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Abstract

The notion of Energy-Dissipation-Principle convergence (EDP-convergence) is used to derive effective evolution equations for gradient systems describing diffusion in a structure consisting of several thin layers in the limit of vanishing layer thickness. The thicknesses of the sublayers tend to zero with different rates and the diffusion coefficients scale suitably. The Fokker–Planck equation can be formulated as gradient-flow equation with respect to the logarithmic relative entropy of the system and a quadratic Wasserstein-type gradient structure. The EDP-convergence of the gradient system is shown by proving suitable asymptotic lower limits of the entropy and the total dissipation functional. The crucial point is that the limiting evolution is again described by a gradient system, however, now the dissipation potential is not longer quadratic but is given in terms of the hyperbolic cosine. The latter describes jump processes across the thin layers and is related to the Marcelin-de Donder kinetics.

1 Introduction

In this text, we study the limit of a diffusion equation on a sandwich-like domain $\Omega_\varepsilon \subset \mathbb{R}^d$ consisting of three thin layers whose thicknesses tend to zero when $\varepsilon \downarrow 0$. In particular, we assume that the middle layer is even thinner than the top and bottom layers, i.e., its thickness is of higher order in ε. The equation that we consider is of Fokker–Planck-type and reads

$$\partial_t u(t,x) = \operatorname{div}\big(A_\varepsilon(x)(\nabla u(t,x) + u(t,x)\nabla V_\varepsilon(x))\big) \quad \text{in } \Omega_\varepsilon. \tag{1}$$

The diffusion matrix $A_\varepsilon(x) \in \mathbb{R}^{d\times d}$ is of order one in the top and bottom layers and satisfies a suitable scaling assumption in the middle layer such that a non-trivial limit arises. The potential V_ε is assumed to be uniformly bounded and converges to a limit. No-flux boundary conditions complement the equation. Such a setting can be observed for example in thin-film organic light-emitting diodes, where organic semiconductor materials with comparably bad conductivity parameter is sandwiched between well conducting electrodes.

The derivation of the limit problem for (1) on the level of PDEs is straightforward, and we refer to e.g. to [AMP$^+$12, AP87, DMFZ18, NRJ07] for related problems. In our case, the effective system of PDEs is given by two reaction-diffusion equations for the top and bottom densities, respectively. The reaction terms are linear exchange reactions which model the transmission of particles through the middle layer, namely

$$\left.\begin{aligned} \partial_t u_0^+ &= \operatorname{div}'\Big(B^+\big(\nabla' u_0^+ + u_0^+\nabla'\bar{V}_0^+\big)\Big) + A^*\Big(\tfrac{u_0^+}{w_0^+} - \tfrac{u_0^-}{w_0^-}\Big) \\ \partial_t u_0^- &= \operatorname{div}'\Big(B^-\big(\nabla' u_0^- + u_0^-\nabla'\bar{V}_0^-\big)\Big) + A^*\Big(\tfrac{u_0^-}{w_0^-} - \tfrac{u_0^+}{w_0^+}\Big) \end{aligned}\right\} \quad \text{on } \Sigma, \tag{2}$$

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where $\Sigma \subset \mathbb{R}^{d-1}$ is the cross-section, u_0^+ and u_0^- are the limit densities on the upper and lower layer and A^* is the effective transmission coefficient, see (49).

Here, however, we are interested in the convergence of the (generalized) gradient systems associated with the equation. By the latter we mean the following: A gradient system for the equation (1) is a triple $(\mathrm{X}_\varepsilon, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, where X_ε is a Riemannian state space, $\mathcal{E}_\varepsilon : \mathrm{X} \to \mathbb{R} \cup \{\infty\}$ is a driving functional, and $\mathcal{R}_\varepsilon : \mathrm{TX} \to [0, \infty]$ is a dissipation potential (non-negative, convex, and lower semicontinuous) on the tangent bundle such that the evolution is equivalently described by $\mathrm{D}\mathcal{E}_\varepsilon(u) + \partial_{\dot{u}} \mathcal{R}_\varepsilon(u; \dot{u}) \ni 0$.

It is well-known since the seminal works [JKO97, Ott98, Ott01a] that the Fokker-Planck equation in (1) can be written as the gradient-flow equation with respect to the driving functional $\mathcal{E}_\varepsilon(u) = \int_{\Omega_\varepsilon} E_{\mathsf{B}}(u/w_\varepsilon) w_\varepsilon \,\mathrm{d}x$, where $E_{\mathsf{B}}(z) = z \log z - z + 1$ and w_ε is the steady state associated with V_ε, and the Wasserstein metric (see also [Lis09] for the case of variable coefficients). The latter can be written in terms of the Legendre transform of $\mathcal{R}_\varepsilon$ in the form $\mathcal{R}_\varepsilon^*(u, \xi) = \frac{1}{2} \int_\varepsilon \nabla \xi \cdot A_\varepsilon(x) \nabla \xi u \,\mathrm{d}x$. In particular, the evolution is entirely formulated in terms of functionals. Hence, the natural question arises, whether we can use variational methods such as Γ-convergence to derive the effective evolution. The umbrella term *evolutionary Γ-convergence* covers several notions of convergence and indicates that evolutionary problems are treated with variational methods, see e.g. [Ste08, Ser11, Bra13, Vis13, Mie16] and the references therein.

In this work, we use the notion of *EDP-convergence* as introduced in [LMPR17] (see also [DFM18, MMP18]). It is based on De Giorgi's *energy-dissipation principle* (in our case it should be called entropy-entropy production principle), which in turn is based on the Legendre–Fenchel equivalences and the chain rule for $t \mapsto \mathcal{E}_\varepsilon(u(t))$, namely

$$\mathcal{E}_\varepsilon(u(t)) + \int_0^t \Big\{ \mathcal{R}_\varepsilon(u; \dot{u}) + \mathcal{R}_\varepsilon^*(u; -\mathrm{D}\mathcal{E}_\varepsilon(u)) \Big\} \,\mathrm{d}t = \mathcal{E}_\varepsilon(u(0)). \tag{3}$$

In particular, it states that the entropy at time t and the entropy production given by the integral term (which we call De Giorgi functional and denote by $\mathfrak{D}_\varepsilon$ in the following) is equal to the initial entropy, we refer to Subsection 2.1 for more details. The notion of EDP-convergence (see Definition 2.2 in Subsection 2.2) requires to establish the Γ-limits of $\mathcal{E}_\varepsilon$ and of $\mathfrak{D}_\varepsilon$ and, in addition, that $\mathfrak{D}_\varepsilon$ is again of $\mathcal{R} \oplus \mathcal{R}^*$ form. By requiring only the Γ-convergence of $\mathfrak{D}_\varepsilon$, instead of separate lower estimates for $\mathcal{R}_\varepsilon(u_\varepsilon; \dot{u}_\varepsilon)$ and $\mathcal{R}_\varepsilon^*(u_\varepsilon; -\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon))$ as in the Sandier–Serfaty approach [SS04], we allow for an interplay of the statics and dynamics (given by $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$, respectively) to obtain an effective dissipation potential $\mathcal{R}_{\mathsf{eff}}$ in the macroscopic limit which is different from the limit of $\mathcal{R}_\varepsilon$. Moreover, we do not work with the solutions of the gradient-flow equation directly but consider convergence along general "fluctuation paths" with bounded entropy and total dissipation. Under the assumption of well-prepared initial conditions, i.e., $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u_0(0)) \in \mathbb{R}$, the convergence of the solutions to (1) then follows from suitable a priori bounds, using the Γ-liminf estimate in (3) and exploiting the chain rule to show that the limits satisfy the differential inclusion $\partial_\xi \mathcal{R}_{\mathsf{eff}}^*(u; -\mathrm{D}\mathcal{E}_0(u)) \ni \dot{u}$, which is formally equivalent to the linear reaction-diffusion system (2).

After introducing the concrete geometric setting for problem (1) in Section 3, we perform the limit passage in Section 4. First, we introduce the gradient system, which is rescaled in Subsection 4.1 by blowing up the domain Ω_ε to a domain of fixed thickness. In Subsection 4.2 we derive a priori bounds. In particular, we rely only on the Wasserstein gradient structure of the equation which gives compactness in the space of measures only. The lower $\liminf$ estimates for the entropies and the De Giorgi functionals is then proven in Subsection 4.3. The crucial point is that in the limit $\varepsilon \to 0$ the time derivative on the middle layer vanishes due to the different time scales. Hence, by performing an inner minimization over all density profiles across the middle layer with fixed boundary conditions, we obtain the effective dissipation functional, whose $\mathcal{R} \oplus \mathcal{R}^*$ structure follows directly from the explicit formula.

Indeed, in the problem given by (1), the effective dissipation potential is not quadratic (in the rates) anymore. Instead, the transmission through the middle layer is given in terms of the function $\mathscr{C}^*(\zeta) = 4(\cosh(\frac{1}{2}\zeta) - 1)$ such that the effective dual dissipation potential reads

$$\mathcal{R}^*_{\text{eff}}(u,\xi) = \frac{1}{2}\int_\Sigma \Big\{\nabla'\xi^+ \cdot B^+\nabla'\xi^+ u^+ + \nabla'\xi^- \cdot B^-\nabla'\xi^- u^-\Big\}\,\mathrm{d}x + \int_\Sigma A^*\mathscr{C}^*(\xi^+ - \xi^-)\sqrt{u^+u^-}\,\mathrm{d}x.$$

The first term describes the lateral diffusion in the upper and lower layer and is of Wasserstein type. The second term gives the dissipation due to jump processes across the middle layer. The linear reaction terms in the effective PDE system arising from this term follow from the calculation rules for the logarithm and the fact, that the derivative of the $\cosh$ function can be written as exponentials.

Finally, in Section 5 we discuss the limit system. In particular, in Subsection 5.2 we rephrase our convergence result in the stronger notion of tilted EDP-convergence which was recently introduced in [MMP18]. In the tilted EDP-convergence arbitrary perturbations of the driving functionals $\mathcal{E}_\varepsilon$ (so-called tilts) are considered. The idea is that due to the arbitrariness of the tilts, we can uniquely recover the $(\mathcal{R}_{\text{eff}}, \mathcal{R}^*_{\text{eff}})$ structure of the effective system. However, we show that in our case the effective dissipation potential depends non-trivially on the tilt. In Subsection 5.3, we compare the EDP-convergence result of the Wasserstein structure with the EDP-convergence of the H^{-1} gradient structure which in the case $V_\varepsilon \equiv 0$ gives a different gradient structure of (1) with quadratic dissipation potentials and driving functionals. We show that in the latter case also the effective dissipation is quadratic. However, let us emphasize that also in the case of the logarithmic entropy we can provide a quadratic gradient structure for the effective equation. This follows from the framework for general reaction-diffusion equations presented in [Mie11], see Subsection 5.4. In Subsection 5.5, we connect our derived effective gradient system to large deviation principles for Markovian jump processes. For the latter it was shown in [MPR14] that the rate functional is given in terms of a generalized gradient system, where the dual dissipation potential is also given in terms of the $\cosh$ function.

2 Abstract setting

2.1 Abstract gradient flow formulation

We call a triple $(\mathrm{X}, \mathcal{E}, \mathcal{R})$ a generalized gradient system, where X is a Riemannian space containing the states of the system, $\mathcal{E} : \mathrm{X} \to \mathbb{R}\cup\{\infty\}$ is a driving functional, and $\mathcal{R} : \mathrm{TX} \to [0,\infty]$ is a dissipation potential defined on the tangent bundle TX. A dissipation potential $\mathcal{R}$ satisfies that $v \mapsto \mathcal{R}(u,v)$ is convex, lower semi-continuous, and $\mathcal{R}(u,0) = 0$. We say that an abstract evolution equation $\dot u = \mathcal{V}(u)$ has a gradient structure if there exists a gradient system $(\mathrm{X}, \mathcal{E}, \mathcal{R})$ such that the evolution can be equivalently written as

$$\dot u = \mathcal{V}(u) \quad \Longleftrightarrow \quad \partial_v\mathcal{R}(u,\dot u) + \mathrm{D}\mathcal{E}(u) \ni 0, \tag{4}$$

where $\partial_{\dot u}\mathcal{R}(u,\dot u) \subset \mathrm{T}^*_u\mathrm{X}$ denotes the usual convex sub-differential containing “frictional” forces and $\mathrm{D}\mathcal{E}(u)$ is a suitable notion of differential of $\mathcal{E}$ giving the driving forces for the evolution.

In case of $\dot{u} \mapsto \mathcal{R}(u, \dot{u})$ being quadratic, i.e., $\mathcal{R}(u, \dot{u}) = \frac{1}{2}\langle \mathbb{G}(u)\dot{u}, \dot{u}\rangle$ with a state-dependent, symmetric, and positive semi definite operator $\mathbb{G}(u) : \mathrm{T}_u\mathrm{X} \to \mathrm{T}^*_u\mathrm{X}$, we speak of a classical gradient system $(\mathrm{X}, \mathcal{E}, \mathcal{R})$. In particular, in this case $\mathbb{G}(u)$ has to be seen as a Riemannian metric, whose inverse $\mathbb{K}(u) = \mathbb{G}(u)^{-1}$ gives the gradient of $\mathcal{E}$, namely,

$$\dot{u} = -\mathbb{K}(u)\mathrm{D}\mathcal{E}(u) = -\nabla_{\mathbb{K}}\mathcal{E}(u).$$

In connection to [Ons31, OM53] we call $\mathbb{K}$ Onsager operator. In many applications it is advantageous to use the Onsager operator $\mathbb{K}$ instead of $\mathbb{G}$, and we refer to [Mie11, Mie13] for a detailed discussion of the above framework for thermodynamic consistent modeling of reaction-diffusion systems. We emphasize, that an evolutionary system $\dot{u} = \mathcal{V}(u)$ can have more than one gradient structure see e.g. Subsection 5.4.

The Legendre transform of $\mathcal{R}$, also called dual dissipation potential, is given via

$$\begin{gathered}\mathcal{R}^* : \mathrm{T}^*\mathrm{X} \to [0, \infty]\\ \mathcal{R}^*(u, \xi) = \sup\big\{\langle \xi, v\rangle - \mathcal{R}(u, v) \,\big|\, v \in \mathrm{T}_u\mathrm{X}\big\}.\end{gathered}$$

We easily check that in the quadratic case it holds $\mathcal{R}^*(u, \xi) = \frac{1}{2}\langle \xi, \mathbb{K}(u)\xi\rangle$. The primal and the dual dissipation potential satisfy the Legendre–Fenchel equivalences, i.e.

$$\begin{aligned}&\text{(i) } \xi \in \partial_{\dot{u}}\mathcal{R}(u, \dot{u}) \quad \Longleftrightarrow \quad \text{(ii) } \dot{u} \in \partial_\xi\mathcal{R}^*(u, \xi)\\ &\Longleftrightarrow \quad \text{(iii) } \mathcal{R}(u, \dot{u}) + \mathcal{R}^*(u, \xi) = \langle \xi, \dot{u}\rangle.\end{aligned} \tag{5}$$

With (ii) we obtain an equivalent formulation of the gradient flow formulation in (4), namely

$$\dot{u} \in \partial_\xi\mathcal{R}^*(u, -\mathrm{D}\mathcal{E}(u)). \tag{6}$$

The notion of evolutionary Γ-convergence used in the subsequent sections is based on a third equivalent formulation of (4), which we call Entropy-Dissipation Balance (EDB). For this, we use (iii) and assume that a chain rule for $t \mapsto \mathcal{E}(u(t))$ holds such that

$$\begin{aligned}\mathcal{E}(u(T)) - \mathcal{E}(u(0)) &= \int_0^T \langle \mathrm{D}\mathcal{E}(u), \dot{u}\rangle \,\mathrm{d}t\\ &\overset{\text{(iii)}}{=} -\int_0^T \Big\{\mathcal{R}(u, \dot{u}) + \mathcal{R}^*(u, -\mathrm{D}\mathcal{E}(u))\Big\}\,\mathrm{d}t.\end{aligned} \tag{7}$$

On the other hand, if (7) and a chain rule holds, it is easy to see by the equilvalences in (5) that also (4) is satisfied.

Note that the Entropy-Dissipation Principle (EDP), also called De Giorgi's $(\mathcal{R}, \mathcal{R}^*)$ formulation, is a scalar identity in contrast to (4) and (6). In particular, the rich toolbox of Calculus of Variations can be exploited to derive effective limits for multiscale systems.

2.2 Evolutionary Γ-convergence

Let us now consider a sequence of functionals $\mathcal{E}_\varepsilon$, $\mathcal{R}_\varepsilon$ depending on a small parameter $\varepsilon > 0$ which describes for example the ratio between the microscopic and macroscopic length scales. We are interested in deriving effective equations for the case $\varepsilon \to 0$. Following the survey paper [Mie16] this derivation is based on evolutionary Γ-convergence of the generalized gradient systems $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, which is defined as follows.

Definition 2.1. For $\varepsilon \geq 0$ let u_ε be the flow induced by $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$. We say that $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ converges in the sense of evolutionary Γ-convergence with well-prepared initial conditions (called pE-convergence in [Mie16]) to $(\mathrm{X}, \mathcal{E}_0, \mathcal{R}_0)$ if from $u_\varepsilon(0) \xrightarrow{\tau} u_0(0)$ and $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}_0(u_0(0)) < \infty$ it follows that $u_\varepsilon(t) \xrightarrow{\tau} u_0(t)$ and $\mathcal{E}_\varepsilon(u_\varepsilon(t)) \to \mathcal{E}_0(u_0(t))$ for all $t \in]0, T]$ and with respect to a topology τ on X.

In [SS04] (see also [Ser11]), abstract conditions for the convergence of the functionals $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ were formulated to establish the evolutionary Γ-convergence of the gradient systems by passing to the limit in the Entropy-Dissipation Balance (7). The crucial conditions are the two separate liminf estimates

$$\int_0^T \mathcal{R}_0(u_0(t), \dot{u}_0(t))\,\mathrm{d}t \leq \liminf_{\varepsilon\to 0} \int_0^T \mathcal{R}_\varepsilon(u_\varepsilon(t), \dot{u}_\varepsilon(t))\,\mathrm{d}t \quad \text{and}$$
$$\mathcal{R}_0^*(u_0, -\mathrm{D}\mathcal{E}_0(u_0)) \leq \liminf_{\varepsilon\to 0} \mathcal{R}_\varepsilon^*(u_\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon)).$$

However, it turns out that these conditions are too strict for our problem of thin heterostructures with a Wasserstein-type gradient structure. Instead, we prove a lower estimate for the De Giorgi functional $\int_0^T \mathcal{M}_\varepsilon(u_\varepsilon, \dot{u}_\varepsilon)\,\mathrm{d}t$ where $\mathcal{M}_\varepsilon$ is of $\mathcal{R}_\varepsilon \oplus \mathcal{R}_\varepsilon^*$ form, i.e., $\mathcal{M}_\varepsilon(u, v) = \mathcal{R}_\varepsilon(u, v) + \mathcal{R}_\varepsilon^*(u, -\mathrm{D}\mathcal{E}_\varepsilon(u))$. Note that in general, it is not clear that the limiting integrand $\mathcal{M}_0$ is again of the form $\mathcal{R} \oplus \mathcal{R}^*$. In the case of the thin heterostructures considered in this text it is possible to establish the $\mathcal{R} \oplus \mathcal{R}^*$ form of the limiting De Giorgi functional, where the disspation potentials are not longer quadratic.

We follow [LMPR17] and define the De Giorgi functional $\mathfrak{D}_\varepsilon : \mathrm{W}^{1,1}([a, b]; \mathrm{X}) \to [0, \infty]$ via

$$\mathfrak{D}_\varepsilon(u; [a, b]) := \int_a^b \Big\{\mathcal{R}_\varepsilon(u, \dot{u}) + \mathcal{R}_\varepsilon^*(u, -\mathrm{D}\mathcal{E}_\varepsilon(u))\Big\}\,\mathrm{d}t. \tag{8}$$

As in [SS04] we consider a sense S of convergence, i.e., $u_\varepsilon \xrightarrow{S} u$. For the context of this paper the sense S is given by the convergence of $R_\varepsilon u_\varepsilon \rightharpoonup u$ for some $R_\varepsilon : X \to Y$. In particular, u_ε and u do not belong to the same space. Since the De Giorgi functional is an fundamental object in our analysis, we need a sense S_s for the static convergence $R_\varepsilon u_\varepsilon(t) \rightharpoonup u(t)$ and a sense S_e for the evolutionary convergence of the curves $\{t \mapsto R_\varepsilon u_\varepsilon(t)\} \rightharpoonup \{t \mapsto u(t)\}$.

Definition 2.2. The generalized gradient systems $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ EDP-converge to $(\mathrm{X}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$ with respect to sense S_s on X and the sense S_e on $\mathrm{L}^\infty(0, T; \mathrm{X})$, respectively, if we have

(i) $(\mathrm{X}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$ satisfies a chain rule,

(ii) $\mathcal{E}_\varepsilon \xrightarrow{S_s\text{-}\Gamma} \mathcal{E}_0$ and $\mathfrak{D}_\varepsilon \xrightarrow{S_e\text{-}\Gamma} \mathfrak{D}_{\text{eff}}$,

(iii) $\mathfrak{D}_{\text{eff}}(u; [a, b]) = \int_a^b \mathcal{R}_{\text{eff}}(u, \dot{u}) + \mathcal{R}_{\text{eff}}^*\big(u, -\mathrm{D}\mathcal{E}_0(u)\big)\,\mathrm{d}t$.

Note that in contrast to [LMPR17], we additionally require the existence of recovery sequences for the De Giorgi functionals $\mathfrak{D}_\varepsilon$. In particular, we do not consider the convergence of the functionals along solutions of the gradient flow equation but $\mathfrak{D}_\varepsilon$ and $\mathfrak{D}_{\text{eff}}$ are evaluated along general "fluctuation paths" $u : [0, T] \to \mathrm{X}$. Moreover, it is natural to restrict ourselves to fluctuation paths with finite entropies and De Giorgi functional. Note that this definition of EDP-convergence is called "simple" EDP-convergence in [MMP18].

Assuming that we are able to extract a subsequence of the solutions u_ε to the gradient system $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ that converge in the same topology in which the Γ-limits are computed, the convergence

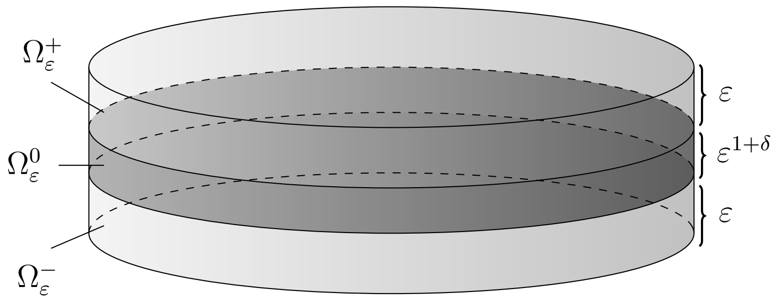

Figure 1: Sketch of the domain $\Omega_\varepsilon \subset \mathbb{R}^d$ with cross section $\Sigma \subset \mathbb{R}^{d-1}$. The diameter of Σ is considered to be large compared to the thickness of the domain Ω_ε. The domain is decomposed into a top layer Ω_ε^+, center layer Ω_ε^0, and bottom layer Ω_ε^-, whose thicknesses are given by ε, $\varepsilon^{1+\delta}$ for a fixed $\delta > 0$, and ε, respectively.

to a solution of the effective problem can be shown as follows: With the Γ-liminf estimate of $\mathfrak{D}_\varepsilon$ and $\mathcal{E}_\varepsilon$ and the well-prepared initial conditions we can pass to the limit in (7) via

$$\begin{aligned}\mathcal{E}_0\big(u(T)\big) + \mathfrak{D}_{\text{eff}}(u;[0,T]) &\leq \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon\big(u_\varepsilon(T)\big) + \liminf_{\varepsilon\to 0} \mathfrak{D}_\varepsilon(u_\varepsilon;[0,T]) \\ &\leq \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon\big(u_\varepsilon(0)\big) = \mathcal{E}_0\big(u(0)\big).\end{aligned}$$

With the Fenchel–Young estimate for the integrand of $\mathfrak{D}_{\text{eff}}$ and the chain rule we conclude the equality $\mathfrak{D}_{\text{eff}}(u;[0,T]) = \mathcal{E}_0\big(u(0)\big) - \mathcal{E}_0\big(u(T)\big)$, i.e., the limit u of solutions u_ε is indeed the flow induced by the gradient system $(X, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$ and we furthermore obtain the evolutionary Γ-convergence in the sense of Definition 2.1.

3 Diffusion in a thin hetero structure

We investigate a drift-diffusion equation in a thin domain, which is given by a cross section $\Sigma \subset \mathbb{R}^{d-1}$ and consists of three thin layers. In particular, the thicknesses of the individual layers are assumed to be small compared to the diameter of the cross section so that we introduce the small parameter $\varepsilon > 0$ related to the thicknesses. The crucial assumptions is that the middle layer scales differently than the upper and lower layer: We define the sets $I_\varepsilon^+ := \varepsilon^{1+\delta}/2 +]0,\varepsilon[$, $I_\varepsilon^0 := \big]{-\varepsilon^{1+\delta}/2}, \varepsilon^{1+\delta}/2\big[$ for a fixed $\delta > 0$, and $I_\varepsilon^- := -\varepsilon^{1+\delta}/2 +]{-\varepsilon}, 0[$. Hence, for $\Omega_\varepsilon = \Sigma \times \big]{-(\varepsilon + \varepsilon^{1+\delta}/2)}, \varepsilon + \varepsilon^{1+\delta}/2\big[$ we have the decomposition into the upper, middle, and lower layer

$$\overline{\Omega}_\varepsilon = \overline{\Omega}_\varepsilon^+ \cup \overline{\Omega}_\varepsilon^0 \cup \overline{\Omega}_\varepsilon^- \quad \text{with} \quad \begin{cases} \Omega_\varepsilon^+ = \Sigma \times I_\varepsilon^+, \\ \Omega_\varepsilon^0 = \Sigma \times I_\varepsilon^0, \\ \Omega_\varepsilon^- = \Sigma \times I_\varepsilon^-. \end{cases}$$

We will write $\hat{x} = (y, \hat{z}) \in \Omega_\varepsilon$ such that $y \in \Sigma$ for the lateral and $-\varepsilon - \varepsilon^{1+\delta}/2 < \hat{z} < \varepsilon + \varepsilon^{1+\delta}/2$ for the vertical variable.

For a given fixed potential $\hat{V}_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ (for the precise assumptions on $\hat{V}_\varepsilon$ see Section 4), we consider in Ω_ε the scalar drift-diffusion equation

$$\partial_t \hat{u}(t,\hat{x}) = \operatorname{div}\big(A_\varepsilon(\hat{x})(\nabla \hat{u}(t,\hat{x}) + \hat{u}(t,\hat{x})\nabla \hat{V}_\varepsilon(\hat{x}))\big) \text{ in } \Omega_\varepsilon \tag{9}$$

with no-flux boundary condition $A_\varepsilon(\nabla\hat{u} + \hat{u}\nabla\hat{V}_\varepsilon)\cdot\nu_\varepsilon = 0$ on $\partial\Omega_\varepsilon$ and initial condition $\hat{u}(0) = \hat{u}_0$. In particular, we have conservation of total mass, and we assume without loss of generality that $\int_{\Omega_\varepsilon}\hat{u}\,\mathrm{d}\hat{x} = 1$.

The symmetric and positive semi-definite coefficient matrix $A_\varepsilon \in \mathrm{L}^\infty(\Omega_\varepsilon;\mathbb{R}^{d\times d}_{\text{spd}})$ has the form

$$A_\varepsilon(\hat{x}) = \begin{pmatrix} B_\varepsilon(\hat{x}) & 0 \\ 0 & a_\varepsilon(\hat{x}) \end{pmatrix}, \tag{10}$$

where $B_\varepsilon \in \mathrm{L}^\infty(\Sigma;\mathbb{R}^{(d-1)\times(d-1)}_{\mathrm{spd}})$ and $a_\varepsilon \in \mathrm{L}^\infty(\Sigma;\mathbb{R}_{>0})$ are assumed to be piecewise constant, namely,

$$B_\varepsilon(\hat{x}) = \begin{cases} B^+ & \text{on } \Omega_\varepsilon^+, \\ \varepsilon^\gamma B^0 & \text{on } \Omega_\varepsilon^0, \\ B^- & \text{on } \Omega_\varepsilon^-, \end{cases} \qquad \text{and} \qquad a_\varepsilon(\hat{x}) = \begin{cases} a^+ & \text{on } \Omega_\varepsilon^+ \\ \varepsilon^{2+\delta}a^0 & \text{on } \Omega_\varepsilon^0 \\ a^- & \text{on } \Omega_\varepsilon^- \end{cases} \tag{11}$$

with $\gamma \geq 0$, B^+, B^0, $B^- \in \mathbb{R}^{(d-1)\times(d-1)}_{\text{spd}}$ and a^+, a^0, $a^- > 0$ being fixed and constant.

Note that the discontinuity set of A_ε is closed and a Lebesgue null set. Thus, A_ε is almost everywhere equal to a matrix $\tilde{A}_\varepsilon$ whose inverse satisfies that the map $x \mapsto \tilde{A}_\varepsilon^{-1}(\hat{x})\xi\cdot\xi$ is lower semicontinuous for all $\xi \in \mathbb{R}^d$. This property is crucial for the purely metric approach presented in [Lis09].

Remark 1. The choice of the scaling ε^γ with $\gamma \geq 0$ in the middle layer Ω_ε^0 for the lateral directions does not matter in the effective system as diffusion and drift will vanish in the limit $\varepsilon \to 0$ (in fact $\gamma > -\delta$ is sufficient). In contrast, the scaling $\varepsilon^{2+\delta}$ in the vertical direction can be justified as follows: The relative densities $\hat{\rho}_\varepsilon$ are of order 1, however the reference density $\hat{w}_\varepsilon$ is of order ε^{-1} and the vertical derivatives of $\hat{\rho}_\varepsilon$ in the middle layer are of order $\varepsilon^{-(1+\delta)}$. Hence, we have that the integrand in the Fischer information $\hat{\mathcal{R}}_\varepsilon(\hat{\nu}_\varepsilon, -\mathrm{D}\hat{\mathcal{E}}_\varepsilon(\hat{\nu}_\varepsilon))$ satisfies in the middle layer the scaling $|\partial_{\hat{z}}\log\hat{\rho}_\varepsilon|^2\hat{\rho}_\varepsilon\hat{w}_\varepsilon \sim \varepsilon^{-(3+2\delta)}$. Since the thickness of the middle layer is $\varepsilon^{1+\delta}$, we arrive at the critical order $\varepsilon^{2+\delta}$.

It is well known, that several gradient systems induce the diffusion equation (9). Below, we consider the Wasserstein gradient system in Section 4 with Boltzmann entropy and the purely quadratic H^{-1} gradient sytem in Section 5.3. For both gradient systems we apply techniques of evolutionary Γ-convergence and derive the variational formulation of the limit flow. However, the De Giorgi functional $\mathfrak{D}_\varepsilon$ is quadratic in the case of the H^{-1} gradient system. By the general theory of Γ-convergence (cf. [Bra06, Prop 2.13]) we expect the effective dissipation potential to be also quadratic. In fact, the methods developed in [SS04] apply and we have $\mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_{\text{eff}}$. In particular, the effective dissipation potential is also quadratic. Whereas the De Giorgi functional $\mathfrak{D}_\varepsilon$ associated with the Wasserstein gradient system is not quadratic. In this case, we obtain that the effective dissipation potential is not quadratic, since $\mathcal{R}^*_{\text{eff}}$ involves exponential terms. In particular, the limiting variational formulation can not be cast into a metric formulation involving the Wasserstein distance.

4 Entropic gradient structure

It is well known since the seminal work of Otto [Ott01b, JKO97], that certain diffusion equations can be interpreted as gradient-flow equations with respect to a driving functional $\mathcal{E}$ and the Wasserstein distance $\mathcal{W}_2$. The rigorous treatment of the evolution equation in (9) as Wasserstein gradient flow with variable coefficient can be found in [Lis09]. Therein, the problem on the full space $\mathbb{R}^d$ was considered, however, our setting is recovered by setting $\hat{V}_\varepsilon(\hat{x}) = +\infty$ outside of Ω_ε (see [Lis08, Ch. 4]). We consider the relative Boltzmann entropy functional defined on the space of probability measures $\mathrm{X}_\varepsilon = \mathrm{Prob}(\overline{\Omega}_\varepsilon)$

$$\hat{\mathcal{E}}_\varepsilon(\hat{\nu}) = \begin{cases} \int_{\Omega_\varepsilon} E_{\mathsf{B}}\big(\hat{\rho}(\hat{x})\big)\,\mathrm{d}\hat{\lambda}_\varepsilon(\hat{x}) & \text{if } \hat{\nu} = \hat{\rho}\hat{\lambda}_\varepsilon, \\ +\infty & \text{otherwise,} \end{cases} \tag{12}$$

where $E_{\mathsf{B}}(z) = z\log z - z + 1$ and $\hat{\rho} = \mathrm{d}\hat{\nu}/\,\mathrm{d}\hat{\lambda}_\varepsilon$ denotes the relative density of $\hat{\nu} \in \mathrm{X}_\varepsilon$ with respect to the reference measure $\hat{\lambda}_\varepsilon \in \mathrm{X}_\varepsilon$. The latter is given by $\hat{\lambda}_\varepsilon = \hat{w}_\varepsilon \mathcal{L}^d$, where $\hat{w}_\varepsilon(\hat{x}) = \exp(-\hat{V}_\varepsilon(\hat{x}))/Z_\varepsilon$ with the normalization constant $Z_\varepsilon := \int_{\Omega_\varepsilon} \exp(-\hat{V}_\varepsilon(\hat{x}))\,\mathrm{d}\hat{x}$.

We assume that the potential $\hat{V}_\varepsilon$ is sufficiently smooth and there exist a constant $C_V > 0$ independent of $\varepsilon > 0$ such that for almost every $\hat{x} \in \Omega_\varepsilon$ we have $-C_V \leq \hat{V}_\varepsilon(x) \leq C_V$. Note that $\mathcal{L}^d(\Omega_\varepsilon) = \varepsilon(2{+}\varepsilon^\delta)\mathcal{L}^{d-1}(\Sigma) =: \varepsilon\beta_\varepsilon$. Thus, we have the estimates

$$\beta_\varepsilon \mathrm{e}^{-C_V} \leq Z_\varepsilon/\varepsilon \leq \beta_\varepsilon \mathrm{e}^{C_V} \quad \text{with } \beta_\varepsilon \to \beta_0 = 2\mathcal{L}^{d-1}(\Sigma). \tag{13}$$

Following [Lis09], the primal dissipation potential $\hat{\mathcal{R}}_\varepsilon$ is defined as

$$\hat{\mathcal{R}}_\varepsilon(\hat{\nu}, \dot{\hat{\nu}}) = \frac{1}{2}\int_{\Omega_\varepsilon} A_\varepsilon(\hat{x})\hat{v}\cdot\hat{v}\,\mathrm{d}\hat{\nu} \quad \text{with} \quad \langle \dot{\hat{\nu}}, \varphi\rangle = \int_{\Omega_\varepsilon} A_\varepsilon(\hat{x})\hat{v}\cdot\nabla\varphi\,\mathrm{d}\hat{\nu} \tag{14}$$

for all $\varphi \in \mathrm{C}^\infty(\Omega_\varepsilon)$ where the velocity field satisfies $\hat{v} \in \mathrm{L}^2(\Omega_\varepsilon; \hat{\nu}(\mathrm{d}\hat{x}))$. With this, we define the De Giorgi functional $\hat{\mathfrak{D}}_\varepsilon$ for a curve $[0,T] \ni t \mapsto \hat{\nu}(t) \in \mathrm{Prob}(\overline{\Omega}_\varepsilon)$ via

$$\hat{\mathfrak{D}}_\varepsilon(\hat{\nu}; [a,b]) = \int_a^b \Big\{\hat{\mathcal{R}}_\varepsilon\big(\hat{\nu}(t), \dot{\hat{\nu}}(t)\big) + \hat{\mathcal{R}}^*_\varepsilon\big(\hat{\nu}(t), -\mathrm{D}\hat{\mathcal{E}}_\varepsilon(\hat{\nu}(t))\big)\Big\}\,\mathrm{d}t,$$

where $\mathrm{D}\hat{\mathcal{E}}_\varepsilon(\hat{\nu}) = \log(\hat{\rho}_\varepsilon) = \log(\mathrm{d}\hat{\nu}/\mathrm{d}\hat{\lambda}_\varepsilon)$. In particular, the so-called Fisher information takes the form

$$\hat{\mathcal{R}}^*_\varepsilon\big(\hat{\nu}, -\mathrm{D}\hat{\mathcal{E}}_\varepsilon(\hat{\nu})\big) = \int_{\Omega_\varepsilon} \frac{1}{2\hat{\rho}^2}\cdot A_\varepsilon(\hat{x})\nabla\hat{\rho}\cdot\nabla\hat{\rho}\,\mathrm{d}\hat{\nu} = 2\int_{\Omega_\varepsilon} A_\varepsilon(\hat{x})\nabla\sqrt{\hat{\rho}}\cdot\nabla\sqrt{\hat{\rho}}\,\mathrm{d}\hat{\lambda}_\varepsilon,$$
$$\text{if } \hat{\rho} = \frac{\mathrm{d}\hat{\nu}}{\mathrm{d}\hat{\lambda}_\varepsilon} \text{ with } \sqrt{\hat{\rho}} \in \mathrm{H}^1(\Omega_\varepsilon).$$

Theorem 1.1 in [Lis09] guarantees for $\varepsilon > 0$ fixed and given initial value $\hat{\nu}^0 = \hat{u}^0\mathcal{L}^d \in \mathrm{X}_\varepsilon$ the existence of a curve $t \mapsto \hat{\nu}_\varepsilon(t) \in \mathrm{X}_\varepsilon$ which is absolutely continuous with respect to the 2-Wasserstein distance and metric time derivative in $\mathrm{L}^2(0,T)$. Moreover, $t \mapsto \hat{\mathcal{E}}_\varepsilon(\hat{\nu}(t))$ is locally absolutely continuous and for almost every in $t \in [0,\infty)$ the local energy identity holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathcal{E}}_\varepsilon(\hat{\nu}_\varepsilon(t)) + \hat{\mathcal{R}}_\varepsilon\big(\hat{\nu}_\varepsilon(t), \dot{\hat{\nu}}_\varepsilon(t)\big) + \hat{\mathcal{R}}^*_\varepsilon\big(\hat{\nu}_\varepsilon(t), -\mathrm{D}\hat{\mathcal{E}}_\varepsilon(\hat{\nu}_\varepsilon(t))\big) = 0.$$

Finally, the Lebesgue density $\hat{u}_\varepsilon$ of $\hat{\nu}_\varepsilon$ satisfies (9) in the distributional sense.

4.1 Transformation of the domain

To make the dependence on the parameter $\varepsilon > 0$ explicit, we rescale the domain Ω_ε in the vertical direction, such that the top, middle, and bottom layer are each of constant thickness 1. For this, we introduce the Lipschitz map $S_\varepsilon : \Omega_\varepsilon \to \Omega_1$, which is defined via $S_\varepsilon(\hat{x}) = (y, \Phi_\varepsilon(\hat{z}))$ with

$$\Phi_\varepsilon(\hat{z}) = \begin{cases} (2\hat{z}+\varepsilon^{1+\delta})/(2\varepsilon) - 1/2 & \text{if } \hat{z} \in I_\varepsilon^- :=]-\varepsilon-\frac{\varepsilon^{1+\delta}}{2}, -\frac{\varepsilon^{1+\delta}}{2}[, \\ \hat{z}/\varepsilon^{1+\delta} & \text{if } \hat{z} \in I_\varepsilon^0 := [-\frac{\varepsilon^{1+\delta}}{2}, \frac{\varepsilon^{1+\delta}}{2}], \\ (2\hat{z}-\varepsilon^{1+\delta})/(2\varepsilon) + 1/2 & \text{if } \hat{z} \in I_\varepsilon^+ :=]\frac{\varepsilon^{1+\delta}}{2}, \varepsilon + \frac{\varepsilon^{1+\delta}}{2}[. \end{cases} \tag{15}$$

We use the push-forward of the reference measure $\hat{\lambda}_\varepsilon \in \mathrm{X}_\varepsilon$ under the map S_ε to obtain the new reference measure $\lambda_\varepsilon \in \mathrm{X}_1 = \mathrm{Prob}(\Omega_1)$, i.e., $\lambda_\varepsilon := (S_\varepsilon)_\# \hat{\lambda}_\varepsilon$. In particular, we have that λ_ε is given by $\lambda_\varepsilon = w_\varepsilon \mathcal{L}^d$ with $w_\varepsilon(x) = (\varepsilon/Z_\varepsilon)\, \mathsf{m}_\varepsilon(x) \exp(-V_\varepsilon(x))$ and $\hat{V}_\varepsilon(\hat{x}) = V_\varepsilon(S_\varepsilon(\hat{x}))$, and the volume factor $\mathsf{m}_\varepsilon : \Omega_1 \to \mathbb{R}_+$ satisfies

$$\mathsf{m}_\varepsilon(x) = \begin{cases} 1 & \text{for } x \in \Omega_1^\pm := \Sigma \times (I_1^+ \cup I_1^-), \\ \varepsilon^\delta & \text{for } x \in \Omega_1^0 := \Sigma \times I_1^0. \end{cases} \tag{16}$$

Obviously, V_ε satisfies the same upper and lower bounds as $\hat{V}_\varepsilon$, and we assume moreover that $V_\varepsilon \to V_0$ in $\mathrm{C}^0(\overline{\Omega}_1)$. Thus, as $Z_\varepsilon/\varepsilon \to Z_0 = \int_{\Omega_1^\pm} \exp(-V_0(x))\,\mathrm{d}x$ for $\varepsilon \to 0$, we have $\lambda_\varepsilon \to \lambda_0$ in X_1, where the limiting reference measure $\lambda_0 \in \mathrm{X}_1$ has the Lebesgue density $w_0(x) = \exp(-V_0(x))/Z_0$ if $x \in \Omega_1^\pm$ and $w_0(x) = 0$ for $x \in \Omega_1^0$.

Clearly, the density w_ε is not continuous at $\{x_d = \pm 1/2\}$ due to the definition of m_ε. Instead, we use the rescaled density W_ε and the associated measure $\Lambda_\varepsilon \in \mathrm{Meas}(\overline{\Omega}_1)$ defined via

$$W_\varepsilon(x) := \frac{\varepsilon}{Z_\varepsilon} \exp(-V_\varepsilon(x)), \qquad \Lambda_\varepsilon = W_\varepsilon \mathcal{L}^d.$$

Obviously, the latter is not a probability measure anymore. We have the convergence $\Lambda_\varepsilon \to \Lambda_0$ in $\mathrm{Meas}(\overline{\Omega}_1)$ where the limiting measure is given by the density $W_0(x) = \exp(-V_0(x))/Z_0$. Moreover, note that due to the assumptions on V_ε we have that

$$\frac{\exp(-2C_V)}{\beta_\varepsilon} \le W_\varepsilon(x) \le \frac{\exp(2C_V)}{\beta_\varepsilon}. \tag{17}$$

For a measure $\hat{\nu} \in \mathrm{X}_\varepsilon$ with relative density $\hat{\rho} \in \mathrm{L}^1(\Omega_\varepsilon)$ with respect to $\hat{\lambda}_\varepsilon$, i.e. $\hat{\nu} = \hat{\rho}\hat{\lambda}_\varepsilon$, we have that the associated transformed measure $\nu = (S_\varepsilon)_\# \hat{\nu} \in \mathrm{X}_1$ has the relative density $\rho = \hat{\rho} \circ S_\varepsilon^{-1} \in \mathrm{L}^1(\Omega_1)$ with respect to the rescaled reference measure λ_ε such that $\nu = \rho\lambda_\varepsilon$. In particular, due to mass conservation the density ρ satisfies

$$\nu(\Omega_1) = \int_{\Omega_1} \rho(x)\,\mathrm{d}\lambda_\varepsilon = \int_{\Omega_1} \rho(x)\, \mathsf{m}_\varepsilon(x) W_\varepsilon(x)\,\mathrm{d}x = 1. \tag{18}$$

Using the transformation above in the driving functional $\hat{\mathcal{E}}_\varepsilon$ and the dissipation potentials $\hat{\mathcal{R}}_\varepsilon$ leads to the new gradient system $(\mathrm{X}_1, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ defined via

$$\mathcal{E}_\varepsilon(\nu) = \hat{\mathcal{E}}_\varepsilon(\hat{\nu}) = \begin{cases} \displaystyle\int_{\Omega_1} E_{\mathsf{B}}(\rho(x))\,\mathrm{d}\lambda_\varepsilon & \text{if } \nu = \rho\lambda_\varepsilon, \\ +\infty & \text{otherwise,} \end{cases}$$

and, with $B_\varepsilon \in \mathrm{L}^\infty(\Omega_1;\mathbb{R}^{(d-1)\times(d-1)})$ and $a_\varepsilon \in \mathrm{L}^\infty(\Omega_1)$ given in (10),

$$\begin{aligned}\mathcal{R}^*_\varepsilon(\nu;\xi) &= \frac{1}{2}\int_{\Omega_1} \Big\{B_\varepsilon(x)\nabla'\xi\cdot\nabla'\xi + a_\varepsilon(x)\Phi'_\varepsilon(z)^2|\partial_z\xi|^2\Big\}\,\mathrm{d}\nu \\ &= \frac{1}{2}\int_{\Omega_1^+}\Big\{B^+\nabla'\xi\cdot\nabla'\xi + \frac{a^+}{\varepsilon^2}|\partial_z\xi|^2\Big\}\,\mathrm{d}\nu + \frac{1}{2}\int_{\Omega_1^-}\Big\{B^-\nabla'\xi\cdot\nabla'\xi + \frac{a^-}{\varepsilon^2}|\partial_z\xi|^2\Big\}\,\mathrm{d}\nu \\ &\quad + \frac{1}{2}\int_{\Omega_1^0}\Big\{\varepsilon^\gamma B^0\nabla'\xi\cdot\nabla'\xi + \frac{a^0}{\varepsilon^\delta}|\partial_z\xi|^2\Big\}\,\mathrm{d}\nu,\end{aligned}$$

where $\nabla'\xi = (\partial_1\xi,\dots,\partial_{d-1}\xi)^\top$ denotes the lateral gradient. In the following we will use the notation $\Omega_1^\pm = \Omega_1^+\cup\Omega_1^-$ as well as the definition $B^\pm(x) = B^+$ if $y\in\Omega_1^+$ and $B^\pm(x) = B^-$ if $y\in\Omega_1^-$ (and analogously for $a^\pm$) for notational simplicity.

In particular, we have for the transformed Fischer information and $\nu = \rho\lambda_\varepsilon$ the formula

$$\begin{aligned}\mathcal{R}^*_\varepsilon\big(\nu;-\mathrm{D}\mathcal{E}_\varepsilon(\nu)\big) &= \frac{1}{2}\int_{\Omega_1}\Big\{B_\varepsilon(x)\nabla'\log\rho\cdot\nabla'\log\rho + a_\varepsilon(x)\Phi'_\varepsilon(z)^2|\partial_z\log\rho|^2\Big\}\,\mathrm{d}\nu \\ &= \frac{1}{2}\int_{\Omega_1^\pm}\Big\{\frac{B^\pm(x)\nabla'\rho\cdot\nabla'\rho}{\rho} + \frac{a^\pm(x)|\partial_z\rho|^2}{\varepsilon^2\rho}\Big\}\,\mathrm{d}\lambda_\varepsilon \\ &\quad + \frac{1}{2}\int_{\Omega_1^0}\Big\{\frac{\varepsilon^\gamma B^0\nabla'\rho\cdot\nabla'\rho}{\rho} + \frac{a^0|\partial_z\rho|^2}{\varepsilon^\delta\rho}\Big\}\,\mathrm{d}\lambda_\varepsilon\end{aligned}$$

The primal dissipation potential $\mathcal{R}_\varepsilon$, defined via the Legendre transform of $\mathcal{R}^*_\varepsilon$, takes the form

$$\mathcal{R}_\varepsilon(\nu;\dot\nu) = \frac{1}{2}\int_{\Omega_1^\pm}\Big\{B^\pm(x)v'\cdot v' + \frac{a^\pm(x)}{\varepsilon^2}|v_d|^2\Big\}\,\mathrm{d}\nu + \frac{1}{2}\int_{\Omega_1^0}\Big\{\varepsilon^\gamma B^0 v'\cdot v' + \frac{a^0}{\varepsilon^\delta}|v_d|^2\Big\}\,\mathrm{d}\nu,$$

where the rate $\dot\nu$ and the velocity field $v = (v', v_d)^\top \in \mathrm{L}^2(\Omega_1;\mathrm{d}\nu)^d$ satisfy the kinetic relation

$$\begin{aligned}\langle\dot\nu,\varphi\rangle = \int_{\Omega_1^\pm}\Big\{B^\pm(x)v'\cdot\nabla'\varphi + \frac{a^\pm(x)}{\varepsilon^2}v_d\,\partial_z\varphi\Big\}\,\mathrm{d}\nu \\ + \int_{\Omega_1^0}\Big\{\varepsilon^\gamma B^0 v'\cdot\nabla'\varphi + \frac{a^0}{\varepsilon^\delta}v_d\,\partial_z\varphi\Big\}\,\mathrm{d}\nu\end{aligned} \tag{19}$$

for all $\varphi\in C^1_{\mathsf{pw}}(\overline{\Omega}_1) := \{\tilde\varphi\in\mathrm{C}^0(\overline{\Omega}_1)\,|\,\tilde\varphi|_{\Omega_1^\pm}\in\mathrm{C}^1(\Omega_1^\pm)$ and $\tilde\varphi|_{\Omega_0^\pm}\in\mathrm{C}^1(\Omega_1^0)\}$.

Remark 2. Note that in the continuity equation (19) we consider test function φ from the larger space $C^1_{\mathsf{pw}}(\overline{\Omega}_1)$ instead of the space $\mathrm{C}^1(\overline{\Omega}_1)$. This is due to the fact, that the transformation S_ε is only Lipschitz continuous and test functions $\hat\varphi$ defined on the domain Ω_ε are mapped to test functions $\varphi = \hat\varphi\circ S_\varepsilon^{-1}$ whose gradient is not defined at $\{y_d = \pm 1/2\}$. However, the measure ν is absolutely continuous if the De Giorgi functional is finite. Thus, (19) is well defined.

Clearly, if $t\mapsto\hat\nu_\varepsilon(t)\in\mathrm{X}_\varepsilon$ is a solution to the gradient flow equation associated with the gradient system $(\mathrm{X}_\varepsilon,\hat{\mathcal{E}}_\varepsilon,\hat{\mathcal{R}}_\varepsilon)$, then the transformed curve $t\mapsto\nu_\varepsilon(t) := (S_\varepsilon)_\#\hat\nu_\varepsilon(t)$ is a solution to the gradient flow equation induced by $(\mathrm{X}_1,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$. In particular, it satisfies De Giorgi's $(\mathcal{R}_\varepsilon,\mathcal{R}^*_\varepsilon)$-formulation for $t>0$

$$\mathcal{E}_\varepsilon(\nu_\varepsilon(t)) + \mathfrak{D}_\varepsilon(\nu_\varepsilon;[0,t]) = \mathcal{E}_\varepsilon(\nu_\varepsilon(0)), \tag{20}$$

where the De Giorgi functional is given via $\mathfrak{D}_\varepsilon(\nu;[a,b]) = \int_a^b \mathcal{R}_\varepsilon\oplus\mathcal{R}^*_\varepsilon\,\mathrm{d}t$ as before (see (8))

In the sequel, we establish the evolutionary Γ-convergence and the EDP-convergence of the rescaled gradient system $(\mathrm{X}_1,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$. However, first we identify the topologies that are used in the computation of the Γ-limits of the entropy and De Giorgi functional by deriving a priori estimates.

4.2 A priori estimates

The crucial point is to establish uniform bounds for the rescaled measures $\nu_\varepsilon/\varepsilon^\delta$ in the middle layer Ω_1^0. In the following we only use the gradient structure of the problem and work in the space of measures. In particular, we do not need L^∞-bounds on the densities.

We assume wellprepared initial conditions, i.e., the initial measures $\nu_\varepsilon(0) = \nu_\varepsilon^0 \in \mathrm{X}_1$ satisfy $\nu_\varepsilon^0 \rightharpoonup \nu_0^0$ in $\mathrm{Prob}(\overline{\Omega}_1)$ and $\mathcal{E}_\varepsilon(\nu_\varepsilon) \to \mathcal{E}_0(\nu_0^0) < \infty$, where the Γ-limit $\mathcal{E}_0$ is given via

$$\mathcal{E}_0(\nu) = \begin{cases} \displaystyle\int_{\Omega_1} E_{\mathsf{B}}\big(\rho(x)\big)\,\mathrm{d}\lambda_0 & \text{if } \nu = \rho\lambda_0, \\ +\infty & \text{otherwise.} \end{cases}$$

In particular, since λ_0 vanishes in Ω_1^0 so must every $\nu \in \mathrm{X}_1$ with finite relative entropy.

Without loss of generality we can assume that $\sup_{\varepsilon>0} \mathcal{E}_\varepsilon(\nu_\varepsilon(0)) < \infty$ such that with (20) we immediately obtain for $T > 0$

$$\sup_{\varepsilon>0} \sup_{t\in[0,T]} \mathcal{E}_\varepsilon(\nu_\varepsilon(t)) < \infty \quad \text{and} \quad \sup_{\varepsilon>0} \mathfrak{D}_\varepsilon(\nu_\varepsilon;[0,T]) < \infty, \tag{21}$$

Lemma 4.1. *Assume that the family of curves $t \mapsto \nu_\varepsilon(t) = \rho_\varepsilon(t)\lambda_\varepsilon \in \mathrm{X}_1$ satisfies the uniform bounds* (21). *Then, ρ_ε satisfies the estimates*

$$\int_0^T\!\!\int_{\Omega_1^0} \big(|\rho_\varepsilon| + |\partial_z\rho_\varepsilon|\big)\,\mathrm{d}x\,\mathrm{d}t < C \quad \textit{and} \quad \int_0^T\!\!\int_{\Omega_1^\pm} \Big(|\nabla'\rho_\varepsilon| + \frac{|\partial_z\rho_\varepsilon|}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}t < C \tag{22}$$

with a constant $C > 0$ independent of ε. Moreover, the family $\{\nabla\rho_\varepsilon|_{\Omega_1^\pm}\}_{\varepsilon>0} \subset \mathrm{L}^1([0,T]\times\Omega_1^\pm)$ is equi-integrable.

Proof. For an arbitrary $\tilde\rho \in \mathrm{C}^1(\Omega_1)$ with $\tilde\rho \geq 0$ and $y \in \Sigma$, $-3/2 < z_1, z_2 < 3/2$ we have

$$\tilde\rho(y,z_2) - \tilde\rho(y,z_1) = \int_{z_1}^{z_2} \partial_z\,\tilde\rho(y,z)\,\mathrm{d}z.$$

Thus, integrating first over $y \in \Sigma$ and then over $z_1 \in I_1^0 = [-1/2, 1/2]$ and $z_2 \in I_1^+ \cup I_1^-$ leads to the estimate

$$2\int_{\Omega_1^0} \tilde\rho(x)\,\mathrm{d}x \leq \int_{\Omega_1^\pm} \tilde\rho(x)\,\mathrm{d}x + 2\int_{\Omega_1} |\partial_z\tilde\rho(x)|\,\mathrm{d}x. \tag{23}$$

Using Young's inequality, we arrive at the estimate

$$\int_{\Omega_1} |\partial_z\tilde\rho(x)|\,\mathrm{d}x = \int_{\Omega_1} \frac{|\partial_z\tilde\rho(x)|}{\sqrt{\tilde\rho(x)}}\sqrt{\tilde\rho(x)}\,\mathrm{d}x \leq \frac{1}{2}\int_{\Omega_1} \frac{|\partial_z\tilde\rho(x)|^2}{\tilde\rho(x)}\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_1} \tilde\rho(x)\,\mathrm{d}x$$

Hence, with (23) we obtain the estimate

$$\int_{\Omega_1^0} \tilde\rho(x)\,\mathrm{d}x \leq 2\int_{\Omega_1^\pm} \tilde\rho(x)\,\mathrm{d}x + \int_{\Omega_1} \frac{|\partial_z\tilde\rho(x)|^2}{\tilde\rho(x)}\,\mathrm{d}x.$$

Obviously, this estimate also holds for almost every $t \in [0,T]$ for the reletive densities $\rho_\varepsilon(t)$ of the measures ν_ε, hence, after integration over time $t \in [0,T]$ we get

$$\int_0^T \int_{\Omega_1^0} \rho_\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t \leq C\big(1 + \mathfrak{D}_\varepsilon(\nu_\varepsilon;[0,T])\big),$$

where we also used the mass conservation in (18) and the lower bound for the rescaled reference density W_ε in (17).

Finally, the estimate (22) for $\partial_z\rho_\varepsilon/\varepsilon$ and the equi-integrability of $\nabla\rho_\varepsilon$ on $\Omega_1^\pm$ follows from standard arguments, see e.g., [Lis09]. ☐

Remark 3. Considering the measures $N_\varepsilon := \rho_\varepsilon\mathcal{L}^1|_{[0,T]}\otimes\Lambda_\varepsilon \in \mathcal{M}([0,T]\times\overline{\Omega}_1)$ as well as $G_\varepsilon := \partial_z\rho_\varepsilon\mathcal{L}^1|_{[0,T]}\otimes\Lambda_\varepsilon \in \mathcal{M}([0,T]\times\overline{\Omega}_1)$, Lemma 4.1 gives the relative compactness of N_ε and G_ε in $\mathcal{M}([0,T]\times\overline{\Omega}_1)$.

Next, we prove uniform estimates for the solutions of the gradient flow equation that allow us to pass to the limit for every $t\in[0,T]$. However, due to the behavior of ν_ε on the middle layer Ω_1^0, we cannot expect uniform estimates in the space of absolutely continuous curves in the 2-Wasserstein space as e.g. in [AMP$^+$12].

Due to the second estimate in (22), the weak limit of ρ_ε is constant in the vertical direction in the upper and lower layers Ω_1^+ and Ω_1^-, respectively. Hence, we define the reduction map $R:\overline{\Omega}_1\to\overline{\Omega}_1^0$ via

$$R(y,z)=\begin{cases}(y,1/2) & \text{for } z\in[1/2,3/2],\\ (y,z) & \text{for } z\in\,]-1/2,1/2[\\ (y,-1/2) & \text{for } z\in[-3/2,-1/2].\end{cases}$$

By considering the push-forward of measures $\nu\in\mathrm{Prob}(\overline{\Omega}_1)$ under the map R we arrive at reduced measures $\eta:=R_\#\nu\in\mathrm{Prob}(\overline{\Omega}_1^0)$ for which we will consider the following decomposition

$$\eta:=R_\#\nu=\eta^+\otimes\delta_{1/2}+\eta^0+\eta^-\otimes\delta_{-1/2}, \tag{24}$$

where $\eta^+,\eta^-\in\mathcal{M}(\Sigma)$ with $\eta^+(A)=\nu(A\times\overline{I}_1^+)$ and $\eta^-=\nu(A\times\overline{I}_1^-)$ for a Borel set $A\subset\Sigma$ and $\eta^0=\nu|_{\Omega_1^0}\in\mathcal{M}(\Omega_1^0)$.

We prove uniform pointwise BV regularity of the curve $t\mapsto\eta_\varepsilon(t)=R_\#\nu_\varepsilon(t)$, where ν_ε is a curve with bounded De Giorgi functional. Using a Helly-type argument we obtain a weak* limit η_0 such that $\eta_\varepsilon(t)\rightharpoonup^*\eta_0(t)$ for every $t\in[0,T]$. Moreover, continuity of the limiting curve $t\mapsto\eta_0(t)$ is concluded a posteriori using the representation of $\dot\eta_0$ via the limiting continuity equation.

Lemma 4.2. *Let $t\mapsto\nu_\varepsilon(t)\in\mathrm{X}_1$ be such that*

$$\sup_\varepsilon\sup_{t\in[0,T]}\mathcal{E}_\varepsilon(\nu_\varepsilon(t))<\infty \quad\textit{and}\quad \sup_\varepsilon\mathfrak{D}_\varepsilon(\nu_\varepsilon;[0,T])<\infty.$$

Then the total variation of the reduced measures $t\mapsto\eta_\varepsilon(t)=R_\#\nu_\varepsilon(t)$ with respect to the 1-Wasserstein metric $\mathcal{W}_1$ on $\mathrm{Prob}(\overline{\Omega}_1^0)$ given via

$$\mathrm{Var}_{\mathcal{W}_1}(\eta_\varepsilon;[0,T]):=\sup\Big\{\sum_{j=1}^N\mathcal{W}_1\big(\eta_\varepsilon(t_j),\eta_\varepsilon(t_{j-1})\big)\,\Big|\,0=t_0<\ldots<t_N=T\Big\}$$

is uniformly bounded, i.e., $\sup_{\varepsilon>0}\mathrm{Var}_{\mathcal{W}_1}(\eta_\varepsilon;[0,T])<\infty$.

Proof. We exploit the well known dual formulation of the 1-Wasserstein distance in terms of 1-Lipschitz continuous function [AGS05], i.e., for probability measures $\eta_1,\ \eta_2\in\mathrm{Prob}(\overline{\Omega}_1^0)$ we have that

$$\mathcal{W}_1(\eta_1,\eta_2)=\sup\Big\{\int_{\overline{\Omega}_1^0}\varphi(x)\,\mathrm{d}\eta_1(x)-\int_{\overline{\Omega}_1^0}\varphi(x)\,\mathrm{d}\eta_2(x)\,\Big|\,\varphi\in\mathrm{C}^{\mathsf{Lip}}(\overline{\Omega}_1^0),\ \mathsf{Lip}(\varphi)\le 1\Big\}.$$

For a given $\varphi \in \mathrm{C}^{\mathsf{Lip}}(\overline{\Omega}_1^0)$ with $\mathsf{Lip}(\varphi) \leq 1$ let us denote by $\bar{\varphi}$ its extension to $\overline{\Omega}_1$, i.e., $\bar{\varphi}(x) := \varphi(R(x))$. Then, by the kinetic relation between $\dot{\nu}_\varepsilon$ and v_ε in (19) we obtain

$$\int_{\overline{\Omega}_1^0} \varphi(x)\,\mathrm{d}\eta_\varepsilon(t_j) - \int_{\Omega_1^0} \varphi(x)\,\mathrm{d}\eta_\varepsilon(t_{j-1}) = \int_{t_{j-1}}^{t_j} \int_{\Omega_1^\pm} B^\pm(x)\nabla'\bar{\varphi}(x)\cdot v_\varepsilon'(t,x)\,\mathrm{d}\nu_\varepsilon(t)\,\mathrm{d}t$$
$$+ \int_{t_{j-1}}^{t_j} \int_{\Omega_1^0} \Big\{ B^0\nabla'\bar{\varphi}(x)\cdot v_\varepsilon'(t,x) + \frac{a^0}{\varepsilon^\delta}\partial_z\bar{\varphi}(x)\,v_{\varepsilon,d}(t,x)\Big\}\,\mathrm{d}\nu_\varepsilon(t)\,\mathrm{d}t.$$

Using the Fenchel–Young inequality, we arrive at the estimate

$$\int_{\Omega_1^0} \varphi(x)\,\mathrm{d}(\eta_\varepsilon(t_j) - \eta_\varepsilon(t_{j-1})) \leq \int_{t_{j-1}}^{t_j} \Big\{ \mathcal{R}_\varepsilon^*(\nu_\varepsilon;\bar{\varphi}) + \mathcal{R}_\varepsilon(\nu_\varepsilon;\dot{\nu}_\varepsilon)\Big\}\,\mathrm{d}t.$$

The time integral of the primal dissipation potential along the curve ν_ε is uniformly bounded by assumption. To estimate the time integral of the slope term, we use that $|\nabla\overline{\varphi}| \leq 1$ almost everywhere in Ω_1 such that

$$\int_{t_{j-1}}^{t_j} \mathcal{R}_\varepsilon^*(\nu_\varepsilon;\overline{\varphi})\,\mathrm{d}t \leq C \int_{t_{j-1}}^{t_j} \int_{\Omega_1} \rho_\varepsilon(t,x)\,\mathrm{d}\Lambda_\varepsilon\,\mathrm{d}t.$$

Hence, we arrive at

$$\sum_{j=1}^N \mathcal{W}_1(\eta_\varepsilon(t_j),\eta_\varepsilon(t_{j-1})) \leq C\bigg(\int_0^T \mathcal{R}_\varepsilon(\nu_\varepsilon(t);\dot{\nu}_\varepsilon(t))\,\mathrm{d}t + \int_0^T \int_{\Omega_1} \rho_\varepsilon(t,x)\,\mathrm{d}\Lambda_\varepsilon\,\mathrm{d}t\bigg).$$

Applying Lemma 4.1 finishes the proof of this lemma. □

The following a priori estimate follows directly from (22) and the assumptions on the coefficients in (11).

Lemma 4.3. *Let $t \mapsto \nu_\varepsilon(t) \in \mathrm{X}_1$ satsify* (22) *then the velocity field $v_\varepsilon = (v_\varepsilon', v_{\varepsilon,d}) : [0,T]\times\Omega_1 \to \mathbb{R}^d$ satisfy*

$$\int_0^T \int_{\Omega_1^\pm} \Big\{ |v_\varepsilon'|^2 + \frac{v_{\varepsilon,d}^2}{\varepsilon^2}\Big\}\,\mathrm{d}\nu_\varepsilon\,\mathrm{d}t + \int_0^T \int_{\Omega_1^0} \Big\{ \varepsilon^{\gamma+\delta}|v_\varepsilon'|^2 + v_{\varepsilon,d}^2 \Big\}\frac{\mathrm{d}\nu_\varepsilon}{\varepsilon^\delta}\,\mathrm{d}t \leq C.$$

4.3 Limit passage

Here we prove the lower Γ-limits for the entropies and the De Giorgi functionals. The construction of the recovery sequence can be found in the thesis [Fre19]. In order to pass to the limit $\varepsilon \to 0$ in (20) we use the a priori estimates in Lemmas 4.1, 4.2, 4.3 to extract converging subsequences. In particular, we find (non-relabeled) subsequences and limits such that

$$\mathcal{L}^1|_{[0,T]} \otimes \nu_\varepsilon \rightharpoonup^* \nu_0 \quad \text{and} \quad N_\varepsilon := \mathcal{L}^1|_{[0,T]} \otimes \rho_\varepsilon\Lambda_\varepsilon \rightharpoonup^* N_0 \quad \text{in } \mathcal{M}([0,T]\times\overline{\Omega}_1). \tag{25}$$

Since $\nu_\varepsilon(t) = N_\varepsilon(t)$ on $\Omega_1^\pm$ we also have that $N_0 = \nu_0$ on $[0,T]\times\Omega_1^\pm$. Moreover, since the relative entropies are superlinear and bounded, there exists $u_0 \in \mathrm{L}^1([0,T]\times\Omega_1)$ such that u_0 is the Lebesgue density of ν_0. With the reference measure λ_ε vanishing on Ω_1^0 we have that $\nu_0 \equiv 0$ and $u_0 \equiv 0$ on Ω_1^0, too.

Exploiting Lemma 4.2, Helly's selection principle (see also [DM09]) gives up to a subsequence a pointwise limit

$$R_{\#}\nu_\varepsilon(t) = \eta_\varepsilon(t) \rightharpoonup^* R_{\#}\nu_0(t) = \eta_0(t) \text{ in } \mathrm{Prob}(\overline{\Omega}_1^0) \text{ for every } t \in [0,T]. \tag{26}$$

We easily see that $\eta_0(t)$ satisfies for all $t \in [0,T]$ the decomposition

$$\eta_0(t) = \eta_0^+(t) \otimes \delta_{1/2} + \eta_0^- \otimes \delta_{-1/2}, \tag{27}$$

where $\eta_0^+(t) \in \mathcal{M}(\Sigma)$ and $\eta_0^-(t) \in \mathcal{M}(\Sigma)$ are absolutely continuous with respect to the Lebesgue measure on Σ and their densities are given by

$$u_0^+(t,y) = \int_{1/2}^{3/2} u_0(t,y,z)\,\mathrm{d}z \quad \text{and} \quad u_0^-(t,y) = \int_{-3/2}^{-1/2} u_0(t,y,z)\,\mathrm{d}z. \tag{28}$$

Moreover, by (22) we have for the relative densities ρ_ε additionally that

$$\rho_\varepsilon|_{(0,T)\times\Omega_1^\pm} \rightharpoonup \rho_0 \quad \text{and} \quad \nabla\rho_\varepsilon|_{(0,T)\times\Omega_1^\pm} \rightharpoonup \nabla\rho_0 \text{ in } \mathrm{L}^1((0,T)\times\Omega_1^\pm) \tag{29}$$

In fact, on the upper and lower layer Ω_1^+ and Ω_1^- the limiting relative densities are constant in the vertical direction as

$$\partial_z\rho_\varepsilon|_{(0,T\times\Omega_1^\pm)} \to 0 \quad \text{in } \mathrm{L}^1((0,T)\times\Omega_1^\pm). \tag{30}$$

Finally, we can assume that also

$$G_\varepsilon := \mathcal{L}^1|_{[0,T]} \otimes (\partial_z\rho_\varepsilon\Lambda_\varepsilon) \rightharpoonup^* G_0 \text{ in } \mathcal{M}([0,T]\times\overline{\Omega}_1). \tag{31}$$

We define the reduced reference measures in $\mathcal{M}(\overline{\Sigma})$ for the upper and lower layer, denoted $\vartheta_0^+ = w_0^+\mathcal{L}^{d-1}$ and $\vartheta_0^- = w_0^+\mathcal{L}^{d-1}$, via their Lebesgue densities w_0^+ and w_0^-, where

$$w_0^+(y) = \int_{1/2}^{3/2} w_0(y,z)\,\mathrm{d}z \quad \text{and} \quad w_0^-(y) = \int_{-3/2}^{-1/2} w_0(y,z)\,\mathrm{d}z. \tag{32}$$

Setting $\vartheta_0 = \vartheta_0^+ \otimes \delta_{1/2} + \vartheta_0^- \otimes \delta_{-1/2} \in \mathrm{Prob}(\overline{\Omega}_1^0)$, such that $\vartheta_0 = \lim_{\varepsilon\to0} R_{\#}\lambda_\varepsilon$, we define the limit entropy functional on $\mathrm{Prob}(\overline{\Omega}_1^0)$

$$\mathcal{E}_0(\eta) = \mathcal{E}_0(\eta^+,\eta^-) := \begin{cases} \displaystyle\int_{\overline{\Omega}_1^0} E_\mathrm{B}(\rho)\,\mathrm{d}\vartheta_0 & \text{if } \eta = \rho\vartheta_0 \\ \infty & \text{else,} \end{cases} \tag{33}$$

In particular, we have the decomposition $\mathcal{E}_0(\eta) = \int_\Sigma E_\mathrm{B}(\rho^+)\,\mathrm{d}\vartheta_0^+ + \int_\Sigma E_\mathrm{B}(\rho^-)\,\mathrm{d}\vartheta_0^-$ for η satisfying $\eta \ll \vartheta_0$.

We arrive at the following lower estimates for the relative entropies $\mathcal{E}_\varepsilon$ and the De Giorgi functional $\mathfrak{D}_\varepsilon$.

Proposition 1 (Lower estimate). *Let the family $t \mapsto \nu_\varepsilon(t)$ and satisfy the convergences in* (25) – (30). *Then, we have the following lower estimates for the relative entropies $\mathcal{E}_\varepsilon$ and the De Giorgi functionals $\mathfrak{D}_\varepsilon$*

$$\forall\, t \in [0,T]: \ \liminf_{\varepsilon\to0} \mathcal{E}_\varepsilon(\nu_\varepsilon(t)) \geq \mathcal{E}_0(\eta_0(t)) \tag{34}$$

$$\liminf_{\varepsilon\to0} \mathfrak{D}_\varepsilon(\nu_\varepsilon;[0,T]) \geq \mathfrak{D}_\mathrm{lat}\big(N_0;[0,T]\big) + \mathfrak{D}_\mathrm{vert}\big((N_0,G_0);[0,T]\big) \tag{35}$$

where the limit of the De Giorgi functional is decomposed into lateral and vertical parts, which are given via

$$\begin{aligned}\mathfrak{D}_{\rm lat}(N_0;[0,T]) &= \frac{1}{2}\int_0^T\int_\Sigma \Big\{B^+v_0^+\cdot v_0^+ + \frac{B^+\nabla'\rho_0^+\cdot\nabla'\rho_0^+}{(\rho_0^+)^2}\Big\}\,\mathrm{d}\eta_0^+\,\mathrm{d}t\\ &\quad+\frac{1}{2}\int_0^T\int_\Sigma \Big\{B^-v_0^-\cdot v_0^- + \frac{B^-\nabla'\rho_0^-\cdot\nabla'\rho_0^-}{(\rho_0^-)^2}\Big\}\,\mathrm{d}\eta_0^-\,\mathrm{d}t \qquad (36)\\ \mathfrak{D}_{\rm vert}\big((N_0,G_0);[0,T]\big) &= \frac{a^0}{2}\int_{[0,T]\times\overline{\Omega}_1^0}\Big\{(\gamma_0^0)^2+\Big|\frac{\mathrm{d}G_0}{\mathrm{d}N_0}\Big|^2\Big\}\,\mathrm{d}N_0.\end{aligned}$$

iff $\partial_z\rho_0\equiv 0$ in $\Omega_1^\pm$ and $+\infty$ otherwise. The triple (v_0^+,v_0^-,γ_0^0), consisting of the vector fields $v_0^\pm(t,y)\in\mathbb{R}^{d-1}$ and the scalar field $\gamma_0^0(t,x)\in\mathbb{R}$, satisfies the reduced continuity equation

$$\begin{aligned}0=\int_0^T\int_\Sigma\Big(\partial_t\varphi^+ + B^+\nabla'\varphi^+\cdot v_0^+\Big)\,\mathrm{d}\eta_0^+\,\mathrm{d}t+\int_0^T\int_\Sigma\Big(\partial_t\varphi^- + B^-\nabla'\varphi^-\cdot v_0^-\Big)\,\mathrm{d}\eta_0^-\,\mathrm{d}t\\ +\int_{[0,T]\times\overline{\Omega}_1^0} a^0\partial_z\varphi(t,x)\gamma_0^0(t,x)\,\mathrm{d}N_0 \qquad (37)\end{aligned}$$

for all $\varphi\in\mathrm{C}^1\big((0,T)\times\Omega_1\big)$ with $(\partial_z\varphi)_{|_{\Omega_1\setminus\Omega_1^0}}\equiv 0$ and $\varphi^\pm(y):=\varphi(y,\pm 1/2)$.

Proof. 1. Lower limit for the entropies. Due to the convexity of $z\mapsto E_{\rm B}(z)$, Jensen's inequality and $E_{\rm B}(z)\geq 0$ lead to the estimate

$$\begin{aligned}\mathcal{E}_\varepsilon(\nu_\varepsilon(t)) &\geq \int_\Sigma E_{\rm B}\Big(\frac{u_\varepsilon^+(t)}{w_\varepsilon^+}\Big)w_\varepsilon^+\,\mathrm{d}y+\int_\Sigma E_{\rm B}\Big(\frac{u_\varepsilon^-(t)}{w_\varepsilon^-}\Big)w_\varepsilon^-\,\mathrm{d}y\\ &=\int_\Sigma E_{\rm B}\Big(\frac{\mathrm{d}\eta_\varepsilon^+(t)}{\mathrm{d}\vartheta_\varepsilon^+}\Big)\,\mathrm{d}\vartheta_\varepsilon^+ + \int_\Sigma E_{\rm B}\Big(\frac{\mathrm{d}\eta_\varepsilon^-(t)}{\mathrm{d}\vartheta_\varepsilon^-}\Big)\,\mathrm{d}\vartheta_\varepsilon^- = \mathcal{E}_0(\eta_\varepsilon),\end{aligned}$$

where the quantities $u_\varepsilon^+,u_\varepsilon^-$ as well as w_ε^+ and w_ε^- are defined as in (28) and (32).

The liminf estimate (34) follows from lower semicontinuity of the relative entropy $(\eta,\vartheta)\mapsto\mathcal{H}(\eta|\vartheta)=\int_\Sigma E_{\rm B}(\,\mathrm{d}\eta/\,\mathrm{d}\vartheta)\,\mathrm{d}\vartheta$ under weak* convergence, which is well known, see e.g., [AGS05, Lemma 9.4.3].

2. Lower limit for the De Giorgi functionals. To prove the lower estimate in (35), we consider first the term arising from the primal dissipation potential, i.e. $\int_0^T\mathcal{R}_\varepsilon(\nu_\varepsilon,\dot\nu_\varepsilon)\,\mathrm{d}t$. Indeed, due to the estimate in 4.3 and $N_\varepsilon\rightharpoonup^* N_0$ in $\mathcal{M}([0,T]\times\overline{\Omega}_1)$, Theorem 5.4.4 in [AGS05] gives the existence of limits $\widetilde{v}_0^+\in\mathrm{L}^2((0,T)\times\Omega_1^+;\mathrm{d}N_0)^{d-1}$, $\widetilde{v}_0^-\in\mathrm{L}^2((0,T)\times\Omega_1^-;\mathrm{d}N_0)^{d-1}$, and $\gamma_0^0\in\mathrm{L}^2((0,T)\times\Omega_1^-;\mathrm{d}N_0)$ such that $v_\varepsilon' N_\varepsilon\rightharpoonup^*\widetilde{v}^\pm N_0$ in $\mathcal{M}([0,T]\times\overline{\Omega}_1^\pm)^{d-1}$ and $v_{\varepsilon,d}N_0\rightharpoonup^*\gamma_0^0 N_0$ in $\mathcal{M}([0,T]\times\overline{\Omega}_1^0)$ as well as the lower estimates

$$\liminf_{\varepsilon\to0}\int_0^T\int_{\Omega_1^\pm} v_\varepsilon'\cdot B^\pm v_\varepsilon'\,\mathrm{d}N_\varepsilon\,\mathrm{d}t\geq\int_{[0,T]\times\overline{\Omega}_1^0}\widetilde{v}_0^\pm\cdot B^\pm\widetilde{v}_0^\pm\,\mathrm{d}N_0, \qquad (38)$$

$$\liminf_{\varepsilon\to0}\int_0^T\int_{\Omega_1^0} a^0|v_{\varepsilon,d}|^2\,\mathrm{d}N_\varepsilon\,\mathrm{d}t\geq\int_{[0,T]\times\overline{\Omega}_1^0} a^0|\gamma_0^0|^2\,\mathrm{d}N_0 \qquad (39)$$

In particular, passing to the limit in the continuity equation (19), we obtain

$$0=\int_{[0,T]\times\overline{\Omega}_1^0}\Big\{\partial_t\varphi+\widetilde{v}^\pm\cdot B^\pm\nabla'\varphi^\pm\Big\}\,\mathrm{d}N_0+\int_{[0,T]\times\overline{\Omega}_1^0} a^0\gamma_0^0\partial_z\varphi\,\mathrm{d}N_0$$

for all $\varphi \in \mathrm{C}^1\big((0,T)\times\Omega_1\big)$ with $(\partial_z\varphi)_{|_{\Omega_1\setminus\Omega_1^0}} \equiv 0$. We easily check, that we can replace $\widetilde{v}_0^\pm$ by the averages

$$v_0^\pm(t,y) = \int_{I_1^\pm} \widetilde{v}_0^\pm(t,y,z)\frac{w_0(y,z)}{w_0^\pm(y)}\,\mathrm{d}z,$$

which solves the continuity equation (37) with respect to the measures $\eta_0^\pm$ defined in (27). Moreover, by Jensen's inequality we have the lower estimate

$$\int_{[0,T]\times\overline{\Omega}_1^0} \widetilde{v}_0^\pm\cdot B^\pm\widetilde{v}_0^\pm\,\mathrm{d}N_0 \geq \int_0^T\int_\Sigma v_0^\pm\cdot B^\pm v_0^\pm\,\mathrm{d}\eta_0^\pm\,\mathrm{d}t. \tag{40}$$

Next, we consider the term in the De Giorgi functionals $\mathfrak{D}_\varepsilon$ arising from the Fischer information, i.e. $\int_0^T \mathcal{R}_\varepsilon^*(\nu_\varepsilon,-\mathrm{D}\mathcal{E}_\varepsilon(\nu_\varepsilon))\,\mathrm{d}t$.

Using the joint convexity of $(\rho,\xi)\mapsto B^\pm\xi\cdot\xi/\rho$ and the convergences in (29) we immediately obtain with a Ioffe-type argument the lower estimate

$$\liminf_{\varepsilon\to0}\frac12\int_0^T\int_{\Omega_1^\pm}\frac{B^\pm\nabla'\rho_\varepsilon\cdot\nabla'\rho_\varepsilon}{\rho_\varepsilon}\,\mathrm{d}\Lambda_\varepsilon\,\mathrm{d}t \geq \frac12\int_0^T\int_{\Omega_1^\pm}\frac{B^\pm\nabla'\rho_0\cdot\nabla'\rho_0}{\rho_0}\,\mathrm{d}\Lambda_0\,\mathrm{d}t.$$

Since $\partial_z\rho_0 \equiv 0$, we can integrate over $z\in I_1^\pm$ to arrive with (38) and (40) at (35) for $\mathfrak{D}_{\mathrm{lat}}$ (the terms containing the vertical derivatives $\partial_z\rho_\varepsilon$ are non-negative and hence can be estimated by 0 from below). Finally, using the second part of Theorem 9.4.3 in [AGS05] we get with (31) and (25)

$$\begin{aligned}\liminf_{\varepsilon\to0}\int_0^T\int_{\Omega_1^0}\frac{a^0}{2}\frac{|\partial_z\rho_\varepsilon|^2}{\rho_\varepsilon}\,\mathrm{d}\Lambda_\varepsilon\,\mathrm{d}t &= \liminf_{\varepsilon\to0}\int_0^T\int_{\Omega_1^0}\frac{a^0}{2}\Big|\frac{\mathrm{d}G_\varepsilon}{\mathrm{d}N_\varepsilon}\Big|^2\,\mathrm{d}N_\varepsilon\\ &\geq \int_{[0,T]\times\overline{\Omega}_1^0}\frac{a^0}{2}\Big|\frac{\mathrm{d}G_0}{\mathrm{d}N_0}\Big|^2\,\mathrm{d}N_0,\end{aligned}$$

which, with (39), is (35) for $\mathfrak{D}_{\mathrm{vert}}$. □

4.4 Identification of the limit gradient structure

In the last subsection, we proved that the limits obtained in (25) – (31) satisfy for $T>0$ the estimate

$$\mathcal{E}_0(\eta_0(T)) + \mathfrak{D}_{\mathrm{lat}}(N_0;[0,T]) + \mathfrak{D}_{\mathrm{vert}}\big((N_0,G_0);[0,T]\big) \leq \mathcal{E}_0(\eta_0(0)). \tag{41}$$

In this section we derive the final form of the limit system. In particular, the evolution on the middle layer Ω_1^0 is given in (41) only via (N_0,G_0) in $\mathfrak{D}_{\mathrm{vert}}$. To arrive at the effective limit system we first identify G_0 to be the vertical derivative of N_0 in a weak sense and second minimize over all profiles $z\mapsto N_0((\cdot,z))$ to obtain the limit dissipation functional describing jumps across the middle layer.

The crucial technique is the disintegration theorem (see also [AGS05, Sect. 5.3]), which allows us to integrate over each fiber $\{y\}\times I_1$ for $y\in\Sigma$. In fact, let us introduce the map $\pi' : [0,T]\times\Omega_1^0\to[0,T]\times\Sigma$ as the projection on the time variable t and the lateral component y of the spatial variable $x=(y,z)$, i.e., $\pi'(t,x)=(t,y)$. With the limiting measure $N_0\in\mathcal{M}([0,T]\times\overline{\Omega}_1^0)$, we associate the averaged measure $\bar\eta = \pi'_\#N_0 \in \mathcal{M}([0,T]\times\overline{\Sigma})$ and consider the related disintegration of N_0, i.e., there is a $\bar\eta$-a.e. uniquely determined family of fiber probability measures $\bar\mu_{t,y}\in\mathrm{Prob}(\overline{I}_1)$ such that for all $\varphi\in\mathrm{C}([0,T]\times\overline{\Omega}_1)$

$$\int_{[0,T]\times\overline{\Omega}_1^0}\varphi(t,x)\,\mathrm{d}N_0(t,x) = \int_{[0,T]\times\overline{\Sigma}}\int_{\overline{I}_1}\varphi(t,(y,z))\,\mathrm{d}\bar\mu_{t,y}(z)\,\mathrm{d}\bar\eta(t,y). \tag{42}$$

Remark 4. If the measure N_0 is absolutely continuous with respect to the Lebesgue measure on $[0,T]\times\overline{\Omega}_1$, i.e., $N_0 = u_0\mathcal{L}^{d+1}$, then we have for the disintegration that

$$\bar\eta = \left(\int_{\bar I_1} u_0(\cdot,\cdot,z)\,\mathrm{d}z\right)\mathcal{L}^d \quad\text{and}\quad \bar\mu_{t,y} = \frac{u_0(t,y,\cdot)}{\int_{\bar I_1} u_0(t,y,z)\,\mathrm{d}z}\mathcal{L}^1.$$

By the boundedness of the De Giorgi functionals $\mathfrak{D}_\varepsilon$, we deduce $\mathrm{W}^{1,1}$-regularity of the fiber measure $\bar\mu_{t,y}$ for $\bar\eta$-a.e. (t,y).

Lemma 4.4. *Let N_0 and G_0 be given by* (25) *and* (31) *such that*

$$\int_{[0,T]\times\overline{\Omega}_1^0} \left|\frac{\mathrm{d}G_0}{\mathrm{d}N_0}\right|^2 \mathrm{d}N_0 < \infty,$$

and let $\int \bar\mu_{t,y}\,\mathrm{d}\bar\eta$ be the disintegration of N_0 as in (42). *Then, for $\bar\eta$-a.e. $(t,y)\in[0,T]\times\overline{\Sigma}$ there exists $\rho_{t,y}\in\mathrm{W}^{1,1}(I_1)$ such that $\bar\mu_{t,y} = \rho_{t,y}W_0(t,y,\cdot)\mathcal{L}^1$ and $\frac{\mathrm{d}G_0}{\mathrm{d}N_0} = \partial_z\rho_{t,y}/\rho_{t,y}$. In particular, we have*

$$\int_{[0,T]\times\overline{\Omega}_1^0} \left|\frac{\mathrm{d}G_0}{\mathrm{d}N_0}\right|^2 \mathrm{d}N_0 = \int_{[0,T]\times\overline{\Sigma}}\int_{I_1^0} \frac{(\partial_z\rho_{t,y})^2}{\rho_{t,y}} W_0\,\mathrm{d}z\,\mathrm{d}\bar\eta. \tag{43}$$

Proof. The relative density ρ_ε satisfies the identity

$$\forall\,\varphi\in\mathrm{C}_c^1((0,T)\times\Omega_1):\quad \int_0^T\int_{\Omega_1}\varphi\partial_z\rho_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_{\Omega_1}\rho_\varepsilon\partial_z\varphi\,\mathrm{d}x\,\mathrm{d}t$$

Using the definition and the convergence of G_ε in (31), of N_ε in (25) and $W_\varepsilon\to W_0$ in $\mathrm{C}(\overline{\Omega_1})$, we arrive for $\varepsilon\to0$ at

$$\int_{[0,T]\times\overline{\Omega}_1}\frac{\varphi}{W_0}\,\mathrm{d}G_0 = -\int_{[0,T]\times\overline{\Omega}_1}\frac{\partial_z\varphi}{W_0}\,\mathrm{d}N_0.$$

Denoting the density of G_0 with respect to N_0 with g_0 and using the disintegration of N_0 as above, we obtain that $\bar\eta$-a.e. in $[0,T]\times\overline{\Sigma}$

$$\int_{\bar I_1}\frac{\varphi g_0}{W_0}\,\mathrm{d}\bar\mu_{t,y} = -\int_{\bar I_1}\frac{\partial_z\varphi}{W_0}\,\mathrm{d}\bar\mu_{t,y}$$

Hence, we have that for $\bar\eta$-a.e. (t,y) the measure $\bar\mu_{t,y}/W_0(t,y,\cdot)$ has a weak derivative in $\mathcal{M}(\bar I_1)$ given by $g_0(t,y,\cdot)/W_0(t,y,\cdot)\bar\mu_{t,y}$. In particular, by [AFP00, Thm 3.30] there exists for $\bar\eta$-a.e. (t,y) a BV-function $\rho_{t,y}: I_1\to\mathbb{R}$ such that $\bar\mu_{t,y}/W_0(t,y,\cdot) = \rho_{t,y}\mathcal{L}^1|_{I_1}$ and the derivative reads $\partial_z\rho_{t,y} = g_0(t,y,\cdot)/W_0(t,y,\cdot)\bar\mu_{t,y}$. However, this means that $\partial_z\rho_{t,y}$ is also absolutely continuous with respect to the Lebesgue measure on I_1 and we infer $\rho_{t,y}\in\mathrm{W}^{1,1}(I_1)$ and $\partial_z\rho_{t,y} = g_0(t,y,\cdot)\rho_{t,y}$.

Clearly, on $I_1^+\cup I_1^-$ we must have $\partial_z\rho_{t,y}\equiv0$ due to (30). On the middle layer given by I_1^0, we conclude that the identity in (43) holds. □

Using the characterization of (G_0,N_0) on $[0,T]\times\overline{\Omega}_1^0$ from the above lemma and choosing in (37) the test function φ such that $\varphi\equiv0$ in $\Omega_1^\pm$ yields

$$\int_{[0,T]\times\overline{\Sigma}}\int_{I_1^0}\gamma_0^0\rho_{t,y}W_0\partial_z\varphi\,\mathrm{d}z\,\mathrm{d}\bar\eta = 0.$$

Hence, we conclude that the vertical flux through the middle layer given by $a^0\gamma_0^0\rho_{t,y}W_0$ is constant in $z \in I_1^0$ for $\overline{\eta}$-a.e. (t,y). Denoting the latter by $\tilde{\kappa}(t,y)$ we can rewrite (37) to get

$$0 = \int_0^T \int_\Sigma \Big(\partial_t\varphi^+ + B^+\nabla'\varphi^+ \cdot v_0^+\Big)\,\mathrm{d}\eta_0^+\,\mathrm{d}t + \int_0^T \int_\Sigma \Big(\partial_t\varphi^- + B^-\nabla'\varphi^- \cdot v_0^-\Big)\,\mathrm{d}\eta_0^-\,\mathrm{d}t \\ + \int_{[0,T]\times\overline{\Sigma}} \tilde{\kappa}(\varphi^+ - \varphi^-)\,\mathrm{d}\overline{\eta}. \quad (44)$$

Moreover, $\mathfrak{D}_{\mathrm{vert}}$ given in (36) can be written as

$$\mathfrak{D}_{\mathrm{vert}}\big((N_0,G_0);[0,T]\big) = \int_{[0,T]\times\overline{\Sigma}} \int_{I_1^0} \left\{ \frac{\tilde{\kappa}^2}{2a^0W_0\rho_{t,y}} + \frac{a^0W_0(\partial_z\rho_{t,y})^2}{2\rho_{t,y}} \right\} \mathrm{d}z\,\mathrm{d}\overline{\eta}. \quad (45)$$

The main structure in the limit model given by $\mathcal{E}_0$ and $\mathfrak{D}_{\mathrm{vert}}$ and $\mathfrak{D}_{\mathrm{lat}}$ is that $\mathfrak{D}_{\mathrm{vert}}$ does not depend on the time derivative $\dot{N}_0|_{[0,T]\times\overline{\Omega}_1^0}$. Moreover, the vertical flux $\tilde{\kappa}$ is constant in z. Hence, typical for Γ-convergence methods, the final step in the derivation of the effective system consists of minimizing (45) over all profiles $\rho_{t,y} : I_1^0 \to \mathbb{R}$ subject to fixed boundary conditions $\rho_{t,y}(\pm 1/2)$: We write the inner integral in (45) as a functional of $\rho_{t,y}$ and denote it by $\mathcal{G}_{\tilde{\kappa}}(\rho_{t,y})$ with $\tilde{\kappa}$ treated as a fixed parameter.

$$\tilde{G}(\tilde{\kappa};r,s) := \min\big\{\mathcal{G}_{\tilde{\kappa}}(\tilde{\rho}) \,\big|\, \tilde{\rho} \in \mathrm{W}^{1,1}(I_1^0) \text{ with } \tilde{\rho}(1/2) = r,\ \tilde{\rho}(-1/2) = s\big\}.$$

It turns out, that this minimization problem can be explicitly solved. We introduce the transmission coefficient via the harmonic mean of W_0 across I_1^0, i.e.

$$K_*(y) = a^0\,\mathrm{harm}_{I_1^0}[W_0]\,(y), \quad \text{where} \quad \mathrm{harm}_{I_1^0}[W_0]^{-1} = \int_{I_1^0} \frac{1}{W_0(\cdot,z)}\,\mathrm{d}z. \quad (46)$$

In particular, for a large barrier $V_0(\cdot,z)$ across the middle layer, the transmission coefficient becomes A_* beomes indeed small.

We refer to [LMPR17, Proposition A.2] for a proof of the following Lemma.

Lemma 4.5. *For fixed $r \geq 0,\ s \geq$ and $\tilde{k} \in \mathbb{R}$ we have the identity*

$$\tilde{G}(\tilde{k};r,s) = K_*\sqrt{rs}\,\mathscr{C}\Big(\frac{\tilde{k}}{K_*\sqrt{rs}}\Big) + K_*\sqrt{rs}\,\mathscr{C}^*\Big(\log\frac{r}{s}\Big)$$

where $\mathscr{C}^*$ is the Legendre transform of $\mathscr{C}$ and given by $\mathscr{C}^*(\xi) = 4\big(\cosh(\xi/2) - 1\big)$.

With the transformation $k := \tilde{k}/(K_*\sqrt{rs})$, we can rewrite the above result as

$$G(k;r,s) := \tilde{G}(\tilde{k};r,s) = K_*\sqrt{rs}\Big(\mathscr{C}(k) + \mathscr{C}^*\big(\log(r/s)\big)\Big).$$

Note that $(\mathscr{C}^*)'(\xi) = 2\sinh(\xi/2) = \mathrm{e}^{\xi/2} - \mathrm{e}^{-\xi/2}$. In particular, with this we compute the crucial identity

$$\sqrt{rs}\,(\mathscr{C}^*)'\big(\log(r/s)\big) = \sqrt{rs}\,\Big(\sqrt{\frac{r}{s}} - \sqrt{\frac{s}{r}}\Big) = r - s \quad (47)$$

We show the following identity for the traces of the measure N_0 at the interfaces $z_\pm = \pm 1/2$

$$\sqrt{\rho_{t,y}(z_+)\rho_{t,y}(z_-)}\mu = \sqrt{\rho_0(\cdot,z_+)\rho_0(\cdot,z_-)}\mathcal{L}|_{[(0,T)\times\Sigma]},$$

i.e., Lemma 4.5 leads to the geometric mean $\mathrm{GM}(\eta^+,\eta^-) \in \mathcal{M}(\Sigma)$ of positive measures $\eta^+ \in \mathcal{M}(\Sigma)$ and $\eta^- \in \mathcal{M}(\Sigma)$. It is defined via

$$\mathrm{GM}(\eta^+,\eta^-) := \sqrt{\frac{\mathrm{d}\eta^-}{\mathrm{d}(\eta^+ + \eta^-)}\frac{\mathrm{d}\eta^+}{\mathrm{d}(\eta^+ + \eta^-)}}(\eta^+ + \eta^-).$$

Lemma 4.6. *Let the assumptions of Lemma 4.4 hold true. Then, we have*

$$\sqrt{\rho_{\cdot,\cdot}(z^+)\rho_{\cdot,\cdot}(z^-)}\bar\eta = \sqrt{\rho^+\rho^-}\mathcal{L}^d \quad \textit{and} \quad \rho^+(t,y)\rho_{t,y}(z^-) = \rho^-(t,y)\rho_{t,y}(z^+).$$

Proof. With the same arguments as in the proof of Lemma 4.4, we find $\rho^1_{t,y} \in \mathrm{W}^{1,1}(I_1)$ such that $N_0 = (\rho^1_{t,y}W_0(t,y,\cdot)\mathcal{L}^1) \otimes \bar\eta^1$ with $\bar\eta^1 = \pi'_\# N_0 \in \mathcal{M}([0,T]\times\overline{\Sigma})$, where $\pi'(t,x) = (t,y)$ for $x \in \overline{\Omega}_1$. By taking the one sided limits we obtain

$$\lim_{h\to 0}\frac{1}{h}N_0(\mathcal{A}\times[z^+ - h, z^+]) = \lim_{h\to 0}\frac{1}{h}N_0(\mathcal{A}\times[z^+, z^+ + h]) = \int_{\mathcal{A}} \rho^1_{t,y}(z^+)W_0(y,z^+)\,\mathrm{d}\bar\eta^1.$$

Moreover, by Lemma 4.4 we also find

$$\lim_{h\to 0}\frac{1}{h}N_0(\mathcal{A}\times[z^+ - h, z^+]) = \int_{\mathcal{A}} \rho_{t,y}(z^+)W_0(y,z^+)\,\mathrm{d}\bar\eta$$

and

$$\lim_{h\to 0}\frac{1}{h}N_0(\mathcal{A}\times[z^+, z^+ + h]) = \int_{\mathcal{A}} \rho^+(t,y)W_0(y,z^+)\,\mathrm{d}(y,t).$$

Since $W_0 \in \mathrm{C}(\overline{\Omega}^1_1)$, we conclude $\rho_{\cdot,\cdot}(z^+)\bar\eta = \rho^+\mathcal{L}^d$. Similarly, we conclude also for the other interface that $\rho_{\cdot,\cdot}(z^-)\bar\eta = \rho^-\mathcal{L}^d$.

We denote $\bar\eta^\pm := \rho_{\cdot,\cdot}(z^\pm)\bar\eta$ and compute

$$\sqrt{\rho_{\cdot,\cdot}(z^+)\rho_{\cdot,\cdot}(z^-)}\bar\eta = \mathrm{GM}(\bar\eta^+,\bar\eta^-) = \mathrm{GM}(\rho^+\mathcal{L}^d, \rho^-\mathcal{L}^d) = \sqrt{\rho^+\rho^-}\mathcal{L}^d.$$

The relation $\rho^\pm(t,y)\rho_{t,y}(z^\mp) = \rho^\mp(t,y)\rho_{t,y}(z^\pm)$ follows from

$$\rho^+\rho_{\cdot,\cdot}(z^-)\mathcal{L}^d = \rho_{\cdot,\cdot}(z^+)\rho_{\cdot,\cdot}(z^-)\bar\eta = \rho^-\rho_{\cdot,\cdot}(z^+)\mathcal{L}^d.$$

□

Introducing the new variable $\kappa(t,y) = \tilde\kappa(t,y)/\big(K_*(y)\sqrt{\rho_{t,y}(z_+)\rho_{t,y}(z_-)}\big)$, the continuity equation (44) reads

$$\begin{aligned} 0 = \int_0^T\int_\Sigma \big(\partial_t\varphi^+ + \nabla'\varphi^+\cdot B^+v_0^+\big)\,\mathrm{d}\eta_0^+\,\mathrm{d}t + \int_0^T\int_\Sigma \big(\partial_t\varphi^- + \nabla'\varphi^-\cdot B^-v_0^-\big)\,\mathrm{d}\eta_0^-\,\mathrm{d}t \\ + \int_0^T\int_\Sigma K_*\kappa(\varphi^+ - \varphi^-)\sqrt{\rho_0^-\rho_0^+}\,\mathrm{d}y\,\mathrm{d}t \end{aligned} \tag{48}$$

for all $\varphi \in \mathrm{C}^1\big((0,T)\times\Sigma;\mathbb{R}^2\big)$ with $\varphi(0,\cdot)_{|\Sigma} = \varphi(T,\cdot)|_\Sigma = 0$.

In the following, we show continuity of $t \mapsto \eta_0(t)$ and that it has a time derivative in a suitable sense. Indeed, with (48) we conclude continuity and weak differentiability of η_0 as follows. We pass to the limit in (19) with fixed $\varphi \in \mathrm{C}^1(\Omega_1)$ such that $(\partial_z\varphi)_{|_{\Omega_1\setminus\overline{\Omega}_1^0}} \equiv 0$ and obtain for $0 \leq t_1 < t_2$ that

$$\begin{aligned}\big\langle \eta_0(t_2) - \eta_0(t_1), \varphi\big\rangle = \int_{t_1}^{t_2}\int_\Sigma \nabla'\varphi^+ \cdot B^+ v_0^+ \,\mathrm{d}\eta_0^+ \,\mathrm{d}t + \int_{t_1}^{t_2}\int_\Sigma \nabla'\varphi^- \cdot B^- v_0^- \,\mathrm{d}\eta_0^- \,\mathrm{d}t \\ + \int_{t_1}^{t_2}\int_\Sigma K_*\kappa(\varphi^+ - \varphi^-)\sqrt{\rho_0^-\rho_0^+}\,\mathrm{d}y\,\mathrm{d}t.\end{aligned}$$

Hence, we infer that $\|\eta_0(t_2) - \eta_0(t_1)\|_* \to 0$ as $(t_2 - t_1) \to 0$, where

$$\|\eta\|_* = \sup\big\{\langle \eta, \varphi\rangle \;\big|\; \|\nabla'\varphi^\pm\|_{\mathrm{L}^\infty(\Sigma)} \leq 1, \|\varphi^+ - \varphi^-\|_{\mathrm{L}^\infty(\Sigma)} \leq 1\big\}.$$

Moreover, we obtain weak differentiability of η_0. To show this, we define X to be the closure of $\mathrm{C}^1\big((0,T)\times\Sigma;\mathbb{R}^2\big)$ with respect to the norm

$$\|\varphi\|_Y = \|\nabla'\varphi^+\|_{\mathrm{L}^2(\eta_0^+,\Sigma_T)} + \|\nabla'\varphi^-\|_{\mathrm{L}^2(\eta_0^-,\Sigma_T)} + \|\varphi^+ - \varphi^-\|_{\mathrm{L}^{\mathscr{C}^*}(\bar\eta_0,\Sigma_T)},$$

where $\Sigma_T := (0,T)\times\Sigma$ and we have set $\bar\eta_0 = \mathrm{GM}(\eta_0^+, \eta_0^-)$ for brevity. By taking the quotient space with respect to the equivalence relation $\sim$ defined via $\varphi_1 \sim \varphi_2 \Leftrightarrow \|\varphi_1 - \varphi_2\|_Y = 0$ we define the Banach space $Y := X/_\sim$. With Hölder's inequality we find that $\dot\eta_0 \in Y^*$. The Orlicz norm is then defined as follows

$$\|f\|_{\mathrm{L}^{\mathscr{C}^*}(\bar\eta_0,(0,T)\times\Sigma)} := \inf\Big\{k > 0 \;\Big|\; \int_{(0,T)\times\Sigma} \mathscr{C}^*(f/k)\,\mathrm{d}\bar\eta_0 \leq 1\Big\}.$$

For an introduction to Orlicz spaces we refer to [RR91]. In addition to $\dot\eta_0 \in Y^*$, we also have $\|\dot\eta_0(t)\|_{Y^*_{\eta_0(t)}} \in \mathrm{L}^1(0,T)$ with

$$\|\varphi\|_{Y_\eta} = \|\nabla'\varphi^+\|_{\mathrm{L}^2(\eta^+,\Sigma)} + \|\nabla'\varphi^-\|_{\mathrm{L}^2(\eta^-,\Sigma)} + \|\varphi^+ - \varphi^-\|_{\mathrm{L}^{\mathscr{C}^*}(\bar\eta,\Sigma)}.$$

This follows from the fact (cf. [RR91]) that

$$\|\kappa\|_{\mathrm{L}^{\mathscr{C}}(\bar\eta_0(t),\Sigma)} \leq 1 + \int_\Sigma \mathscr{C}(\kappa)\,\mathrm{d}\bar\eta_0(t).$$

Note that the continuity equation (48) gives rise to the natural decomposition into lateral and transmission rates $\dot\eta = \dot\eta_v + \dot\eta_\kappa$ where in the distributional sense

$$\langle \dot\eta_v, \varphi\rangle = \int_\Sigma v^- \cdot B^-\nabla'\varphi^-\,\mathrm{d}\eta^- + \int_\Sigma w^+ \cdot B^+\nabla'\varphi^+\,\mathrm{d}\eta^+$$

and

$$\langle \dot\eta_\kappa, \varphi\rangle = \int_\Sigma A_*[V_0]\kappa(\varphi^+ - \varphi^-)\,\mathrm{dGM}(\eta^-, \eta^+)$$

for all $\varphi = (\varphi^-, \varphi^+) \in \mathrm{C}^1(\Sigma;\mathbb{R}^2)$ with the effective coefficient A_* (written here as a nonlinear operator $A_* : \mathrm{L}^\infty(\Omega_1) \to \mathrm{L}^\infty(\Sigma)$)

$$A_*[V_0] := \frac{K_*}{\sqrt{w_0^+ w_0^-}} = \frac{a^0\big(\int_{I_1^0} \mathrm{e}^{V_0(\cdot,z)}\,\mathrm{d}z\big)^{-1}}{\big(\int_{I_1^+} \mathrm{e}^{-V_0(\cdot,z)}\,\mathrm{d}z\big)^{1/2}\big(\int_{I_1^-} \mathrm{e}^{-V_0(\cdot,z)}\,\mathrm{d}z\big)^{1/2}}. \tag{49}$$

Note that, indeed by Corollary 4.6 we have

$$K_* \sqrt{\rho_{\cdot,\cdot}(z_-)\rho_{\cdot,\cdot}(z_+)}\bar{\eta} = K_* \sqrt{\rho_0^+ \rho_0^-}\,\mathcal{L}^d = A_*[V_0]\,\mathrm{dGM}(\eta^-,\eta^+).$$

In particular, it holds that

$$\int_{[0,T]\times\Sigma} G\big(\tilde{\kappa};\rho_{t,y}(1/2),\rho_{t,y}(-1/2)\big)\,\mathrm{d}\bar{\eta} = \int_0^T \int_\Sigma G(\kappa;\rho_0^+,\rho_0^-)\,\mathrm{d}y\,\mathrm{d}t.$$

Moreover, with Lemma 4.5 we obtain that

$$\int_\Sigma G(\kappa;\rho_0^+,\rho_0^-)\,\mathrm{d}y = \mathcal{R}_{\mathrm{memb}}(\eta,\dot{\eta}_\kappa) + \mathcal{R}^*_{\mathrm{memb}}\Big(\eta, -\log\Big(\frac{\rho_0^+}{\rho_0^-}\Big)\Big)$$

with

$$\mathcal{R}_{\mathrm{memb}}(\eta,\dot{\eta}) = \begin{cases} \displaystyle\int_\Sigma A_*[V_0]\mathscr{C}(\kappa)\,\mathrm{dGM}(\eta^-,\eta^+) & \text{if } \dot{\eta} = \dot{\eta}_\kappa, \\ \infty & \text{else.} \end{cases} \tag{50}$$

Since $\mathfrak{D}_{\mathrm{lat}}(N_0;[0,T])$ given in (36) is of $\mathcal{R}_{\mathrm{bulk}} \oplus \mathcal{R}^*_{\mathrm{bulk}}$ form with

$$\mathcal{R}_{\mathrm{bulk}}(\eta,\dot{\eta}) = \begin{cases} \displaystyle\int_\Sigma \frac{1}{2} w^+ \cdot B^+ w^+ \,\mathrm{d}\eta^+ + \int_\Sigma \frac{1}{2} w^- \cdot B^- w^- \,\mathrm{d}\eta^- & \text{if } \dot{\eta} = \dot{\eta}_v, \\ \infty & \text{else,} \end{cases}$$

we conclude that the effective dissipation potential is then given by the inf-convolution of $\mathcal{R}_{\mathrm{memb}}$ and $\mathcal{R}_{\mathrm{bulk}}$, that is

$$\mathcal{R}_{\mathrm{eff}}(\eta,\dot{\eta}) = \inf\big\{\mathcal{R}_{\mathrm{bulk}}(\eta,\dot{\eta}_v) + \mathcal{R}_{\mathrm{memb}}(\eta,\dot{\eta}_\kappa)\,\big|\,\dot{\eta} = \dot{\eta}_v + \dot{\eta}_\kappa\big\}.$$

In particular, we obtain from Proposition 1 the Γ-liminf for the De Giorgi functionals $\mathfrak{D}_\varepsilon$ and as a consequence pE-convergence.

Theorem 4.7. *Let the family of measures* $\nu_\varepsilon \in \mathcal{M}_+([0,T]\times\overline{\Omega}_1)$ *satisfy* (21) *and converge in the sense of* (25) – (31)*. Then, we have the following* Γ*-liminf estimate for the De Giorgi functionals* $\mathfrak{D}_\varepsilon$

$$\liminf_{\varepsilon\to 0} \mathfrak{D}_\varepsilon(\nu_\varepsilon;[0,T]) \geq \mathfrak{D}_{\mathrm{eff}}(\eta_0;[0,T])$$

with

$$\mathfrak{D}_{\mathrm{eff}}(\eta_0;[0,T]) = \int_0^T \mathcal{R}_{\mathrm{eff}}(\eta_0,\dot{\eta}_0) + \mathcal{R}^*_{\mathrm{eff}}\big(\eta_0, -\mathrm{D}\mathcal{E}_0(\eta_0)\big)\,\mathrm{d}t.$$

The remainder of this subsection is devoted to the chain rule, which is used below to conclude the differential inclusion (53) for the limit of solutions. The chain rule is proved by time regularization.

Lemma 4.8 (Chain rule)**.** *For* $t \mapsto \eta(t) \in \mathcal{M}(\overline{\Sigma}) \times \mathcal{M}(\overline{\Sigma})$ *assume*

$$\mathfrak{D}_{\mathrm{eff}}(\eta;[0,T]) < \infty \quad \textit{and} \quad \sup_{t\in[0,T]} |\mathcal{E}_0(\eta(t))| < \infty.$$

Then, the chain rule holds, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_0\big(\eta(t)\big) = \langle \mathrm{D}\mathcal{E}_0\big(\eta(t)\big), \dot{\eta}(t)\rangle \quad \textit{for a.a. } t \in (0,T).$$

Proof. 1. Regularization of entropy functional. For $\alpha > 0$, we replace the Boltzmann entropy density $E_{\mathsf{B}}(z) = z\log z - z + 1$ by $E_{\mathsf{B}}^{(\alpha)}(z) := E_{\mathsf{B}}(z+\alpha)$. By the dominated convergence theorem, we check that for the related driving functionals we have $\mathcal{E}_0^{(\alpha)}(\eta) \to \mathcal{E}_0(\eta)$ as $\alpha \to 0$

Step 2. Time regularization. Fix $0 < t_1 < t_2 < T$ and define $\eta_n(t) \in \mathcal{M}(\overline{\Sigma})\times\mathcal{M}(\overline{\Sigma})$ via convolution of η with the kernel $\delta_n(t) = n\delta(nt)$, where $\delta(t) = 1 - \max\{2|t|, 1\}$, namely

$$\eta_n(t) = \int_0^T \delta_n(\tau - t)\eta(\tau)\,\mathrm{d}\tau \quad \text{for } t \in [t_1, t_2]$$

and constantly extended for $t \in [0,T] \setminus [t_1, t_2]$. Note that due to the boundedness of the De Giorgi functionals we have $\nabla\sqrt{\rho^\pm} \in \mathrm{L}^2(\Sigma_T)$. In particular, $\rho^\pm \in \mathrm{L}^p(\Sigma_T)$ for some $p > 1$. However, this also gives $\eta_n \in \mathrm{L}^p(\Sigma_T;\mathbb{R}^2)$ and $\dot\eta_n \in \mathrm{L}^p(\Sigma_T;\mathbb{R}^2)$. In particular, with [MRS13, Prop. 2.4] it follows that for $0 \le s < t \le T$

$$\int_s^t \int_\Sigma \big\{\dot\eta_n^+ \log(\alpha + \rho_n^+) + \dot\eta_n^- \log(\alpha + \rho_n^-)\big\}\,\mathrm{d}y\,\mathrm{d}r = \mathcal{E}_0^{(\alpha)}\big(\eta_n(t)\big) - \mathcal{E}_0^{(\alpha)}\big(\eta_n(s)\big). \tag{51}$$

Note that if (v^+, v^-, κ) satisfy (48) for η then

$$v_n^\pm(t) = \begin{cases} \dfrac{(v^\pm(t)\eta^\pm) * \delta_n(t)}{\eta_n^\pm(t)} & \text{for } t \in [t_1, t_2], \\ 0 & \text{for } t \in [0,T] \setminus [t_1, t_2] \end{cases}$$

and

$$\kappa_n(t) = \begin{cases} \dfrac{\big(\kappa(t)\sqrt{\eta^+(t)\eta^-(t)}\big) * \delta_n(t)}{\sqrt{\eta_n^+(t)\eta_n^-(t)}} & \text{for } t \in [t_1, t_2], \\ 0 & \text{for } t \in [0,T] \setminus [t_1, t_2] \end{cases}$$

satisfy (48) for η_n.

Step 3. Passing to the limit $\alpha \to 0$. Using the continuity equation and the uniform bound on the De Giorgi functionals, and exploiting that $u/(u + \alpha w)^2 \le 1/u$ for $u > 0$ and $\alpha \mapsto \mathscr{C}^*(\frac12 \log((\rho^+ + \alpha)/(\rho^- + \alpha)))$ is non-increasing, we conclude again by Young's inequality and the dominated convergence theorem that we can pass to the limit $\alpha \to 0$ in the identity (51) to arrive at

$$\begin{aligned} \mathcal{E}_0^{(\alpha)}\big(\eta_n(s)\big) - \mathcal{E}_0^{(\alpha)}\big(\eta_n(t)\big) &= \int_s^t \int_\Sigma \nabla' \log\rho_n^+ \cdot B^+ v_n^+ \,\mathrm{d}\eta_n^+\,\mathrm{d}\tau \\ &\quad + \int_s^t \int_\Sigma \nabla' \log\rho_n^- \cdot B^- v_n^- \,\mathrm{d}\eta_n^-\,\mathrm{d}\tau + \int_s^t \int_\Sigma K_*\kappa(\varphi^+ - \varphi^-)\sqrt{\rho_n^-\rho_n^+}\,\mathrm{d}y\,\mathrm{d}\tau. \end{aligned}$$

Step 4. Passing to the limit $n \to \infty$. We carry out the limit passage in terms of the continuity equation. By convexity of the maps

$$\mathbb{R}^{d-1} \times \mathbb{R}_+ \ni (a,b) \mapsto \left|\frac{a}{b}\right|^2 b \quad \text{and} \quad \mathbb{R}_+^3 \ni (a,b,c) \mapsto \mathscr{C}\left(\frac{a}{\sqrt{bc}}\right)\sqrt{bc}$$

and convexity of the slope term $\eta \mapsto \mathcal{R}_{\mathsf{eff}}(\eta, -\mathrm{D}\mathcal{E}_0(\eta))$, we have the estimate (cf. [AGS05, Lem. 8.1.9, 8.1.10])

$$\mathcal{R}_{\mathrm{eff}}(\eta_n, \dot\eta_n) \le \mathcal{R}_{\mathrm{eff}}(\eta, \dot\eta) \quad \text{and} \quad \mathcal{R}^*_{\mathrm{eff}}\big(\eta_n, -\mathrm{D}\mathcal{E}_0(\eta_n)\big) \le \mathcal{R}^*_{\mathrm{eff}}\big(\eta, -\mathrm{D}\mathcal{E}_0(\eta)\big).$$

Hence, we conclude

$$\int_s^t \langle \dot\eta_n, \log\rho_n\rangle\,\mathrm{d}\tau \to \int_s^t \langle \dot\eta, \log\rho\rangle\,\mathrm{d}\tau,$$

where we used again the dominated convergence theorem. □

5 Discussion

In this section, we discuss the effective limit system obtained in the previous section. In particular, we derive the system of PDEs that is formally equivalent to the evolutionary system given by $(\mathrm{X}, \mathcal{E}_0, \mathcal{R}_{\mathrm{eff}})$. It consists of two drift-diffusion equations for the upper and lower layer coupled by a linear exchange reaction term which models jump processes from the upper to the lower layer and vice versa, see (54).

Moreover, in [MMP18] a stronger notion of EDP-convergence, called tilted EDP-convergence, is introduced. Hence, the question arises whether this stronger convergence also holds in our case. We answer this question in Subsection 5.2.

In the case that we do not have drift, i.e. $V_\varepsilon \equiv 0$, the equation in (9) can be written as a gradient-flow equation with respect to an H^{-1}-type gradient structure with quadratic driving functional. We show that also in this case EDP-convergence can be shown. However, in contrast to the logarithmic entropy functional and the Wasserstein gradient structure, the resulting effective limit gradient system still features a quadratic dissipation potential, see Subsection 5.3. Though the effective PDE system is the same. In particular, there is no unique gradient structure for the limit PDE model. In fact, we show in Subsection 5.4 that also for the logarithmic relative entropy in (33) with potential V_0 a different, quadratic dissipation potential exists which leads to the same evolution equation.

Finally, we connect our limit derivation to recent results for stochastic Markovian jump processes [MPR14]. In particular, we highlight in Subsection 5.5 that our limit problem with non-quadratic dissipation arises in a natural way from large deviation principles, see also [LMPR17].

5.1 Linear drift-diffusion-reaction system

Note that ν_ε satisfy (21). Hence, we conclude up to a subsequence that ν_ε converges in the sense (25) – (31). In particular, for well prepared initial conditions $\nu_\varepsilon(0)$ with $R_\# \nu_\varepsilon(0) \rightharpoonup^* \eta_0(0)$ such that $\mathcal{E}_\varepsilon\big(\nu_\varepsilon(0)\big) \to \mathcal{E}_0\big(\eta_0(0)\big)$ we obtain pointwise convergence $R_\# \nu_\varepsilon(t) \rightharpoonup^* \eta_0(t)$ for all $t \in (0,T)$. Passing to the limit in (20) we obtain

$$\mathcal{E}_0\big(\eta_0(T)\big) + \mathfrak{D}_{\mathrm{eff}}(\eta_0;[0,T]) \leq \mathcal{E}_0\big(\eta_0(0)\big). \tag{52}$$

By the Fenchel–Young estimate for $\mathfrak{D}_\varepsilon$ and by the chain rule (see Lemma 4.8) we conclude equality in (52) and $\mathcal{E}_\varepsilon\big(\nu_\varepsilon(t)\big) \to \mathcal{E}_0\big(\eta_0(t)\big)$ for all $t \in (0,T]$. Moreover, it follows that η_0 satisfies the differential inclusion

$$\dot\eta_0(t) \in \partial\mathcal{R}^*_{\mathrm{eff}}\big(\eta_0(t), -\mathrm{D}\mathcal{E}_0(\eta_0(t))\big) \tag{53}$$

for almost all $t \in (0,T)$. Thus, we conclude EDP-convergence $(\mathrm{X}_1, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ to $(X_0, \mathcal{E}_0, \mathcal{R}_{\mathrm{eff}})$ in the sense of Definition 2.2, where

$$X_0 = \big\{(\eta^-,\eta^+) \in \mathcal{M}_{\geq 0}(\overline{\Sigma}) \times \mathcal{M}_{\geq 0}(\overline{\Sigma}) \,\big|\, \eta^- + \eta^+ \in \mathrm{Prob}(\overline{\Sigma})\big\}.$$

Using (47), we infer that (53) is formally equivalent to the limiting system of PDEs

$$\begin{aligned} \partial_t u_0^+ &= \mathrm{div}'\Big(w_0^+ B^+ \nabla' \frac{u_0^+}{w_0^+}\Big) + A^*\Big(\frac{u_0^+}{w_0^+} - \frac{u_0^-}{w_0^-}\Big), \\ \partial_t u_0^- &= \mathrm{div}'\Big(w_0^- B^- \nabla' \frac{u_0^-}{w_0^-}\Big) + A^*\Big(\frac{u_0^-}{w_0^-} - \frac{u_0^+}{w_0^+}\Big), \end{aligned}$$

where the reference states w_0^+ and w_0^- are defined in (32). In particular, introducing the effective potentials $\bar{V}_0^\pm = -\log(\bar{Z}_0 w_0^\pm)$, with $\bar{Z}_0 = \int_\Sigma (\mathrm{e}^{-\bar{V}_0^+} + \mathrm{e}^{-\bar{V}_0^-})\,\mathrm{d}y$, and assuming $\bar{V}_0^\pm$ to be sufficiently smooth, we arrive at the drift-diffusion-reaction system

$$\begin{aligned} \partial_t u_0^+ &= \mathrm{div}'\Big(B^+\big(\nabla' u_0^+ + u_0^+ \nabla' \bar{V}_0^+\big)\Big) + A^*\Big(\frac{u_0^+}{w_0^+} - \frac{u_0^-}{w_0^-}\Big), \\ \partial_t u_0^- &= \mathrm{div}'\Big(B^-\big(\nabla' u_0^- + u_0^- \nabla' \bar{V}_0^-\big)\Big) + A^*\Big(\frac{u_0^-}{w_0^-} - \frac{u_0^+}{w_0^+}\Big). \end{aligned} \tag{54}$$

On the boundary $\partial\Sigma$, we have no-flux conditions.

5.2 EDP-convergence with tilting

In [MMP18] a stronger version of EDP convergence was introduced, which guarantees the EDP convergence to an effective gradient system with the same effective dissipation potential $\mathcal{R}_{\text{eff}}$ for all "tilts", i.e. perturbations, of the driving functionals $\mathcal{E}_\varepsilon$. The latter is defined via $\mathcal{E}_\varepsilon^\zeta(\nu) = \mathcal{E}_\varepsilon(\nu) - \langle \zeta, \nu \rangle$ for a tilt $\zeta \in \mathrm{X}^*$. The point of introducing the tilts ζ is that the space in which gradient systems are explored is enlarged. In particular, by the arbitrariness of the tilts ζ we can uniquely recover the $(\mathcal{R}_{\text{eff}}, \mathcal{R}_{\text{eff}}^*)$ structure of the Γ-limit $\mathfrak{D}_0^\zeta(\nu;[a,b]) = \int_a^b \mathcal{N}_0(\nu,\dot{\nu},\zeta)\,\mathrm{d}t$ of the tilted De Giorgi functionals

$$\mathfrak{D}_\varepsilon^\zeta(\nu;[a,b]) = \int_a^b \mathcal{R}_\varepsilon(\nu,\dot{\nu}) + \mathcal{R}_\varepsilon^*(\nu, \zeta - \mathrm{D}\mathcal{E}_\varepsilon(\nu))\,\mathrm{d}t.$$

In stochastic fluctuation theory, the system is pushed out of equilibrium by an external force ζ to explore the solutions away from the deterministic limit. In this sense tilts are a counterpart to stochastic fluctuations. The resulting dissipation potential may be different than those obtained by more classical methods, and in some cases better represent the modeling aspects of the limit.

Definition 5.1. We say the generalized gradient system $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ EDP-converges with tilting to $(\mathrm{X}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$ with respect to the sense S_s on X, the sense S_e on $\mathrm{L}^\infty(0,T;\mathrm{X})$ and with respect to the class $\mathfrak{C} \subset \mathrm{X}^*$ of tilts $\zeta \in \mathfrak{C}$ if

(i) $(\mathrm{X}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$ is the evolutionary Γ-limit of $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$

(ii) $\mathcal{E}_\varepsilon \xrightarrow{S_s\text{-}\Gamma} \mathcal{E}_0$ and $\mathfrak{D}_\varepsilon^\zeta \xrightarrow{S_e\text{-}\Gamma} \mathfrak{D}_{\text{eff}}^\zeta$ for all $\zeta \in \mathfrak{C}$

(iii) $\mathfrak{D}_{\text{eff}}^\zeta(u;[a,b]) = \int_a^b \mathcal{R}_{\text{eff}}(u,\dot{u}) + \mathcal{R}_{\text{eff}}^*\big(u, -\mathrm{D}\mathcal{E}_0(u) + \zeta\big)\,\mathrm{d}t.$

Note that this definition is more demanding than Definition 2.2 since the choice of $\mathcal{R}_{\text{eff}}$ must be independent of ζ. In [DFM18, MMP18] it is shown that in general, we may have $\mathcal{M}_0(u,v,\xi) \neq \mathcal{R}(u,v) + \mathcal{R}^*(u,\xi)$ for any dissipation potential $\mathcal{R}$, where $\mathcal{M}_0$ is given by the integrand of the Γ-limit of $\mathfrak{D}_\varepsilon$ Note that Definition 2.2 requires the identity

$$\mathcal{M}_0(u,v,-\mathrm{D}\mathcal{E}(u)) = \mathcal{R}(u,v) + \mathcal{R}^*(u,-\mathrm{D}\mathcal{E}(u))$$

only for the equilibrium driving force $\xi = -\mathrm{D}\mathcal{E}(u)$.

In our setting, the tilt ζ corresponds to the change of the reference measure $\lambda_\varepsilon = w_\varepsilon \mathcal{L}^d$, i.e.,

$$\mathcal{E}_\varepsilon^\zeta(\nu) = \begin{cases} \displaystyle\int_{\Omega_1} E_{\mathsf{B}}\Big(\frac{u\,\mathrm{e}^{-\zeta}}{w_\varepsilon}\Big)\frac{w_\varepsilon}{\mathrm{e}^{-\zeta}}\,\mathrm{d}x & \text{if } \nu = u\mathcal{L}^d, \\ +\infty & \text{otherwise.} \end{cases}$$

We have two choices to introduce the tilt: (i) in the original gradient system in (12) and (14), and (ii) in its transformed counterpart introduced in Subsection 4.1. The limit passage for both cases can be carried out as in the last section. However, since the tilt ζ must not develop a microstructure in the limit $\varepsilon \to 0$, i.e. is not allowed to depend on ε, we obtain $\zeta^+ = \zeta^-$. This means, that the tilt only “sees” the lateral transport in the upper and lower layer but not the coupling of both via $\mathcal{R}_{\mathsf{memb}}$. Hence, we do not have tilted EDP-convergence in this case.

In the second case, the tilt ζ enters the effective coefficient A_* defined in (49) via the harmonic mean on I_1^0 and the arithmetic means on I_1^+ and I_1^- of $W_0 \mathrm{e}^\zeta$. In particular, the dissipation potential $\mathcal{R}_{\mathrm{eff}}$ cannot be chosen indepedent of ζ. Hence, also in this case we do not have tilted EDP-convergence.

However, when restricting to the class of tilts $\zeta \in \mathrm{C}^1(\Omega_1)$ such that $A_*[V_0] = A_*[V_0 + \zeta]$, we have that the effective dissipation potential is independent of ζ and thus the tilted EDP-convergence with respect to this class.

5.3 Effective limit for H^{-1} gradient structures

Let us consider the case $V_\varepsilon \equiv 0$. It is well known, that in this case the diffusion equation in (9) has a gradient structure of H^{-1} type with the state space $\mathrm{X}_\varepsilon = \mathrm{H}^1(\Omega_\varepsilon)^*$. The energy and the dual dissipation potential are given by

$$\hat{\mathcal{E}}_\varepsilon(\hat{u}) = \begin{cases} \dfrac{|\Omega_\varepsilon|}{2} \displaystyle\int_{\Omega_\varepsilon} \hat{u}^2 \, \mathrm{d}\hat{x} & \text{if } \hat{u} \in \mathrm{L}^2(\Omega_\varepsilon), \\ \infty & \text{if } \hat{u} \in \mathrm{H}^1(\Omega_\varepsilon)^* \setminus \mathrm{L}^2(\Omega_\varepsilon), \end{cases}$$

and

$$\hat{\mathcal{R}}^*_\varepsilon(\hat{\xi}) = \frac{1}{2|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} \nabla\hat{\xi} \cdot A_\varepsilon(\hat{x}) \nabla\hat{\xi} \, \mathrm{d}\hat{x}.$$

In relation to the logarithmic gradient systems, we also define the relative density $\hat{\rho} = |\Omega_\varepsilon|\hat{u}$. The equation (9) is the gradient-flow equation induced by $(\mathrm{X}_\varepsilon, \hat{\mathcal{E}}_\varepsilon, \hat{\mathcal{R}}_\varepsilon)$, namely

$$\dot{u}(t) \in \partial\mathcal{R}^*_\varepsilon\big(-\mathrm{D}\mathcal{E}_\varepsilon(u(t))\big).$$

We observe that both, the energy and the dissipation potential are quadratic. In particular, the De Giorgi functional $\hat{\mathfrak{D}}_\varepsilon(u; [a,b]) = \int_a^b \hat{\mathcal{R}}_\varepsilon(\dot{u}) + \hat{\mathcal{R}}^*_\varepsilon(-\mathrm{D}\mathcal{E}_\varepsilon(u)) \, \mathrm{d}t$ is also quadratic. Hence, by the general theory of Γ-convergence (cf. [Bra06, Prop 2.13]), we expect the effective dissipation potential to be quadratic as well in contrast to the Wasserstein case.

As before, we rescale the domain Ω_ε via the map $\Phi_\varepsilon : \Omega_\varepsilon \to \Omega_1$ defined in (15) and introduce the transformed variables $\xi = \hat{\xi} \circ \Phi_\varepsilon^{-1}$ and $u = \varepsilon \mathsf{m}_\varepsilon(\hat{u} \circ \Phi_\varepsilon^{-1})$, where m_ε is defined as in (16). The transformed gradient system reads

$$\mathcal{E}_\varepsilon(u) = \begin{cases} \dfrac{1}{2} \displaystyle\int_{\Omega_1} \dfrac{|\Omega_\varepsilon|}{\varepsilon \mathsf{m}_\varepsilon} u^2 \, \mathrm{d}x & \text{if } u \in \mathrm{L}^2(\Omega_1), \\ \infty & \text{if } u \in \mathrm{H}^1(\Omega_1)^* \setminus \mathrm{L}^2(\Omega_1) \end{cases}$$

with relative density $\rho = (|\Omega_\varepsilon|/(\varepsilon \mathsf{m}_\varepsilon))u$ and

$$\mathcal{R}^*_\varepsilon(\xi) = \frac{1}{2} \int_{\Omega_1^\pm} \frac{\varepsilon}{|\Omega_\varepsilon|} \Big\{ \nabla'\xi \cdot B^\pm \nabla'\xi + \frac{a^\pm}{\varepsilon^2} |\partial_z \xi|^2 \Big\} \, \mathrm{d}x \\ + \frac{1}{2} \int_{\Omega_1^0} \frac{\varepsilon}{|\Omega_\varepsilon|} \Big\{ \varepsilon^{\delta+\gamma} \nabla'\xi \cdot B^0 \nabla'\xi + a^0 |\partial_z \xi|^2 \Big\} \, \mathrm{d}x.$$

In particular, for $\hat{\rho} \in \mathrm{H}^1(\Omega_\varepsilon)$ we have that $\rho \in \mathrm{H}^1(\Omega_1)$ as well while the same does not hold for $\hat{u}$ and u as m_ε is discontinuous. The De Giorgi functionals are defined by inserting the driving force $\xi = -\mathrm{D}\mathcal{E}_\varepsilon(u) = -\rho$, i.e.,

$$\mathfrak{D}_\varepsilon(u;[0,T]) = \int_0^T \Big\{ \mathcal{R}_\varepsilon(\dot{u}) + \mathcal{R}_\varepsilon^*\big(-\mathrm{D}\mathcal{E}_\varepsilon(u)\big) \Big\} \,\mathrm{d}t.$$

Note that the functionals $\mathfrak{D}_\varepsilon$ are quadratic with respect to curves $t \mapsto u(t)$. To derive the limiting gradient system, we use the Sandier-Serfaty approach [Ser11] to evolutionary Γ-convergence. We introduce the limit driving functional given by

$$\mathcal{E}_0(u) = \begin{cases} \dfrac{|\Sigma|}{2} \displaystyle\int_\Sigma (u^-)^2 + (u^+)^2 \,\mathrm{d}y & \text{if } u \in \mathrm{L}^2(\Sigma;\mathbb{R}^2), \\ \infty & \text{if } u \in \mathrm{H}^1(\Sigma;\mathbb{R}^2)^* \setminus \mathrm{L}^2(\Sigma;\mathbb{R}^2). \end{cases}$$

Next, for $\xi = (\xi^+,\xi^-) \in \mathrm{H}^1(\Sigma;\mathbb{R}^2)$, we introduce the effective dual dissipation potential

$$\mathcal{R}^*_{\mathrm{eff}}(\xi) = \frac{1}{2|\Sigma|} \int_\Sigma \Big\{ \nabla'\xi^\pm \cdot B^\pm \nabla'\xi^\pm + a^0(\xi^+ - \xi^-)^2 \Big\} \,\mathrm{d}y.$$

and prove that this gives rise to the limiting gradient structure.

In the following, we denote by $Y_0 \subset \mathrm{L}^2(\Omega_1)$ the space given by

$$Y_0 = \{\xi \in \mathrm{L}^2(\Omega_1) \mid \partial_z \xi \in \mathrm{L}^2(\Omega_1),\, \xi|_{\Omega_1^\pm} \in \mathrm{H}^1(\Omega_1^\pm), \partial_z\xi|_{\Omega_1^\pm} = 0\}.$$

Theorem 5.2. *Let the family of curves $t \mapsto u_\varepsilon(t) \in \mathrm{X}$ be such that and*

$$\sup_{\varepsilon>0} \Big\{ \mathfrak{D}(u_\varepsilon;[0,T]) + \sup_{t\in[0,T]} \mathcal{E}_\varepsilon(t) \Big\} < \infty.$$

Then, we have for all $t \in (0,T]$

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon\big(u_\varepsilon(t)\big) \geq \mathcal{E}_0\big(u(t)\big) \tag{55}$$

and

$$\liminf_{\varepsilon\to 0} \int_0^T \mathcal{R}_\varepsilon(\dot{u}_\varepsilon)\,\mathrm{d}t \geq \int_0^T \mathcal{R}_{\mathrm{eff}}(\dot{u})\,\mathrm{d}t \tag{56}$$

$$\liminf_{\varepsilon\to 0} \int_0^T \mathcal{R}^*_\varepsilon\big(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon(t))\big)\,\mathrm{d}t \geq \int_0^T \mathcal{R}^*_{\mathrm{eff}}\big(-\mathrm{D}\mathcal{E}_0(u(t))\big)\,\mathrm{d}t. \tag{57}$$

Proof. The lower estimate for the driving functionals $\mathcal{E}_\varepsilon$ is straightforward. For $\varepsilon > 0$ we define the average $u_\varepsilon^\pm(t,y) = \int_{I_1^\pm} u_\varepsilon(t,y,z)\,\mathrm{d}z$ and exploit standard estimates allowing us to extract a subsequence such that $u_\varepsilon^\pm(t) \to u_0^\pm(t)$ in $\mathrm{L}^2(\Sigma)$ for each $t \in (0,T]$. Jensen's inequality gives

$$\mathcal{E}_\varepsilon(u_\varepsilon(t)) \geq \frac{|\Omega_\varepsilon|}{2\varepsilon} \int_\Sigma (u_\varepsilon^+(t))^2 + (u_\varepsilon^-(t))^2 \,\mathrm{d}y.$$

For $\varepsilon \to 0$ the $\liminf$ estimate follows. Note that due to the uniform bound on the driving functionals $\mathcal{E}_\varepsilon(u_\varepsilon(t))$ we have that $u_\varepsilon(t)|_{\Omega_1^0} \to 0$ in $\mathrm{L}^2(\Omega_1^0)$.

Next, we prove the lower estimate for the primal dissipation potential in (56). Again by standard estimates, we have that $\dot{u}_\varepsilon \rightharpoonup \dot{u}_0$ in $\mathrm{L}^2(0,T;Y_0^*)$. In particular, there exists a $\xi_0 \in Y_0$ such that

$\dot u_0 = -\mathrm{div}'(B^\pm \nabla' \xi_0)$ in $\Omega_1^\pm$ and $\dot u_0 = -a^0 \partial_z^2 \xi_0$ in Ω_1^0. Since $u_0 \equiv 0$ in $(0,T)\times\Omega_1^0$, we have that ξ_0 is affine in z and thus, we obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0} \int_0^T \mathcal{R}_\varepsilon(\dot u_\varepsilon)\,\mathrm{d}t \geq \frac12 \int_0^T \int_{\Omega_1^\pm} \nabla'\xi_0 \cdot B^\pm \nabla' \xi_0 \,\mathrm{d}t \\ + \frac{a^0}{2}\int_0^T \int_\Sigma (\xi_0(\cdot,z_+) - \xi_0(\cdot,z_-))^2\,\mathrm{d}t = \int_0^T \mathcal{R}_{\mathsf{eff}}(\dot u)\,\mathrm{d}t.\end{aligned}$$

Finally, we show the liminf estimate for the dual part of the De Giorgi functionals. Due to the boundedness of the dissipation, we can find subsequences and a limit ρ_0 such that for $\rho_\varepsilon = |\Omega_1|/(\varepsilon \mathsf{m}_\varepsilon) u_\varepsilon$ we have $\nabla'\rho_\varepsilon|_{(0,T)\times\Omega_1^\pm} \rightharpoonup \nabla'\rho_0|_{(0,T)\times\Omega_1^\pm}$ in $\mathrm{L}^2((0,T)\times\Omega_1^\pm)$ and $\partial_z\rho_\varepsilon \rightharpoonup \partial_z \rho_0$ in $\mathrm{L}^2((0,T)\times\Omega_1)$, where $\partial_z\rho_0 = 0$ in $(0,T)\times\Omega_1^\pm$. Moreover, the limit ρ_0 satisfies $\rho_0(t,y,z) = u_0^\pm(t,y)$ for almost all $(t,y,z)\in(0,T)\times\Omega_1^\pm$. By weak lower semicontinuity and Jensen's inequality we arrive at

$$\begin{aligned}\liminf_{\varepsilon\to 0} \int_0^T \mathcal{R}_\varepsilon^*(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon))\,\mathrm{d}t \geq \frac12 \int_0^T \int_{\Omega_1^\pm} \nabla'\rho_0 \cdot B^\pm \nabla'\rho_0\,\mathrm{d}t \\ + \frac{a^0}{2}\int_0^T \int_{\Omega_1^0} (\partial_z \rho_0)^2\,\mathrm{d}t\end{aligned}$$

Using Jensen's inequality once again with respect to $z\in(-1/2,1/2)$ in the last term gives (57) and concludes the proof. ☐

Since solutions u_ε to the gradient-flow equation associated with $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ are precompact with respect to the topologies used in the proof of Theorem 5.2, we can pass to the limit in the EDB and obtain

$$\mathcal{E}_0\big(u_0(T)\big) + \mathfrak{D}_{\mathrm{eff}}(u_0;[0,T]) \leq \mathcal{E}_0\big(u_0(0)\big) \tag{58}$$

for well prepared initial conditions and $\mathcal{E}_\varepsilon\big(u_\varepsilon(0)\big) \to \mathcal{E}_0\big(u(0)\big)$. Moreover, with the chain rule we conclude equality in (58) and that

$$\mathfrak{D}_\varepsilon(u_\varepsilon;[0,T]) \to \mathfrak{D}_{\mathrm{eff}}(u_0;[0,T]) \quad \text{and} \quad \mathcal{E}_\varepsilon\big(u_\varepsilon(t)\big) \to \mathcal{E}_0\big(u_0(t)\big) \text{ for all } t\in[0,T].$$

In particular, u_0 is a solution to the flow induced by $(\mathrm{H}^1(\Sigma;\mathbb{R}^2)^*, \mathcal{E}_0, \mathcal{R}_{\mathrm{eff}})$. This associated effective PDE system reads as

$$\begin{aligned}\partial_t u^+ &= \mathrm{div}\big(B^+\nabla' u^+\big) - a^0(u^+ - u^-),\\ \partial_t u^- &= \mathrm{div}\big(B^-\nabla' u^-\big) + a^0(u^+ - u^-),\end{aligned}$$

with homogeneous Neumann boundary conditions. Hence, the effective PDE is the same as in (54).

5.4 A quadratic gradient structure for the limit equation

The crucial feature of the effective limit system $(\mathrm{X}, \mathcal{E}_0, \mathcal{R}_{\mathsf{eff}})$ is the non-quadratic dependence of the dissipation potential $\mathcal{R}_{\mathsf{eff}}$ on the thermodynamic driving force ξ. However, it was shown in [Mie11] that for reaction-diffusion systems with reactions following the mass-action law and fulfilling the detailed-balance condition gradient systems with quadratic dissipation potential exist. In particular, with the same driving functional $\mathcal{E}_0$ defined in (33) we introduce for $u = (u^+, u^-)$ and $\xi = (\xi^+, \xi^-)$

$$\mathcal{R}^*_{\mathsf{quad}}(u;\xi) = \frac{1}{2}\int_\Sigma \Big\{ \nabla\xi^+ \cdot B^+\nabla\xi^+ u^+ + \nabla\xi^- \cdot B^-\nabla\xi^- u^- \\ + A_*\Lambda\Big(\frac{u^+}{w_0^+}, \frac{u^-}{w_0^-}\Big)(\xi^+ - \xi^-)^2 \Big\}\,\mathrm{d}y$$

where $\Lambda(a,b) = (a-b)/\log(a/b)$ for $a \neq b$ and $\Lambda(a,a) = 0$ denotes the logarithmic mean of $a \geq 0$ and $b \geq 0$. Indeed, with $\Lambda(a,b)\log(a/b) = (a-b)$ we easily check that the equation $\dot{u} = \partial_\xi \mathcal{R}^*_{\mathsf{quad}}(u; -\log(u/w_0))$ is (formally) equivalent to (54).

5.5 Connection to large deviation principles

The large deviation principles for stochastic processes offer a method to generate gradient structures (see [MPR14]). In particular, the dissipation potential $\mathcal{R}_{\mathsf{memb}}$ in (50) describing the jump across the vanishing middle layer is directly linked to a large deviation principle for a Markovian jump process on a finite state space. Here we briefly recall the results of [MPR14, Section 4.1] (see also [LMPR17, Section 2.4.2]).

We introduce the state space $\mathcal{S} = \{z_+, z_-\}$ and, with K_* from (46) and reference states w^+, w^-, we define the rates $Q_{+-} = K_*/w^+$ and $Q_{-+} = K_*/w^-$ for a jump from z_+ to z_- and vice versa.

Let $X_1(t), X_2(t), \ldots, X_n(t) \in \mathcal{S}$ be independent realizations of the underlying Markov process, and define the associated empirical measure via

$$u^{(n)}(t) = \frac{1}{n}\sum_{j=1}^n \delta_{X_j(t)} \in \mathcal{P}(\mathcal{S}).$$

Under suitable assumptions it can be shown, that the empirical process $u^{(n)}$ satisfies the large deviation principle

$$\text{Prob}\big(u^{(n)}(\cdot) \approx u(\cdot)\big) \underset{n\to\infty}{\sim} \mathrm{e}^{-nI(u)} \quad \text{with} \quad I(u) = \int_0^T \mathscr{L}\big(u(t), \dot{u}(t)\big)\,\mathrm{d}t,$$

where for $\kappa = \dot{u}^+/\sqrt{u^+u^-Q_{+-}Q_{-+}}$ we have

$$\mathscr{L}(u,\dot{u}) = \frac{1}{2}K_*\sqrt{\tfrac{u^+u^-}{w^+w^-}}\big(\mathscr{C}(\kappa) + \mathscr{C}^*(-[\![\log(u/w)]\!])\big).$$

Hence, the tilted EDP-limit derived in Section 4 is consistent with the gradient structure arising from the large deviation principle.

Finally, let us emphasize that the form of the dissipation potential is related to the so-called Marcelin-de Donder kinetics in chemistry, see [Fei72, Def. 3.3], [GKZD00, Eqn. (11)], and [Grm10, Eqn. (69)]. The latter states that chemical reaction rates are given via exponentials of the thermodynamic driving forces ξ. This further highlights that the gradient structure derived in this work has a more physical relevance than the quadratic one described in Subsection 5.4.

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