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Abstract

In this paper we consider the Anderson Hamiltonian with white noise potential on the box $[0, L]^2$ with Dirichletg boundary conditions. We show that all the eigenvalues divided by $\log L$ converge as $L \to \infty$ almost surely to the same deterministic constant, which is given by a variational formula.

1 Introduction

We consider the Anderson Hamiltonian (also called random Schrödinger operator), formally defined by

$$\mathscr{H} = \Delta + \xi,$$

under Dirichlet boundary conditions on the two-dimensional box $[0, L]^2$, where ξ is considered to be white noise. We are interested in the behaviour of this operator as the size of the box, L, tends to infinity. In this paper we prove the following asymptotics of the eigenvalues. Let $\lambda(L) = \lambda_1(L) \geq \lambda_2(L) \geq \lambda_3(L) \cdots$ be the eigenvalues of the Anderson Hamiltonian on $[0, L]^2$. For all $n \in \mathbb{N}$, almost surely

$$\lim_{L\to\infty} \frac{\lambda_n(L)}{\log L} = 4 \sup_{\substack{V \in C_c^\infty(\mathbb{R}^2) \\ \|V\|_{L^2}^2 \leq 1}} \sup_{\substack{\psi \in C_c^\infty(\mathbb{R}^2) \\ \|\psi\|_{L^2}^2 = 1}} \int_{\mathbb{R}^2} -|\nabla \psi|^2 + V\psi^2.$$

1.1 Main challenge and literature

In the one dimensional setting, i.e., on the box $[0, L]$, the Anderson Hamiltonian can be defined using the associated Dirichlet form as the white noise is sufficiently regular, see Fukushima and Nakao [11] (see [31] for the regularity of white noise). In dimension two the regularity of white noise is too small to allow for the same approach. A naive way to tackle the problem of the construction is to take a smooth approximation of the white noise ξ_ε so that the operator $\mathscr{H}_\varepsilon = \Delta + \xi_\varepsilon$ is well defined as an unbounded self-adjoint operator, and then take the limit $\varepsilon \downarrow 0$. However, $\mathscr{H}_\varepsilon$ does not converge, but $\mathscr{H}_\varepsilon - c_\varepsilon$ does converge to an operator $\mathscr{H}$ for certain renormalisation constants $c_\varepsilon \nearrow_{\varepsilon\downarrow 0} \infty$. This has been shown by Allez and Chouk [1] for periodic boundary conditions, using the techniques of paracontrolled distributions introduced by Gubinelli, Imkeller and Perkowski [14] in order to study singular stochastic partial differential equations. In this paper we extend this to Dirichlet boundary conditions.

Recently, also Labbé [18] constructed the Anderson Hamiltonian with both periodic and Dirichlet boundary conditions, using the tools of regularity structures. Gubinelli, Ugurcan and Zachhuber [13]

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extend the work of Allez and Chouk to define the Anderson Hamiltonian with periodic boundary conditions also for dimension 3.

One of the main interests in the study of this operator is due to its universal property, precisely it was proved by Chouk, Gairing and Perkowski [6, Theorem 6.1] that under periodic boundary conditions the operator $\mathscr{H}$ is the limit under a suitable renormalisation of the discrete Anderson Hamiltonian

$$\mathscr{H}_N = \Delta_N + \tfrac{1}{N}\eta_N$$

defined on the periodic lattice $(\frac{1}{N}\mathbb{Z}/N\mathbb{Z})^2$ where Δ_N is discrete Laplacian and $(\eta_N(i), i \in \mathbb{Z}^2)$ are centred I.I.D. random variables with normalised variance and finite p-th moment, for some $p > 6$.

Recently, Dumaz and Labbé [10] proved the Anderson localization for the one dimensional case for the largest eigenvalues and they obtain the exact fluctuation of the eigenvalue and the exact behaviour of the eigenfunctions near their maxima. Unfortunately, their approach used to tackle the Anderson localization in the one dimensional setting is strongly attached to the SDE obtained by the so-called Riccati transform and cannot be adapted to the two dimensional setting. Also Chen [5] considers the one dimensional setting for the white noise (and shows $\lambda(L) \approx (\log L)^{\frac{2}{3}}$), but also a higher dimensional setting for the more regular fractional white noise (where $\lambda(L) \approx (\log L)^{\beta}$ for some $\beta \in (\frac{1}{2}, 1)$ (and $\beta \in (\frac{1}{2}, \frac{2}{3})$ for $d = 1$), where β is a function of the degree of singularity of the covariance at zero). The techniques in his work do not allow for an extension to a higher dimensional setting with a white noise potential.

The asymptotics of the principal eigenvalue is of particular interest for the asymptotics of the total mass of the solution to the parabolic Anderson model: $\partial_t u = \Delta u + \xi u = \mathscr{H} u$. Chen [5] shows that with $U(t)$ the total mass of $u(t, \cdot)$, one has $\log U(t) \approx t\lambda_{L_t}$ for some almost linear L_t, so that the asymptotics of $\lambda(L)$ leads to asymptotics of $\log U(t)$: In $d = 1$ with ξ white noise, $\log U(t) \approx t(\log t)^{\frac{2}{3}}$; for $d \geq 1$ with ξ a fractional white noise $\log U(t) \approx t(\log t)^{\beta}$, with β as above. For smooth Gaussian fields ξ, Carmona and Molchanov [4] show $\log U(t) \sim t(\log t)^{\frac{1}{2}}$. In a future work by W. König, N. Perkowski and W. van Zuijlen, the following asymptotics of the total mass of the solution to the parabolic Anderson model with white noise potential in two dimensions will be shown: $\log U(t) \approx t \log t$.

For a general overview about the parabolic Anderson model and the Anderson Hamiltonian we refer to the book by König [17].

Let us mention that our main result is already applied in [24] to prove that the super Brownian motion in static random environment is almost surely super-exponentially persistent.

1.2 Outline

In Section 2 we state the main results of this paper. In Section 3 we give a proof of the tail bounds of the eigenvalues using the other ingredients presented in Section 2, and use to prove the main theorem. The definitions of our Dirichlet and Neumann (Besov) spaces and para- and resonance products between those spaces are given in Section 4. With the definitions given we can properly define the Anderson Hamiltonian on its Dirichlet domain and state the spectral properties in Section 5. In Section 6 we prove the convergence to enhanced white noise, that will be used to extend properties for smooth potentials to analogue properties where enhanced white noise is taken; for scaling and translation properties in Section 7, to compare eigenvalues on boxes of different size in Section 8 and to prove the large deviation principle of the enhanced white noise in Section 9 that leads to the large deviation principle for the eigenvalues. In Section 10 we study infima over the large deviation rate function, which are used the express the limit of the eigenvalues. The more cumbersome calculations needed to prove convergence to enhanced white noise are postponed to Section 11.

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1.3 Notation

$\mathbb{N} = \{1, 2, \dots\}$. $\delta_{k,l}$ is the Kronecker delta, i.e., $\delta_{k,k} = 1$ and $\delta_{k,l} = 0$ for $k \neq l$. $\mathrm{i} = \sqrt{-1}$. For $f, g \in L^2(D)$, for some domain $D \subset \mathbb{R}^d$ we write $\langle f, g\rangle_{L^2(D)} = \int_D f\overline{g}$. We write $\mathbb{T}_L^d$ for the d-dimensional torus of length $L > 0$, i.e., $\mathbb{R}^d/L\mathbb{Z}^d$. $(\Omega, \mathbb{P})$ will be our underlying complete probability space. In order to avoid cumbersome administration of constants, we also write $a \lesssim b$ to denote that there exists a $C > 0$ such that $a \le Cb$. We write $C_c^\infty(A)$ for those functions in $C^\infty(A)$ that have compact support in A°.

2 Main results

In this section we give the main results of this paper without the technical details and definitions; the main theorem is Theorem 2.8.

We build on the results on the spectrum of the Anderson Hamiltonian in [1]. They consider the operator on the torus, i.e., periodic boundary conditions. In Section 4 we construct the framework of Dirichlet and Neumann Besov spaces. In this way we generalise the main theorem of [1] to Dirichlet boundary conditions on $Q_L = [0, L]^2$.

Theorem 2.1 (Summary of Theorem 5.4)**.** *Let $\alpha \in (-\frac{4}{3}, -1)$. Let $y \in \mathbb{R}^2, L > 0$ and $\Gamma = y + Q_L$. For a Neumann rough distribution $\boldsymbol{\xi} = (\xi, \Xi) \in \mathfrak{X}_{\mathfrak{n}}^\alpha(\Gamma)$ one can construct a stongly paracontrolled Dirichlet domain $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}}(\Gamma)$, such that the Anderson Hamiltonian on $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}}(\Gamma)$ maps in $L^2(\Gamma)$ and is self-adjoint as an operator on $L^2(\Gamma)$ with a countable spectrum given by eigenvalues $\lambda(\Gamma, \boldsymbol{\xi}) = \lambda_1(\Gamma, \boldsymbol{\xi}) \ge \lambda_2(\Gamma, \boldsymbol{\xi}) \ge \cdots$ (counting multiplicities). For all $n \in \mathbb{N}$ the map $\boldsymbol{\xi} \mapsto \lambda_n(\Gamma, \boldsymbol{\xi})$ is locally Lipschitz. Moreover, a Courant-Fischer formula is given for λ_n (see* (43)*).*

In Section 6 we show that there exists a canonical enhanced white noise in $\mathfrak{X}_{\mathfrak{n}}^\alpha$:

Theorem 2.2 (See Theorem 6.4 and 6.5)**.** *Let $\alpha \in (-\frac{4}{3}, -1)$. For all $y \in \mathbb{R}^2$ and $L > 0$ there exists a canonical $\boldsymbol{\xi}_L^y = (\xi_L^y, \Xi_L^y) \in \mathfrak{X}_{\mathfrak{n}}^\alpha(y + Q_L)$ such that ξ_L^y is a white noise (in the sense that is described in that theorem). We write $\boldsymbol{\xi}_L = \boldsymbol{\xi}_L^0, \xi_L = \xi_L^0, \Xi_L = \Xi_L^0$ and $\boldsymbol{\lambda}_n(y + Q_L, \beta) = \lambda_n(y + Q_L, (\beta\xi_L^y, \beta^2\Xi_L^y))$ for $\beta \in \mathbb{R}$ and $\boldsymbol{\lambda}_n(y + Q_L) = \boldsymbol{\lambda}_n(y + Q_L, 1)$.*

Now we have the framework set and can get to the key ingredients, of which two are given in Section 7:

2.3. (a) (Lemma 7.3) For $L, \beta > 0$,

$$\boldsymbol{\lambda}_n(Q_L) \overset{d}{=} \tfrac{1}{\beta^2}\boldsymbol{\lambda}_n(Q_{\frac{L}{\beta}}, \beta) + \tfrac{1}{2\pi}\log\beta.$$

(b) (Lemma 7.4) For $y \in \mathbb{R}^2$ and $L, \beta > 0$,

$$\boldsymbol{\lambda}_n(Q_L, \beta) \overset{d}{=} \boldsymbol{\lambda}_n(y + Q_L, \beta).$$

Moreover, if $y + Q_L^\circ \cap Q_L^\circ = \emptyset$, then $\boldsymbol{\lambda}_n(Q_L, \beta)$ and $\boldsymbol{\lambda}_n(y + Q_L, \beta)$ are independent.

In [12, Proposition 1] and [3, Lemma 4.6] the principal eigenvalue on a large box are bounded by maxima of principal eigenvalues on smaller boxes. We extend these results from smooth potentials to rough ones:

Theorem 2.4 (Consequence of Theorem 8.6[1]). *There exists a $K > 0$ such that for all $\varepsilon > 0$ and $L > r > 0$ with $\frac{L}{r} \in 2\mathbb{N}$, the following inequalities hold almost surely*

$$\max_{k \in \mathbb{N}_0^2, |k|_\infty < \frac{L}{r} - 1} \boldsymbol{\lambda}(rk + Q_r, \varepsilon) \leq \boldsymbol{\lambda}(Q_L, \varepsilon) \leq \max_{k \in \mathbb{N}_0^2, |k|_\infty < \frac{L}{r} + 1} \boldsymbol{\lambda}(rk + Q_{\frac{3}{2}r}, \varepsilon) + \tfrac{K}{a^2}.$$

Moreover, for $n \in \mathbb{N}$; if $y, y_1, \ldots, y_n \in \mathbb{R}^d$ are such that $(y_i + Q_r)_{i=1}^n$ are pairwise disjoint subsets of $y + Q_L$, then almost surely $\boldsymbol{\lambda}_n(y + Q_L, \varepsilon) \geq \min_{i \in \{1, \ldots, n\}} \boldsymbol{\lambda}(y_i + Q_r, \varepsilon)$.

Another important tool that we prove is the large deviations of the eigenvalues, which –by the contraction principle and continuity of the eigenvalues in terms of its rough distribution– is a consequence of the large deviations of $(\sqrt{\varepsilon}\xi_L, \varepsilon\Xi_L)$, proven in Section 9.

Theorem 2.5 (See Corollary 9.3). *$\boldsymbol{\lambda}_n(Q_L, \sqrt{\varepsilon}) = \lambda_n(Q_L, (\sqrt{\varepsilon}\xi_L, \varepsilon\Xi_L))$ satisfies the large deviation principle with rate ε and rate function $I_{L,n} : \mathbb{R} \to [0, \infty]$ given by*

$$I_{L,n}(x) = \inf_{\substack{V \in L^2(Q_L) \\ \lambda_n(Q_L, V) = x}} \tfrac{1}{2}\|V\|_{L^2}^2.$$

In Section 10 we study infima over the large deviation rate function over half-lines, in terms of which the almost sure limits of the eigenvalues will be described.

Theorem 2.6 (See Lemma 10.4, Lemma 10.5 and Theorem 10.7). *There exists a $C > 0$ such that for all $n \in \mathbb{N}$,*

$$\rho_n = \inf_{L > 0} \inf I_{L,n}[1, \infty) = \lim_{L \to \infty} \inf I_{L,n}[1, \infty) > C$$

and

$$\frac{2}{\rho_n} = 4 \sup_{\substack{V \in C_c^\infty(\mathbb{R}^2) \\ \|V\|_{L^2}^2 \leq 1}} \sup_{\substack{F \sqsubset C_c^\infty(\mathbb{R}^2) \\ \dim F = n}} \inf_{\substack{\psi \in F \\ \|\psi\|_{L^2}^2 = 1}} \int_{\mathbb{R}^2} -|\nabla\psi|^2 + V\psi^2.$$

Using the scaling and translation properties of 2.3, the comparison of the eigenvalue with maxima of eigenvalues of smaller boxes in Theorem 2.4 and the large deviations in Theorem 2.5 we obtain the following tail bounds in Section 3.

Theorem 2.7. *Let $K > 0$ be as in Theorem 8.6. Let $r > 0$. We will abbreviate $I_{r,1}$ by I_r. For all $\mu > \inf I_r(1, \infty)$ and $\kappa < \inf I_{\frac{3}{2}r}[1 - \frac{16K}{r^2})$ there exists an $M > 0$ such that for all $L, x > 0$ with $L\sqrt{x} > M$*

$$\mathbb{P}\left(\boldsymbol{\lambda}(Q_L) \leq x\right) \leq \exp\left(-\frac{e^{2\log L - \mu x} x}{8r^2}\right), \tag{1}$$

$$\mathbb{P}\left(\boldsymbol{\lambda}(Q_L) \geq x\right) \leq \tfrac{8}{r^2} x e^{2\log L - \kappa x}. \tag{2}$$

[1] In this statement we have choosen $a = \frac{1}{2}r$.

Using the tail bounds and the limit in Theorem 2.6 we obtain our main result by a Borel-Cantelli argument and the 'moreover' part of Theorem 2.4 (see Section 3).

Theorem 2.8. *With* $2^{\mathbb{N}} = \{2^n : n \in \mathbb{N}\}$*, for any sequence* $(y_L)_{L\in 2^{\mathbb{N}}}$ *in* $\mathbb{R}^2$*,*

$$\lim_{L\in 2^{\mathbb{N}}, L\to\infty} \frac{\boldsymbol{\lambda}_n(y_L+Q_L)}{\log L} = \frac{2}{\rho_1} = 4 \sup_{\substack{V\in C_c^\infty(\mathbb{R}^2)\\ \|V\|_{L^2}^2\le 1}} \sup_{\substack{\psi\in C_c^\infty(\mathbb{R}^2)\\ \|\psi\|_{L^2}^2=1}} \int_{\mathbb{R}^2} -|\nabla\psi|^2 + V\psi^2 \qquad a.s.$$

3 Proof of Theorem 2.7 and Theorem 3.4

In this section we prove Theorem 2.7 and Theorem 3.4 by using 2.1–2.6.

3.1. Let $K>0$ be as in Theorem 2.4. By consecutively applying the scaling in 2.3(a), the bounds in Theorem 2.4 and then the independence and translation properties in 2.3(b), we get for $L,r,\varepsilon>0$ with $L>\varepsilon r$ and $\frac{L}{\varepsilon r}\in\mathbb{N}$

$$\begin{aligned}
\mathbb{P}\left(\varepsilon^2\boldsymbol{\lambda}(Q_L)\le 1\right) &= \mathbb{P}\left(\boldsymbol{\lambda}(Q_{\frac{L}{\varepsilon}},\varepsilon) + \tfrac{\varepsilon^2}{2\pi}\log\varepsilon \le 1\right)\\
&\le \mathbb{P}\left(\max_{k\in\mathbb{N}_0^2, |k|_\infty<\frac{L}{\varepsilon r}-1} \boldsymbol{\lambda}(rk+Q_r,\varepsilon) \le 1-\tfrac{\varepsilon^2}{2\pi}\log\varepsilon\right)\\
&= \mathbb{P}\left(\boldsymbol{\lambda}(Q_r,\varepsilon)\le 1-\tfrac{\varepsilon^2}{2\pi}\log\varepsilon\right)^{\#\{k\in\mathbb{N}_0^2:|k|_\infty<\frac{L}{\varepsilon r}-1\}},
\end{aligned} \tag{3}$$

and similarly

$$\begin{aligned}
&\mathbb{P}\left(\varepsilon^2\boldsymbol{\lambda}(Q_L)\ge 1\right) \le \mathbb{P}\left(\boldsymbol{\lambda}(Q_{\frac{L}{\varepsilon}},\varepsilon) + \tfrac{\varepsilon^2}{2\pi}\log\varepsilon \ge 1\right)\\
&\le \mathbb{P}\left(\max_{k\in\mathbb{N}_0^2, |k|_\infty<\frac{L}{\varepsilon r}+1} \boldsymbol{\lambda}(rk+Q_{\frac{3}{2}r},\varepsilon) + \tfrac{K}{a^2} + \tfrac{\varepsilon^2}{2\pi}\log\varepsilon \ge 1\right)\\
&\le \#\{k\in\mathbb{N}_0^2 : |k|_\infty < \tfrac{L}{\varepsilon r}+1\}\mathbb{P}\left(\boldsymbol{\lambda}(Q_{\frac{3}{2}r},\varepsilon)\ge 1-\tfrac{4K}{r^2}-\tfrac{\varepsilon^2}{2\pi}\log\varepsilon\right).
\end{aligned} \tag{4}$$

Observe that there exists an $M>0$ such that for all $L,r,\varepsilon>0$ with $\frac{L}{\varepsilon r}>M$

$$\tfrac{1}{2}(\tfrac{L}{\varepsilon r})^2 \le \#\{k\in\mathbb{N}_0^2 : |k|_\infty < \tfrac{L}{\varepsilon r}\pm 1\} \le 2(\tfrac{L}{\varepsilon r})^2.$$

By combining the above observations we have obtained the following.

Lemma 3.2. *Let* $K>0$ *be as in Theorem 8.6. There exists an* $M>0$ *such that for all* $L,r,\varepsilon>0$ *with* $\frac{L}{\varepsilon r}\in\mathbb{N}$ *and* $\frac{L}{\varepsilon r}>M$

$$\mathbb{P}\left(\varepsilon^2\boldsymbol{\lambda}_n(Q_L)\le 1\right) \le \mathbb{P}\left(\boldsymbol{\lambda}_n(Q_r,\varepsilon)\le 1-\tfrac{\varepsilon^2}{2\pi}\log\varepsilon\right)^{\frac{1}{2}\left(\frac{L}{\varepsilon r}\right)^2}, \tag{5}$$

$$\mathbb{P}\left(\varepsilon^2\boldsymbol{\lambda}_n(Q_L)\ge 1\right) \le 2\left(\tfrac{L}{\varepsilon r}\right)^2\mathbb{P}\left(\boldsymbol{\lambda}_n(Q_{\frac{3}{2}r},\varepsilon)\ge 1-\tfrac{4K}{r^2}-\tfrac{\varepsilon^2}{2\pi}\log\varepsilon\right). \tag{6}$$

3.3. Let $r>0$. Let us now use the large deviation principle in Corollary 9.3. First, observe that as $\lim_{\varepsilon\downarrow 0}\frac{\varepsilon^2}{2\pi}\log\varepsilon = 0$, also $\boldsymbol{\lambda}_n(Q_r,\varepsilon)+\frac{\varepsilon^2}{2\pi}\log\varepsilon$ satisfies the large deviation principle with the same rate function (by exponential equivalence, see [7, Theorem 4.2.13]). Hence for all $\mu > \inf I_{r,n}(1,\infty)$

and $\kappa < \inf I_{\frac{3}{2}r,n}[1-\frac{4K}{r^2},\infty)$ there exists a ε_0 such that for $\varepsilon \in (0,\varepsilon_0)$ we have the following bound on the probability appearing in (3) (using that $1-x \le e^{-x}$ for $x \ge 0$):

$$\mathbb{P}\left(\boldsymbol{\lambda}_n(Q_r,\varepsilon) \le 1 - \tfrac{\varepsilon^2}{2\pi}\log\varepsilon\right) \le 1 - e^{-\frac{\mu}{\varepsilon^2}} \le e^{-e^{-\frac{\mu}{\varepsilon^2}}}, \tag{7}$$

$$\mathbb{P}\left(\boldsymbol{\lambda}_n(Q_{\frac{3}{2}r},\varepsilon) \ge 1 - \tfrac{4K}{r^2} - \tfrac{\varepsilon^2}{2\pi}\log\varepsilon\right) \le e^{-\frac{\kappa}{\varepsilon^2}}. \tag{8}$$

Proof of Theorem 2.7. This now follows by Lemma 3.2 and the bounds (7) and (8): We obtain

$$\begin{aligned}
&\forall r>0\, \forall \mu > \inf I_r(1,\infty)\, \exists M>0\, \forall L,x>0 \text{ with } \tfrac{L\sqrt{x}}{r}\in\mathbb{N} \text{ and } \tfrac{L\sqrt{x}}{r} > M:\\
&\mathbb{P}(\boldsymbol{\lambda}_n(Q_L) \le x) \le e^{-\frac{L^2x}{2r^2}e^{-\mu x}}, && (9)\\
&\forall L,r>0\, \forall \kappa < \inf I_{\frac{3}{2}r}[1-\tfrac{4K}{r^2},\infty)\, \exists M>0\, \forall L,x>0 \text{ with } \tfrac{L\sqrt{x}}{r}\in\mathbb{N} \text{ and } \tfrac{L\sqrt{x}}{r} > M:\\
&\mathbb{P}(\boldsymbol{\lambda}_n(Q_L) \ge x) \le 2\tfrac{L^2x}{r^2}e^{-\kappa x}. && (10)
\end{aligned}$$

We want to get rid of the condition that the quotient $\frac{L\sqrt{x}}{r}$ has to be a positive integer. For this we use that for all $a \ge 1$ one has $[a,2a]\cap\mathbb{N} \ne \emptyset$.

Fix $L,r>0$, $\mu > \inf I_r(1,\infty)$ and $\kappa < \inf I_{\frac{3}{2}r}[1-\frac{16K}{r^2},\infty)$. Let $M>2$ be as in the above statements. For all $L,x>0$ such that $L\sqrt{x} > rM$ there exist $r_1 \in [r,2r]$ such that $\frac{L\sqrt{x}}{r_1}\in\mathbb{N}$ and $r_2 \in [\frac{r}{2},r]$ such that $\frac{L\sqrt{x}}{r_2}\in\mathbb{N}$. As

$$\mu > \inf I_r(1,\infty) \ge \inf I_{r_1}(1,\infty) \text{ and } \kappa < \inf I_{\frac{3}{2}r}[1-\tfrac{16K}{r^2},\infty) \le \inf I_{\frac{3}{2}r_2}[1-\tfrac{4K}{r_2^2},\infty),$$

we obtain (9) for $r=r_1$ and (10) for $r=r_2$. The right hand side can then be bounded by the right hand sides in (1) and (2). □

Let us first prove the convergence of the principle eigenvalue, before proving Theorem 2.8.

Theorem 3.4. *For any sequence $(y_L)_{L\in 2^{\mathbb{N}}}$ in $\mathbb{R}^2$.*

$$\lim_{L\in 2^{\mathbb{N}},L\to\infty} \frac{\boldsymbol{\lambda}(y_L+Q_L)}{\log L} = \frac{2}{\rho_1} = 4 \sup_{\substack{V\in C_c^\infty(\mathbb{R}^2)\\ \|V\|_{L^2}^2\le 1}} \sup_{\substack{\psi\in C_c^\infty(\mathbb{R}^2)\\ \|\psi\|_{L^2}^2=1}} \int_{\mathbb{R}^2} -|\nabla\psi|^2 + V\psi^2 \quad a.s.$$

Proof. Without loss of generality we may assume $y_L = 0$ for all $L \in 2^{\mathbb{N}}$. Let $p,q\in\mathbb{R}$ be such that $p < \frac{2}{\rho_1} < q$. We show that

$$\liminf_{m\to\infty} \frac{\boldsymbol{\lambda}(Q_{2^m})}{\log 2^m} > p \quad \text{a.s.}, \qquad \limsup_{m\to\infty} \frac{\boldsymbol{\lambda}(Q_{2^m})}{\log 2^m} < q \quad \text{a.s.}$$

By the lemma of Borel-Cantelli it is sufficient to show that

$$\sum_{m=1}^\infty \mathbb{P}\left[\frac{\boldsymbol{\lambda}(Q_{2^m})}{\log 2^m} < p\right] < \infty, \qquad \sum_{m=1}^\infty \mathbb{P}\left[\frac{\boldsymbol{\lambda}(Q_{2^m})}{\log 2^m} > q\right] < \infty.$$

By Lemma 10.1

$$\lim_{r\to\infty} \inf I_r(1,\infty) = \lim_{r\to\infty} \inf I_{\frac{3}{2}r}[1-\tfrac{16K}{r^2},\infty) = \rho_n.$$

Let $r > 0$ be large enough such that

$$p \inf I_r(1,\infty) < 2 < q \inf I_{\frac{3}{2}r}[1 - \tfrac{16K}{r^2}, \infty).$$

Let $\mu > \inf I_r(1,\infty)$ be such that $p\mu < 2$ and $\kappa < \inf I_{\frac{3}{2}r}[1 - \frac{16K}{r^2}, \infty)$ be such that $q\kappa > 2$. By Theorem 2.7 for $M \in \mathbb{N}$ large enough

$$\sum_{m=M}^{\infty} \mathbb{P}\left[\frac{\boldsymbol{\lambda}(Q_{2^m})}{\log 2^m} < p\right] \le \sum_{m=M}^{\infty} 2^{-m\frac{p2^{(2-p\mu)m}}{8r^2}} < \infty,$$

which is finite because $\frac{p2^{(2-p\mu)m}}{8r^2} > 1$ for large m, as $2 - p\mu > 0$. Also

$$\sum_{m=M}^{\infty} \mathbb{P}\left[\frac{\boldsymbol{\lambda}(Q_{2^m})}{\log 2^m} > q\right] \le \sum_{m=M}^{\infty} \frac{8m\log 2}{r^2} 2^{(2-\kappa q)m},$$

which is finite as $2 - \kappa q < 0$ (and because $2^{-\alpha m} m \to 0$ for $\alpha > 0$). □

Proof of Theorem 2.8. Without loss of generality we take $y_L = 0$ for all $L \in 2^{\mathbb{N}}$. Let $n \in \mathbb{N}$. Let us first observe that as $\boldsymbol{\lambda}_n(Q_{2^m}) \le \boldsymbol{\lambda}(Q_{2^m})$, we have $\limsup_{m\to\infty} \frac{\boldsymbol{\lambda}_n(Q_{2^m})}{\log 2^m} \le \frac{2}{\rho_1}$. Let $x_1, \dots, x_n \in Q_{2^n}$ such that $(x_i + Q_1)_{i=1}^n$ are disjoint, then by Theorem 2.4 we obtain almost surely

$$\lim_{m\to\infty} \frac{\boldsymbol{\lambda}_n(Q_{2^{n+m}})}{\log 2^{n+m}} \ge \min_{i\in\{1,\dots,n\}} \lim_{m\to\infty} \frac{\boldsymbol{\lambda}_n(2^m x_i + Q_{2^m})}{\log 2^n + \log 2^m} \ge \frac{2}{\rho_1}.$$

□

4 Dirichlet and Neumann Besov spaces, para- and resonance products

Let $d \in \mathbb{N}$. Let $L > 0$. We will first introduce Dirichlet and Neumann spaces on $Q_L = [0,L]^d$. In order to do this we use 3 different bases of $L^2([0,L]^d)$, one standard (the e_k's), one as an underlying basis for Dirichlet spaces (the $\mathfrak{d}_k$'s) and one as an underlying basis for Neumann spaces (the $\mathfrak{n}_k$'s). After defining these spaces (in Definition 4.9) we prove a few results that compare Besov and Sobolev spaces. Later, in Definition 4.18 we show how to generalize this to spaces on general boxes of the form $\prod_{i=1}^d [a_i, b_i]$. Then we present bounds on Fourier multipliers (Theorem 4.19) and define para- and resonance products (Definition 4.23) and state their Bony estimates (Theorem 4.25).

In the following we will introduce some notation. For $\mathfrak{q} \in \{-1,1\}^d$ and $x \in \mathbb{R}^d$ we use the following short hand notation ($\mathfrak{q} \circ x$ is known as the Hadamard product)

$$(\prod \mathfrak{q}) = \prod_{i=1}^d \mathfrak{q}_i, \qquad \mathfrak{q} \circ x = (\mathfrak{q}_1 x_1, \dots, \mathfrak{q}_d x_d).$$

We call a function $f : [-L,L]^d \to \mathbb{C}$ *odd* if $f(x) = (\prod \mathfrak{q}) f(\mathfrak{q} \circ x)$ for all $\mathfrak{q} \in \{-1,1\}^d$, and similarly we call f *even* if $f(x) = f(\mathfrak{q} \circ x)$ for all $\mathfrak{q} \in \{-1,1\}^d$. Note that if f is odd, then $f = 0$ on $\partial[0,L]^d$. For any $f : [0,L]^d \to \mathbb{C}$ we write $\tilde{f} : [-L,L]^d \to \mathbb{C}$ for its odd extension (the $\sim$ notation is taken as it looks like the graph of an odd function) and $\overline{f} : [-L,L]^d \to \mathbb{C}$ for its even extension

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(similarly, the notation – is taken as it looks like the graph of an even function), i.e., for the functions that satisfy

$$\tilde{f}(\mathfrak{q}\circ x) = (\prod \mathfrak{q}) f(x), \quad \overline{f}(\mathfrak{q}\circ x) = f(x) \quad \text{for all } x\in[0,L]^d, \mathfrak{q}\in\{-1,1\}^d.$$

If a function $f:[-L,L]^d \to \mathbb{C}$ is periodic in the sense that it can be extended periodically on $\mathbb{R}^d$ (with period $2L$) we will also consider it to be a function on the domain $\mathbb{T}^d_{2L}$.

For $k=(k_1,\dots,k_d)\in\mathbb{N}_0^d$ let $\nu_k = 2^{-\frac{1}{2}\#\{i:k_i=0\}}$ and write $\mathfrak{d}_{k,L}$ and $\mathfrak{n}_{k,L}$ or simply $\mathfrak{d}_k$ and $\mathfrak{n}_k$ for the functions $[0,L]^d\to\mathbb{C}$ and $e_{k,2L}$ or simply e_k for the function $[-L,L]^d\to\mathbb{C}$ given by

$$\mathfrak{d}_{k,L}(x) = \mathfrak{d}_k(x) = (\tfrac{2}{L})^{\frac{d}{2}} \prod_{i=1}^d \sin(\tfrac{\pi}{L}k_i x_i), \tag{11}$$

$$\mathfrak{n}_{k,L}(x) = \mathfrak{n}_k(x) = \nu_k(\tfrac{2}{L})^{\frac{d}{2}} \prod_{i=1}^d \cos(\tfrac{\pi}{L}k_i x_i), \tag{12}$$

$$e_{k,2L}(x) = e_k(x) = (\tfrac{1}{2L})^{\frac{d}{2}} e^{\frac{\pi \mathrm{i}}{L}\langle k,x\rangle}. \tag{13}$$

Note that $\tilde{\mathfrak{d}}_k(x)$ equals the right-hand side of (11) and $\overline{\mathfrak{n}}_k(x)$ equals the right-hand side of (12) for $x\in[-L,L]^d$, so that $\tilde{\mathfrak{d}}_k$ and $\overline{\mathfrak{n}}_k$ are elements of $C^\infty(\mathbb{T}^d_{2L})$. We can also write $\tilde{\mathfrak{d}}_k$ and $\overline{\mathfrak{n}}_k$ as follows

$$\tilde{\mathfrak{d}}_k(x) = (\tfrac{2}{L})^{\frac{d}{2}} \prod_{i=1}^d \frac{e^{\frac{\pi\mathrm{i}}{L}k_i x_i} - e^{-\frac{\pi\mathrm{i}}{L}k_i x_i}}{2\mathrm{i}} = (-\mathrm{i})^d \sum_{\mathfrak{q}\in\{-1,1\}^d} (\prod \mathfrak{q}) e_{\mathfrak{q}\circ k}(x), \tag{14}$$

$$\overline{\mathfrak{n}}_k(x) = \nu_k(\tfrac{2}{L})^{\frac{d}{2}} \prod_{i=1}^d \frac{e^{\frac{\pi\mathrm{i}}{L}k_i x_i} + e^{-\frac{\pi\mathrm{i}}{L}k_i x_i}}{2} = \nu_k \sum_{\mathfrak{q}\in\{-1,1\}^d} e_{\mathfrak{q}\circ k}(x). \tag{15}$$

For an integrable function $f:\mathbb{T}^d_{2L}\to\mathbb{C}$ its k-th Fourier coefficient is defined by

$$\mathscr{F} f(k) = \langle f, e_k\rangle = \frac{1}{(2L)^{\frac{d}{2}}} \int_{\mathbb{T}^d_{2L}} f(x) e^{-\frac{\pi\mathrm{i}}{L}\langle k,x\rangle}\,\mathrm{d}x \qquad (k\in\mathbb{Z}^d).$$

4.1. It is not difficult to see that for $\varphi,\psi\in L^2([0,L]^d)$, the following equalities hold:

$$\mathscr{F}(\tilde{\varphi})(k) = (\prod \mathfrak{q})\mathscr{F}(\tilde{\varphi})(\mathfrak{q}\circ k) \text{ for all } k\in\mathbb{Z}^d, \mathfrak{q}\in\{-1,1\}^d, \tag{16}$$

$$\mathscr{F}(\tilde{\varphi})(k) = 0 \text{ for all } k\in\mathbb{Z}^d \text{ with } k_i = 0 \text{ for some } i, \tag{17}$$

$$\mathscr{F}(\overline{\varphi})(k) = \mathscr{F}(\overline{\varphi})(\mathfrak{q}\circ k) \text{ for all } k\in\mathbb{Z}^d, \mathfrak{q}\in\{-1,1\}^d, \tag{18}$$

$$\langle\tilde{\varphi},\tilde{\psi}\rangle_{L^2[-L,L]^d} = 2^d\langle\varphi,\psi\rangle_{L^2[0,L]^d} = \langle\overline{\varphi},\overline{\psi}\rangle_{L^2[-L,L]^d}, \tag{19}$$

$$\langle\varphi,\mathfrak{d}_k\rangle = \mathrm{i}^{-d}\mathscr{F}(\tilde{\varphi})(k) \text{ for all } k\in\mathbb{N}^d, \tag{20}$$

$$\langle\varphi,\mathfrak{n}_k\rangle = \mathscr{F}(\overline{\varphi})(k) \text{ for all } k\in\mathbb{N}_0^d. \tag{21}$$

4.2. By partial integration one obtains that

$$\mathscr{F}(\partial^\alpha f)(k) = (\tfrac{\pi\mathrm{i}}{L}k)^\alpha \mathscr{F}(f)(k).$$

So that $\mathscr{F}(\Delta f)(k) = -|\frac{\pi}{L}k|^2\mathscr{F}(f)(k)$. Consequently $\langle\Delta f,\mathfrak{d}_k\rangle = -|\frac{\pi}{L}k|^2\langle f,\mathfrak{d}_k\rangle$ and $\langle\Delta f,\mathfrak{n}_k\rangle = -|\frac{\pi}{L}k|^2\langle f,\mathfrak{n}_k\rangle$. This will be used later to define $(a-\Delta)^{-1}$ for $a\in\mathbb{R}\setminus\{0\}$.

Moreover, from this one obtains that the spectrum of $-\Delta$ is given by $\{\frac{\pi^2}{L^2}|k|^2 : k\in\mathbb{Z}^d\}$ and that every e_k is an eigenvector.

Lemma 4.3. *$\{\mathfrak{d}_k : k \in \mathbb{N}^d\}$ and $\{\mathfrak{n}_k : k \in \mathbb{N}_0^d\}$ form orthonormal bases for $L^2([0,L]^d)$.*

Proof. We leave it to the reader to check that those sets are orthonormal. Let $\varphi \in L^2([0,L]^d)$. By expressing $\tilde{\varphi}$ and $\overline{\varphi}$ in terms of the basis $\{e_k : k \in \mathbb{Z}^d\}$ and using 4.1 one obtains $\tilde{\varphi} = \sum_{k\in\mathbb{N}^d} \langle \tilde{\varphi}, \tilde{\mathfrak{d}}_k \rangle \tilde{\mathfrak{d}}_k$ and $\overline{\varphi} = \sum_{k\in\mathbb{N}_0^d} \langle \overline{\varphi}, \overline{\mathfrak{n}}_k \rangle \overline{\mathfrak{n}}_k$. □

Definition 4.4. We define the set of test functions on $[0,L]^d$ that oddly and evenly extend to smooth functions on $\mathbb{T}_{2L}^d$ (here $\mathscr{S}(\mathbb{T}_{2L}^d) = C^\infty(\mathbb{T}_{2L}^d)$):

$$\mathscr{S}_0([0,L]^d) := \{\varphi \in C^\infty([0,L]^d) : \tilde{\varphi} \in \mathscr{S}(\mathbb{T}_{2L}^d)\},$$
$$\mathscr{S}_\mathfrak{n}([0,L]^d) := \{\varphi \in C^\infty([0,L]^d) : \overline{\varphi} \in \mathscr{S}(\mathbb{T}_{2L}^d)\}.$$

We equip $\mathscr{S}_0([0,L]^d)$, $\mathscr{S}_\mathfrak{n}([0,L]^d)$ and $\mathscr{S}(\mathbb{T}_{2L}^d)$ with the Schwarz–seminorms. Note that[2] $C_c^\infty([0,L]^d)$ is a subset of both $\mathscr{S}_0([0,L]^d)$ and $\mathscr{S}_\mathfrak{n}([0,L]^d)$.

In the following theorem we state how one can represent elements of $\mathscr{S}$, $\mathscr{S}_0$ and $\mathscr{S}_\mathfrak{n}$ and of $\mathscr{S}'$, $\mathscr{S}_0'$ and $\mathscr{S}_\mathfrak{n}'$ in terms of series in terms of e_k, $\mathfrak{d}_k$ and $\mathfrak{n}_k$.

Theorem 4.5. (a) *Every $\omega \in \mathscr{S}(\mathbb{T}_{2L}^d)$, $\varphi \in \mathscr{S}_0([0,L]^d)$ and $\psi \in \mathscr{S}_\mathfrak{n}([0,L]^d)$ can be represented by*

$$\omega = \sum_{k\in\mathbb{Z}^d} a_k e_k, \quad \varphi = \sum_{k\in\mathbb{N}^d} b_k \mathfrak{d}_k, \quad \psi = \sum_{k\in\mathbb{N}_0^d} c_k \mathfrak{n}_k, \tag{22}$$

where $(a_k)_{k\in\mathbb{Z}^d}$, $(b_k)_{k\in\mathbb{N}^d}$ and $(c_k)_{k\in\mathbb{N}_0^d}$ in $\mathbb{C}$ are such that

$$\forall n \in \mathbb{N}: \sup_{k\in\mathbb{Z}^d} (1+|k|)^n |a_k| < \infty, \ \sup_{k\in\mathbb{N}^d} (1+|k|)^n |b_k| < \infty, \ \sup_{k\in\mathbb{N}_0^d} (1+|k|)^n |c_k| < \infty, \tag{23}$$

and $a_k = \langle \omega, e_k \rangle$, $b_k = \langle \varphi, \mathfrak{d}_k \rangle$ and $c_k = \langle \psi, \mathfrak{n}_k \rangle$.

Conversely, if $(a_k)_{k\in\mathbb{Z}^d}$, $(b_k)_{k\in\mathbb{N}^d}$ and $(c_k)_{k\in\mathbb{N}_0^d}$ satisfy (23) *then $\sum_{k\in\mathbb{Z}^d} a_k e_k$, $\sum_{k\in\mathbb{N}^d} b_k \mathfrak{d}_k$ and $\sum_{k\in\mathbb{N}_0^d} c_k \mathfrak{n}_k$ converge in $\mathscr{S}(\mathbb{T}_{2L}^d)$, $\mathscr{S}_0([0,L]^d)$ and $\mathscr{S}_\mathfrak{n}([0,L]^d)$, respectively.*

(b) *Every $w \in \mathscr{S}'(\mathbb{T}_{2L}^d)$, $u \in \mathscr{S}_0'([0,L]^d)$ and $v \in \mathscr{S}_\mathfrak{n}'([0,L]^d)$ can be represented by*

$$w = \sum_{k\in\mathbb{Z}^d} a_k e_k, \quad u = \sum_{k\in\mathbb{N}^d} b_k \mathfrak{d}_k, \quad v = \sum_{k\in\mathbb{N}_0^d} c_k \mathfrak{n}_k, \tag{24}$$

where $(a_k)_{k\in\mathbb{Z}^d}$, $(b_k)_{k\in\mathbb{N}^d}$ and $(c_k)_{k\in\mathbb{N}_0^d}$ in $\mathbb{C}$ are such that

$$\exists n \in \mathbb{N}: \sup_{k\in\mathbb{Z}^d} \frac{|a_k|}{(1+|k|)^n} < \infty, \ \sup_{k\in\mathbb{N}^d} \frac{|b_k|}{(1+|k|)^n} < \infty, \ \sup_{k\in\mathbb{N}_0^d} \frac{|c_k|}{(1+|k|)^n} < \infty, \tag{25}$$

and $a_k = \langle w, e_k \rangle$, $b_k = \langle u, \mathfrak{d}_k \rangle$ and $c_k = \langle v, \mathfrak{n}_k \rangle$.

Conversely, if $(a_k)_{k\in\mathbb{Z}^d}$, $(b_k)_{k\in\mathbb{N}^d}$ and $(c_k)_{k\in\mathbb{N}_0^d}$ satisfy (25) *then $\sum_{k\in\mathbb{Z}^d} a_k e_k$, $\sum_{k\in\mathbb{N}^d} b_k \mathfrak{d}_k$ and $\sum_{k\in\mathbb{N}_0^d} c_k \mathfrak{n}_k$ converge in $\mathscr{S}'(\mathbb{T}_{2L}^d)$, $\mathscr{S}_0'([0,L]^d)$ and $\mathscr{S}_\mathfrak{n}'([0,L]^d)$, respectively.*

[2]For the notation see Section 1.3.

Proof. Let $\omega \in \mathscr{S}(\mathbb{T}^d_{2L})$. As one has the relation $\mathscr{F}(\Delta^n \omega)(k) = (-\frac{\pi^2}{L^2}|k|^2)^n \mathscr{F}(\omega)(k)$ for all $n \in \mathbb{N}_0$, we have (23) and $\sum_{k\in\mathbb{Z}^d:|k|\le N} \mathscr{F}(\omega)(k) e_k \xrightarrow{N\to\infty} \omega$ in $\mathscr{S}(\mathbb{T}^d_{2L})$, see also [29, Corollary 2.2.4].

Let $\varphi \in \mathscr{S}_0([0,L]^d)$. Using the shown convergence above for $\omega = \tilde\varphi$, by (14), (16), (17) and (20)

$$\sum_{\substack{k\in\mathbb{Z}^d\\|k|\le N}} \mathscr{F}(\tilde\varphi)(k)e_k = \sum_{\substack{k\in\mathbb{N}^d\\|k|\le N}} \sum_{\mathfrak{q}\in\{-1,1\}^d} \mathscr{F}(\tilde\varphi)(\mathfrak{q}\circ k)e_{\mathfrak{q}\circ k} = \sum_{\substack{k\in\mathbb{N}^d\\|k|\le N}} \langle \varphi, \mathfrak{d}_k\rangle \tilde{\mathfrak{d}}_k.$$

Hence $\sum_{k\in\mathbb{N}^d:|k|\le N} \langle \varphi, \mathfrak{d}_k\rangle \mathfrak{d}_k$ converges to φ in $\mathscr{S}_0([0,L]^d)$.

Let $\psi \in \mathscr{S}_\mathfrak{n}([0,L]^d)$. Using the shown convergence above for $\overline{\psi}$, by (15), (18) and (21)

$$\sum_{\substack{k\in\mathbb{Z}^d\\|k|\le N}} \mathscr{F}(\overline{\psi})(k)e_k = \sum_{\substack{k\in\mathbb{N}_0^d\\|k|\le N}} 2^{-\#\{i:k_i=0\}} \sum_{\mathfrak{q}\in\{-1,1\}^d} \mathscr{F}(\tilde\varphi)(\mathfrak{q}\circ k)e_{\mathfrak{q}\circ k} = \sum_{\substack{k\in\mathbb{N}_0^d\\|k|\le N}} c_k \overline{\mathfrak{n}}_k.$$

Hence $\sum_{k\in\mathbb{N}^d:|k|\le N} \langle \psi, \mathfrak{n}_k\rangle \mathfrak{n}_k$ converges to ψ in $\mathscr{S}_\mathfrak{n}([0,L]^d)$.

(b) follows from (a).

□

For $\varphi \in \mathscr{S}_0([0,L]^d)$, note that $\tilde\varphi = \sum_{k\in\mathbb{N}^d} \langle\varphi, \mathfrak{d}_k\rangle \tilde{\mathfrak{d}}_k$. Moreover, note that $\omega \in \mathscr{S}(\mathbb{T}^d_{2L})$ is odd if and only if $\langle \omega, e_{\mathfrak{q}\circ k}\rangle = (\prod \mathfrak{q})\langle \omega, e_k\rangle$ for all $k \in \mathbb{Z}^d$ and $\mathfrak{q} \in \{-1,1\}^d$. This motivates the following definition.

Definition 4.6. For $u \in \mathscr{S}_0'([0,L]^d)$ we write $\tilde u$ for the distribution in $\mathscr{S}'(\mathbb{T}^d_{2L})$ given by $\tilde u = \sum_{k\in\mathbb{N}^d} \langle u, \mathfrak{d}_k\rangle \tilde{\mathfrak{d}}_k$. For $v \in \mathscr{S}_\mathfrak{n}'([0,L]^d)$ we write $\overline{v}$ for the distribution in $\mathscr{S}'(\mathbb{T}^d_{2L})$ given by $\overline{v} = \sum_{k\in\mathbb{N}_0^d} \langle u, \mathfrak{n}_k\rangle \overline{\mathfrak{n}}_k$. An $w \in \mathscr{S}'$ is called *odd* if $\langle w, e_{\mathfrak{q}\circ k}\rangle = (\prod \mathfrak{q})\langle w, e_k\rangle$ for all $k \in \mathbb{Z}^d$ and $\mathfrak{q} \in \{-1,1\}^d$. If instead $\langle w, e_{\mathfrak{q}\circ k}\rangle = \langle w, e_k\rangle$ for all $k \in \mathbb{Z}^d$ and $\mathfrak{q} \in \{-1,1\}^d$, then w is called *even*.

Note that $\tilde u$ is odd and $\overline{v}$ is even.

By (19) and Theorem 4.5, for $u \in \mathscr{S}_0'([0,L]^d)$, $\varphi \in \mathscr{S}_0([0,L]^d)$ and $v \in \mathscr{S}_\mathfrak{n}'([0,L]^d)$, $\psi \in \mathscr{S}_\mathfrak{n}([0,L]^d)$

$$\langle u, \varphi\rangle = 2^{-d}\langle \tilde u, \tilde\varphi\rangle, \qquad \langle v, \psi\rangle = 2^{-d}\langle \overline{v}, \overline{\psi}\rangle. \tag{26}$$

Theorem 4.7. (a) *We have*

$$\tilde{\mathscr{S}_0}(\mathbb{T}^d_{2L}) := \{\tilde\varphi : \varphi \in \mathscr{S}_0([0,L]^d)\} = \{\psi \in \mathscr{S}(\mathbb{T}^d_{2L}) : \psi \text{ is odd}\},$$
$$\overline{\mathscr{S}}_\mathfrak{n}(\mathbb{T}^d_{2L}) := \{\overline{\varphi} : \varphi \in \mathscr{S}_\mathfrak{n}([0,L]^d)\} = \{\psi \in \mathscr{S}(\mathbb{T}^d_{2L}) : \psi \text{ is even}\},$$

and $\tilde{\mathscr{S}_0}(\mathbb{T}^d_{2L})$ *and* $\overline{\mathscr{S}}_\mathfrak{n}(\mathbb{T}^d_{2L})$ *are closed in* $\mathscr{S}(\mathbb{T}_{2L})$.

(b) $\mathscr{S}(\mathbb{T}^d_{2L})$, $\mathscr{S}_0([0,L]^d)$ *and* $\mathscr{S}_\mathfrak{n}([0,L]^d)$ *are complete.*

(c) *We have*

$$\tilde{\mathscr{S}_0'}(\mathbb{T}^d_{2L}) := \{\tilde u : u \in \mathscr{S}_0'([0,L]^d)\} = \{w \in \mathscr{S}'(\mathbb{T}^d_{2L}) : w \text{ is odd}\},$$
$$\overline{\mathscr{S}}'_\mathfrak{n}(\mathbb{T}^d_{2L}) := \{\overline{v} : v \in \mathscr{S}_\mathfrak{n}'([0,L]^d)\} = \{w \in \mathscr{S}'(\mathbb{T}^d_{2L}) : w \text{ is even}\},$$

and $\tilde{\mathscr{S}_0'}(\mathbb{T}^d_{2L})$ *and* $\overline{\mathscr{S}}'_\mathfrak{n}(\mathbb{T}^d_{2L})$ *are closed in* $\mathscr{S}'(\mathbb{T}^d_{2L})$.

(d) $\mathscr{S}'(\mathbb{T}^d_{2L})$, $\mathscr{S}'_0([0,L]^d)$ *and* $\mathscr{S}'_{\mathfrak{n}}([0,L]^d)$ *are (weak*) sequentially complete.*

Proof. (a) follows as convergence in $\mathscr{S}$ implies pointwise convergence and therefore the limit of odd and even functions is again odd and even, respectively. (b) follows from (a) as $\mathscr{S}(\mathbb{T}^d_{2L})$ is complete (see [9, Page 134]). (c) If a net $(w_\iota)_{\iota\in\mathbb{I}}$ in $\tilde{\mathscr{S}}'_0$ converges in $\mathscr{S}'$ to some w, then $\langle w_\iota, e_k\rangle \to \langle w, e_k\rangle$ for all k, so that w is odd. (d) follows from (c) as $\mathscr{S}'(\mathbb{T}^d_{2L})$ is weak* sequentially complete (see [9, Page 137]). □

As we index the basis e_k, $\mathfrak{d}_k$ and $\mathfrak{n}_k$ by integers and not by $k \in \frac{1}{L}\mathbb{Z}^d$, in the next definition of a Fourier multiplier we have an additional $\frac{1}{L}$ factor in the argument of the functions τ and σ.

Definition 4.8. Let $\tau : \mathbb{R}^d \to \mathbb{R}$, $\sigma : [0,\infty)^d \to \mathbb{R}$, $w \in \mathscr{S}'(\mathbb{T}^d_{2L})$, $u \in \mathscr{S}'_0([0,L]^d)$ and $v \in \mathscr{S}'_{\mathfrak{n}}([0,L]^d)$. We define (at least formally)

$$\begin{aligned} \tau(\mathrm{D})w &= \sum_{k\in\mathbb{Z}^d} \tau(\tfrac{k}{L})\langle w, e_k\rangle e_k, \\ \sigma(\mathrm{D})u &= \sum_{k\in\mathbb{N}^d} \sigma(\tfrac{k}{L})\langle u, \mathfrak{d}_k\rangle \mathfrak{d}_k, \quad \sigma(\mathrm{D})v = \sum_{k\in\mathbb{N}_0^d} \sigma(\tfrac{k}{L})\langle v, \mathfrak{n}_k\rangle \mathfrak{n}_k. \end{aligned} \tag{27}$$

Let (χ,ρ) form a *dyadic partition of unity*, i.e., χ and ρ are C^∞ radial functions on $\mathbb{R}^d$, where χ is supported in a ball and ρ is supported in an annulus, such that for $\rho_{-1} := \chi$ and $\rho_j := \rho(2^{-j}\cdot)$ for $j \geq 0$ one has

$$\sum_{j\geq -1} \rho_j(y) = 1, \qquad \frac{1}{2} \leq \sum_{j\geq -1} \rho_j(y)^2 \leq 1 \qquad (y \in \mathbb{R}^d), \tag{28}$$

$$|j-k| \geq 2 \Longrightarrow \operatorname{supp}\rho_j \cap \operatorname{supp}\rho_k = \emptyset \qquad (j,k \in \mathbb{N}_0). \tag{29}$$

Let $w \in \mathscr{S}'(\mathbb{T}^d_{2L})$, $u \in \mathscr{S}'_0([0,L]^d)$ and $v \in \mathscr{S}'_{\mathfrak{n}}([0,L]^d)$. We define the Littlewood-Paley blocks $\Delta_j w$, $\Delta_j u$ and $\Delta_j v$ for $j \geq -1$ by $\Delta_j w = \rho_j(L\mathrm{D})w$, $\Delta_j u = \rho_j(L\mathrm{D})u$, $\Delta_j v = \rho_j(L\mathrm{D})v$, i.e.,

$$\Delta_j w = \sum_{k\in\mathbb{Z}^d} \langle w, e_k\rangle \rho_j(k) e_k, \quad \Delta_j u = \sum_{k\in\mathbb{N}^d} \langle u, \mathfrak{d}_k\rangle \rho_j(k)\mathfrak{d}_k, \quad \Delta_j v = \sum_{k\in\mathbb{N}_0^d} \langle v, \mathfrak{n}_k\rangle \rho_j(k)\mathfrak{n}_k.$$

Let $\overline{\sigma} : \mathbb{R}^d \to \mathbb{R}$ be the even extension of σ, i.e., $\overline{\sigma}(\mathfrak{q}\circ x) = \sigma(x)$ for all $x \in [0,\infty)^d$ and $\mathfrak{q} \in \{-1,1\}^d$. As $\sigma(\mathrm{D})\mathfrak{d}_k = \sigma(\frac{k}{L})\mathfrak{d}_k$ and $\overline{\sigma}(\mathrm{D})\tilde{\mathfrak{d}}_k = \sigma(\frac{k}{L})\tilde{\mathfrak{d}}_k$, by Theorem 4.5 we obtain that for all $u \in \mathscr{S}'_0([0,L]^d)$ and $v \in \mathscr{S}'_{\mathfrak{n}}([0,L]^d)$,

$$\widetilde{\sigma(\mathrm{D})u} = \overline{\sigma}(\mathrm{D})\tilde{u}, \qquad \overline{\sigma(\mathrm{D})v} = \overline{\sigma}(\mathrm{D})\overline{v}. \tag{30}$$

Moreover, with $a_{d,p} = 2^{-\frac{d}{p}}$ for $p < \infty$ and $a_{d,\infty} = 2^{-d}$ we have for all $p \in [1,\infty]$

$$\begin{aligned} \|\sigma(\mathrm{D})u\|_{L^p([0,L]^d)} &= a_{d,p}\|\widetilde{\sigma(\mathrm{D})u}\|_{L^p(\mathbb{T}^d_{2L})} = a_{d,p}\|\overline{\sigma}(\mathrm{D})\tilde{u}\|_{L^p(\mathbb{T}^d_{2L})}, \\ \|\sigma(\mathrm{D})v\|_{L^p([0,L]^d)} &= a_{d,p}\|\overline{\sigma(\mathrm{D})v}\|_{L^p(\mathbb{T}^d_{2L})} = a_{d,p}\|\overline{\sigma}(\mathrm{D})\overline{v}\|_{L^p(\mathbb{T}^d_{2L})}. \end{aligned}$$

Therefore, by applying the above to $\sigma = \rho_j$,

$$a_{d,p}\|\tilde{u}\|_{B^\alpha_{p,q}} = \|(2^{i\alpha}\|\Delta_i u\|_{L^p})_{i\geq -1}\|_{\ell^q}, \qquad a_{d,p}\|\overline{v}\|_{B^\alpha_{p,q}} = \|(2^{i\alpha}\|\Delta_i v\|_{L^p})_{i\geq -1}\|_{\ell^q}.$$

This motivates the following definition.

Definition 4.9. Let $\alpha \in \mathbb{R}, p, q \in [1,\infty]$. We define the *Dirichlet Besov space* $B^{\mathfrak{d},\alpha}_{p,q}([0,L]^d)$ to be the space of $u \in \mathscr{S}_0'([0,L]^d)$ for which $\|u\|_{B^{\mathfrak{d},\alpha}_{p,q}} := a_{d,p}\|\tilde{u}\|_{B^\alpha_{p,q}} < \infty$. Similarly, we define the *Neumann Besov space* $B^{\mathfrak{n},\alpha}_{p,q}([0,L]^d)$ as the space of $v \in \mathscr{S}_\mathfrak{n}'([0,L]^d)$ for which $\|v\|_{B^{\mathfrak{n},\alpha}_{p,q}} := \|\overline{v}\|_{B^\alpha_{p,q}} < \infty$.

We will also use the following notation: $H_0^\alpha = B^{\mathfrak{d},\alpha}_{2,2}$, $H_\mathfrak{n}^\alpha = B^{\mathfrak{n},\alpha}_{2,2}$, $\mathscr{C}_0^\alpha = B^{\mathfrak{d},\alpha}_{\infty,\infty}$, $\mathscr{C}_\mathfrak{n}^\alpha = B^{\mathfrak{n},\alpha}_{\infty,\infty}$. In Theorem 4.16 we justify the use of the notation $H_0^\alpha = B^{\mathfrak{d},\alpha}_{2,2}$.

As $B^\alpha_{p,q}$ is a Banach space, $\|\cdot\|_{B^{\mathfrak{d},\alpha}_{p,q}}$ is a norm on $B^{\mathfrak{d},\alpha}_{p,q}([0,L]^d)$ under which it is a Banach space. Similarly, $\|\cdot\|_{B^{\mathfrak{n},\alpha}_{p,q}}$ is a norm on $B^{\mathfrak{n},\alpha}_{p,q}([0,L]^d)$ under which it is a Banach space.

4.10. Observe that by Lemma 4.3 $H_0^0 = H_\mathfrak{n}^0 = L^2$ and $\|\cdot\|_{H_0^0} \eqsim \|\cdot\|_{H_\mathfrak{n}^0} \eqsim \|\cdot\|_{L^2}$.

4.11. By 4.2 we have $(a-\Delta)^{-1} f = \sigma(\mathrm{D}) f$ for $\sigma(x) = (a + \pi^2|x|^2)^{-1}$.

4.12. For any function φ and $\lambda \in \mathbb{R}$ we write $l_\lambda \varphi$ for the function $x \mapsto \varphi(\lambda x)$. For a distribution u we write $l_\lambda u$ for the distribution given by $\langle l_\lambda u, \varphi\rangle = \lambda^{-d}\langle u, l_{\frac{1}{\lambda}}\varphi\rangle$. As $l_\lambda e_{k,2L} = \lambda^{-\frac{d}{2}} e_{k,\frac{2L}{\lambda}}$, and $\langle l_\lambda u, e_{k,\frac{2L}{\lambda}}\rangle = \lambda^{-\frac{d}{2}}\langle u, e_{k,2L}\rangle$, we have for $u \in \mathscr{S}'(\mathbb{T}^d_{2L})$

$$l_\lambda[\sigma(\lambda \mathrm{D})u] = \sigma(\mathrm{D})[l_\lambda u]. \tag{31}$$

Similarly, (31) holds for $u \in \mathscr{S}_0'([0,L]^d)$ and $u \in \mathscr{S}_\mathfrak{n}'([0,L]^d)$ (use e.g. 4.1).

Theorem 4.13. *$C_c^\infty([0,L]^d)$ is dense in $B^{\mathfrak{d},\alpha}_{p,q}([0,L]^d)$ for all $\alpha \in \mathbb{R}$, $p, q \in [1,\infty)$.*

Proof. The proof follows the same strategy as the proof of [2, Proposition 2.74]. ☐

Theorem 4.14. *With either $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d_{2L}$, for $\alpha > 0$, $H^\alpha(\Omega) = B^\alpha_{2,2}(\Omega) = \Lambda^\alpha_{2,2}(\Omega)$ (for the definitions see [30, p. 36] and [27, p. 168]) and their norms are equivalent.*

Proof. For $H^\alpha(\mathbb{R}^d) = F^\alpha_{2,2}(\mathbb{R}^d)$ see [30, p.88], for $F^\alpha_{2,2}(\mathbb{R}^d) = B^\alpha_{2,2}(\mathbb{R}^d)$ see [30, p.47] and for $B^\alpha_{2,2}(\mathbb{R}^d) = \Lambda^\alpha_{2,2}(\mathbb{R}^d)$ see [30, p.90]. The statements for the torus can be found in [27, p.164,168,169], where it is mentioned that the proofs are similar as for the $\mathbb{R}^d$ space. ☐

The following is a consequence of the fact that the norms of $H(\mathbb{T}^d_{2L})$ (see [27, p. 168]) and $B^\alpha_{2,2}(\mathbb{T}^d_{2L})$ are equivalent.

Theorem 4.15. *For all $\alpha \in \mathbb{R}$ we have*

$$\|f\|_{H^\alpha_\mathfrak{n}} \eqsim \sqrt{\sum_{k\in\mathbb{N}_0^2}(1+|\tfrac{k}{L}|^2)^\alpha \langle f, \mathfrak{n}_k\rangle^2}, \quad \|f\|_{H^\alpha_0} \eqsim \sqrt{\sum_{k\in\mathbb{N}_0^2}(1+|\tfrac{k}{L}|^2)^\alpha \langle f, \mathfrak{d}_k\rangle^2}.$$

Theorem 4.16. *For $\alpha > 0$ the spaces $B^{\mathfrak{d},\alpha}_{2,2}([0,L]^d)$ and $H_0^\alpha([0,L]^d)$ are equal with equivalent norms, where $H_0^\alpha([0,L]^d)$ is the closure of $C_c^\infty([0,L]^d)$ in $H^\alpha(\mathbb{R}^d)$.*

Proof. As $C_c^\infty([0,L]^d)$ is dense in $B^{\mathfrak{d},\alpha}_{2,2}([0,L]^d)$ (Theorem 4.13) it is sufficient to prove the equivalence of the norms on $C_c^\infty([0,L]^d)$. Let $f \in C_c^\infty([0,L]^d)$. By definition of the $\Lambda^\alpha_{2,2}$ norm, $\|f\|_{\Lambda^\alpha_{2,2}(\mathbb{T}_L)} = \|f\|_{\Lambda^\alpha_{2,2}(\mathbb{R}^d)}$. As $D^\beta \tilde{f} = \widetilde{D^\beta f}$ we have $\|\tilde{f}\|_{\Lambda^\alpha_{2,2}(\mathbb{T}^d_{2L})} = 2^{\frac{d}{2}}\|f\|_{\Lambda^\alpha_{2,2}(\mathbb{T}_L)}$. Because $\|\tilde{f}\|_{B^\alpha_{2,2}(\mathbb{T}^d_{2L})} = 2^{\frac{d}{2}}\|f\|_{B^{\mathfrak{d},\alpha}_{2,2}([0,L]^d)}$ (by definition), the proof follows by Theorem 4.14. ☐

Theorem 4.17. *Let $p, q \in [1, \infty]$ and $\beta, \gamma \in \mathbb{R}$, $\gamma < \beta$. Then $B^\beta_{p,q}(\mathbb{T}^d_{2L})$ is compactly embedded in $B^\gamma_{p,q}(\mathbb{T}^d_{2L})$, i.e., every bounded set in $B^\beta_{p,q}(\mathbb{T}^d_{2L})$ is compact in $B^\gamma_{p,q}(\mathbb{T}^d_{2L})$. The analogues statement holds for $B^{\mathfrak{d},\beta}_{p,q}([0, L]^d)$ and $B^{\mathfrak{n},\beta}_{p,q}([0, L]^d)$ spaces. In particular, the injection $j : H^\beta_0([0, L]^d) \to H^\gamma_0([0, L]^d)$ is a compact operator.*

Proof. We consider the underlying space to be $\mathbb{T}^d_{2L}$; the other cases follow by Theorem 4.7. Suppose that $u_n \in B^\beta_{p,q}$ and $\|u_n\|_{B^\beta_{p,q}} \le 1$ for all $n \in \mathbb{N}$. We prove that there is a subsequence of $(u_n)_{n\in\mathbb{N}}$ that converges in $B^\gamma_{p,q}$. By [2, Theorem 2.72] there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$, which we assume to be the sequence itself, such that $u_n \to u$ in $\mathscr{S}'$ and $\|u\|_{B^\beta_{p,q}} \le 1$. As $\langle u_n, e_k\rangle \to \langle u, e_k\rangle$ for all $k \in \mathbb{Z}^d$, we have $\|\Delta_j(u_n - u)\|_{L^p} \to 0$ for all $j \ge -1$. Let $\varepsilon > 0$. Choose $J \in \mathbb{N}$ large enough such that $2^{(\gamma-\beta)J} < \varepsilon$, so that for all $n \in \mathbb{N}$

$$\begin{aligned}\|(2^{\gamma j}\|\Delta_j(u_n - u)\|_{L^p})_{j=J+1}^\infty\|_{\ell^q} &\le 2^{(\gamma-\beta)J}\|(2^{\beta j}\|\Delta_j(u_n - u)\|_{L^p})_{j=J+1}^\infty\|_{\ell^q}\\ &\le 2^{(\gamma-\beta)J}(\|u_n\|_{B^\beta_{p,q}} + \|u\|_{B^\beta_{p,q}}) < 2\varepsilon.\end{aligned}$$

Then, by choosing $N \in \mathbb{N}$ large enough such that $\|(2^{\gamma j}\|\Delta_j(u_n-u)\|_{L^p})_{j=-1}^J\|_{\ell^q} < \varepsilon$ for all $n \ge N$, one has with the above bound that $\|u_n - u\|_{B^\gamma_{p,q}} < 3\varepsilon$ for all $n \ge N$. □

Definition 4.18. Let $y \in \mathbb{R}^d$, $s \in (0, \infty)^d$ and

$$\Gamma = y + \prod_{i=1}^d [0, s_i].$$

Let $l : \prod_{i=1}^d [0, s_i] \to [0, 1]^d$ be given by $l(x) = (\frac{x_1}{s_1}, \dots, \frac{x_d}{s_d})$. For a function φ we define new functions $l\varphi$ and $\mathscr{T}_y\varphi$ by $l\varphi(x) = \varphi \circ l(x)$ and $\mathscr{T}_y\varphi(x) = \varphi(x - y)$ and for a distribution u we define the distributions lu and $\mathscr{T}_y u$ by by $\langle lu, \varphi\rangle = |\det l|^{-1}\langle u, l^{-1}\varphi\rangle$ and $\langle \mathscr{T}_y u, \varphi\rangle = \langle u, \mathscr{T}_y^{-1}\varphi\rangle$. We define

$$\begin{aligned}&\mathscr{S}_0(\Gamma) := \mathscr{T}_y l[\mathscr{S}_0([0, 1]^d)], \qquad \mathscr{S}'_0(\Gamma) := \mathscr{T}_y l(\mathscr{S}'_0([0, 1]^d)),\\ &\sigma(\mathrm{D})u := \mathscr{T}_y l[(l\sigma)(\mathrm{D})((\mathscr{T}_y l)^{-1}u)].\end{aligned} \tag{32}$$

Note that the definition of $\sigma(\mathrm{D})u$ is consistent with (27) by 4.12. Moreover, we define

$$\|u\|_{B^{\mathfrak{d},\alpha}_{p,q}}(\Gamma) := |\det l|^{-\frac{1}{p}} \left\|(\mathscr{T}_y l)^{-1}u\right\|_{B^{\mathfrak{d},\alpha}_{p,q}([0,1]^d)},$$

where for $p = \infty$ we make the convention that $|\det l|^{-\frac{1}{p}} = 1$. Observe that this agrees with our definition of the $B^{\mathfrak{d},\alpha}_{p,q}$ on $[0, L]^d$ for all $L > 0$ (see also 4.12). Similarly, we define $\mathscr{S}_{\mathfrak{n}}(\Gamma)$, $\mathscr{S}'_{\mathfrak{n}}(\Gamma)$, $B^{\mathfrak{n},\alpha}_{p,q}(\Gamma)$ and $\|\cdot\|_{B^{\mathfrak{n},\alpha}_{p,q}(\Gamma)}$.

The following theorem gives a bound on Fourier multipliers, similar as in [2, Theorem 2.78]. However, considering the particular choice $H^\gamma = B^\gamma_{2,2}$ allows us to reduce condition to control all derivatives of σ to a condition that only controls the growth of σ itself.

Theorem 4.19. *Let $\gamma, m \in \mathbb{R}$ and $M > 0$. Let $\sigma : \mathbb{R}^d \to \mathbb{R}$ be C^∞ on $\mathbb{R}^d \setminus \{0\}$ and such that $|\sigma(x)| \le M(1 + |x|)^{-m}$ for all $x \in \mathbb{R}^d$. Then*

$$\|\sigma(\mathrm{D})w\|_{H^{\gamma+m}(\mathbb{T}^d_{2L})} \lesssim \|w\|_{H^\gamma(\mathbb{T}^d_{2L})}. \tag{33}$$

By (30) *one may replace "$H^\gamma(\mathbb{T}^d_{2L})$" by "$H^\gamma_0([0,L]^d)$".*

If, moreover, for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le 2\lfloor 1+\frac{d}{2}\rfloor$ there exist a $C_\alpha > 0$ such that $|\partial^\alpha \sigma(x)| \le C_\alpha |x|^{m-|\alpha|}$ for all $x \neq 0$, then

$$\|\sigma(\mathrm{D})w\|_{\mathscr{C}^{\gamma+m}(\mathbb{T}^d_{2L})} \lesssim \|w\|_{\mathscr{C}^{\gamma}(\mathbb{T}^d_{2L})}. \tag{34}$$

By (30) *one may replace "$\mathscr{C}^\gamma(\mathbb{T}^d_{2L})$" by "$\mathscr{C}^\gamma_{\mathfrak{n}}([0,L]^d)$".*

Proof. Let $a > 0$ be such that $\rho(k) = 0$ if $|k| < a$. Then for $j \ge 0$ one has $|\rho_j(k)\sigma(k)| \le M(1+\frac{a2^j}{L})^{-m}\rho_j(k) \le ML^m a^{-m} 2^{-jm}\rho_j(k)$ for all $k \in \mathbb{Z}^d$. As σ is bounded on the support of ρ_{-1}, there exists a $C > 0$ such that for all $j \ge -1$

$$\|\sigma(\mathrm{D})\Delta_j w\|_{L^2} = \sqrt{\sum_{k\in\mathbb{Z}^d} |w(e_k)|^2 |\sigma(\tfrac{k}{L})|^2 |\rho_j(k)|^2} \le C2^{-jm}\|\Delta_j w\|_{L^2}.$$

(34) follows from [2, Lemma 2.2]. □

4.20. Using the multivariate chain rule (Faà di Bruno's formula) one can prove that $\sigma(x) = (1+\pi^2|x|^2)^{-1}$ satisfies the conditions in Theorem 4.19 (those needed for (34)).

One other bound that we will refer to is a special case of [2, Proposition 2.71]:

Theorem 4.21. *For all $\alpha \in \mathbb{R}$,*

$$\|w\|_{\mathscr{C}^\alpha_{\mathfrak{n}}} \lesssim \|w\|_{H_{\mathfrak{n}}^{\alpha+\frac{d}{2}}}.$$

Now we consider (para- and resonance-) products between elements of $\mathscr{S}'_0([0,L]^d)$ and $\mathscr{S}'_{\mathfrak{n}}([0,L]^d)$, and between elements of $\mathscr{S}'_{\mathfrak{n}}([0,L]^d)$.

4.22. Let $w_1, w_2 \in \mathscr{S}'(\mathbb{T}^d_{2L})$ be represented by $w_1 = \sum_{k\in\mathbb{Z}^d} a_k e_k$ and $w_2 = \sum_{l\in\mathbb{Z}^d} b_l e_l$. Then formally $w_1 w_2 = \sum_{m\in\mathbb{Z}^d} c_m e_m$, with $c_m = \sum_{k,l\in\mathbb{Z}^d, k+l=m} a_k b_l$.

Of course this series is not always convergent (e.g. take $a_k = b_k = |k|^n$ for some $n \in \mathbb{N}$ and see (25)). But if it does, then due to the identities

$$(2L)^{\frac{d}{2}}\tilde{\mathfrak{d}}_k \overline{\mathfrak{n}}_l = \nu_l \sum_{\mathfrak{p}\in\{-1,1\}^d} \tilde{\mathfrak{d}}_{k+\mathfrak{p}\circ l}, \tag{35}$$

$$(2L)^{\frac{d}{2}}\tilde{\mathfrak{d}}_k \tilde{\mathfrak{d}}_l = (-1)^d \sum_{\mathfrak{p}\in\{-1,1\}^d} \nu_{k+\mathfrak{p}\circ l}^{-1} \left(\prod \mathfrak{p}\right) \overline{\mathfrak{n}}_{k+\mathfrak{p}\circ l}, \tag{36}$$

$$(2L)^{\frac{d}{2}}\overline{\mathfrak{n}}_k \overline{\mathfrak{n}}_l = \sum_{\mathfrak{p}\in\{-1,1\}^d} \frac{\nu_k \nu_l}{\nu_{k+\mathfrak{p}\circ l}} \overline{\mathfrak{n}}_{k+\mathfrak{p}\circ l}, \tag{37}$$

the product obeys the following rules

$$\text{even} \times \text{even} = \text{even}, \quad \text{odd} \times \text{even} = \text{odd}, \quad \text{odd} \times \text{odd} = \text{even}.$$

For example, if $u \in \mathscr{S}'_0$ and $v \in \mathscr{S}'_{\mathfrak{n}}$ and uv exists in a proper sense, then $uv \in \mathscr{S}'_0$.

Definition 4.23. For $u \in \mathscr{S}'_0([0,L]^d) \cup \mathscr{S}'_{\mathfrak{n}}([0,L]^d)$ and $v \in \mathscr{S}'_{\mathfrak{n}}([0,L]^d)$ we write (at least formally)

$$u \varolessthan v = v \varogreaterthan u = \sum_{\substack{i,j\ge -1\\ i\le j-1}} \Delta_i u \Delta_j v, \qquad u \odot v = \sum_{\substack{i,j\ge -1\\ |i-j|\le 1}} \Delta_i u \Delta_j v. \tag{38}$$

4.24. As $\widetilde{\mathfrak{d}_k \mathfrak{n}_m} = \tilde{\mathfrak{d}}_k \overline{\mathfrak{n}}_m$ and $\overline{\mathfrak{n}_k \mathfrak{n}_m} = \overline{\mathfrak{n}}_k \overline{\mathfrak{n}}_m$, we have (at least formally)

$$\widetilde{u \otimes v} = \tilde{u} \otimes \overline{v}, \qquad \widetilde{u \ominus v} = \tilde{u} \ominus \overline{v}, \qquad \widetilde{u \odot v} = \tilde{u} \odot \overline{v}, \tag{39}$$

$$\overline{u \otimes v} = \overline{u} \otimes \overline{v}, \qquad \overline{u \ominus v} = \overline{u} \ominus \overline{v}, \qquad \overline{u \odot v} = \overline{u} \odot \overline{v}. \tag{40}$$

With this one can extend the Bony estimates on the (para-/resonance) products on the torus to Bony estimates between elements of $B^{\mathfrak{d},\alpha}_{p,q}([0,L]^d)$ and $B^{\mathfrak{n},\beta}_{p,q}([0,L]^d)$ and between elements of $B^{\mathfrak{n},\beta}_{p,q}([0,L]^d)$. We list some Bony estimates in Theorem 4.25.

Theorem 4.25. *(Bony estimates) With $H^\gamma = H^\gamma(\mathbb{T}^d_{2L})$ and $\mathscr{C}^\alpha = \mathscr{C}^\alpha(\mathbb{T}^d_{2L})$,*

(a) *For $\alpha < 0$, $\gamma \in \mathbb{R}$, $\|f \ominus \xi\|_{H^{\alpha+\gamma}} \lesssim_{\alpha,\gamma} \|f\|_{H^\gamma} \|\xi\|_{\mathscr{C}^\alpha}$,*

(b) *For all $\delta > 0$, $\gamma \geq -\delta$, $\beta \in \mathbb{R}$, $\|f \otimes \xi\|_{H^{\beta-\delta}} \lesssim_{\beta,\delta} \|f\|_{H^\gamma} \|\xi\|_{\mathscr{C}^\beta}$,*

(c) *For all $\alpha, \gamma \in \mathbb{R}$ with $\alpha + \gamma > 0$, $\|f \odot \xi\|_{H^{\alpha+\gamma}} \lesssim_{\alpha,\gamma} \|f\|_{H^\gamma} \|\xi\|_{\mathscr{C}^\alpha}$ and $\|\theta \odot \xi\|_{\mathscr{C}^{\alpha+\gamma}} \lesssim_{\alpha,\gamma} \|\theta\|_{\mathscr{C}^\gamma} \|\xi\|_{\mathscr{C}^\alpha}$.*

(d) *For all $\alpha, \gamma \in \mathbb{R}$ with $\alpha + \gamma > 0$, $\delta > 0$, $\|f\xi\|_{H^{\alpha \wedge \gamma - \delta}} \lesssim_{\alpha,\gamma,\delta} \|f\|_{H^\gamma} \|\xi\|_{\mathscr{C}^\alpha}$.*

By 4.24 one may replace "H^γ" by "$H^\gamma_0([0,L]^d)$" and "$\mathscr{C}^\alpha$" by "$\mathscr{C}^\alpha_{\mathfrak{n}}([0,L]^d)$.

Proof. For (a) and (b) see [25, Lemma 2.1] and [2, Proposition 2.82] where the underlying space is $\mathbb{R}^d$ rather than the torus. For (c) see [2, Proposition 2.85]. (d) follows from the rest. □

5 The operator $\Delta + \xi$ with Dirichlet boundary conditions

We define the Anderson Hamiltonian and study its spectral properties that will be used in the rest of the paper. **In this section we assume $d = 2$, $y \in \mathbb{R}^2$ and $s \in (0,\infty)^2$ and write $\Gamma = y + \prod_{i=1}^2 [0, s_i]$. Moreover, we let $\alpha \in (-\frac{4}{3}, -1)$ and $\xi \in \mathscr{C}^\alpha_{\mathfrak{n}}(\Gamma)$. We abbreviate $\mathscr{C}^\alpha_{\mathfrak{n}}(\Gamma)$ by $\mathscr{C}^\alpha_{\mathfrak{n}}$, $H^\gamma_0(\Gamma)$ by H^γ_0, etc. We write $\sigma : \mathbb{R}^2 \to (0,\infty)$ for the function given by**

$$\sigma(x) = \frac{1}{1 + \pi^2 |x|^2}.$$

Additional assumptions are given in 5.10. Remember, see 4.11, that $\sigma(\mathrm{D}) = (1 - \Delta)^{-1}$.

Definition 5.1. For $\beta \in \mathbb{R}$, we define the *space of Neumann rough distributions*, written $\mathfrak{X}^\beta_{\mathfrak{n}}$, to be the closure in $\mathscr{C}^\beta_{\mathfrak{n}} \times \mathscr{C}^{2\beta+2}_{\mathfrak{n}}$ of the set

$$\{(\zeta, \zeta \odot \sigma(\mathrm{D})\zeta - c) : \zeta \in \mathscr{S}_{\mathfrak{n}}, c \in \mathbb{R}\}.$$

We equip $\mathfrak{X}^\beta_{\mathfrak{n}}$ with the relative topology with respect to $\mathscr{C}^\beta_{\mathfrak{n}} \times \mathscr{C}^{2\beta+2}_{\mathfrak{n}}$.

We will now define the Dirichlet domain of the Anderson Hamiltonian analogously to [1] did on the torus.

Definition 5.2. Let $\boldsymbol{\xi} = (\xi, \Xi) \in \mathfrak{X}^\alpha_{\mathfrak{n}}$. For $0 < \gamma < \alpha + 2$ we define $\mathscr{D}^{\mathfrak{d},\gamma}_{\boldsymbol{\xi}} = \{f \in H^\gamma_0 : f^{\sharp\boldsymbol{\xi}} \in H^{2\gamma}_0\}$, where $f^{\sharp\boldsymbol{\xi}} := f - f \otimes \sigma(\mathrm{D})\xi$. Moreover, we define an inner product on $\mathscr{D}^{\mathfrak{d},\gamma}_{\boldsymbol{\xi}}$, written $\langle \cdot, \cdot \rangle_{\mathscr{D}^{\mathfrak{d},\gamma}_{\boldsymbol{\xi}}}$, by $\langle f, g \rangle_{\mathscr{D}^{\mathfrak{d},\gamma}_{\boldsymbol{\xi}}} = \langle f, g \rangle_{H^\gamma_0} + \langle f^{\sharp\boldsymbol{\xi}}, g^{\sharp\boldsymbol{\xi}} \rangle_{H^{2\gamma}_0}$.

For $-\frac{\alpha}{2} < \gamma < \alpha + 2$ we define the space of *strongly paracontrolled distributions* by $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma} = \{f \in H_0^\gamma : f^{\flat\boldsymbol{\xi}} \in H_0^2\}$, where $f^{\flat\boldsymbol{\xi}} := f^{\sharp\xi} - B(f, \boldsymbol{\xi})$ and $B(f, \boldsymbol{\xi}) = \sigma(\mathrm{D})(f\Xi + f \circledcirc \xi - ((\Delta - 1)f) \otimes \sigma(\mathrm{D})\xi - 2\sum_{i=1}^d \partial_{x_i} f \otimes \partial_{x_i}\sigma(\mathrm{D})\xi)$. We define an inner product on $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma}$, written $\langle \cdot, \cdot \rangle_{\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma}}$, by $\langle f, g \rangle_{\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma}} = \langle f, g \rangle_{H_0^\gamma} + \langle f^{\flat\boldsymbol{\xi}}, g^{\flat\boldsymbol{\xi}} \rangle_{H_0^2}$. As in the periodic setting, one has $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d},\gamma} \subset H_0^{\alpha+2-}$ for all $\gamma \in (-\frac{\alpha}{2}, \alpha + 2)$. We write $\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}} = \{f \in H_0^{\alpha+2-} : f^{\flat\boldsymbol{\xi}} \in H_0^2\}$.

We will define the Anderson Hamiltonian on the Dirichlet domain in a similar sense as is done on the periodic domain, however we choose to change the sign in front of the Laplacian as this is more common in literature on the parabolic Anderson model.

Definition 5.3. Let $-\frac{\alpha}{2} < \gamma < \alpha + 2$, $\boldsymbol{\xi} \in \mathfrak{X}_{\mathfrak{n}}^\alpha$. We define[3] the operator $\mathscr{H}_{\boldsymbol{\xi}} : \mathscr{D}_\xi^{\mathfrak{d},\gamma} \to H_0^{\gamma-2}$ by

$$\mathscr{H}_{\boldsymbol{\xi}} f = \Delta f + f \otimes \xi + f^{\sharp\xi} \odot \xi + \mathscr{R}(f, \sigma(\mathrm{D})\xi, \xi) + f\Xi + f \circledcirc \xi,$$

where $\mathscr{R}(f, g, h) := (f \otimes g) \odot h - f(g \odot h)$.

We state the main results about the spectrum of the Anderson Hamiltonian, on its Dirichlet domain. These results are analogous to the Anderson Hamiltonian on the torus [1]. Moreover, they are similar to the results of [18], which proof is based on the theory of regularity structures.

Theorem 5.4. *Let $\boldsymbol{\xi} \in \mathfrak{X}_{\mathfrak{n}}^\alpha$. For $-\frac{\alpha}{2} < \gamma < \alpha + 2$*

$$\|\mathscr{H}_{\boldsymbol{\xi}} f\|_{H_0^{\gamma-2}} \lesssim \|f\|_{\mathscr{D}_\xi^{\mathfrak{d},\gamma}} (1 + \|\boldsymbol{\xi}\|_{\mathfrak{X}_{\mathfrak{n}}^\alpha})^2. \tag{41}$$

$\mathscr{H}_{\boldsymbol{\xi}}(\mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}}) \subset L^2$ and $\mathscr{H}_{\boldsymbol{\xi}} : \mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}} \to L^2$ is closed and self-adjoint as an operator on L^2. There exist $\lambda_1(\Gamma, \boldsymbol{\xi}) \geq \lambda_2(\Gamma, \boldsymbol{\xi}) \geq \cdots$ such that $\sigma(\mathscr{H}_{\boldsymbol{\xi}}) = \sigma_p(\mathscr{H}_{\boldsymbol{\xi}}) = \{\lambda_n(\Gamma, \boldsymbol{\xi}) : n \in \mathbb{N}\}$ and $\#\{n \in \mathbb{N} : \lambda_n(\Gamma, \boldsymbol{\xi}) = \lambda\} = \dim\ker(\lambda - \mathscr{H}_{\boldsymbol{\xi}}) < \infty$ for all $\lambda \in \sigma(\mathscr{H}_{\boldsymbol{\xi}})$. One has

$$\mathfrak{D}_{\boldsymbol{\xi}} = \bigoplus_{\lambda \in \sigma(\mathscr{H}_{\boldsymbol{\xi}})} \ker(\lambda - \mathscr{H}_{\boldsymbol{\xi}}).$$

There exists an $M > 0$ such that for all $n \in \mathbb{N}$ and $\boldsymbol{\xi}, \boldsymbol{\theta} \in \mathfrak{X}^\alpha$

$$|\lambda_n(\Gamma, \boldsymbol{\xi}) - \lambda_n(\Gamma, \boldsymbol{\theta})| \lesssim \|\boldsymbol{\xi} - \boldsymbol{\theta}\|_{\mathfrak{X}_{\mathfrak{n}}^\alpha} (1 + \|\boldsymbol{\xi}\|_{\mathfrak{X}_{\mathfrak{n}}^\alpha} + \|\boldsymbol{\theta}\|_{\mathfrak{X}_{\mathfrak{n}}^\alpha})^M. \tag{42}$$

With the notation $\sqsubset$ for "is a linear subspace of",

$$\lambda_n(\Gamma, \boldsymbol{\xi}) = \sup_{\substack{F \sqsubset \mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}} \\ \dim F = n}} \inf_{\substack{\psi \in F \\ \|\psi\|_{L^2} = 1}} \langle \mathscr{H}_{\boldsymbol{\xi}} \psi, \psi \rangle_{L^2} \tag{43}$$

In particular,

$$\lambda_1(\Gamma, \boldsymbol{\xi}) = \sup_{\psi \in \mathfrak{D}_{\boldsymbol{\xi}}^{\mathfrak{d}} : \|\psi\|_{L^2} = 1} \langle \mathscr{H}_{\boldsymbol{\xi}} \psi, \psi \rangle_{L^2}$$

.

Remark 5.5. Let us mention that in an analogous way one can state (and prove) the same statement for the operator with Neumann boundary conditions by replacing "$\mathfrak{d}$" by "$\mathfrak{n}$" and "H_0" by "$H_{\mathfrak{n}}$".

[3] The definition needs of course justification to show $H_0^{\gamma-2}$ is really the codomain, this is shown in Theorem 5.4.

Remark 5.6. In [1] it is pointed out that in (43) one may replace $\mathfrak{D}^{\mathfrak{d}}_{\boldsymbol{\xi}}$ by $\mathscr{D}^{\gamma}_{\xi}$ for $\gamma \in (\frac{2}{3}, \alpha+2)$, and $\langle \mathscr{H}_{\boldsymbol{\xi}}\psi, \psi\rangle_{L^2}$ by ${}_{H_0^{-\gamma}}\langle \mathscr{H}_{\boldsymbol{\xi}}\psi, \psi\rangle_{H_0^{\gamma}}$, where ${}_{H_0^{-\gamma}}\langle \cdot, \cdot\rangle_{H_0^{\gamma}} : H_0^{-\gamma} \times H_0^{\gamma} \to \mathbb{R}$ is the continuous bilinear map (see [2, Theorem 2.76]) given by

$$ {}_{H_0^{-\gamma}}\langle f, g\rangle_{H_0^{\gamma}} = \sum_{\substack{i,j\ge -1\\ |i-j|\le 1}} \langle \Delta_i f, \Delta_j g\rangle_{L^2}. $$

This is done for the periodic setting, but the arguments can easily be adapted to our setting.

5.7. Let $\eta \in L^2$ (which equals $H^0_{\mathfrak{n}}$, see 4.10). By Theorem 4.19 $\sigma(\mathrm{D})\eta \in H^2_{\mathfrak{n}}$, which is included in $\mathscr{C}^1_{\mathfrak{n}}$ by Theorem 4.21. Then by Theorem 4.25, $\eta \odot \sigma(\mathrm{D})\eta \in H^1_{\mathfrak{n}}$. Moreover, if $\eta_\varepsilon \to \eta$ in L^2, then $\eta_\varepsilon \odot \sigma(\mathrm{D})\eta_\varepsilon \to \eta \odot \sigma(\mathrm{D})\eta$ in $H^1_{\mathfrak{n}}$ (by the same theorems). Hence, by Theorem 4.21 we obtain the following convergence in $\mathfrak{X}^\alpha_{\mathfrak{n}}$ for all $\alpha \le -1$

$$ (\eta_\varepsilon, \eta_\varepsilon \odot \sigma(\mathrm{D})\eta_\varepsilon) \to (\eta, \eta \odot \sigma(\mathrm{D})\eta). $$

We write $\lambda_n(\Gamma, \eta) = \lambda_n(\Gamma, (\eta, \eta \odot \sigma(\mathrm{D})\eta))$.

By 5.7 and the continuity of $\boldsymbol{\xi} \mapsto \lambda_n(\Gamma, \boldsymbol{\xi})$, see (42) in Theorem 5.4, we obtain the following lemma.

Lemma 5.8. *The map $L^2(\Gamma) \to \mathbb{R}, \eta \mapsto \lambda_n(\Gamma, \eta)$ is continuous.*

5.9. Let $\zeta \in \mathscr{S}^\infty_{\mathfrak{n}}$. Then $\boldsymbol{\zeta} := (\zeta, \zeta \odot \sigma(\mathrm{D})\zeta) \in \mathfrak{X}^\beta_{\mathfrak{n}}$, $f \otimes \sigma(\mathrm{D})\zeta \in H^\beta$ and $B(f, \boldsymbol{\zeta}) \in H^\beta$ for all $\beta \in \mathbb{R}$ (use Theorems 4.19, 4.20 and 4.25). Therefore, for all $\gamma < 1$, $\mathscr{D}^{\mathfrak{d},\gamma}_{\zeta} = H_0^{2\gamma}$ and $\mathfrak{D}^{\mathfrak{d},\gamma}_{\boldsymbol{\zeta}} = H_0^2$ and for $f \in H_0^\gamma$, $f \odot \zeta = f^{\sharp\zeta} \odot \zeta + \mathscr{R}(f, \sigma(\mathrm{D})\zeta, \zeta) + f(\zeta \odot \sigma(\mathrm{D})\zeta)$, so that

$$ \mathscr{H}_\zeta f := \Delta f + f\zeta = \mathscr{H}_{\boldsymbol{\zeta}} f. \tag{44} $$

Now suppose $\zeta \in L^\infty \subset \mathscr{C}^\infty_{\mathfrak{n}}$. Then $\boldsymbol{\zeta} := (\zeta, \zeta \odot \sigma(\mathrm{D})\zeta) \in \mathfrak{X}^0_{\mathfrak{n}}$, but the Bony estimates give $f \otimes \sigma(\mathrm{D})\zeta \in H_0^{2-}$ (and not $\in H_0^2$). Nevertheless, by the Kato-Rellich theorem [26, Theorem X.12] on the domain H_0^2 the operator $\mathscr{H}_\zeta$ defined as in (44) is self-adjoint. As the injection map $H_0^2 \to L^2$ is compact (see Theorem 4.17), every resolvent is compact. Hence by the Riesz-Schauder theorem [26, Theorem VI.15] and the Hilbert-Schmidt theorem [26, Theorem VI.16] there exist $\lambda_1(\Gamma, \zeta) \ge \lambda_2(\Gamma, \zeta) \ge \cdots$ such that $\sigma(\mathscr{H}_\zeta) = \sigma_p(\mathscr{H}_\zeta) = \{\lambda_n(\Gamma, \zeta) : n \in \mathbb{N}\}$ and $\#\{n \in \mathbb{N} : \lambda_n(\Gamma, \zeta) = \lambda\} = \dim\ker(\lambda - \mathscr{H}_\zeta) < \infty$ for all $\lambda \in \sigma(\mathscr{H}_\zeta)$. Moreover, by Fischer's principle [20, Section 28, Theorem 4, p. 318][4] and Lemma A.2

$$ \begin{aligned} \lambda_n(\Gamma, \zeta) &= \sup_{\substack{F \sqsubset H_0^2\\ \dim F = n}} \inf_{\substack{\psi \in F\\ \|\psi\|_{L^2}=1}} \langle \mathscr{H}_\zeta \psi, \psi\rangle_{L^2} \\ &= \sup_{\substack{F \sqsubset C_c^\infty\\ \dim F = n}} \inf_{\substack{\psi \in F\\ \|\psi\|_{L^2}=1}} \int -|\nabla\psi|^2 + \zeta\psi^2. \end{aligned} \tag{45} $$

The proof of Theorem 5.4 follows from the results of the Anderson Hamiltonian on the torus with the help of Lemma 5.12. The proof is written below Lemma 5.12. We may restrict us to the case $\Gamma = Q_L$.

5.10. For the rest of the section $y = 0$ and $b_i = L$ for all i, i.e., $\Gamma = Q_L = [0, L]^2$.

[4] In this reference the operator is actually assumed to be compact and symmetric, whereas we apply it to $\mathscr{H}_{\boldsymbol{\xi}}$. But the compactness is only assumed to guarantee that the spectrum is countable and ordered, so that the arguments still hold.

5.11. For $\mathfrak{q} \in \{-1,1\}^d$ and $w \in \mathscr{S}'$ we write $l_{\mathfrak{q}} w$ for the element in $\mathscr{S}'$ given by $\langle l_{\mathfrak{q}} w, \varphi \rangle = \langle w, \varphi(\mathfrak{q} \circ \cdot) \rangle$ for $\varphi \in \mathscr{S}$. Then w is odd if and only if $w = (\prod \mathfrak{q}) l_{\mathfrak{q}} w$ for all $\mathfrak{q} \in \{-1,1\}^d$ and w is even if and only if $w = l_{\mathfrak{q}} w$ for all $\mathfrak{q} \in \{-1,1\}^d$.

Lemma 5.12. *Let $\boldsymbol{\xi} \in \mathfrak{X}^\alpha_{\mathfrak{n}}$. Let $\frac{2}{3} < \gamma < \alpha + 2$. Write $\overline{\boldsymbol{\xi}} = (\overline{\xi}, \overline{\Xi})$, $\mathscr{D}^\gamma_{\overline{\xi}} = \mathscr{D}^\gamma_{\overline{\xi}}(\mathbb{T}^d_{2L})$, $\mathfrak{D}^\gamma_{\overline{\boldsymbol{\xi}}} = \mathfrak{D}^\gamma_{\overline{\boldsymbol{\xi}}}(\mathbb{T}^d_{2L})$.*

(a) $\widetilde{\mathscr{D}^{\mathfrak{d},\gamma}_{\xi}} = \{w \in \mathscr{D}^\gamma_{\overline{\xi}} : w \text{ is odd}\}$, $\widetilde{\mathfrak{D}^{\mathfrak{d},\gamma}_{\boldsymbol{\xi}}} = \{w \in \mathfrak{D}^\gamma_{\overline{\boldsymbol{\xi}}} : w \text{ is odd}\}$, $\widetilde{\mathscr{H}_{\boldsymbol{\xi}} f} = \mathscr{H}_{\overline{\boldsymbol{\xi}}} \tilde{f}$ *and* $\|f\|_{\mathscr{D}^{\mathfrak{d},\gamma}_{\xi}} \eqsim \|\tilde{f}\|_{\mathscr{D}^\gamma_{\overline{\xi}}}$ *for all* $f \in \mathscr{D}^{\mathfrak{d},\gamma}_{\xi}$ *and* $\|f\|_{\mathfrak{D}^{\mathfrak{d},\gamma}_{\boldsymbol{\xi}}} \eqsim \|\tilde{f}\|_{\mathfrak{D}^\gamma_{\overline{\boldsymbol{\xi}}}}$ *for all* $f \in \mathfrak{D}^{\mathfrak{d},\gamma}_{\boldsymbol{\xi}}$.

(b) $\mathscr{H}_{\boldsymbol{\xi}}(\mathscr{D}^{\mathfrak{d},\gamma}_{\xi}) \subset H^{\gamma-2}_0$, $\mathscr{H}_{\boldsymbol{\xi}}(\mathfrak{D}^{\mathfrak{d},\gamma}_{\boldsymbol{\xi}}) \subset L^2$.

(c) $\mathscr{H}_{\overline{\boldsymbol{\xi}}}(l_{\mathfrak{q}} f) = l_{\mathfrak{q}} \mathscr{H}_{\overline{\boldsymbol{\xi}}} f$ *for all* $f \in \mathscr{D}^\gamma_{\overline{\xi}}$ *and* $\mathfrak{q} \in \{-1,1\}^2$.

(d) $\sigma(\mathscr{H}_{\boldsymbol{\xi}}) \subset \sigma(\mathscr{H}_{\overline{\boldsymbol{\xi}}})$ *(for the operators either on the $\mathscr{D}$ or $\mathfrak{D}$ domains) and for all $a \in \mathbb{C} \setminus \sigma(\mathscr{H}_{\overline{\boldsymbol{\xi}}})$ the inverse of $a - \mathscr{H}_{\boldsymbol{\xi}} : \mathfrak{D}^{\mathfrak{d}}_{\boldsymbol{\xi}} \to L^2$ is self-adjoint and compact.*

Proof. (a) follows from the identities (39), $\widetilde{f^{\sharp\xi}} = \tilde{f}^{\sharp\overline{\xi}}$, $\widetilde{B(f,\boldsymbol{\xi})} = B(\tilde{f}, \overline{\boldsymbol{\xi}})$, $\widetilde{f^{\flat\boldsymbol{\xi}}} = \tilde{f}^{\flat\overline{\boldsymbol{\xi}}}$ and because $\|\tilde{g}\|_{H^\gamma} \eqsim \|g\|_{H^\gamma_0}$ for all $\gamma \in \mathbb{R}$ and $g \in H^\gamma_0([0,L]^d)$ (indeed, $\|g\|_{B^{\mathfrak{d},\gamma}_{2,2}} = \|\tilde{g}\|_{B^\gamma_{2,2}}$ by definition and $\|\cdot\|_{H^\gamma_0} \eqsim \|\cdot\|_{B^{\mathfrak{d},\gamma}_{2,2}}$ and $\|\cdot\|_{B^\gamma_{2,2}} \eqsim \|\cdot\|_{H^\gamma}$ by Theorems 4.14 and 4.16).

(b) follows from (a) as $\mathscr{H}_{\overline{\boldsymbol{\xi}}}(\mathscr{D}^\gamma_{\overline{\xi}}) \subset H^{\gamma-2}$ and $\mathscr{H}_{\overline{\boldsymbol{\xi}}}(\mathfrak{D}_{\overline{\boldsymbol{\xi}}}) \subset H^0$ (see [1]).

(c) follows by a straightforward calculation; use that $\mathscr{F}(l_{\mathfrak{q}} f) = l_{\mathfrak{q}} \mathscr{F}(f)$, $l_{\mathfrak{q}} \rho_i = \rho_i$, $l_{\mathfrak{q}} \overline{\xi} = \overline{\xi}$ and $l_{\mathfrak{q}} \overline{\Xi} = \overline{\Xi}$ for $\mathfrak{q} \in \{-1,1\}^2$.

(d) Let $a \in \mathbb{C}$ be such that $a - \mathscr{H}_{\overline{\boldsymbol{\xi}}}$ has a bounded inverse $\mathscr{R}_a$. By (c) $(a - \mathscr{H}_{\overline{\boldsymbol{\xi}}}) f$ is odd if and only if f is odd, indeed, if $(a - \mathscr{H}_{\overline{\boldsymbol{\xi}}}) f$ is odd, then $(a - \mathscr{H}_{\overline{\boldsymbol{\xi}}})[f - (\prod \mathfrak{q}) l_{\mathfrak{q}} f] = 0$ (see 5.11) and thus $f = (\prod \mathfrak{q}) l_{\mathfrak{q}} f$. Hence $a - \mathscr{H}_{\boldsymbol{\xi}}$ has a bounded inverse $\mathscr{R}^{\mathfrak{d}}_a$ such that $\widetilde{\mathscr{R}^{\mathfrak{d}}_a h} = \mathscr{R}_a \tilde{h}$. From the fact that $\mathscr{R}_a$ is self-adjoint and compact it follows that $\mathscr{R}^{\mathfrak{d}}_a$ is too.

□

Proof of Theorem 5.4. By Lemma 5.12 it follows that $\mathscr{H}_{\boldsymbol{\xi}}$ is a closed densely defined symmetric operator and that $\sigma(\mathscr{H}_{\boldsymbol{\xi}}) \subset \sigma(\mathscr{H}_{\overline{\boldsymbol{\xi}}})$ so that $\mathscr{H}_{\boldsymbol{\xi}}$ is indeed self-adjoint. As the resolvents are compact, the statements in Theorem 5.4 up to (42) follow by the Riesz-Schauder theorem [26, Theorem VI.15] and the Hilbert-Schmidt theorem [26, Theorem VI.16] because of the following identity, where $R_\mu = (\mu - \mathscr{H}_{\boldsymbol{\xi}})^{-1}$,

$$\sigma(\mathscr{H}_{\boldsymbol{\xi}}) = \sigma_p(\mathscr{H}_{\boldsymbol{\xi}}) = \{\mu - \tfrac{1}{\lambda} : \lambda \in \sigma_p(R_\mu) \setminus \{0\}\},$$

this means that $\lambda - R_\mu$ is boundedly invertible (or injective) if and only if $\mu - \frac{1}{\lambda} - \mathscr{H}_{\boldsymbol{\xi}}$ is, and in turn follows from the identity

$$\begin{aligned} \lambda(\mu - \tfrac{1}{\lambda} - \mathscr{H}_{\boldsymbol{\xi}}) &= \lambda(\mu - \mathscr{H}_{\boldsymbol{\xi}}) - 1 = (\lambda - R_\mu)(\mu - \mathscr{H}_{\boldsymbol{\xi}}) \\ &= (\mu - \mathscr{H}_{\boldsymbol{\xi}})\lambda - 1 = (\mu - \mathscr{H}_{\boldsymbol{\xi}})(\lambda - R_\mu). \end{aligned}$$

As every eigenvalue of $\mathscr{H}_{\boldsymbol{\xi}}$ is an eigenvalue of $\mathscr{H}_{\overline{\boldsymbol{\xi}}}$ which is locally lipschitz in the analogues sense of (42), also (42) holds by the equivalences of norms in Lemma 5.12(a). (43) follows from Fischer's principle [20, Section 28, Theorem 4, p. 318]. □

6 Enhanced white noise

In this section we prove Theorem 6.4; we first recall a definition and introduce notation.

Definition 6.1. A *white noise* on $\mathbb{R}^d$ is a random variable $W : \Omega \to \mathscr{S}'(\mathbb{R}^d)$ such that for all $f \in \mathscr{S}$ the random variable $\langle W, f\rangle$ is a centered complex Gaussian random variable with $\langle W, \overline{f}\rangle = \overline{\langle W, f\rangle}$ and $\mathbb{E}[\langle W, f\rangle\overline{\langle W, g\rangle}] = \langle f, g\rangle_{L^2}$ for $f, g \in \mathscr{S}$.

6.2. Then $f \mapsto \langle W, f\rangle$ is linear [22, Remark below Definition 1.1.1] and as $\|\langle W, f\rangle\|_{L^2(\Omega)} = \|f\|_{L^2}$, the map $f \mapsto \langle W, f\rangle$ extends to a bounded linear operator $\mathscr{W} : L^2(\mathbb{R}^d) \to L^2(\Omega)$ such that for all $f \in L^2(\mathbb{R}^d)$, $\mathscr{W} f$ is a complex Gaussian random variable, $\mathscr{W}\overline{f} = \overline{\mathscr{W} f}$ and $\mathbb{E}[\mathscr{W} f\overline{\mathscr{W} g}] = \langle f, g\rangle_{L^2}$ for all $f, g \in L^2(\mathbb{R}^d)$.

6.3. Let W be a white noise on $\mathbb{R}^2$ and $\mathscr{W}$ as in 6.2. For the rest of this section we fix $L > 0$. Unless mentioned otherwise $\tau \in C_c^\infty(\mathbb{R}^d, [0, 1])$ is an even function that is equal to 1 on a neighbourhood of 0. Define $\xi_{L,\varepsilon} \in \mathscr{S}_{\mathfrak{n}}([0, L]^d)$ by (for $\langle \mathscr{W}, \mathfrak{n}_{k,L}\rangle$, we interpret $\mathfrak{n}_{k,L}$ to be the function in $L^2(\mathbb{R}^2)$ being equal to $\mathfrak{n}_{k,L}$ on $[0, L]^d$ and equal to 0 elsewhere)

$$\xi_{L,\varepsilon} = \sum_{k\in\mathbb{N}_0^d} \tau(\tfrac{\varepsilon}{L}k)\langle \mathscr{W}, \mathfrak{n}_{k,L}\rangle \mathfrak{n}_{k,L}. \tag{46}$$

For $k \in \mathbb{N}_0^d$ define $Z_k := \langle \mathscr{W}, \mathfrak{n}_{k,L}\rangle$. Then Z_k is a (real) normal random variable with

$$\mathbb{E}[Z_k] = 0, \qquad \mathbb{E}[Z_k Z_l] = \delta_{k,l}. \tag{47}$$

Theorem 6.4. *Let $d = 2$. For all $\alpha < -1$ there exists a $\boldsymbol{\xi}_L \in \mathfrak{X}_{\mathfrak{n}}^\alpha$ such that the following convergence holds almost surely in $\mathfrak{X}_{\mathfrak{n}}^\alpha$, i.e., on a measurable set Ω_L with $\mathbb{P}(\Omega_L) = 1$*

$$\lim_{\varepsilon\downarrow 0,\varepsilon\in\mathbb{Q}\cap(0,\infty)} (\xi_{L,\varepsilon}, \xi_{L,\varepsilon} \odot \sigma(\mathrm{D})\xi_{L,\varepsilon} - c_\varepsilon) = \boldsymbol{\xi}_L, \tag{48}$$

where $c_\varepsilon = \frac{2}{\pi}\log(\frac{1}{\varepsilon})$. The ξ_L is a white noise in the sense that for $\varphi, \psi \in \mathscr{S}_{\mathfrak{n}}(Q_L)$, $\xi_L(\varphi)$ and $\xi_L(\psi)$ are normal random variables with

$$\mathbb{E}[\xi_L(\varphi)] = 0, \qquad \mathbb{E}[\xi_L(\varphi)\xi_L(\psi)] = \langle \varphi, \psi\rangle_{L^2([0,L]^d)}. \tag{49}$$

Moreover, for $\varphi \in C_c^\infty(Q_L)$ one has almost surely (i.e., on Ω_L)

$$\langle \xi_L, \varphi\rangle = \lim_{\varepsilon\downarrow 0}\langle \xi_{L,\varepsilon}, \varphi\rangle = \sum_{k\in\mathbb{N}_0^d} \langle \mathscr{W}, \mathfrak{n}_{k,L}\rangle\langle \mathfrak{n}_{k,L}, \varphi\rangle = \langle W, \varphi\rangle.$$

Hence, for every $L > 0$ the W viewed as an element of $\mathscr{D}'(Q_L)$ extends almost surely uniquely to a ξ_L in $\mathscr{C}_{\mathfrak{n}}^\alpha$.

Instead of taking Q_L as an underlying space, we can also take a shift of the box, i.e., $y + Q_L$:

6.5. For $y \in \mathbb{R}^d$ we define

$$\xi_{L,\varepsilon}^y = \mathscr{T}_y\Big[\sum_{k\in\mathbb{N}_0^d} \tau(\tfrac{\varepsilon}{L}k)\langle \mathscr{T}_y^{-1}\mathscr{W}, \mathfrak{n}_{k,L}\rangle \mathfrak{n}_{k,L}\Big].$$

If $d = 2$, by Theorem 6.4 there exists a $\boldsymbol{\xi}_L^y = (\xi_L^y, \Xi_L^y) \in \mathfrak{X}_{\mathfrak{n}}^\alpha(y + Q_L)$ such that almost surely

$$\lim_{\varepsilon\downarrow 0,\varepsilon\in\mathbb{Q}\cap(0,\infty)} (\xi_{L,\varepsilon}^y, \xi_{L,\varepsilon}^y \odot \sigma(\mathrm{D})\xi_{L,\varepsilon}^y - \tfrac{1}{2\pi}\log(\tfrac{1}{\varepsilon})) = \boldsymbol{\xi}_L^y, \tag{50}$$

and such that ξ_L^y is a white noise in the sense described in Theorem 6.4 (i.e. $\mathscr{T}_{-y}\xi_L^y$ satisfies (49)).

For the rest of this section we fix $L > 0$ and drop the subindex L; we write $\xi_\varepsilon = \xi_{L,\varepsilon}$ and $\mathfrak{n}_k = \mathfrak{n}_{k,L}$.

Definition 6.6. Define $\Xi_\varepsilon \in \mathscr{S}_\mathfrak{n}(Q_L)$ by

$$\Xi_\varepsilon(x) = \xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x) - \mathbb{E}[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x)]. \tag{51}$$

The strategy of the proof of the following theorem is rather similar to the proof on the torus in [1], but due to the differences of the Dirichlet setting and for the sake of self-containedness we provide the proof.

Theorem 6.7. *For all $\alpha < -\frac{d}{2}$, ξ_ε converges almost surely as $\varepsilon \downarrow 0$ in $\mathscr{C}_\mathfrak{n}^\alpha$, to the white noise ξ_L (as in Theorem 6.4). Moreover, for $d = 2$ and all $\alpha < -1$, Ξ_ε converges almost surely as $\varepsilon \downarrow 0$ in $\mathscr{C}_\mathfrak{n}^{2\alpha+2}$. We denote its limit by Ξ.*

Proof. The proof relies on the Kolmogorov-Chentsov theorem (Theorem 6.8). Lemma 6.10 shows that the required bound for this theorem can be reduced to bounds on the second moments of $\Delta_i(\xi_\varepsilon - \xi_\delta)(x)$ and $\Delta_i(\Xi_\varepsilon - \Xi_\delta)(x)$, given in 6.11 (the proofs of these bounds are lengthy and therefore postponed to Section 11). (49) follows from

$$\mathbb{E}[\langle \xi_\varepsilon, \varphi\rangle\langle \xi_\varepsilon, \psi\rangle] = \sum_{k\in\mathbb{N}_0^d} \tau(\varepsilon k)^2 \langle \varphi, \mathfrak{n}_k\rangle\langle \psi, \mathfrak{n}_k\rangle \xrightarrow{\varepsilon\downarrow 0} \sum_{k\in\mathbb{N}_0^d} \langle \varphi, \mathfrak{n}_k\rangle\langle \psi, \mathfrak{n}_k\rangle = \langle \varphi, \psi\rangle.$$

□

Theorem 6.8 (Kolmogorov-Chentsov theorem)**.** *[16, Theorem 2.23] Let ζ_ε be a random variable with values in a Banach space $\mathfrak{X}$ for all $\varepsilon > 0$. Suppose there exist $a, b, C > 0$ such that for all $\varepsilon, \delta > 0$,*

$$\mathbb{E}\left[\|\zeta_\varepsilon - \zeta_\delta\|_\mathfrak{X}^a\right] \le C|\varepsilon - \delta|^{1+b}.$$

Then there exists a random variable ζ with values in $\mathfrak{X}$ such that in $L^a(\Omega, \mathfrak{X})$ and almost surely

$$\lim_{\varepsilon\downarrow 0, \varepsilon\in\mathbb{Q}\cap(0,\infty)} \zeta_\varepsilon = \zeta.$$

Proof. This follows from the proof of [16, Theorem 2.23]. □

It is generally known that the p-th moment of a centered Gaussian random variable Z can be bounded by its second moment, as $\mathbb{E}[|Z|^p] = (p-1)!!\mathbb{E}[|Z|^2]^{\frac{p}{2}}$ (see [23, p.110]). We will use the generalisation of this bound, which is a consequence of the so-called hypercontractivity.

Lemma 6.9. *[22, Theorem 1.4.1 and equation (1.71)] Suppose that Z_n for $n \in \mathbb{N}$ are independent Gaussian random variables. If Z is a random variable of the form $\sum_{n\in\mathbb{N}} a_n Z_n$ or $\sum_{n,m\in\mathbb{N}} a_{n,m} Z_n Z_m$ with $a_n, a_{n,m} \in \mathbb{C}$, then for $p > 1$*

$$\mathbb{E}[|Z|^p] \le p^p \mathbb{E}[|Z|^2]^{\frac{p}{2}}.$$

Lemma 6.10. *Let $A > 0$ and $a \in \mathbb{R}$.*

(a) *Suppose ζ is a random variable with values in $\mathscr{S}'_{\mathfrak{n}}([0,L]^d)$ such that $\Delta_i\zeta(x)$ is a random variable of the form as Z is, as in Lemma 6.9 for all $i \geq -1$ and $x \in [0,L]^d$. Then, if for all $i \geq -1$, $x \in [0,L]^d$*

$$\mathbb{E}[|\Delta_i\zeta(x)|^2] \leq A2^{ai}, \tag{52}$$

then for all $\kappa > 0$ there exists a $C > 0$ independent of ζ such that for all $p > 1$

$$\mathbb{E}[\|\zeta\|^p_{\mathscr{C}_{\mathfrak{n}}^{-\frac{a}{2}-\kappa-\frac{2}{p}}}] \leq Cp^pL^dA^{\frac{p}{2}}. \tag{53}$$

(b) *Suppose that $(\zeta_\varepsilon)_{\varepsilon>0}$ is a family of such random variables for which* (52) *holds, such that for all $k \in \mathbb{N}_0^d$*

$$\mathbb{E}[|\langle\zeta_\varepsilon, \mathfrak{n}_k\rangle|^2] \to 0. \tag{54}$$

Then for all $\kappa > 0$ and $p > 1$

$$\mathbb{E}[\|\zeta_\varepsilon\|^p_{\mathscr{C}_{\mathfrak{n}}^{-\frac{a}{2}-\kappa-\frac{2}{p}}}] \to 0,$$

and thus we have $\zeta_\varepsilon \xrightarrow{\mathbb{P}} 0$ (convergence in probability) in $\mathscr{C}_{\mathfrak{n}}^{-\frac{a}{2}-\kappa-\frac{2}{p}}$.

Proof. (a) For $\kappa > 0$, by Lemma 6.9, with $C_\kappa = \sum_{i\geq -1} 2^{-\kappa i}$,

$$\mathbb{E}[\|\zeta\|^p_{B_{p,p}^{\mathfrak{n},-\frac{a}{2}-\kappa}}] = \sum_{i\geq -1} 2^{(-\frac{a}{2}-\kappa)pi}\mathbb{E}\left[\|\Delta_i\zeta\|^p_{L^p}\right] \leq p^pL^d\left(\sum_{i\geq -1}2^{-p\kappa i}\right)A^{\frac{p}{2}} \leq C_\kappa p^pL^dA^{\frac{p}{2}}.$$

Using the embedding property of Besov spaces [2, Proposition 2.71], i.e., there exists a $C > 0$ such that $\|\cdot\|_{\mathscr{C}_{\mathfrak{n}}^{-\frac{a}{2}-\kappa-\frac{2}{p}}} \leq C\|\cdot\|_{B_{p,p}^{\mathfrak{n},-\frac{a}{2}-\kappa}}$, one obtains (53).
(b) By Lemma 6.9 (and Fubini)

$$\mathbb{E}[\|\Delta_i\zeta_\varepsilon\|^p_{L^p}] \leq p^p\int\mathbb{E}[|\Delta_i\zeta_\varepsilon(x)|^2]^{\frac{p}{2}}\,\mathrm{d}x \lesssim p^pL^d\left(\sum_{k\in\mathbb{N}_0^d}\rho_i(k)^2\mathbb{E}[|\langle\zeta_\varepsilon,\mathfrak{n}_k\rangle|^2]\right)^{\frac{p}{2}}.$$

and so

$$\mathbb{E}[\|\zeta_\varepsilon\|^p_{B_{p,p}^{\mathfrak{n},-\frac{a}{2}-\kappa}}] \leq p^pL^d\left(\sum_{i=-1}^{I}2^{(-\frac{a}{2}-\kappa)pi}\left(\sum_{k\in\mathbb{N}_0^d}\rho_i(k)^2\mathbb{E}[|\langle\zeta_\varepsilon,\mathfrak{n}_k\rangle|^2]\right)^{\frac{p}{2}} + A^{\frac{p}{2}}\sum_{i\geq I+1}2^{-\kappa i}\right).$$

The latter becomes arbitrarily small by choosing I large and subsequently ε small. □

6.11. The following two statements are proved in Section 11.

(a) (Lemma 11.1) For all $\gamma \in (0,1)$ there exists a $C > 0$ such that for all $i \geq -1$, $\varepsilon, \delta > 0$,

$$\mathbb{E}[\|\Delta_i(\xi_\varepsilon - \xi_\delta)\|^2_\infty] \leq C2^{(d+2\gamma)i}|\varepsilon-\delta|^\gamma.$$

(b) (Lemma 11.17) Let $d = 2$. For all $\gamma \in (0,1)$ there exists a $C > 0$ such that for all $i \geq -1$, $\varepsilon, \delta > 0$

$$\mathbb{E}[\|\Delta_i(\Xi_\varepsilon - \Xi_\delta)\|^2_{L^\infty}] \leq C2^{2\gamma i}|\varepsilon-\delta|^\gamma.$$

Definition 6.12. Define $c_{\varepsilon,L} \in \mathbb{R}$ by

$$c_{\varepsilon,L} = \frac{1}{L^2} \sum_{k\in\mathbb{Z}^2} \frac{\tau(\frac{\varepsilon}{L}k)^2}{1+\frac{\pi^2}{L^2}|k|^2}. \tag{55}$$

In the periodic setting one has that with ξ_ε defined as in [1], $\mathbb{E}[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x)] = c_{\varepsilon,L}$. Observe that it is independent of x. In our setting, the Dirichlet setting, we have (remember (47) and use that $\sum_{i,j\ge -1, |i-j|\le 1} \rho_i(k)\rho_j(k) = 1$)

$$\mathbb{E}[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x)] = \sum_{k\in\mathbb{N}_0^2} \frac{\tau(\frac{\varepsilon}{L}k)^2}{1+\frac{\pi^2}{L^2}|k|^2}\mathfrak{n}_k(x)^2.$$

Note that

$$\sum_{k\in\mathbb{N}_0^2} \frac{\tau(\frac{\varepsilon}{L}k)^2}{1+\frac{\pi^2}{L^2}|k|^2}\mathfrak{n}_k(0)^2 = \sum_{k\in\mathbb{N}_0^2} \frac{\tau(\frac{\varepsilon}{L}k)^2}{1+\frac{\pi^2}{L^2}|k|^2}\frac{4\nu_k^2}{L^2} = c_{\varepsilon,L}. \tag{56}$$

Lemma 6.15 deals with this x dependence of $\mathbb{E}[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x)]$.

The following observation will be used multiple times.

6.13. As $0 \le \rho_i \le 1$ and there is a $b \ge 1$ such that ρ_i is supported in a ball of radius $2^i b$ for all $i \ge -1$, one has for all $i \ge -1$, $k \in \mathbb{N}_0^d$ and $\gamma > 0$

$$\rho_i(k) \le \left(2b\frac{2^i}{1+|k|}\right)^\gamma. \tag{57}$$

Theorem 6.14. *Let $\tau : \mathbb{R}^2 \to [0,1]$ be a compactly supported even function that equals 1 on a neighbourhood of 0. Let $\gamma \in \mathbb{R}$. For all $h \in H_\mathfrak{n}^\gamma$ we have $\|h - \tau(\varepsilon\mathrm{D})h\|_{H_\mathfrak{n}^\gamma} \to 0$ and for $\beta < \gamma$*

$$\|h - \tau(\varepsilon\mathrm{D})h\|_{H_\mathfrak{n}^\beta} \lesssim \varepsilon^{\gamma-\beta}\|h\|_{H_\mathfrak{n}^\gamma}.$$

Proof. By assumption on τ there exists an $a > 0$ such that $\tau = 1$ on $B(0,a)$. Then

$$\begin{cases} 1-\tau(\frac{\varepsilon}{L}k) = 0 & |k| < \frac{La}{\varepsilon}, \\ (1+|\frac{k}{L}|^2)^{\beta-\gamma} \le (1+|\frac{k}{L}|^2)^{\beta-\gamma} \lesssim \varepsilon^{2(\gamma-\beta)} & |k| \ge \frac{La}{\varepsilon}. \end{cases}$$

By the following bounds the theorem is proved; by Theorem 4.15

$$\|h - \tau(\varepsilon\mathrm{D})h\|_{H_\mathfrak{n}^\beta} \lesssim \sqrt{\sum_{k\in\mathbb{N}_0^d} (1+|\tfrac{k}{L}|^2)^\beta (1-\tau(\tfrac{\varepsilon}{L}k))^2 \langle h, \mathfrak{n}_k\rangle^2} \lesssim \varepsilon^{\gamma-\beta}\|h\|_{H_\mathfrak{n}^\gamma}.$$

□

Lemma 6.15. *Let $\tau : \mathbb{R}^2 \to [0,1]$ be a compactly supported even function that equals 1 on a neighbourhood of 0. Then*

$$x \mapsto \sum_{k\in\mathbb{N}_0^2} \frac{\tau(\frac{\varepsilon}{L}k)^2}{1+\frac{\pi^2}{L^2}|k|^2}\mathfrak{n}_k(x)^2 - c_{\varepsilon,L}$$

is an element of $\mathscr{S}_\mathfrak{n}$ and converges in $\mathscr{C}_\mathfrak{n}^{-\gamma}$ to a limit that is independent of τ, as $\varepsilon \downarrow 0$ for all $\gamma > 0$. In particular, $x \mapsto \mathbb{E}[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x)] - c_{\varepsilon,L}$ converges in $\mathscr{C}_\mathfrak{n}^{-\gamma}$ as $\varepsilon \downarrow 0$ for all $\gamma > 0$.

Proof. As there are only finitely many $k \in \mathbb{N}_0^2$ for which $\tau(\frac{\varepsilon}{L}k) \neq 0$, $x \mapsto \mathbb{E}[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon(x)] - c_{\varepsilon,L}$ is smooth. By (37), as $\mathfrak{n}_k(0) = \frac{2}{L}\nu_k$ and $\nu_{2k} = \nu_k$,

$$\mathfrak{n}_k(x)^2 = \frac{2}{L}\mathfrak{n}_{2k}(x) + \frac{2}{L}\frac{\nu_k^2}{\nu_{(k_1,0)}}\mathfrak{n}_{(2k_1,0)}(x) + \frac{2}{L}\frac{\nu_k^2}{\nu_{(0,k_2)}}\mathfrak{n}_{(0,2k_2)}(x) + \tfrac{4\nu_k^2}{L^2}.$$

Moreover, as $\delta_0 \in H_{\mathfrak{n}}^{-1}$ and $\langle \delta_0, \mathfrak{n}_k \rangle = \frac{2}{L}$ for all $k \in \mathbb{N}_0^2$, and as $\mathfrak{n}_{2k}(x) = \mathfrak{n}_k(2x)$

$$\frac{2}{L}\sum_{k\in\mathbb{N}_0^2} \frac{\tau(\frac{\varepsilon}{L}k)^2}{1+\frac{\pi^2}{L^2}|k|^2}\mathfrak{n}_{2k}(x) = \tfrac{1}{4}[\tau(\varepsilon\mathrm{D})^2\sigma(\mathrm{D})\delta_0](2x).$$

By Theorem 4.19 $\sigma(\mathrm{D})\delta_0 \in H_{\mathfrak{n}}^1$, so that by Theorem 6.14 $\tau(\varepsilon\mathrm{D})^2\sigma(\mathrm{D})\delta_0 \to \sigma(\mathrm{D})\delta_0$ in $H_{\mathfrak{n}}^1$ and thus in $\mathscr{C}_{\mathfrak{n}}^0$ (by [2, Theorem 2.71]). This convergence is 'stable' under 'multiplying the argument by 2' (see also 4.12).

Let us write h_ε for

$$h_\varepsilon(x) = \sum_{l,m\in\mathbb{N}_0} \frac{\tau(\frac{\varepsilon}{L}(l,m))^2}{1+\frac{\pi^2}{L^2}(l^2+m^2)}\frac{\nu_{(l,m)}^2}{\nu_{(l,0)}}\mathfrak{n}_{(l,0)}(x).$$

Let $\gamma > 0$. With (57) $\|\Delta_i \mathfrak{n}_{(l,0)}\|_{L^\infty} \lesssim |\rho_i(l,0)| \lesssim 2^{\gamma i}(1+l^2)^{-\frac{\gamma}{2}}$, hence

$$\sup_{i\geq -1} 2^{-\gamma i}\|\Delta_i(h_\varepsilon - h_0)\|_{L^\infty} \lesssim \sum_{l,m\in\mathbb{N}_0} (1+l^2)^{-\frac{\gamma}{2}} \frac{\left|\tau(\frac{\varepsilon}{L}(l,m))^2 - 1\right|}{1+\frac{\pi^2}{L^2}(l^2+m^2)}.$$

By Lebesgue's dominated convergence theorem and the next bound it follows that $h_0 \in \mathscr{C}^{-\gamma}$ and $h_\varepsilon \to h_0$ in $\mathscr{C}^{-\gamma}$

$$\begin{aligned}\sum_{l,m\in\mathbb{N}_0} \frac{(1+l^2)^{-\frac{\gamma}{2}}}{1+\frac{\pi^2}{L^2}(l^2+m^2)} &\lesssim \int_0^\infty\int_0^\infty (1+\lfloor r\rfloor^2)^{-\frac{\gamma}{2}}\frac{1}{1+\lfloor r\rfloor^2+\lfloor s\rfloor^2}\,\mathrm{d}s\,\mathrm{d}r \\ &\lesssim C + \int_1^\infty\int_0^\infty r^{-\gamma}\frac{1}{r^2+s^2}\,\mathrm{d}s\,\mathrm{d}r \\ &\lesssim C + \int_1^\infty r^{-1-\gamma}\,\mathrm{d}r \int_0^\infty \frac{1}{1+u^2}\,\mathrm{d}u < \infty,\end{aligned}$$

where we used the substitution $u = \frac{s}{r}$. In the same fashion the term with "$(0,m)$" instead of "$(0,l)$" converges. By these convergences and by plugging in the factor 2 also here the convergence is proved (remember (56)). □

Before we give the proof of Theorem 6.4, we study the behaviour of $c_{\varepsilon,L}$.

Lemma 6.16. *Let $L \geq \sqrt{2}$. Let $\tau : \mathbb{R}^2 \to [0,1]$ be almost everywhere continuous, be equal to 1 on $B(0,a)$ and zero outside $B(0,b)$ for some a, b with $0 < a < b$. For all $L > 0$,*

$$c_{\varepsilon,L} - \tfrac{2}{\pi}\log\tfrac{1}{\varepsilon}$$

converges in $\mathbb{R}$ as $\varepsilon \downarrow 0$.

Proof. We define $\lfloor y \rfloor = (\lfloor y_1 \rfloor, \lfloor y_2 \rfloor)$ and $h_L(y) = (L^2 + \pi^2 |y|^2)^{-1}$ for $y \in \mathbb{R}^2$. Then

$$c_{\varepsilon,L} = \int_{\mathbb{R}^2} \tau(\tfrac{\varepsilon}{L} \lfloor y \rfloor)^2 h_L(\lfloor y \rfloor) \, \mathrm{d}y.$$

We first show that

$$c_{\varepsilon,L} - \int_{\mathbb{R}^2} \tau(\tfrac{\varepsilon}{L} y)^2 h_L(y) \, \mathrm{d}y \to 0.$$

Write $A(s,t)$ for the annulus $\{y \in \mathbb{R}^2 : s \le |y| \le t\}$. To shorten notation, we write $\delta = \frac{\varepsilon}{L}$. As $|[y] - y| \le \sqrt{2}$

$$\begin{aligned} c_{\varepsilon,L} - \int_{\mathbb{R}^2} \tau(\tfrac{\varepsilon}{L} y)^2 h_L(y) \, \mathrm{d}y &= \int_{B(0, \frac{a}{\delta} - \sqrt{2})} h_L([y]) - h_L(y) \, \mathrm{d}y \\ &\quad + \int_{A(\frac{a}{\delta} - \sqrt{2}, \frac{b}{\delta} + \sqrt{2})} \tau(\delta \lfloor y \rfloor)^2 h_L(\lfloor y \rfloor) - \tau(\delta y)^2 h_L(y) \, \mathrm{d}y. \end{aligned}$$

As $h_L([y]) - h_L(y) = h_L([y]) h_L(y) (|y|^2 - |[y]|^2)$, $h_L([y]) \lesssim h_L(y)$ and $(|y|^2 - |[y]|^2) \lesssim 1 + |y|$, we have $h_L([y]) - h_L(y) \lesssim (1 + |y|) h_L(y)^2$. As the latter function is integrable over $\mathbb{R}^2$, it follows by Lebesgue's dominated convergence theorem that $\int_{B(0, \frac{a}{\delta} - \sqrt{2})} h_L([y]) - h_L(y) \, \mathrm{d}y$ converges in $\mathbb{R}$. On the other hand, the integral over the annulus can be written as

$$\int_{A(a - \sqrt{2}\delta, b + \sqrt{2}\delta)} \frac{\tau(\delta \lfloor \frac{x}{\delta} \rfloor)^2}{\delta^2 L^2 + \delta^2 |\lfloor \frac{x}{\delta} \rfloor|^2} - \frac{\tau(x)^2}{\delta^2 L^2 + |x|^2} \, \mathrm{d}x.$$

Again by a domination argument (note that $\frac{1}{|x|^2}$ is integrable over annuli), using that $\frac{1}{2} |\frac{x}{\delta}|^2 \le 2 + |\lfloor \frac{x}{\delta} \rfloor|^2 \le L^2 + |\lfloor \frac{x}{\delta} \rfloor|^2$, we conclude that this integral converges to 0. By some substitutions (remember $\delta = \frac{\varepsilon}{L}$)

$$\frac{1}{2\pi} \int_{B(0, \frac{a}{\delta} - \sqrt{2})} h_L(y) \, \mathrm{d}y = \int_0^1 \frac{s}{1 + \pi^2 s^2} \, \mathrm{d}s + \int_1^{\frac{a}{\varepsilon}} \frac{s}{1 + \pi^2 s^2} \, \mathrm{d}s - \int_{a - \frac{\sqrt{2}\varepsilon}{L}}^{a} \frac{s}{\varepsilon^2 + \pi^2 s^2} \, \mathrm{d}s.$$

The last integral converges as $\varepsilon \downarrow 0$ to zero. For the second integral we consider

$$\int_1^{\frac{a}{\varepsilon}} \frac{s}{1 + \pi^2 s^2} - \frac{1}{\pi^2 s} \, \mathrm{d}s = \int_1^{\frac{a}{\varepsilon}} \frac{-1}{\pi^2 s (1 + \pi^2 s^2)}, \qquad \int_1^{\frac{a}{\varepsilon}} \frac{1}{\pi^2 s} \, \mathrm{d}s = \frac{1}{\pi^2} \log(\frac{a}{\varepsilon}).$$

Hence by the above calculations we obtain that $c_{\varepsilon,L} - \frac{2}{\pi} \log \frac{1}{\varepsilon}$ converges in $\mathbb{R}$. □

Proof of Theorem 6.4. This is a consequence of Theorem 6.7 and Lemmas 6.15 and 6.16. □

7 Scaling and translation

In this section we prove the scaling properties of the eigenvalues, by scaling the size of the box and the noise. **In this section we fix $L > 0$ and $n \in \mathbb{N}$.**

Lemma 7.1. *Suppose that $V \in L^\infty([0, L]^d)$. For all $\beta > 0$*

$$\lambda_n([0, L]^d, V) = \tfrac{1}{\beta^2} \lambda_n([0, \tfrac{L}{\beta}]^d, \beta^2 V(\beta \cdot)).$$

Proof. Fix $n \in \mathbb{N}$ and write $\lambda = \lambda_n([0,L]^d, V)$. Suppose that $g \in H_0^2$ (see 5.9) is an eigenfunction for λ of $\Delta + V$. With $g_\beta(x) := g(\beta x)$ we have for allmost all x

$$\Delta g_\beta(x) + \beta^2 V(\beta x) = \beta^2 (\Delta g)(\beta x) + \beta^2 V(\beta x) = \beta^2 \lambda g_\beta(x).$$

So that $\beta^2\lambda$ is an eigenvalue of $\Delta + \beta^2 V(\beta\cdot)$ on $[0, \frac{L}{\beta}]^d$. As the multiplicities of the eigenvalues on $[0,L]^d$ and $[0,\frac{L}{\beta}]^d$ are the same, $\beta^2\lambda = \lambda_n([0,\frac{L}{\beta}]^d, \beta^2 V(\beta\cdot))$. □

7.2. For $y \in \mathbb{R}^2$, $L > 0$ and $\beta \in \mathbb{R}$ we write

$$\boldsymbol{\lambda}_n(y + Q_L, \beta) = \lambda_n(y + Q_L, (\beta\xi_L^y, \beta^2\Xi_L^y)), \quad \boldsymbol{\lambda}_n(y + Q_L) = \boldsymbol{\lambda}_n(y + Q_L, 1),$$

where $\boldsymbol{\xi}_L^y = (\xi_L^y, \Xi_L^y)$ is as in 6.5.

Lemma 7.3. *For $\beta > 0$*

$$\boldsymbol{\lambda}_n\left(Q_L\right) \stackrel{d}{=} \tfrac{1}{\beta^2}\boldsymbol{\lambda}_n(Q_{\frac{L}{\beta}}, \beta) + \tfrac{2}{\pi}\log\beta.$$

Proof. $\beta l_\beta \xi_L$ is a white noise on $Q_{\frac{L}{\beta}}$, so that $\langle \beta l_\beta \xi_L, \mathfrak{n}_k\rangle \stackrel{d}{=} \langle \xi_{\frac{L}{\beta}}, \mathfrak{n}_k\rangle$ for all $k \in \mathbb{N}_0^2$ and thus $\frac{1}{\beta}\xi_{\frac{L}{\beta}} \stackrel{d}{=} l_\beta\xi_L$. By 4.12 $l_\beta\xi_{L,\varepsilon} = \tau(\frac{\varepsilon}{\beta}\mathrm{D})[l_\beta\xi_L] \stackrel{d}{=} \frac{1}{\beta}\tilde{\xi}_{\frac{L}{\beta},\frac{\varepsilon}{\beta}}$. So that by Lemma 7.1

$$\begin{aligned}
\lambda_n &\left(Q_L, (\xi_{L,\varepsilon}, \xi_{L,\varepsilon} \odot \sigma(\mathrm{D})\xi_{L,\varepsilon} - \tfrac{2}{\pi}\log(\tfrac{1}{\varepsilon}))\right) = \tfrac{1}{\beta^2}\lambda_n\left(Q_L, \xi_{L,\varepsilon}\right) - \tfrac{2}{\pi}\log(\tfrac{1}{\varepsilon}) \\
&\stackrel{d}{=} \tfrac{1}{\beta^2}\lambda_n\left(Q_{\frac{L}{\beta}}, \beta\xi_{\frac{L}{\beta},\frac{\varepsilon}{\beta}}\right) - \tfrac{2}{\pi}\log(\tfrac{1}{\varepsilon}) \\
&\stackrel{d}{=} \tfrac{1}{\beta^2}\lambda_n\left(Q_{\frac{L}{\beta}}, (\beta\xi_{\frac{L}{\beta},\frac{\varepsilon}{\beta}}, \beta^2\left[\xi_{\frac{L}{\beta},\frac{\varepsilon}{\beta}} \odot \sigma(\mathrm{D})\xi_{\frac{L}{\beta},\frac{\varepsilon}{\beta}} - \tfrac{2}{\pi}\log(\tfrac{\beta}{\varepsilon})\right])\right) + \tfrac{2}{\pi}\log\beta.
\end{aligned}$$

□

Lemma 7.4. *For $y \in \mathbb{R}^2$ and $\beta > 0$*

$$\boldsymbol{\lambda}_n(Q_L, \beta) \stackrel{d}{=} \boldsymbol{\lambda}_n(y + Q_L, \beta).$$

Moreover, if $y + Q_L \cap Q_L = \emptyset$, then $\boldsymbol{\lambda}_n(Q_L,\beta)$ and $\boldsymbol{\lambda}_n(y + Q_L, \beta)$ are independent.

Proof. As (see also Definition 4.18, in particular (32)) $\mathscr{H}_{\boldsymbol{\xi}_L^y} f = \mathscr{T}_y(\mathscr{H}_{\mathscr{T}_{-y}\boldsymbol{\xi}_L^y}(\mathscr{T}_{-y}f))$, it is sufficient to show $\boldsymbol{\xi}_L \stackrel{d}{=} \mathscr{T}_{-y}\boldsymbol{\xi}_L^y$. As $\mathscr{T}_{-y}\mathscr{W} \stackrel{d}{=} \mathscr{W}$, we have $\mathscr{T}_{-y}\xi_{L,\varepsilon}^y \stackrel{d}{=} \xi_{L,\varepsilon}$ and hence obtain $\boldsymbol{\xi}_L \stackrel{d}{=} \mathscr{T}_{-y}\boldsymbol{\xi}_L^y$ by (48) and (50).

For the "moreover"; note that $(\langle \mathscr{T}_y^{-1}\mathscr{W}, \mathfrak{n}_{k,L}\rangle)_{k\in\mathbb{N}_0^2}$ and $(\langle \mathscr{W}, \mathfrak{n}_{k,L}\rangle)_{k\in\mathbb{N}_0^2}$ are independent when $y + Q_L^\circ \cap Q_L^\circ = \emptyset$ (as $\mathbb{E}[\langle \mathscr{T}_y^{-1}\mathscr{W}, \mathfrak{n}_{k,L}\rangle\langle \mathscr{W}, \mathfrak{n}_{m,L}\rangle] = \langle \mathscr{T}_y(\mathfrak{n}_{k,L}\mathbb{1}_{Q_L}), \mathfrak{n}_{k,L}\mathbb{1}_{Q_L}\rangle = 0$). □

8 Comparing eigenvalues on boxes of different size

8.1 Bounded potentials

In this section we prove the bounds comparing eigenvalues on large boxes with eigenvalues on smaller boxes for bounded potentials, see Lemma 8.1, Theorem 8.4 and Theorem 8.5. In Section 8.2, Theorem 8.6, we extend this for white noise potentials. We fix $d \in \mathbb{N}$ and use the notation $|k|_\infty = \max_{i\in\{1,\dots,d\}}|k_i|$.

Lemma 8.1. *Let $L > r > 0$ and $\zeta \in L^\infty([0,L]^d)$. For all $y \in \mathbb{R}^2$ such that $y + [0,r]^d \subset [0,L]^d$, we have*

$$\lambda_n(y + [0,r]^d, \zeta) \le \lambda_n([0,L]^d, \zeta).$$

Proof. This follows from (45) as one can identify a finite dimensional $F \sqsubset H_0^2(y + [0,r]^d)$ with a linear subspace of $H_0^2([0,L]^d)$ with the same dimension. □

We will now prove an upper bound for $\lambda_n(Q_L, \zeta)$ in terms of a maximum over smaller boxes. For this we cover Q_L by smaller boxes that overlap and correct the potential with a function that takes into account the overlaps. We use the following lemma.

Lemma 8.2. *Let $r > a > 0$. There exists a smooth function $\eta : \mathbb{R}^d \to [0,1]$ with $\eta = 1$ on $[0, r-a]^d$ and $\operatorname{supp} \eta \subset [-a, r]^d$ such that $\|\nabla \eta\|_\infty \le \frac{K}{a^d}$ for some $K > 0$ that does not depend on r and a, and*

$$\sum_{k \in \mathbb{Z}^d} \eta(x - rk)^2 = 1 \qquad (x \in \mathbb{R}^d). \tag{58}$$

Proof. We adapt the proof of [12, Proposition 1] and [3, Lemma 4.6]. Let $\varphi : \mathbb{R} \to [0,1]$ be smooth, $\varphi = 0$ on $(-\infty, -1]$ and $\varphi = 1$ on $[1, \infty)$ for all $x \in \mathbb{R}$. Let

$$\zeta(x) = \sqrt{\varphi(\tfrac{2x}{a} + 1)(1 - \varphi(\tfrac{2(x-r)}{a} + 1))}$$

Then $\zeta = 0$ outside $[-a, r]$, $\zeta = 1$ on $[0, r-a]$ and $\sum_{k \in \mathbb{Z}} \zeta(x - rk)^2 = 1$. Moreover, $\|\zeta'\|_\infty \le \frac{2}{a}[\|\sqrt{\varphi}'\|_\infty + \|\sqrt{1-\varphi}'\|_\infty]$. Hence with $\eta : \mathbb{R}^d \to [0,1]$ defined by $\eta(x) = \prod_{i=1}^d \zeta(x_i)$ we have (58) and $\|\nabla \eta\|_\infty \le \frac{C}{a^d}$ for some $C > 0$. □

8.3 (IMS formula)**.** Write $\eta_k(x) = \eta(x - rk)$. Then

$$\eta_k^2 \Delta \psi + \Delta(\eta_k^2 \psi) - 2\eta_k \Delta(\eta_k \psi) = \psi |\nabla \eta_k|^2.$$

Consequently, with $\mathscr{H}_k \psi = \mathscr{H}(\eta_k \psi)$ ($\mathscr{H} = \mathscr{H}_\zeta$) and $\Phi = \sum_{k \in \mathbb{Z}^d} |\nabla \eta_k|^2$

$$\mathscr{H} - \Phi = \sum_{k \in \mathbb{Z}^d} \eta_k H_k. \tag{59}$$

(59) is also called the IMS-formula, see also [28, Lemma 3.1] with references to first works in which it appears. The technique to prove [12, Proposition 1], which we slightly generalize, is basically the IMS-formula.

Theorem 8.4. *For all $r > a > 0$ there is a smooth function $\Phi_{a,r} : \mathbb{R}^d \to [0, \infty)$ whose support is contained in the a-neighbourhood of the grid $r\mathbb{Z}^d + \partial[0,r]^d$, is periodic in each coordinate with period r, with $\|\Phi_{a,r}\|_\infty \le \frac{K}{a^d}$ for some $K > 0$ that does not depend on a and r, such that $\zeta \in L^\infty(\mathbb{R}^d)$ and $L > r$*

$$\lambda([0,L]^d, \zeta) - \tfrac{K}{a^d} \le \lambda([0,L]^d, \zeta - \Phi_{a,r}) \le \max_{k \in \mathbb{N}_0^d, |k|_\infty < \frac{L}{r} + 1} \lambda(rk + [-a, r]^d, \zeta). \tag{60}$$

Proof. Let η be as in Lemma 8.2, $\eta_k(x) = \eta(x - rk)$ and $\Phi_{a,r} = \Phi = \sum_{k\in\mathbb{Z}^d} |\nabla\eta_k|^2$. By Lemma 8.2 it follows that $\|\Phi\|_\infty \le \frac{K}{a^d}$ for some $K > 0$ that does not depend on a and r. Observe that $\sum_{k\in\mathbb{N}_0^d:|k|_\infty<\frac{L}{r}+1} \eta_k$ equals 1 on $[0,L]^d$. With H_k as in 8.3 we have $H_k \le \lambda(rk + [-a,r]^d)\eta_k$ for all $k \in \mathbb{Z}^d$. Hence we have by the IMS-formula (59) on $H_0^2([0,L]^d)$

$$\mathscr{H} - \Phi \le \sum_{k\in\mathbb{N}_0^d,|k|_\infty<\frac{L}{r}+1} \lambda(rk + [-a,r]^d)\eta_k^2 \le \max_{k\in\mathbb{N}_0^d,|k|_\infty<\frac{L}{r}+1} \lambda(rk + [-a,r]^d).$$

□

Theorem 8.5. *Let $\zeta \in L^\infty(\mathbb{R}^d)$. Let $y, y_1, \dots, y_n \in \mathbb{R}^d$, $L > r > 0$ be such that $(y_i + [0,r]^d)_{i=1}^n$ are pairwise disjoint subsets of $y + [0,L]^d$. Then*

$$\lambda_n(y + [0,L]^d, \zeta) \ge \min_{i\in\{1,\dots,n\}} \lambda(y_i + [0,r]^d, \zeta). \tag{61}$$

Proof. By (45) (see also (111))

$$\lambda_n(y + [0,L]^d, \zeta) \ge \sup_{\substack{f_1,\dots,f_n,\\ f_i\in C_c^\infty(y_i+[0,r]^d),\|f_i\|_{L^2}=1}} \min_{i\in\{1,\dots,n\}} \int -|\nabla f_i|^2 + \zeta f_i^2,$$

which proves (61) by (45) with $n = 1$. □

8.2 White noise as potential

In this section we prove analogous bounds to those in Lemma 8.1, Theorem 8.4 and Theorem 8.5 by replacing the bounded potential ζ by white noise, i.e., we prove Theorem 8.6.

Theorem 8.6. (a) *For all $\kappa > 0$, $L > r > 0$ with $\frac{L}{r} \in \mathbb{N}$ and $y \in \mathbb{R}^2$ such that $y + Q_r \subset Q_L$*

$$\boldsymbol{\lambda}_n(y + Q_r, \kappa) \le \boldsymbol{\lambda}_n(Q_L, \kappa) \qquad \textit{a.s.} \tag{62}$$

(b) *There exists a $K > 0$ such that for all $\kappa > 0$ and $L > r > a > 0$ with $\frac{L}{r+a} \in \mathbb{N}$,*

$$\boldsymbol{\lambda}(Q_L, \kappa) \le \max_{k\in\mathbb{N}_0^2,|k|_\infty<\frac{L}{r}+1} \boldsymbol{\lambda}(rk + Q_{r+a}, \kappa) + \tfrac{K}{a^2} \qquad \textit{a.s.} \tag{63}$$

(c) *For $\kappa > 0$, $L > r > 0$ with $\frac{L}{r} \in \mathbb{N}$ and $y, y_1, \dots, y_n \in \mathbb{R}^2$ such that $(y_i + Q_r)_{i=1}^n$ are pairwise disjoint subsets of $y + Q_L$*

$$\boldsymbol{\lambda}_n(y + Q_L, \kappa) \ge \min_{i\in\{1,\dots,n\}} \boldsymbol{\lambda}(y_i + Q_r, \kappa) \qquad \textit{a.s.} \tag{64}$$

Let us describe how one can prove Theorem 8.6. Suppose $L, r, \kappa > 0$ and $\frac{L}{r} \in \mathbb{N}$. It is sufficient to show that for all $y \in \mathbb{R}^2$ such that $y + Q_r \subset Q_L$ almost surely one has the following convergences

$$\lambda_n(y + Q_r, \kappa\xi'_{L,\varepsilon}) - c_\varepsilon \to \boldsymbol{\lambda}_n(y + Q_r, \kappa), \qquad \lambda_n(Q_L, \kappa\xi'_{L,\varepsilon}) - c_\varepsilon \to \boldsymbol{\lambda}_n(Q_L, \kappa),$$

for an appropriate $\xi'_{L,\varepsilon}$. We choose $\xi'_{L,\varepsilon}$ to be as $\xi'_{L,\varepsilon}$ in (46) but here $\tau = \mathbb{1}_{(-1,1)^2}$. Then

$$\lambda_n(y + Q_r, \kappa\xi'_{L,\varepsilon}) = \lambda_n(y + Q_r, \kappa\theta_\varepsilon^y) = \lambda_n(y + Q_r, (\kappa\theta_\varepsilon^y, \kappa^2\theta_\varepsilon^y \odot \sigma(\mathrm{D})\theta_\varepsilon^y)),$$

for θ^y_ε (which equals $\xi'_{L,\varepsilon}|_{y+Q_r}$ in $L^2(y+Q_r)$) given by

$$\begin{aligned}\theta^y_\varepsilon &= \sum_{k\in\mathbb{N}_0^2} \langle \xi'_{L,\varepsilon}, \mathscr{T}_y \mathfrak{n}_{k,r}\rangle_{L^2(y+Q_r)} \mathscr{T}_y \mathfrak{n}_{k,r} \\ &= \sum_{k\in\mathbb{N}_0^2}\sum_{\mathfrak{k}\in\mathbb{N}_0^2} \tau(\tfrac{\varepsilon}{L}\mathfrak{k})\langle \mathscr{W}, \mathfrak{n}_{\mathfrak{k},L}\rangle \langle \mathfrak{n}_{\mathfrak{k},L}, \mathscr{T}_y\mathfrak{n}_{k,r}\rangle_{L^2(y+Q_r)} \mathscr{T}_y \mathfrak{n}_{k,r}. \end{aligned} \tag{65}$$

Therefore the following theorem resembles the missing part of the proof.

Theorem 8.7. *Let $L > r > 0$ with $\frac{L}{r} \in \mathbb{N}$ and $y \in \mathbb{R}^2$ be such that $y + Q_r \subset Q_L$. Let $\tau = \mathbb{1}_{(-1,1)^2}$ and θ^y_ε as in* (65) *($c_\varepsilon = \frac{2}{\pi}\log(\frac{1}{\varepsilon})$), we have $(\theta^y_\varepsilon, \theta^y_\varepsilon \odot \sigma(\mathrm{D})\theta^y_\varepsilon - c_\varepsilon) \xrightarrow{\mathbb{P}} \boldsymbol{\xi}^y_r$ in $\mathfrak{X}^\alpha_\mathfrak{n}(y+Q_r)$.*

Remark 8.8. The conditions $\frac{L}{r} \in \mathbb{N}$ and $\tau = \mathbb{1}_{(-1,1)^2}$ in Theorem 8.7 are of computational reasons; $\tau = \mathbb{1}_{(-1,1)^2}$ assures that G_ε as in (95) equals a sum and $\frac{L}{r} \in \mathbb{N}$ assures cancellations in sums, see also Remark 11.21.

In order to prove Theorem 8.7, we summarize some results from Section 11. The proofs are lengthy and rely on similar techniques as the bounds in 6.11 and are therefore postponed to Section 11.

8.9. Let $d = 2$, $L > r > 0$ such that $\frac{L}{r} \in \mathbb{N}$. Let $\xi_\varepsilon = \xi_{\varepsilon,r}$ be as in Section 6 (i.e., defined with a $\tau \in C_c^\infty(\mathbb{R}^2, [0,1])$). Let ξ'_ε be as in (46) with $L = r$ and $\tau = \mathbb{1}_{(-1,1)^2}$, Ξ'_ε be as in (51) with "ξ'_ε" instead of "ξ_ε" and let $\theta_\varepsilon = \theta^0_\varepsilon$ equal the right-hand side in (65) with $\tau = \mathbb{1}_{(-1,1)^2}$.

(a) (Lemma's 11.1 and 11.9) For all $\gamma \in (0,1)$ there exists a $C > 0$ such that for all $i \geq -1$, $\varepsilon > 0$,

$$\mathbb{E}[\|\Delta_i \xi_\varepsilon\|^2_{L^\infty}] \vee \mathbb{E}[\|\Delta_i \xi'_\varepsilon\|^2_{L^\infty}] \vee \mathbb{E}[\|\Delta_i \theta_\varepsilon\|^2_{L^\infty}] \leq C 2^{(d+\gamma)i}.$$

(b) (Lemma 11.10) $\mathbb{E}[|\langle \theta_\varepsilon - \xi'_\varepsilon, \mathfrak{n}_k\rangle|^2] \to 0$ for all $k \in \mathbb{N}_0^2$. Along the same lines of the proof $\mathbb{E}[|\langle \xi_\varepsilon - \xi'_\varepsilon, \mathfrak{n}_k\rangle|^2] \to 0$ for all $k \in \mathbb{N}_0^2$.

(c) (Lemma's 11.17 and 11.18) Let $\Theta_\varepsilon = \theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon - \mathbb{E}[\theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon]$. For all $\gamma \in (0,\infty)$ there exists a $C > 0$ such that for all $i \geq -1$, $\varepsilon > 0$

$$\mathbb{E}[\|\Delta_i \Xi_\varepsilon\|^2_{L^\infty}] \vee \mathbb{E}[\|\Delta_i \Xi'_\varepsilon\|^2_{L^\infty}] \vee \mathbb{E}[\|\Delta_i \Theta_\varepsilon\|^2_{L^\infty}] \leq C 2^{\gamma i}.$$

(d) (Lemma 11.26) $\mathbb{E}[\theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon - \xi'_\varepsilon \odot \sigma(\mathrm{D})\xi'_\varepsilon] \to 0$ in $\mathscr{C}^{-\gamma}_\mathfrak{n}$ for all $\gamma > 0$.

(e) (Lemma 6.15) $\mathbb{E}[\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon - \xi'_\varepsilon \odot \sigma(\mathrm{D})\xi'_\varepsilon] \to 0$ in $\mathscr{C}^{-\gamma}_\mathfrak{n}$ for all $\gamma > 0$.

(f) (Lemma 11.27) $\mathbb{E}[|\langle \theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon - \xi'_\varepsilon \odot \sigma(\mathrm{D})\xi'_\varepsilon, \mathfrak{n}_z\rangle|^2] \to 0$ for all $z \in \mathbb{N}_0^2$.

(g) (Lemma 11.29) $\mathbb{E}[|\langle \Xi_\varepsilon - \Xi'_\varepsilon, \mathfrak{n}_z\rangle|^2] \to 0$ for all $z \in \mathbb{N}_0^2$.

Proof of Theorem 8.7. We give the proof for $y = 0$, for general y we refer to Remark 11.11 and Remark 11.28 on how to extend the statements in 8.9 for θ_ε to θ^y_ε.

By Theorem 6.4 it is sufficient to show that in $\mathfrak{X}^\alpha_\mathfrak{n}$

$$\begin{aligned}(\xi_\varepsilon - \xi'_\varepsilon, \xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon - \xi'_\varepsilon \odot \sigma(\mathrm{D})\xi'_\varepsilon) &\xrightarrow{\mathbb{P}} 0, \\ (\theta_\varepsilon - \xi'_\varepsilon, \theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon - \xi'_\varepsilon \odot \sigma(\mathrm{D})\xi'_\varepsilon) &\xrightarrow{\mathbb{P}} 0.\end{aligned}$$

This follows by applying Lemma 6.10(b) for which the ingredients are given in 8.9. Let us mention that for $\zeta_\varepsilon = \theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon - \xi'_\varepsilon \odot \sigma(\mathrm{D})\xi'_\varepsilon$ the bound (52) follows by 8.9(c) and (d) as $\zeta_\varepsilon = \Theta_\varepsilon + \mathbb{E}[\theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon - \xi'_\varepsilon \odot \sigma(\mathrm{D})\xi'_\varepsilon] - \Xi'_\varepsilon$. □

9 Large deviation principle of the enhancement of white noise

In this section we assume $L > 0$ and write $\boldsymbol{\xi} = (\xi, \Xi)$ for the limit $\boldsymbol{\xi}_L$ as in Theorem 6.4. We prove the following theorem.

Theorem 9.1. $(\sqrt{\varepsilon}\xi, \varepsilon\Xi)$ *satisfies the large deviation principle with rate* ε *and rate function* $\mathfrak{X}^\alpha_{\mathfrak{n}} \to [0,\infty]$ *given by* $(\psi_1, \psi_2) \mapsto \frac{1}{2}\|\psi_1\|^2_{L^2}$.

Remark 9.2. Analogously, by some lines of the proof in a straightforward way, the statement in Theorem 9.1 holds with underlying space the torus and (ξ, Ξ) being the analogue limit as in Theorem 6.4 as is considered in [1].

As a direct consequence of this large deviation principle and the continuity of the eigenvalues in the (enhanced) noise (see (42)), we obtain the following by an application of the contraction principle (see [7, Theorem 4.2.1]).

Corollary 9.3. $\boldsymbol{\lambda}_n(Q_L, \varepsilon) = \lambda_n(Q_L, (\varepsilon\xi_L, \varepsilon^2\Xi_L))$ *satisfies the large deviation principle with rate* ε^2 *and rate function* $I_{L,n} : \mathbb{R} \to [0,\infty]$ *given by*

$$I_{L,n}(x) = \inf_{\substack{V \in L^2(Q_L) \\ \lambda_n(Q_L, V) = x}} \tfrac{1}{2}\|V\|^2_{L^2}. \tag{66}$$

Theorem 9.1 is an extension of the following theorem. A proof can be given by using [8, Theorem 3.4.5], but as our proof is rather simple and – to our knowledge – different from proofs in literature, we include it.

Theorem 9.4. $\sqrt{\varepsilon}\xi$ *satisfies the large deviation principle with rate function* $\mathscr{C}^\alpha_{\mathfrak{n}}([0,L]^d) \to [0,\infty]$ *given by* $\psi \mapsto \frac{1}{2}\|\psi\|^2_{L^2}$.

Proof. We use the Dawson-Gärtner projective limit theorem [7, Theorem 4.6.1] and the inverse contraction principle [7, Theorem 4.2.4]. Let $J = \mathbb{N}$ with its natural ordering. Let $\mathscr{Y}_i = \mathbb{R}^i$ for all $i \in J$. Let p_{ij} be the projection $\mathscr{Y}_j \to \mathscr{Y}_i$ on the first i-coordinates. Let $\mathscr{Y}$ be the projective limit $\lim_{\leftarrow} \mathscr{Y}_j$ (see [7, above Theorem 4.6.1], it is a subset of $\prod_{j\in J} \mathscr{Y}_j$). Let $p_j : \mathscr{Y} \to \mathscr{Y}_i$ be the canonical projection.

Let $\mathfrak{s} : \mathbb{N} \to \mathbb{N}_0^d$ be a bijection. Write $\mathfrak{d}'_n = \mathfrak{d}_{\mathfrak{s}(n)}$. Let $\Phi : \mathscr{C}^\alpha_{\mathfrak{n}}(\mathbb{T}^d_{2L}) \to \mathscr{Y}$ be given by $\Phi(u) = (\langle u, \mathfrak{d}'_1\rangle, \dots, \langle u, \mathfrak{d}'_n\rangle)_{n\in\mathbb{N}}$. This Φ is continuous and injective. We first prove that $\Phi \circ \xi$ satisfies the large deviation principle.

For every $n \in \mathbb{N}$ the vector $(\langle \xi, \mathfrak{d}'_1\rangle, \dots, \langle \xi, \mathfrak{d}'_n\rangle)$ is an n-dimensional standard normal variable, whence $\sqrt{\varepsilon}(\langle \xi, \mathfrak{d}'_1\rangle, \dots, \langle \xi, \mathfrak{d}'_n\rangle) = (\langle \sqrt{\varepsilon}\xi, \mathfrak{d}'_1\rangle, \dots, \langle \sqrt{\varepsilon}\xi, \mathfrak{d}'_n\rangle)$ satisfies a large deviation principle on $\mathbb{R}^n$ with rate function given by $I_n(y) := \frac{1}{2}|y|^2 = \frac{1}{2}\sum_{i=1}^n y_i^2$. By the Dawson-Gärtner projective limit theorem the sequence $\sqrt{\varepsilon}(\langle \xi, \mathfrak{d}'_1\rangle, \dots, \langle \xi, \mathfrak{d}'_n\rangle)_{n\in\mathbb{N}}$ satisfies the large deviation principle on $\mathscr{Y}$ with rate function

$$I((y_1, \dots, y_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} I_n(y_1, \dots, y_n) = \sup_{n\in\mathbb{N}} \frac{1}{2}\sum_{i=1}^n y_i^2.$$

The image of $\mathscr{C}^\alpha_{\mathfrak{n}}$ under Φ is measurable, which follows from the following identity

$$\Phi(\mathscr{C}^\alpha_{\mathfrak{n}}) = \Big\{ (a_1, \dots, a_n)_{n\in\mathbb{N}} : \sup_{i\geq -1} \Big\| \sum_{n\in\mathbb{N}} \rho_i(\mathfrak{s}(n)) a_n \mathfrak{d}'_n \Big\|_\infty < \infty \Big\}.$$

As $\mathbb{P}(\Phi(\sqrt{\varepsilon}\xi) \in \Phi(\mathscr{C}_{\mathfrak{n}}^{\alpha})) = 1$, and the domain on which I is finite is contained in $\Phi(\mathscr{C}_{\mathfrak{n}}^{\alpha})$, i.e., $\{y \in \mathscr{Y} : I(y) < \infty\} \subset \Phi(\mathscr{C}_{\mathfrak{n}}^{\alpha})$, by [7, Theorem 4.1.5] $\Phi(\sqrt{\varepsilon}\xi)$ satisfies the large deviation principle on $\Phi(\mathscr{C}_{\mathfrak{n}}^{\alpha})$ with rate function I (restricted to $\Phi(\mathscr{C}_{\mathfrak{n}}^{\alpha})$).

Now we apply the inverse contraction principle. $\Phi : \mathscr{C}_{\mathfrak{n}}^{\alpha} \to \Phi(\mathscr{C}_{\mathfrak{n}}^{\alpha})$ is a continuous bijection. Also $I \circ \Phi(\psi) = \frac{1}{2}\|\psi\|_{L^2}^2$ (by Parseval's identity). Hence the proof is finished by showing that $\sqrt{\varepsilon}\xi$ is exponentially tight in $\mathscr{C}_{\mathfrak{n}}^{\alpha}$. Let $m > 0$ and $K_m := \{\psi \in \mathscr{C}_{\mathfrak{n}}^{\alpha} : I \circ \Phi(\psi) \le m\}$. As L^2 is compactly embedded in $H_{\mathfrak{n}}^{\alpha+1}$ by Theorem 4.17, which is continuously embedded in $\mathscr{C}_{\mathfrak{n}}^{\alpha}$ (by [2, Theorem 2.71], K_m is relative compact in $\mathscr{C}_{\mathfrak{n}}^{\alpha}$. By the large deviation principle on $\Phi(\mathscr{C}_{\mathfrak{n}}^{\alpha})$ it follows that

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{P}(\sqrt{\varepsilon}\xi \in \overline{K_m}^c) \le \limsup_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{P}(\sqrt{\varepsilon}\xi \in K_m^c) \le -m.$$

This proves the exponential tightness of $\sqrt{\varepsilon}\xi$ in $\mathscr{C}_{\mathfrak{n}}^{\alpha}$, which finishes the proof. □

To prove Theorem 9.1 we use Theorem 9.4 and the extension of the contraction principle, [7, Theorem 4.2.23]:

Theorem 9.5. *[7, Theorem 4.2.23] Let $\mathscr{X}$ be a Hausdorf space and $(\mathscr{Y}, d)$ be a metric space. Suppose that $(\eta_\varepsilon)_{\varepsilon>0}$ are random variables with values in $\mathscr{X}$ that satisfy the large deviation principle with (rate ε and) rate function $I : \mathscr{X} \to [0,\infty]$. Suppose furthermore that $F_\delta : \mathscr{X} \to \mathscr{Y}$ is a continuous map for all $\delta > 0$, $F : \mathscr{X} \to \mathscr{Y}$ is measurable and that for all $q \in [0,\infty)$*

$$\lim_{\delta \downarrow 0} \sup_{x \in \mathscr{X} : I(x) \le q} d(F_\delta(x), F(x)) = 0, \tag{67}$$

and that $F_\delta(\eta_\varepsilon)$ are exponential good approximations for $F(\eta_\varepsilon)$, i.e., if for all $\delta > 0$

$$\lim_{\delta \downarrow 0} \limsup_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{P}(d(F_\delta(\eta_\varepsilon), F(\eta_\varepsilon)) > \delta) = -\infty. \tag{68}$$

Then $F(\eta_\varepsilon)$ satisfies the large deviation principle with rate function $\mathscr{Y} \to [0,\infty]$ given by

$$y \mapsto \inf_{x \in \mathscr{X} : F(x) = y} I(x).$$

Lemma 9.6. *Let $\alpha \in (-\frac{4}{3}, -1)$. Let τ be as in 6.3 and write $h_\delta = \tau(\delta \mathrm{D})h$. There exists a $C > 0$ such that for all $\delta > 0$ and $h \in L^2$*

$$\|h_\delta \odot \sigma(\mathrm{D})h_\delta - h \odot \sigma(\mathrm{D})h\|_{\mathscr{C}_{\mathfrak{n}}^{2\alpha+2}} \le C\delta^{-\alpha-1}\|h\|_{L^2}^2. \tag{69}$$

Proof. By Theorem 4.25 (note $2\alpha + 4 > 0$) and Theorem 4.21 (also using $\|h_\delta\|_{H_{\mathfrak{n}}^{\alpha+1}} \lesssim \|h\|_{H_{\mathfrak{n}}^{\alpha+1}}$)

$$\begin{aligned} &\|h_\delta \odot \sigma(\mathrm{D})h_\delta - h \odot \sigma(\mathrm{D})h\|_{\mathscr{C}_{\mathfrak{n}}^{2\alpha+2}} \\ &\qquad \le \|(h - h_\delta) \odot \sigma(\mathrm{D})h_\delta\|_{H_{\mathfrak{n}}^{2\alpha+4}} + \|h \odot \sigma(\mathrm{D})(h_\delta - h)\|_{H_{\mathfrak{n}}^{2\alpha+4}} \\ &\qquad \lesssim \|h - h_\delta\|_{H_{\mathfrak{n}}^{\alpha+1}} \|h\|_{H_{\mathfrak{n}}^{\alpha+1}}, \end{aligned}$$

so that (69) follows by Theorem 6.14 as $\|h\|_{H_{\mathfrak{n}}^{\alpha+1}} \lesssim \|h\|_{L^2}$ (see also 4.10). □

Proof of Theorem 9.1. For $\delta > 0$ we write h_δ as in Lemma 9.6 and define $F_\delta : \mathscr{C}_{\mathfrak{n}}^{\alpha}(Q_L) \to \mathfrak{X}_{\mathfrak{n}}^{\alpha}(Q_L)$ by

$$F_\delta(h) = (h, h_\delta \odot \sigma(\mathrm{D})h_\delta).$$

We define $F : \mathscr{C}^{\alpha}_{\mathfrak{n}}(Q_L) \to \mathfrak{X}^{\alpha}_{\mathfrak{n}}(Q_L)$ as follows. If for $h \in \mathscr{C}^{\alpha}_{\mathfrak{n}}(Q_L)$ the function $h_\delta \odot \sigma(\mathrm{D})h_\delta$ converges in $\mathscr{C}^{2\alpha+2}_{\mathfrak{n}}$, then $F(h) = (h, h_\delta \odot \sigma(\mathrm{D})h_\delta)$; if $h_\delta \odot \sigma(\mathrm{D})h_\delta$ does not converge, but $h_\delta \odot \sigma(\mathrm{D})h_\delta - c_\delta$ does (where $c_\delta = \frac{2}{\pi}\log(\frac{1}{\delta})$), then define $F(h) = \lim_{\delta\downarrow 0}(h, h_\delta \odot \sigma(\mathrm{D})h_\delta - c_\delta)$; whereas if $h_\delta \odot \sigma(\mathrm{D})h_\delta - c_\delta$ does not converge for all c_δ, then $F(h) = 0$.

With $\mathscr{X} = \mathscr{C}^{\alpha}_{\mathfrak{n}}(Q_L)$ and $\mathscr{Y} = \mathfrak{X}^{\alpha}_{\mathfrak{n}}(Q_L)$ and $\eta_\varepsilon = \sqrt{\varepsilon}\xi$, by Theorem 9.4 and Theorem 9.5 it is sufficient to prove that (67) and (68) hold because when $F(\phi) = (\psi_1, \psi_2) \neq 0$ then $\phi = \psi_1$.

- First we check (67). By Lemma 9.6 we have ($F(h) = (h, h \odot \sigma(\mathrm{D})h)$ and)

$$\sup_{h \in \mathscr{C}^{\alpha}_{\mathfrak{n}}(Q_L) : \|h\|_{L^2} \le q} \|F_\delta(h) - F(h)\|_{\mathfrak{X}^{\alpha}_{\mathfrak{n}}} \lesssim \delta^{-\alpha-1} q^2,$$

for all $q \ge 0$, i.e., (67) holds.

- Now we check (68). Let $\delta > 0$. We have that $\Xi := \lim_{\delta\downarrow 0} \xi_\delta \odot \sigma(\mathrm{D})\xi_\delta - c_\delta$ exists almost surely by Theorem 6.4. Hence, for $p > 1$

$$\begin{aligned}\mathbb{P}\left(\|F_\delta(\sqrt{\varepsilon}\xi) - F(\sqrt{\varepsilon}\xi)\|_{\mathfrak{X}^{\alpha}_{\mathfrak{n}}} > \delta\right) &\le \frac{\varepsilon^p}{\delta^p}\mathbb{E}\left[\|\xi_\delta \odot \sigma(\mathrm{D})\xi_\delta - \Xi\|^p_{\mathscr{C}^{2\alpha+2}_{\mathfrak{n}}}\right] \\ &\le \frac{\varepsilon^p 2^p}{\delta^p}(c_\delta^p + \mathbb{E}\left[\|\xi_\delta \odot \sigma(\mathrm{D})\xi_\delta - c_\delta - \Xi\|^p_{\mathscr{C}^{2\alpha+2}_{\mathfrak{n}}}\right])\end{aligned}$$

Let $\eta = -(2\alpha + 2)$. By Lemmas 6.10, 6.15, 6.16 and 11.17 there exists a $C > 0$ such that for all $p > 1$

$$\mathbb{E}\left[\|\xi_\delta \odot \sigma(\mathrm{D})\xi_\delta - c_\delta - \Xi\|^p_{\mathscr{C}^{2\alpha+2}_{\mathfrak{n}}}\right] \le C^p p^p \delta^{\eta p}.$$

Therefore (using that $a^p + b^p \le (a+b)^p$)

$$\mathbb{P}\left(\|F_\delta(\sqrt{\varepsilon}\xi) - F(\sqrt{\varepsilon}\xi)\|_{\mathfrak{X}^{\alpha}_{\mathfrak{n}}} > \delta\right) \le \left[\frac{2\varepsilon}{\delta}(c_\delta + Cp\delta^\eta)\right]^p$$

Hence with $p = \frac{1}{\varepsilon}$ we obtain

$$\begin{aligned}\limsup_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{P}\left(\|F_\delta(\sqrt{\varepsilon}\xi) - F(\sqrt{\varepsilon}\xi)\|_{\mathfrak{X}^{\alpha}_{\mathfrak{n}}} > \delta\right) &\le \limsup_{\varepsilon\downarrow 0} \log\left[\tfrac{2}{\delta}(\varepsilon c_\delta + C\delta^\eta)\right] \\ &\le \log(\tfrac{2C}{\delta}\delta^\eta).\end{aligned}$$

So that

$$\lim_{\delta\downarrow 0}\limsup_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{P}\left(\|F(\sqrt{\varepsilon}\xi_\delta) - F(\sqrt{\varepsilon}\xi)\|_{\mathfrak{X}^{\alpha}_{\mathfrak{n}}} > \delta\right) = -\infty,$$

i.e., (68) holds. □

10 Infima over the large deviation rate function

In this section we consider infima over sets of the rate function $I_{L,n}$ as in (66). We prove the results summarized in Theorem 2.6.

Lemma 10.1. *For $a, b \in \mathbb{R}$ and all $\tau > 0$*

$$(1-\tau)\inf I_{L,n}[b,\infty) + \frac{1}{2}(1-\tfrac{1}{\tau})L^2a^2 \le \inf I_{L,n}[b+a,\infty)$$
$$\le (1+\tau)\inf I_{L,n}[b,\infty) + \frac{1}{2}(1+\tfrac{1}{\tau})L^2a^2.$$

Consequently, if $(a_L)_{L>0}$ in $\mathbb{R}$ with $\lim_{L\to\infty} La_L = 0$,

$$\lim_{L\to\infty} \inf I_{L,n}[b,\infty) = \lim_{L\to\infty} \inf I_{L,n}[b+a_L,\infty)$$
$$= \lim_{L\to\infty} \inf I_{L,n}(b+a_L,\infty) = \lim_{L\to\infty} \inf I_{L,n}(b,\infty).$$

Proof. As $\lambda_n(Q_L,V) + a = \lambda_n(Q_L, V + a\mathbb{1}_{Q_L})$, and $\|a\mathbb{1}_{Q_L}\|_{L^2} = aL$, and $2\langle V, a\mathbb{1}_{Q_L}\rangle \le \tau\|V\|_{L^2}^2 + \frac{1}{\tau}a^2L^2$ for all $\tau > 0$;

$$\inf I_{L,n}[b+a,\infty) = \inf_{\substack{V\in L^2(Q_L);\\ \lambda_n(Q_L,V)\ge b}} \tfrac{1}{2}\|V + a\mathbb{1}_{Q_L}\|_{L^2(Q_L)}^2$$
$$\le (1+\tau) \inf_{\substack{V\in L^2(Q_L);\\ \lambda_n(Q_L,V)\ge b}} \tfrac{1}{2}\|V\|_{L^2(Q_L)}^2 + \tfrac{1}{2}(1+\tfrac{1}{\tau})a^2L^2.$$

The lower bound can be proven similarly. □

We define

$$\mu_{L,n} := \inf I_{L,n}[1,\infty), \qquad \rho_n := \inf_{L>0} \mu_{L,n}. \tag{70}$$

We prove that ρ_n is bounded away from 0 uniformly in n (Lemma 10.4) and give an alternative variational formula for ρ_n (Lemma 10.5 and Theorem 10.7).

Lemma 10.2. $\mu_{L,n} = \inf I_{L,n}(1,\infty) = \inf_{\substack{V\in C_c^\infty(Q_L)\\ \lambda_n(Q_L,V)\ge 1}} \frac{1}{2}\|V\|_{L^2}^2.$

Proof. The first equality follows by Lemma 10.1. The second follows by Lemma 5.8. □

We will use Ladyzhenskaya's inequality [19], which is a special case of the Gagliardo– Nirenberg interpolation inequality [21].

Lemma 10.3 (Ladyzhenskaya's inequality)**.** *There exists a $C > 0$ such that for all $L > 0$ and $f \in H^1(Q_L)$,*

$$\|f\|_{L^4}^2 \le C\|\nabla f\|_{L^2}\|f\|_{L^2} \tag{71}$$

Lemma 10.4. *There exists a $C > 0$ such that $\rho_n \ge C$ for all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. Let $L > 0$ and $\varepsilon > 0$. Let $V \in C_c^\infty(Q_L)$ be such that $\lambda_n(Q_L,V) \ge 1$ and $\frac{1}{2}\|V\|_{L^2}^2 \le \mu_{L,n} + \varepsilon$. By (45) there is a $\psi \in C_c^\infty(Q_L)$ with $\|\psi\|_{L^2} = 1$ such that (by partial integration)

$$1 - \varepsilon \le -\|\nabla\psi\|_{L^2}^2 + \int V\psi^2 \le -\|\nabla\psi\|_{L^2}^2 + \|V\|_{L^2}^2\|\psi\|_{L^4}^2,$$

and so using Ladyzhenskaya's inequality (71), which implies $\|\nabla\psi\|_{L^2}^2 \ge \frac{1}{C}\|\psi\|_{L^4}^4$,

$$\|V\|_{L^2}^2 \ge \frac{1-\varepsilon+\|\nabla\psi\|_{L^2}^2}{\|\psi\|_{L^4}^2} \ge \frac{1-\varepsilon}{\|\psi\|_{L^4}^2} + \frac{1}{C}\|\psi\|_{L^4}^2$$

As $a^2+b^2 \ge 2ab$ we have $\mu_{L,n}+\varepsilon \ge \frac{1}{2}\|V\|_{L^2}^2 \ge \sqrt{\frac{1-\varepsilon}{C}}$. As this holds for all $\varepsilon > 0$ we conclude that $\mu_{L,n} \ge \frac{1}{\sqrt{C}}$ for all $L > 0$. Hence $\rho_n > \frac{1}{\sqrt{C}}$. □

Lemma 10.5. *For all $n \in \mathbb{N}$, $a > 0$,*

$$\inf_{L>0} \inf_{\substack{V\in C_c^\infty(Q_L);\\ \lambda_n(Q_L,V)\ge a}} \tfrac{1}{2}\|V\|_{L^2(Q_L)}^2 = \inf_{L>0} \inf_{\substack{V\in C_c^\infty(Q_L)\\ \|V\|_{L^2}^2 \le \frac{1}{a}}} \tfrac{1}{2\lambda_n(Q_L,V)}. \tag{72}$$

Moreover, $\mu_{L,n}$ is decreasing in L, and one could replace "$\inf_{L>0}$" in (72) by "$\lim_{L\to\infty}$". In particular, $\rho_n = \lim_{L\to\infty}\mu_{L,n}$.

Proof. With $W = L^2V(L\cdot)$ we have $W \in C_c^\infty(Q_1)$, $\|W\|_{L^2(Q_1)}^2 = L^2\|V\|_{L^2(Q_L)}^2$ and by Theorem 7.1 $\lambda_n(Q_L,V) = \lambda_n(Q_L, \frac{1}{L^2}W(\frac{1}{L}\cdot)) = \frac{1}{L^2}\lambda_n(Q_1,W)$. Therefore

$$\inf_{\substack{V\in C_c^\infty(Q_L);\\ \lambda_n(Q_L,V)\ge a}} \tfrac{1}{2}\|V\|_{L^2(Q_L)}^2 = \inf_{\substack{W\in C_c^\infty(Q_1);\\ \lambda_n(Q_1,W)\ge aL^2}} \tfrac{1}{2}\tfrac{1}{L^2}\|W\|_{L^2(Q_1)}^2, \tag{73}$$

$$\inf_{\substack{V\in C_c^\infty(Q_L)\\ \|V\|_{L^2}^2\le\frac{1}{a}}} \tfrac{1}{2\lambda_n(Q_L,V)} = \inf_{\substack{W\in C_c^\infty(Q_1)\\ \|W\|_{L^2}^2\le\frac{L^2}{a}}} \tfrac{L^2}{2\lambda_n(Q_1,W)}. \tag{74}$$

With this, (72) follows directly from Lemma 10.6. That $\mu_{L,n}$ and

$$\inf_{\substack{V\in L^2(Q_L)\\ \|V\|_{L^2}^2\le a}} \tfrac{1}{\lambda_n(Q_L,V)}$$

are decreasing in L follows from Lemma 8.1. □

Lemma 10.6. *Let $\mathscr{Y}$ be a topological space and $f, g : \mathscr{Y} \to \mathbb{R}$ be continuous functions. Let $a > 0$ and suppose that $\rho := \inf_{L>0}\inf_{w\in\mathscr{Y}: f(w)\ge aL} \frac{g(w)}{L} > 0$. Then*

$$\inf_{L>0} \inf_{\substack{w\in\mathscr{Y}\\ f(w)\ge aL}} \frac{g(w)}{L} = \inf_{L>0} \inf_{\substack{w\in\mathscr{Y}\\ g(w)\le\frac{L}{a}}} \frac{L}{f(w)}.$$

Proof. By definition we have $\forall L > 0\, \forall w \in \mathscr{Y} : \frac{1}{L}g(w) < \rho \implies f(w) < aL$, by continuity of f and g we obtain (by taking $K = L\rho a$)

$$\forall K > 0\, \forall w \in \mathscr{Y} : g(w) \le \frac{K}{a} \implies \frac{f(w)}{K} \le \frac{1}{\rho}.$$

Let $\varepsilon > 0$. Then there exists an L_0 such that $\forall L \ge L_0\, \exists w_L \in \mathscr{Y}$ such that $f(w_L) \ge aL$ and $\frac{1}{L}g(w_L) \le \rho+\varepsilon$. Then with $K_0 = L_0a(\rho+\varepsilon)$ for all $K \ge K_0$ there exists a $w \in \mathscr{Y}$ (namely w_L for $L = \frac{K}{\rho+\varepsilon}$) such that $\frac{g(w)}{K} \le \frac{1}{a}$ and $\frac{f(w)}{K} \ge \frac{1}{\rho+\varepsilon}$. So that

$$\sup_{K>0} \sup_{\substack{w\in\mathscr{Y}\\ g(w)\le\frac{K}{a}}} \tfrac{f(w)}{K} = \frac{1}{\rho}.$$

□

By (45) and Lemma 10.5 (for $a = 1$) we obtain the following theorem, which expresses ρ_n in another variational form.

Theorem 10.7.

$$\frac{2}{\rho_n} = 4 \sup_{\substack{V \in C_c^\infty(\mathbb{R}^2) \\ \|V\|_{L^2}^2 \le 1}} \sup_{\substack{F \sqsubset C_c^\infty(\mathbb{R}^2) \\ \dim F = n}} \inf_{\substack{\psi \in F \\ \|\psi\|_{L^2}^2 = 1}} \int_{\mathbb{R}^2} -|\nabla \psi|^2 + V\psi^2.$$

11 Bounds on second moments and convergence of Gaussians

In this section $L > r > 0$. We will consider $\tau : \mathbb{R}^d \to [0,1]$ to be either $\mathbb{1}_{(-1,1)^2}$ or an even C_c^∞ function that is equal to 1 on a neighbourhood of 0. We prove the bounds and convergences mentioned in 6.11 and in 8.9. We will use the following notation:

$$Z_{\mathfrak{k}} = \langle \mathscr{W}, \mathfrak{n}_{\mathfrak{k},L} \rangle, \qquad \mathscr{Z}_k = \langle \mathscr{W}, \mathfrak{n}_{k,r} \rangle = \sum_{\mathfrak{k} \in \mathbb{N}_0^2} Z_{\mathfrak{k}} \langle \mathfrak{n}_{\mathfrak{k},L}, \mathfrak{n}_{k,r} \rangle_{L^2(Q_r)},$$

$$\xi_\varepsilon = \xi_{r,\varepsilon}, \qquad X_k^\varepsilon = \sum_{\mathfrak{k} \in \mathbb{N}_0^2} \tau(\tfrac{\varepsilon}{L}\mathfrak{k}) Z_{\mathfrak{k}} \langle \mathfrak{n}_{\mathfrak{k},L}, \mathfrak{n}_{k,r} \rangle_{L^2(Q_r)},$$

$$\rho^\odot : \mathbb{N}_0^d \times \mathbb{N}_0^d \to \mathbb{R}, \qquad \rho^\odot(k,l) = \sum_{\substack{i,j \ge -1 \\ |i-j| \le 1}} \rho_i(k)\rho_j(l).$$

so that (see also (65))

$$\theta_\varepsilon = \sum_{k \in \mathbb{N}_0^2} X_k^\varepsilon \mathfrak{n}_{k,r}, \qquad \xi_\varepsilon = \sum_{k \in \mathbb{N}_0^2} \tau(\tfrac{\varepsilon}{r}k) \mathscr{Z}_k \mathfrak{n}_{k,r},$$

$$\xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon = \sum_{k,l \in \mathbb{N}_0^2} \frac{\rho^\odot(k,l)}{1 + \frac{\pi^2}{r^2}|l|^2} \tau(\tfrac{\varepsilon}{r}k) \mathscr{Z}_k \tau(\tfrac{\varepsilon}{r}l) \mathscr{Z}_l \mathfrak{n}_{k,r}\mathfrak{n}_{l,r},$$

$$\theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon = \sum_{k,l \in \mathbb{N}_0^2} \frac{\rho^\odot(k,l)}{1 + \frac{\pi^2}{r^2}|l|^2} X_k^\varepsilon X_l^\varepsilon \mathfrak{n}_{k,r}\mathfrak{n}_{l,r}.$$

11.1 Terms in the first Wiener chaos

In this section we consider only the terms in the first Wiener chaos, i.e., ξ_ε and θ_ε.

Lemma 11.1. *For all $\gamma \in (0,1)$ there exists a $C > 0$ such that for all $i \ge -1$, $\varepsilon, \delta > 0$,*

$$\mathbb{E}[\|\Delta_i \xi_\varepsilon\|_\infty^2] \le C 2^{(d+\gamma)i}, \tag{75}$$

and if $\tau \in C_c^\infty(\mathbb{R}^2, [0,1])$ there exists a $C > 0$ such that for all $i \ge -1$, $\varepsilon, \delta > 0$,

$$\mathbb{E}[\|\Delta_i(\xi_\varepsilon - \xi_\delta)\|_\infty^2] \le C 2^{(d+2\gamma)i} |\varepsilon - \delta|^\gamma. \tag{76}$$

Proof. As $\Delta_i(\xi_{r\varepsilon} - \xi_{r\delta})(x) = \sum_{k \in \mathbb{N}_0^2} \rho_i(k)(\tau(\varepsilon k) - \tau(\delta k)) Z_k \mathfrak{n}_{k,r}(x)$, and $\|\mathfrak{n}_{k,r}\|_\infty^2 \le (\frac{2}{r})^d$ by (57) we have for all $\eta > 0$

$$\mathbb{E}[\|\Delta_i(\xi_{r\varepsilon} - \xi_{r\delta})\|_{L^\infty}^2] \lesssim 2^{(d+\eta)i} \sum_{k \in \mathbb{N}_0^d} \frac{(\tau(\varepsilon k) - \tau(\delta k))^2}{(1+|k|)^{d+\eta}}.$$

On the other hand, as $|\tau| \le 1$, using (57)

$$\mathbb{E}[\|\Delta_i \xi_{r\varepsilon}\|_{L^\infty}^2] \lesssim 2^{(d+\eta)i} \sum_{k\in\mathbb{N}_0^d} \frac{1}{(1+|k|)^{d+\eta}} \lesssim 2^{(d+\eta)i},$$

which proves (75). Suppose $\tau \in C_c^\infty$. As $|\tau(\varepsilon k) - \tau(\delta k)| \le \|\tau'\|_\infty |\varepsilon - \delta||k|$ and $\|\tau\|_\infty = 1$,

$$(\tau(\varepsilon k) - \tau(\delta k))^2 \lesssim \|\tau'\|_\infty^\gamma |\varepsilon - \delta|^\gamma |k|^\gamma. \tag{77}$$

Therefore, as $\sum_{k\in\mathbb{Z}^d} \frac{|k|^\gamma}{(1+|k|)^{d+2\gamma}} < \infty$, we obtain (76) by taking $\eta = 2\gamma$. □

11.2. Let us introduce a bit more notation. We write and introduce for $\mathfrak{k}, k, l \in \mathbb{N}_0^d$, $\varepsilon > 0$

$$B_{\mathfrak{k},k} = \langle \mathfrak{n}_{\mathfrak{k},L}, \mathfrak{n}_{k,r}\rangle_{L^2([0,r]^d)}, \qquad F_\varepsilon(k,l) = \sum_{\mathfrak{k}\in\mathbb{N}_0^2} \tau(\tfrac{\varepsilon}{L}\mathfrak{k})^2 B_{\mathfrak{k},k} B_{\mathfrak{k},l} = \mathbb{E}[X_k^\varepsilon X_l^\varepsilon].$$

In order to bound $\mathbb{E}[\|\Delta_i\theta_\varepsilon\|_{L^\infty}^2] = \sum_{k,l\in\mathbb{N}_0^d} \|\Delta_i \mathfrak{n}_{k,r}\|_{L^\infty} \|\Delta_i \mathfrak{n}_{l,r}\|_{L^\infty} \mathbb{E}[X_k^\varepsilon X_l^\varepsilon]$, we first give a bound for $F_\varepsilon(k,l)$. In order to do that we first find an expression for $b_{m,l} = \langle \mathfrak{n}_{m,L}, \mathfrak{n}_{l,r}\rangle_{L^2([0,r])}$ where $m, l \in \mathbb{N}_0$ as $B_{\mathfrak{k},k} = \prod_{i=1}^d b_{\mathfrak{k}_i,k_i}$.

11.3. In this paper we make the following convention: $\frac{\sin(\pi k)}{0} = 1$ for all $k \in \mathbb{Z}$.

11.4. Due to the identities $2\cos(a)\cos(b) = \cos(a-b) + \cos(a+b)$ for $a, b \in \mathbb{R}$ and

$$\sin(\pi(a \pm l)) = (-1)^l \sin(\pi a) \qquad \text{for } l \in \mathbb{Z} \text{ and } a \in \mathbb{R}, \tag{78}$$

we obtain

$$\begin{aligned} b_{m,l} := \langle \mathfrak{n}_{m,L}, \mathfrak{n}_{l,r}\rangle_{L^2([0,r])} &= \frac{2}{\sqrt{Lr}} \nu_m \nu_l \int_0^r \cos(\tfrac{\pi}{L} m x) \cos(\tfrac{\pi}{r} l x)\, \mathrm{d}x \\ &= \sqrt{\frac{r}{L}} \frac{1}{\pi} \nu_m \nu_l \left[\frac{\sin(\pi(\frac{mr}{L} - l))}{\frac{mr}{L} - l} + \frac{\sin(\pi(\frac{mr}{L} + l))}{\frac{mr}{L} + l} \right] \\ &= \sqrt{\frac{r}{L}} \frac{1}{\pi} (-1)^l \nu_m \nu_l \sum_{\mathfrak{p}\in\{-1,1\}} \frac{\sin(\pi(\frac{mr}{L}))}{\frac{mr}{L} + \mathfrak{p} l}. \end{aligned} \tag{79}$$

Similar to (79), for $y \in \mathbb{R}$

$$\begin{aligned} b_{m,l}^y := \langle \mathfrak{n}_{m,L}, \mathscr{T}_y \mathfrak{n}_{l,r}\rangle_{L^2(y+[0,r])} &= \frac{2}{\sqrt{Lr}} \nu_m \nu_l \int_0^r \cos(\tfrac{\pi}{L} m x + \tfrac{\pi}{L} m y) \cos(\tfrac{\pi}{r} l x)\, \mathrm{d}x \\ = \sqrt{\frac{r}{L}} \frac{1}{\pi} (-1)^l \nu_m \nu_l \sum_{\mathfrak{p}\in\{-1,1\}} &\left[\frac{\sin(\pi(\frac{mr}{L}))}{\frac{mr}{L} + \mathfrak{p} l} \sin(\tfrac{\pi m}{L} y) + \mathbb{1}_{\frac{mr}{L} + \mathfrak{p} l \neq 0} \frac{\cos(\pi(\frac{mr}{L}))}{\frac{mr}{L} + \mathfrak{p} l} \cos(\tfrac{\pi m}{L} y) \right]. \end{aligned}$$

See Remark 11.11 and Remark 11.28 on how this expression is used to prove the statements in this section for general y and not just $y = 0$.

11.5. We will use the following bound in the following without further mentioning

$$\tfrac{\sin(x)}{x} \le \tfrac{3}{1+x} \qquad \text{for } x \ge 0. \tag{80}$$

Moreover, uniformly over $a, b \ge 0$

$$(1+a)(1+b) = (1 + a + b + ab) \ge (1 + a + b) \gtrsim (1 + \sqrt{a^2 + b^2}).$$

Let us also state bounds of sums by integrals, that we will use numerous times in this section.

Lemma 11.6. *Let $M \in \mathbb{N}$ and $f : [0,M] \to \mathbb{R}$ be a decreasing measurable function. Then $\sum_{m=1}^{M} f(m) \le \int_0^M f(x)\,\mathrm{d}x \le \sum_{m=0}^{M-1} f(m)$. If f instead is increasing, then $\sum_{m=0}^{M-1} f(m) \le \int_0^M f(x)\,\mathrm{d}x \le \sum_{m=1}^{M} f(m)$.*

11.7. In some other cases we bound a sum by an integral using that the following holds. For $k \in \mathbb{Z}^d$ and $x \in B_\infty(k, \frac{1}{2})$, i.e., $|x-k|_\infty < \frac{1}{2}$ and thus $|x-k| \le \frac{\sqrt{d}}{2}$,

$$\big||x|-|l|\big| \le \big||k|-|l|\big| + \big||x|-|k|\big| \le \big||k|-|l|\big| + \tfrac{\sqrt{d}}{2}. \tag{81}$$

The following bound will be used multiple times and is due to the above argument and the bound on the integral as in Lemma B.1: For $\gamma, \delta > 0$ such that $\delta < \gamma$, for all $u, v \in \mathbb{R}$,

$$\sum_{l \in \mathbb{N}_0} \frac{1}{(1+|l-u|)^\gamma} \frac{1}{(1+|l-v|)^{1-\delta}} \lesssim (1+|u-v|)^{\delta-\gamma}. \tag{82}$$

Theorem 11.8. *For all $\delta > 0$ there exists a $C > 0$ such that for all $k, l \in \mathbb{N}_0^d$*

$$|F_\varepsilon(k,l)| \le C \prod_{i=1}^{d} (1+|k_i-l_i|)^{\delta-1}. \tag{83}$$

Proof. One has $|F_\varepsilon(k,l)| \lesssim \prod_{i=1}^d \sum_{m\in\mathbb{N}_0} |b_{m,k_i} b_{m,l_i}|$. Because $|b_{m,n}| \lesssim \frac{1}{1+|\frac{m}{N}-n|} \lesssim \frac{N}{1+|m-Nn|}$ for $n \in \mathbb{N}_0$ where $N = \frac{L}{r}$ (see (80)),

$$|F_\varepsilon(k,l)| \lesssim \prod_{i=1}^{d} \sum_{m \in \mathbb{N}_0} \frac{1}{1+|m-Nk_i|} \frac{1}{1+|m-Nl_i|}.$$

For all $\delta > 0$ one has $1+|m-u| \ge (1+|m-u|)^{1-\frac{\delta}{2}}$, so that (83) follows by (82). □

Lemma 11.9. *For all $\gamma \in (0,1)$ there exists a $C > 0$ such that for all $i \ge -1$, $\varepsilon > 0$,*

$$\mathbb{E}[\|\Delta_i \theta_\varepsilon\|_{L^\infty}^2] \le C 2^{(d+\gamma)i}. \tag{84}$$

Proof. By (57) $2^{-\beta i}\|\Delta_i \mathfrak{n}_{k,r}\|_{L^\infty} \lesssim \frac{1}{(1+|k|)^\beta} \le \prod_{i=1}^d \frac{1}{(1+k_i)^{\frac{\beta}{d}}}$ and as for $\delta > 0$, $|\mathbb{E}[X_k^\varepsilon X_l^\varepsilon]| = |F_\varepsilon(k,l)| \lesssim \prod_{i=1}^2 (1+|k_i-l_i|)^{\delta-1}$, we have

$$2^{-(d+d\gamma)i}\mathbb{E}[\|\Delta_i\theta_\varepsilon\|_{L^\infty}^2] \lesssim \Bigg(\sum_{k,l\in\mathbb{N}_0} \frac{1}{(1+k)^{\frac{1+\gamma}{2}}} \frac{1}{(1+l)^{\frac{1+\gamma}{2}}} (1+|k-l|)^{\delta-1} \Bigg)^d.$$

Let $\delta < \gamma$ (in particular $\delta < \frac{1+\gamma}{2}$). By (82) we obtain

$$2^{-(d+d\gamma)i}\mathbb{E}[\|\Delta_i\theta_\varepsilon\|_{L^\infty}^2] \lesssim \sum_{k\in\mathbb{N}_0} \frac{1}{(1+k)^{1+\gamma-\delta}} < \infty.$$

□

Lemma 11.10. $\mathbb{E}[|\langle \theta_\varepsilon - \xi_\varepsilon, \mathfrak{n}_k\rangle|^2] \to 0$ *for all $k \in \mathbb{N}_0^2$.*

Proof. $\mathbb{E}[|\langle \theta_\varepsilon - \xi_\varepsilon, \mathfrak{n}_k\rangle|^2] = \mathbb{E}[|X_k^\varepsilon - \tau(\frac{\varepsilon}{r}k)\mathscr{Z}_k|^2] \lesssim \sum_{\mathfrak{k}\in\mathbb{N}_0^d} B_{\mathfrak{k},k}^2(\tau(\frac{\varepsilon}{L}\mathfrak{k}) - \tau(\frac{\varepsilon}{r}k))^2$. By Lebesgue's dominated convergence theorem this converges to zero. □

Remark 11.11. Let $y \in \mathbb{R}^2$ be such that $y + Q_r \subset Q_L$. Proving Lemma 11.10 for "$\theta_\varepsilon^y, \xi_\varepsilon^y$" instead of "$\theta_\varepsilon, \xi_\varepsilon$" is straightforward. To adapt Lemma 11.9 as mentioned above, it is sufficient to adapt the bound in (83), for which one uses that $|b_{m,l}^y| \lesssim \frac{1}{1+|\frac{m}{N}-l|}$, which holds because $\mathbb{1}_{\frac{m}{N}+\mathfrak{p}l\neq 0}\frac{\cos(\pi(\frac{m}{N}))}{\frac{m}{N}+\mathfrak{p}l} \lesssim \frac{1}{1+|\frac{m}{N}-l|}$ for all $m, l \in \mathbb{N}_0$ and $\mathfrak{p} \in \{-1,1\}$.

11.2 Terms in the second Wiener chaos

In this section we consider only the terms in the second Wiener chaos, i.e., Ξ_ε (Lemma 11.17), Θ_ε (Lemma 11.18) and $\theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon - \xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon$ (Lemma's 11.26 and 11.27). In 11.15 we make an assumption for the rest of the section. We start by presenting auxiliary lemma's and observations.

Lemma 11.12. *There exist $b > 0$ and $c > 1$ such that*

$$\operatorname{supp} \rho^\odot \subset B(0,b)^2 \cup \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \tfrac{1}{c}|x| \le |y| \le c|x|\}$$

Consequently, uniformly in $k, l \in \mathbb{Z}^d$

$$\frac{\rho^\odot(k,l)}{(1+|l|^2)} \eqsim \frac{\rho^\odot(k,l)}{(1+|k|^2)}. \tag{85}$$

Proof. Let $0 < a < b$ be such that $\operatorname{supp} \rho_0 \subset \{x \in \mathbb{R}^d : a \le |x| \le b\}$ and $\operatorname{supp} \rho_{-1} \subset B(0,b)$. If $i, j \in \{-1, 0\}$, then $x, y \in B(0,b)$. Let $i, j \ge 0$ and $|i-j| \le 1$. Suppose that $x, y \in \mathbb{R}^2$ are such that $\rho_i(x)\rho_j(y) \neq 0$. Then $|x| \in [2^i a, 2^i b]$ and $|y| \in [2^j a, 2^j b] \subset [2^{i-1}a, 2^{i+1}b]$. This in turn implies

$$\tfrac{a}{2b}|x| \le \tfrac{a}{2b}2^i b = 2^{i-1}a \le |y| \le 2^{i+1}b \le \tfrac{2b}{a}2^i a \le \tfrac{2b}{a}|x|.$$

□

11.13. Let $k, l, z \in \mathbb{N}_0^d$. We write $\mathfrak{n}_k = \mathfrak{n}_{k,r}$ here. By (37) (and using (26)) and as $\mathfrak{n}_{\mathfrak{q}\circ k} = \mathfrak{n}_k$ for all $\mathfrak{q} \in \{-1,1\}^d$,

$$\begin{aligned}\langle \mathfrak{n}_k\mathfrak{n}_l, \mathfrak{n}_z\rangle_{L^2(Q_r)} &= (2r)^{-\frac{d}{2}} \sum_{\mathfrak{p}\in\{-1,1\}^d} \frac{\nu_k\nu_l}{\nu_{k+\mathfrak{p}\circ l}} \langle \mathfrak{n}_{k+\mathfrak{p}\circ l}, \mathfrak{n}_z\rangle_{L^2(Q_r)} \\ &= (2r)^{-\frac{d}{2}} \sum_{\mathfrak{p},\mathfrak{q}\in\{-1,1\}^d} \frac{\nu_k\nu_l}{\nu_{k+\mathfrak{p}\circ l}} \delta_{\mathfrak{q}\circ k+\mathfrak{p}\circ l, z}.\end{aligned} \tag{86}$$

By combining this with (57) we have for $x \in (0,r)^d$ and $\gamma > 0$

$$|\Delta_i(\mathfrak{n}_k\mathfrak{n}_l)(x)| \lesssim \sum_{\mathfrak{p},\mathfrak{q}\in\{-1,1\}^d} \rho_i(\mathfrak{q}\circ k + \mathfrak{p}\circ l)|\mathfrak{n}_{\mathfrak{q}\circ k+\mathfrak{p}\circ l}(x)| \lesssim \frac{2^{\gamma i}}{(1+|k-l|)^\gamma}. \tag{87}$$

11.14. By Wick's theorem [15, Theorem 1.28] (as $\mathbb{E}[Z_k Z_l] = \delta_{k,l}$)

$$\mathbb{E}[Z_k Z_l Z_m Z_n] = \delta_{k,l}\delta_{m,n} + \delta_{k,m}\delta_{l,n} + \delta_{k,n}\delta_{m,l}, \tag{88}$$

$$\mathbb{E}\left([Z_k Z_l - \delta_{k,l}]\,[Z_m Z_n - \delta_{m,n}]\right) = \delta_{k,m}\delta_{l,n} + \delta_{k,n}\delta_{m,l}. \tag{89}$$

11.15. From here on, in this section we assume $d = 2$.

11.16. As $||k|-|l|| \leq |k-l|$, we have $(1+|k-l|)^{-\gamma} \lesssim (1+||k|-|l||)^{-\gamma}$ and therefore have the following bound by 11.7 and Lemma 11.12 for $\gamma \in (0,1)$ and $l \in \mathbb{N}_0^2$

$$\sum_{k\in\mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)}{(1+|k-l|)^{\gamma}} \lesssim 1 + 2\pi \int_{\frac{1}{c}|l|}^{c|l|} \frac{x}{(1+|x-|l||)^{\gamma}}\,\mathrm{d}x \lesssim (1+|l|)^{2-\gamma}. \tag{90}$$

Lemma 11.17. *Let Ξ_ε be as in Definition 6.6 (for $\xi_\varepsilon = \xi_{\varepsilon,r}$) and $\gamma \in (0,1)$. There exists a $C>0$ such that for all $i \geq -1$, $\varepsilon > 0$*

$$\mathbb{E}[\|\Delta_i \Xi_\varepsilon\|_{L^\infty}^2] \leq C 2^{\gamma q}, \tag{91}$$

and if $\tau \in C_c^\infty(\mathbb{R}^2,[0,1])$ there exists a $C>0$ such that for all $i \geq -1$, $\varepsilon, \delta > 0$

$$\mathbb{E}[\|\Delta_i(\Xi_\varepsilon - \Xi_\delta)\|_{L^\infty}^2] \leq C|\varepsilon-\delta|^{\gamma} 2^{2\gamma q}. \tag{92}$$

Proof. First observe $\Xi_{r\varepsilon} = \sum_{k,l\in\mathbb{N}_0^2} \rho^{\odot}(k,l) \frac{\tau(\varepsilon k)\tau(\varepsilon l)}{1+\frac{\pi^2}{r^2}|l|^2} [\mathscr{Z}_k \mathscr{Z}_l - \delta_{k,l}] \mathfrak{n}_k \mathfrak{n}_l$. By (89) and (87) (as both contributions $\delta_{k,m}\delta_{l,n}$ and $\delta_{k,n}\delta_{m,l}$ can be bounded by the same expression by Lemma 11.12)

$$\begin{aligned} &2^{-2\gamma i}\mathbb{E}[\|\Delta_i(\Xi_{r\varepsilon} - \Xi_{r\delta})\|_{L^\infty}^2] \\ &\lesssim \sum_{k,l\in\mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)^2}{(1+\frac{\pi^2}{r^2}|l|^2)^2} \frac{[\tau(\varepsilon k)\tau(\varepsilon l) - \tau(\delta k)\tau(\delta l)]^2}{(1+|k-l|)^{2\gamma}}. \end{aligned}$$

On the other hand, as $|\tau| \leq 1$, using (90)

$$2^{-2\gamma i}\mathbb{E}[\|\Delta_i \Xi_{r\varepsilon}\|_{L^\infty}^2] \lesssim \sum_{k,l\in\mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)^2}{(1+\frac{\pi^2}{r^2}|l|^2)^2} \frac{1}{(1+|k-l|)^{2\gamma}} \lesssim \sum_{l\in\mathbb{N}_0^2} \frac{1}{(1+|l|)^{6-\gamma}} < \infty.$$

Suppose $\tau \in C_c^\infty$. As

$$\begin{aligned} &\tau(\varepsilon k)\tau(\varepsilon l) - \tau(\delta k)\tau(\delta l) \\ &= \big(\tau(\varepsilon k) - \tau(\delta k)\big)\big(\tau(\varepsilon l) + \tau(\delta l)\big) + \big(\tau(\varepsilon k) + \tau(\delta k)\big)\big(\tau(\varepsilon l) - \tau(\delta l)\big), \end{aligned}$$

we can use the bound (77) as in the proof of Lemma 11.1 to obtain

$$|\tau(\varepsilon k)\tau(\varepsilon l) - \tau(\delta k)\tau(\delta l)|^2 \leq 4\|\tau'\|_\infty^\gamma |\varepsilon-\delta|^\gamma (|k|^\gamma + |l|^\gamma).$$

Using Lemma 11.12 and (90) we obtain

$$\begin{aligned} 2^{-2\gamma i}\mathbb{E}[\|\Delta_i(\Xi_{r\varepsilon} - \Xi_{r\delta})\|_{L^\infty}^2] &\lesssim |\varepsilon-\delta|^\gamma \sum_{l\in\mathbb{N}_0^2} \frac{1}{(1+|l|)^{4-\gamma}} \sum_{k\in\mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)^2}{(1+|k-l|)^{2\gamma}} \\ &\lesssim |\varepsilon-\delta|^\gamma \sum_{l\in\mathbb{N}_0^2} \frac{1}{(1+|l|)^{2+\gamma}}. \end{aligned}$$

□

Lemma 11.18. *For all $\gamma \in (0,\infty)$ there exists a $C>0$ such that for all $i \geq -1$, $\varepsilon > 0$*

$$\mathbb{E}[\|\Delta_i \Theta_\varepsilon\|_{L^\infty}^2] \leq C 2^{\gamma i}. \tag{93}$$

Proof. First note that $\Theta_\varepsilon = \sum_{k,l\in\mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)}{1+\frac{\pi^2}{r^2}|l|^2}\mathfrak{n}_{k,r}\mathfrak{n}_{l,r}[X_k^\varepsilon X_l^\varepsilon - \mathbb{E}[X_k^\varepsilon X_l^\varepsilon]]$. The identity (89) extends to

$$\mathbb{E}\left([X_k^\varepsilon X_l^\varepsilon - \mathbb{E}[X_k^\varepsilon X_l^\varepsilon]][X_m^\varepsilon X_n^\varepsilon - \mathbb{E}[X_m^\varepsilon X_n^\varepsilon]]\right) = F_\varepsilon(k,m)F_\varepsilon(l,n) + F_\varepsilon(k,n)F_\varepsilon(l,m).$$

By exploiting symmetries, by Lemma 11.12 and by (87) we have

$$2^{-2\gamma i}\mathbb{E}[\|\Delta_i\Theta_\varepsilon\|_{L^\infty}^2] \lesssim \sum_{k,l,m,n\in\mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)}{(1+|k-l|)^\gamma}\frac{\rho^{\odot}(m,n)}{(1+|m-n|)^\gamma}\frac{|F_\varepsilon(k,m)F_\varepsilon(l,n)|}{(1+|l|^2)(1+|m|^2)}.$$

We will bound the $\rho^{\odot}$ function by 1, bound F_ε as in Theorem 11.8 for some $\delta > 0$ (will be chosen small enough later) and we 'split the dimensions' by using that $1+|k|^2 \gtrsim (1+k_1)(1+k_2)$ and $(1+|k-l|)^\gamma \gtrsim (1+|k_1-l_1|)^{\frac{\gamma}{2}}(1+|k_2-l_2|)^{\frac{\gamma}{2}}$ and we obtain

$$2^{-2\gamma i}\mathbb{E}[\|\Delta_i\Theta_\varepsilon\|_{L^\infty}^2] \lesssim \left(\sum_{k,l,m,n\in\mathbb{N}_0} \frac{(1+|k-m|)^{\delta-1}}{(1+|k-l|)^{\frac{\gamma}{2}}}\frac{(1+|l-n|)^{\delta-1}}{(1+|m-n|)^{\frac{\gamma}{2}}}\frac{1}{(1+l)(1+m)}\right)^2. \quad (94)$$

For $\delta < \frac{\gamma}{2}$ we have by (82)

$$\sum_{n\in\mathbb{N}_0} \frac{(1+|l-n|)^{\delta-1}}{(1+|m-n|)^{\frac{\gamma}{2}}} \vee \sum_{k\in\mathbb{N}_0} \frac{(1+|k-m|)^{\delta-1}}{(1+|k-l|)^{\frac{\gamma}{2}}} \lesssim \frac{1}{(1+|m-l|)^{\frac{\gamma}{2}-\delta}},$$

and for $\delta < \frac{\gamma}{3}$

$$\sum_{m\in\mathbb{N}_0} \frac{1}{(1+|m-l|)^{\gamma-2\delta}}\frac{1}{1+m} \lesssim \frac{1}{(1+l)^{\gamma-3\delta}},$$

so that for $\delta < \frac{\gamma}{3}$ the right-hand side of (94) is finite. □

11.19. Observe that

$$\theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon - \xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon = \sum_{k,l\in\mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)}{1+\frac{\pi^2}{r^2}|l|^2}\mathfrak{n}_{k,r}\mathfrak{n}_{l,r}[X_k^\varepsilon X_l^\varepsilon - \tau(\tfrac{\varepsilon}{r}k)\tau(\tfrac{\varepsilon}{r}l)\mathscr{Z}_k\mathscr{Z}_l].$$

We write for $k,l\in\mathbb{N}_0^d$ and $\varepsilon > 0$

$$G_\varepsilon(k,l) = F_\varepsilon(k,l) - \tau(\tfrac{\varepsilon}{r}k)^2\delta_{k,l}. \quad (95)$$

Observe that

$$G_\varepsilon(k,l) = \mathbb{E}[X_k^\varepsilon X_l^\varepsilon - \tau(\tfrac{\varepsilon}{r}k)\tau(\tfrac{\varepsilon}{r}l)\mathscr{Z}_k\mathscr{Z}_l].$$

By Theorem 11.8 we also have the existence of a $C > 0$ such that for all $\varepsilon > 0$ and $k,l\in\mathbb{N}_0^2$

$$|G_\varepsilon(k,l)| \le C\prod_{i=1}^2 (1+|k_i-l_i|)^{\delta-1}. \quad (96)$$

However, we will also use another bound to prove Lemma 11.26 and Lemma 11.27. The bound will be given in Theorem 11.25. We will first prove the auxiliary Lemma's 11.22, 11.23 and 11.24.

11.20. From here on we take $\tau = \mathbb{1}_{(-1,1)^2}$, write $N = \frac{L}{r}$ and assume $N \in \mathbb{N}$ and $N \geq 2$.

Remark 11.21. That we assume $N \in \mathbb{N}$ is due to the fact that this guarantees cancellations in (99), whereas $N \geq 2$ is used in (97) (to ensure $1 - \frac{1}{N} \gtrsim 1$).

Lemma 11.22. *For all $k, l, M \in \mathbb{N}_0$ with $k \neq l$ or for $k = l \leq \frac{M}{N}$*

$$\sum_{\mathfrak{p},\mathfrak{q} \in \{-1,1\}} \left| \sum_{m=M}^{\infty} \frac{\sin(\pi \frac{m}{N})^2}{(\frac{m}{N} + \mathfrak{q} l)(\frac{m}{N} + \mathfrak{p} k)} \right| \lesssim \frac{1}{1 + |k - \frac{M}{N}|} + \frac{1}{1 + |l - \frac{M}{N}|}.$$

Proof. We separate the cases $\mathfrak{p} = \mathfrak{q} = 1$, with $\mathfrak{p}\mathfrak{q} = -1$ and $\mathfrak{p} = \mathfrak{q} = -1$.

• In case $\mathfrak{p} = \mathfrak{q} = 1$ we have

$$\sum_{\mathfrak{p},\mathfrak{q} \in \{-1,1\}} \sum_{m=M}^{\infty} \frac{\sin(\pi \frac{m}{N})^2}{(\frac{m}{N} + k)(\frac{m}{N} + l)} \lesssim \int_{M-1}^{\infty} \frac{1}{(1 + \frac{x}{N} + k \wedge l)^2} \,\mathrm{d}x \lesssim \frac{1}{1 + \frac{M}{N} + k \wedge l}. \tag{97}$$

We will now consider calculations for which at least one of $\mathfrak{p}$ and $\mathfrak{q}$ equals -1.

For $P \geq M$ and $\mathfrak{p}, \mathfrak{q} \in \{-1, 1\}$ and $k, l \in \mathbb{N}_0$ such that $\mathfrak{q} l - \mathfrak{p} k \neq 0$ (we use (78))

$$\begin{aligned}\sum_{m=M}^{P} \frac{\sin(\pi \frac{m}{N})^2}{(\frac{m}{N} + \mathfrak{q} l)(\frac{m}{N} + \mathfrak{p} lk)} &= \frac{1}{\mathfrak{q} l - \mathfrak{p} k} \sum_{m=M}^{P} \left[\frac{\sin(\pi(\frac{m}{N} + \mathfrak{p} k))^2}{\frac{m}{N} + \mathfrak{p} k} - \frac{\sin(\pi(\frac{m}{N} + \mathfrak{q} l))^2}{\frac{m}{N} + \mathfrak{q} l} \right] \\ &= \frac{1}{\mathfrak{q} l - \mathfrak{p} k} \left[\sum_{m=M+\mathfrak{p} Nk}^{P+\mathfrak{p} Nk} \frac{\sin(\pi \frac{m}{N})^2}{\frac{m}{N}} - \sum_{m=M+\mathfrak{q} Nl}^{P+\mathfrak{q} Nl} \frac{\sin(\pi \frac{m}{N})^2}{\frac{m}{N}} \right].\end{aligned} \tag{98}$$

• Let us consider $\mathfrak{p}\mathfrak{q} = -1$ by taking $\mathfrak{p} = -1$ and $\mathfrak{q} = 1$. And let us note that if $k = l = 0$, then (97) is valid, so we consider $k + l \neq 0$. By cancellations of terms we obtain for $P \geq M + N(k + l)$ (so that $P - Nk \geq M + Nl$)

$$\sum_{m=M}^{P} \frac{\sin(\pi \frac{m}{N})^2}{(\frac{m}{N} + l)(\frac{m}{N} - k)} = \frac{1}{k + l} \left[\sum_{m=M-Nk}^{M+Nl-1} \frac{\sin(\pi \frac{m}{N})^2}{\frac{m}{N}} - \sum_{m=P-Nk+1}^{P+Nl} \frac{\sin(\pi \frac{m}{N})^2}{\frac{m}{N}} \right]. \tag{99}$$

By taking $P \to \infty$ we obtain

$$\sum_{m=M}^{\infty} \frac{\sin(\pi \frac{m}{N})^2}{(\frac{m}{N} + l)(\frac{m}{N} - k)} = \frac{1}{k + l} \sum_{m=M-Nk}^{M+Nl-1} \frac{\sin(\pi \frac{m}{N})^2}{\frac{m}{N}}. \tag{100}$$

By distinguishing between the cases $M \leq Nk$ and $M > Nk$ we obtain by (100)

$$\left| \sum_{m=M}^{\infty} \frac{\sin(\pi \frac{m}{N})^2}{(\frac{m}{N} + l)(\frac{m}{N} - k)} \right| \leq \frac{1}{1 + |k - \frac{m}{N}|}.$$

• We are left with the case $\mathfrak{p} = \mathfrak{q} = -1$. We consider $k \geq l$. For $M \geq Nk \geq Nl$ we have

$$\begin{aligned}\sum_{m=M}^{\infty} \frac{\sin(\pi \frac{m}{N})^2}{(\frac{m}{N} - l)(\frac{m}{N} - k)} &\leq \sum_{m=M}^{\infty} \frac{\sin(\pi \frac{m}{N})^2}{(\frac{m}{N} - k)^2} \lesssim \sum_{m=M}^{\infty} \frac{1}{(1 + \frac{m}{N} - k)^2} \\ &\lesssim \int_{M-1}^{\infty} \frac{1}{(1 + \frac{x}{N} - k)^2} \,\mathrm{d}x \lesssim \frac{1}{1 + \frac{M}{N} - k}.\end{aligned} \tag{101}$$

Hence we derive the desired bound for $l \leq k \leq \frac{M}{N}$.

For $k > l$ and $P \geq M + N(k-l)$ (so that $P - Nk \geq M - Nl$), we have – similarly to (99) –

$$\sum_{m=M}^{P} \frac{\sin(\pi \frac{m}{N})^2}{(\frac{m}{N} - l)(\frac{m}{N} - k)} = \frac{1}{k-l} \left[\sum_{m=M-Nk}^{M-Nl-1} \frac{\sin(\pi \frac{m}{N})^2}{\frac{m}{N}} - \sum_{m=P-Nk+1}^{P-Nl} \frac{\sin(\pi \frac{m}{N})^2}{\frac{m}{N}} \right], \tag{102}$$

and thus

$$\sum_{m=M}^{\infty} \frac{\sin(\pi \frac{m}{N})^2}{(\frac{m}{N} - l)(\frac{m}{N} - k)} = \frac{1}{k-l} \sum_{m=M-Nk}^{M-Nl-1} \frac{\sin(\pi \frac{m}{N})^2}{\frac{m}{N}}. \tag{103}$$

It is sufficient to show the following (of which the case $k \leq \frac{M}{N}$ is proved in (101)).

$$\left| \sum_{m=M}^{\infty} \frac{\sin(\pi \frac{m}{N})^2}{(\frac{m}{N} - l)(\frac{m}{N} - k)} \right| \lesssim \begin{cases} \frac{1}{1+l-\frac{M}{N}} & \frac{M}{N} \leq l, \\ \frac{1}{1+\frac{M}{N}-l} & l \leq \frac{M}{N} \leq \frac{k+l}{2}, \\ \frac{1}{1+k-\frac{M}{N}} & \frac{k+l}{2} \leq \frac{M}{N} \leq k, \\ \frac{1}{1+\frac{M}{N}-k} & k \leq \frac{M}{N}. \end{cases} \tag{104}$$

$\star$ If $0 \leq M \leq Nl$, then $M - Nk \leq M - Nl - 1 < 0$ and so

$$\frac{1}{k-l} \sum_{m=M-Nk}^{M-Nl-1} \frac{\sin(\pi \frac{m}{N})^2}{\frac{m}{N}} \lesssim \frac{1}{1 + l - \frac{M}{N}}.$$

$\star$ If $Nl < M \leq N\frac{k+l}{2}$, then $M - Nk \leq Nl + 1 - M \leq 0$ and thus

$$\left| \frac{1}{k-l} \sum_{m=M-Nk}^{M-Nl-1} \frac{\sin(\pi \frac{m}{N})^2}{\frac{m}{N}} \right| = \frac{1}{k-l} \sum_{m=M-Nk}^{Nl+1-M} - \frac{\sin(\pi \frac{m}{N})^2}{\frac{m}{N}} \lesssim \frac{1}{1 + \frac{M}{N} - l}.$$

$\star$ Whereas, if $N\frac{k+l}{2} < M \leq Nk$, then $0 \leq Nk - M \leq M - Nl - 1$ and thus

$$\left| \frac{1}{k-l} \sum_{m=M-Nk}^{M-Nl-1} \frac{\sin(\pi \frac{m}{N})^2}{\frac{m}{N}} \right| = \frac{1}{k-l} \sum_{m=Nk-M+1}^{Nl+1-M} \frac{\sin(\pi \frac{m}{N})^2}{\frac{m}{N}} \lesssim \frac{1}{1 + k - \frac{M}{N}}.$$

□

Lemma 11.23. *For all* $k \geq \frac{M}{N}$,

$$\sum_{\mathfrak{p},\mathfrak{q} \in \{-1,1\}} \sum_{m=0}^{M-1} \left| \frac{\sin(\pi \frac{m}{N})^2}{(\frac{m}{N} + \mathfrak{q}k)(\frac{m}{N} + \mathfrak{p}k)} \right| \lesssim \frac{1}{1 + |k - \frac{M}{N}|}.$$

Proof. As $k + \frac{m}{N} \geq k - \frac{m}{N} \geq 0$,

$$\sum_{\mathfrak{p},\mathfrak{q} \in \{-1,1\}} \sum_{m=0}^{M-1} \left| \frac{\sin(\pi \frac{m}{N})^2}{(\frac{m}{N} + \mathfrak{q}k)(\frac{m}{N} + \mathfrak{p}k)} \right| \lesssim \int_0^M \frac{1}{(1 + k - \frac{x}{N})^2} \,\mathrm{d}x \lesssim \frac{1}{1 + |k - \frac{M}{N}|}.$$

□

As a consequence of Lemmas 11.22 and 11.23 and the expression of $b_{m,l}$ in 11.4, we obtain the following lemma.

Lemma 11.24. *For all $k,l \in \mathbb{N}_0$ with either $k \neq l$ or $k = l \leq \frac{M}{N}$ and $\varepsilon > 0$*

$$\Big| \sum_{m\in\mathbb{N}_0, m \geq \frac{L}{\varepsilon}} b_{m,k} b_{m,l} \Big| \lesssim \frac{1}{1+|k-\frac{r}{\varepsilon}|} + \frac{1}{1+|l-\frac{r}{\varepsilon}|}. \tag{105}$$

For all $k \in \mathbb{N}_0$ with $k \geq \frac{M}{N}$ and $\varepsilon > 0$

$$\Big| \sum_{m\in N_0, m < \frac{L}{\varepsilon}} b_{m,k}^2 \Big| \lesssim \frac{1}{1+|k-\frac{r}{\varepsilon}|}. \tag{106}$$

Proof. Let $M \in \mathbb{N}_0$ be such that $m \geq \frac{L}{\varepsilon}$ if and only if $m \geq M$, then $|M - \frac{L}{\varepsilon}| < 1$ and $|\frac{M}{N} - \frac{r}{\varepsilon}| < \frac{1}{N}$. By using that $N \geq 2$ we have $1 + |k - \frac{M}{N}| \gtrsim 1 + |k - \frac{r}{\varepsilon}|$. Therefore (105) follows from Lemma 11.22 and similarly (106) follows from Lemma 11.23. □

Theorem 11.25. *Let $\tau = \mathbb{1}_{(-1,1)^2}$. There exists a $C > 0$ such that for all $\varepsilon > 0$ and $k,l \in \mathbb{N}_0^2$*

$$|G_\varepsilon(k,l)| \lesssim \begin{cases} \prod_{i=1}^2 \frac{1}{1+|k_i-\frac{r}{\varepsilon}|} + \frac{1}{1+|l_i-\frac{r}{\varepsilon}|} & \text{if for } i \in \{1,2\} \text{ either } k_i \neq l_i \\ & \text{or } k_i = l_i \geq \frac{r}{\varepsilon}, \\ \frac{1}{1+|k_i-\frac{r}{\varepsilon}|} + \frac{1}{1+|l_i-\frac{r}{\varepsilon}|} & \text{if either } k_i \neq l_i \text{ or } k_i = l_i \geq \frac{r}{\varepsilon} \\ & \text{and } k_{3-i} = l_{3-i} < \frac{r}{\varepsilon}, \\ \frac{1}{1+|k_1-\frac{r}{\varepsilon}|} + \frac{1}{1+|k_2-\frac{r}{\varepsilon}|} & k_i = l_i < \frac{r}{\varepsilon} \text{ for } i \in \{1,2\}. \end{cases} \tag{107}$$

Proof. Let $k,l \in \mathbb{N}_0^2$. If $k = l$ with $|k|_\infty < \frac{r}{\varepsilon}$, then

$$|G_\varepsilon(k,l)| = \Big| \sum_{m\in\mathbb{N}_0^2 : |m|_\infty \geq \frac{L}{\varepsilon}} B_{m,k} B_{m,l} \Big| \lesssim \Big| \sum_{m\in\mathbb{N}_0, m\geq\frac{L}{\varepsilon}} b_{m,k_1}^2 \Big| + \Big| \sum_{m\in\mathbb{N}_0, m\geq\frac{L}{\varepsilon}} b_{m,k_2}^2 \Big|.$$

If k and l are not like that, then

$$G_\varepsilon(k,l) = \Big(\sum_{m\in\mathbb{N}_0, m<\frac{L}{\varepsilon}} b_{m,k_1} b_{m,l_1} \Big) \Big(\sum_{m\in\mathbb{N}_0, m<\frac{L}{\varepsilon}} b_{m,k_2} b_{m,l_2} \Big).$$

By using that when $k_i \neq l_i$

$$\Big(\sum_{m\in\mathbb{N}_0, m<\frac{L}{\varepsilon}} b_{m,k_i} b_{m,l_i} \Big) = \Big(\sum_{m\in\mathbb{N}_0, m\geq\frac{L}{\varepsilon}} b_{m,k_i} b_{m,l_i} \Big),$$

the bound (107) then follows from Lemma 11.24. □

Lemma 11.26. *Let $\tau = \mathbb{1}_{(-1,1)^2}$. $\mathbb{E}[\theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon - \xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon] \to 0$ in $\mathscr{C}_{\mathfrak{n}}^{-\gamma}$ for all $\gamma > 0$.*

Proof. By 11.19 and (87)

$$\sup_{i\geq -1} 2^{-\gamma i} \|\Delta_i \mathbb{E}[\theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon - \xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon]\|_\infty \lesssim \sum_{k,l\in\mathbb{N}_0^2} \frac{\rho^\odot(k,l)}{(1+|l|^2)} \frac{|G_\varepsilon(k,l)|}{(1+|k-l|)^\gamma}.$$

We use (107) and split the sum in the three regions for which we have the bounds:

$R_1 = \{(k,l) \in \mathbb{N}_0^2 \times \mathbb{N}_0^2 : \forall i \in \{1,2\}$ either $k_i \neq l_i$ or $k_i = l_i \geq \frac{r}{\varepsilon}\}$,
$R_2 = \{(k,l) \in \mathbb{N}_0^2 \times \mathbb{N}_0^2 : \exists i \in \{1,2\}$ either $k_i \neq l_i$ or $k_i = l_i \geq \frac{r}{\varepsilon}$ and $k_{3-i} = l_{3-i} < \frac{r}{\varepsilon}\}$,
$R_3 = \{(k,l) \in \mathbb{N}_0^2 \times \mathbb{N}_0^2 : \forall i \in \{1,2\}\ k_i = l_i < \frac{r}{\varepsilon}\}$.

• **[Sum over R_1]** By exploiting symmetries using Lemma 11.12

$$\sum_{(k,l)\in R_1} \frac{\rho^{\odot}(k,l)}{(1+|l|^2)} \frac{|G_\varepsilon(k,l)|}{(1+|k-l|)^\gamma} \lesssim \mathscr{S}_\varepsilon^1 + \mathscr{S}_\varepsilon^2,$$
$$\mathscr{S}_\varepsilon^1 = \sum_{k,l\in\mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)}{(1+|l|^2)} \frac{1}{(1+|k-l|)^\gamma} \frac{1}{1+|\frac{r}{\varepsilon}-l_1|} \frac{1}{1+|\frac{r}{\varepsilon}-l_2|}$$
$$\mathscr{S}_\varepsilon^2 = \sum_{k,l\in\mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)}{(1+|l|^2)} \frac{1}{(1+|k-l|)^\gamma} \frac{1}{1+|\frac{r}{\varepsilon}-l_1|} \frac{1}{1+|\frac{r}{\varepsilon}-k_2|}.$$

By (90) and as $(1+|l|)^\gamma \geq (1+l_1)^{\frac{\gamma}{2}}(1+l_2)^{\frac{\gamma}{2}}$, secondly using (82) with $\delta = \frac{\gamma}{4}$

$$\mathscr{S}_\varepsilon^1 \lesssim \left(\sum_{l\in\mathbb{N}_0} \frac{1}{(1+l)^{\frac{\gamma}{2}}} \frac{1}{1+|\frac{r}{\varepsilon}-l|} \right)^2 \lesssim (1+\tfrac{r}{\varepsilon})^{-\frac{\gamma}{4}} \lesssim \varepsilon^{\frac{\gamma}{4}}.$$

For $\mathscr{S}_\varepsilon^2$ by Lemma 11.12 there exist $b>0, c>1$ such that (using that $|k-l| \geq |k_1 - l_1|$)

$$\sum_{k\in\mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)}{(1+|k-l|)^\gamma} \frac{1}{1+|\frac{r}{\varepsilon}-k_2|}$$
$$\lesssim \sum_{\substack{k\in\mathbb{N}_0^2\\ |k|\leq b}} \frac{1}{1+|\frac{r}{\varepsilon}-k_2|} + \sum_{\substack{k_1\in\mathbb{N}_0\\ k_1\leq c|l|}} \frac{1}{(1+|k_1-l_1|)^\gamma} \sum_{\substack{k_2\in\mathbb{N}_0\\ k_2\leq c|l|}} \frac{1}{1+|\frac{r}{\varepsilon}-k_2|}.$$

We will bound the second sum on the right hand side by its corresponding integrals and will bound these to get a bound on the sum over k. Straightforward calculations show

$$\int_0^{c|l|} \frac{1}{(1+|x-l_1|)^\gamma}\, \mathrm{d}x \lesssim (1+|l|)^{1-\gamma}.$$

On the other hand, for $\delta > 0$ and $z > 0$

$$\int_0^z \frac{1}{1+|\frac{r}{\varepsilon}-x|}\, \mathrm{d}x \lesssim \log(1+\tfrac{r}{\varepsilon})^2(1+z) \lesssim (1+\tfrac{r}{\varepsilon})^{2\delta}(1+z)^\delta.$$

Hence for all $\delta > 0$ (we use (82) for the last inequality)

$$\mathscr{S}_\varepsilon^2 \lesssim \sum_{l\in\mathbb{N}_0^2} \frac{1}{(1+|l|^2)} \frac{1}{1+|\frac{r}{\varepsilon}-l_1|} (1+l_1+l_2)^{1-\gamma+\delta}(1+\tfrac{1}{\varepsilon})^{2\delta}$$
$$\lesssim \sum_{l\in\mathbb{N}_0^2} \frac{1}{(1+l_1+l_2)^{1+\gamma-\delta}} \frac{1}{1+|\frac{r}{\varepsilon}-l_1|} (1+\tfrac{1}{\varepsilon})^{2\delta}$$
$$\lesssim \sum_{l_2\in\mathbb{N}_0} \frac{1}{(1+l_2)^{1+\delta}} \sum_{l_1\in\mathbb{N}_0} \frac{1}{(1+l_1)^{\gamma-2\delta}} \frac{1}{1+|\frac{r}{\varepsilon}-l_1|} (1+\tfrac{1}{\varepsilon})^{2\delta} \lesssim (1+\tfrac{1}{\varepsilon})^{5\delta-\gamma}.$$

Therefore, by choosing $\delta < \frac{\gamma}{5}$ we obtain also $\mathscr{S}_\varepsilon^2 \to 0$.

• **[Sum over R_2]** Again by exploiting symmetries using Lemma 11.12 (we bound the sum over R_2 by the sum over all $l \in \mathbb{N}_0^2, k_2 \in \mathbb{N}_0$ and take $k_1 = l_1$), using (82) for $\delta < \frac{\gamma}{2}$

$$\sum_{(k,l)\in R_2} \frac{\rho^{\odot}(k,l)}{(1+|l|^2)} \frac{|G_\varepsilon(k,l)|}{(1+|k-l|)^\gamma} \lesssim \sum_{l\in\mathbb{N}_0^2} \frac{1}{(1+|l|^2)} \sum_{k_2\in\mathbb{N}_0^2} \frac{1}{(1+|k_2-l_2|)^\gamma} \frac{1}{1+|\frac{r}{\varepsilon}-k_2|}$$
$$\lesssim \sum_{l\in\mathbb{N}_0} \frac{1}{(1+l_1)^{1+\delta}} \frac{1}{(1+l_2)^{1-\delta}} \frac{1}{(1+|\frac{r}{\varepsilon}-l_2|)^{\gamma-\delta}} \lesssim (1+\tfrac{1}{\varepsilon})^{2\delta-\gamma}.$$

• **[Sum over R_3]** Again by symmetries

$$\sum_{(k,l)\in R_3} \frac{\rho^{\odot}(k,l)}{(1+|l|^2)} \frac{|G_\varepsilon(k,l)|}{(1+|k-l|)^\gamma} \lesssim \sum_{l_1\in\mathbb{N}_0} \frac{1}{(1+l_1)^{1+\delta}} \sum_{l_2\in\mathbb{N}_0} \frac{1}{(1+l_2)^{1-\delta}} \frac{1}{1+|\frac{r}{\varepsilon}-l_2|},$$

which by (82) is $\lesssim (1+\frac{1}{\varepsilon})^{2\delta-1}$ for all $\delta > 0$. □

Lemma 11.27. *Let* $\tau = \mathbb{1}_{(-1,1)^2}$. $\mathbb{E}[|\langle \theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon - \xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon, \mathfrak{n}_z\rangle|^2] \to 0$ *for all* $z \in \mathbb{N}_0^2$.

Proof. By (86), as $\frac{1}{4} \le \nu_k \le 1$ for all $k \in \mathbb{N}_0^2$,

$$\begin{aligned}&\mathbb{E}[|\langle \theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon - \xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon, \mathfrak{n}_z\rangle|^2] \\ &\lesssim \sum_{k,l,m,n\in\mathbb{N}_0^2} \frac{\rho^{\odot}(k,l)}{1+\frac{\pi^2}{r^2}|l|^2} \frac{\rho^{\odot}(m,n)}{1+\frac{\pi^2}{r^2}|n|^2} \sum_{\mathfrak{p},\mathfrak{r},\mathfrak{q},\mathfrak{s}\in\{-1,1\}^2} \delta_{\mathfrak{r}\circ k+\mathfrak{p}\circ l,z}\delta_{\mathfrak{s}\circ m+\mathfrak{q}\circ n,z} \\ &\quad \cdot \mathbb{E}\left([X_k^\varepsilon X_l^\varepsilon - \tau(\tfrac{\varepsilon}{r}k)\tau(\tfrac{\varepsilon}{r}l)\mathscr{Z}_k\mathscr{Z}_l][X_m^\varepsilon X_n^\varepsilon - \tau(\tfrac{\varepsilon}{r}m)\tau(\tfrac{\varepsilon}{r}n)\mathscr{Z}_m\mathscr{Z}_n]\right)\end{aligned}$$

Let us first find a bound for the expectation in the above expression. By 11.14, (88)

$$\begin{aligned}&\mathbb{E}\left([X_k^\varepsilon X_l^\varepsilon - \tau(\tfrac{\varepsilon}{r}k)\tau(\tfrac{\varepsilon}{r}l)\mathscr{Z}_k\mathscr{Z}_l][X_m^\varepsilon X_n^\varepsilon - \tau(\tfrac{\varepsilon}{r}m)\tau(\tfrac{\varepsilon}{r}n)\mathscr{Z}_m\mathscr{Z}_n]\right) \\ &= \sum_{\mathfrak{k},\mathfrak{l},\mathfrak{m},\mathfrak{n}\in\mathbb{N}_0^2} [\delta_{\mathfrak{k},\mathfrak{l}}\delta_{\mathfrak{m},\mathfrak{n}} + \delta_{\mathfrak{k},\mathfrak{m}}\delta_{\mathfrak{l},\mathfrak{n}} + \delta_{\mathfrak{k},\mathfrak{n}}\delta_{\mathfrak{m},\mathfrak{l}}] B_{\mathfrak{k},k}B_{\mathfrak{l},l}B_{\mathfrak{m},m}B_{\mathfrak{n},n} \\ &\qquad \times [\tau(\tfrac{\varepsilon}{L}\mathfrak{m})\tau(\tfrac{\varepsilon}{L}\mathfrak{n}) - \tau(\tfrac{\varepsilon}{r}m)\tau(\tfrac{\varepsilon}{r}n)][\tau(\tfrac{\varepsilon}{L}\mathfrak{k})\tau(\tfrac{\varepsilon}{L}\mathfrak{l}) - \tau(\tfrac{\varepsilon}{r}k)\tau(\tfrac{\varepsilon}{r}l)].\end{aligned} \tag{108}$$

The $\delta_{\mathfrak{k},\mathfrak{l}}\delta_{\mathfrak{m},\mathfrak{n}}$ contribution to (108) is given by $G_\varepsilon(k,l)G_\varepsilon(m,n)$.

Let us consider the $\delta_{\mathfrak{k},\mathfrak{m}}\delta_{\mathfrak{l},\mathfrak{n}}$ contribution; the contribution by $\delta_{\mathfrak{k},\mathfrak{n}}\delta_{\mathfrak{m},\mathfrak{l}}$ is the same by interchanging 'm' with 'n'. Using that $\tau^2 = \tau$ we compute

$$\begin{aligned}&[\tau(\tfrac{\varepsilon}{L}\mathfrak{k})\tau(\tfrac{\varepsilon}{L}\mathfrak{l}) - \tau(\tfrac{\varepsilon}{r}m)\tau(\tfrac{\varepsilon}{r}n)][\tau(\tfrac{\varepsilon}{L}\mathfrak{k})\tau(\tfrac{\varepsilon}{L}\mathfrak{l}) - \tau(\tfrac{\varepsilon}{r}k)\tau(\tfrac{\varepsilon}{r}l)] \\ &= [1 - \tau(\tfrac{\varepsilon}{r}m)\tau(\tfrac{\varepsilon}{r}n) - \tau(\tfrac{\varepsilon}{r}k)\tau(\tfrac{\varepsilon}{r}l)]\tau(\tfrac{\varepsilon}{L}\mathfrak{k})\tau(\tfrac{\varepsilon}{L}\mathfrak{l}) + \tau(\tfrac{\varepsilon}{r}m)\tau(\tfrac{\varepsilon}{r}n)\tau(\tfrac{\varepsilon}{r}k)\tau(\tfrac{\varepsilon}{r}l),\end{aligned}$$

and use this to obtain that the contribution by $\delta_{\mathfrak{k},\mathfrak{m}}\delta_{\mathfrak{l},\mathfrak{n}}$ equals

$$\begin{aligned}&\sum_{\mathfrak{k},\mathfrak{l}\in\mathbb{N}_0^2} B_{\mathfrak{k},k}B_{\mathfrak{k},m}B_{\mathfrak{l},l}B_{\mathfrak{l},n}[\tau(\tfrac{\varepsilon}{L}\mathfrak{k})\tau(\tfrac{\varepsilon}{L}\mathfrak{l}) - \tau(\tfrac{\varepsilon}{r}m)\tau(\tfrac{\varepsilon}{r}n)][\tau(\tfrac{\varepsilon}{L}\mathfrak{k})\tau(\tfrac{\varepsilon}{L}\mathfrak{l}) - \tau(\tfrac{\varepsilon}{r}k)\tau(\tfrac{\varepsilon}{r}l)] \\ &= [1 - \tau(\tfrac{\varepsilon}{r}m)\tau(\tfrac{\varepsilon}{r}n) - \tau(\tfrac{\varepsilon}{r}k)\tau(\tfrac{\varepsilon}{r}l)](G_\varepsilon(k,m) + \tau(\tfrac{\varepsilon}{r}k)\delta_{k,m})(G_\varepsilon(l,n) + \tau(\tfrac{\varepsilon}{r}l)\delta_{l,n}) \\ &\quad + \tau(\tfrac{\varepsilon}{r}m)\tau(\tfrac{\varepsilon}{r}n)\tau(\tfrac{\varepsilon}{r}k)\tau(\tfrac{\varepsilon}{r}l)\delta_{k,m}\delta_{l,n} \\ &= [1 - \tau(\tfrac{\varepsilon}{r}m)\tau(\tfrac{\varepsilon}{r}n) - \tau(\tfrac{\varepsilon}{r}k)\tau(\tfrac{\varepsilon}{r}l)] \\ &\quad \times \left(G_\varepsilon(k,m)G_\varepsilon(l,n) + \tau(\tfrac{\varepsilon}{r}l)\delta_{l,n}G_\varepsilon(k,m) + \tau(\tfrac{\varepsilon}{r}k)\delta_{k,m}G_\varepsilon(l,n)\right).\end{aligned}$$

By the previously mentioned symmetry of the contributions of $\delta_{\mathfrak{k},\mathfrak{m}}\delta_{\mathfrak{l},\mathfrak{n}}$ and $\delta_{\mathfrak{k},\mathfrak{n}}\delta_{\mathfrak{m},\mathfrak{l}}$, we can leave the contribution by $\delta_{\mathfrak{k},\mathfrak{n}}\delta_{\mathfrak{m},\mathfrak{l}}$ out in the following bound:

$$\begin{aligned}&\mathbb{E}[|\langle \theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon - \xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon, \mathfrak{n}_z\rangle|^2] \\ &\lesssim \sum_{k,l,m,n\in\mathbb{N}_0^2} \frac{1}{1+|l|^2}\frac{1}{1+|n|^2} \sum_{\mathfrak{p},\mathfrak{r}\in\{-1,1\}^d} \delta_{\mathfrak{r}\circ k+\mathfrak{p}\circ l,z} \sum_{\mathfrak{q},\mathfrak{s}\in\{-1,1\}^d} \delta_{\mathfrak{s}\circ m+\mathfrak{q}\circ n,z} \\ &\quad \times \left|G_\varepsilon(k,l)G_\varepsilon(m,n) + G_\varepsilon(k,n)G_\varepsilon(l,m) + \tau(\tfrac{\varepsilon}{r}l)\delta_{l,m}G_\varepsilon(k,n) + \tau(\tfrac{\varepsilon}{r}k)\delta_{k,n}G_\varepsilon(l,m)\right|\end{aligned}$$

By extending G_ε evenly to $\mathbb{Z}^2$ in the sense that $\overline{G}_\varepsilon(k,l) = G_\varepsilon((|k_1|,|k_2|),(|l_1|,|l_2|))$, we can sum over $k\in\mathbb{Z}^2$ instead of $\mathfrak{q}\circ k$ with $\mathfrak{q}\in\{-1,1\}^2$ and $k\in\mathbb{N}_0^2$, and obtain

$$\begin{aligned}&\mathbb{E}[|\langle \theta_\varepsilon \odot \sigma(\mathrm{D})\theta_\varepsilon - \xi_\varepsilon \odot \sigma(\mathrm{D})\xi_\varepsilon, \mathfrak{n}_z\rangle|^2] \\ &\lesssim \sum_{l,n\in\mathbb{Z}^2} \frac{1}{1+|l|^2}\frac{1}{1+|n|^2}\Big(|\overline{G}_\varepsilon(z-l,l)\overline{G}_\varepsilon(z-n,n)| + |\overline{G}_\varepsilon(z-l,n)\overline{G}_\varepsilon(l,z-n)| \\ &\qquad + \delta_{|l_1|,|z_1-n_1|}\delta_{|l_2|,|z_2-n_2|}[|\overline{G}_\varepsilon(z-l,n)| + |\overline{G}_\varepsilon(l,z-n)|]\Big) \lesssim \mathscr{A}_{\varepsilon,z}^2 + \mathscr{B}_{\varepsilon,z} + \mathscr{C}_{\varepsilon,z},\end{aligned}$$

where

$$\begin{aligned}\mathscr{A}_{\varepsilon,z} &= \sum_{l\in\mathbb{Z}^2} \frac{1}{1+|l|^2}|\overline{G}_\varepsilon(z-l,l)|, \\ \mathscr{B}_{\varepsilon,z} &= \sum_{l,n\in\mathbb{Z}^2} \frac{1}{1+|l|^2}\frac{1}{1+|n|^2}|\overline{G}_\varepsilon(z-l,n)\overline{G}_\varepsilon(l,z-n)|, \\ \mathscr{C}_{\varepsilon,z} &= \sum_{l\in\mathbb{Z}^2} \frac{1}{1+|l|^2}\frac{1}{1+|n|^2}\delta_{|l_1|,|z_1-n_1|}\delta_{|l_2|,|z_2-n_2|}[|\overline{G}_\varepsilon(z-l,n)| + |\overline{G}_\varepsilon(l,z-n)|].\end{aligned}$$

We will now show that $\lim_{\varepsilon\downarrow 0}\mathscr{A}_{\varepsilon,z} = \lim_{\varepsilon\downarrow 0}\mathscr{B}_{\varepsilon,z} = \lim_{\varepsilon\downarrow 0}\mathscr{C}_{\varepsilon,z} = 0$. For $\mathscr{B}_{\varepsilon,z}$ and $\mathscr{C}_{\varepsilon,z}$ we show this by showing the summands are bounded by a summable function, which is sufficient by Lebesgue's dominated convergence theorem as $\lim_{\varepsilon\downarrow 0}\overline{G}_\varepsilon(l,n)\to 0$ for all $l,n\in\mathbb{Z}$.

- For $\mathscr{C}_{\varepsilon,z}$ it is sufficient to note that (as $|\overline{G}_\varepsilon|\lesssim 1$)

$$\mathscr{C}_{\varepsilon,z} \lesssim \sum_{n\in\mathbb{Z}^2} \frac{1}{1+|z-n|^2}\frac{1}{1+|n|^2} < \infty.$$

- By (96)

$$\begin{aligned}\mathscr{B}_{\varepsilon,z} &\lesssim \sum_{l,n\in\mathbb{Z}^2} \frac{1}{1+|l|^2}\frac{1}{1+|n|^2}\prod_{i=1}^2 \frac{1}{(1+||z_i-l_i|-|n_i||)^{1-\delta}} \\ &\lesssim \prod_{i=1}^2\left(\sum_{l,n\in\mathbb{Z}} \frac{1}{1+||z_i|-|l||}\frac{1}{1+|n|}\frac{1}{(1+||l|-|n||)^{1-\delta}}\right).\end{aligned}$$

By (82) we obtain

$$\begin{aligned}\mathscr{B}_{\varepsilon,z} &\lesssim \prod_{i=1}^2 \int_0^\infty \frac{1}{1+||z_i|-x|}\int_0^\infty \frac{1}{1+y}\frac{1}{(1+|x-y|)^{1-\delta}}\,\mathrm{d}y\,\mathrm{d}x \\ &\lesssim \prod_{i=1}^2 \int_0^\infty \frac{1}{1+||z_i|-x|}\frac{1}{(1+|x|)^{1-2\delta}}\,\mathrm{d}x \lesssim \prod_{i=1}^2 \frac{1}{(1+|z_i|)^{1-2\delta}}.\end{aligned}$$

• For $\mathscr{A}_{\varepsilon,z}$ we use the bound given in Theorem 11.25 (the bound (96) does not work here). For all $z \in \mathbb{N}_0^2$ we have

$$\overline{G}_\varepsilon(z-l,l) \lesssim \sum_{i=1}^{2} \left(\frac{1}{1+||l_i|-\frac{r}{\varepsilon}|} + \frac{1}{1+\left|z_i - |l_i|\right| - \frac{r}{\varepsilon}|} \right).$$

Hence, by letting $\delta \in (0, \frac{1}{2})$ and bounding $(1+|l|^2) \gtrsim (1+l_1)^{1-\delta}(1+l_2)^{1+\delta}$, by (82),

$$\begin{aligned}\mathscr{A}_{\varepsilon,z} &\lesssim \sum_{l \in \mathbb{N}_0} \left(\frac{1}{(1+l)^{1-\delta}} + \frac{1}{(1+|l-z_1|)^{1-\delta}} \right) \frac{1}{1+|l-\frac{r}{\varepsilon}|} \\ &\lesssim \frac{1}{(1+\frac{r}{\varepsilon})^{1-2\delta}} + \frac{1}{(1+|z_1-\frac{r}{\varepsilon}|)^{1-2\delta}} \to 0.\end{aligned}$$

□

Remark 11.28. Let $y \in \mathbb{R}^2$ be such that $y + Q_r \subset Q_L$ (as in Remark 11.11). We show how to prove certain lemmas for "$\theta_\varepsilon^y, \xi_\varepsilon^y, \Theta_\varepsilon^y, \Xi_\varepsilon^y$" instead of "$\theta_\varepsilon, \xi_\varepsilon, \Theta_\varepsilon, \Xi_\varepsilon$". Lemma 11.18 can be adapted as only (83) is used (and can be adapted as is mentioned in Remark 11.11). For Lemma 11.26 and Lemma 11.27 it is sufficient to adapt the bound (107) in Theorem 11.25. By replacing $\sin(\pi\frac{m}{N})^2$ either by $\sin(\pi\frac{m}{N})\cos(\pi\frac{m}{N})\mathbb{1}_{m\neq 0}$ or $\cos(\pi\frac{m}{N})^2\mathbb{1}_{m\neq 0}$, one can still follow the lines of the proofs of Lemma 11.22 and Lemma 11.23, because for (98) the equality in (78) is used, but this still holds by replacing "$\sin$" by "$\cos$". This then provides the extension of Lemma 11.24 and thus of Theorem 11.25.

Lemma 11.29. *Let $\tau, \tau' : \mathbb{R}^2 \to [0,1]$ be compactly supported functions that are equal to 1 on a neighbourhood of 0. Let Ξ'_ε be as in Definition 6.6 with "τ'" instead of "τ". $\mathbb{E}[|\langle \Xi_\varepsilon - \Xi'_\varepsilon, \mathfrak{n}_z\rangle|^2] \to 0$ for all $z \in \mathbb{N}_0^2$.*

Proof. By (86) and 11.14, (88), as $\frac{1}{4} \le \nu_k \le 1$ for all $k \in \mathbb{N}_0^2$,

$$\begin{aligned}&\mathbb{E}[|\langle \Xi_\varepsilon - \Xi'_\varepsilon, \mathfrak{n}_z\rangle|^2] \\ &\lesssim \sum_{k,l,m,n \in \mathbb{N}_0^2} \frac{\rho^\odot(k,l)}{1+\frac{\pi^2}{r^2}|l|^2} \frac{\rho^\odot(m,n)}{1+\frac{\pi^2}{r^2}|n|^2} \sum_{\mathfrak{p},\mathfrak{r},\mathfrak{q},\mathfrak{s} \in \{-1,1\}^2} \delta_{\mathfrak{r}\circ k + \mathfrak{p}\circ l, z}\delta_{\mathfrak{s}\circ m + \mathfrak{q}\circ n, z}(\delta_{k,m}\delta_{l,n} + \delta_{k,n}\delta_{l,m}) \\ &\quad \cdot [\tau(\tfrac{\varepsilon}{r}k)\tau(\tfrac{\varepsilon}{r}l) - \tau'(\tfrac{\varepsilon}{r}k)\tau'(\tfrac{\varepsilon}{r}l)] \cdot [\tau(\tfrac{\varepsilon}{r}m)\tau(\tfrac{\varepsilon}{r}n) - \tau'(\tfrac{\varepsilon}{r}m)\tau'(\tfrac{\varepsilon}{r}n)].\end{aligned}$$

As both $\tau(\frac{\varepsilon}{r}k) \to 1$ and $\tau'(\frac{\varepsilon}{r}k) \to 1$ as $\varepsilon \downarrow 0$ for all $k \in \mathbb{N}_0^2$, it is sufficient to show that the summand can be bounded by something summable. By symmetries and Lemma 11.12 we can bound the summands (in the $\lesssim$ sense) by

$$\sum_{\mathfrak{p},\mathfrak{r},\mathfrak{q},\mathfrak{s} \in \{-1,1\}^2} \frac{1}{(1+\frac{\pi^2}{r^2}|l|^2)^2} \delta_{k, \mathfrak{r}\circ(z-\mathfrak{p}\circ l)},$$

which is clearly summable over $k, l \in \mathbb{N}_0^2$. □

A The min-max formula for smooth potentials

Lemma A.1. *Let $f_1, \dots, f_n$ be pairwise orthogonal in H_0^2. There exist pairwise orthogonal $f_{1,k}, \dots, f_{n,k}$ in C_c^∞ for $k \in \mathbb{N}$ such that for all i*

$$f_{i,k} \xrightarrow{k\to\infty} f_i \quad \text{in } H_0^2. \tag{109}$$

Proof. Let $g_{i,k} \in C_c^\infty$ be such that $g_{i,k} \to f_i$ in H_0^2 for all i. By doing a Gram-Schmidt procedure on $g_{1,k}, \dots, g_{n,k}$ we can give the proof by induction. We prove the induction step, assuming that $f_{1,k} = g_{1,k}, \dots, f_{n-1,k} = g_{n-1,k}$ are pairwise independent. We define

$$f_{n,k} = g_{n,k} - \sum_{i=1}^{n-1} \frac{\langle g_{n,k}, f_{i,k} \rangle}{\langle f_{i,k}, f_{i,k} \rangle} f_{i,k}.$$

Then $f_{n,k}$ is pairwise independent from $f_{1,k}, \dots, f_{n-1,k}$. As for $i \in \{1, \dots, n-1\}$ we have

$$\langle g_{n,k}, f_{i,k} \rangle \to \langle f_n, f_i \rangle = 0,$$

it follows that $f_{n,k} \to f_n$. □

Lemma A.2. *Let $\zeta \in L^\infty$. Then (for notation see 5.4)*

$$\lambda_n(Q_L, \zeta) = \sup_{\substack{F \sqsubset H_0^2 \\ \dim F = n}} \inf_{\substack{\psi \in F \\ \|\psi\|_{L^2}=1}} \langle \mathscr{H}_\zeta \psi, \psi \rangle = \sup_{\substack{F \sqsubset C_c^\infty \\ \dim F = n}} \inf_{\substack{\psi \in F \\ \|\psi\|_{L^2}=1}} \langle \mathscr{H}_\zeta \psi, \psi \rangle. \tag{110}$$

Proof. First observe that

$$\lambda_n(Q_L, \zeta) = \sup_{\substack{f_1, \dots, f_n \in H_0^2 \\ \langle f_i, f_j \rangle_{H_0^2} = \delta_{ij}}} \inf_{\substack{\psi = \sum_{i=1}^n \alpha_i f_i \\ \alpha_i \in [0,1], \sum_{i=1}^n \alpha_i^2 = 1}} \langle \mathscr{H}_\zeta \psi, \psi \rangle.$$

Let $f_1, \dots, f_n \in H_0^2$ with $\langle f_i, f_j \rangle_{H_0^2} = \delta_{ij}$. By Lemma A.1 there exist $f_{1,k}, \dots, f_{n,k}$ in C_c^∞ with $\langle f_{i,k}, f_{j,k} \rangle_{H_0^2} = \delta_{ij}$ (by renormalising) such that (109) holds. Then

$$\begin{aligned}
&\left| \inf_{\substack{\psi = \sum_{i=1}^n \alpha_i f_i \\ \alpha_i \in [0,1], \sum_{i=1}^n \alpha_i^2 = 1}} \langle \mathscr{H}_\zeta \psi, \psi \rangle - \inf_{\substack{\psi = \sum_{i=1}^n \alpha_i f_{i,k} \\ \alpha_i \in [0,1], \sum_{i=1}^n \alpha_i^2 = 1}} \langle \mathscr{H}_\zeta \psi, \psi \rangle \right| \\
&\le \sup_{\substack{\psi = \sum_{i=1}^n \alpha_i f_i, \varphi = \sum_{i=1}^n \alpha_i f_{i,k} \\ \alpha_i \in [0,1], \sum_{i=1}^n \alpha_i^2 = 1}} |\langle \mathscr{H}_\zeta \psi, \psi \rangle_{L^2} - \langle \mathscr{H}_\zeta \varphi, \varphi \rangle_{L^2}| \\
&\lesssim \sup_{\alpha_i \in [0,1], \sum_{i=1}^n \alpha_i^2 = 1} \left\| \sum_{i=1}^n \alpha_i f_i - \sum_{i=1}^n \alpha_i f_{i,k} \right\|_{H_0^2} \le \sum_{i=1}^n \|f_i - f_{i,k}\|_{H_0^2} \to 0.
\end{aligned}$$

This proves

$$\lambda_n(Q_L, \zeta) = \sup_{\substack{f_1, \dots, f_n \in C_c^\infty \\ \langle f_i, f_j \rangle_{H_0^2} = \delta_{ij}}} \inf_{\substack{\psi = \sum_{i=1}^n \alpha_i f_i \\ \alpha_i \in [0,1], \sum_{i=1}^n \alpha_i^2 = 1}} \langle \mathscr{H}_\zeta \psi, \psi \rangle, \tag{111}$$

and therefore (110). □

B Useful bound on an integral

Lemma B.1. *Let $\gamma, \theta \in (0,1)$ and $\gamma + \theta > 1$. There exists a $C > 0$ such that for all $u \in \mathbb{R}$*

$$\int_0^\infty \frac{1}{(1+|x-u|)^\gamma} \frac{1}{(1+x)^\theta} \, \mathrm{d}x \le C(1+|u|)^{1-\gamma-\theta}. \tag{112}$$

Consequently, there exists a $C > 0$ such that for all $u, v \in \mathbb{R}$

$$\int_{\mathbb{R}} \frac{1}{(1+|x-u|)^\gamma} \frac{1}{(1+|x-v|)^\theta} \, \mathrm{d}x \le C(1+|u-v|)^{1-\gamma-\theta}. \tag{113}$$

Proof. We have uniformly in $a \in (0,1)$

$$\int_0^\infty \frac{1}{(a+x)^\gamma} \frac{1}{(1+x)^\theta} \,\mathrm{d}x \le \int_1^\infty \frac{1}{x^{\gamma+\theta}} \,\mathrm{d}x + \int_0^1 \frac{1}{(a+x)^\gamma} \,\mathrm{d}x \lesssim 1 + (1+a)^{1-\gamma} \lesssim 1.$$

Hence for all $u \ge 0$

$$\begin{aligned} \int_u^\infty \frac{1}{(1+x-u)^\gamma} \frac{1}{(1+x)^\theta} \,\mathrm{d}x &= \int_0^\infty \frac{1}{(1+x)^\gamma} \frac{1}{(1+u+x)^\theta} \,\mathrm{d}x \\ &= (1+u)^{1-\gamma-\theta} \int_0^\infty \frac{1}{(\frac{1}{1+u}+x)^\gamma} \frac{1}{(1+x)^\theta} \,\mathrm{d}x \lesssim (1+u)^{1-\gamma-\theta}. \end{aligned} \tag{114}$$

On the other hand we have

$$\int_0^{\frac{u}{2}} \frac{1}{(1+u-x)^\gamma} \frac{1}{(1+x)^\theta} \,\mathrm{d}x \le (1+\frac{u}{2})^{-\gamma} \int_0^{\frac{u}{2}} \frac{1}{(1+x)^\theta} \,\mathrm{d}x \lesssim (1+u)^{1-\gamma-\theta},$$

and similarly $\int_{\frac{u}{2}}^u \frac{1}{(1+u-x)^\gamma} \frac{1}{(1+x)^\theta} \,\mathrm{d}x \lesssim (1+u)^{1-\gamma-\theta}$. In case u is negative, the bound is already proved in (114) (by interchanging θ and γ).

For (113) it is sufficient to observe that

$$\int_v^\infty \frac{1}{(1+|x-u|)^\gamma} \frac{1}{(1+|x-v|)^\theta} \,\mathrm{d}x = \int_0^\infty \frac{1}{(1+|x+v-u|)^\gamma} \frac{1}{(1+x)^\theta} \,\mathrm{d}x,$$
$$\int_{-\infty}^v \frac{1}{(1+|x-u|)^\gamma} \frac{1}{(1+|x-v|)^\theta} \,\mathrm{d}x = \int_0^\infty \frac{1}{(1+|x+u-v|)^\gamma} \frac{1}{(1+x)^\theta} \,\mathrm{d}x.$$

□

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