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An existence result for a class of electrothermal drift-diffusion models with Gauss–Fermi statistics for organic semiconductors

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Abstract

This work is concerned with the analysis of a drift-diffusion model for the electrothermal behavior of organic semiconductor devices. A “generalized Van Roosbroeck” system coupled to the heat equation is employed, where the former consists of continuity equations for electrons and holes and a Poisson equation for the electrostatic potential, and the latter features source terms containing Joule heat contributions and recombination heat. Special features of organic semiconductors like Gauss–Fermi statistics and mobilities functions depending on the electric field strength are taken into account. We prove the existence of solutions for the stationary problem by an iteration scheme and Schauder’s fixed point theorem. The underlying solution concept is related to weak solutions of the Van Roosbroeck system and entropy solutions of the heat equation. Additionally, for data compatible with thermodynamic equilibrium, the uniqueness of the solution is verified. It was recently shown that self-heating significantly influences the electronic properties of organic semiconductor devices. Therefore, modeling the coupled electric and thermal responses of organic semiconductors is essential for predicting the effects of temperature on the overall behavior of the device. This work puts the electrothermal drift-diffusion model for organic semiconductors on a sound analytical basis.

1 Introduction

One of the most significant and technological breakthroughs of the second half of the *twentieth century* is the discovery of *semiconductors*. Nowadays *semiconductors* are found virtually in every electronic device imaginable and their continuous development over the years has led to a new class of materials, commonly known as *organic semiconductors*.

The first model describing drift, diffusion, and reaction processes in semiconductor devices was derived by *Van Roosbroeck* (1950) from physical laws. In many modern electronic devices that employ *organic semiconductors*, such as *organic solar cells*, *organic transistors*, etc. the charge transport properties are significantly influenced by the device’s temperature [11]. Since in organic semiconductor materials the conductivity increases with temperature, self-heating effects caused by the high electric fields and strong recombination have a potent impact on the device’s performance and must be included in mathematical models [34], [21].

Self-heating effects were shown to lead to interesting nonlinear phenomena, like the S-shaped relation between current and voltage resulting in regions where a decrease in voltage across the device resulted in an increase in current through it, commonly denoted as regions of *negative differential resistance* [10], [9]. Moreover, the synergy of self-heating and temperature activated hopping transport in combination with the heat balance results in spatially inhomogeneous current flow and temperature distribution which cause, for instance, spatially inhomogeneous luminance in large-area organic light-emitting diodes (OLEDs) [9], [12].

In [25], a stationary thermistor model based on the heat equation for the temperature coupled to a $p(x)$-Laplace type equation for the electric potential with mixed type boundary conditions has been used to model the electrothermal effects in *organic semiconductors*, where the p-Laplacian described the non-Ohmic electrical behavior of the organic material. In the analysis of the aformentioned thermistor model [16], [4], the electric current and heat flow with a resulting source term from *Joule heat* that depended on the total current, were balanced. Additionally, an Arrhenius-like temperature dependence was taken into account. Numerically the authors of [20] showed that the model is able to capture the spatial inhomogeneities that appear in large area OLEDs.

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Since the establishment of the drift-diffusion model by *Van Roosbroeck* the statistical relation between carrier densities and chemical potentials in *semiconductors* has been of fundamental importance to the mathematical treatment of the model. For classical *semiconductors* equations, the existence (and under certain assumptions uniqueness) of solutions was established provided the behavior of the *semiconductor* device can be described by either the *Boltzmann* statistics or, the physically more realistic, *Dirac-Fermi* statistics [29], [13]. In *organic semiconductors* in contrast, charge transport is realized by hopping of electrons (and holes) between discrete energy levels of molecular sites nearby (see Fig. 1). This hopping transport intensifies as the temperature of the device increases. Organic molecules have two energy states: the *Highest Occupied Molecular Orbital* (HOMO, energy E_H) and the *Lowest Unoccupied Molecular Orbital* (LUMO, energy E_L). The HOMO states describe the electrons in the localized electron pair-bindings between the atoms of the molecule, whereas the LUMO states describe de-localized electrons in the π-bindings. By crossing the HOMO-LUMO-gap (e.g. by optical excitation) electrons in the molecule can change from the HOMO state into the LUMO state. Thereby arises a positively charged cavity in the charge cloud of the molecule which is called a hole. Both types of charge carriers, electrons and holes, can move by hopping transport between energy levels of neighboring molecules. In this respect *organic semiconductor* materials behave like amorphous *semiconductors*. The random alignment

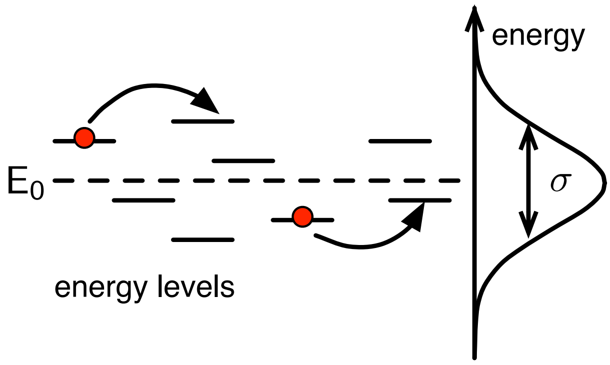

Figure 1.1: Schematic image describing hopping-transport phenonmenon between Gaussian distributed energy levels of neigboring molecules. Then mean energy level is denoted by E_0 and the standard deviation by σ (also called disorder parameter).

of the molecules leads to a disordered system with *Gaussian* distributed energy levels. As a consequence, the statistial relation between carrier densities and chemical potentials cannot be described by the classical *Boltzmann* or *Dirac-Fermi* statistics and instead *Gauss-Fermi* statistics [31] should be used. Additionally, for *organic semiconductors* one has to consider further special features like the dependence of mobility functions on the electric field strength. For a complete analysis of the isothermal stationary drift-diffusion model for *organic semiconductors* based on the *Gauss-Fermi* statistics, the interested reader can refer to the work in [7], while the isothermal, non-stationary case was treated in [17].

In this work, we are interested in a more detailed drift-diffusion-type description of the charge transport balancing currents of both electrons and holes in contrast to the net current flow in the $p(x)$-Laplace thermistor model discussed in [25, 3]. Furthermore, we want to evaluate the heat produced by both types of current flow and the generation-recombination of charge carriers and balance it by means of a heat flow equation. Since the mobilities of the charge carriers and the reaction rate constant depend on the local device temperature, we obtain an electrothermal feedback. For this purpose, we use a drift-diffusion model consisting of a *"generalized Van Roosbroeck"* system for *organic semiconductors* coupled to the heat flow equation. For classical *semiconductors* a similar energy-drift-diffusion model (based on the ideas in [33, 35]) with, in principle, the same model frame was studied in [18]. However, the special features of the drift-diffusion models for organic devices, especially the dependence of the mobility on the electric field strength, will not allow to proceed in the analysis as in [18].

The paper is organized as follows: In section 2 we collect and prove certain properties of the *Gauss–Fermi* statistics, mobility functions, and generation/recombination of electrons and holes. In section 3 we introduce the "*generalized Van Roosbroeck system* for *organic semiconductors*" coupled to the heat flow equation, we describe the notion of weak and entropy solutions, and state the main result of the paper. Section 4 is devoted to proving existence of a solution to the system introduced in section 3 using *Schauder's fixed point theorem*. This is done by first introducing the iteration scheme where we define the fixed point map, and through a series of lemmas we collect relevant assertions for certain solutions of subproblems that help us to verify the continuity of the fixed point map and assert the existence result. Section 5 is allocated to concluding remarks and discussion.

2 Drift-diffusion models for organic semiconductors

2.1 Statistical relation between densities and chemical potentials

In organic semiconductor materials, the energy positions are Gaussian distibuted, such that both, the electrons and holes, can be described by a Gaussian density of state

$$N_{Gauss}(E) = \frac{N_0}{\sqrt{2\pi\sigma^2}} \exp\Big[-\Big(\frac{E-E_0}{\sqrt{2}\sigma}\Big)^2\Big],$$

(see Fig. 1), where N_0 is the total density of transport states. E_0 denotes the corresponding average HOMO- and LUMO-levels, respectively, and σ^2 their variance. σ is also called the disorder parameter which characterizes the disorder of the organic material. For organic semiconductors which do not underly an outer field the density of electrons is given by the Gauss-Fermi integral

$$\begin{aligned} n &= \int_{-\infty}^{\infty} N_{Gauss}(E) \frac{1}{\exp\big(\frac{E-E_F}{k_B T}\big)+1}\,\mathrm{d}E \\ &= \frac{N_{n0}}{\sigma_n\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(E-E_L)^2}{2\sigma_n^2}\Big) \frac{1}{\exp\big(\frac{E-E_F}{k_B T}\big)+1}\,\mathrm{d}E, \end{aligned}$$

where E_F denotes the Fermi energy and the Fermi function $f(E,T) = \big(\exp\big(\frac{E-E_F}{k_B T}\big)+1\big)^{-1}$ gives the probability that an electron is in the quantum state with the energy E. The notation k_B stands for Boltzmann's constant and E_L is the LUMO-energy. Using the variable $\xi = \frac{E-E_L}{\sigma_n}$ we obtain

$$\begin{aligned} n &= \frac{N_{n0}}{\sigma_n\sqrt{2\pi}}\,\sigma_n \int_{-\infty}^{\infty} \exp\Big(-\frac{\xi^2}{2}\Big) \frac{1}{\exp\big(\frac{\sigma_n}{k_B T}\xi - \frac{E_F-E_L}{k_B T}\big)+1}\,\mathrm{d}\xi \\ &= \frac{N_{n0}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\Big(-\frac{\xi^2}{2}\Big) \frac{1}{\exp\big(z_n\xi - \eta_n^0\big)+1}\,\mathrm{d}\xi \\ &= N_{n0}\mathcal{G}\big(\eta_n^0; z_n\big), \qquad \eta_n^0 := \frac{E_F-E_L}{k_B T} \quad z_n := \frac{\sigma_n}{k_B T} \end{aligned} \tag{2.1}$$

with the dimensionless quantities η_n^0 and z_n. Relation (2.1) is valid for homogeneous semiconductors in absence of an outer field. It can be generalized to the situation that in the semiconductor an electric field $-\nabla\psi$ with a spatially weakly varying potential ψ is present. Then the concept of bent bands is applied and the energy level E_L has to be replaced by $E_L - q\psi$ and ξ by $\widetilde{\xi} = \frac{E-E_L+q\psi}{\sigma_n}$ and the electron density n is expressed by

$$\begin{aligned} n &= \frac{N_{n0}}{\sigma_n\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(E-E_L+q\psi)^2}{2\sigma_n^2}\Big) \frac{1}{\exp\big(\frac{E-E_F}{k_B T}\big)+1}\,\mathrm{d}E \\ &= \frac{N_{n0}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\Big(-\frac{\widetilde{\xi}^2}{2}\Big) \frac{1}{\exp\big(\frac{\sigma_n}{k_B T}\widetilde{\xi} - \frac{E_F-E_L+q\psi}{k_B T}\big)+1}\,\mathrm{d}\widetilde{\xi} \\ &= N_{n0}\mathcal{G}\big(\eta_n; z_n\big), \qquad \eta_n := \frac{E_F-E_L+q\psi}{k_B T} = \frac{q(\psi-\varphi_n)-E_L}{k_B T}, \quad z_n := \frac{\sigma_n}{k_B T}. \end{aligned} \tag{2.2}$$

Similarly, using the the HOMO energy E_H, the hole density p is obtained as function of the renormalized chemical potential η_p of the holes,

$$p = N_{p0}\mathcal{G}(\eta_p; z_p), \qquad \eta_p := \frac{E_H - q(\psi - \varphi_p)}{k_B T}, \quad z_p := \frac{\sigma_p}{k_B T}.$$

The densities of transport states N_{i0}, $i = n, p$, the average HOMO- and LUMO-levels E_H and E_L, and the disorder parameters σ_n, σ_p are considered as temperature dependent quantities. We will assume that $N_{i0}(\cdot, T) \le \overline{N_{i0}}$ in the device for all $T > 0$. Since the Fermi function f takes only values between 0 and 1, relation (2.2) ensures

$$0 < n = N_{n0}\mathcal{G}(\eta_n; z_n) < \frac{N_{n0}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\Big(-\frac{\xi^2}{2}\Big)\, \mathrm{d}\xi = N_{n0} \quad \forall \eta_n \in \mathbb{R}, \forall z_n \ge 0$$

such that the carrier density in organic materials remains bounded for all values of η_n. Considering the infinite narrow distribution $\sigma \to 0$, the density of state $N_{Gauss}(E)$ converges to $N_{n0}\delta(E - E_0)$. Let

$$\mathcal{G}(\eta; 0) = \frac{1}{\exp\{-\eta\} + 1} \qquad \Big(= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\Big\{-\frac{\xi^2}{2}\Big\} \frac{1}{\exp\{-\eta\} + 1}\, \mathrm{d}\xi \Big)$$

denote the statistical relation for the density of state $N_{n0}\delta(E - E_0)$. From

$$\begin{aligned} \int_{-\infty}^{\infty} \exp\Big\{-\frac{\xi^2}{2}\Big\} \frac{1}{\exp\{z\xi\} + 1}\, \mathrm{d}\xi &= \int_0^{\infty} \exp\Big\{-\frac{\xi^2}{2}\Big\} \Big(\frac{1}{\exp\{z\xi\} + 1} + \frac{1}{\exp\{-z\xi\} + 1} \Big)\, \mathrm{d}\xi \\ &= \int_0^{\infty} \exp\Big\{-\frac{\xi^2}{2}\Big\}\, \mathrm{d}\xi = \sqrt{\frac{\pi}{2}} \end{aligned}$$

it results $\mathcal{G}(0; 0) = \frac{1}{2}$ and $\mathcal{G}(0; z) = \frac{1}{2}$ for all $z > 0$.

Additionally, the mapping $\eta \mapsto \mathcal{G}(\eta; z)$ is strictly monotone increasing, $\mathcal{G}$ is differentiable w.r.t. η and

$$\frac{d\mathcal{G}}{d\eta}(\eta; z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\Big(-\frac{\xi^2}{2}\Big) \frac{\exp\big(z\xi - \eta\big)}{\big(\exp(z\xi - \eta) + 1\big)^2}\, \mathrm{d}\xi.$$

Note that the fraction in the integrand takes only values between 0 and 1. Therefore

$$\frac{d\mathcal{G}}{d\eta}(\eta; z) \in (0, 1) \quad \text{and} \quad \lim_{\eta \to +\infty} \frac{d\mathcal{G}}{d\eta}(\eta; z) = \lim_{\eta \to -\infty} \frac{d\mathcal{G}}{d\eta}(\eta; z) = 0.$$

Lemma 2.1 *The map $z \mapsto \mathcal{G}(\eta; z)$ is continuously differentiable at all $z > 0$ for all $\eta \in \mathbb{R}$. The derivative follows the estimate*

$$\frac{\partial}{\partial z}\mathcal{G}(\eta; z) \begin{cases} > 0 & \text{if } \eta < 0 \\ = 0 & \text{if } \eta = 0 \\ < 0 & \text{if } \eta > 0 \end{cases} \quad \text{and} \quad \Big| \frac{\partial}{\partial z}\mathcal{G}(\eta; z) \Big| \le \frac{1}{z}(1 + \exp|\eta|) \qquad \forall z > 0, \quad \forall \eta \in \mathbb{R}.$$

Proof. For all $z > 0$ we calculate

$$\frac{\partial}{\partial z}\mathcal{G}(\eta; z) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\Big\{-\frac{\xi^2}{2}\Big\} \frac{\exp(z\xi - \eta)\xi}{(\exp(z\xi - \eta) + 1)^2}\, \mathrm{d}\xi. \tag{2.3}$$

We write

$$\frac{\exp(z\xi - \eta)\xi}{(\exp(z\xi - \eta) + 1)^2} = \frac{\xi \exp(z\xi)}{(\exp(z\xi) + 1)^2} \, \frac{\exp(-\eta)(\exp(z\xi) + 1)^2}{(\exp(z\xi - \eta) + 1)^2}, \tag{2.4}$$

where the second factor is positive and bounded by $1+\exp|\eta|$. To see this, we consider with $a := \exp(z\xi) > 0$ and $b := \exp(-\eta)$ the expression $I := \frac{b(a+1)^2}{(ab+1)^2} = \frac{b(a^2+2a+1)}{a^2b^2+2ab+1}$, which can be estimated form above in the case $b > 1$ by

$$I \leq \frac{b^2a^2 + 2ab + b}{a^2b^2 + 2ab + 1} \leq \frac{b^2a^2 + 2ab + 1 + b}{a^2b^2 + 2ab + 1} \leq 1 + b$$

and in the case $b < 1$ by

$$I = \frac{1}{b}\frac{b^2a^2 + 2ab^2 + b^2}{b^2a^2 + 2ab + 1} \leq \frac{1}{b}\frac{b^2a^2 + 2ab + 1}{b^2a^2 + 2ab + 1} = \frac{1}{b}.$$

Next, we take a look at the absolute value of the first factor in (2.4): Since $\frac{|y \exp y|}{(\exp y+1)^2} < 1$ for $y \in \mathbb{R}$, we estimate $\frac{|\xi \exp(z\xi)|}{(\exp(z\xi)+1)^2} < \frac{1}{z}$. In summary, the second factor in the integrand of (2.3) for all $\xi \in \mathbb{R}$ can be uniformly estimated by a constant depending only on z and η, and the integral in (2.3) exists, meaning that the derivative $\frac{\partial}{\partial z}\mathcal{G}(\eta; z)$ exists for all $z > 0$ and for all $\eta \in \mathbb{R}$.

On the other hand, writing the integral in (2.3) as integral over $(0, \infty)$, we calculate

$$\begin{aligned}
\frac{\partial}{\partial z}\mathcal{G}(\eta; z) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\Big\{-\frac{\xi^2}{2}\Big\}\Big\{-\frac{\exp(z\xi-\eta)\xi}{(\exp(z\xi-\eta)+1)^2} + \frac{\exp(-z\xi-\eta)\xi}{(\exp(-z\xi-\eta)+1)^2}\Big\}\,\mathrm{d}\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\Big\{-\frac{\xi^2}{2}\Big\}\Big\{-\frac{\exp(z\xi-\eta)\xi}{(\exp(z\xi-\eta)+1)^2} + \frac{\exp(z\xi-\eta)\xi}{(\exp(-\eta)+\exp(z\xi))^2}\Big\}\,\mathrm{d}\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\Big\{-\frac{\xi^2}{2}\Big\}\frac{\exp(z\xi-\eta)\xi}{(\exp(z\xi-\eta)+1)^2}\Big[-1+\Big(\frac{\exp(z\xi-\eta)+1}{\exp(-\eta)+\exp(z\xi)}\Big)^2\Big]\,\mathrm{d}\xi.
\end{aligned}$$

For arbitrary $\exp(z\xi) > 1$ we have,

$$\frac{\exp(z\xi-\eta)+1}{\exp(-\eta)+\exp(z\xi)}\begin{cases}\in(0,1) & \text{if } \eta > 0,\\ = 1 & \text{if } \eta = 0,\\ > 1 & \text{if } \eta < 0.\end{cases}$$

Therefore, $\frac{\partial}{\partial z}\mathcal{G}(\eta; z)$ is negative for $\eta > 0$, is zero for $\eta = 0$ and positive for $\eta < 0$. □

2.2 Mobility and recombination laws

Due to the energetic disorder in organic semiconductor materials with Gaussian density of state, charge transport and recombination processes are different from their inorganic counterparts. In particular, the mobility functions μ_n, μ_p show a positive feedback with respect to temperature T, densities n or p, and with respect to electrical field strength $F = |\nabla\psi|$. Numerical solutions of the master equation for hopping transport in organic materials were used e.g. in [32] to determine these dependencies. In case of the electron mobility (and analogously for the hole mobility), [32] obtained the following product form as extension of the Gaussian disorder model

$$\mu_n(T, n, F) = \mu_{n0}(T) \times g_1(n, T) \times g_2(F, T),$$

see also the discussion in [7]. To treat also heterostructures, we additionally take spatial dependence into account. For the analysis we suppose that $\mu_n : \Omega\times(0,\infty)\times[0,\overline{N_{n0}}]\times\mathbb{R}_+ \to \mathbb{R}$, $\mu_p : \Omega\times(0,\infty)\times[0,\overline{N_{p0}}]\times\mathbb{R}_+ \to \mathbb{R}$ are Caratheodory functions fulfilling

$$\begin{aligned}
&\text{for all } \tau > 0 \text{ there exists } \overline{\mu}_\tau \text{ such that } \mu_n(\cdot, T, n, F),\ \mu_p(\cdot, T, p, F) \leq \overline{\mu}_\tau < \infty \\
&\text{for all } (T, n, p, F) \in [\tau, \infty) \times [0, \overline{N_{n0}}] \times [0, \overline{N_{p0}}] \times \mathbb{R}_+ \text{ a.e. in } \Omega, \\
&0 < \underline{\mu} \leq \mu_n(\cdot, T, n, F),\ \mu_p(\cdot, T, p, F) \\
&\text{for all } (T, n, p, F) \in [T_a, \infty) \times [0, \overline{N_{n0}}] \times [0, \overline{N_{p0}}] \times \mathbb{R}_+ \text{ a.e. in } \Omega,
\end{aligned} \tag{2.5}$$

where T_a is the ambient temperature. However, one fundamental difficulty for the analytical treatment remains, namely the dependence of the mobility functions on $F = |\nabla\psi|$.

Concerning the recombination of charge carriers, we follow [8] and write the generation-recombination laws for electrons and holes in the form

$$R = r(\cdot, n, p, T)\Big(1 - \exp\frac{q(\varphi_n - \varphi_p)}{k_\mathrm{B}T}\Big), \tag{2.6}$$

where we suppose for the analytical investigations $r(\cdot, n, p, T) : \Omega \times [0, \overline{N_{n0}}] \times [0, \overline{N_{p0}}] \times (0, \infty) \to \mathbb{R}_+$ is a Caratheodory function such that for all $\tau > 0$ there exists an $\overline{r}_\tau$ with $0 \le r(\cdot, n, p, T) \le \overline{r}_\tau$ for all $(n, p, T) \in [0, \overline{N_{n0}}] \times [0, \overline{N_{p0}}] \times (\tau, \infty)$ and a.a. $x \in \Omega$.

Typically, the function r has the form $r(\cdot, n, p, T) = r_0(\cdot, n, p, T)np$ with $r_0(\cdot, n, p, T) : \Omega \times [0, \overline{N_{n0}}] \times [0, \overline{N_{p0}}] \times (0, \infty) \to \mathbb{R}_+$. The particular form of the rate in (2.6) guarantees compatibility with thermodynamic equilibrium. Especially in equilibrium it reflects that the quasi Fermi levels of electrons and holes have to coincide.

3 Stationary energy model

According to the discussion in the introduction, we consider the following energy-drift-diffusion model for the interplay of electronic and heat transport in organic semiconductor devices. We use the notation from the last section and take into account the electrothermal effects of Joule heating resulting from both electron and hole current, and the reaction heat as source terms in the heat flow equation. A corresponding model frame was already formulated in [33, p. 42, p. 121] for classical semiconductors. The paper [18] deals with analytical investigations of this problem for the classical, inorganic situation. We study the coupled system

$$\begin{aligned}
-\nabla\cdot(\varepsilon_0\varepsilon_r\nabla\psi) &= q(C - n + p),\\
-\nabla\cdot j_n &= -qR, \qquad j_n = -qn\mu_n(T, n, |\nabla\psi|)\nabla\varphi_n,\\
\nabla\cdot j_p &= -qR, \qquad j_p = -qp\mu_p(T, p, |\nabla\psi|)\nabla\varphi_p,\\
-\nabla\cdot(\lambda\nabla T) &= qn\mu_n(T, n, |\nabla\psi|)|\nabla\varphi_n|^2 + qp\mu_p(T, p, |\nabla\psi|)|\nabla\varphi_p|^2 + qR(\varphi_p - \varphi_n),
\end{aligned} \tag{3.1}$$

where

$$n = N_{n0}\mathcal{G}\Big(\frac{q(\psi - \varphi_n) - E_L}{k_\mathrm{B}T}; \frac{\sigma_n}{k_\mathrm{B}T}\Big), \quad p = N_{p0}\mathcal{G}\Big(\frac{E_H - q(\psi - \varphi_p)}{k_\mathrm{B}T}; \frac{\sigma_p}{k_\mathrm{B}T}\Big),$$

$$R = r(\cdot, n, p, T)\Big(1 - \exp\frac{q(\varphi_n - \varphi_p)}{k_\mathrm{B}T}\Big).$$

We remark that additional thermoelectric effects (Peltier, Thomson, and Seebeck) are not included in the above model. In [22, Sect. II.D] it is argued that in the case of organic semiconductors such effects are neglible as the thermal voltages are small compared to the applied voltage.

3.1 Rescaling of the model equation

To simplify notation in the analytical investigations, we perform the following rescaling of the problem. We define the new quantities

$$\widetilde{\psi} := q\psi, \quad \widetilde{\psi}^D := q\psi^D, \quad \widetilde{\varphi}_i := q\varphi_i, \quad \widetilde{\varphi}_i^D := q\varphi_i^D, \quad \widetilde{V}_G := qV_G, \quad \widetilde{T} := k_\mathrm{B}T, \quad \widetilde{T}_a := k_\mathrm{B}T_a,$$

$i = n, p$, and new coefficients

$$\widetilde{\varepsilon} := \frac{\varepsilon_0\varepsilon_r}{q^2}, \quad \widetilde{\lambda} := \frac{\lambda q}{k_\mathrm{B}}.$$

We consider (3.1) in the rescaled form in Ω

$$\begin{aligned}
-\nabla\cdot(\widetilde{\varepsilon}\nabla\widetilde{\psi}) &= C-n+p,\\
-\nabla\cdot\widetilde{j}_n = -\widetilde{R}, \qquad \widetilde{j}_n &= -n\widetilde{\mu}_n(\widetilde{T},n,|\nabla\widetilde{\psi}|)\nabla\widetilde{\varphi}_n,\\
\nabla\cdot\widetilde{j}_p = -\widetilde{R}, \qquad \widetilde{j}_p &= -p\widetilde{\mu}_p(\widetilde{T},p,|\nabla\widetilde{\psi}|)\nabla\widetilde{\varphi}_p,\\
-\nabla\cdot(\widetilde{\lambda}\nabla\widetilde{T}) &= n\widetilde{\mu}_n(\widetilde{T},n,|\nabla\widetilde{\psi}|)|\nabla\widetilde{\varphi}_n|^2+p\widetilde{\mu}_p(\widetilde{T},p,|\nabla\widetilde{\psi}|)|\nabla\widetilde{\varphi}_p|^2\ +\widetilde{R}(\widetilde{\varphi}_p-\widetilde{\varphi}_n)
\end{aligned} \tag{3.2}$$

with suitable functions $\widetilde{\mu}_n,\ \widetilde{\mu}_p,\ \widetilde{r}$ and

$$n=N_{n0}\mathcal{G}\Big(\frac{\widetilde{\psi}-\widetilde{\varphi}_n-E_L}{\widetilde{T}};\frac{\sigma_n}{\widetilde{T}}\Big),\quad p=N_{p0}\mathcal{G}\Big(\frac{E_H-(\widetilde{\psi}-\widetilde{\varphi}_p)}{\widetilde{T}};\frac{\sigma_p}{\widetilde{T}}\Big),$$
$$\widetilde{R}=\widetilde{r}(\cdot,n,p,\widetilde{T})\Big(1-\exp\frac{\widetilde{\varphi}_n-\widetilde{\varphi}_p}{\widetilde{T}}\Big).$$

Finally, we omit the tildes above all quantities in (3.2) and obtain the model equations in Ω for the analytical investigations

$$\begin{aligned}
-\nabla\cdot(\varepsilon\nabla\psi) &= C-n+p,\\
-\nabla\cdot j_n = -R, \qquad j_n &= -n\mu_n(T,n,|\nabla\psi|)\nabla\varphi_n,\\
\nabla\cdot j_p = -R, \qquad j_p &= -p\mu_p(T,p,|\nabla\psi|)\nabla\varphi_p,\\
-\nabla\cdot(\lambda\nabla T) &= n\mu_n(T,n,|\nabla\psi|)|\nabla\varphi_n|^2+p\mu_p(T,p,|\nabla\psi|)|\nabla\varphi_p|^2\ +R(\varphi_p-\varphi_n)
\end{aligned} \tag{3.3}$$

with

$$n=N_{n0}\mathcal{G}\Big(\frac{\psi-\varphi_n-E_L}{T};\frac{\sigma_n}{T}\Big),\quad p=N_{p0}\mathcal{G}\Big(\frac{E_H-(\psi-\varphi_p)}{T};\frac{\sigma_p}{T}\Big),$$
$$R=r(\cdot,n,p,T)\Big(1-\exp\frac{\varphi_n-\varphi_p}{T}\Big).$$

For the formulation of boundary conditions we decompose $\partial\Omega$ into Ohmic contacts $\Gamma_D=\cup_{i=1}^{I}\Gamma_{Di}$, a gate contact Γ_G and the semiconductor-insulator interface Γ_N, i.e. Ohmic contacts like semiconductor-metal interfaces are modeled by Dirichlet boundary conditions, semiconductor-insulator interfaces are realized by no flux boundary conditions, Gate contacts are described by Robin boundary conditions for the electrostatic potential ψ and no-flux boundary conditions in the continuity equations. For the heat flow equation boundary conditions of third kind are used. Let ν denote the outer normal vector. Then, in summary we close our system by the following set of boundary conditions

$$\begin{aligned}
\psi=\psi^D,\quad \varphi_n=\varphi_n^D,\quad \varphi_p=\varphi_p^D \quad &\text{on } \Gamma_D,\\
\varepsilon\nabla\psi\cdot\nu=j_n\cdot\nu=j_p\cdot\nu=0 \quad &\text{on } \Gamma_N,\\
\varepsilon\nabla\psi\cdot\nu+\alpha_{\mathrm{ox}}(\psi-V_G)=0,\quad j_n\cdot\nu=j_p\cdot\nu=0 \quad &\text{on } \Gamma_G,\\
\lambda\nabla T\cdot\nu+\kappa(T-T_a)=0 \quad &\text{on } \partial\Omega
\end{aligned} \tag{3.4}$$

with suitable coefficient functions α_{ox} and κ, a scaled gate voltage V_G and ambient temperature T_a.

3.2 Notation and assumptions

We work with the Lebesgue spaces $L^p(\Omega)$ and the Sobolev spaces $W^{1,q}(\Omega)$. Moreover, we use the closed subspace of $H^1(\Omega)=W^{1,2}(\Omega)$,

$$H_D^1(\Omega):=\{u\in H^1(\Omega):\ u|_{\Gamma_D}=0\}.$$

In our estimates, positive constants, which may depend at most on the data of our problem, are denoted by c. In particular, we allow them to change from line to line.

We discuss the stationary energy model for organic semiconductor devices (3.3), (3.4) under the general **Assumptions (A):**

- $\Omega \in \mathbb{R}^d$, $d = 2, 3$, is a bounded Lipschitz domain, $\Gamma_D,\ \Gamma_N,\ \Gamma_G \subset \Gamma := \partial\Omega$ are disjoint subsets such that $\overline{\Gamma_D \cup \Gamma_N \cup \Gamma_G} = \Gamma$ and $\mathrm{mes}(\Gamma_D) > 0$,
- $\psi^D \in W^{1,\infty}(\Omega),\ \varphi_i^D \in H^1(\Omega) \cap L^\infty(\Omega)$, $\|\varphi_i^D\|_{L^\infty} \le K$, $i = n, p$, $\varepsilon \in L^\infty(\Omega)$, $0 < \underline{\varepsilon} \le \varepsilon$ a.e. in Ω, $C \in L^\infty(\Omega)$, $V_G \in L^\infty(\Gamma_G)$, $\alpha_{\mathrm{ox}} \in L^\infty_+(\Gamma_G)$, $\lambda \in L^\infty(\Omega)$, $0 < \underline{\lambda} \le \lambda$ a.e. in Ω, $\kappa \in L^\infty_+(\Gamma)$, $\|\kappa\|_{L^1(\Gamma)} > 0$, $T_a = \mathrm{const} > 0$,
- $E_J,\ \sigma_i,\ N_{i0} : \Omega \times (0,\infty) \to \mathbb{R}_+$ are Caratheodory functions with $|E_J(\cdot,T)| \le \widehat{E}$, $0 < \sigma_i(\cdot,T) \le \overline{\sigma}$, $0 < \underline{N_{i0}} \le N_{i0}(\cdot,T) \le \overline{N_{i0}}$ a.e. in Ω for all $T \in (0,\infty)$, $|E_J(\cdot,T_1) - E_J(\cdot,T_2)| \le c|T_1 - T_2|$, $J = E, H$, $|\sigma_i(\cdot,T_1) - \sigma_i(\cdot,T_2)| \le c|T_1 - T_2|$, $|N_{i0}(\cdot,T_1) - N_{i0}(\cdot,T_2)| \le c|T_1 - T_2|$ a.e. in Ω for all $T_1, T_2 \in [T_a,\infty)$, $i = n, p$,
- $r(\cdot,n,p,T) : \Omega \times [0,\overline{N_{n0}}] \times [0,\overline{N_{p0}}] \times (0,\infty) \to \mathbb{R}_+$ is a Caratheodory function such that for all $\tau > 0$ there exists an $\overline{r}_\tau$ with $0 \le r(\cdot,n,p,T) \le \overline{r}_\tau$ for all $(n,p,T) \in [0,\overline{N_{n0}}] \times [0,\overline{N_{p0}}] \times [\tau,\infty)$ and a.a. $x \in \Omega$
- $\mu_i : \Omega \times (0,\infty) \times [0,\overline{N_{i0}}] \times \mathbb{R}_+ \to \mathbb{R}_+$, $i = n, p$, are Caratheodory functions such that for all $\tau > 0$ there exists $\overline{\mu}_\tau$ with $\mu_n(\cdot,T,n,F),\ \mu_p(\cdot,T,p,F) \le \overline{\mu}_\tau < \infty$ for all $(T,n,p,F) \in [\tau,\infty) \times [0,\overline{N_{n0}}] \times [0,\overline{N_{p0}}] \times \mathbb{R}_+$ a.e. in Ω and $0 < \underline{\mu} \le \mu_n(\cdot,T,n,F),\ \mu_p(\cdot,T,p,F)$ for all $(T,n,p,F) \in [T_a,\infty) \times [0,\overline{N_{n0}}] \times [0,\overline{N_{p0}}] \times \mathbb{R}_+$ a.e. in Ω.

In the following we set $\overline{r} := \overline{r}_{T_a}$ as well as $\overline{\mu} := \overline{\mu}_{T_a}$.

3.3 Concept of solution

First let us note that the right hand side of the heat flow equation in (3.3) is a priori only an L^1 function. Therefore we prefer to follow the concept of entropy solutions for the temperature distribution T, see e.g. [2, 30, 23] (for the case of Dirichlet boundary conditions) or [4] (for Robin boundary conditions). For the remaining three equations in (3.3) a weak formulation is appropriate. Thus in the present paper, we work with the following concept of solutions for the stationary energy model for organic semiconductor devices (3.3), (3.4): Find $(\psi,\varphi_n,\varphi_p,T) \in (\psi^D + H^1_D(\Omega)) \times (\varphi_n^D + H^1_D(\Omega) \cap L^\infty(\Omega)) \times (\varphi_p^D + H^1_D(\Omega) \cap L^\infty(\Omega)) \times \{T \in W^{1,q}(\Omega) : (\ln T)^- \in L^\infty(\Omega)\}$, where $q \in [1, \frac{d}{d-1})$ such that

$$
\begin{aligned}
&\int_\Omega \varepsilon\nabla\psi\cdot\nabla\overline{\psi}\,\mathrm{d}x + \int_{\Gamma_G} \alpha_{\mathrm{ox}}(\psi - V_G)\overline{\psi}\,\mathrm{d}\Gamma = \int_\Omega (C - n + p)\overline{\psi}\,\mathrm{d}x,\\
&\int_\Omega \big(n\mu_n(T,n,|\nabla\psi|)\nabla\varphi_n\cdot\nabla\overline{\varphi}_n + p\mu_p(T,p,|\nabla\psi|)\nabla\varphi_p\cdot\nabla\overline{\varphi}_p\big)\,\mathrm{d}x\\
&\quad = \int_\Omega r(n,p,T)\Big(1 - \exp\frac{\varphi_n - \varphi_p}{T}\Big)(\overline{\varphi}_n - \overline{\varphi}_p)\,\mathrm{d}x \qquad \forall\overline{\psi},\ \overline{\varphi}_n,\ \overline{\varphi}_p \in H^1_D(\Omega),\\
&\int_\Omega \lambda\nabla T\cdot\nabla(C_m(T-\omega))\,\mathrm{d}x + \int_\Gamma \kappa(T - T_a)C_m(T-\omega)\,\mathrm{d}\Gamma\\
&\quad \le \int_\Omega h\,C_m(T-\omega)\,\mathrm{d}x \quad \forall\omega \in H^1(\Omega)\cap L^\infty(\Omega),\ \forall m \ge 0,
\end{aligned}
\tag{3.5}
$$

where the densities n and p and the right hand side h in the heat flow equation have to be determined pointwise by

$$
\begin{aligned}
&n = N_{n0}(T)\mathcal{G}\Big(\frac{\psi - \varphi_n - E_L(T)}{T};\frac{\sigma_n(T)}{T}\Big), \quad p = N_{p0}(T)\mathcal{G}\Big(\frac{E_H(T) - (\psi - \varphi_p)}{T};\frac{\sigma_p(T)}{T}\Big),\\
&h = n\mu_n(T,n,|\nabla\psi|)|\nabla\varphi_n|^2 + p\mu_p(T,p,|\nabla\psi|)|\nabla\varphi_p|^2\\
&\qquad + r(n,p,T)\Big(\exp\frac{\varphi_n - \varphi_p}{T} - 1\Big)(\varphi_n - \varphi_p).
\end{aligned}
\tag{3.6}
$$

The notation C_m in the entropy formulation of the heat flow equation in (3.5) means the truncation function $C_m : \mathbb{R} \to [-m, m]$ realized by $C_m(s) := \max\{-m, \min\{s, m\}\}$.

The choice of spaces guarantees that all terms in (3.5) are well-defined. The following lemma ensures that we only have to consider admissible temperatures T satisfying $T \geq T_a$ (see the definition of the set $\mathcal{N}$ in (4.1)).

Lemma 3.1 *We assume (A). Then any solution $(\psi, \varphi_n, \varphi_p, T)$ to (3.5) satisfies $T \geq T_a$ a.e. in Ω.*

Proof. Let $(\psi, \varphi_n, \varphi_p, T)$ be a solution to (3.5), then h defined in (3.6) is a nonnegative L^1 function. Therefore, the result follows from Lemma 3.5 in [4] (see also Lemma 4.6). □

3.4 Main result

The main result of our paper is the following existence theorem for the stationary energy-drift-diffusion model for organic semiconductors which we prove in Section 4.

Theorem 3.1 *Under Assumption (A) there exists a solution $(\psi, \varphi_n, \varphi_p, T)$ to Problem (3.5) with $\psi \in \psi^D + H^1_D(\Omega)$, $\varphi_i \in \varphi_i^D + H^1_D(\Omega)$, $i = n, p$, and $T \in W^{1,q}(\Omega)$, $q \in [1, \frac{d}{d-1})$. There are positive constants c_{ψ,L^∞}, c_{ψ,H^1}, c_{H^1}, $\underline{c_c}$, $\overline{c_c}$, $c_{W^{1,q}}$ depending only on the data such that*

$$\begin{aligned} &\|\psi\|_{L^\infty} \leq c_{\psi,L^\infty}, \quad \|\psi\|_{H^1} \leq c_{\psi,H^1}, \\ &\|\varphi_i\|_{L^\infty} \leq K, \quad \|\varphi_i\|_{H^1} \leq c_{H^1}, \quad i = n, p, \\ &T \geq T_a \text{ a.e. } \Omega, \quad \|T\|_{W^{1,q}} \leq c_{W^{1,q}}, \end{aligned}$$

and the by (3.6) related densities n and p are bounded by $\underline{c_c} \leq n,\ p \leq \overline{c_c}$ a.e. on Ω.

Corollary 3.1 *Supposing additionally to Assumption (A) that*

$$\varphi_i^D = \text{const}, \quad i = n, p, \quad \varphi_n^D = \varphi_p^D \text{ on } \Omega \tag{3.7}$$

then there is a unique solution to Problem (3.5). This solution is the thermodynamic equilibrium and has the form $(\psi^*, \varphi_n^D, \varphi_p^D, T_a)$, where ψ^* is the unique weak solution to the nonlinear Poisson equation

$$\begin{aligned} -\nabla \cdot (\varepsilon \nabla \psi^*) &= C - N_{n0}(T_a)\mathcal{G}\Big(\frac{\psi^* - \varphi_n^D - E_L(T_a)}{T_a}; \frac{\sigma_n(T_a)}{T_a}\Big) \\ &\quad + N_{p0}(T_a)\mathcal{G}\Big(\frac{E_H(T_a) - (\psi^* - \varphi_p^D)}{T_a}; \frac{\sigma_p(T_a)}{T_a}\Big) \text{ in } \Omega, \\ \psi^* = \psi^D \text{ on } \Gamma_D, &\quad \varepsilon \nabla \psi^* \cdot \nu = 0 \text{ on } \Gamma_N, \quad \varepsilon \nabla \psi^* \cdot \nu + \alpha_{\text{ox}}(\psi^* - V_G) = 0 \text{ on } \Gamma_G. \end{aligned} \tag{3.8}$$

The proof of the corollary is given in Subsection 4.3. A general uniqueness result is not to be expected. Even in the classical, isothermal case, where the stationary van Roosbroeck system applies it is well known that certain devices admit multiple stationary states for certain biasing conditions, see [27].

4 Existence proof

This section is devoted to the proof of Theorem 3.1 which is based on Schauder's fixed point theorem. First, in Subsection 4.1 we shortly introduce the iteration scheme used to define the fixed point map $\mathcal{Q}$. In Subsection 4.2 we supply and prove results for relevant subproblems with frozen arguments that are needed to ensure the well-posedness of the iteration scheme and additionally guarantee the bounds of solutions stated in Theorem 3.1. This concerns results for the Poisson equation (Lemma 4.1), the continuity equations (Lemma 4.3, Lemma 4.4) and the heat equation with right-hand side only in L^1 (Theorem 4.1, Lemma 4.5, Lemma 4.6). Finally, in the Subsection 4.3, we show the continuity of the fixed point map $\mathcal{Q}$, see Lemma 4.7 (for the sub problem of the Poisson equation) and Lemma 4.8 (for the final result).

4.1 Iteration scheme

Let

$$\mathcal{N} := \Big\{(\varphi_n, \varphi_p, T) \in L^2(\Omega)^3 : \|\varphi_n\|_{H^1},\ \|\varphi_p\|_{H^1} \le c_{H^1},\ \|T\|_{W^{1,4/3}} \le c_{T,W^{1,4/3}}, \\ - K \le \varphi_n,\ \varphi_p \le K, \text{ and } T \ge T_a \text{ a.e. in } \Omega\Big\}, \tag{4.1}$$

where $c_{H^1} > 0$ will be given in Lemma 4.4 and $c_{T,W^{1,4/3}} > 0$ will be fixed in (4.9). The fixed point map $\mathcal{Q} : \mathcal{N} \to \mathcal{N}$ with $(\varphi_n, \varphi_p, T) = \mathcal{Q}(\widetilde{\varphi}_n, \widetilde{\varphi}_p, \widetilde{T})$ will be defined by the following four steps:

1. For given $(\widetilde{\varphi}_n, \widetilde{\varphi}_p, \widetilde{T}) \in \mathcal{N}$ we consider the nonlinear Poisson equation

$$\begin{aligned} -\nabla\cdot(\varepsilon\nabla\psi) = C - N_{n0}(\widetilde{T})\mathcal{G}\Big(\frac{\psi-\widetilde{\varphi}_n-E_L(\widetilde{T})}{\widetilde{T}}, \frac{\sigma_n(\widetilde{T})}{\widetilde{T}}\Big) \\ + N_{p0}(\widetilde{T})\mathcal{G}\Big(\frac{E_H(\widetilde{T})-(\psi-\widetilde{\varphi}_p)}{\widetilde{T}}; \frac{\sigma_p(\widetilde{T})}{\widetilde{T}}\Big) \quad \text{in } \Omega, \\ \psi = \psi^D \quad \text{on } \Gamma_D, \quad \varepsilon\nabla\psi\cdot\nu + \alpha_{\rm ox}(\psi - V_G) = 0 \quad \text{on } \Gamma_G, \\ \varepsilon\nabla\psi\cdot\nu = 0 \quad \text{on } \Gamma_N \end{aligned} \tag{4.2}$$

and obtain by Lemma 4.1 a unique weak solution $\psi \in \psi^D + H^1_D(\Omega)$. It fulfills $\|\psi\|_{L^\infty} \le c_{\psi,L^\infty}$ and $\|\psi\|_{H^1} \le c_{\psi,H^1}$.

2. We set now

$$\widetilde{n} := N_{n0}(\widetilde{T})\mathcal{G}\Big(\frac{\psi-\widetilde{\varphi}_n-E_L(\widetilde{T})}{\widetilde{T}}; \frac{\sigma_n(\widetilde{T})}{\widetilde{T}}\Big), \quad \widetilde{p} := N_{p0}(\widetilde{T})\mathcal{G}\Big(\frac{E_H(\widetilde{T})-(\psi-\widetilde{\varphi}_p)}{\widetilde{T}}; \frac{\sigma_p(\widetilde{T})}{\widetilde{T}}\Big). \tag{4.3}$$

Lemma 4.3 ensures the uniform estimates

$$0 < \underline{c_n} \le \widetilde{n} \le \overline{c_n},\ 0 < \underline{c_p} \le \widetilde{p} \le \overline{c_p}, \tag{4.4}$$

$$0 < c_u \le \widetilde{n}\mu_n(\widetilde{T}, \widetilde{n}, |\nabla\psi|),\ \widetilde{p}\mu_p(\widetilde{T}, \widetilde{p}, |\nabla\psi|) \le c_o. \tag{4.5}$$

3. Next, we solve the continuity equations with frozen electron density $\widetilde{n}$ and mobility $\mu_n(\widetilde{T}, \widetilde{n}, |\nabla\psi|)$ as well as hole density $\widetilde{p}$ and mobility $\mu_p(\widetilde{T}, \widetilde{p}, |\nabla\psi|)$ and reaction rate coefficient $r(\widetilde{n}, \widetilde{p}, \widetilde{T})$ for a weak solution (φ_n, φ_p) to

$$\begin{aligned} \nabla\cdot(\widetilde{n}\mu_n(\widetilde{T}, \widetilde{n}, |\nabla\psi|)\nabla\varphi_n) = r(\widetilde{n}, \widetilde{p}, \widetilde{T})\Big(\exp\frac{\varphi_n-\varphi_p}{\widetilde{T}} - 1\Big) \quad \text{in } \Omega, \\ \nabla\cdot(\widetilde{p}\mu_p(\widetilde{T}, \widetilde{p}, |\nabla\psi|)\nabla\varphi_p) = -r(\widetilde{n}, \widetilde{p}, \widetilde{T})\Big(\exp\frac{\varphi_n-\varphi_p}{\widetilde{T}} - 1\Big) \quad \text{in } \Omega, \\ \varphi_n = \varphi_n^D \quad \text{on } \Gamma_D, \quad \widetilde{n}\mu_n(\widetilde{T}, \widetilde{n}, |\nabla\psi|)\nabla\varphi_n\cdot\nu = 0 \quad \text{on } \Gamma_N\cup\Gamma_G, \\ \varphi_p = \varphi_p^D \quad \text{on } \Gamma_D, \quad \widetilde{p}\mu_p(\widetilde{T}, \widetilde{p}, |\nabla\psi|)\nabla\varphi_p\cdot\nu = 0 \quad \text{on } \Gamma_N\cup\Gamma_G. \end{aligned} \tag{4.6}$$

By Lemma 4.4 there exists a unique weak solution $(\varphi_n, \varphi_p) \in (\varphi_n^D + H^1_D(\Omega)) \times (\varphi_p^D + H^1_D(\Omega))$ to (4.6). It fulfills the uniform estimates

$$\|\varphi_n\|_{L^\infty},\ \|\varphi_p\|_{L^\infty} \le K, \quad \|\varphi_n\|_{H^1},\ \|\varphi_p\|_{H^1} \le c_{H^1},$$

$$\|\widetilde{n}\mu_n(\widetilde{T}, \widetilde{n}, |\nabla\psi|)|\nabla\varphi_n|^2\|_{L^1},\ \|\widetilde{p}\mu_p(\widetilde{T}, \widetilde{p}, |\nabla\psi|)|\nabla\varphi_p|^2\|_{L^1} \le c_J,$$

$$\|r(\widetilde{n}, \widetilde{p}, \widetilde{T})\Big(\exp\frac{\varphi_n-\varphi_p}{\widetilde{T}} - 1\Big)(\varphi_n - \varphi_p)\|_{L^1} \le c_R$$

with the constant K from Assumption (A) and $c_{H^1},\ c_J,\ c_R > 0$.

4. These estimates ensure, that the source term for the heat equation

$$\begin{aligned} h := \widetilde{n}\mu_n(\widetilde{T},\widetilde{n},|\nabla\psi|)|\nabla\varphi_n|^2 + \widetilde{p}\mu_p(\widetilde{T},\widetilde{p},|\nabla\psi|)|\nabla\varphi_p|^2 \\ + r(\widetilde{n},\widetilde{p},\widetilde{T})\Big(\exp\frac{\varphi_n-\varphi_p}{\widetilde{T}}-1\Big)(\varphi_n-\varphi_p) \end{aligned} \tag{4.7}$$

resulting from $(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T}) \in \mathcal{N}$ is uniformly bounded in $L^1(\Omega)$, $\|h\|_{L^1} \le c_h$. With the source terms $h \in L^1(\Omega)$ and $g := \kappa T_a \in L^1(\Gamma)$ we solve now the linear heat flow equation

$$\begin{aligned} -\nabla\cdot(\lambda\nabla T) &= h \qquad \text{in } \Omega, \\ \lambda\nabla T\cdot\nu + \kappa T &= g \qquad \text{on } \Gamma. \end{aligned} \tag{4.8}$$

According to Theorem 4.1, there exists exactly one entropy solution T to (4.8). Moreover, it fulfills the estimate

$$\|T\|_{W^{1,4/3}} \le c_{E4/3}(c_h + \|\kappa T_a\|_{L^1(\Gamma)}) =: c_{T,W^{1,4/3}}. \tag{4.9}$$

The continuous imbedding $W^{1,4/3}(\Omega) \hookrightarrow L^2(\Omega)$ guarantees $T \in L^2(\Omega)$, and Lemma 4.6 ensures $T \ge T_a$. In summary, this yields that $(\varphi_n,\varphi_p,T) = \mathcal{Q}(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T}) \in \mathcal{N}$.

4.2 Solvability and properties of solutions to subproblems

(1) Poisson equation. Let

$$\mathcal{N}^* := \Big\{(\varphi_n,\varphi_p,T) \in (L^2(\Omega))^3 :\ -K \le \varphi_n,\ \varphi_p \le K,\ T \ge T_a,\ \text{a.e. in } \Omega\Big\}. \tag{4.10}$$

For given $(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T}) \in \mathcal{N}^* \supset \mathcal{N}$ we are looking for weak solutions to the nonlinear Poisson equation (4.2).

Lemma 4.1 *We assume (A). Let $(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T}) \in \mathcal{N}^*$ be arbitrarily given. Then there exists a unique weak solution $\psi \in \psi^D + H^1_D(\Omega)$ to the nonlinear Poisson equation* (4.2). *There are constants $c_{\psi,L^\infty},\ c_{\psi,H^1} > 0$ not depending on the choice of $(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T}) \in \mathcal{N}^*$ such that*

$$\|\psi\|_{L^\infty} \le c_{\psi,L^\infty}, \quad \|\psi\|_{H^1} \le c_{\psi,H^1}.$$

Proof. Due to the properties of the function $\mathcal{G}$, for given $(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T})$ the operator $B_{(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T})} : \psi^D + H^1_D(\Omega) \to (H^1_D(\Omega))^*$,

$$\begin{aligned} \langle B_{(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T})}\psi,\overline{\psi}\rangle := \int_\Omega \varepsilon\nabla\psi\cdot\nabla\overline{\psi}\,\mathrm{d}x + \int_{\Gamma_G}\alpha_{\mathrm{ox}}(\psi - V_G)\overline{\psi}\,\mathrm{d}\Gamma \\ + \int_\Omega\Big(N_{n0}(\widetilde{T})\mathcal{G}\Big(\frac{\psi-\widetilde{\varphi}_n-E_L(\widetilde{T})}{\widetilde{T}};\frac{\sigma_n(\widetilde{T})}{\widetilde{T}}\Big) - N_{p0}(\widetilde{T})\mathcal{G}\Big(\frac{E_H(\widetilde{T})-(\psi-\widetilde{\varphi}_p)}{\widetilde{T}};\frac{\sigma_p(\widetilde{T})}{\widetilde{T}}\Big) - C\Big)\overline{\psi}\,\mathrm{d}x, \end{aligned}$$

$\overline{\psi} \in H^1_D(\Omega)$, is strongly monotone and Lipschitz continuous (note that $\|\nabla\cdot\|_{L^2}$ is an equivalent norm on $H^1_D(\Omega)$, that $\frac{\partial\mathcal{G}}{\partial\eta}(\eta;z) \in (0,1)$ for all $\eta \in \mathbb{R}$, $z > 0$ and that $\widetilde{T} \ge T_a$). Thus, the unique solution $\psi \in \psi^D + H^1_D(\Omega)$ to $B_{(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T})}\psi = 0$ is the unique weak solution to (4.2). Since $\mathcal{G}(\eta,z) \in (0,1)$ for all $\eta \in \mathbb{R}$, $z > 0$ we find for

$$f := C - N_{n0}(\widetilde{T})\mathcal{G}\Big(\frac{\psi-\widetilde{\varphi}_n-E_L(\widetilde{T})}{\widetilde{T}};\frac{\sigma_n(\widetilde{T})}{\widetilde{T}}\Big) + N_{p0}(\widetilde{T})\mathcal{G}\Big(\frac{E_H(\widetilde{T})-(\psi-\widetilde{\varphi}_p)}{\widetilde{T}};\frac{\sigma_p(\widetilde{T})}{\widetilde{T}}\Big)$$

that according to Assumption (A)

$$\|f\|_{L^\infty} \le \|C\|_{L^\infty} + \overline{N_{n0}} + \overline{N_{p0}} < c.$$

With this estimate of the right hand side the following Lemma 4.2 which is proved in [7] completes the proof of Lemma 4.1. □

Lemma 4.2 *We assume (A) and $f \in L^\infty(\Omega)$. Then the weak solution to*

$$\begin{aligned} &-\nabla\cdot(\varepsilon\nabla\psi) = f \quad \text{in}\,\Omega,\\ &\psi = \psi^D \ \text{on}\,\Gamma_D, \quad \varepsilon\nabla\psi\cdot\nu + \alpha_{\text{ox}}(\psi - V_G) = 0 \ \text{on}\,\Gamma_G, \quad \varepsilon\nabla\psi\cdot\nu = 0 \ \text{on}\,\Gamma_N \end{aligned} \tag{4.11}$$

belongs to $H^1(\Omega)\cap L^\infty(\Omega)$ and

$$\|\psi\|_{L^\infty} \le c_1(\|f\|_{L^\infty}, \|V_G\|_{L^\infty(\Gamma_G)}, \|\psi^D\|_{W^{1,\infty}}), \quad \|\psi\|_{H^1} \le c_2(\|f\|_{L^\infty}, \|V_G\|_{L^\infty(\Gamma_G)}, \|\psi^D\|_{H^1}).$$

(2) Bounds for densities and mobilities.

Lemma 4.3 *We assume (A). Let $(\widetilde{\varphi}_n, \widetilde{\varphi}_p, \widetilde{T}) \in \mathcal{N}^*$ and let ψ be the weak solution to (4.2) and $\widetilde{n}$ and $\widetilde{p}$ be defined by (4.3). Then there are constants $\underline{c_n},\ \overline{c_n},\ \underline{c_p},\ \overline{c_p},\ c_u,\ c_o > 0$ such that the defined densities and the mobility functions fulfill the uniform estimates*

$$\underline{c_n} \le \widetilde{n} \le \overline{c_n},\ \underline{c_p} \le \widetilde{p} \le \overline{c_p}, \quad c_u \le \widetilde{n}\mu_n(\widetilde{T}, \widetilde{n}, |\nabla\psi|),\, \widetilde{p}\mu_p(\widetilde{T}, \widetilde{p}, |\nabla\psi|) \le c_o \quad \text{a.e. in}\,\Omega.$$

Proof. Since $E_L(\widetilde{T})$, $E_H(\widetilde{T})$, $\widetilde{\varphi}_n$, $\widetilde{\varphi}_p$, and ψ have upper and lower bounds and $\widetilde{T}$ is bounded from below by T_a we obtain the estimates for $\widetilde{\eta}_n$ and $\widetilde{\eta}_p$

$$\begin{aligned} -\frac{c_{\psi,L^\infty} + K + \widehat{E}}{T_a} &\le \widetilde{\eta}_n := \frac{\psi - \widetilde{\varphi}_n - E_L(\widetilde{T})}{\widetilde{T}} \le \frac{c_{\psi,L^\infty} + K + \widehat{E}}{T_a},\\ -\frac{\widehat{E} + c_{\psi,L^\infty} + K}{T_a} &\le \widetilde{\eta}_p := \frac{E_H(\widetilde{T}) - (\psi - \widetilde{\varphi}_p)}{\widetilde{T}} \le \frac{\widehat{E} + c_{\psi,L^\infty} + K}{T_a}. \end{aligned} \tag{4.12}$$

For $\eta < 0$ the function $z \mapsto \mathcal{G}(\eta, z)$ is strictly monotone increasing, see Lemma 2.1. Thus we obtain $\mathcal{G}(\eta; z) > \mathcal{G}(\eta; 0)$ for all $\eta < 0$ and all $z > 0$. Because of the strict monotonicity of the mapping $\eta \mapsto \mathcal{G}(\eta; z)$ we find for $\widetilde{\eta}_i$, $i = n, p$, in the range of relation (4.12) and all $z > 0$ an estimate from below by

$$N_{i0}\mathcal{G}(\widetilde{\eta}_i, z) \ge \underline{N_{i0}}\mathcal{G}\Big(-\frac{\widehat{E} + c_{\psi,L^\infty} + K}{T_a}; z\Big) > \underline{N_{i0}}\mathcal{G}\Big(-\frac{\widehat{E} + c_{\psi,L^\infty} + K}{T_a}; 0\Big) =: \underline{c_i}$$

which leads to the estimates

$$\begin{aligned} \underline{c_n} < \widetilde{n} = N_{n0}(\widetilde{T})\mathcal{G}(\widetilde{\eta}_n; \sigma_n(\widetilde{T})/\widetilde{T}) \le \overline{N_{n0}} =: \overline{c_n},\\ \underline{c_p} < \widetilde{p} = N_{p0}(\widetilde{T})\mathcal{G}(\widetilde{\eta}_p; \sigma_p(\widetilde{T})/\widetilde{T}) \le \overline{N_{p0}} =: \overline{c_p}. \end{aligned}$$

Exploiting (2.5) we therefore obtain

$$0 < c_u \le \widetilde{n}\mu_n(\widetilde{T}, \widetilde{n}, |\nabla\psi|),\, \widetilde{p}\mu_p(\widetilde{T}, \widetilde{p}, |\nabla\psi|) \le c_o \quad \text{a.e. in}\,\Omega \tag{4.13}$$

with positive constants c_u, c_o. □

(3) Continuity equations.

Lemma 4.4 *We assume (A). Let $(\widetilde{\varphi}_n, \widetilde{\varphi}_p, \widetilde{T}) \in \mathcal{N}^*$ and let ψ be the weak solution to (4.2) and $\widetilde{n}$ and $\widetilde{p}$ be given by (4.3). Then there exists a unique weak solution $(\varphi_n, \varphi_p) \in (\varphi_n^D + H^1_D(\Omega)) \times (\varphi_p^D + H^1_D(\Omega))$ to (4.6). It fulfills the estimates*

$$\begin{gathered} -K \le \varphi_n,\, \varphi_p \le K \quad \text{a.e. on}\,\Omega, \quad \|\varphi_n\|_{H^1},\, \|\varphi_p\|_{H^1} \le c_{H^1},\\ \|\widetilde{n}\mu_n(\widetilde{T}, \widetilde{n}, |\nabla\psi|)|\nabla\varphi_n|^2\|_{L^1},\, \|\widetilde{p}\mu_p(\widetilde{T}, \widetilde{p}, |\nabla\psi|)|\nabla\varphi_p|^2\|_{L^1} \le c_J,\\ \|r(\widetilde{n}, \widetilde{p}, \widetilde{T})\Big(\exp\frac{\varphi_n - \varphi_p}{\widetilde{T}} - 1\Big)(\varphi_n - \varphi_p)\|_{L^1} \le c_R \end{gathered}$$

with K from Assumption (A) and constants $c_{H^1}, c_R, c_J > 0$ independent of the special choice of $(\widetilde{\varphi}_n, \widetilde{\varphi}_p, \widetilde{T}) \in \mathcal{N}^*$ (and the resulting ψ, $\widetilde{n}$ and $\widetilde{p}$).

Proof. 1. We use the notation

$$\widetilde{a}_n := \widetilde{n}\mu_n(\widetilde{T}, \widetilde{n}, |\nabla\psi|), \quad \widetilde{a}_p := \widetilde{p}\mu_p(\widetilde{T}, \widetilde{p}, |\nabla\psi|), \quad \widetilde{r} := r(\widetilde{n}, \widetilde{p}, \widetilde{T}).$$

Let $\rho_M : \mathbb{R}^2 \to [0,1]$ be a fixed Lipschitz continuous function with

$$\rho_M(y,z) := \begin{cases} 0 & \text{if } \max\{|y|,|z|\} \geq M, \\ 1 & \text{if } \max\{|y|,|z|\} \leq \frac{M}{2}. \end{cases}$$

The operator $A^M_{(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T})} : (\varphi_n^D + H^1_D(\Omega)) \times (\varphi_p^D + H^1_D(\Omega)) \to (H^1_D(\Omega)^*)^2$,

$$A^M_{(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T})}(\varphi_n,\varphi_p) = \widehat{A}^M_{(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T})}\big((\varphi_n,\varphi_p),(\varphi_n,\varphi_p)\big)$$

with the argument splitting

$$\begin{aligned}&\Big\langle \widehat{A}^M_{(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T})}\big((\varphi_n,\varphi_p),(\widehat{\varphi}_n,\widehat{\varphi}_p)\big),(\overline{\varphi}_n,\overline{\varphi}_p)\Big\rangle := \\ &\int_\Omega \Big(\sum_{i=n,p} \widetilde{a}_i \nabla\widehat{\varphi}_i \cdot \nabla\overline{\varphi}_i + \rho_M(\varphi_n,\varphi_p)\,\widetilde{r}\Big(\exp\frac{\varphi_n-\varphi_p}{\widetilde{T}} - 1\Big)(\overline{\varphi}_n - \overline{\varphi}_p)\Big)\,\mathrm{d}x, \quad \overline{\varphi}_i \in H^1_D(\Omega),\end{aligned}$$

is an operator of variational type (see [26, p. 182]). Have in mind that by (4.13) the main part (in the arguments $\widehat{\varphi}_n$, $\widehat{\varphi}_p$) is bounded, continuous, and monotone. Furthermore, the regularized reaction term is bounded and the map $(\varphi_n,\varphi_p) \mapsto \rho_M(\varphi_n,\varphi_p)\big(\exp\{\frac{\varphi_n-\varphi_p}{\widetilde{T}}\} - 1\big)$ is Lipschitz continuous. Since, additionally the operator $A^M_{(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T})}(\varphi_n,\varphi_p)$ is coercive, the problem $A^M_{(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T})}(\varphi_n,\varphi_p) = 0$ has at least one solution $(\varphi_n^M,\varphi_p^M) \in (\varphi_n^D + H^1_D(\Omega)) \times (\varphi_p^D + H^1_D(\Omega))$.

2. Using the test function $((\varphi_n^M - K)^+, (\varphi_p^M - K)^+) \in H^1_D(\Omega)^2$ for $A^M_{(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T})}(\varphi_n^M,\varphi_p^M) = 0$ with K from Assumption (A) we obtain

$$\begin{aligned} 0 = &\int_\Omega \sum_{i=n,p} \widetilde{a}_i |\nabla(\varphi_i^M - K)^+|^2\,\mathrm{d}x \\ &+ \int_\Omega \rho_M(\varphi_n^M,\varphi_p^M)\widetilde{r}\Big(\exp\frac{\varphi_n^M - \varphi_p^M}{\widetilde{T}} - 1\Big)\big((\varphi_n^M - K)^+ - (\varphi_p^M - K)^+\big)\,\mathrm{d}x.\end{aligned}$$

Examining the four different cases φ_n^M $(\varphi_p^M) > K$ $(\leq K)$, we find that the integrand in the last line is always non-negative (note that ρ_M and r are also non-negative). Thus, (4.13) ensures that φ_n^M, $\varphi_p^M \leq K$ a.e. in Ω. On the other hand, testing by $(-(\varphi_n^M + K)^-, -(\varphi_p^M + K)^-)$ gives the estimates φ_n^M, $\varphi_p^M \geq -K$ a.e. in Ω. Therefore, if we choose $M \geq 2K$, each solution to $A^M_{(\widetilde{\varphi}_n,\widetilde{\varphi}_p,\widetilde{T})}(\varphi_n,\varphi_p) = 0$ is a weak solution to (4.6), too. The estimates of Step 2 can be done in exactly the same way but leaving out the factor ρ_M to obtain the upper and lower bounds for all weak solutions (φ_n,φ_p) to (4.6), such that $\|\varphi_i\|_{L^\infty} \leq K$, $i = n, p$.

3. To verify the uniqueness of the weak solution to (4.6) we suppose there would be two different solutions (φ_n,φ_p) and $(\widehat{\varphi}_n,\widehat{\varphi}_p)$. Using the test function $(\varphi_n - \widehat{\varphi}_n, \varphi_p - \widehat{\varphi}_p) \in H^1_D(\Omega)^2$ for (4.6) leads to

$$\begin{aligned} 0 = &\int_\Omega \sum_{i=n,p} \widetilde{a}_i |\nabla(\varphi_i - \widehat{\varphi}_i)|^2\,\mathrm{d}x \\ &+ \int_\Omega \widetilde{r}\Big(\exp\frac{\varphi_n - \varphi_p}{\widetilde{T}} - \exp\frac{\widehat{\varphi}_n - \widehat{\varphi}_p}{\widetilde{T}}\Big)\big(\varphi_n - \varphi_p - (\widehat{\varphi}_n - \widehat{\varphi}_p)\big)\,\mathrm{d}x.\end{aligned}$$

Since $\operatorname{mes}\Gamma_D > 0$, the estimate (4.13), the monotonicity of the exponential function, and $\widetilde{r} \geq 0$ ensure $(\varphi_n,\varphi_p) = (\widehat{\varphi}_n,\widehat{\varphi}_p)$.

4. Finally, we establish the uniform H^1-estimate for the weak solution to (4.6) by testing with $(\varphi_n - \varphi_n^D, \varphi_p - \varphi_p^D) \in H_D^1(\Omega)^2$, using Hölder's and Young's inequalities and the fact that $\widetilde{T} \geq T_a$ and $\varphi_n, \varphi_p \in [-K, K]$ a.e. in Ω from Step 2:

$$\int_\Omega \Big(\sum_{i=n,p} \widetilde{a}_i |\nabla(\varphi_i - \varphi_i^D)|^2 + \widetilde{r}\Big(\exp \frac{\varphi_n - \varphi_p}{\widetilde{T}} - 1\Big)(\varphi_n - \varphi_p)\Big)\,\mathrm{d}x$$
$$\leq \int_\Omega \frac{1}{2} \sum_{i=n,p} \widetilde{a}_i \big(|\nabla(\varphi_i - \varphi_i^D)|^2 + |\nabla \varphi_i^D|^2\big)\,\mathrm{d}x + 2\overline{r}\,K \exp \frac{2K}{T_a} \,\mathrm{mes}(\Omega).$$

Again taking advantage from (4.13), the non-negativity of the function r, the monotonicity of the exponential function, and that $\varphi_i^D \in H^1(\Omega)$ are given functions, and using the constants $\overline{r}$ and K from Assumption (A), we obtain the estimates $\|\varphi_i\|_{H^1} \leq c_{H^1}$, $i = n, p$, where the constant c_{H^1} is independent of the special choice of $(\widetilde{\varphi}_n, \widetilde{\varphi}_p, \widetilde{T}) \in \mathcal{N}^*$ (and the resulting ψ, $\widetilde{n}$ and $\widetilde{p}$). Together with Lemma 4.3 we additionally derive the uniform estimates

$$\|\widetilde{n}\mu_n(\widetilde{T}, \widetilde{n}, |\nabla\psi|)|\nabla\varphi_n|^2\|_{L^1},\ \|\widetilde{p}\mu_p(\widetilde{T}, \widetilde{p}, |\nabla\psi|)|\nabla\varphi_p|^2\|_{L^1} \leq c_J.$$

5. Finally, we find by the upper bound of r in Assumption (A) and the bounds for φ_n, φ_p, and the lower estimate $\widetilde{T} \geq T_a$ that

$$\|r(\widetilde{n}, \widetilde{p}, \widetilde{T})\Big(\exp \frac{\varphi_n - \varphi_p}{\widetilde{T}} - 1\Big)(\varphi_n - \varphi_p)\|_{L^1} \leq c_R. \qquad \square$$

The estimates in Step 4 and 5 of the previous proof ensure for the function h defined in (4.7) that $\|h\|_{L^1} \leq 2c_J + c_R$.

(4) Entropy solutions of the heat flow equation. Under the first two assumptions in (A), we consider the stationary linear heat flow equation with Robin boundary conditions and right hand side $h \in L^1(\Omega)$ as well as boundary data $g \in L^1(\Gamma)$, namely

$$\begin{aligned} -\nabla \cdot (\lambda(x)\nabla T) &= h(x) \quad \text{in } \Omega, \\ -\lambda(x)\nabla T \cdot \nu &= \kappa(x)T - g(x) \quad \text{on } \Gamma. \end{aligned} \tag{4.14}$$

The following three results are proven in [4].

Theorem 4.1 *Let $h \in L^1(\Omega)$, $g \in L^1(\Gamma)$. Then there exists a unique entropy solution T to* (4.14)*. This entropy solution belongs to $W^{1,q}(\Omega)$, for all $1 \leq q < \frac{d}{d-1}$. Especially, there are constants $c_{Eq} > 0$ not depending on f and g such that*

$$\|T\|_{W^{1,q}} \leq c_{Eq}(\|h\|_{L^1} + \|g\|_{L^1(\Gamma)}), \quad 1 \leq q < \tfrac{d}{d-1}.$$

Lemma 4.5 *Let $h^l \to h$ in $L^1(\Omega)$, $g^l \to g$ in $L^1(\Gamma)$. Then the corresponding entropy solutions T^l to* (4.14) *converge weakly in $W^{1,q}(\Omega)$ to the entropy solution T for data h and g, $1 \leq q < \frac{d}{d-1}$.*

Lemma 4.6 *Let $h \in L^1_+(\Omega)$ and $g = \kappa T_a$ with $T_a =$ const > 0. Then the entropy solution T to* (4.14) *fulfills $T \geq T_a$ a.e. on Ω.*

4.3 Continuity of the fixed point map $\mathcal{Q}$ and existence result

We prove the continuity of the fixed point map $\mathcal{Q} : \mathcal{N} \mapsto \mathcal{N}$ in two steps, first we verify the continuous dependence of the solution to the nonlinear Poisson equation on the arguments $(\widetilde{\varphi}_n, \widetilde{\varphi}_p, \widetilde{T})$ and in the second step the continuity of $\mathcal{Q}$ itself is established.

Lemma 4.7 *We assume (A). Let $(\widetilde{\varphi}_n^l, \widetilde{\varphi}_p^l, \widetilde{T}^l)$, $(\widetilde{\varphi}_n, \widetilde{\varphi}_p, \widetilde{T}) \in \mathcal{N}$ with $\widetilde{\varphi}_n^l \to \widetilde{\varphi}_n$, $\widetilde{\varphi}_p^l \to \widetilde{\varphi}_p$, and $\widetilde{T}^l \to \widetilde{T}$ in $L^2(\Omega)$, let ψ^l and ψ denote the corresponding unique weak solutions to* (4.2)*. Then $\psi^l \to \psi$ in $H^1(\Omega)$.*

Proof. Since mes $\Gamma_D > 0$, the expression $\|\nabla\cdot\|_{L^2}$ is an equivalent norm on $H^1_D(\Omega)$ such that $\underline{\alpha}\|w\|^2_{H^1(\Omega)} \le \underline{\varepsilon}\|\nabla w\|^2_{L^2}$ for all $w \in H^1_D(\Omega)$. We use the short notation $\widetilde{N}_{i0} := N_{i0}(\widetilde{T})$, $\widetilde{N}^l_{i0} := N_{i0}(\widetilde{T}^l)$, $\widetilde{\sigma}_i := \sigma_i(\widetilde{T})$, $\widetilde{\sigma}^l_i := \sigma_i(\widetilde{T}^l)$, $i = n, p$, $\widetilde{E}_J := E_J(\widetilde{T})$, $\widetilde{E}^l_J := E_J(\widetilde{T}^l)$, $J = L, H$. We test (4.2) (for the corresponding solutions ψ^l and ψ) by $\psi^l - \psi \in H^1_D(\Omega)$ and obtain

$$
\begin{aligned}
&\underline{\alpha}\|\psi^l - \psi\|^2_{H^1}\\
&\le \int_\Omega \Big[\widetilde{N}_{n0}\mathcal{G}\Big(\frac{\psi - \widetilde{\varphi}_n - \widetilde{E}_L}{\widetilde{T}}; \frac{\widetilde{\sigma}_n}{\widetilde{T}}\Big) - \widetilde{N}^l_{n0}\mathcal{G}\Big(\frac{\psi^l - \widetilde{\varphi}^l_n - \widetilde{E}^l_L}{\widetilde{T}^l}; \frac{\widetilde{\sigma}^l_n}{\widetilde{T}^l}\Big)\Big](\psi^l - \psi)\,\mathrm{d}x\\
&- \int_\Omega \Big[\widetilde{N}_{p0}\mathcal{G}\Big(\frac{\widetilde{E}_H - (\psi - \widetilde{\varphi}_p)}{\widetilde{T}}; \frac{\widetilde{\sigma}_p}{\widetilde{T}}\Big) - \widetilde{N}^l_{p0}\mathcal{G}\Big(\frac{\widetilde{E}^l_H - (\psi^l - \widetilde{\varphi}^l_p)}{\widetilde{T}^l}; \frac{\widetilde{\sigma}^l_p}{\widetilde{T}^l}\Big)\Big](\psi^l - \psi)\,\mathrm{d}x\\
&= \sum_{j=1}^4 I_j,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 := &\int_\Omega \Big(\widetilde{N}_{n0} - \widetilde{N}^l_{n0}\Big)\mathcal{G}\Big(\frac{\psi - \widetilde{\varphi}_n - \widetilde{E}_L}{\widetilde{T}}; \frac{\widetilde{\sigma}_n}{\widetilde{T}}\Big)(\psi^l - \psi)\,\mathrm{d}x\\
&- \int_\Omega \Big(\widetilde{N}_{p0} - \widetilde{N}^l_{p0}\Big)\mathcal{G}\Big(\frac{\widetilde{E}_H - (\psi - \widetilde{\varphi}_p)}{\widetilde{T}}; \frac{\widetilde{\sigma}_p}{\widetilde{T}}\Big)(\psi^l - \psi)\,\mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
I_2 := &\int_\Omega \widetilde{N}^l_{n0}\Big[\mathcal{G}\Big(\frac{\psi - \widetilde{\varphi}_n - \widetilde{E}_L}{\widetilde{T}}; \frac{\widetilde{\sigma}_n}{\widetilde{T}}\Big) - \mathcal{G}\Big(\frac{\psi - \widetilde{\varphi}^l_n - \widetilde{E}^l_L}{\widetilde{T}^l}; \frac{\widetilde{\sigma}_n}{\widetilde{T}}\Big)\Big](\psi^l - \psi)\,\mathrm{d}x\\
&- \int_\Omega \widetilde{N}^l_{p0}\Big[\mathcal{G}\Big(\frac{\widetilde{E}_H - (\psi - \widetilde{\varphi}_p)}{\widetilde{T}}; \frac{\widetilde{\sigma}_p}{\widetilde{T}}\Big) - \mathcal{G}\Big(\frac{\widetilde{E}^l_H - (\psi - \widetilde{\varphi}^l_p)}{\widetilde{T}^l}; \frac{\widetilde{\sigma}_p}{\widetilde{T}}\Big)\Big](\psi^l - \psi)\,\mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
I_3 := &\int_\Omega \widetilde{N}^l_{n0}\Big[\mathcal{G}\Big(\frac{\psi - \widetilde{\varphi}^l_n - \widetilde{E}^l_L}{\widetilde{T}^l}; \frac{\widetilde{\sigma}_n}{\widetilde{T}}\Big) - \mathcal{G}\Big(\frac{\psi - \widetilde{\varphi}^l_n - \widetilde{E}^l_L}{\widetilde{T}^l}; \frac{\widetilde{\sigma}^l_n}{\widetilde{T}^l}\Big)\Big](\psi^l - \psi)\,\mathrm{d}x\\
&- \int_\Omega \widetilde{N}^l_{p0}\Big[\mathcal{G}\Big(\frac{\widetilde{E}^l_H - (\psi - \widetilde{\varphi}^l_p)}{\widetilde{T}^l}; \frac{\widetilde{\sigma}_p}{\widetilde{T}}\Big) - \mathcal{G}\Big(\frac{\widetilde{E}^l_H - (\psi - \widetilde{\varphi}^l_p)}{\widetilde{T}^l}; \frac{\widetilde{\sigma}^l_p}{\widetilde{T}^l}\Big)\Big](\psi^l - \psi)\,\mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
I_4 := &\int_\Omega \widetilde{N}^l_{n0}\Big[\mathcal{G}\Big(\frac{\psi - \widetilde{\varphi}^l_n - \widetilde{E}^l_L}{\widetilde{T}^l}; \frac{\widetilde{\sigma}^l_n}{\widetilde{T}^l}\Big) - \mathcal{G}\Big(\frac{\psi^l - \widetilde{\varphi}^l_n - \widetilde{E}^l_L}{\widetilde{T}^l}; \frac{\widehat{\sigma}^l_n}{\widetilde{T}^l}\Big)\Big](\psi^l - \psi)\,\mathrm{d}x\\
&- \int_\Omega \widetilde{N}^l_{p0}\Big[\mathcal{G}\Big(\frac{\widetilde{E}^l_H - (\psi - \widetilde{\varphi}^l_p)}{\widetilde{T}^l}; \frac{\widetilde{\sigma}^l_p}{\widetilde{T}^l}\Big) - \mathcal{G}\Big(\frac{\widetilde{E}^l_H - (\psi^l - \widetilde{\varphi}^l_p)}{\widetilde{T}^l}; \frac{\widetilde{\sigma}^l_p}{\widetilde{T}^l}\Big)\Big](\psi^l - \psi)\,\mathrm{d}x.
\end{aligned}
$$

Due to the monotonicity of $\eta \mapsto \mathcal{G}(\eta, z)$ the term I_4 is non-positive and can be omitted in our estimates. According to Assumption (A) and the boundedness of $\mathcal{G}$ we find

$$|I_1| \le c\|\widetilde{T} - \widetilde{T}^l\|_{L^2}\|\psi^l - \psi\|_{L^2}.$$

Since

$$
\begin{aligned}
|I_2| \le &\int_\Omega c\Big|\mathcal{G}\Big(\frac{\psi - \widetilde{\varphi}_n - \widetilde{E}_L}{\widetilde{T}}; \frac{\widetilde{\sigma}_n}{\widetilde{T}}\Big) - \mathcal{G}\Big(\frac{\psi - \widetilde{\varphi}^l_n - \widetilde{E}^l_L}{\widetilde{T}^l}; \frac{\widetilde{\sigma}_n}{\widetilde{T}}\Big)\Big||\psi^l - \psi|\,\mathrm{d}x\\
&+ \int_\Omega c\Big|\mathcal{G}\Big(\frac{\widetilde{E}_H - (\psi - \widetilde{\varphi}_p)}{\widetilde{T}}; \frac{\widetilde{\sigma}_p}{\widetilde{T}}\Big) - \mathcal{G}\Big(\frac{\widetilde{E}^l_H - (\psi - \widetilde{\varphi}^l_p)}{\widetilde{T}^l}; \frac{\widetilde{\sigma}_p}{\widetilde{T}}\Big)\Big||\psi^l - \psi|\,\mathrm{d}x,
\end{aligned}
$$

we estimate the term with the large absolute value bars by the derivative of $\mathcal{G}$ with respect to the first variable in an intermediate point (which is uniformly bounded) times $|\frac{\psi-\widetilde{\varphi}_n-\widetilde{E}_L}{\widetilde{T}} - \frac{\psi-\widetilde{\varphi}_n^l-\widetilde{E}_L^l}{\widetilde{T}^l}|$ and $|\frac{\widetilde{E}_H-(\psi-\widetilde{\varphi}_p)}{\widetilde{T}} - \frac{\widetilde{E}_H^l-(\psi-\widetilde{\varphi}_p^l)}{\widetilde{T}^l}|$, respectively, which then due to the Lipschitz continuity of E_L and E_H, and the lower bound T_a of the temperature for these absolute values gives a bound

$$c\Big(\Big|\frac{1}{\widetilde{T}^l} - \frac{1}{\widetilde{T}}\Big| + \frac{|\widetilde{T}^l - \widetilde{T}|}{T_a} + \sum_{i=n,p} |\widetilde{\varphi}_i^l - \widetilde{\varphi}_i|\Big) \le c\Big(\frac{1}{T_a}|\widetilde{T}^l - \widetilde{T}| + \sum_{i=n,p} |\widetilde{\varphi}_i^l - \widetilde{\varphi}_i|\Big).$$

Moreover, using Lemma 2.1 we derive

$$\begin{aligned}
&\Big|\mathcal{G}(\eta; \frac{\widetilde{\sigma}_i}{\widetilde{T}}) - \mathcal{G}(\eta; \frac{\widetilde{\sigma}_i}{\widetilde{T}^l})\Big| + \Big|\mathcal{G}(\eta; \frac{\widetilde{\sigma}_i}{\widetilde{T}^l}) - \mathcal{G}(\eta; \frac{\widetilde{\sigma}_i^l}{\widetilde{T}^l})\Big| \\
&\le \max_{z_{\theta_1} \in [\frac{\widetilde{\sigma}_i}{\widetilde{T}}, \frac{\widetilde{\sigma}_i}{\widetilde{T}^l}]} \Big|\frac{\partial \mathcal{G}}{\partial z}(\eta; z_{\theta_1})\Big|\widetilde{\sigma}_i\Big|\frac{1}{\widetilde{T}} - \frac{1}{\widetilde{T}^l}\Big| + \max_{z_{\theta_2} \in [\frac{\widetilde{\sigma}_i}{\widetilde{T}^l}, \frac{\widetilde{\sigma}_i^l}{\widetilde{T}^l}]} \Big|\frac{\partial \mathcal{G}}{\partial z}(\eta; z_{\theta_2})\Big| \frac{1}{\widetilde{T}^l}\, |\widetilde{T} - \widetilde{T}^l| \\
&\le \max\{\widetilde{T}, \widetilde{T}^l\}(1 + \mathrm{e}^{|\eta|})\Big|\frac{1}{\widetilde{T}} - \frac{1}{\widetilde{T}^l}\Big| + \max\Big\{\frac{1}{\widetilde{\sigma}_i}, \frac{1}{\widetilde{\sigma}_i^l}\Big\}(1 + \mathrm{e}^{|\eta|})|\widetilde{T} - \widetilde{T}^l| \\
&\le (\frac{1}{T_a} + \frac{1}{\underline{\sigma}})(1 + \mathrm{e}^{|\eta|})|\widetilde{T} - \widetilde{T}^l|.
\end{aligned}$$

With this, we can continue the estimate for I_3

$$\begin{aligned}
|I_3| \le &\int_\Omega c\Big|\mathcal{G}\Big(\frac{\psi - \widetilde{\varphi}_n^l - \widetilde{E}_L^l}{\widetilde{T}^l}; \frac{\widetilde{\sigma}_n}{\widetilde{T}}\Big) - \mathcal{G}\Big(\frac{\psi - \widetilde{\varphi}_n^l - \widetilde{E}_L^l}{\widetilde{T}^l}; \frac{\widetilde{\sigma}_n^l}{\widetilde{T}^l}\Big)\Big||\psi^l - \psi|\,\mathrm{d}x \\
&+ \int_\Omega c\Big|\mathcal{G}\Big(\frac{\widetilde{E}_H^l - (\psi - \widetilde{\varphi}_p^l)}{\widetilde{T}^l}; \frac{\widetilde{\sigma}_p}{\widetilde{T}}\Big) - \mathcal{G}\Big(\frac{\widetilde{E}_H^l - (\psi - \widetilde{\varphi}_p^l)}{\widetilde{T}^l}; \frac{\widetilde{\sigma}_p^l}{\widetilde{T}^l}\Big)\Big||\psi^l - \psi|\,\mathrm{d}x
\end{aligned}$$

by finding the upper bound for the two terms with the large absolute value bars

$$c\Big(1 + \exp\Big|\frac{\psi - \widetilde{\varphi}_n^l - \widetilde{E}_L^l}{\widetilde{T}^l}\Big|\Big)|\widetilde{T}^l - \widetilde{T}|, \quad \text{and} \quad c\Big(1 + \exp\Big|\frac{\widetilde{E}_H^l - (\psi - \widetilde{\varphi}_p^l)}{\widetilde{T}^l}\Big|\Big)|\widetilde{T}^l - \widetilde{T}|,$$

respectively. Note that the argument in the exponential is uniformly bounded since $(\widetilde{\varphi}_n^l, \widetilde{\varphi}_p^l, \widetilde{T}^l)$, $(\widetilde{\varphi}_n, \widetilde{\varphi}_p, \widetilde{T}) \in \mathcal{N}$, $|\widetilde{E}_L^l|$, $|\widetilde{E}_H^l| \le \widehat{E}$, and $\|\psi\|_{L^\infty} \le c_{\psi,L^\infty}$ by Lemma 4.1. Collecting now all the previous arguments we arrive at

$$\begin{aligned}
\|\psi^l - \psi\|_{H^1}^2 &\le c\int_\Omega \Big(\sum_{i=n,p} |\widetilde{\varphi}_i^l - \widetilde{\varphi}_i| + |\widetilde{T}^l - \widetilde{T}|\Big)|\psi^l - \psi|\,\mathrm{d}x \\
&\le c\Big(\sum_{i=n,p} \|\widetilde{\varphi}_i^l - \widetilde{\varphi}_i\|_{L^2} + \|\widetilde{T}^l - \widetilde{T}\|_{L^2}\Big)\|\psi^l - \psi\|_{L^2}
\end{aligned}$$

which ensures that $\|\psi^l - \psi\|_{H^1} \to 0$ since $\widetilde{\varphi}_n^l \to \widetilde{\varphi}_n$, $\widetilde{\varphi}_p^l \to \widetilde{\varphi}_p$, and $\widetilde{T}^l \to \widetilde{T}$ in $L^2(\Omega)$. □

Lemma 4.8 *Let the Assumption (A) be fulfilled. Then the map $\mathcal{Q} : \mathcal{N} \to \mathcal{N}$ is continuous.*

Proof. 1. Let $(\widetilde{\varphi}_n^l, \widetilde{\varphi}_p^l, \widetilde{T}^l)$, $(\widetilde{\varphi}_n, \widetilde{\varphi}_p, \widetilde{T}) \in \mathcal{N}$ with $\widetilde{\varphi}_n^l \to \widetilde{\varphi}_n$, $\widetilde{\varphi}_p^l \to \widetilde{\varphi}_p$, and $\widetilde{T}^l \to \widetilde{T}$ in $L^2(\Omega)$, let ψ^l and ψ denote the corresponding unique weak solutions to (4.2), $\widetilde{n}^l$, $\widetilde{n}$ and let $\widetilde{p}^l$, $\widetilde{p}$ be the corresponding quantities in (4.3). We have to show that $(\varphi_n^l, \varphi_p^l, T^l) := \mathcal{Q}(\widetilde{\varphi}_n^l, \widetilde{\varphi}_p^l, \widetilde{T}^l) \to (\varphi_n, \varphi_p, T) := \mathcal{Q}(\widetilde{\varphi}_n, \widetilde{\varphi}_p, \widetilde{T})$ in $L^2(\Omega)^3$. Lemma 4.7 ensures that $\psi^l \to \psi$ in $H^1(\Omega)$.

2. By Lemma 4.4 we have $\|\varphi_n^l\|_{H^1}$, $\|\varphi_p^l\|_{H^1} \le c_{H^1}$. We show that all weakly convergent subsequences of $\{(\varphi_n^l, \varphi_p^l)\}$ in $H^1(\Omega)^2$ converge weakly to the same limit (φ_n, φ_p). Then using [14, Lemma 5.4] we have

$(\varphi_n^l, \varphi_p^l) \rightharpoonup (\varphi_n, \varphi_p)$ in $H^1(\Omega)^2$ for the entire sequence and as a consequence $\varphi_n^l \to \varphi_n$, $\varphi_p^l \to \varphi_p$ in $L^2(\Omega)$.

Let $\{(\varphi_n^{l_k}, \varphi_p^{l_k})\}$ be some subsequence in $H^1(\Omega)^2$ that converges weakly to some $(\varphi_n^*, \varphi_p^*) \in H^1(\Omega)^2$. Our goal is to verify that $\varphi_n^* = \varphi_n$ and $\varphi_p^* = \varphi_p$. Since $\widetilde{\varphi}_n^{l_k} \to \widetilde{\varphi}_n$, $\widetilde{\varphi}_p^{l_k} \to \widetilde{\varphi}_p$, and $\widetilde{T}^{l_k} \to \widetilde{T}$ in $L^2(\Omega)$ and $\psi^{l_k} \to \psi$ in $H^1(\Omega)$ we obtain, for a further, non-relabeled subsequence, that $\widetilde{\varphi}_n^{l_k} \to \widetilde{\varphi}_n$, $\widetilde{\varphi}_p^{l_k} \to \widetilde{\varphi}_p$, $\widetilde{T}^{l_k} \to \widetilde{T}$, $\psi^{l_k} \to \psi$, and $\nabla\psi^{l_k} \to \nabla\psi$ a.e. in Ω. Because of the continuity of the functions $(\psi, \varphi_n, T) \mapsto N_{n0}(T)\mathcal{G}(\frac{\psi-\varphi_n-E_L(T)}{T}; \frac{\sigma_n(T)}{T})$, $(\psi, \varphi_p, T) \mapsto N_{p0}(T)\mathcal{G}(\frac{E_H(T)-(\psi-\varphi_p)}{T}; \frac{\sigma_p(T)}{T})$ for $T \geq T_a$ as well as that of the mobility functions μ_n and μ_p (with respect to T, n, p and $|\nabla\psi|$) we find for that subsequence

$$\begin{aligned}
&\widetilde{n}^{l_k} := N_{n0}(\widetilde{T}^{l_k})\mathcal{G}\Big(\tfrac{\psi^{l_k}-\widetilde{\varphi}_n^{l_k}-E_L(\widetilde{T}^{l_k})}{\widetilde{T}^{l_k}}; \tfrac{\sigma_n(\widetilde{T}^{l_k})}{\widetilde{T}^{l_k}}\Big) \to \widetilde{n} := N_{n0}(\widetilde{T})\mathcal{G}\Big(\tfrac{\psi-\widetilde{\varphi}_n-E_L(\widetilde{T})}{\widetilde{T}}; \tfrac{\sigma_n(\widetilde{T})}{\widetilde{T}}\Big),\\
&\widetilde{p}^{l_k} := N_{p0}(\widetilde{T}^{l_k})\mathcal{G}\Big(\tfrac{E_H(\widetilde{T}^{l_k})-(\psi^{l_k}-\widetilde{\varphi}_p^{l_k})}{\widetilde{T}^{l_k}}; \tfrac{\sigma_p(\widetilde{T}^{l_k})}{\widetilde{T}^{l_k}}\Big) \to \widetilde{p} := N_{p0}(\widetilde{T})\mathcal{G}\Big(\tfrac{E_H(\widetilde{T})-(\psi-\widetilde{\varphi}_p)}{\widetilde{T}}; \tfrac{\sigma_p(\widetilde{T})}{\widetilde{T}}\Big),\\
&\mu_n(\widetilde{T}^{l_k}, \widetilde{n}^{l_k}, |\nabla\psi^{l_k}|) \to \mu_n(\widetilde{T}, \widetilde{n}, |\nabla\psi|), \quad \mu_p(\widetilde{T}^{l_k}, \widetilde{p}^{l_k}, |\nabla\psi^{l_k}|) \to \mu_p(\widetilde{T}, \widetilde{p}, |\nabla\psi|) \text{ a.e. in } \Omega.
\end{aligned}$$

The pointwise a.e. convergences for this subsequence and Assumption (A) additionally ensure the following convergences

$$\begin{aligned}
\widetilde{a}_n^{l_k} &:= \widetilde{n}^{l_k}\mu_n(\widetilde{T}^{l_k}, \widetilde{n}^{l_k}, |\nabla\psi^{l_k}|) \to \widetilde{a}_n := \widetilde{n}\mu_n(\widetilde{T}, \widetilde{n}, |\nabla\psi|),\\
\widetilde{a}_p^{l_k} &:= \widetilde{p}^{l_k}\mu_p(\widetilde{T}^{l_k}, \widetilde{p}^{l_k}, |\nabla\psi^{l_k}|) \to \widetilde{a}_p := \widetilde{p}\mu_p(\widetilde{T}, \widetilde{p}, |\nabla\psi|),\\
\widetilde{r}^{l_k} &:= r(\widetilde{n}^{l_k}, \widetilde{p}^{l_k}, \widetilde{T}^{l_k}) \to \widetilde{r} := r(\widetilde{n}, \widetilde{p}, \widetilde{T}) \quad \text{a.e. in } \Omega.
\end{aligned} \tag{4.15}$$

Using $(\varphi_n^{l_k} - \varphi_n, \varphi_p^{l_k} - \varphi_p) \in H_D^1(\Omega)^2$ as a test function in (4.6) we obtain

$$\begin{aligned}
&\int_\Omega \sum_{i=n,p} \big\{\widetilde{a}_i^{l_k}\nabla\varphi_i^{l_k} - \widetilde{a}_i\nabla\varphi_i\big\} \cdot \nabla(\varphi_i^{l_k} - \varphi_i)\,\mathrm{d}x\\
&= \int_\Omega \Big(\widetilde{r}\Big(\exp\frac{\varphi_n-\varphi_p}{\widetilde{T}} - 1\Big) - \widetilde{r}^{l_k}\Big(\exp\frac{\varphi_n^{l_k}-\varphi_p^{l_k}}{\widetilde{T}^{l_k}} - 1\Big)\Big)\Big(\varphi_n^{l_k}-\varphi_n - \varphi_p^{l_k}+\varphi_p\Big)\,\mathrm{d}x.
\end{aligned} \tag{4.16}$$

We introduce the following decomposition

$$\begin{aligned}
\widetilde{a}_i^{l_k}\nabla\varphi_i^{l_k} &= \widetilde{a}_i^{l_k}\nabla(\varphi_i^{l_k} - \varphi_i) + \widetilde{a}_i^{l_k}\nabla\varphi_i\,,\\
\widetilde{r}^{l_k}\exp\frac{\varphi_n^{l_k} - \varphi_p^{l_k}}{\widetilde{T}^{l_k}} &= (\widetilde{r}^{l_k} - \widetilde{r})\exp\frac{\varphi_n^{l_k} - \varphi_p^{l_k}}{\widetilde{T}^{l_k}}\\
&\quad + \widetilde{r}\Big[\exp\frac{\varphi_n^{l_k} - \varphi_p^{l_k}}{\widetilde{T}^{l_k}} - \exp\frac{\varphi_n^{l_k} - \varphi_p^{l_k}}{\widetilde{T}}\Big] + \widetilde{r}\exp\frac{\varphi_n^{l_k} - \varphi_p^{l_k}}{\widetilde{T}}.
\end{aligned}$$

Then, the fact that $\widetilde{T}$, $\widetilde{T}^{l_k} \geq T_a$ and $\varphi_n^{l_k}$, $\varphi_p^{l_k} \in [-K, K]$ a.e. in Ω due to Lemma 4.4 with $\exp\frac{\varphi_n^{l_k}-\varphi_p^{l_k}}{\widetilde{T}^{l_k}} \leq c$ together with Lipschitz continuity of the map $(\varphi_n, \varphi_p, T) \mapsto \exp\frac{\varphi_n-\varphi_p}{T}$ on $[-K, K]^2 \times [T_a, \infty)$, the bounds from (4.13) and mes$(\Gamma_D) > 0$ yield from (4.16)

$$\begin{aligned}
&c\sum_{i=n,p}\|\varphi_i^{l_k} - \varphi_i\|_{H^1}^2 + \int_\Omega \widetilde{r}\Big(\exp\frac{\varphi_n^{l_k} - \varphi_p^{l_k}}{\widetilde{T}} - \exp\frac{\varphi_n - \varphi_p}{\widetilde{T}}\Big)\Big(\varphi_n^{l_k} - \varphi_n - \varphi_p^{l_k} + \varphi_p\Big)\,\mathrm{d}x\\
&\quad \leq c\sum_{i=n,p}\|\nabla(\varphi_i^{l_k} - \varphi_i)\|_{L^2}\Big(\int_\Omega \big|\widetilde{a}_i^{l_k} - \widetilde{a}_i\big|^2|\nabla\varphi_i|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\\
&\qquad + c\sum_{i=n,p}\|\varphi_i^{l_k} - \varphi_i\|_{L^2}\Big(\Big(\int_\Omega \big|\widetilde{r}^{l_k} - \widetilde{r}\big|^2\mathrm{d}x\Big)^{\frac{1}{2}} + \|\widetilde{T}^{l_k} - \widetilde{T}\|_{L^2}\Big).
\end{aligned}$$

Due to (4.13) and $\|\varphi_i\|_{H^1} \leq c_{H^1}$ the first integral on the right hand side has an integrable majorant. Since by assumption, the function r is bounded also the integrand of the integral in the last line has an integrable majorant. Thus we can apply for both integrals Lebesgue's dominated convergence theorem to show that both integrals tend to zero, and $\widetilde{T}^{l_k} \to \widetilde{T}$ in $L^2(\Omega)$ by assumption. Therefore, in summary it follows $\|\varphi_i^{l_k} - \varphi_i\|_{H^1} \to 0$ for the subsequence related to the a.e. convergence of $\widetilde{\varphi}_n^{l_k}, \widetilde{\varphi}_p^{l_k}, \widetilde{T}^{l_k}, \psi^{l_k}$ and $\nabla\psi^{l_k}$. Since by assumption this subsequence also weakly converges to φ_i^*, we find that $\varphi_i^* = \varphi_i$ and that the entire subsequence converges weakly to φ_i, $i = n, p$.

Since the subsequence was arbitrary, we verified that all weakly convergent subsequences of $\{(\varphi_n^l, \varphi_p^l)\}$ converge weakly to (φ_n, φ_p). Thus by [14, Lemma 5.4] follows $(\varphi_n^l, \varphi_p^l) \rightharpoonup (\varphi_n, \varphi_p)$ in $H^1(\Omega)^2$ for the entire sequence and therefore $\varphi_i^l \to \varphi_i$ in $L^2(\Omega)$, $i = n, p$.

3. It remains to show for the corresponding solutions to (4.8) that $T^l \to T$ in $L^2(\Omega)$. For this purpose we show $T^l \rightharpoonup T$ in $W^{1,4/3}(\Omega)$. According to Theorem 4.1 and (4.9) we have $\|T^l\|_{W^{1,4/3}} \leq c_{T,W^{1,4/3}}$ for all l. We show that all weakly convergent subsequences of $\{T^l\}$ in the reflexive Banach space $W^{1,4/3}(\Omega)$ converge weakly to T. Then it results $T^l \rightharpoonup T$ in $W^{1,4/3}(\Omega)$ for the entire sequence and therefore $T^l \to T$ in $L^2(\Omega)$ which we finally aim to prove. Let for some subsequence $\{T^{l_k}\}$ and some $T^* \in W^{1,4/3}(\Omega)$ hold true that $T^{l_k} \rightharpoonup T^*$ in $W^{1,4/3}(\Omega)$. We verify that $T^* = T$.

For this purpose we start with further convergences of non-relabeled subsequences, where especially $\varphi_i^{l_k} \to \varphi_i$ in $H^1(\Omega)$, $i = n, p$. Since

$$\Big|\widetilde{a}_i^{l_k}|\nabla\varphi_i^{l_k}|^2 - \widetilde{a}_i|\nabla\varphi_i|^2\Big| \leq \widetilde{a}_i^{l_k}|\nabla(\varphi_i^{l_k} - \varphi_i)||\nabla\varphi_i^{l_k}| + \widetilde{a}_i^{l_k}|\nabla\varphi_i||\nabla(\varphi_i^{l_k} - \varphi_i)| + |\widetilde{a}_i^{l_k} - \widetilde{a}_i||\nabla\varphi_i|^2$$

it follows with (4.5)

$$\begin{aligned}\int_\Omega \Big|\widetilde{a}_i^{l_k}|\nabla\varphi_i^{l_k}|^2 - \widetilde{a}_i|\nabla\varphi_i|^2\Big|\,\mathsf{d}x &\leq c\|\varphi_i^{l_k} - \varphi_i\|_{H^1}\|\varphi_i^{l_k}\|_{H^1} + c\|\varphi_i\|_{H^1}\|\varphi_i^{l_k} - \varphi_i\|_{H^1} \\ &\quad + \Big(\int_\Omega |\widetilde{a}_i^{l_k} - \widetilde{a}_i|^2|\nabla\varphi_i|^2\,\mathsf{d}x\Big)^{\frac{1}{2}}\|\varphi_i\|_{H^1}.\end{aligned}$$

Note that $\|\varphi_i^{l_k}\|_{H^1}, \|\varphi_i\|_{H^1} \leq c_{H^1}$. Due to (4.15), Lebesgue's dominated convergence theorem gives the convergence to zero of the last integral for a further, non-relabeled subsequence. Hence the right hand side in the estimate tends to zero.

Moreover, Step 2 ensures $\varphi_i^{l_k} \to \varphi_i$ in $L^2(\Omega)$, and therefore, for a further, non-relabeled subsequence $\varphi_i^{l_k} \to \varphi_i$ a.e. in Ω, $i = n,\ p$. Together with

$$\widetilde{r}^{l_k} \to \widetilde{r} \text{ and } \exp\frac{\varphi_n^{l_k} - \varphi_p^{l_k}}{\widetilde{T}^{l_k}} \to \exp\frac{\varphi_n - \varphi_p}{\widetilde{T}} \text{ a.e. in } \Omega,$$

and the integrable majorant $4K\overline{r}\exp\frac{2K}{T_a}$ Lebesgue's dominated convergence theorem gives for this subsequence

$$\int_\Omega \Big|\widetilde{r}^{l_k}\Big(\exp\frac{\varphi_n^{l_k} - \varphi_p^{l_k}}{\widetilde{T}^{l_k}} - 1\Big)(\varphi_n^{l_k} - \varphi_p^{l_k}) - \widetilde{r}\Big(\exp\frac{\varphi_n - \varphi_p}{\widetilde{T}} - 1\Big)(\varphi_n - \varphi_p)\Big|\,\mathsf{d}x \to 0$$

such that in summary

$$\begin{aligned} h^{l_k} &:= \sum_{i=n,p}\widetilde{a}_i^{l_k}|\nabla\varphi_i^{l_k}|^2 + \widetilde{r}^{l_k}\Big(\exp\frac{\varphi_n^{l_k} - \varphi_p^{l_k}}{\widetilde{T}^{l_k}} - 1\Big)(\varphi_n^{l_k} - \varphi_p^{l_k}) \\ &\to h := \sum_{i=n,p}\widetilde{a}_i|\nabla\varphi_i|^2 + \widetilde{r}\Big(\exp\frac{\varphi_n - \varphi_p}{\widetilde{T}} - 1\Big)(\varphi_n - \varphi_p) \quad \text{in } L^1(\Omega).\end{aligned}$$

According to Lemma 4.5 we find for the entropy solutions T^{l_k} and T of (4.8) with right hand sides h^{l_k} and h, respectively, that $T^{l_k} \rightharpoonup T$ in $W^{1,4/3}(\Omega)$. By Theorem 4.1 the solution to (4.8) with right hand side h is unique, it follows that $T^{l_k} \rightharpoonup T^* = T$ in $W^{1,4/3}(\Omega)$, for this subsequence.

4. Since we verified for arbitrary weakly convergent subsequences $T^{l_k} \rightharpoonup T^*$ in $W^{1,4/3}(\Omega)$ that $T^* = T$ we obtain the weak convergence of the entire sequence $T^l \rightharpoonup T$ in $W^{1,4/3}(\Omega)$. And by the compact embedding (for $d \leq 3$) of $W^{1,4/3}(\Omega)$ into $L^2(\Omega)$ we obtain the strong convergence of the entire sequence in $L^2(\Omega)$ which proves in summary the continuity of the operator Q. $\square$

Proof of Theorem 3.1. The set $\mathcal{N}$ is nonempty, convex, and precompact in $L^2(\Omega)^3$. Thus, by Lemma 4.8, Schauder's fixed point theorem ensures the existence of a fixed point $(\varphi_n, \varphi_p, T) \in \mathcal{N}$ of $\mathcal{Q}$. For a fixed point $(\varphi_n, \varphi_p, T) \in \mathcal{N}$ of the mapping $\mathcal{Q}$ we solve $B_{(\varphi_n,\varphi_p,T)}\psi = 0$ and calculate $n = N_{n0}\mathcal{G}\Big(\frac{\psi-\varphi_n-E_L}{T}; \frac{\sigma_n}{T}\Big)$ and $p = N_{p0}\mathcal{G}\Big(\frac{E_H-(\psi-\varphi_p)}{T}; \frac{\sigma_p}{T}\Big)$. Then the quadruple $(\psi, \varphi_n, \varphi_p, T)$ is a solution to problem (3.5) in the sense introduced in Subsection 3.3. The bounds of the solution stated in Theorem 3.1 result from Lemma 4.1, Lemma 4.4, Lemma 4.6, Theorem 4.1, and Lemma 4.3. Thus, the proof of Theorem 3.1 is complete. $\square$

Proof of Corollary 3.1. Let $(\psi, \varphi_n, \varphi_p, T)$ be a solution to (3.5) with boundary conditions fulfilling (3.7). Testing the continuity equations by $(\varphi_n - \varphi_n^D, \varphi_p - \varphi_p^D) \in H^1_D(\Omega)^2$ and taking into account the relations (3.7) it results

$$\int_\Omega \Big(n\mu_n |\nabla(\varphi_n - \varphi_n^D)|^2 + p\mu_p |\nabla(\varphi_p - \varphi_p^D)|^2 + r\big(\exp\frac{\varphi_n - \varphi_p}{T} - 1\big)(\varphi_n - \varphi_p)\Big)\,\mathrm{d}x = 0.$$

Since $\mathrm{mes}(\Gamma_D) > 0$, the lower bounds for μ_i and r as well as the lower bounds for n and p from Theorem 3.1 and the strict monotonicity of the exponential function ensure that $\varphi_n = \varphi_n^D = \varphi_p^D = \varphi_p =$ const. Therefore the source term h in the heat flow equation is zero. Thus the Robin boundary condition and the unique solvability of the heat flow equation lead to $T = T_a$. Finally with all these information and substituting n and p by the statistical relation, ψ has to fulfill the nonlinear Poisson equation (3.8). The unique solvability of the nonlinear Poisson equation relies on the Lipschitz continuity and strict monotonicity of the function $y \mapsto N_{n0}(T_a)\mathcal{G}\Big(\frac{y-\varphi_n^D-E_L(T_a)}{T_a}; \frac{\sigma_n(T_a)}{T_a}\Big) - N_{p0}(T_a)\mathcal{G}\Big(\frac{E_H(T_a)-(y-\varphi_p^D)}{T_a}; \frac{\sigma_p(T_a)}{T_a}\Big)$. $\square$

5 Concluding remarks

Theorem 3.1 gives an existence result for the stationary energy-drift-diffusion model (3.3), (3.4) for organic semiconductors in two and three spatial dimensions. Here the components electrostatic potential ψ and the quasi Fermi potentials φ_n, φ_p are to be understood as weak solutions to the van Roosbroeck system, whereas the temperature distribution T is an entropy solution to the heat equation.

In two spatial dimensions the following higher regularity of the solutions can be verified.

Corollary 5.1 *In the case $d = 2$, let Assumption (A) and $\varphi_n^D, \varphi_p^D \in W^{1,\infty}(\Omega)$ be fulfilled. Then there exist q_ψ, q_φ, $q_T > 2$ and c_{q_ψ}, c_{q_φ}, $c_{q_T} > 0$ such that for all solutions $(\psi, \varphi_n, \varphi_p, T)$ to* (3.5)

$$\|\psi\|_{W^{1,q_\psi}} \leq c_{q_\psi}, \quad \|\varphi_n\|_{W^{1,q_\varphi}} \leq c_{q_\varphi}, \quad \|\varphi_p\|_{W^{1,q_\varphi}} \leq c_{q_\varphi}, \quad \|T\|_{W^{1,q_T}} \leq c_{q_T}.$$

In particular, all four components ψ, φ_n, φ_p, T are continuous functions and T is also a weak solution of the heat flow equation.

Proof. Let $(\psi, \varphi_n, \varphi_p, T)$ be any solution to problem (3.5) and

$$n = N_{n0}\mathcal{G}\Big(\frac{\psi - \varphi_n - E_L}{T}; \frac{\sigma_n}{T}\Big), \quad p = N_{p0}\mathcal{G}\Big(\frac{E_H - (\psi - \varphi_p)}{T}; \frac{\sigma_p}{T}\Big)$$

be the corresponding densities. According to (4.13) we obtain

$$0 < c_u \le a_n := n\mu_n(T, n, |\nabla\psi|),\ a_p := p\mu_p(T, p, |\nabla\psi|) \le c_o \quad \text{a.e. in } \Omega, \quad a_n,\ a_p \in L^\infty(\Omega).$$

Due to $\|\varphi_n\|_{L^\infty},\ \|\varphi_p\|_{L^\infty} \le K$, also the expression $R = R(n, p, T)$ has a fixed L^∞ bound. With the supposed regularity of φ_n^D, φ_p^D and Gröger's regularity result for elliptic equations with nonsmooth data in 2D (see [19]) applied to

$$-\nabla\cdot(a_n\nabla\varphi_n) = R, \quad -\nabla\cdot(a_p\nabla\varphi_p) = -R,$$

we find an exponent $q_\varphi > 2$ and $c_{q_\varphi} > 0$ with $\|\varphi_n\|_{W^{1,q_\varphi}},\ \|\varphi_p\|_{W^{1,q_\varphi}} \le c_{q_\varphi}$. Using the same regularity result for the Poisson equation with a fixed L^∞ bound for the right hand side gives a $q_\psi > 2$ and $c_{q_\psi} > 0$ with $\|\psi\|_{W^{1,q_\psi}} \le c_{q_\psi}$. Have in mind the Assumption (A) for the coefficients ε and λ. The previous arguments ensure for $h := n\mu_n(T, n, |\nabla\psi|)|\nabla\varphi_n|^2 + p\mu_p(T, p, |\nabla\psi|)|\nabla\varphi_p|^2 + R(\varphi_p - \varphi_n)$ that $h \in L^{\frac{q_\varphi}{2}}(\Omega)$ with a fixed bound for the norm (and with $\frac{q_\varphi}{2} > 1$). Thus, for some $\widetilde{q} > 2$ we have $h \in W^{1,(\widetilde{q})'}(\Omega)^* \subset H^1(\Omega)^*$ with $\frac{1}{\widetilde{q}} + \frac{1}{\widetilde{q}'} = 1$. Therefore there exists a unique weak solution $\widetilde{T} \in H^1(\Omega)$ to the heat equation

$$-\nabla\cdot(\lambda\nabla T) = h$$

with Robin boundary conditions. Since $C_m(\widetilde{T} - \omega) \in H^1(\Omega)$ for all $\omega \in H^1(\Omega) \cap L^\infty(\Omega)$, $m \ge 0$, is an admissible test function in the weak formulation, we find that $\widetilde{T}$ is an entropy solution (compare (3.5)) to the heat equation, too. By Theorem 4.1 the entropy solution is unique and we have $\widetilde{T} = T$. Additionally, again by Gröger's regularity result for elliptic equations in 2D in [19] - now with Robin boundary conditions - there exists a $q_T \in (2, \widetilde{q})$ and $c_{q_T} > 0$ with $\|T\|_{W^{1,q_T}} \le c_{q_T}$. Finally, since $d = 2$ and $q_\psi,\ q_\varphi,\ q_T > 2$ Sobolev's embedding theorem ensures that $\psi,\ \varphi_n,\ \varphi_p\, T$ are continuous functions. $\square$

In analogy to the classical, inorganic situation, uniqueness is not generally to be expected. In particular, measured S-shaped current-voltage characteristics with regions of negative differential resistance for organic n-i-n resistors in [10] and organic LEDs in [9] underline the thermistor like behavior of organic semiconductor devices. Even the simplified $p(x)-$Laplace thermistor model introduced in [25] reproduces this behavior such that for a certain range of applied voltages there exist different solutions, see [24].

For a similar model frame in the inorganic situation (but assuming also a Dirichlet boundary of positive measure for the heat flow equation and not Robin type boundary conditions as in the present paper), [18] applies the implicit function theorem in the scale of Sobolev-Campanato spaces to verify local existence and uniqueness of solutions for data nearly compatible with thermodynamic equilibrium. Moreover in the inorganic setting, local uniqueness of the solution near thermodynamic equilibrium has been verified in [15] for models with cross diffusion and under consideration of thermoelectric powers. There also Dirichlet boundary parts for all equations are assumed and the heat flow equation was substituted by a balance equation for the total energy. The used variables are the electrostatic potential, electrochemical potentials divided by temperature and minus the inverse temperature. The techniques work in 2D and are based on the implicit function theorem and regularity results in $W^{1,p}$ with $p > 2$ for systems of strongly coupled elliptic differential equations with mixed boundary conditions and non-smooth data.

In the simplified model for organic semiconductor devices investigated in the present paper thermoelectric forces due to Thomson, Peltier, and Seebeck effects, leading e.g. to cross diffusion terms, were completely neglected. The next modeling steps for organic semiconductors should also include theses effects and the resulting coupling terms in the balance equations. In this direction, a reformulation of the heat flow equation as an entropy balance or balance for the total energy as done in [1, 28] or discussed in the earlier papers [5, 6] – all for the inorganic situation – is desired in order to thermodynamically design energy models for organic semiconductor devices.

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