}\tau \\
&\quad\le \max\{c(a,\delta)T, \delta\}|||(\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2, \hat{n}_1 - \hat{n}_2, \hat{p}_1 - \hat{p}_2)||| + \delta\int_0^t \|\nabla(n_1 - n_2)\|_2^2 \,\mathrm{d}\tau.
\end{aligned}$$

For the second term on the right-hand side of (6), using the definition (6) of the recombination term and the assumption (7), we get

$$\begin{aligned}\int_0^t \|\hat{R}_1^+ - \hat{R}_2^+\|_2 \|n_1 - n_2\|_2 \,\mathrm{d}\tau &\leq \int_0^t L_R(\|\hat{n}_1 - \hat{n}_2\|_2 + \|\hat{p}_1 - \hat{p}_2\|_2)\|n_1 - n_2\|_2 \,\mathrm{d}\tau \\ &\leq \int_0^t \left[c(\delta)(\|\hat{n}_1 - \hat{n}_2\|_2^2 + \|\hat{p}_1 - \hat{p}_2\|_2^2) + \delta\|n_1 - n_2\|_2^2\right] \mathrm{d}\tau \\ &\leq c(\delta)T|||(\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2, \hat{n}_1 - \hat{n}_2, \hat{p}_1 - \hat{p}_2)||| + \delta \int_0^t \|n_1 - n_2\|_2^2 \,\mathrm{d}\tau.\end{aligned}$$

Combining the previous estimates, we get

$$\begin{aligned}&\frac{1}{2}\|n_1 - n_2\|_2^2 + D_n \int_0^t \|\nabla(n_1 - n_2)\|_2^2 \mathrm{d}\tau \\ &\quad \leq \max\{c(a,\delta)T, \delta\}|||(\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2, \hat{n}_1 - \hat{n}_2, \hat{p}_1 - \hat{p}_2)||| + \delta \int_0^t \|n_1 - n_2\|_{1,2}^2 \,\mathrm{d}\tau.\end{aligned} \tag{9}$$

An analogous estimate holds for $p_1 - p_2$, that is,

$$\begin{aligned}&\frac{1}{2}\|p_1 - p_2\|_2^2 + D_p \int_0^t \|\nabla(p_1 - p_2)\|_2^2 \mathrm{d}\tau \\ &\quad \leq \max\{c(a,\delta)T, \delta\}|||(\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2, \hat{n}_1 - \hat{n}_2, \hat{p}_1 - \hat{p}_2)||| + \delta \int_0^t \|p_1 - p_2\|_{1,2}^2 \,\mathrm{d}\tau.\end{aligned} \tag{10}$$

Next, we write (1a) for $\boldsymbol{y} = \boldsymbol{y}_1, \boldsymbol{y} = \boldsymbol{y}_2$, subtract the resulting equations, and multiply the result by $\boldsymbol{y}_1 - \boldsymbol{y}_2$. We note that

$$\boldsymbol{P}^\top \boldsymbol{E}_1 \boldsymbol{P} = \boldsymbol{P}^\top \boldsymbol{E}_1^\top \boldsymbol{P}, \tag{11}$$

which, together with $\boldsymbol{y}_1 - \boldsymbol{y}_2 = \boldsymbol{P}(\boldsymbol{y}_1 - \boldsymbol{y}_2)$, implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}(\boldsymbol{y}_1 - \boldsymbol{y}_2)^\top \boldsymbol{E}_1(\boldsymbol{y}_1 - \boldsymbol{y}_2)\right) &= (\boldsymbol{y}_1 - \boldsymbol{y}_2)^\top \boldsymbol{E}_1 \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{y}_1 - \boldsymbol{y}_2) \\ = (\boldsymbol{y}_1 - \boldsymbol{y}_2)^\top \boldsymbol{E}_1 &\left\{\boldsymbol{P}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\hat{\boldsymbol{y}}_1 + \mathcal{F}_D(\boldsymbol{P}\hat{\boldsymbol{y}}_1)\right) - \boldsymbol{P}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\hat{\boldsymbol{y}}_2 + \mathcal{F}_D(\boldsymbol{P}\hat{\boldsymbol{y}}_2)\right)\right\} \\ = (\boldsymbol{y}_1 - \boldsymbol{y}_2)^\top &\left\{\boldsymbol{E}_1\boldsymbol{P}\boldsymbol{E}_1^{-1}\boldsymbol{A}_1\left(\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2\right) + \mathcal{F}_D(\boldsymbol{P}\hat{\boldsymbol{y}}_1) - \mathcal{F}_D(\boldsymbol{P}\hat{\boldsymbol{y}}_2)\right\}.\end{aligned}$$

After integrating with respect to time, we find

$$\begin{aligned}&\frac{1}{2}(\boldsymbol{y}_1 - \boldsymbol{y}_2)^\top \boldsymbol{E}_1(\boldsymbol{y}_1 - \boldsymbol{y}_2) \\ &\quad = \int_0^t (\boldsymbol{y}_1 - \boldsymbol{y}_2)^\top \left\{\left(\boldsymbol{E}_1\boldsymbol{P}\boldsymbol{E}_1^{-1}\boldsymbol{A}_1(\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2) + \mathcal{F}_D(\boldsymbol{P}\hat{\boldsymbol{y}}_1) - \mathcal{F}_D(\boldsymbol{P}\hat{\boldsymbol{y}}_2)\right)\right\} \mathrm{d}\tau \\ &\quad \leq \int_0^t |\boldsymbol{y}_1 - \boldsymbol{y}_2|^2 \,\mathrm{d}\tau + c\int_0^t \left(|\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2|^2 + |\mathcal{F}_D(\boldsymbol{P}\hat{\boldsymbol{y}}_1) - \mathcal{F}_D(\boldsymbol{P}\hat{\boldsymbol{y}}_2)|^2\right) \mathrm{d}\tau.\end{aligned}$$

From the definition of $\mathcal{F}_D$, we have

$$|\mathcal{F}_D(\boldsymbol{P}\hat{\boldsymbol{y}}_1) - \mathcal{F}_D(\boldsymbol{P}\hat{\boldsymbol{y}}_2)|^2 \leq c\sum_{i=1}^{m_D} |\mathcal{J}_{D,i}(\hat{\boldsymbol{J}}_1) - \mathcal{J}_{D,i}(\hat{\boldsymbol{J}}_2)|^2,$$

with

$$\hat{\boldsymbol{J}}_\ell = q(D_n\nabla\hat{n}_\ell + \mu_n\hat{n}_\ell\hat{\mathbf{E}}_\ell - D_p\nabla\hat{p}_\ell + \mu_p\hat{p}_\ell\hat{\mathbf{E}}_\ell), \quad \ell = 1, 2.$$

We notice that since $\hat{n}_1 - \hat{n}_2 = 0$ on Γ_D and $\boldsymbol{\nu} \cdot \nabla(\hat{n}_1 - \hat{n}_2) = 0$ on Γ_N we have that

$$\mathcal{L}(\Delta(\hat{n}_1 - \hat{n}_2)) = -\frac{\hat{n}_1 - \hat{n}_2}{\epsilon_s}.$$

Using also the linearity of $\mathcal{J}_{D,i}(\boldsymbol{J})$ with respect to $\boldsymbol{J}$, we find that

$$\mathcal{J}_{D,i}(D_n \nabla(\hat{n}_1 - \hat{n}_2)) = \mathcal{J}_{D,i}(D_p \nabla(\hat{p}_1 - \hat{p}_2)) = 0,$$

and then

$$|\mathcal{J}_{D,i}(\hat{\boldsymbol{J}}_1) - \mathcal{J}_{D,i}(\hat{\boldsymbol{J}}_2)| \leq q\mu_n |\mathcal{J}_{D,i}(\hat{n}_1 \hat{\mathbf{E}}_1 - \hat{n}_2 \hat{\mathbf{E}}_2)| + q\mu_p |\mathcal{J}_{D,i}(\hat{p}_1 \hat{\mathbf{E}}_1 - \hat{p}_2 \hat{\mathbf{E}}_2)|. \tag{12}$$

Moreover, for any vector function $\boldsymbol{J}(\mathbf{x})$ such that $\boldsymbol{\nu} \cdot \boldsymbol{J} = 0$ on Γ_N, we have

$$\begin{aligned} |\mathcal{J}_{D,i}(\boldsymbol{J})| &= \left| \int_\Omega \nabla w_i \cdot (\epsilon_s \nabla \mathcal{L}(\nabla \cdot \boldsymbol{J}) + \boldsymbol{J}) \, \mathsf{dx} \right| \\ &\leq \|\nabla w_i\|_2 \big(\|\epsilon_s \nabla \mathcal{L}(\nabla \cdot \boldsymbol{J})\|_2 + \|\boldsymbol{J}\|_2 \big) \leq c \|\boldsymbol{J}\|_2. \end{aligned} \tag{13}$$

In fact, if we pose $\varphi = \mathcal{L}(\nabla \cdot \boldsymbol{J})$, we have by definition

$$-\int_\Omega \varphi \nabla \cdot (\epsilon_s \nabla \varphi) \, \mathbf{dx} = \int_\Omega \varphi \nabla \cdot \boldsymbol{J} \, \mathbf{dx},$$

which, integrating by parts and using the boundary conditions for φ, and the condition for $\boldsymbol{J}$, yields

$$\int_\Omega \epsilon_s |\nabla \varphi|^2 \, \mathbf{dx} = -\int_\Omega (\nabla \varphi) \cdot \boldsymbol{J} \, \mathbf{dx} \leq \delta \|\nabla \varphi\|_2^2 + c(\delta) \|\boldsymbol{J}\|_2^2.$$

Choosing the positive constant δ small enough, we get

$$\|\nabla \varphi\|_2 \leq c \|\boldsymbol{J}\|_2, \tag{14}$$

which implies the last inequality in (13). We can apply this result to (12), obtaining

$$|\mathcal{J}_{D,i}(\hat{\boldsymbol{J}}_1) - \mathcal{J}_{D,i}(\hat{\boldsymbol{J}}_2)|^2 \leq c(\|\hat{n}_1 \hat{\mathbf{E}}_1 - \hat{n}_2 \hat{\mathbf{E}}_2\|_2^2 + \|\hat{p}_1 \hat{\mathbf{E}}_1 - \hat{p}_2 \hat{\mathbf{E}}_2\|_2^2), \tag{15}$$

which, in turns, yields

$$\begin{aligned} |\mathcal{F}_D(\boldsymbol{P}\hat{\boldsymbol{y}}_1) - \mathcal{F}_D(\boldsymbol{P}\hat{\boldsymbol{y}}_2)|^2 \;\leq\; & c(a)\big(c|\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2|^2 + c(\delta)(\|\hat{n}_1 - \hat{n}_2\|_2^2 + \|\hat{p}_1 - \hat{p}_2\|_2^2) \\ & + \delta(\|\hat{n}_1 - \hat{n}_2\|_{1,2}^2 + \|\hat{p}_1 - \hat{p}_2\|_{1,2}^2)\big). \end{aligned}$$

Since the matrix $\boldsymbol{E}_1$ is positive definite, there exist a constant $c_E > 0$ such that

$$\boldsymbol{y}^\top \boldsymbol{E}_1 \boldsymbol{y} \geq c_E |\boldsymbol{y}|^2.$$

Then, using the previous estimates, we find

$$\begin{aligned} c_E |\boldsymbol{y}_1 - \boldsymbol{y}_2|^2 &\leq \frac{1}{2} (\boldsymbol{y}_1 - \boldsymbol{y}_2)^\top \boldsymbol{E}_1 (\boldsymbol{y}_1 - \boldsymbol{y}_2) \\ &\leq \max\{c(a,\delta)T, \delta\} |||(\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2, \hat{n}_1 - \hat{n}_2, \hat{p}_1 - \hat{p}_2)||| + T \max_{0 \leq t \leq T} |\boldsymbol{y}_1 - \boldsymbol{y}_2|^2. \end{aligned} \tag{16}$$

Finally, we can combine (9), (10) and (16), and choose T and δ small enough so that

$$|||(\boldsymbol{y}_1 - \boldsymbol{y}_2, n_1 - n_2, p_1 - p_2)||| \leq c_{\mathcal{Q}} |||(\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2, \hat{n}_1 - \hat{n}_2, \hat{p}_1 - \hat{p}_2)|||,$$

for some positive constant $c_Q < 1$. So we have proved that there exists a time $T > 0$, such that Q becomes a contraction on S_a, then the existence of a unique fixed point $(n_*, p_*, \boldsymbol{y}_*) \in S_a$ it is ensured by the Banach's fixed point theorem. To show that this fixed point is the unique solution to (2)-(4), it is sufficient to prove the nonnegativity of n_* and p_*. To this end, we consider $n_*^- = \min(n_*, 0) \in Y$. Writing (5a), for the test function $\xi_n = n_*^-$, we obtain

$$\begin{aligned}\frac{1}{2}\|n_*^-(t)\|_2^2 + D_n \int_0^t \|\nabla n_*^-\|_2^2 \mathsf{d}\tau &= \int_0^t \int_\Omega (-\mu_n n_*^+ \mathbf{E}_* \cdot \nabla n_*^- - R(n_*^+, p_*^+) n_*^-) \mathsf{d}\boldsymbol{x} \mathsf{d}\tau \\ &= \int_0^t \int_\Omega r(n_*^+, p_*^+) n_\mathsf{i}^2 n_*^- \mathsf{d}\boldsymbol{x} \mathsf{d}\tau \le 0.\end{aligned}$$

This means that n_*^- vanishes almost everywhere; that is, n_* is nonnegative almost everywhere. In the same way we can prove the nonnegativity of p_*. ■

To prove the general existence and uniqueness of solution of the problem (1) we need to ensure that it is possible to prolong to arbitrary time intervals the solution, whose existence is proved in theorem 1.

3.2 Global existence and uniqueness

In this section we prove some a priori estimates which allow for the prolongation of solution of problem (1). We start introducing the definition of the energy of the system which is the sum of two contributions one is the energy associated to the device and the other is the energy associated to the circuit.

Energy associated to the device

The local physical energy associated to the device is defined as

$$w_D = qU_T \Big\{ n\Big(\log\frac{n}{n_\mathsf{i}} - 1\Big) + p\Big(\log\frac{p}{n_\mathsf{i}} - 1\Big)\Big\} + \frac{\epsilon_s}{2}|\nabla\psi|^2. \tag{17}$$

Using the continuity equations in (1e) and (1f), the time derivative of the local energy is given by

$$\begin{aligned}\partial_t w_D = &-\nabla\cdot\Big\{-U_T\log\frac{n}{n_\mathsf{i}}\boldsymbol{J}_n + U_T\log\frac{p}{n_\mathsf{i}}\boldsymbol{J}_p\Big\} - \boldsymbol{J}_n\cdot\Big\{U_T\frac{1}{n}\nabla n\Big\} + \boldsymbol{J}_p\cdot\Big\{U_T\frac{1}{p}\nabla p\Big\} \\ &- qU_T R\log\frac{np}{n_\mathsf{i}^2} + \epsilon_s\nabla\psi\cdot\partial_t\nabla\psi.\end{aligned}$$

Considering the equations in (1d)-(1f) and the Einstein's relations (5), we get

$$\begin{aligned}&\partial_t w_D + \nabla\cdot\Big\{-U_T\log\frac{n}{n_\mathsf{i}}\boldsymbol{J}_n + U_T\log\frac{p}{n_\mathsf{i}}\boldsymbol{J}_p\Big\} = \\ &\quad -\frac{1}{qn\mu_n}\boldsymbol{J}_n^2 - \frac{1}{qp\mu_p}\boldsymbol{J}_p^2 - \nabla\psi\cdot\Big\{\boldsymbol{J}_n + \boldsymbol{J}_p - \epsilon_s\partial_t\nabla\psi\Big\} - qU_T R\log\frac{np}{n_\mathsf{i}^2}.\end{aligned} \tag{18}$$

The total energy associated to the device is defined by

$$W_D = \int_\Omega w_D \mathsf{d}\mathbf{x}.$$

Integrating (18) on the domain Ω we have

$$\begin{aligned}\frac{\mathrm{d}W_D}{\mathrm{d}t} &+ \int_\Omega \nabla\cdot\Big\{-U_T\log\frac{n}{n_\mathrm{i}}\boldsymbol{J}_n + U_T\log\frac{p}{n_\mathrm{i}}\boldsymbol{J}_p\Big\}\mathrm{d}\mathbf{x}\\ &= -\int_\Omega\Big(\frac{\boldsymbol{J}_n^2}{q\mu_n n}+\frac{\boldsymbol{J}_p^2}{q\mu_p p}\Big)\mathrm{d}\mathbf{x} - \int_\Omega qU_T R\log\frac{np}{n_\mathrm{i}^2}\mathrm{d}\mathbf{x}\\ &\quad - \int_\Omega \nabla\psi\cdot\Big(\boldsymbol{J}_n+\boldsymbol{J}_p-\epsilon_s\partial_t\nabla\psi\Big)\mathrm{d}\mathbf{x}.\end{aligned}\tag{19}$$

We apply the Gauss theorem on the second term, we use $n_D p_D = n_\mathrm{i}^2$ and the boundary conditions

$$\begin{aligned}&\int_\Omega \nabla\cdot\Big(-U_T\log\frac{n}{n_\mathrm{i}}\boldsymbol{J}_n + U_T\log\frac{p}{n_\mathrm{i}}\boldsymbol{J}_p\Big)\mathrm{d}\mathbf{x}\\ &= \sum_{i=0}^{m_D}\int_{\Gamma_{D,i}} \boldsymbol{\nu}_i\cdot\Big(-U_T\log\frac{n_D}{n_\mathrm{i}}\boldsymbol{J}_n + U_T\log\frac{p_D}{n_\mathrm{i}}\boldsymbol{J}_p\Big)\mathrm{d}\sigma\\ &= \sum_{i=0}^{m_D}\int_{\Gamma_{D,i}} -U_T\log\frac{n_D}{n_\mathrm{i}}\Big(\boldsymbol{J}_n+\boldsymbol{J}_p\Big)\cdot\boldsymbol{\nu}_i\mathrm{d}\sigma = -\sum_{i=0}^{m_D}\int_{\Gamma_{D,i}}\psi_\mathrm{bi}\Big(\boldsymbol{J}_n+\boldsymbol{J}_p\Big)\cdot\boldsymbol{\nu}_i\mathrm{d}\sigma.\end{aligned}$$

Integrating by parts the last term in (19) and using (11) and (13), we obtain

$$\begin{aligned}&-\int_\Omega \nabla\psi\cdot\Big(\boldsymbol{J}_n+\boldsymbol{J}_p-\epsilon_s\partial_t\nabla\psi\Big)\mathrm{d}\mathbf{x}\\ &= -\int_{\partial\Omega}\psi\Big(\boldsymbol{J}_n+\boldsymbol{J}_p-\epsilon_s\partial_t\nabla\psi\Big)\cdot\boldsymbol{\nu}\mathrm{d}\sigma + \int_\Omega \psi\nabla\cdot\Big(\boldsymbol{J}_n+\boldsymbol{J}_p-\epsilon_s\partial_t\nabla\psi\Big)\mathrm{d}\mathbf{x}\\ &= -\sum_{i=0}^{m_D}\int_{\Gamma_{D,i}}(\psi_\mathrm{bi}+u_{D,i})\Big(\boldsymbol{J}_n+\boldsymbol{J}_p-\epsilon_s\partial_t\nabla\psi\Big)\cdot\boldsymbol{\nu}_i\mathrm{d}\sigma\\ &= -\sum_{i=0}^{m_D}\int_{\Gamma_{D,i}}\psi_\mathrm{bi}\Big(\boldsymbol{J}_n+\boldsymbol{J}_p-\epsilon_s\partial_t\nabla\psi\Big)\cdot\boldsymbol{\nu}_i\mathrm{d}\sigma + \sum_{i=0}^{m_D} u_{D,i}J_{D,i}.\end{aligned}$$

Then we can rewrite (19)

$$\begin{aligned}\frac{\mathrm{d}W_D}{\mathrm{d}t} &= -\int_\Omega\Big(\frac{\boldsymbol{J}_n^2}{q\mu_n n}+\frac{\boldsymbol{J}_p^2}{q\mu_p p}\Big)\mathrm{d}\mathbf{x} - \int_\Omega qU_T R\log\frac{np}{n_\mathrm{i}^2}\,\mathrm{d}\mathbf{x}\\ &\quad + \sum_{i=0}^{m_D}\int_{\Gamma_{D,i}}\psi_\mathrm{bi}(\epsilon_s\partial_t\nabla\psi)\cdot\boldsymbol{\nu}_i\,\mathrm{d}\sigma + \boldsymbol{u}_D^\top\boldsymbol{J}_D.\end{aligned}$$

Energy associated to the circuit

The energy associated to the circuit without the device is

$$W_C = \frac{1}{2}\boldsymbol{y}^\top\boldsymbol{E}_1\boldsymbol{y} - \frac{1}{2}\boldsymbol{y}^\top\boldsymbol{\pi}^\top\boldsymbol{A}_D\boldsymbol{C}_D\boldsymbol{A}_D^\top\boldsymbol{\pi}\boldsymbol{y} = \frac{1}{2}\boldsymbol{u}^\top\boldsymbol{A}_C\boldsymbol{C}\boldsymbol{A}_C^\top\boldsymbol{u} + \frac{1}{2}\boldsymbol{i}_L^\top\boldsymbol{L}\boldsymbol{i}_L.\tag{20}$$

Using the circuit equations (1a)-(1b), and the symmetry property (11), we get

$$\begin{aligned}\frac{\mathrm{d}W_C}{\mathrm{d}t} &= \boldsymbol{y}^\top\boldsymbol{E}_1\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} - \boldsymbol{y}^\top\boldsymbol{\pi}^\top\boldsymbol{A}_D\boldsymbol{C}_D\boldsymbol{A}_D^\top\boldsymbol{\pi}\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t}\\ &= \boldsymbol{y}^\top\boldsymbol{E}_1\boldsymbol{P}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y}+\mathcal{F}_D(\boldsymbol{P}\boldsymbol{y})+\boldsymbol{s}(t)\right) - \boldsymbol{y}^\top\boldsymbol{\pi}^\top\boldsymbol{A}_D\boldsymbol{C}_D\boldsymbol{A}_D^\top\boldsymbol{\pi}\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t}.\end{aligned}$$

From the definition of $\boldsymbol{s}_D$ and of $\mathcal{F}_D$, equations (31) and (33), we have

$$\boldsymbol{\pi}^\top \boldsymbol{A}_D \boldsymbol{C}_D \boldsymbol{A}_D^\top \boldsymbol{\pi} \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \boldsymbol{s}_D + \mathcal{F}_D(\boldsymbol{P}\boldsymbol{y}).$$

Moreover, recalling (42), we have

$$\boldsymbol{E}_1 \boldsymbol{P} \boldsymbol{E}_1^{-1} \mathcal{F}_D(\boldsymbol{P}\boldsymbol{y}) = \boldsymbol{E}_1(\boldsymbol{P}+\boldsymbol{Q})\boldsymbol{E}_1^{-1}\mathcal{F}_D(\boldsymbol{P}\boldsymbol{y}) = \mathcal{F}_D(\boldsymbol{P}\boldsymbol{y}).$$

Then we get

$$\frac{\mathrm{d}W_C}{\mathrm{d}t} = \boldsymbol{y}^\top \boldsymbol{E}_1 \boldsymbol{P} \boldsymbol{E}_1^{-1} \left(\boldsymbol{A}_1 \boldsymbol{y} + \boldsymbol{s}(t)\right) - \boldsymbol{y}^\top \boldsymbol{s}_D.$$

Recalling (28) and (30), we recognize that

$$\boldsymbol{y}^\top \boldsymbol{s}_D = \boldsymbol{y}^\top \boldsymbol{\pi}^\top \boldsymbol{S}_D \boldsymbol{J}_D = \boldsymbol{u}_D^\top \boldsymbol{J}_D.$$

In conclusion, we find

$$\frac{\mathrm{d}W_C}{\mathrm{d}t} = \boldsymbol{y}^\top \boldsymbol{E}_1 \boldsymbol{P} \boldsymbol{E}_1^{-1} \left(\boldsymbol{A}_1 \boldsymbol{y} + \boldsymbol{s}(t)\right) - \boldsymbol{u}_D^\top \boldsymbol{J}_D. \tag{21}$$

The total energy of the coupled system

For the total energy of the system $W = W_D + W_C$ we obtain

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}t} = &- \int_\Omega \Big(\frac{\boldsymbol{J}_n^2}{q\mu_n n} + \frac{\boldsymbol{J}_p^2}{q\mu_p p}\Big)\mathrm{d}\mathbf{x} - \int_\Omega qU_T R \log\frac{np}{n_\mathrm{i}^2}\,\mathrm{d}\mathbf{x} \\ &+ \sum_{i=0}^{m_D} \int_{\Gamma_{D,i}} \psi_{\mathrm{bi}}(\epsilon_s \partial_t \nabla\psi)\cdot \boldsymbol{\nu}_i \,\mathrm{d}\sigma + \boldsymbol{y}^\top \boldsymbol{E}_1 \boldsymbol{P} \boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y} + \boldsymbol{s}(t)\right).\end{aligned}$$

An alternative definition for the total energy where no boundary terms appear in the total energy balance equation can be given using the steady-state solution $(n^\mathrm{e}, p^\mathrm{e}, \psi^\mathrm{e})$ defined via (2) and (3). We consider a shifted local energy w_D^*:

$$w_D^* = qU_T\Big\{n\Big(\log\frac{n}{n^\mathrm{e}} - 1\Big) + n^\mathrm{e} + p\Big(\log\frac{p}{p^\mathrm{e}} - 1\Big) + p^\mathrm{e}\Big\} + \frac{\epsilon_s}{2}|\nabla(\psi - \psi^\mathrm{e})|^2, \tag{22}$$

whose time derivative is

$$\begin{aligned}\partial_t w_D^* &= \partial_t w_D - qU_T \log\frac{n^\mathrm{e}}{n_\mathrm{i}}\partial_t n - qU_T\log\frac{p^\mathrm{e}}{n_\mathrm{i}}\partial_t p - \epsilon_s \nabla\psi^\mathrm{e}\cdot\partial_t\nabla\psi \\ &= \partial_t w_D - q\psi^\mathrm{e}\partial_t n + \psi^\mathrm{e}\partial_t p + \psi^\mathrm{e}\nabla\cdot(\epsilon_s\partial_t\nabla\psi) - \nabla\cdot(\psi^\mathrm{e}\epsilon_s\partial_t\nabla\psi) \\ &= \partial_t w_D + \psi^\mathrm{e}\partial_t(-qn + qp + \nabla\cdot(\epsilon_s\nabla\psi)) - \nabla\cdot(\psi^\mathrm{e}\epsilon_s\partial_t\nabla\psi) \\ &= \partial_t w_D - \nabla\cdot(\psi^\mathrm{e}\epsilon_s\partial_t\nabla\psi).\end{aligned}$$

Defining the shifted total energy associated to the device as

$$W_D^* = \int_\Omega w_D^* \mathrm{d}\,\mathbf{x}$$

we get

$$\frac{\mathrm{d}W_D^*}{\mathrm{d}t} = \frac{\mathrm{d}W_D}{\mathrm{d}t} - \int_\Omega \nabla\cdot(\psi^{\mathrm{e}}\epsilon_s\partial_t\nabla\psi)\mathrm{d}\mathbf{x} = \frac{\mathrm{d}W_D}{\mathrm{d}t} - \sum_{i=0}^{m_D}\int_{\Gamma_{D,i}}\psi_{\mathrm{bi}}(\epsilon_s\partial_t\nabla\psi)\cdot\boldsymbol{\nu}\mathrm{d}\mathbf{x}.$$

Then, the shifted total energy $W^* = W_D^* + W_C$ satisfies

$$\frac{\mathrm{d}W^*}{\mathrm{d}t} = -\int_\Omega\Big(\frac{\boldsymbol{J}_n^2}{q\mu_n n} + \frac{\boldsymbol{J}_p^2}{q\mu_p p}\Big)\mathrm{d}\mathbf{x} - \int_\Omega qU_T R\log\frac{np}{n_{\mathrm{i}}^2}\,\mathrm{d}\mathbf{x} + \boldsymbol{y}^\top\boldsymbol{E}_1\boldsymbol{P}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y}+\boldsymbol{s}(t)\right).$$

We introduce the following notation: for any function $f(n,p,\psi,\boldsymbol{y})$ we write $f^{\mathrm{e}} = f(n^{\mathrm{e}},p^{\mathrm{e}},\psi^{\mathrm{e}},\boldsymbol{y}^{\mathrm{e}})$, where $\boldsymbol{y}^{\mathrm{e}} = 0$.

For any $\alpha > 0$, we define the Liapunov functional

$$\begin{aligned}\mathcal{H}_\alpha &= \int_\Omega qU_T\Big\{(n+\alpha)\Big(\log\frac{n+\alpha}{n^{\mathrm{e}}+\alpha}-1\Big)+n^{\mathrm{e}}\Big\}\mathrm{d}\mathbf{x}\\ &\quad+\int_\Omega qU_T\Big\{(p+\alpha)\Big(\log\frac{p+\alpha}{p^{\mathrm{e}}+\alpha}-1\Big)+p^{\mathrm{e}}\Big\}\mathrm{d}\mathbf{x}+\int_\Omega\frac{\epsilon_s}{2}|\nabla(\psi-\psi^{\mathrm{e}})|^2\mathrm{d}\mathbf{x}+W_C.\end{aligned} \tag{23}$$

The parameter α is needed to ensure that $\mathcal{H}_\alpha$ is well defined, since n and p may vanish locally. Formally, as α tends to zero, the Liapunov functional $\mathcal{H}_\alpha$ tends to the total physical energy W^* of the coupled system. This assertion can be stated in a precise way by observing that the function

$$g(z) = \begin{cases} z\log z & \text{if } z>0,\\ 0 & \text{if } z=0\end{cases} \tag{24}$$

is continuous for $z\in[0,\infty)$. Then the functional $\mathcal{H} = \lim_{\alpha\to 0}\mathcal{H}_\alpha$ is well defined, and we have $\mathcal{H} = W^*$. We define the following functions $\psi_n(\mathbf{x},t)$, $\psi_p(\mathbf{x},t)$, $\psi_n^{\mathrm{e}}(\mathbf{x})$ and $\psi_p^{\mathrm{e}}(\mathbf{x})$:

$$\psi_n = \psi - U_T\log\frac{n+\alpha}{n_{\mathrm{i}}}, \qquad \psi_p = \psi + U_T\log\frac{p+\alpha}{n_{\mathrm{i}}}, \tag{25}$$

$$\psi_n^{\mathrm{e}} = \psi^{\mathrm{e}} - U_T\log\frac{n^{\mathrm{e}}+\alpha}{n_{\mathrm{i}}}, \qquad \psi_p^{\mathrm{e}} = \psi^{\mathrm{e}} + U_T\log\frac{p^{\mathrm{e}}+\alpha}{n_{\mathrm{i}}}. \tag{26}$$

Lemma 1 *For the functional $\mathcal{H}_\alpha$ defined in* (23)*, we have*

$$\begin{aligned}\mathcal{H}_\alpha(t) &= \mathcal{H}_\alpha(0) + \int_0^t\int_\Omega\big(\boldsymbol{J}_n\cdot\nabla(\psi_n-\psi_n^e)+\boldsymbol{J}_p\cdot\nabla(\psi_p-\psi_p^e)\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau\\ &\quad-\int_0^t\int_\Omega qU_T R\log\frac{(n+\alpha)(p+\alpha)}{(n^e+\alpha)(p^e+\alpha)}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau+\int_0^t\boldsymbol{y}^\top\boldsymbol{E}_1\boldsymbol{P}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y}+\boldsymbol{s}(\tau)\right)\,\mathrm{d}\tau.\end{aligned} \tag{27}$$

Proof. First we compute the time derivative of the functional $\mathcal{H}_\alpha$ which, for any solution of the full problem, is a function of time only and its time derivative is

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}_\alpha}{\mathrm{d}t} &= \int_\Omega qU_T\Big(\log\frac{n+\alpha}{n^{\mathrm{e}}+\alpha}\partial_t n+\log\frac{p+\alpha}{p^{\mathrm{e}}+\alpha}\partial_t p\Big)\mathrm{d}\mathbf{x}\\ &\quad+\int_\Omega\epsilon_s\nabla(\psi-\psi^{\mathrm{e}})\cdot\partial_t\nabla\psi\,\mathrm{d}\mathbf{x}+\boldsymbol{y}^\top\boldsymbol{E}_1\boldsymbol{P}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y}+\boldsymbol{s}(t)\right)-\boldsymbol{u}_D^\top\boldsymbol{J}_D\\ &= \int_\Omega U_T\Big(\log\frac{n+\alpha}{n^{\mathrm{e}}+\alpha}\nabla\cdot\boldsymbol{J}_n-\log\frac{p+\alpha}{p^{\mathrm{e}}+\alpha}\nabla\cdot\boldsymbol{J}_p\Big)\mathrm{d}\mathbf{x}\\ &\quad-\int_\Omega qU_T R\log\frac{(n+\alpha)(p+\alpha)}{(n^{\mathrm{e}}+\alpha)(p^{\mathrm{e}}+\alpha)}\mathrm{d}\mathbf{x}+\int_\Omega\epsilon_s\nabla(\psi-\psi^{\mathrm{e}})\cdot\partial_t\nabla\psi\,\mathrm{d}\mathbf{x}\\ &\quad+\boldsymbol{y}^\top\boldsymbol{E}_1\boldsymbol{P}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y}+\boldsymbol{s}(t)\right)-\boldsymbol{u}_D^\top\boldsymbol{J}_D,\end{aligned} \tag{28}$$

in which we have used the model equations in (1).

We integrate by parts the first term on the right-hand side of (28), and using the definitions (25) and (26) we obtain:

$$\int_\Omega U_T \left(\log \frac{n+\alpha}{n^{\mathsf{e}}+\alpha} \nabla \cdot \boldsymbol{J}_n - \log \frac{p+\alpha}{p^{\mathsf{e}}+\alpha} \nabla \cdot \boldsymbol{J}_p \right) \mathsf{dx}$$
$$= \int_\Omega \big(\boldsymbol{J}_n \cdot \nabla(\psi_n - \psi_n^{\mathsf{e}}) + \boldsymbol{J}_p \cdot \nabla(\psi_p - \psi_p^{\mathsf{e}}) \big) \, \mathsf{dx} - \int_\Omega (\boldsymbol{J}_n + \boldsymbol{J}_p) \cdot \nabla(\psi - \psi^{\mathsf{e}}) \, \mathsf{dx}.$$

Inserting this in (28) we get

$$\begin{aligned} \frac{\mathsf{d}\mathcal{H}_\alpha}{\mathsf{d}t} &= \int_\Omega \big(\boldsymbol{J}_n \cdot \nabla(\psi_n - \psi_n^{\mathsf{e}}) + \boldsymbol{J}_p \cdot \nabla(\psi_p - \psi_p^{\mathsf{e}}) \big) \, \mathsf{dx} \\ &\quad - \int_\Omega q U_T R \log \frac{(n+\alpha)(p+\alpha)}{(n^{\mathsf{e}}+\alpha)(p^{\mathsf{e}}+\alpha)} \mathsf{dx} \\ &\quad - \int_\Omega (-\epsilon_s \partial_t \nabla \psi + \boldsymbol{J}_n + \boldsymbol{J}_p) \cdot \nabla(\psi - \psi^{\mathsf{e}}) \, \mathsf{dx} \\ &\quad + \boldsymbol{y}^\top \boldsymbol{E}_1 \boldsymbol{P} \boldsymbol{E}_1^{-1} \left(\boldsymbol{A}_1 \boldsymbol{y} + \boldsymbol{s}(t) \right) - \boldsymbol{u}_D^\top \boldsymbol{J}_D. \end{aligned} \tag{29}$$

Integrating by parts the last integral and using (11) and the definition of $\boldsymbol{J}_D$, we obtain

$$\begin{aligned} \frac{\mathsf{d}\mathcal{H}_\alpha}{\mathsf{d}t} &= \int_\Omega \big(\nabla(\psi_n - \psi_n^{\mathsf{e}}) \cdot \boldsymbol{J}_n + \nabla(\psi_p - \psi_p^{\mathsf{e}}) \cdot \boldsymbol{J}_p \big) \, \mathsf{dx} \\ &\quad - \int_\Omega q U_T R \log \frac{(n+\alpha)(p+\alpha)}{(n^{\mathsf{e}}+\alpha)(p^{\mathsf{e}}+\alpha)} \mathsf{dx} + \boldsymbol{y}^\top \boldsymbol{E}_1 \boldsymbol{P} \boldsymbol{E}_1^{-1} \left(\boldsymbol{A}_1 \boldsymbol{y} + \boldsymbol{s}(t) \right). \end{aligned} \tag{30}$$

Integrating on time, we get the thesis. ■

3.3 A priori estimates

In the following lemmas we prove the a priori bounds for the solution.

Lemma 2 *Assume that the recombination is of the type*

$$R(n,p) = r(n,p)\Big(np - n_i^2\Big), \text{ with } 0 \le r(n,p)(1+|n|+|p|) \le \bar{r} \tag{31}$$

were $\bar{r}$ is a constant, and the mobilities are bounded. Then there exist constants C_1, C_2 independent of t such that for any solution n, p of the system it holds that

$$\|n(\cdot,t)\|_1 + \|p(\cdot,t)\|_1 + \|\mathbf{E}(\cdot,t)\|_2^2 + |\boldsymbol{y}(t)|^2 \le C_1 e^{C_2 t}, \quad t \ge 0. \tag{32}$$

Proof. We estimate all the terms on the right-hand side of (27). We start observing that, thanks to (25), we have $n = n_{\mathsf{i}} \exp\left(\frac{\psi - \psi_n}{U_T}\right) - \alpha$, which leads to

$$\nabla n = n_{\mathsf{i}} \exp\Big(\frac{\psi - \psi_n}{U_T}\Big) \frac{1}{U_T} \nabla(\psi - \psi_n) = \frac{n+\alpha}{U_T} \nabla(\psi - \psi_n).$$

Then, the electron current can be written as

$$\begin{aligned} \boldsymbol{J}_n &= q\mu_n(U_T \nabla n - n \nabla \psi) = q\mu_n((n+\alpha)\nabla(\psi - \psi_n) - n\nabla\psi) \\ &= q\mu_n(-(n+\alpha)\nabla\psi_n + \alpha\nabla\psi), \end{aligned} \tag{33}$$

and for the first term on the right-hand side of (27) we find

$$\begin{aligned}
\int_\Omega \boldsymbol{J}_n \cdot \nabla(\psi_n - \psi_n^{\mathrm{e}})\,\mathbf{dx} &= \int_\Omega q\mu_n(-(n+\alpha)\nabla\psi_n + \alpha\nabla\psi)\cdot\nabla(\psi_n - \psi_n^{\mathrm{e}})\,\mathbf{dx} \\
&= \int_\Omega q\mu_n\Big(-(n+\alpha)|\nabla\psi_n|^2 + (n+\alpha)\nabla\psi_n\cdot\nabla\psi_n^{\mathrm{e}} + \alpha\nabla\psi\cdot(\nabla\psi_n - \nabla\psi_n^{\mathrm{e}})\Big)\,\mathbf{dx} \\
&\le \int_\Omega q\mu_n\Big(-(n+\alpha)|\nabla\psi_n|^2 + (n+\alpha)|\nabla\psi_n||\nabla\psi_n^{\mathrm{e}}| + \alpha|\nabla\psi|(|\nabla\psi_n| + |\nabla\psi_n^{\mathrm{e}}|)\Big)\,\mathbf{dx} \\
&\le \int_\Omega q\mu_n\Big((n+\alpha)|\nabla\psi_n^{\mathrm{e}}|^2 + \frac{\alpha^2}{(n+\alpha)}|\nabla\psi|^2\Big)\,\mathbf{dx} \le c(1 + \|n\|_1 + \|\psi\|_2^2).
\end{aligned}$$

In a similar way, we can estimate

$$\int_\Omega \boldsymbol{J}_p \cdot \nabla(\psi_p - \psi_p^{\mathrm{e}})\,\mathbf{dx} \le c(1 + \|p\|_1 + \|\psi\|_2^2).$$

For the recombination term, we have

$$\begin{aligned}
&\int_\Omega qU_T R(n,p)\log\frac{(n+\alpha)(p+\alpha)}{(n^{\mathrm{e}}+\alpha)(p^{\mathrm{e}}+\alpha)}\,\mathbf{dx} \\
&\qquad = \int_\Omega qU_T r(n,p)\Big(np - n_{\mathrm{i}}^2\Big)\log\frac{(n+\alpha)(p+\alpha)}{(n^{\mathrm{e}}+\alpha)(p^{\mathrm{e}}+\alpha)}\,\mathbf{dx}.
\end{aligned}$$

Observing that $n^{\mathrm{e}}p^{\mathrm{e}} = n_{\mathrm{i}}^2$, we have the identity:

$$np - n_{\mathrm{i}}^2 = (n+\alpha)(p+\alpha) - (n^{\mathrm{e}}+\alpha)(p^{\mathrm{e}}+\alpha) - \alpha(n + p - n^{\mathrm{e}} - p^{\mathrm{e}}), \tag{34}$$

which leads to

$$\begin{aligned}
&-\int_\Omega qU_T R\log\frac{(n+\alpha)(p+\alpha)}{(n^{\mathrm{e}}+\alpha)(p^{\mathrm{e}}+\alpha)}\,\mathbf{dx} \\
&= -\,qU_T\int_\Omega r\Big((n+\alpha)(p+\alpha) - (n^{\mathrm{e}}+\alpha)(p^{\mathrm{e}}+\alpha)\Big)\log\frac{(n+\alpha)(p+\alpha)}{(n^{\mathrm{e}}+\alpha)(p^{\mathrm{e}}+\alpha)}\,\mathbf{dx} \\
&\quad + qU_T\int_\Omega \alpha r(n + p - n^{\mathrm{e}} - p^{\mathrm{e}})\log\frac{(n+\alpha)(p+\alpha)}{(n^{\mathrm{e}}+\alpha)(p^{\mathrm{e}}+\alpha)}\,\mathbf{dx} \\
&\le qU_T\int_\Omega \alpha r(n + p - n^{\mathrm{e}} - p^{\mathrm{e}})\log\frac{(n+\alpha)(p+\alpha)}{(n^{\mathrm{e}}+\alpha)(p^{\mathrm{e}}+\alpha)}\,\mathbf{dx} \\
&\le c\int_\Omega \left| r\log\frac{(n+\alpha)(p+\alpha)}{(n^{\mathrm{e}}+\alpha)(p^{\mathrm{e}}+\alpha)}\right| |n + p + n^{\mathrm{e}} + p^{\mathrm{e}}|\,\mathbf{dx} \\
&\le c(1 + \|n\|_1 + \|p\|_1).
\end{aligned}$$

The last inequality follows from the hypothesis on $r(n,p)$ (31), and from the inequality

$$\left|\frac{\log(z+a)}{z+a}\right| \le \max\left(\frac{1}{e}, \left|\frac{\log a}{a}\right|\right), \quad \forall z \ge 0, a > 0.$$

Finally, for the last term, recalling (1b), we have immediately

$$\boldsymbol{y}^\top \boldsymbol{E}_1 \boldsymbol{P}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y} + \boldsymbol{s}(t)\right) \le c(1 + |\boldsymbol{y}|^2).$$

Combining the previous estimates, we get

$$\mathcal{H}_\alpha(t) \leq c_1 + c_2 \int_0^t \Big(1 + \|n\|_1 + \|p\|_1 + \|\nabla\psi\|_2^2 + |\boldsymbol{y}|^2\Big)\mathsf{d}s. \tag{35}$$

The thesis follows after noting that

$$\mathcal{H}_\alpha(t) \geq c\Big(1 + \|n\|_1 + \|p\|_1 + \|\nabla\psi\|_2^2 + |\boldsymbol{y}|^2\Big),$$

with the help of Gronwall's lemma. ∎

Lemma 3 *Under the same hypothesis of Lemma 2, there exists a constant* $c = c(T)$ *such that for all* $t \leq T$ *we have*

$$\|n(\cdot,t)\|_2^2 + \|p(\cdot,t)\|_2^2 + \int_0^t \Big(\|\nabla n(\cdot,s)\|_2^2 + \|\nabla p(\cdot,s)\|_2^2\Big)d\tau \leq c. \tag{36}$$

Proof. We consider the weak formulation (4a) with $\xi_n = n - n^{\mathsf{e}}$ and integrate with respect to time

$$\frac{1}{2}\Big(\|n(\cdot,t) - n^{\mathsf{e}}\|_2^2 - \|n(\cdot,0) - n^{\mathsf{e}}\|_2^2\Big) + D_n \int_0^t \int_\Omega |\nabla(n - n^{\mathsf{e}})|^2 \mathsf{d}x\mathsf{d}\tau$$
$$= -D_n \int_0^t \int_\Omega \nabla n^{\mathsf{e}} \cdot \nabla(n - n^{\mathsf{e}})\mathsf{d}x\mathsf{d}\tau + \int_0^t (\mu_n n\nabla\psi, \nabla(n - n^{\mathsf{e}}))\mathsf{d}\tau - \int_0^t (R, n - n^{\mathsf{e}})\mathsf{d}\tau.$$

We have the estimate

$$-\int_0^t \int_\Omega \nabla n^{\mathsf{e}} \cdot \nabla(n - n^{\mathsf{e}})\mathsf{d}x\mathsf{d}\tau \leq \frac{1}{2}\int_0^t \Big(\|\nabla n(\cdot,\tau)\|_2^2 + \|\nabla n^{\mathsf{e}}\|_2^2\Big)\mathsf{d}\tau.$$

Furthermore, integrating by parts and using the Poisson equation, we obtain

$$\int_0^t \int_\Omega n\nabla\psi \cdot \nabla(n - n^{\mathsf{e}})\,\mathsf{d}x\mathsf{d}\tau$$
$$= -\int_0^t \int_\Omega (n - n^{\mathsf{e}})(\nabla\cdot\nabla\psi)(n - n^{\mathsf{e}})\,\mathsf{d}x\mathsf{d}\tau - \int_0^t \int_\Omega n^{\mathsf{e}}\mathbf{E}\cdot\nabla(n - n^{\mathsf{e}})\,\mathsf{d}x\mathsf{d}\tau$$
$$\leq \frac{q}{2\epsilon_s}\int_0^t \int_\Omega (C - n + p)(n - n^{\mathsf{e}})^2\,\mathsf{d}x\mathsf{d}\tau + \int_0^t \Big(c\|\mathbf{E}\|_2^2 + \frac{U_T}{4}\|\nabla(n - n^{\mathsf{e}})\|_2^2\Big)\,\mathsf{d}\tau$$
$$\leq \frac{q}{2\epsilon_s}\int_0^t \int_\Omega n^2(p - n)\,\mathsf{d}x\mathsf{d}\tau + \int_0^t c\|\mathbf{E}\|_2^2\,\mathsf{d}\tau + \frac{U_T}{4}\int_0^t \|\nabla(n - n^{\mathsf{e}})\|_2^2\,\mathsf{d}\tau$$
$$\leq c(T)\Big\{1 + \|n\|_2^2 + \|p\|_2^2\Big\} + \frac{q}{2\epsilon_s}\int_0^t \int_\Omega n^2(p - n)\,\mathsf{d}x\mathsf{d}\tau + \frac{U_T}{4}\int_0^t \|\nabla(n - n^{\mathsf{e}})\|_2^2\,\mathsf{d}\tau.$$

We have used the result in Lemma 2 to estimate all linear terms in n and p and the L^2-norm of $\mathbf{E}$. For the recombination generation term whose form is in (6) and for which we have assumed (31) we have

$$|(R, n - n^{\mathsf{e}})| \leq \int_\Omega \big|r(n,p)(np - n_{\mathsf{i}}^2)(n - n^{\mathsf{e}})\mathsf{d}x\big| \leq c\left(1 + \|n\|_2^2 + \|p\|_2^2\right)$$

Finally, we get the estimate

$$\|n(t) - n^{\mathsf{e}}\|_2^2 + \frac{D_n}{2}\int_0^t \int_\Omega |\nabla(n - n^{\mathsf{e}})|^2\mathsf{d}x\mathsf{d}\tau$$
$$\leq c(T)\left\{1 + \int_0^t \left(\|n\|_2^2 + \|p\|_2^2\right)\mathsf{d}\tau\right\} + \frac{q}{\epsilon_s}\int_0^t \int_\Omega n^2(p - n)\,\mathsf{d}x\mathsf{d}\tau,$$

which leads to

$$\|n(t)\|_2^2 + \int_0^t \|\nabla n\|_2^2 \,\mathrm{d}\tau \le c(T)\left\{1 + \int_0^t \left(\|n\|_2^2 + \|p\|_2^2\right)\mathrm{d}\tau\right\} + c_n \int_0^t \int_\Omega n^2(p-n)\,\mathrm{d}x\mathrm{d}\tau,$$

for some positive constant c_n. Similarly for p we have

$$\|p(t)\|_2^2 + \int_0^t \|\nabla p\|_2^2 \,\mathrm{d}\tau \le c(T)\left\{1 + \int_0^t \left(\|n\|_2^2 + \|p\|_2^2\right)\mathrm{d}\tau\right\} - c_p \int_0^t \int_\Omega p^2(p-n)\,\mathrm{d}x\mathrm{d}\tau,$$

for some positive constant c_p. Summing up we get

$$\begin{aligned}
\|n(t)\|_2^2 + \|p(t)\|_2^2 + \int_0^t \left(\|\nabla n\|_2^2 + \|\nabla p\|_2^2\right)\,\mathrm{d}\tau \\
&\le c(T)\left\{1 + \int_0^t \left(\|n\|_2^2 + \|p\|_2^2\right)\mathrm{d}\tau\right\} + c_1\left\{\int_0^t \int_\Omega (n^2-p^2)(p-n)\,\mathrm{d}x\mathrm{d}\tau\right\} \\
&\le c(T)\left\{1 + \int_0^t \left(\|n\|_2^2 + \|p\|_2^2\right)\mathrm{d}\tau\right\},
\end{aligned}$$

in which we applied $(n^2 - p^2)(p-n) = -(n+p)(n-p)^2 \le 0$. Using Gronwall's lemma, we get the thesis. ■

Combining the results in the previous Lemmas we have

Lemma 4 *There exists a constant $c = c(T)$ such that for all $t \le T$ we have*

$$\|n(\cdot,t)\|_2^2 + \|p(\cdot,t)\|_2^2 + |\boldsymbol{y}|^2 + \int_0^t \Big(\|\nabla n(\cdot,\tau)\|_2^2 + \|\nabla p(\cdot,\tau)\|_2^2\Big) d\tau \le c. \tag{37}$$

The main result of this paper is the following global existence and uniqueness theorem.

Theorem 2 *[Global existence and uniqueness] Let the source functions $\boldsymbol{i}_I(t)$ and $\boldsymbol{v}_V(t)$ be continuous, let the network matrices be symmetric, positive definite and the topological conditions* (28) *and* (31) *be fulfilled, and let diffusivities and mobilities be constant. Then problem* (2)-(4) *admits a unique solution on the time interval $[0,T]$ for any $T \in (0,\infty)$.*

Proof. The local solution proved in Theorem 1 can be prolonged using the results in Lemmas 2, 3 and 4 ■

4 Asymptotic behaviour of the solution

In this section, we study the asymptotic behaviour of the solution to (1) as t tends to infinity. For a single device without coupling with an electric circuit, it is known that all solutions relax to equilibrium [26] if the applied potential vanishes. Therefore, in the coupled system, we need to demand appropriate topological conditions on the electric network to enable such a result. We derive sufficient conditions on the network topology during the development of this section.

Theorem 3 (Asymptotic behaviour of the solution) *Under the assumption of Theorem 2, let the network system satisfy the additional topological conditions*

$$\ker \boldsymbol{A}_R^\top \subset \ker \boldsymbol{A}_C^\top \subset \ker \boldsymbol{A}_L^\top. \tag{1}$$

Moreover let the sources satisfy the decaying conditions

$$\lim_{t\to\infty} |\boldsymbol{s}(t)|^2 = 0, \qquad \int_0^t |\boldsymbol{s}(\tau)|^2 d\tau \le K \text{ for all } t \ge 0. \tag{2}$$

Then as time tends to infinity, the solution of the problem (1) *approaches the equilibrium states meaning that there exists* $\bar{\boldsymbol{i}}_L \in \ker \boldsymbol{A}_L^\top \subset \mathbb{R}^{n_L}$ *such that*

$$\begin{array}{ll} n \to n^e, \quad p \to p^e & \text{in } L^1, \\ \psi \to \psi^e, & \text{in } H^1, \\ \boldsymbol{u} \to 0, \quad \boldsymbol{i}_V \to 0, \quad \boldsymbol{i}_L \to \bar{\boldsymbol{i}}_L & \text{in Euclidean norm.} \end{array} \tag{3}$$

Proof. The proof follows along the line of Gajewski's analogous result in [26], modified to take into account the coupling of the device with the circuit.

Summing and subtracting the term $q\mu_n(n+\alpha)\nabla\psi_n^{\mathrm{e}}$ in the definition of the current densities $\boldsymbol{J}_n$ rewritten as in (33), we get

$$\begin{aligned} \boldsymbol{J}_n &= -q\mu_n(n+\alpha)\nabla(\psi_n - \psi_n^{\mathrm{e}}) - q\mu_n(n+\alpha)\nabla\psi_n^{\mathrm{e}} + q\mu_n\alpha\nabla\psi \\ &= -q\mu_n(n+\alpha)\nabla(\psi_n - \psi_n^{\mathrm{e}}) + q\mu_n\alpha\nabla\psi - q\mu_n\alpha\frac{n+\alpha}{n^{\mathrm{e}}+\alpha}\nabla\psi^{\mathrm{e}} \\ &= -q\mu_n(n+\alpha)\nabla(\psi_n - \psi_n^{\mathrm{e}}) - q\mu_n\alpha\frac{n-n^{\mathrm{e}}}{n^{\mathrm{e}}+\alpha}\nabla\psi^{\mathrm{e}} + q\mu_n\alpha\nabla(\psi - \psi^{\mathrm{e}}), \end{aligned}$$

in which we have used that from (26) we find $\nabla\psi_n^{\mathrm{e}} = \frac{\alpha}{n^{\mathrm{e}}+\alpha}\nabla\psi^{\mathrm{e}}$ and $\nabla\psi_p^{\mathrm{e}} = \frac{\alpha}{p^{\mathrm{e}}+\alpha}\nabla\psi^{\mathrm{e}}$. Similarly, for $\boldsymbol{J}_p$ we get

$$\boldsymbol{J}_p = -q\mu_p(p+\alpha)\nabla(\psi_p - \psi_p^{\mathrm{e}}) - q\mu_p\alpha\frac{p-p^{\mathrm{e}}}{p^{\mathrm{e}}+\alpha}\nabla\psi^{\mathrm{e}} + q\mu_p\alpha\nabla(\psi - \psi^{\mathrm{e}}).$$

Then we find, for arbitrary time $t_1 \ge t_0 \ge 0$,

$$\begin{aligned} \int_{t_0}^{t_1}\int_\Omega \boldsymbol{J}_n\cdot\nabla(\psi_n - \psi_n^{\mathrm{e}})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau &= -\int_{t_0}^{t_1}\int_\Omega q\mu_n(n+\alpha)|\nabla(\psi_n - \psi_n^{\mathrm{e}})|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau \\ &+ \int_{t_0}^{t_1}\int_\Omega q\mu_n\alpha\left\{\nabla(\psi - \psi^{\mathrm{e}}) - \frac{n-n^{\mathrm{e}}}{n^{\mathrm{e}}+\alpha}\nabla\psi^{\mathrm{e}}\right\}\cdot\nabla(\psi_n - \psi_n^{\mathrm{e}})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau \\ &\le -\frac{q}{2}\int_{t_0}^{t_1}\int_\Omega \mu_n(n+\alpha)|\nabla(\psi_n - \psi_n^{\mathrm{e}})|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau \\ &+ q\alpha\int_{t_0}^{t_1}\int_\Omega \frac{\mu_n}{2}\left|\mathbf{E} - \mathbf{E}^{\mathrm{e}} - \frac{\mathbf{E}^{\mathrm{e}}(n-n^{\mathrm{e}})}{n^{\mathrm{e}}+\alpha}\right|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau. \end{aligned} \tag{4}$$

In a similar way we get

$$\begin{aligned} \int_{t_0}^{t_1}\int_\Omega \boldsymbol{J}_p\cdot\nabla(\psi_p - \psi_p^{\mathrm{e}})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau &\le -\frac{q}{2}\int_{t_0}^{t_1}\int_\Omega \mu_p(p+\alpha)|\nabla(\psi_p - \psi_p^{\mathrm{e}})|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau \\ &+ q\alpha\int_{t_0}^{t_1}\int_\Omega \frac{\mu_p}{2}\left|\mathbf{E} - \mathbf{E}^{\mathrm{e}} - \frac{\mathbf{E}^{\mathrm{e}}(p-p^{\mathrm{e}})}{p^{\mathrm{e}}+\alpha}\right|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau. \end{aligned} \tag{5}$$

d'

Using (4) and (5) in Lemma 1, considering the definition (6) and (34), for arbitrary time $t_1 \geq t_0 \geq 0$, we can write

$$\begin{aligned}
&\mathcal{H}_\alpha(t_1) + \frac{q}{2}\int_{t_0}^{t_1}\int_\Omega \left(\mu_n(n+\alpha)|\nabla(\psi_n - \psi_n^{\mathsf{e}})|^2 + \mu_p(p+\alpha)|\nabla(\psi_p - \psi_p^{\mathsf{e}})|^2\right) \mathrm{d}\mathbf{x}\,\mathrm{d}\tau \\
&\quad + q\int_{t_0}^{t_1}\int_\Omega U_T r[(n+\alpha)(p+\alpha) - (n^{\mathsf{e}}+\alpha)(p^{\mathsf{e}}+\alpha)] \log\frac{(n+\alpha)(p+\alpha)}{(n^{\mathsf{e}}+\alpha)(p^{\mathsf{e}}+\alpha)} \mathrm{d}\mathbf{x}\,\mathrm{d}\tau \\
&\quad \leq \mathcal{H}_\alpha(t_0) + \int_{t_0}^{t_1} \boldsymbol{y}^\top \boldsymbol{E}_1 \boldsymbol{P} \boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y} + \boldsymbol{s}\right)\,\mathrm{d}\tau \\
&\qquad + \alpha q \int_{t_0}^{t_1}\int_\Omega \left[\frac{\mu_n}{2}\left|\mathbf{E} - \mathbf{E}^{\mathsf{e}} - \frac{\mathbf{E}^{\mathsf{e}}(n-n^{\mathsf{e}})}{n^{\mathsf{e}}+\alpha}\right|^2 + \frac{\mu_p}{2}\left|\mathbf{E} - \mathbf{E}^{\mathsf{e}} - \frac{\mathbf{E}^{\mathsf{e}}(p-p^{\mathsf{e}})}{p^{\mathsf{e}}+\alpha}\right|^2\right. \\
&\qquad\qquad \left. + U_T r(n+p-n^{\mathsf{e}}-p^{\mathsf{e}})\log\frac{(n+\alpha)(p+\alpha)}{(n^{\mathsf{e}}+\alpha)(p^{\mathsf{e}}+\alpha)}\right] \mathrm{d}\mathbf{x}\,\mathrm{d}\tau.
\end{aligned} \tag{6}$$

The last integral term in (6) is bounded, so we can pass to the limit for $\alpha \to 0$ and find

$$\begin{aligned}
&\mathcal{H}(t_1) + \frac{q}{2}\int_{t_0}^{t_1}\int_\Omega \left(\mu_n n|\nabla(\phi_n - \phi_n^{\mathsf{e}})|^2 + \mu_p p|\nabla(\phi_p - \phi_p^{\mathsf{e}})|^2\right)\mathrm{d}\mathbf{x}\,\mathrm{d}\tau \\
&\quad \leq \mathcal{H}(t_0) + \int_{t_0}^{t_1} \boldsymbol{y}^\top \boldsymbol{E}_1\boldsymbol{P}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y} + \boldsymbol{s}\right)\,\mathrm{d}\tau
\end{aligned} \tag{7}$$

where ϕ_n, ϕ_n^{e}, ϕ_p and ϕ_p^{e} are the physical quasi-Fermi levels which means

$$\phi_n = \psi - U_T \log\frac{n}{n_{\mathsf{i}}}, \qquad \phi_p = \psi + U_T\log\frac{p}{n_{\mathsf{i}}}, \tag{8}$$

$$\phi_n^{\mathsf{e}} = \psi^{\mathsf{e}} - U_T\log\frac{n^{\mathsf{e}}}{n_{\mathsf{i}}}, \qquad \phi_p^{\mathsf{e}} = \psi^{\mathsf{e}} + U_T\log\frac{p^{\mathsf{e}}}{n_{\mathsf{i}}}, \tag{9}$$

and

$$\begin{aligned}
\mathcal{H}(t) &= \int_\Omega qU_T\left[n\left(\log\frac{n}{n^{\mathsf{e}}} - 1\right) + n^{\mathsf{e}} + p\left(\log\frac{p}{p^{\mathsf{e}}} - 1\right) + p^{\mathsf{e}}\right]\mathrm{d}\mathbf{x} \\
&\quad + \int_\Omega \frac{\epsilon_s}{2}|\mathbf{E} - \mathbf{E}^{\mathsf{e}}|^2\mathrm{d}\mathbf{x} + \frac{1}{2}\boldsymbol{u}^\top\boldsymbol{A}_C\boldsymbol{C}\boldsymbol{A}_C^\top\boldsymbol{u} + \frac{1}{2}\boldsymbol{i}_L^\top\boldsymbol{L}\boldsymbol{i}_L \\
&= \mathcal{H}^*(t) + \frac{1}{2}\boldsymbol{u}^\top(t)\boldsymbol{A}_C\boldsymbol{C}\boldsymbol{A}_C^\top\boldsymbol{u}(t) + \frac{1}{2}\boldsymbol{i}_L^\top(t)\boldsymbol{L}\boldsymbol{i}_L(t).
\end{aligned} \tag{10}$$

Using Definition 1, and the definitions of $\boldsymbol{y}$, $\boldsymbol{z}$, $\boldsymbol{E}_1$ and $\boldsymbol{A}_1$, we can write

$$\begin{aligned}
&\boldsymbol{y}^\top\boldsymbol{E}_1\boldsymbol{P}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y} + \boldsymbol{s}\right) \\
&\quad = \boldsymbol{y}^\top\boldsymbol{E}_1(\boldsymbol{P}+\boldsymbol{Q})\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y}+\boldsymbol{s}\right) - \boldsymbol{y}^\top\boldsymbol{E}_1\boldsymbol{Q}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y}+\boldsymbol{s}\right) \\
&\quad = \boldsymbol{y}^\top\left(\boldsymbol{A}_1\boldsymbol{y}+\boldsymbol{s}\right) - \boldsymbol{y}^\top\boldsymbol{E}_1\boldsymbol{z} = \boldsymbol{y}^\top\left(\boldsymbol{A}\boldsymbol{y}+\boldsymbol{s}\right) + \boldsymbol{y}^\top\boldsymbol{A}\boldsymbol{z}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
-\boldsymbol{z}^\top\boldsymbol{A}\boldsymbol{z} &= \boldsymbol{z}^\top\boldsymbol{E}_1\boldsymbol{z} = \boldsymbol{z}^\top\boldsymbol{E}_1\boldsymbol{Q}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y}+\boldsymbol{s}\right) \\
&= \boldsymbol{z}^\top\boldsymbol{E}_1(\boldsymbol{P}+\boldsymbol{Q})\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y}+\boldsymbol{s}\right) - \boldsymbol{z}^\top\boldsymbol{E}_1\boldsymbol{P}\boldsymbol{E}_1^{-1}\left(\boldsymbol{A}_1\boldsymbol{y}+\boldsymbol{s}\right) \\
&= \boldsymbol{z}^\top\left(\boldsymbol{A}_1\boldsymbol{y}+\boldsymbol{s}\right) - \boldsymbol{z}^\top\boldsymbol{E}_1\boldsymbol{y} = \boldsymbol{z}^\top\left(\boldsymbol{A}\boldsymbol{y}+\boldsymbol{s}\right),
\end{aligned}$$

since $\boldsymbol{Q}^\top \boldsymbol{E}_1 \boldsymbol{P} = \boldsymbol{Q}^\top(\boldsymbol{E} - \boldsymbol{A}\boldsymbol{Q})\boldsymbol{P} = \boldsymbol{O}$. Summing up, we obtain the identity

$$\begin{aligned}\boldsymbol{y}^\top \boldsymbol{E}_1 \boldsymbol{P} \boldsymbol{E}_1^{-1} (\boldsymbol{A}_1 \boldsymbol{y} + \boldsymbol{s}) &= (\boldsymbol{y} + \boldsymbol{z})^\top [\boldsymbol{A}(\boldsymbol{y} + \boldsymbol{z}) + \boldsymbol{s}] \\ &= -\boldsymbol{u}^\top \boldsymbol{A}_R \boldsymbol{G} \boldsymbol{A}_R^\top \boldsymbol{u} + \boldsymbol{i}_V^\top \boldsymbol{v}_V + \boldsymbol{u}^\top \boldsymbol{A}_I \boldsymbol{i}_I.\end{aligned}$$

Next, recalling the definition of $\boldsymbol{y}$ and $\boldsymbol{z}$, we write

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}' \\ \boldsymbol{y}'' \\ 0 \end{pmatrix} := \begin{pmatrix} \boldsymbol{P}_{CD}\boldsymbol{u} \\ \boldsymbol{i}_L \\ 0 \end{pmatrix}, \quad \boldsymbol{z} = \begin{pmatrix} \boldsymbol{z}' \\ 0 \\ \boldsymbol{z}'' \end{pmatrix} := \begin{pmatrix} \boldsymbol{Q}_{CD}\boldsymbol{u} \\ 0 \\ \boldsymbol{i}_V \end{pmatrix}.$$

The condition $\ker \boldsymbol{A}_R^\top \subset \ker \boldsymbol{A}_C^\top$, and the positive-definiteness of $\boldsymbol{G}$, imply

$$\boldsymbol{u}^\top \boldsymbol{A}_R \boldsymbol{G} \boldsymbol{A}_R^\top \boldsymbol{u} \geq c_G |\boldsymbol{P}_R \boldsymbol{u}|^2 \geq c_G |\boldsymbol{P}_C \boldsymbol{u}|^2 = c_G |\boldsymbol{y}'|^2.$$

Also, we have

$$\boldsymbol{i}_V^\top \boldsymbol{v}_V + \boldsymbol{u}^\top \boldsymbol{A}_I \boldsymbol{i}_I = \boldsymbol{y}'^\top \boldsymbol{A}_I \boldsymbol{i}_I + \boldsymbol{z}'^\top \boldsymbol{A}_I \boldsymbol{i}_I + \boldsymbol{z}_2^\top \boldsymbol{v}_V \leq |\boldsymbol{y}'||\boldsymbol{s}| + |\boldsymbol{z}||\boldsymbol{s}|.$$

Recalling once more the expression of $\boldsymbol{z}$ in terms of $\boldsymbol{y}$, we find

$$\boldsymbol{z} = \boldsymbol{Q}\boldsymbol{E}_1^{-1} (\boldsymbol{A}_1 \boldsymbol{y} + \boldsymbol{s}) = \bar{\boldsymbol{E}}_1^{-1} \boldsymbol{Q}^\top (\boldsymbol{A}_1 \boldsymbol{y} + \boldsymbol{s}).$$

Using the condition $\ker \boldsymbol{A}_C^\top \subset \ker \boldsymbol{A}_L^\top$, we can compute explicitely

$$\boldsymbol{Q}^\top \boldsymbol{A}_1 \boldsymbol{y} = \begin{pmatrix} -\boldsymbol{Q}_{CD}^\top \boldsymbol{A}_R \boldsymbol{G} \boldsymbol{A}_R^\top \\ \boldsymbol{O} \\ \boldsymbol{A}_V^\top \end{pmatrix} \boldsymbol{P}_{CD}^\top \boldsymbol{u} = \boldsymbol{M}_1 \boldsymbol{y}',$$

which implies

$$|\boldsymbol{z}| \leq c(|\boldsymbol{y}'| + |\boldsymbol{s}|). \tag{11}$$

In conclusion, we find

$$\begin{aligned}\boldsymbol{y}^\top \boldsymbol{E}_1 \boldsymbol{P} \boldsymbol{E}_1^{-1} (\boldsymbol{A}_1 \boldsymbol{y} + \boldsymbol{s}) &\leq -c_G |\boldsymbol{y}'|^2 + |\boldsymbol{y}'||\boldsymbol{s}| + c(|\boldsymbol{y}'| + |\boldsymbol{s}|)|\boldsymbol{s}| \\ &\leq -\frac{c_G}{2} |\boldsymbol{y}'|^2 + c|\boldsymbol{s}|^2,\end{aligned} \tag{12}$$

in which we have used the weighted Schwarz's inequality. Inserting (12) in (7), we obtain

$$\begin{aligned}&\mathcal{H}(t_1) + \frac{q}{2} \int_{t_0}^{t_1} \int_\Omega \left(\mu_n n |\nabla(\phi_n - \phi_n^{\mathsf{e}})|^2 + \mu_p p |\nabla(\phi_p - \phi_p^{\mathsf{e}})|^2 \right) \mathrm{d}\mathbf{x}\, \mathrm{d}\tau \\ &\quad + \frac{c_G}{2} \int_{t_0}^{t_1} |\boldsymbol{y}'|^2 \,\mathrm{d}\tau \leq \mathcal{H}(t_0) + c \int_{t_0}^{t_1} |\boldsymbol{s}(\tau)|^2 \,\mathrm{d}\tau.\end{aligned} \tag{13}$$

which implies for all $t_1 \geq t_0 \geq 0$ that

$$\mathcal{H}(t_1) - c \int_0^{t_1} |\boldsymbol{s}(\tau)|^2 \,\mathrm{d}\tau \leq \mathcal{H}(t_0) - c \int_0^{t_0} |\boldsymbol{s}(\tau)|^2 \,\mathrm{d}\tau, \tag{14}$$

and, using (2), that

$$\begin{aligned}&\mathcal{H}(t_1) + \frac{q}{2} \int_0^{t_1} \int_\Omega \left(\mu_n n |\nabla(\phi_n - \phi_n^{\mathsf{e}})|^2 + \mu_p p |\nabla(\phi_p - \phi_p^{\mathsf{e}})|^2 \right) \mathrm{d}\mathbf{x}\, \mathrm{d}\tau \\ &\quad + \frac{c_G}{2} \int_0^{t_1} |\boldsymbol{y}'|^2 \,\mathrm{d}\tau \leq \mathcal{H}(0) + cK.\end{aligned} \tag{15}$$

This means that there exists a sequence $t_j \to \infty$ such that

$$\lim_{j\to\infty} \left[\int_\Omega \left(n|\nabla\phi_n|^2 + p|\nabla\phi_p|^2 \right) \mathrm{d}\mathbf{x} \right]_{t=t_j} = 0. \tag{16}$$

We show that (16) yields a bound on the norm of $\nabla\sqrt{n}$ and $\nabla\sqrt{p}$, thanks to the identity

$$\begin{aligned} \int_\Omega \left(n|\nabla\phi_n|^2 + p|\nabla\phi_p|^2 \right) \mathrm{d}\mathbf{x} &= \int_\Omega \left(n \left| -\mathbf{E} - U_T \frac{\nabla n}{n} \right|^2 + p \left| -\mathbf{E} + U_T \frac{\nabla p}{p} \right|^2 \right) \mathrm{d}\mathbf{x} \\ &= \int_\Omega \left((n+p)|\mathbf{E}|^2 + 2U_T \mathbf{E} \cdot \nabla(n-p) + 4U_T^2 \left(\left|\nabla\sqrt{n}\right|^2 + |\nabla\sqrt{p}|^2 \right) \right) \mathrm{d}\mathbf{x}. \end{aligned} \tag{17}$$

Rearranging the terms in (17), we find, for $t = t_j$,

$$\begin{aligned} 4U_T^2 \left(\|\nabla\sqrt{n}\|^2 + \|\nabla\sqrt{p}\|^2 \right) &+ \int_\Omega (n+p)|\mathbf{E}|^2 \,\mathrm{d}\mathbf{x} \\ &= \int_\Omega \left(n|\nabla\phi_n|^2 + p|\nabla\phi_p|^2 \right) \mathrm{d}\mathbf{x} - \int_\Omega 2U_T \mathbf{E} \cdot \nabla(n-p) \,\mathrm{d}\mathbf{x} \le c, \end{aligned} \tag{18}$$

since, integrating by parts, considering the boundary conditions (1g)-(1i) and the Poisson equation for the electrostatic potential, we have

$$\begin{aligned} -\int_\Omega \mathbf{E} \cdot \nabla(n-p) \,\mathrm{d}\mathbf{x} &= -\frac{q}{\epsilon_s} \int_\Omega (n-p-C)(n-p) \,\mathrm{d}\mathbf{x} + \int_{\Gamma_D} (n-p)\nabla\psi \cdot \boldsymbol{\nu} \,\mathrm{d}\sigma \\ &= \frac{q}{\epsilon_s} \int_\Omega C(n-p) \,\mathrm{d}\mathbf{x} - \frac{q}{\epsilon_s} \int_\Omega (n-p)^2 \,\mathrm{d}\mathbf{x} + \int_{\Gamma_D} (n-p)\nabla\psi \cdot \boldsymbol{\nu} \,\mathrm{d}\sigma \\ &\le \frac{q}{2\epsilon_s} \|C\|^2 - \frac{q}{2\epsilon_s} \|n-p\|^2 + \frac{1}{4\delta} \int_{\Gamma_D} |n_D - p_D|^2 \,\mathrm{d}\sigma + \delta \|\nabla\psi\|^2_{L^2(\partial\Omega)} \\ &\le \frac{3q}{4\epsilon_s} \|C\|^2 + \frac{1}{4\delta} \int_{\Gamma_D} |n_D - p_D|^2 \,\mathrm{d}\sigma. \end{aligned}$$

Here, we have used $\|\nabla\psi\|^2_{L^2(\partial\Omega)} \le c\|\psi\|^2_{H^2} \le c(\|C\|^2 + \|n-p\|^2)$, and chosen δ small enough, so that $c\delta \le \frac{q}{4\epsilon_s}$. Then from (15) we can write

$$\|\sqrt{n(t_j)}\|_{1,2} + \|\sqrt{p(t_j)}\|_{1,2} + \|\psi(t_j)\|_{1,2} \le c. \tag{19}$$

Following [26], we use the compactness of the embeddings of W_2^1 into $L_{2\beta}$ with $\frac{3}{2} < \beta < 3$ and $L_{2\gamma}(\partial\Omega)$ with $1 < \gamma < 2$, we can suppose that there are two functions $\bar{n}$ and $\bar{p}$ belonging to L_β and $L_\gamma(\partial\Omega)$ such that $\sqrt{\bar{n}}, \sqrt{\bar{p}} \in W_2^1$ and

$$\lim_{j\to\infty} n(t_j) = \bar{n}, \qquad \lim_{j\to\infty} p(t_j) = \bar{p} \quad \text{in} \quad L_\beta \quad \text{and} \quad L_\gamma(\partial\Omega). \tag{20}$$

Next, we prove the convergence of $\boldsymbol{y}$ and $\boldsymbol{z}$. Using (15), we can write

$$\frac{1}{2}\boldsymbol{u}^\top(t)\boldsymbol{A}_C\boldsymbol{C}\boldsymbol{A}_C^\top\boldsymbol{u}(t) + \frac{1}{2}\boldsymbol{i}_L^\top(t)\boldsymbol{L}\boldsymbol{i}_L(t) + \frac{c_G}{2}\int_0^t |\boldsymbol{y}'|^2 \,\mathrm{d}\tau \le \mathcal{H}(0) + cK. \tag{21}$$

Since $\boldsymbol{C}$ is positive definite, and $\boldsymbol{y}'$ does not belong to the nullspace of $\boldsymbol{A}_C^\top$, we have $\boldsymbol{u}^\top\boldsymbol{A}_C\boldsymbol{C}\boldsymbol{A}_C^\top\boldsymbol{u} \ge c_C|\boldsymbol{y}'|^2$, which yields

$$\frac{c_C}{2}|\boldsymbol{y}'|^2 + \frac{c_G}{2}\int_0^t |\boldsymbol{y}'|^2 \,\mathrm{d}\tau \le \mathcal{H}(0) + cK.$$

Gronwal's lemma implies that $\boldsymbol{y}'(t) \to 0$, with asymptotic decay as $t \to \infty$. Then, estimate (11) implies that $\boldsymbol{z}(t) \to 0$ as $t \to \infty$. In particular, we have shown that $\boldsymbol{u} = \boldsymbol{y}' + \boldsymbol{z}'$ tends to zero, that is,

$$\lim_{j\to\infty} \boldsymbol{u}(t_j) = 0. \tag{22}$$

Finally, (21) implies the boundedness of $\boldsymbol{y}'' = \boldsymbol{i}_L$, due to the positive definiteness of $\boldsymbol{L}$, which implies the convergence of $\boldsymbol{i}_L(t) \to \bar{\boldsymbol{i}}_L$.

From (20) and (22) it follows that

$$\lim_{j\to\infty} \psi(t_j) = \bar{\psi} = \mathcal{L}(\bar{p} - \bar{n} + C) \quad \text{in} \quad W^2_\beta, \tag{23}$$

and, since $W^2_\beta \subset C$, also that

$$\lim_{j\to\infty} n(t_j) \exp\left(-\frac{\psi(t_j)}{U_T}\right) = \bar{n}e^{-\bar{\psi}/U_T}, \quad \lim_{j\to\infty} p(t_j) \exp\left(\frac{\psi(t_j)}{U_T}\right) = \bar{p}e^{\bar{\psi}/U_T} \quad \text{in } L_\beta.$$

Moreover, recalling (8), we have the identities

$$n \exp\left(-\frac{\psi}{U_T}\right) = n_i \exp\left(-\frac{\phi_n}{U_T}\right), \quad p \exp\left(\frac{\psi}{U_T}\right) = n_i \exp\left(\frac{\phi_p}{U_T}\right),$$

which imply

$$\nabla(ne^{-\psi/U_T}) = -\frac{n}{U_T}e^{-\psi/U_T}\nabla\phi_n, \quad \nabla(pe^{\psi/U_T}) = \frac{p}{U_T}e^{\psi/U_T}\nabla\phi_p.$$

Then, using (16) and (20), it is possible to show that $\bar{n}e^{-\bar{\psi}/U_T}$ and $\bar{p}e^{\bar{\psi}/U_T}$ are constant, and from the boundary conditions (1g) and (1h), written with $\boldsymbol{u}_D = 0$, because of (22), we find

$$\bar{n}e^{-\bar{\psi}/U_T} = n_D e^{-\psi_D/U_T} = n_\mathrm{i}, \quad \bar{p}e^{\bar{\psi}/U_T} = p_D e^{\psi_D/U_T} = n_\mathrm{i}.$$

This shows that

$$\bar{n} = n_\mathrm{i} e^{\bar{\psi}/U_T}, \quad \bar{p} = n_\mathrm{i} e^{-\bar{\psi}/U_T},$$

and, thanks to (23), $\bar{\psi}$ satisfies the defining equations for the equilibrium potential, with the same boundary conditions, thus

$$\bar{\psi} = \psi^\mathsf{e}, \quad \bar{n} = n^\mathsf{e}, \quad \bar{p} = p^\mathsf{e}. \tag{24}$$

In particular, this implies that the limit current from the device to the circuit becomes zero as t goes to infinity. Thus, from the cicuit equations (1), recalling that $\boldsymbol{y}'$ and $\boldsymbol{z}$ tend to zero as time increases, the limit value $\bar{\boldsymbol{i}}_L$ for $\boldsymbol{y}'' = \boldsymbol{i}_L$ satisfies the condition $\boldsymbol{A}_L^\top \bar{\boldsymbol{i}}_L = 0$, that is, $\bar{\boldsymbol{i}}_L \in \ker \boldsymbol{A}_L^\top$.

We can finally pass from the sequence $\{t_j\}$ to the continuous limit for $t \to \infty$, by recalling that the function $\mathcal{H}(t) - c\int_0^t |\boldsymbol{s}(\tau)|^2$ is decreasing, as shown in (14). Then, we have

$$\lim_{t\to\infty} \left\{\mathcal{H}(t) - c\int_0^t |\boldsymbol{s}(\tau)|^2\right\} = \lim_{j\to\infty} \left\{\mathcal{H}(t_j) - c\int_0^{t_j} |\boldsymbol{s}(\tau)|^2\right\} = -c\lim_{j\to\infty}\int_0^{t_j} |\boldsymbol{s}(\tau)|^2,$$

which yields

$$\lim_{t\to\infty} \mathcal{H}(t) = 0. \tag{25}$$

The estimates (3) follow as in [26]. ■

"Multi-dimensional Modeling and Simulation of electrically pumped semiconductor-based Emitters".

References

[1] E. Hairer, G. Wanner. *Solving Ordinary Differential Equation: II, Stiff and Differential-Algebraic Problems*, 2nd ed., Springer, (2002).

[2] R. Lamour, R. März, C. Tischendorf *Differential-Algebraic Equations: A Projector Based Analysis*. Springer-Verlag, Heidelberg, (2013).

[3] A. Bartel. *Partial differential-algebraic models in chip design – Thermal and semiconductor problems*. Ph.D. Thesis, Bergische UniversitŁt Wuppertal, (2003).

[4] M.Brunk, A.Jüngel. *Simulation of thermal effects in optoelectronic devices using coupled energy-transport and circuit models* , Math. Models Methods Appl. Sci. 18, pp.1–26 (2008).

[5] M. Culpo. *Numerical algorithms for system level electro-thermal simulation*. Ph.D. Thesis, Bergische UniversitŁt Wuppertal, (2009).

[6] H. de Gersem, K. Hameyer, T. Weiland. *Field-circuit coupled models in electromagnetic simulation*. J. Comput. Appl. Math. 168, (2004) special issue: selected papers from the 2nd international conference on advanced computational methods in engineering (ACOMEN2002) Liege University, Belgium, 27Ð31 May 2002.

[7] D. Ioan, W.H.A. Schilders, G. Ciuprina, N. van der Meijs, W. Schoenmaker. *Models for integrated components coupled with their EM environment*. COMPEL 27 (4), pp. 820–829 (2008).

[8] G. Alì, A. Bartel, M. Günther, C. Tischendorf. *Elliptic partial differential-algebraic multiphysics models in electrical network design*.Methods Appl. Sci. 13 (9), pp.1261–1278 (2003).

[9] G. Alì, A. Bartel, M. Günther. *Parabolic differential-algebraic models in electrical network design*. Multiscale Model. Simul. 4 no. 3, pp. 813–838 (2005).

[10] C. Tischendorf. *Coupled systems of differential algebraic and partial differential equations in circuit and device simulation. Modeling and Numerical Analysis*. Habilitationsschrift, Inst. für Math., Humboldt-Univ. zu Berlin, (2004).

[11] G. Alì, N. Rotundo, *An existence result for elliptic partial differential-algebraic equations arising in semiconductor modeling*, *Nonlinear Anal.* **72** (2010) 4666–4681.

[12] G. Alì, A. Bartel, N. Rotundo, *Index-2 elliptic partial differential-algebraic models for circuits and devices*, *J. Math. Anal. Appl.* **423** 2 (2015) 1348–1369.

[13] G. Alì, A. Bartel, M. Günther. *Existence and uniqueness for an elliptic PDAE model of integrated circuits*. SIAM J. Appl. Math. 70 no. 5, pp.1587–1610 (2009/10).

[14] G. Alì, N. Rotundo, *On the tractability index of a class of partial differential-algebraic equations*, *Acta Appl. Math.* **122** (2012) 3–17.

[15] M. Bodestedt, C. Tischendorf, *PDAE models of integrated circuits and perturbation analysis*, Math. Comput. Model. Dyn. Syst. 13 (1), pp. 1–17 (2007).

[16] M. Selva Soto, C. Tischendorf, *Numerical analysis of DAEs from coupled circuit and semiconductor simulation*, Appl. Numer. Math. 53 (2-4), pp. 471–488 (2005).

[17] A. Bartel, M. Günther *PDAEs in refined electrical network modeling*, SIAM Rev. 60 , no. 1, pp. 56Ð91(2018).

[18] C. W. Ho, A. E. Ruehli, P. A.Brennan, *The modified nodal approach to network analysis*, *IEEE Trans. Circuits and Systems, CAS* **22** (1975) 505–509.

[19] W. J. McCalla, *Fundamentals of Computer Aided Circuit Simulation* (Acad. Publ. Group, Dordrecht (1988) Kluwer).

[20] P. A. Markowich, C. A. Ringhofer and C. Schmeiser, *Semiconductor Equations* (Springer, 1990).

[21] A. Jüngel, *Transport Equations for Semiconductors* Lecture Notes in Physics, vol. 773. Springer, Berlin (2009).

[22] C. Tischendorf, *Topological index calculation of differential-algebraic equations in circuit simulation*, *Surv. Math. Ind.* **8** (1999) 187–199.

[23] M. Günther, *A PDAE model for interconnected linear RLC networks*, *Mathematical and Computer Modelling of Dynamical Systems.* **7** No. 2 (2001) 189–203.

[24] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva, *Linear and quasilinear equations of parabolic type* Providence, R.I. : American Mathematical Society, (1968).

[25] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order* (Springer-Verlag, Berlin 1983).

[26] H. Gajewski, *On existence, uniqueness and asymptotic behavior of solutions of the basic equations for carrier transport in semiconductors*, *Z. Angew. Math. Mech.*, **65** (1985), pp. 101–108.