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On the L^p-theory for second-order elliptic operators in divergence form with complex coefficients

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Abstract

Given a complex, elliptic coefficient function we investigate for which values of p the corresponding second-order divergence form operator, complemented with Dirichlet, Neumann or mixed boundary conditions, generates a strongly continuous semigroup on $\mathrm{L}^p(\Omega)$. Additional properties like analyticity of the semigroup, H^∞-calculus and maximal regularity arealso discussed. Finally we prove a perturbation result for real coefficients that gives the whole range of p's for small imaginary parts of the coefficients. Our results are based on the recent notion of p-ellipticity, reverse Hölder inequalities and Gaussian estimates for the real coefficients.

1 Introduction

One of the central items when considering elliptic operators is their 'parabolic behaviour', such as the generator property of (analytic) semigroups ([46], [33]) or maximal parabolic regularity ([18], [41]). In case of second-order divergence operators and *real measurable* coefficients very satisfactory results are available even in case of non-smooth domains and mixed boundary conditions. This is due to the fact that one can prove upper Gaussian estimates for the semigroup on L^2, see [28]. From this one can deduce that the semigroup extrapolates to a consistent semigroup on L^p for all $p \in [1, \infty]$, see [45, Ch. 7]. In addition, maximal parabolic regularity on L^p for all $p \in (1, \infty)$ can be shown ([37] and [14]) and even a bounded H^∞-calculus is obtained [23]. Moreover, it can be shown that these semigroups on L^p are all contraction semigroups ([45, Ch. 4] or [39]). This then allows for another proof of a bounded H^∞-calculus via [15] and for maximal parabolic regularity via [42].

Unfortunately nearly all of this breaks down when admitting complex coefficients. The only thing that obviously remains true is the fact that the L^2 semigroup extrapolates consistently to strongly continuous semigroups on the spaces L^p for all $p \in (2_*, 2^*)$, where 2^* is the first Sobolev exponent and $2_* = (2^*)'$ is the dual exponent. This is a consequence of the inclusion $\mathrm{W}^{1,2} \subset \mathrm{L}^p$ if $p \in [2, 2^*)$.

Apart of this, compared to the case of real-valued measurable coefficient functions several severe obstructions appear. We list four of the most striking ones. First, even if there is a consistent semigroup on L^∞, this need not be a contraction semigroup [5]. Secondly, the 'parabolic maximum principle' does not hold, as was pointed out in [6]. Furthermore, an ingenious example in [43] shows that a distributional solution for the elliptic equation with right-hand side 0 is not necessarily locally bounded. Finally, thesemigroups may even cease to exist on an L^p space with finite p, see [17].

The aim of this paper is to investigate the following two questions: First, given a bounded open set $\Omega \subset \mathbb{R}^d$ with $d \geq 2$, and a strongly elliptic coefficient matrix $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$, for which p does the elliptic divergence form operator $-A := \nabla \cdot \mu \nabla$ complemented with appropriate boundary conditions generate a strongly continuous semigroup on L^p? Secondly, if this is true, what additional properties, such as analyticity of the semigroup, bounded holomorphic functional calculus, and maximal parabolic regularity does the operator have?

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It is well-known that there exists an $\varepsilon_0 > 0$ such that $-A$ generates an analytic semigroup on $\mathrm{L}^p(\Omega)$ for all $p \in (1, \infty)$ with

$$\Big|\frac{1}{p} - \frac{1}{2}\Big| < \frac{1}{d} + \varepsilon_0, \tag{1}$$

see [16, 8, 3, 49]. In the case $d = 2$, this condition already covers the L^p-spaces for all $p \in (1, \infty)$. In general, however, condition (1) is sharp, i.e., for each $p \in [1, \infty)$ that satisfies $|1/p - 1/2| > 1/d$ there exists a strongly elliptic coefficient function $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ such that $-A$ does not generate an analytic semigroup on $\mathrm{L}^p(\Omega)$, see [38]. This raises the issue of quantifying the largeness of ε_0 for given coefficients.

In their pioneering works [10, 11, 12], Cialdea and Maz'ya found a purely algebraic condition between μ and p that implies that $-A$ is accretive on $\mathrm{L}^p(\Omega)$. Subsequently, this condition was elegantly reformulated by Carbonaro and Dragičević [9] as follows. A coefficient function $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ is called p-*elliptic* if there exists a $\lambda_p > 0$ such that

$$\operatorname{Re}\langle \mu(x)\xi, \mathcal{J}_p(\xi)\rangle \geq \lambda_p |\xi|^2$$

for almost every $x \in \Omega$ and $\xi \in \mathbb{C}^d$, where

$$\mathcal{J}_p(\xi) := 2\bigg(\frac{\operatorname{Re}(\xi)}{p'} + \frac{\mathrm{i}\operatorname{Im}(\xi)}{p}\bigg) \qquad (\xi \in \mathbb{C}^d).$$

It is shown in [9], that given a strongly elliptic matrix $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ there exists a unique number $2 < p_0(\mu) \leq \infty$ such that μ is p-elliptic if and only if $p \in (p_0(\mu)', p_0(\mu))$, where $p_0(\mu)'$ denotes the Hölder conjugate exponent to $p_0(\mu)$.

If A is complemented with mixed Dirichlet/Neumann boundary conditions we show in Theorem 3.1 that under very general conditions on Ω the operator $-A$ generates a bounded analytic semigroup on $\mathrm{L}^p(\Omega)$ if

$$\frac{p_0(\mu)d}{d(p_0(\mu) - 1) + 2} < p < \frac{p_0(\mu)d}{d - 2}. \tag{2}$$

This gives a lower bound on ε_0, namely

$$\varepsilon_0 \geq \frac{(p_0(\mu) - 2)(d - 2)}{2p_0(\mu)d}.$$

The proof of Theorem 3.1 relies on the verification of weak reverse Hölder estimates for the resolvent operators $(\lambda + A)^{-1}$ which combined with Shen's L^p-extrapolation theorem [48] extrapolates L^2-resolvent estimates to L^p. The proof of these weak reverse Hölder estimates relies on a Moser-type iteration scheme and it was the insight of Cialdea and Maz'ya [10] that the p-ellipticity condition is just the right condition that allows to test the equation with a testfunction of the form $|u|^{p-2}u$. Subsequently, a localized version of this testfunction was used by Dindoš and Pipher [19] to prove the validity of weak reverse Hölder estimates of solutions u that satisfy $-\nabla \cdot \mu\nabla u = 0$ in interior balls. In Theorem 4.2, we give an adapted argument of how to establish weak reverse Hölder estimates for balls centred at the boundary and also for the resolvent equation.

Notice that Theorem 3.1 was independently proven by Egert in [26] by a different approach. Another version of Theorem 3.1 was proved in [27] with a probabilistic viewpoint instead of explicit boundary conditions.

In Theorem 3.8 we present a perturbation result for real-valued coefficients. We show that, given a real-valued elliptic matrix μ, there exists an $\varepsilon > 0$ such that for every $\nu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ with $\|\nu\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))} < \varepsilon$ the operator $-A$ associated to the matrix $\mu + \nu$ still generates an analytic semigroup on $\mathrm{L}^p(\Omega)$ for all $p \in [1, \infty)$. As a corollary one obtains the existence of a 'threshold' $p_c > 2$ depending only on the geometry and the ellipticity constants of a complex-valued μ such that, whenever $p_0(\mu) > p_c$, the operator $-A$ generates an analytic semigroup on $\mathrm{L}^p(\Omega)$ for all $p \in (1, \infty)$.

In our approach one of the the central instruments are De Giorgi estimates. In view of our results, it seems notaccidental that in [43] the lack of L^∞-bounds for the solution of the elliptic equation is brought in connection with the breakdown of the classical De Giorgi arguments.

The bounded analyticity of the semigroup $(\mathrm{e}^{-tA})_{t\geq 0}$ combined with results in [24] have twofold consequences. One is that A viewed as an operator on $\mathrm{L}^p(\Omega)$ with p subject to (2) admits the property of maximal L^q-regularity for all $1 < q < \infty$. The second consequence is that the operator A viewed as an operator on the negative scale $\mathrm{W}^{-1,p}_D(\Omega)$ admits maximal L^q-regularity for all $1 < q < \infty$ and $p \geq 2$ subject to (2). This is described in Corollary 3.3 and Section 7, respectively.

Finally, in Theorem 3.4, we give optimal bounds for the operator $\nabla(\lambda + A)^{-1}$ in $\mathrm{L}^p(\Omega)$ and the operator $\nabla(\lambda + A)^{-1} \operatorname{div}$ in $\mathrm{L}^p(\Omega, \mathbb{C}^d)$, where $2 \leq p < 2 + \varepsilon$ for some $\varepsilon > 0$. These estimates imply gradient estimates for the semigroup operators. Notice that this is an analogue of the higher integrability statement of Meyers in [44]. We remark that the proof also works in two dimensions and for elliptic systems. It is already known (in the case $\lambda = 0$) that the bound $p < 2 + \varepsilon$ is sharp, see, e.g., the striking counterexample in the plane in [1].

The article is organised as follows. In Section 2 we introduce the geometric setup, discuss the p-ellipticity condition and present some preparatory lemmas. In Section 3 we formulate our main results and in Section 4 we prove all weak reverse Hölder estimates that are needed to extrapolate estimates from $\mathrm{L}^2(\Omega)$ to $\mathrm{L}^p(\Omega)$. In Section 5 we prove Theorems 3.1 and 3.4, and in Section 6 we prove the perturbation result of real-valued matrices, i.e., Theorem 3.8. Finally, in Section 7 we discuss the transference of the maximal regularity to the $\mathrm{W}^{-1,q}_D$-scale.

2 Notation and preliminary results

Throughout this paper the space dimension $d \geq 3$ is fixed. All Sobolev and Lebesgue spaces are considered as Banach spaces over the complex field. If $A \subset \mathbb{R}^d$ is bounded and Lebesgue measurable with its Lebesgue measure $|A| > 0$ and if $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$, then denote by $f_A := |A|^{-1} \int_A f \, \mathrm{d}x$ the mean value of f over A. The characteristic function of the set A is denoted by χ_A.

In the following, we consider a bounded and open set $\Omega \subset \mathbb{R}^d$ along with a closed subset $D \subset \partial\Omega$ of its boundary. The subset D corresponds to the boundary part where Dirichlet boundary conditions are prescribed and will be called the Dirichlet boundary. The complementary part $N := \partial\Omega \setminus D$ is called the Neumann boundary. We denote by

$$\mathrm{C}^\infty_D(\Omega) := \{\varphi|_\Omega : \varphi \in \mathrm{C}^\infty_c(\mathbb{R}^d) \text{ and } \operatorname{supp}(\varphi) \cap D = \emptyset\}$$

the space of all smooth functions that vanish on D. For all $1 \leq p < \infty$ we denote by p' the conjugate exponent to p and we denote by

$$\mathrm{W}^{1,p}_D(\Omega) := \overline{\mathrm{C}^\infty_D(\Omega)}^{\mathrm{W}^{1,p}(\Omega)}$$

the first-order Sobolev space of functions that vanish on D. By the very definition, it is clear that $\mathrm{W}^{1,p}_D(\Omega)$ is invariant under multiplication by smooth and compactly supported functions. For the record we state the following two lemmas.

Lemma 2.1. *Let* $1 \le p < \infty$ *and* $u \in \mathrm{W}^{1,p}_D(\Omega)$. *If* $\eta \in \mathrm{C}^\infty_c(\mathbb{R}^d)$, *then* $\eta u \in \mathrm{W}^{1,p}_D(\Omega)$.

Lemma 2.2. *Let* $1 < p < \infty$ *and* $u \in \mathrm{W}^{1,p}_D(\Omega)$. *Then* $|u| \in \mathrm{W}^{1,p}_D(\Omega)$.

Proof. Let $n \in \mathbb{N}$. There exists a $\varphi_n \in \mathrm{C}^\infty_c(\mathbb{R}^d)$ such that $D \cap \operatorname{supp} \varphi_n = \emptyset$ and, moreover, $\|\varphi_n|_\Omega - u\|_{\mathrm{W}^{1,p}(\Omega)} \le \frac{1}{n}$. Let $\varepsilon_n > 0$ and define $\psi_n = \sqrt{|\varphi_n|^2 + \varepsilon_n^2} - \varepsilon_n$. Clearly $\psi_n \in \mathrm{C}^\infty(\mathbb{R}^d)$ and $\operatorname{supp} \psi_n = \operatorname{supp} \varphi_n$. So $\psi_n \in \mathrm{C}^\infty_c(\mathbb{R}^d)$. Write $w_n = \psi_n|_\Omega$. Then $w_n \in \mathrm{W}^{1,p}_D(\Omega)$. Choose $\varepsilon_n > 0$ such that $\|\psi_n - |\varphi_n|\|_{\mathrm{L}^p(\mathbb{R}^d)} \le \frac{1}{n}$. Now $\nabla \psi_n = \frac{\mathrm{Re}(\overline{\varphi_n} \nabla \varphi_n)}{\sqrt{|\varphi_n|^2 + \varepsilon_n^2}}$, so $|\nabla \psi_n| \le |\nabla \varphi_n|$ and $\|w_n\|_{\mathrm{W}^{1,p}_D(\Omega)} \le \|u\|_{\mathrm{W}^{1,p}(\Omega)} + \frac{2}{n}$. Therefore the sequence $(w_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{W}^{1,p}_D(\Omega)$ and it has a weakly convergent subsequence in $\mathrm{W}^{1,p}_D(\Omega)$. The weak limit is an element of $\mathrm{W}^{1,p}_D(\Omega)$. But $\lim_{n\to\infty} w_n = |u|$ in $\mathrm{L}^p(\Omega)$. Hence the weak limit is equal to $|u|$. Consequently $|u| \in \mathrm{W}^{1,p}_D(\Omega)$. □

In the following, we define as usual $\mathrm{W}^{-1,p}_D(\Omega)$ to be the antidual space of $\mathrm{W}^{1,p'}_D(\Omega)$ whenever $1 < p < \infty$.

2.1 The geometric setup and related inequalities

The set Ω under consideration is supposed to satisfy the following condition at the closure of the Neumann boundary.

Assumption N. There exists a constant $M \ge 1$ such that for every $x \in \overline{N}$ there exist an open neighbourhood $U_x \subset \mathbb{R}^d$ of x and a bi-Lipschitz homeomorphism Φ_x from an open neighbourhood of $\overline{U_x}$ onto an open subset of $\mathbb{R}^d$ such that $\Phi_x(x) = 0$,

$$\begin{aligned} \Phi_x(U_x) &= (-1,1)^d, \\ \Phi_x(U_x \cap \Omega) &= (-1,1)^{d-1} \times (0,1), \\ \Phi_x(U_x \cap \partial\Omega) &= (-1,1)^{d-1} \times \{0\}, \end{aligned}$$

and such that the Lipschitz constants of Φ_x and Φ_x^{-1} are both less than M.

We emphasise that the constant M is independent of the point x.

In the following, we will frequently intersect Ω with a ball $B(x,r)$ so that we introduce the short-hand notation

$$\Omega(x,r) \coloneqq \Omega \cap B(x,r).$$

Remark 2.3. Let $x \in \overline{N}$. Then Assumption N allows to construct a local extension operator E in a neighbourhood of x by reflection at the Lipschitz boundary, see [49, Prop. 2.3 and Rem. 2.2]. More precisely, given $0 < r \le 1/4$ there exists a linear operator E that maps measurable functions on Ω to measurable functions on $\Omega \cup B(x, r/(M\sqrt{d}))$ which satisfies for all $1 \le p < \infty$ the estimates

$$\begin{aligned} \|Ef\|_{\mathrm{L}^p(B(x,r/(M\sqrt{d})))} &\le C\|f\|_{\mathrm{L}^p(\Omega(x,Mr))} && (f \in \mathrm{L}^p(\Omega)) \\ \|\nabla Ef\|_{\mathrm{L}^p(B(x,r/(M\sqrt{d}));\mathbb{C}^d)} &\le C\|\nabla f\|_{\mathrm{L}^p(\Omega(x,Mr);\mathbb{C}^d)} && (f \in \mathrm{W}^{1,p}_D(\Omega)). \end{aligned}$$

Here the constant $C > 0$ depends only on d, p and M.

In some situations it is desirable to have Sobolev's embedding theorem for a function $u \in \mathrm{W}^{1,p}_D(\Omega)$ on sets of the form $\Omega(x_0, r)$ available, where $x_0 \in \partial\Omega$ and $r > 0$ is small enough. The set $\Omega(x_0, r)$, however, might be very irregular as it might contain cuspidal boundary points at boundary parts where u does not vanish. A way out is guaranteed if one allows the domain of integration in the integral on the right-hand side of Sobolev's inequality to be slightly enlarged. Indeed, this allows to introduce a suitable superset of $\Omega(x_0, r)$ which is regular enough to employ Sobolev's embedding theorem there. A quantitative version of this argument is presented in the following lemma whose proof can be found in [49, Lem. 5.4].

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and $D \subset \partial\Omega$ be subject to Assumption N. Let $p \in [1, \infty)$ and $u \in \mathrm{W}^{1,p}_D(\Omega)$. Let $x_0 \in \overline{\Omega}$ and $0 < r \leq 1/(4M\sqrt{d})$ be such that either $B(x_0, r) \subset \Omega$ or $x_0 \in \partial\Omega$. If $q \in [1, \infty)$ is such that $0 \leq 1/p - 1/q \leq 1/d$, then there exists a constant $C_{\mathrm{Sob}} > 0$, depending only on p, q, d and M, such that*

$$\left(\frac{1}{r^d}\int_{\Omega(x_0,r)} |u|^q \,\mathrm{d}x\right)^{1/q} \leq C_{\mathrm{Sob}}\left\{r\left(\frac{1}{r^d}\int_{\Omega(x_0,\alpha r)} |\nabla u|^p \,\mathrm{d}x\right)^{1/p} + \left(\frac{1}{r^d}\int_{\Omega(x_0,r)} |u|^p \,\mathrm{d}x\right)^{1/p}\right\},$$

where $\alpha := M^2\sqrt{d}$.

Remark 2.5. We will use Sobolev's embedding only in the cases $p = 2$ and $q = 2d/(d-2)$, or $p = 2d/(d+2)$ and $q = 2$. In these cases C_{Sob} depends only on d and M.

To obtain higher integrability properties of the gradient of the solution to elliptic equations a further regularity property of Ω will be required in the proof. This property is the so-called corkscrew or plumpness condition of Ω^c.

Assumption Ω^c. There exist $r_0 > 0$ and $\kappa \in (0, 1)$ such that for all $x \in \partial\Omega$ and $0 < r < r_0$ there exists an $x^* \in \Omega^c$ such that $B(x^*, \kappa r) \subset \Omega^c \cap B(x, r)$.

Remark 2.6. Notice that if Ω satisfies Assumption N, then the plumpness condition is automatically satisfied in a neighbourhood of $\overline{N}$ due to the existence of the bi-Lipschitz coordinate charts. Thus, in this case, Assumption Ω^c only introduces a condition to the behaviour of Ω at the Dirichlet boundary D.

Another condition that is needed for the higher integrability property of the gradient is a plumpness condition for the Dirichlet boundary. This prevents the interface that separates D and N to have cusps that reach into the Neumann part.

Assumption D. There exist $s_0 > 0$ and $\iota \in (0, 1)$ such that for all $x \in D \cap \overline{N}$ and $0 < r < s_0$ there exists an $x^* \in D \cap B(x, r)$ such that $B(x^*, \iota r) \cap N = \emptyset$.

Under Assumptions N, Ω^c and D one can prove the following Poincaré-type inequality close to the Dirichlet boundary.

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, open set and $D \subset \partial\Omega$ be closed and subject to Assumptions N, Ω^c and D. Let $p \in [1, \infty)$. Then there exists a constant $C > 0$ such that*

$$\|u\|_{\mathrm{L}^p(\Omega(x,r))} \leq Cr\|\nabla u\|_{\mathrm{L}^p(\Omega(x,\beta r);\mathbb{C}^d)},$$

for all $u \in \mathrm{W}^{1,p}_D(\Omega)$, $x \in \partial\Omega$ and $0 < r < \min\{s_0/2, r_0/(2\iota), r_0, 1/(8M\sqrt{d})\}$ with $B(x, r) \cap D \neq \emptyset$, where $\beta := 4M^2\sqrt{d}$. Here $C > 0$ depends only on d, M, p, κ and ι.

Proof. First of all, let $x \in \partial\Omega$ and $0 < r < r_0$ be such that $B(x,r) \cap N = \emptyset$. Let $u \in \mathrm{W}^{1,p}_D(\Omega)$. Let $\widetilde{u}$ be the extension by zero to $B(x,r)$ of $u|_{\Omega(x,r)}$. Then $\widetilde{u} \in \mathrm{W}^{1,p}_D(\Omega)$ by [29, Lem. 2.2(b)]. Assumption Ω^c implies the existence of a point $x^* \in \Omega^c$ such that $B(x^*, \kappa r) \subset \Omega^c \cap B(x,r)$. By [32, Lem. 7.16] one deduces that

$$|\widetilde{u}(y) - \widetilde{u}_{B(x^*,\kappa r)}| \leq \frac{2^d}{d|B(0,1)|\kappa^d} \int_{B(x,r)} |y-z|^{-(d-1)} |\nabla \widetilde{u}(z)| \, \mathrm{d}z$$

for almost every $y \in B(x,r)$. Notice that $\widetilde{u}$ is zero on $B(x^*, \kappa r)$ so that the mean value integral on the left-hand side is zero. Taking L^p-norms with respect to the variable $y \in B(x,r)$ together with the boundedness of the Riesz potential [32, Lem. 7.12] yields

$$\|u\|_{\mathrm{L}^p(\Omega(x,r))} = \|\widetilde{u}\|_{\mathrm{L}^p(B(x,r))} \leq \frac{2^d}{\kappa^d} r \|\nabla \widetilde{u}\|_{\mathrm{L}^p(B(x,r);\mathbb{C}^d)} = \frac{2^d}{\kappa^d} r \|\nabla u\|_{\mathrm{L}^p(\Omega(x,r);\mathbb{C}^d)} \tag{3}$$

as required.

For all $x \in \partial\Omega$ and $0 < r < \min\{s_0/2, r_0/(2\iota), r_0, 1/(8M)\}$ that satisfy $B(x,r) \cap N \neq \emptyset$ and $B(x,r) \cap D \neq \emptyset$ the situation is reduced to the previous case as follows. First of all, notice that there exists a $z \in \overline{N} \cap D \cap B(x,r)$ and that $B(x,r) \subset B(z,2r)$. Employ Assumption D to obtain a point $z^* \in D \cap B(z,2r)$ such that $B(z^*, 2\iota r) \cap N = \emptyset$. Moreover, let E denote the local extension operator described in Remark 2.3. By employing the fact $Eu = u$ on $\Omega(x,r)$ and the triangle and Hölder's inequality together with $B(x,r) \subset B(z,2r)$ one estimates

$$\|u\|_{\mathrm{L}^p(\Omega(x,r))} \leq \|Eu - (Eu)_{\Omega(z^*,2\iota r)}\|_{\mathrm{L}^p(B(z,2r))} + \frac{|B(x,r)|^{1/p}}{|\Omega(z^*,2\iota r)|^{1/p}} \|u\|_{\mathrm{L}^p(\Omega(z^*,2\iota r))}. \tag{4}$$

Note that Assumption N implies that

$$\frac{|B(x,r)|^{1/p}}{|\Omega(z^*,2\iota r)|^{1/p}} \leq C \tag{5}$$

with a constant $C > 0$ that depends only on ι, d, M and p. Then the second term on the right-hand side of (4) is estimated by (3). It remains to control the term

$$\|Eu - (Eu)_{\Omega(z^*,2\iota r)}\|_{\mathrm{L}^p(B(z,2r))}.$$

This is done as in the first part of the proof by virtue of [32, Lem. 7.12/7.16] and gives together with (5) a constant $C > 0$ that depends only on d, p, M and ι such that

$$\|Eu - (Eu)_{\Omega(z^*,2\iota r)}\|_{\mathrm{L}^p(B(z,2r))} \leq Cr \|\nabla Eu\|_{\mathrm{L}^p(B(z,2r);\mathbb{C}^d)}.$$

Employing Remark 2.3 and $B(z, 2M^2\sqrt{d}r) \subset B(x, 4M^2\sqrt{d}r)$ concludes the proof. □

For later purposes, we introduce some more geometric concepts. To do so, denote by $\mathcal{H}^{d-1}$ the $(d-1)$-dimensional Hausdorff measure defined on $\mathbb{R}^d$.

Definition 2.8. 1 An open or closed set $\Xi \subset \mathbb{R}^d$ is called a d-set if there exists a $c > 0$ such that $|\Xi \cap B(x,r)| \geq cr^d$ for all $x \in \Xi$ and $0 < r < 1$.

2 A closed set $E \subset \mathbb{R}^d$ is called a $(d-1)$-set if there exist $C, c > 0$ such that $cr^{d-1} \leq \mathcal{H}^{d-1}(E \cap B(x,r)) \leq Cr^{d-1}$ for all $x \in E$ and $0 < r < 1$.

Moreover, we formulate another assumption. To do so, denote by $B'(x', r)$ the open ball in $\mathbb{R}^{d-1}$ with radius $r > 0$ and centre x'.

Assumption P. Let $\Omega \subset \mathbb{R}^d$ be open, $D \subset \partial\Omega$ be closed, and set $N := \partial\Omega \setminus D$. If Ω is subject to Assumption N and $x_0 \in \overline{N} \cap D$ let Φ_{x_0} denote the corresponding bi-Lipschitz homeomorphism with corresponding set U_{x_0}. Then there are $c_0 \in (0, 1)$ and $c_1 > 0$ such that

$$\mathcal{H}_{d-1}\{y' \in B'(x', r) : \operatorname{dist}((y', 0), \Phi_{x_0}(U_{x_0} \cap N)) > c_0 r\} \geq c_1 r^{d-1}$$

for all $r \in (0, 1]$ and $x' \in \mathbb{R}^{d-1}$ with $(x', 0) \in \Phi_{x_0}(U_{x_0} \cap D \cap \overline{N})$.

Remark 2.9. Notice that Assumption D implies Assumption P.

Remark 2.10. All stated geometric conditions are fulfilled in case that Ω is a bounded Lipschitz domain and in case that the interface separating D and N is Lipschitz as well. In particular the cases $D = \emptyset$ and $D = \partial\Omega$ are included, which will give Neumann and Dirichlet boundary conditions for the elliptic operator below.

2.2 The elliptic operator

The operator under consideration is an elliptic operator $-\nabla \cdot \mu\nabla$ in divergence form. The matrix of coefficients is assumed to satisfy the following standard conditions.

Assumption 2.11. Let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ be such that there exist numbers $c_\bullet, c^\bullet > 0$ with

$$\|\mu\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))} \leq c^\bullet \quad \text{and} \quad \operatorname{Re}\langle \mu(x)\xi, \xi\rangle \geq c_\bullet |\xi|^2 \quad (\xi \in \mathbb{C}^d, \text{ a.e. } x \in \Omega).$$

Let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ be subject to Assumption 2.11. Define the sesquilinear form

$$\mathfrak{t}\colon \mathrm{W}^{1,2}_D(\Omega) \times \mathrm{W}^{1,2}_D(\Omega) \to \mathbb{C}, \quad \mathfrak{t}[u, v] = \int_\Omega \langle \mu\nabla u, \nabla v\rangle \, \mathrm{d}x. \tag{6}$$

Let A_2 be the operator in $\mathrm{L}^2(\Omega)$ associated to $\mathfrak{t}$. It is defined as follows. Let $u, f \in \mathrm{L}^2(\Omega)$. Then by definition $u \in \operatorname{dom}(A_2)$ and $A_2 u = f$ if and only if $u \in \mathrm{W}^{1,2}_D(\Omega)$ and $\mathfrak{t}[u, v] = \int_\Omega f\overline{v} \, \mathrm{d}x$ for all $v \in \mathrm{W}^{1,2}_D(\Omega)$. It is classical that A_2 is a sectorial operator in $\mathrm{L}^2(\Omega)$, i.e., there exists a $\theta \in (\pi/2, \pi)$ such that

$$\mathrm{S}_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\} \subset \rho(-A_2)$$

and there exists a constant $C > 0$ such that

$$\|\lambda(\lambda + A_2)^{-1} f\|_{\mathrm{L}^2(\Omega)} \leq C\|f\|_{\mathrm{L}^2(\Omega)} \qquad (\lambda \in \mathrm{S}_\theta, \ f \in \mathrm{L}^2(\Omega)).$$

Here the constants C and θ can be chosen to depend only on $c_\bullet$, $c^\bullet$ and d.

To investigate this operator on $\mathrm{L}^p(\Omega)$ one defines for all $p > 2$ the operator A_p by

$$\begin{aligned} \operatorname{dom}(A_p) &:= \{u \in \operatorname{dom}(A_2) \cap \mathrm{L}^p(\Omega) : A_2 u \in \mathrm{L}^p(\Omega)\}, \\ A_p u &:= A_2 u \quad (u \in \operatorname{dom}(A_p)), \end{aligned}$$

i.e., A_p is the part of A_2 in $\mathrm{L}^p(\Omega)$.

If $1 < p < 2$, then A_p is defined as the closure of A_2 whenever A_2 is closable in $\mathrm{L}^p(\Omega)$. To decide whether A_2 is closable in $\mathrm{L}^p(\Omega)$ one can check whether the adjoint operator on $\mathrm{L}^{p'}(\Omega)$ is densely defined by using the following lemma [49, Lem. 2.8]).

Lemma 2.12. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded open set, $p \in (1,2)$ and let B be a densely defined operator in $\mathrm{L}^2(\Lambda)$. Then $\mathrm{dom}(B)$ is dense in $\mathrm{L}^p(\Lambda)$ and B is closable in $\mathrm{L}^p(\Lambda)$ if and only if the part of B^* in $\mathrm{L}^{p'}(\Lambda)$ is densely defined. In this case $(B_p)^* = (B^*)_{p'}$, where B_p denotes the closure of B in $\mathrm{L}^p(\Lambda)$ and $(B^*)_{p'}$ denotes the part of B^* in $\mathrm{L}^{p'}(\Lambda)$.*

To obtain information about the numbers $p \in (1,\infty)$ for which a reasonable L^p-theory of the elliptic operator can be established, we introduce the notion of p-ellipticity. The origin of p-ellipticity is contained in the pioneering works of Cialdea and Maz'ya [10, 11, 12, 13]. In [10], Cialdea and Maz'ya show that the algebraic condition

$$\frac{4}{pp'}\langle \mathrm{Re}(\mu(x))\alpha,\alpha\rangle + \langle \mathrm{Re}(\mu(x))\beta,\beta\rangle + \frac{2}{p}\langle \mathrm{Im}(\mu(x))\alpha,\beta\rangle - \frac{2}{p'}\langle \alpha, \mathrm{Im}(\mu(x))\beta\rangle \geq 0$$

for almost every $x \in \Omega$ and all $\alpha,\beta \in \mathbb{R}^d$ is sufficient for the operator A_2 to be an accretive operator in $\mathrm{L}^p(\Omega)$. Here $\mathrm{Re}(\mu(x))$ is the matrix obtained by taking the real part of each matrix element of $\mu(x)$. Similarly $\mathrm{Im}(\mu(x))$ is defined. The term on the left-hand side can be written as

$$\frac{4}{pp'}\langle \mathrm{Re}(\mu(x))\alpha,\alpha\rangle + \langle \mathrm{Re}(\mu(x))\beta,\beta\rangle + \frac{2}{p}\langle \mathrm{Im}(\mu(x))\alpha,\beta\rangle - \frac{2}{p'}\langle \alpha, \mathrm{Im}(\mu(x))\beta\rangle \\ = p\mathrm{Re}\Big\langle \mu(x)(\alpha' + \mathrm{i}\beta), \frac{\alpha'}{p'} + \frac{\mathrm{i}\beta}{p}\Big\rangle, \tag{7}$$

where $\alpha' = 2\alpha/p$. The term on the right-hand side was investigated thoroughly by Carbonaro and Dragičević in [9] and we next introduce their concepts.

Definition 2.13. Let $p \in (1,\infty)$. Define $\mathcal{J}_p \colon \mathbb{C}^d \to \mathbb{R}^d$ by

$$\mathcal{J}_p(\xi) = 2\Big(\frac{\alpha}{p'} + \frac{\mathrm{i}\beta}{p}\Big),$$

where $\xi = \alpha + \mathrm{i}\beta$ with $\alpha,\beta \in \mathbb{R}^d$. Following [9] the matrix μ is called *p-elliptic* if there exists a number $\lambda_p > 0$ such that

$$\mathrm{Re}\langle \mu(x)\xi, \mathcal{J}_p(\xi)\rangle \geq \lambda_p|\xi|^2 \qquad (\xi \in \mathbb{C}^d,\ \text{a.e. } x \in \Omega). \tag{8}$$

Obviously, a matrix μ subject to Assumption 2.11 is always 2-elliptic and if μ is in additional real-valued, then μ is p-elliptic for all $p \in (1,\infty)$, see Lemma 2.15 below. Next, define

$$\Delta_p(\mu) := \operatorname*{essinf}_{x\in\Omega} \min_{|\xi|=1} \mathrm{Re}\langle \mu(x)\xi, \mathcal{J}_p(\xi)\rangle$$

to be the *p-ellipticity constant* of μ. Then μ is p-elliptic if and only if $\Delta_p(\mu) > 0$. Moreover,

$$\Delta_p(\mu) = \Delta_{p'}(\mu)$$

by Proposition 5.8 in [9]. As a consequence, μ is p-elliptic if and only if it is p'-elliptic. Moreover, by [9, Cor. 5.17] the conjugate coefficient function μ^* is p-elliptic if and only if μ is p-elliptic.

Define

$$\delta(\mu) := \operatorname*{essinf}_{(x,\xi)\in\Omega\times\mathbb{C}^d\setminus\{0\}} \mathrm{Re}\frac{\langle \mu(x)\xi,\xi\rangle}{|\langle \mu(x)\xi,\overline{\xi}\rangle|}.$$

In [9, Prop. 5.15] Carbonaro and Dragičević proved that

$$\frac{\Delta_p(\mu)}{c^\bullet} \leq \delta(\mu) - \Big|1 - \frac{2}{p}\Big| \leq \frac{\Delta_p(\mu)\delta(\mu)}{c_\bullet}.$$

This relation separates μ and p and it implies that μ is p-elliptic if and only if

$$\Big|1 - \frac{2}{p}\Big| < \delta(\mu).$$

Since any μ that satisfies Assumption 2.11 is always 2-elliptic it follows that $\delta(\mu) > 0$. Consequently, there always exists an open interval (p_0', p_0) for some $p_0 \in (2, \infty]$, such that μ is p-elliptic for all $p \in (p_0', p_0)$. On the other hand, [9, Cor. 5.16] states that

$$[2, \infty) \ni p \mapsto \Delta_p(\mu)$$

is Lipschitz continuous and decreasing. Consequently, there is a unique $p_0 \in (2, \infty]$ such that μ is p-elliptic *if and only if* $p \in (p_0', p_0)$.

Notation 2.14. For a matrix μ subject to Assumption 2.11 denote by $p_0(\mu) \in (2, \infty]$ the unique number such that μ is p-elliptic if and only if $p \in (p_0(\mu)', p_0(\mu))$.

Next, we mention how to quantify a lower bound on (7) by means of the p-ellipticity constant. Let $p \in (1, \infty)$ and $\alpha, \beta \in \mathbb{R}^d$. If $\alpha' = 2\alpha/p$, then

$$\mathrm{Re}\Big\langle \mu(x)\Big(\frac{2\alpha}{p} + \mathrm{i}\beta\Big), \frac{2\alpha}{p'} + \mathrm{i}\beta\Big\rangle = p\mathrm{Re}\Big\langle \mu(x)(\alpha' + \mathrm{i}\beta), \frac{\alpha'}{p'} + \frac{\mathrm{i}\beta}{p}\Big\rangle \geq \frac{\lambda_p}{2}\Big(\frac{4|\alpha|^2}{p} + p|\beta|^2\Big) \tag{9}$$

for almost every $x \in \Omega$ and any number $0 < \lambda_p \leq \Delta_p(\mu)$. In particular, this shows that

$$\Big|\Big\langle \mu(x)\Big(\frac{2\alpha}{p} + \mathrm{i}\beta\Big), \frac{2\alpha}{p'} + \mathrm{i}\beta\Big\rangle\Big| \leq C \cdot \mathrm{Re}\Big\langle \mu(x)\Big(\frac{2\alpha}{p} + \mathrm{i}\beta\Big), \frac{2\alpha}{p'} + \mathrm{i}\beta\Big\rangle,$$

for some constant $C > 0$ that only depends on $c^\bullet$, p and λ_p. It follows that there exists some angle $\omega \in (0, \pi/2)$ that only depends on d, $c^\bullet$, p and λ_p such that

$$\Big\langle \mu(x)\Big(\frac{2\alpha}{p} + \mathrm{i}\beta\Big), \frac{2\alpha}{p'} + \mathrm{i}\beta\Big\rangle \in \overline{\mathrm{S}_\omega} \tag{10}$$

for almost every $x \in \Omega$ and all $\alpha, \beta \in \mathbb{R}^d$.

To conclude this preparatory section, we state a short result that quantifies the size of the imaginary part of μ by the real part of μ of a p-elliptic matrix, where $p > 2$.

Lemma 2.15. *Let μ be p-elliptic for some $p > 2$. Then*

$$|\mathrm{Im}(\mu(x))|_{\mathcal{L}(\mathbb{R}^d)} \leq \frac{(p-1)^{1/2}}{p-2} |\mathrm{Re}(\mu(x))|_{\mathcal{L}(\mathbb{R}^d)} \tag{11}$$

for almost every $x \in \Omega$. In particular, if μ is subject to Assumption 2.11 then μ is p-elliptic for all $p \in (1, \infty)$ if and only if $\mathrm{Im}(\mu) = 0$.

Proof. Estimating the right-hand side of (8) from below by zero implies that

$$|\langle \mathrm{Im}(\mu(x))\beta, \alpha\rangle| \leq p'\Big(|\mathrm{Re}(\mu(x))|_{\mathcal{L}(\mathbb{R}^d)}\Big(\frac{|\alpha|^2}{p'} + \frac{|\beta|^2}{p}\Big) + |\mathrm{Im}(\mu(x))|_{\mathcal{L}(\mathbb{R}^d)}\frac{|\alpha||\beta|}{p}\Big)$$

for almost every $x \in \Omega$ and all $\alpha, \beta \in \mathbb{R}^d$ Taking the supremum over all $\alpha \in \mathbb{R}^d$ with $|\alpha| = 1$ gives

$$|\mathrm{Im}(\mu(x))\beta| \leq p'\Big(|\mathrm{Re}(\mu(x))|_{\mathcal{L}(\mathbb{R}^d)}\Big(\frac{1}{p'} + \frac{|\beta|^2}{p}\Big) + |\mathrm{Im}(\mu(x))|_{\mathcal{L}(\mathbb{R}^d)}\frac{|\beta|}{p}\Big).$$

Let $t > 0$, replace β by $t\beta$, divide by t, and minimize the right-hand side with respect to the parameter t to obtain

$$|\mathrm{Im}(\mu(x))\beta| \leq 2|\mathrm{Re}(\mu(x))|_{\mathcal{L}(\mathbb{R}^d)}\Big(\frac{p'}{p}\Big)^{1/2}|\beta| + \frac{p'}{p}|\mathrm{Im}(\mu(x))|_{\mathcal{L}(\mathbb{R}^d)}|\beta|.$$

Taking the supremum over all $\beta \in \mathbb{R}^d$ with $|\beta| = 1$ shows (11).

Notice that for $p \to \infty$ one concludes that $\mathrm{Im}(\mu) = 0$ if μ is p-elliptic for all $p \in (1, \infty)$. On the other hand, if $\mathrm{Im}(\mu) = 0$ a direct calculation shows that μ is p-elliptic for all $p \in (1, \infty)$. □

3 Formulation of the main results

The first main result provides an interval in $(1, \infty)$ in which one obtains resolvent bounds for the operator A_p.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, where $d \geq 3$. Let $D \subset \partial\Omega$ be closed and set $N = \partial\Omega \setminus D$. Adopt Assumption N and let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ be subject to Assumption 2.11. Then for all $p \in (1, \infty)$ that satisfy*

$$\frac{p_0(\mu)d}{d(p_0(\mu) - 1) + 2} < p < \frac{p_0(\mu)d}{d - 2} \tag{12}$$

the operator A_p is sectorial. This means, there exist $\theta \in (\pi/2, \pi)$ and $C > 0$ such that $\mathrm{S}_\theta \subset \rho(-A_p)$ and

$$\|\lambda(\lambda + A_p)^{-1}f\|_{\mathrm{L}^p(\Omega)} \leq C\|f\|_{\mathrm{L}^p(\Omega)} \qquad (\lambda \in \mathrm{S}_\theta,\ f \in \mathrm{L}^p(\Omega)).$$

Finally, given $\gamma_0 > 0$, the constants θ and C can be chosen to depend only on $d, p, M, c_\bullet, c^\bullet$ and γ_0, whenever $\Delta_{\max(2, \frac{d-2}{d}p)}(\mu) \geq \gamma_0$ if $p \geq 2$ and with p replaced by p' if $p < 2$.

We shall prove Theorem 3.1 in Section 5. This theorem has the following direct corollary.

Corollary 3.2. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, where $d \geq 3$. Let $D \subset \partial\Omega$ be closed and set $N = \partial\Omega \setminus D$. Adopt Assumption N and let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ be subject to Assumption 2.11. Then for every $p \in (1, \infty)$ that satisfies condition* (12) *the operator $-A_p$ generates a bounded analytic semigroup $(\mathrm{e}^{-tA_p})_{t\geq 0}$ on $\mathrm{L}^p(\Omega)$.*

Let X be a Banach space, $-\mathcal{A}$ the generator of an analytic semigroup on X, $0 < T \leq \infty$ and $1 < q < \infty$. Consider the problem

$$\begin{cases} u'(t) + \mathcal{A}u(t) = f(t) \qquad (0 < t < T), \\ \qquad\qquad u(0) = 0. \end{cases}$$

We say that $\mathcal{A}$ has maximal L^q-regularity if for every $f \in \mathrm{L}^q(0,T;X)$ the unique mild solution

$$u(t) := \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}s \qquad (0 < t < T)$$

satisfies $u(t) \in \mathrm{dom}(\mathcal{A})$ for almost every $0 < t < T$ and $\mathcal{A}u, u' \in \mathrm{L}^q(0,T;X)$. Since maximal L^q-regularity is independent of q, we will simply say maximal regularity.

Under Assumption N, Egert has shown in [25, Thm. 1.3] that the H^∞-calculus of A_p is bounded with corresponding H^∞-angle less than $\pi/2$ whenever $-A_p$ generates a bounded analytic semigroup. Since the boundedness of the H^∞-calculus with angle less than $\pi/2$ in turn implies that A_p has maximal regularity for every $0 < T \leq \infty$ by [40, p. 340], we have the following corollary.

Corollary 3.3. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, where $d \geq 3$. Let $D \subset \partial\Omega$ be closed and set $N = \partial\Omega \setminus D$. Adopt Assumption N and let $\mu \in \mathrm{L}^\infty(\Omega;\mathbb{C}^{d\times d})$ be subject to Assumption 2.11. Then for every $p \in (1,\infty)$ that satisfies condition* (12) *and for every $0 < T \leq \infty$ the operator A_p has maximal regularity.*

The next theorem provides optimal estimates for gradients of resolvents. Hence it gives optimal estimates of the gradient of the solution to the resolvent equation. In particular, a higher-integrability estimate for right-hand sides in $\mathrm{L}^p(\Omega)$ with $2 < p < 2+\varepsilon$ is derived. In the case $\lambda = 0$ and for Dirichlet boundary conditions this resembles a classical result due to Meyers [44]. The result of Meyers essentially states that there exists an $\varepsilon > 0$ such that for all $2 < p < 2+\varepsilon$ the gradient of a solution to an elliptic equation with right-hand side in $\mathrm{W}^{-1,p}(\Omega)$ lies in $\mathrm{L}^p(\Omega)$ as well. The theorem below gives a corresponding result for the resolvent problem including quantitative estimates with respect to the resolvent parameter λ. Note that additional geometric assumptions are required in order to obtain the existence of this number $\varepsilon > 0$.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, where $d \geq 3$. Let $D \subset \partial\Omega$ be closed and set $N = \partial\Omega \setminus D$. Adopt Assumption N and let $\mu \in \mathrm{L}^\infty(\Omega;\mathbb{C}^{d\times d})$ be subject to Assumption 2.11. Let $p \in (1,\infty)$.*

(a) *If*

$$\frac{p_0(\mu)d}{d(p_0(\mu)-1)+2} < p \leq 2 \tag{13}$$

then $\mathrm{dom}(A_p) \subset \mathrm{W}^{1,p}_D(\Omega)$. Moreover, given $\gamma_0 > 0$ there exists a constant $C > 0$ such that

$$|\lambda|^{1/2}\|\nabla(\lambda + A_p)^{-1}f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \leq C\|f\|_{\mathrm{L}^p(\Omega)} \qquad (\lambda \in \mathrm{S}_\theta,\ f \in \mathrm{L}^p(\Omega)). \tag{14}$$

Here $C > 0$ depends only on d, p, M, $c_\bullet$, $c^\bullet$ and γ_0, whenever $\Delta_{\max(2,\frac{d-2}{d}p')}(\mu) \geq \gamma_0$ and θ denotes the angle as in Theorem 3.1.

(b) *If in addition Assumptions Ω^c and D are satisfied, then there exists an $\varepsilon > 0$ such that $\mathrm{dom}(A_p) \subset \mathrm{W}^{1,p}_D(\Omega)$ for all*

$$2 \leq p < 2+\varepsilon. \tag{15}$$

Moreover, there exists a constant $C > 0$ such that (14) *is valid. Here $C, \varepsilon > 0$ depend only on d, p, κ, ι, r_0, s_0, M, $c_\bullet$ and $c^\bullet$.*

(c) *If in addition Assumptions Ω^c and D are satisfied, then there exists an $\varepsilon > 0$ such that the operator $\nabla(\lambda + A_p)^{-1} \operatorname{div}$ extends from $C_c^\infty(\Omega; \mathbb{C}^d)$ to a bounded operator on $L^p(\Omega; \mathbb{C}^d)$ for all*

$$(2+\varepsilon)' < p < 2 + \varepsilon$$

and $\lambda \in S_\theta$. Furthermore, there exists a $C > 0$ such that

$$\|\nabla(\lambda + A_p)^{-1} \operatorname{div} f\|_{L^p(\Omega;\mathbb{C}^d)} \leq C\|f\|_{L^p(\Omega;\mathbb{C}^d)} \qquad (\lambda \in S_\theta, f \in L^p(\Omega; \mathbb{C}^d)).$$

Here $\varepsilon > 0$ is as in (b) and $C > 0$ depends only on d, p, κ, ι, r_0, s_0, M, $c_\bullet$ and $c^\bullet$.

We shall prove Theorem 3.4 in Section 5.

Remark 3.5. An analogue of Theorem 3.4(b) and (c) for elliptic systems with L^∞-coefficients follows literally by the same lines of proof and it holds also in two space-dimensions.

The representation of e^{-tA_p} by the Cauchy integral formula directly leads to the following corollary.

Corollary 3.6. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, where $d \geq 3$. Let $D \subset \partial\Omega$ be closed and set $N = \partial\Omega \setminus D$. Adopt Assumption N and let $\mu \in L^\infty(\Omega; \mathbb{C}^{d\times d})$ be subject to Assumption 2.11. Let $p \in \big(\frac{p_0(\mu)d}{d(p_0(\mu)-1)+2}, 2\big]$. Then there exists a constant $C > 0$ such that*

$$\|\nabla e^{-tA_p} f\|_{L^p(\Omega;\mathbb{C}^d)} \leq Ct^{-1/2}\|f\|_{L^p(\Omega)} \qquad (f \in L^p(\Omega),\ t > 0).$$

If in addition Assumptions Ω^c and D are satisfied, the same estimate holds for all $p \in [2, 2+\varepsilon)$, where $\varepsilon > 0$ is as in Theorem 3.4(b). Moreover, there exists a $C > 0$ such that

$$\|\nabla e^{-tA_p} \operatorname{div} f\|_{L^p(\Omega;\mathbb{C}^d)} \leq Ct^{-1}\|f\|_{L^p(\Omega;\mathbb{C}^d)} \qquad (f \in L^p(\Omega; \mathbb{C}^d),\ t > 0)$$

for all $(2+\varepsilon)' < p < 2 + \varepsilon$.

Remark 3.7. Let $\Omega \subset \mathbb{R}^d$ open, $D \subset \partial\Omega$ closed and $N := \partial\Omega \setminus D$ subject to Assumption N. In [25], Egert proved under the assumptions that Ω is a d-set and D a $(d-1)$-set that there exists an $\varepsilon > 0$ such that for all numbers p that satisfy (15) or (13) the operator $A_p^{1/2} \colon W_D^{1,p}(\Omega) \to L^p(\Omega)$ is an isomorphism. Direct consequences of this isomorphism property are the gradient estimates in Corollary 3.6. We emphasise that the gradient estimates in Theorem 3.4(a) are valid for all $p \leq 2$ subject to (13) without any additional assumptions to Ω and D. We note that for the estimates in Theorem 3.4(c) for the case $p \in [2, 2+\varepsilon)$ neither Assumptions Ω^c and D imply the assumptions made by Egert nor do the assumptions made by Egert imply the Assumptions Ω^c and D. Vividly spoken, our Assumptions Ω^c and D allow for the presence of outward cusps in the Dirichlet part while Egert's assumptions allow for the presence of inward cusps and slits in the Dirichlet part. Note that under Assumption N these cusps cannot accumulate towards the Neumann boundary.

Our final result concerns the perturbation of real-valued matrices of coefficients μ by complex-valued matrices. We obtain semigroups which are consistent on $L^p(\Omega)$ for all $p \in [1, \infty)$. As a consequence also all resolvent operators are consistent on $L^p(\Omega)$ for the full range $p \in [1, \infty)$. We emphasise that the semigroups are no longer quasi-contractive for all $p \in [1, \infty)$.

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, where $d \geq 3$. Let $D \subset \partial\Omega$ be closed and set $N = \partial\Omega \backslash D$. Assume that Ω^c is a d-set and adopt Assumptions N and P. Finally let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{R}^{d\times d})$ be a real-valued coefficient function subject to Assumption 2.11.*

Then there exists an $\varepsilon > 0$ such that for all (complex valued) $\nu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ satisfying $\|\nu\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))} < \varepsilon$ the L^2-realisation $A_2 = -\nabla \cdot (\mu + \nu)\nabla$ is m-sectorial. Moreover, for all $p \in (1, \infty)$ the operator $-A_p$ generates an analytic C_0-semigroup on $\mathrm{L}^p(\Omega)$. These semigroups also extend to an analytic C_0-semigroup on $\mathrm{L}^1(\Omega)$. The semigroups have a kernel with Gaussian bounds, that is, there are $b, c, \omega > 0$ such that for all $t > 0$ there exists a measurable $K_t \colon \Omega \times \Omega \to \mathbb{C}$ that satisfies

$$|K_t(x, y)| \leq ct^{-d/2}\mathrm{e}^{-b\frac{|x-y|^2}{t}}\mathrm{e}^{\omega t}$$

for all $t > 0$ and $x, y \in \Omega$, and

$$(\mathrm{e}^{-tA_p}u)(x) = \int_\Omega K_t(x, y)u(y)\,\mathrm{d}y$$

for all $t > 0$, $p \in (1, \infty)$, $u \in \mathrm{L}^p(\Omega)$ and $x \in \Omega$. The constants ε, b, c and ω can be chosen to depend only on d, M, $c_\bullet$, $c^\bullet$, α, c_0 and c_1. Here $\alpha > 0$ is such that $|B(x, r) \cap \Omega^c| \geq \alpha r^d$ for all $x \in \partial\Omega$ and $r \in (0, 1]$.

We prove Theorem 3.8 in Section 6. A combination of Theorem 3.8 with Lemma 2.15 establishes the following corollary.

Corollary 3.9. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, where $d \geq 3$. Let $D \subset \partial\Omega$ be closed and set $N = \partial\Omega \setminus D$. Assume that Ω^c is a d-set and adopt Assumptions N and P. Given $c_\bullet, c^\bullet > 0$ there exists a $p_c > 2$, depending only on $c_\bullet$, $c^\bullet$, d, M, α, c_0 and c_1, such that for every $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$, that is subject to Assumption 2.11 with constants $c_\bullet$ and $c^\bullet$ and satisfies $p_0(\mu) > p_c$, the operator $-A_p$ generates an analytic C_0-semigroup for all $p \in (1, \infty)$, which is consistent with an analytic C_0-semigroup on $\mathrm{L}^1(\Omega)$. Here $\alpha > 0$ is such that $|B(x, r) \cap \Omega^c| \geq \alpha r^d$ for all $x \in \partial\Omega$ and $r \in (0, 1]$.*

4 Weak reverse Hölder estimates for resolvent equations

Throughout this section let $\Omega \subset \mathbb{R}^d$ be open and bounded, where $d \geq 3$. Let $D \subset \partial\Omega$ be closed and set $N = \partial\Omega \setminus D$. Adopt Assumption N and let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ be subject to Assumption 2.11. In order to prove Theorem 3.1, we show the validity of weak reverse Hölder estimates for the resolvent. More precisely, let $u \in \mathrm{W}^{1,2}_D(\Omega)$ be a solution to

$$\lambda \int_\Omega u\overline{v}\,\mathrm{d}x + \int_\Omega \langle \mu\nabla u, \nabla v\rangle\,\mathrm{d}x = 0,$$

for all $v \in \mathrm{W}^{1,2}_D(\Omega)$ with $\mathrm{supp}(v) \subset \overline{B(x_0, 2r)}$. Then for any $p \in (2, dp_0(\mu)/(d-2))$, we prove that there exist constants $C > 0$ and $c \in (0, 1)$ such that

$$\left(\frac{1}{r^d}\int_{\Omega(x_0,cr)} |u|^p\,\mathrm{d}x\right)^{1/p} \leq C\left(\frac{1}{r^d}\int_{\Omega(x_0,2r)} |u|^2\,\mathrm{d}x\right)^{1/2}.$$

For interior balls and in the case $\lambda = 0$ this inequality was proven by Dindoš and Pipher in [19] which follows the ideas of Cialdea and Maz'ya [10]. We choose, however, a slightly different approach that allows to treat balls centred at the boundary.

To begin with, we introduce the following cut-off function. For all $0 < \delta < L$ define $|\cdot|_{\delta,L}\colon \mathbb{R} \to (0,\infty)$ by $|x|_{\delta,L} = \delta \vee |x| \wedge L$. So

$$|x|_{\delta,L} = \begin{cases} \delta & \text{if } |x| \leq \delta, \\ |x| & \text{if } \delta < |x| < L, \\ L & \text{if } |x| \geq L. \end{cases}$$

In order to prove the weak reverse Hölder estimate, we will test the resolvent equation with the test-function

$$v := \eta^2 |u|_{\delta,L}^{p-2} u,$$

where η is a suitable cut-off function. To do so, we need the following lemma.

Lemma 4.1. *If $p \in [1,\infty)$, $0 < \delta < L$ and $u \in \mathrm{W}^{1,2}_D(\Omega)$, then $w := |u|_{\delta,L}^{p-2} u \in \mathrm{W}^{1,2}_D(\Omega)$.*

Proof. There exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{C}_D^\infty(\Omega)$ such that $u_n \to u$ as $n \to \infty$ in $\mathrm{W}^{1,2}(\Omega)$. Passing to a subsequence if necessary, we may without loss of generality assume that $u_n \to u$ and $\nabla u_n \to \nabla u$ almost everywhere as $n \to \infty$. For all $n \in \mathbb{N}$ define $w_n := |u_n|_{\delta,L}^{p-2} u_n$. Then $\operatorname{dist}(\operatorname{supp}(w_n), D) > 0$. If $n \in \mathbb{N}$, then

$$\begin{aligned} \|w_n - w\|_{\mathrm{L}^2(\Omega)} &\leq \|(|u_n|_{\delta,L}^{p-2} - |u|_{\delta,L}^{p-2})u\|_{\mathrm{L}^2(\Omega)} + \||u_n|_{\delta,L}^{p-2}(u_n - u)\|_{\mathrm{L}^2(\Omega)} \\ &\leq \|(|u_n|_{\delta,L}^{p-2} - |u|_{\delta,L}^{p-2})u\|_{\mathrm{L}^2(\Omega)} + \max\{\delta^{p-2}, L^{p-2}\}\|u_n - u\|_{\mathrm{L}^2(\Omega)}. \end{aligned} \tag{16}$$

Hence $\lim_{n\to\infty} w_n = w$ in $\mathrm{L}^2(\Omega)$ by the L^2-convergence of $(u_n)_{n\in\mathbb{N}}$ to u and the dominated convergence theorem.

Notice that w is weakly differentiable with weak derivative

$$\partial_j w = |u|_{\delta,L}^{p-2} \partial_j u + (p-2)\chi_{\{\delta < |u| < L\}} |u|^{p-4} u \operatorname{Re}(\overline{u}\partial_j u).$$

The same formula is valid for $\partial_j w_n$ with u replaced by u_n. Write shortly $\chi_n := \chi_{\{\delta<|u_n|<L\}}$ and $\chi := \chi_{\{\delta<|u|<L\}}$ in the following. The convergence of the sequence $(|u_n|_{\delta,L}^{p-2}\partial_j u_n)_{n\in\mathbb{N}}$ to $|u|_{\delta,L}^{p-2}\partial_j u$ in $\mathrm{L}^2(\Omega)$ follows as in (16). Hence it remains to estimate

$$\|\chi_n |u_n|^{p-4} u_n \operatorname{Re}(\overline{u_n}\partial_j u_n) - \chi |u|^{p-4} u \operatorname{Re}(\overline{u}\partial_j u)\|_{\mathrm{L}^2(\Omega)}$$

This is bounded, using the triangle inequality, by

$$L^2 \max\{\delta^{p-4}, L^{p-4}\}\|\partial_j u_n - \partial_j u\|_{\mathrm{L}^2(\Omega)} + \|\chi_n |u_n|^{p-4} u_n \operatorname{Re}(\overline{u_n}\partial_j u) - \chi |u|^{p-4} u \operatorname{Re}(\overline{u}\partial_j u)\|_{\mathrm{L}^2(\Omega)}.$$

The first term converges to zero as $n \to \infty$ by the L^2-convergence of $(\nabla u_n)_{n\in\mathbb{N}}$ to ∇u. For the second term, introduce the sets

$$\begin{aligned} C_0 &:= \{x \in \Omega : |u(x)| \in (\delta, L)\}, \quad C_1 := \{x \in \Omega : |u(x)| \in [0,\delta) \cup (L,\infty]\}, \\ C_2 &:= \{x \in \Omega : |u(x)| = \delta\}, \quad C_3 := \{x \in \Omega : |u(x)| = L\} \end{aligned}$$

and decompose the domain of integration to deduce the inequality

$$\begin{aligned} &\|\chi_n |u_n|^{p-4} u_n \operatorname{Re}(\overline{u_n}\partial_j u) - \chi |u|^{p-4} u \operatorname{Re}(\overline{u}\partial_j u)\|_{\mathrm{L}^2(\Omega)} \\ &\leq \sum_{k=1}^3 \|\chi_n |u_n|^{p-4} u_n \operatorname{Re}(\overline{u_n}\partial_j u)\|_{\mathrm{L}^2(C_k)} + \|\chi_n |u_n|^{p-4} u_n \operatorname{Re}(\overline{u_n}\partial_j u) - \chi |u|^{p-4} u \operatorname{Re}(\overline{u}\partial_j u)\|_{\mathrm{L}^2(C_0)}. \end{aligned}$$

Since $\lim_{n\to\infty} \chi_n |u_n|^{p-4} u_n \mathrm{Re}(\overline{u_n}\partial_j u) = \chi |u|^{p-4} u \mathrm{Re}(\overline{u}\partial_j u)$ almost everywhere on C_0, the last term tends to zero for $n \to \infty$ by the dominated convergence theorem. Similarly the integral on C_1 tends to zero. Next, if C_2 is not a set of measure zero, then $\partial_j |u| = 0$ on C_2 almost everywhere by [32, Lem. 7.7]. This implies that $\mathrm{Re}(\overline{u}\partial_j u) = 0$ on C_2 and hence

$$\mathrm{Re}(\overline{u_n}\partial_j u) = \mathrm{Re}\biggl(\frac{\overline{u_n}}{\overline{u}}\overline{u}\partial_j u\biggr) = -\mathrm{Im}\biggl(\frac{\overline{u_n - u}}{\overline{u}}\biggr)\mathrm{Im}(\overline{u}\partial_j u) \quad \text{on} \quad C_2.$$

The same holds on C_3. Now, the dominated convergence theorem implies that the remaining terms on the right-hand side converge to zero.

We proved that $\lim_{n\to\infty} w_n = w$ in $\mathrm{W}^{1,2}(\Omega)$. It remains to prove that $w_n \in \mathrm{W}^{1,2}_D(\Omega)$ for all $n \in \mathbb{N}$. Since $\mathrm{W}^{1,2}_D(\Omega)$ is a Banach space this implies that $w \in \mathrm{W}^{1,2}_D(\Omega)$.

Let $n \in \mathbb{N}$. By definition of the space $\mathrm{C}^\infty_D(\Omega)$ there exists a $\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^d)$ such that $\mathrm{supp}(\varphi_n) \cap D = \emptyset$ and $u_n = \varphi|_\Omega$. Let $(\rho_k)_{k\in\mathbb{N}}$ be a standard approximation of the identity. For all $k \in \mathbb{N}$ define

$$f_k \coloneqq (|\varphi|^{p-2}_{\delta,L}\varphi) * \rho_k.$$

Then $f_k \in \mathrm{C}^\infty_c(\mathbb{R}^d)$ and if k is large enough then $\mathrm{supp}(f_k) \cap D = \emptyset$. So $f_k|_\Omega \in \mathrm{W}^{1,2}_D(\Omega)$ if k is large enough. Since $\lim_{k\to\infty} f_k|_\Omega = |u_n|^{p-2}_{\delta,L} u_n = w_n$ one deduces that $w_n \in \mathrm{W}^{1,2}_D(\Omega)$. The proof is complete. □

Now, we are in the position to prove the validity of the weak reverse Hölder estimates of the resolvent problem.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, where $d \geq 3$. Let $D \subset \partial\Omega$ be closed and set $N = \partial\Omega \setminus D$. Adopt Assumption N and let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ be subject to Assumption 2.11. If $p \in [2, p_0(\mu))$, then there exists a $\theta \in (\pi/2, \pi)$ such that for all $\lambda \in \mathrm{S}_\theta$, $x_0 \in \overline{\Omega}$ and $r \in (0, M/4)$ the following is valid. Let $u \in \mathrm{W}^{1,2}_D(\Omega)$ with*

$$\lambda \int_\Omega u\overline{v} \,\mathrm{d}x + \mathfrak{t}[u, v] = 0$$

for all $v \in \mathrm{W}^{1,2}_D(\Omega)$ with $\mathrm{supp}(v) \subset \overline{B(x_0, 2r)}$. Then there exist $\mathcal{C} > 0$ and $c \in (0, 1)$ such that $u \in \mathrm{L}^{pd/(d-2)}(\Omega(x_0, cr))$ and

$$\biggl(\frac{1}{r^d}\int_{\Omega(x_0,cr)} |u|^{\frac{pd}{d-2}} \,\mathrm{d}x\biggr)^{\frac{d-2}{pd}} \leq \mathcal{C}\biggl(\frac{1}{r^d}\int_{\Omega(x_0,2r)} |u|^2 \,\mathrm{d}x\biggr)^{\frac{1}{2}}.$$

For given $\gamma_0 > 0$ the constants $\mathcal{C}$ and θ can be chosen to depend only on d, p, M, $c_\bullet$, $c^\bullet$ and γ_0, as long as $\Delta_p(\mu) \geq \gamma_0$ and c can be chosen to depend only on d, p and M.

Proof. Fix $\tau \in \mathrm{C}^\infty_c(\mathbb{R}^d)$ such that $\tau|_{B(0,1)} = \mathbb{1}$, $0 \leq \tau \leq 1$ and $\mathrm{supp}\,\tau \subset B(0, 2)$. One can arrange that $\|\nabla\tau\|_{\mathrm{L}^\infty} \leq 2$. Let $x_0 \in \mathbb{R}^d$ and $r > 0$. Define $\eta \in \mathrm{C}^\infty_c(\mathbb{R}^d)$ by $\eta(x) = \tau(r^{-1}(x - x_0))$. Then $\eta \in \mathrm{C}^\infty_c(B(x_0, 2r))$, $0 \leq \eta \leq 1$, $\eta|_{B(x_0,r)} = \mathbb{1}$ and $\|\nabla\eta\|_{\mathrm{L}^\infty} \leq 2/r$. Let $0 < \delta < L$. By Lemmas 2.1 and 4.1 the testfunction

$$v \coloneqq \eta^2 |u|^{p-2}_{\delta,L} u$$

satisfies $v \in \mathrm{W}^{1,2}_D(\Omega)$. Next, define $\varphi \coloneqq |u|^{\frac{p-2}{2}} u$ and $\Omega_{\delta,L} \coloneqq \{\delta < |u| < L\}$. Then $|\varphi|^{\frac{p-2}{p}}\varphi = |u|^{p-2}u$. On the set $\Omega_{\delta,L}$ one establishes with [32, Lem. 7.6] that

$$\nabla(|\varphi|^{\frac{p-2}{p}}\varphi) = \nabla(|u|^{p-2}u) = u\nabla|u|^{p-2} + |u|^{p-2}\nabla u = u\nabla|u|^{p-2}_{\delta,L} + |u|^{p-2}_{\delta,L}\nabla u = \nabla(|u|^{p-2}_{\delta,L}u).$$

Consequently, $\nabla v = \nabla(\eta^2 |\varphi|^{\frac{p-2}{p}} \varphi)$. Using [32, Cor. 7.7], the definition of φ and the product rule one deduces that

$$\begin{aligned}\mathfrak{t}[u,v] &= \int_\Omega \langle \mu \nabla u, \nabla(\eta^2 |u|_{\delta,L}^{p-2} u)\rangle \,\mathrm{d}x \\ &= \delta^{p-2} \int_{\{|u|\leq\delta\}} \eta^2 \langle \mu\nabla u, \nabla u\rangle \,\mathrm{d}x + 2\delta^{p-2} \int_{\{|u|\leq\delta\}} \eta \langle \mu\nabla u, u\nabla\eta\rangle \,\mathrm{d}x \\ &\quad + L^{p-2} \int_{\{|u|\geq L\}} \eta^2 \langle \mu\nabla u, \nabla u\rangle \,\mathrm{d}x + 2L^{p-2} \int_{\{|u|\geq L\}} \eta \langle \mu\nabla u, u\nabla\eta\rangle \,\mathrm{d}x \\ &\quad + \int_{\Omega_{\delta,L}} \langle \mu \nabla(|\varphi|^{-\frac{p-2}{p}} \varphi), \nabla(\eta^2 |\varphi|^{\frac{p-2}{p}} \varphi)\rangle \,\mathrm{d}x\end{aligned}$$

Introduce the functions $\Phi := \mathrm{Re}(\overline{\mathrm{sgn}\,\varphi} \nabla\varphi)$ and $\Psi := \mathrm{Im}(\overline{\mathrm{sgn}\,\varphi}\nabla\varphi)$, where $\mathrm{sgn}\, z = \frac{z}{|z|}$ if $z \in \mathbb{C}\setminus\{0\}$ and $\mathrm{sgn}\, 0 = 0$. Then

$$\nabla|\varphi| = \Phi. \tag{17}$$

Relying only on the definitions of Φ and Ψ, the product rule and (17) one calculates

$$\begin{aligned}&\int_{\Omega_{\delta,L}} \langle \mu \nabla(|\varphi|^{-\frac{p-2}{p}} \varphi), \nabla(\eta^2 |\varphi|^{\frac{p-2}{p}} \varphi)\rangle \,\mathrm{d}x \\ &= \int_{\Omega_{\delta,L}} \eta^2 \Big\langle \mu\Big(\frac{2\Phi}{p} + \mathrm{i}\Psi\Big), \Big(\frac{2\Phi}{p'} + \mathrm{i}\Psi\Big)\Big\rangle \,\mathrm{d}x + \frac{4}{p}\int_{\Omega_{\delta,L}} \eta\langle \mu\Phi, |\varphi|\nabla\eta\rangle \,\mathrm{d}x \\ &\qquad + 2\mathrm{i} \int_{\Omega_{\delta,L}} \eta\langle\mu\Psi, |\varphi|\nabla\eta\rangle \,\mathrm{d}x.\end{aligned}$$

Since $\lambda \int_\Omega u\overline{v} \,\mathrm{d}x + \mathfrak{t}[u,v] = 0$, a combination of the previous calculations and a rearrangement of the terms gives

$$\begin{aligned}&\lambda \int_\Omega \eta^2 |u|_{\delta,L}^{p-2} |u|^2 \,\mathrm{d}x + \delta^{p-2}\int_{\{|u|\leq\delta\}} \eta^2\langle\mu\nabla u, \nabla u\rangle \,\mathrm{d}x \\ &\quad + L^{p-2} \int_{\{|u|\geq L\}} \eta^2 \langle\mu\nabla u,\nabla u\rangle \,\mathrm{d}x + \int_{\Omega_{\delta,L}} \eta^2 \Big\langle \mu\Big(\frac{2\Phi}{p} + \mathrm{i}\Psi\Big), \Big(\frac{2\Phi}{p'} + \mathrm{i}\Psi\Big)\Big\rangle \,\mathrm{d}x \\ &= -\frac{4}{p}\int_{\Omega_{\delta,L}} \eta\langle\mu\Phi, |\varphi|\nabla\eta\rangle \,\mathrm{d}x - 2\mathrm{i}\int_{\Omega_{\delta,L}} \eta\langle\mu\Psi, |\varphi|\nabla\eta\rangle \,\mathrm{d}x \\ &\quad - 2\delta^{p-2}\int_{\{|u|\leq\delta\}} \eta\langle\mu\nabla u, u\nabla\eta\rangle \,\mathrm{d}x - 2L^{p-2}\int_{\{|u|\geq L\}} \eta\langle\mu\nabla u, u\nabla\eta\rangle \,\mathrm{d}x.\end{aligned} \tag{18}$$

We next determine the angle θ for the sector S_θ for λ. Ellipticity of μ gives the estimate $|\mathrm{Im}\langle\mu(x)\xi,\xi\rangle| \leq c^\bullet|\xi|^2 \leq \frac{c^\bullet}{c_\bullet}\mathrm{Re}\langle\mu(x)\xi,\xi\rangle$ for all $\xi\in\mathbb{C}^d$ and almost every $x\in\Omega$. Define

$$\theta_0 := \arctan\left(\frac{2c^\bullet}{c_\bullet}\right) \in (0, \tfrac{\pi}{2}).$$

Then $\langle\mu(x)\xi,\xi\rangle \in \overline{\mathrm{S}_{\theta_0}}$ for all $\xi\in\mathbb{C}^d$ and almost every $x\in\Omega$. Hence

$$\delta^{p-2}\int_{\{|u|\leq\delta\}} \eta^2\langle\mu\nabla u,\nabla u\rangle \,\mathrm{d}x \in \overline{\mathrm{S}_{\theta_0}} \quad \text{and} \quad L^{p-2}\int_{\{|u|\geq L\}} \eta^2\langle\mu\nabla u,\nabla u\rangle \,\mathrm{d}x \in \overline{\mathrm{S}_{\theta_0}}.$$

Furthermore, by virtue of (10) there exists an $\omega \in (0, \frac{\pi}{2})$, depending only on $c^\bullet$, p and γ_0, such that

$$\int_{\Omega_{\delta,L}} \eta^2 \Big\langle \mu\Big(\frac{2\Phi}{p} + \mathrm{i}\Psi\Big), \Big(\frac{2\Phi}{p'} + \mathrm{i}\Psi\Big) \Big\rangle \,\mathrm{d}x \in \overline{\mathrm{S}_\omega}.$$

Choose

$$\theta := \frac{3\pi}{4} - \frac{\max\{\theta_0, \omega\}}{2},$$

then $\theta > \pi/2$ and

$$\theta + \max\{\theta_0, \omega\} = \frac{3\pi}{4} + \frac{\max\{\theta_0, \omega\}}{2} < \pi.$$

Notice that this implies that there exists a $C_{\theta,\omega} > 0$ such that

$$|z_2| + |z_3| + |z_4| \le |z_1| + |z_2| + |z_3| + |z_4| \le C_{\theta,\omega}|z_1 + z_2 + z_3 + z_4|$$

for all $z_1 \in \overline{\mathrm{S}_\theta}$ and $z_2, z_3, z_4 \in \overline{\mathrm{S}_{\max(\theta_0,\omega)}}$. Hence if $\lambda \in \mathrm{S}_\theta$, then

$$\begin{aligned}
&\delta^{p-2}\Big|\int_{\{|u|\le\delta\}} \eta^2 \langle \mu\nabla u, \nabla u\rangle \,\mathrm{d}x\Big| \\
&\quad + L^{p-2}\Big|\int_{\{|u|\ge L\}} \eta^2 \langle \mu\nabla u, \nabla u\rangle \,\mathrm{d}x\Big| + \Big|\int_{\Omega_{\delta,L}} \eta^2 \Big\langle \mu\Big(\frac{2\Phi}{p} + \mathrm{i}\Psi\Big), \Big(\frac{2\Phi}{p'} + \mathrm{i}\Psi\Big) \Big\rangle \,\mathrm{d}x\Big| \\
&\le C_{\theta,\omega}\Big|\lambda \int_\Omega \eta^2 |u|_{\delta,L}^{p-2}|u|^2 \,\mathrm{d}x + \delta^{p-2}\int_{\{|u|\le\delta\}} \eta^2 \langle \mu\nabla u, \nabla u\rangle \,\mathrm{d}x \\
&\quad + L^{p-2}\int_{\{|u|\ge L\}} \eta^2 \langle \mu\nabla u, \nabla u\rangle \,\mathrm{d}x + \int_{\Omega_{\delta,L}} \eta^2 \Big\langle \mu\Big(\frac{2\Phi}{p} + \mathrm{i}\Psi\Big), \Big(\frac{2\Phi}{p'} + \mathrm{i}\Psi\Big) \Big\rangle \,\mathrm{d}x\Big|.
\end{aligned}$$

Using the ellipticity of μ together with (9) and the condition $\Delta_p(\mu) \ge \gamma_0$ on the left-hand side and (18) on the right-hand side yields

$$\begin{aligned}
&c_\bullet\delta^{p-2}\int_{\{|u|\le\delta\}} \eta^2|\nabla u|^2 \,\mathrm{d}x + c_\bullet L^{p-2}\int_{\{|u|\ge L\}} \eta^2|\nabla u|^2 \,\mathrm{d}x \\
&\quad + \frac{2\gamma_0}{p}\int_{\Omega_{\delta,L}} \eta^2|\Phi|^2 \,\mathrm{d}x + \frac{p\gamma_0}{2}\int_{\Omega_{\delta,L}} \eta^2|\Psi|^2 \,\mathrm{d}x \\
&\le c^\bullet C_{\theta,\omega}\Big\{\frac{4}{p}\int_{\Omega_{\delta,L}} |\eta\Phi||\varphi\nabla\eta| \,\mathrm{d}x + 2\int_{\Omega_{\delta,L}} |\eta\Psi||\varphi\nabla\eta| \,\mathrm{d}x \\
&\quad + 2\delta^{p-2}\int_{\{|u|\le\delta\}} |\eta\nabla u||u\nabla\eta| \,\mathrm{d}x + 2L^{p-2}\int_{\{|u|\ge L\}} |\eta\nabla u||u\nabla\eta| \,\mathrm{d}x\Big\}.
\end{aligned} \tag{19}$$

Use the properties of the cut-off function η as well as Young's inequality to derive the estimates

$$\begin{aligned}
\frac{4}{p}c^\bullet C_{\theta,\omega}\int_{\Omega_{\delta,L}} |\eta\Phi||\varphi\nabla\eta| \,\mathrm{d}x &\le \frac{\gamma_0}{p}\int_{\Omega_{\delta,L}} \eta^2|\Phi|^2 \,\mathrm{d}x + \frac{4C_{\theta,\omega}^2}{p\gamma_0 r^2}c^{\bullet 2}\int_{\Omega_{\delta,L}\cap B(x_0,2r)} |\varphi|^2 \,\mathrm{d}x, \\
2c^\bullet C_{\theta,\omega}\int_{\Omega_{\delta,L}} |\eta\Psi||\varphi\nabla\eta| \,\mathrm{d}x &\le \frac{p\gamma_0}{2}\int_{\Omega_{\delta,L}} \eta^2|\Psi|^2 \,\mathrm{d}x + \frac{2C_{\theta,\omega}^2}{p\gamma_0 r^2}c^{\bullet 2}\int_{\Omega_{\delta,L}\cap B(x_0,2r)} |\varphi|^2 \,\mathrm{d}x, \\
2\delta^{p-2}c^\bullet C_{\theta,\omega}\int_{\{|u|\le\delta\}} |\eta\nabla u||u\nabla\eta| \,\mathrm{d}x &\le \frac{\delta^{p-2}c_\bullet}{2}\int_{\{|u|\le\delta\}} \eta^2|\nabla u|^2 \,\mathrm{d}x
\end{aligned}$$

$$+\frac{2\delta^{p-2}c^{\bullet 2}C_{\theta,\omega}^2}{c_\bullet r^2}\int_{\{|u|\le\delta\}\cap B(x_0,2r)}|u|^2\,\mathrm{d}x$$

and

$$2L^{p-2}c^\bullet C_{\theta,\omega}\int_{\{|u|\ge L\}}|\eta\nabla u||u\nabla\eta|\,\mathrm{d}x\le\frac{c_\bullet L^{p-2}}{2}\int_{\{|u|\ge L\}}\eta^2|\nabla u|^2\,\mathrm{d}x$$
$$+\frac{2L^{p-2}c^{\bullet 2}C_{\theta,\omega}^2}{c_\bullet r^2}\int_{\{|u|\ge L\}\cap B(x_0,2r)}|u|^2\,\mathrm{d}x.$$

Combining these estimates with (19), while incorporating the fact that $\eta\equiv 1$ on $B(x_0,r)$ and rearranging yields

$$\begin{aligned}&c_\bullet\delta^{p-2}\int_{\{|u|\le\delta\}\cap B(x_0,r)}|\nabla u|^2\,\mathrm{d}x+c_\bullet L^{p-2}\int_{\{|u|\ge L\}\cap B(x_0,r)}|\nabla u|^2\,\mathrm{d}x+\frac{2\gamma_0}{p}\int_{\Omega_{\delta,L}\cap B(x_0,r)}|\Phi|^2\,\mathrm{d}x\\&\le\frac{4C_{\theta,\omega}^2c^{\bullet 2}}{r^2}\Big\{\frac{3}{p\gamma_0}\int_{\Omega_{\delta,L}\cap B(x_0,2r)}|\varphi|^2\,\mathrm{d}x\\&\qquad+\frac{\delta^{p-2}}{c_\bullet}\int_{\{|u|\le\delta\}\cap B(x_0,2r)}|u|^2\,\mathrm{d}x+\frac{L^{p-2}}{c_\bullet}\int_{\{|u|\ge L\}\cap B(x_0,2r)}|u|^2\,\mathrm{d}x\Big\}.\end{aligned}\tag{20}$$

Next, define analogously to the function φ the function $\varphi_{\delta,L}:=|u|_{\delta,L}^{\frac{p-2}{2}}u$. Then Lemmas 4.1 and 2.2 imply that $|\varphi_{\delta,L}|\in\mathrm{W}_D^{1,2}(\Omega)$. Define $\alpha:=M^2\sqrt{d}$ and recall that $2^*=2d/(d-2)$. An application of the local embedding Lemma 2.4 to $|\varphi_{\delta,L}|$ together with (17) gives

$$\begin{aligned}&\Big(\frac{1}{r^d}\int_{\Omega(x_0,r/\alpha)}|u|_{\delta,L}^{\frac{2^*}{2}(p-2)}|u|^{2^*}\,\mathrm{d}x\Big)^{\frac{1}{2^*}}=\Big(\frac{1}{r^d}\int_{\Omega(x_0,r/\alpha)}|\varphi_{\delta,L}|^{2^*}\,\mathrm{d}x\Big)^{\frac{1}{2^*}}\\&\le C_{\mathrm{Sob}}\Big\{r\Big(\frac{1}{r^d}\int_{\Omega_{\delta,L}\cap B(x_0,r)}|\Phi|^2\,\mathrm{d}x\Big)^{\frac12}+r\Big(\frac{1}{r^d}\int_{\{|u|\le\delta\}\cap B(x_0,r)}\delta^{p-2}|\nabla u|^2\,\mathrm{d}x\Big)^{\frac12}\\&\qquad+r\Big(\frac{1}{r^d}\int_{\{|u|\ge L\}\cap B(x_0,r)}L^{p-2}|\nabla u|^2\,\mathrm{d}x\Big)^{\frac12}+\alpha\Big(\frac{1}{r^d}\int_{\Omega(x_0,r)}|u|_{\delta,L}^{p-2}|u|^2\,\mathrm{d}x\Big)^{\frac12}\Big\},\end{aligned}$$

where we used that $|\nabla|u||\le|\nabla u|$. Apply (20) to the first three terms of the right hand side and split the integral in the fourth term in three parts. Then

$$\begin{aligned}&\Big(\frac{1}{r^d}\int_{\Omega(x_0,r/\alpha)}|u|_{\delta,L}^{\frac{2^*}{2}(p-2)}|u|^{2^*}\,\mathrm{d}x\Big)^{\frac{1}{2^*}}\le C\Big\{\Big(\frac{1}{r^d}\int_{\Omega_{\delta,L}\cap B(x_0,2r)}|u|^p\,\mathrm{d}x\Big)^{\frac12}\\&\quad+\Big(\frac{1}{r^d}\int_{\{|u|\le\delta\}\cap B(x_0,2r)}\delta^{p-2}|u|^2\,\mathrm{d}x\Big)^{\frac12}+\Big(\frac{1}{r^d}\int_{\{|u|\ge L\}\cap B(x_0,2r)}L^{p-2}|u|^2\,\mathrm{d}x\Big)^{\frac12}\Big\},\end{aligned}\tag{21}$$

for some constant $C>0$ that only depends on d, p, M, $c_\bullet$, $c^\bullet$ and γ_0. Now assume for a moment that $u\in\mathrm{L}^p(\Omega(x_0,2r))$ and use Fatou's lemma to deduce from (21)

$$\begin{aligned}\Big(\frac{1}{r^d}\int_{\Omega(x_0,r/\alpha)}|u|^{\frac{2^*p}{2}}\,\mathrm{d}x\Big)^{\frac{1}{2^*}}&\le\liminf_{\substack{\delta\to0\\L\to\infty}}\Big(\frac{1}{r^d}\int_{\Omega(x_0,r/\alpha)}|u|_{\delta,L}^{\frac{2^*}{2}(p-2)}|u|^{2^*}\,\mathrm{d}x\Big)^{\frac{1}{2^*}}\\&\le C\Big(\frac{1}{r^d}\int_{\Omega(x_0,2r)}|u|^p\,\mathrm{d}x\Big)^{\frac12}.\end{aligned}$$

For further reference, we rephrase this fact with a slight modification of the notation. We proved that

$$\left(\frac{1}{r^d}\int_{\Omega(x_0,R/\alpha)} |u|^{\frac{2^*q}{2}}\,\mathrm{d}x\right)^{\frac{2}{2^*q}} \le C\left(\int_{\Omega(x_0,2R)} |u|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \tag{22}$$

for all $x_0 \in \overline{\Omega}$, $R > 0$ and $q \in [2, p_0(\mu))$

Now we iterate (22) to obtain the desired weak reverse Hölder estimate. For this purpose, let $x_0 \in \overline{\Omega}$,$r \in (0, \frac{M}{4})$ and $p \in [2, p_0(\mu))$ with $\Delta_p(\mu) \ge \gamma_0$. Let $n_0 = \min\{k \in \mathbb{N} : p \le 2 \cdot \left(\frac{2^*}{2}\right)^k\}$ and define $p_0, \ldots, p_{n_0+1} \in [2,\infty)$ by

$$\begin{cases} p_k := 2\cdot\left(\frac{2^*}{2}\right)^k, & \text{if } k \in \{0,\ldots,n_0-1\},\\ p_{n_0} = p, & \\ p_{n_0+1} = \frac{2^*}{2}p. & \end{cases}$$

Since $[2,\infty) \ni q \mapsto \Delta_q(\mu)$ is decreasing by [9, Cor. 5.16] it follows that $\Delta_{p_k}(\mu) \ge \Delta_p(\mu) \ge \gamma_0$ and in particular μ is p_k-elliptic for all $k \in \{0,\ldots,n_0\}$. Since $u \in \mathrm{W}^{1,2}_D(\Omega) \subset \mathrm{L}^{p_0}(\Omega)$, we deduce from (22) with $q = p_0$ and $R = r$ that

$$\left(\frac{1}{r^d}\int_{\Omega(x_0,r/\alpha)} |u|^{p_1}\,\mathrm{d}x\right)^{\frac{1}{p_1}} \le C\left(\int_{\Omega(x_0,2r)} |u|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

It follows that $u \in \mathrm{L}^{p_1}(\Omega(x_0, r/\alpha))$.

Next we perform a second iteration step. Another application of (22) applied in the situation where $q = p_1$ and $R = (2\alpha)^{-1}r$ gives

$$\left(\frac{1}{r^d}\int_{\Omega(x_0,(2\alpha)^{-1}r/\alpha)} |u|^{p_2}\,\mathrm{d}x\right)^{\frac{1}{p_2}} \le C\left(\frac{1}{r^d}\int_{\Omega(x_0,r/\alpha)} |u|^{p_1}\,\mathrm{d}x\right)^{\frac{1}{p_1}}.$$

Applying the zeroth iteration step, it follows

$$\left(\frac{1}{r^d}\int_{\Omega(x_0,(2\alpha)^{-1}r/\alpha)} |u|^{p_2}\,\mathrm{d}x\right)^{\frac{1}{p_2}} \le C\left(\frac{1}{r^d}\int_{\Omega(x_0,2r)} |u|^2\,\mathrm{d}x\right)^{\frac{1}{2}},$$

where C is the product of the constants in the respective inequalities. In particular, it follows that $u \in \mathrm{L}^{p_2}(\Omega(x_0, \alpha^{-2}2^{-1}r))$. Continuing this procedure iteratively yields

$$\left(\frac{1}{r^d}\int_{\Omega(x_0,(2\alpha)^{-n_0}r/\alpha)} |u|^{p_{n_0+1}}\,\mathrm{d}x\right)^{\frac{1}{p_{n_0+1}}} \le C'\left(\frac{1}{r^d}\int_{\Omega(x_0,2r)} |u|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

Altogether, this proves the validity of the desired weak reverse Hölder estimates. □

Following a classical idea, which can be found for example in the book of Giaquinta [30, p. 119], we establish a weak reverse Hölder estimate for ∇u. Its proof essentially relies on a subsequent application of Caccioppoli's inequality, Sobolev's embedding and Poincaré's inequality. We start with an adapted version of Caccioppoli's inequality for mixed boundary conditions and the resolvent problem.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $D \subset \partial\Omega$ be closed, and let*

$$0 < \theta < \pi - \arctan\left(\frac{2c^\bullet}{c_\bullet}\right) =: \pi - \theta_0.$$

Then there exists a constant $C_{\theta_0,\theta} > 0$ such that for all $\lambda \in \mathrm{S}_\theta$, $x_0 \in \overline{\Omega}$, $r > 0$ and $u \in \mathrm{W}^{1,2}_D(\Omega)$ that satisfy

$$\lambda \int_\Omega u\overline{v} \, \mathrm{d}x + \mathfrak{t}[u,v] = 0$$

for all $v \in \mathrm{W}^{1,2}_D(\Omega)$ with $\mathrm{supp}(v) \subset \overline{B(x_0,2r)}$ the following is valid.

(a) *If $B(x_0,2r) \cap D = \emptyset$, then for all $c \in \mathbb{C}$*

$$\begin{aligned}|\lambda| \int_{\Omega(x_0,r)} |u|^2 \, \mathrm{d}x &+ \frac{c_\bullet}{2} \int_{\Omega(x_0,r)} |\nabla u|^2 \, \mathrm{d}x \\ &\leq C_{\theta_0,\theta} \Big\{ |\lambda||c| \int_{\Omega(x_0,2r)} |u| \, \mathrm{d}x + \frac{2d^2 C_{\theta_0,\theta} C_d^2 c^{\bullet 2}}{c_\bullet} \frac{1}{r^2} \int_{\Omega(x_0,2r)} |u+c|^2 \, \mathrm{d}x \Big\}.\end{aligned}$$

(b) *If $B(x_0,2r) \cap D \neq \emptyset$, then*

$$|\lambda| \int_{\Omega(x_0,r)} |u|^2 \, \mathrm{d}x + \frac{c_\bullet}{2} \int_{\Omega(x_0,r)} |\nabla u|^2 \, \mathrm{d}x \leq \frac{2d^2 C_{\theta_0,\theta}^2 C_d^2 c^{\bullet 2}}{c_\bullet} \frac{1}{r^2} \int_{\Omega(x_0,2r)} |u|^2 \, \mathrm{d}x.$$

Proof. '(a)'. Since $\theta + \theta_0 < \pi$ there exists a $C_{\theta_0,\theta} > 0$ such that $|z_1| + |z_2| \leq C_{\theta_0,\theta}|z_1 + z_2|$ for all $z_1 \in \overline{\mathrm{S}_\theta}$ and $z_2 \in \overline{\mathrm{S}_{\theta_0}}$. Let $\eta \in \mathrm{C}_c^\infty(B(x_0,2r))$ be such that $0 \leq \eta \leq 1$, $\eta|_{B(x_0,r)} = \mathbb{1}$ and $\|\nabla\eta\|_{\mathrm{L}^\infty} \leq 2/r$, cf. the proof of Theorem 4.2. Let $c \in \mathbb{C}$ and define $v := \eta^2(u+c)$. Then $v \in \mathrm{W}^{1,2}_D(\Omega)$ since $B(x_0,2r) \cap D = \emptyset$.Hence

$$\begin{aligned}0 &= \lambda \int_\Omega \eta^2 u \overline{(u+c)} \, \mathrm{d}x + \int_\Omega \langle \mu\nabla u, \nabla\big(\eta^2(u+c)\big)\rangle \, \mathrm{d}x \\ &= \lambda \int_\Omega \eta^2 |u|^2 \, \mathrm{d}x + \lambda\overline{c} \int_\Omega \eta^2 u \, \mathrm{d}x + \int_\Omega \langle \eta^2 \mu\nabla u, \nabla u\rangle \, \mathrm{d}x + 2\int_\Omega \eta \langle \mu\nabla u, (u+c)\nabla\eta\rangle \, \mathrm{d}x.\end{aligned}$$

The ellipticity of μ implies that

$$\int_\Omega \eta^2 \langle \mu\nabla u, \nabla u\rangle \, \mathrm{d}x \in \overline{\mathrm{S}_{\theta_0}}.$$

Therefore

$$|\lambda| \int_\Omega \eta^2 |u|^2 \, \mathrm{d}x + c_\bullet \int_\Omega \eta^2 |\nabla u|^2 \, \mathrm{d}x \leq C_{\theta_0,\theta} \Big\{ |\lambda||c| \int_\Omega \eta^2 |u| \, \mathrm{d}x + 2dc^\bullet \int_\Omega \eta |\nabla u||\nabla\eta||u+c| \, \mathrm{d}x \Big\}.$$

Consequently, by Young's inequality

$$2dC_{\theta_0,\theta} c^\bullet \int_\Omega \eta |\nabla u||\nabla\eta||u+c| \, \mathrm{d}x \leq \frac{c_\bullet}{2} \int_\Omega \eta^2 |\nabla u|^2 \, \mathrm{d}x + \frac{2d^2 C_{\theta_0,\theta}^2 c^{\bullet 2}}{c_\bullet} \int_\Omega |\nabla\eta|^2 |u+c|^2 \, \mathrm{d}x$$

and the term involving the gradient of u can be absorbed into the left-hand side. The properties of η then yield

$$\begin{aligned}|\lambda| \int_{\Omega(x_0,r)} |u|^2 \, \mathrm{d}x &+ \frac{c_\bullet}{2} \int_{\Omega(x_0,r)} |\nabla u|^2 \, \mathrm{d}x \\ &\leq C_{\theta_0,\theta} \Big\{ |\lambda||c| \int_{\Omega(x_0,2r)} |u| \, \mathrm{d}x + \frac{2d^2 C_{\theta_0,\theta} C_d^2 c^{\bullet 2}}{c_\bullet} \frac{1}{r^2} \int_{\Omega(x_0,2r)} |u+c|^2 \, \mathrm{d}x \Big\}\end{aligned}$$

as required.

'(b)'. The argument is almost as above, but now one has to choose $c = 0$, i.e., $v = \eta^2 u$. □

We continue by combining Lemma 4.3 with the local Sobolev embedding (Lemma 2.4) and the local Poincaré inequality (Lemma 2.7).

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, where $d \geq 3$. Let $D \subset \partial\Omega$ be closed and set $N = \partial\Omega \setminus D$. Adopt Assumptions N, Ω^c and D, and let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ be subject to Assumption 2.11. Let $0 < \theta < \pi$ be chosen as in Lemma 4.3. Then there exists a constant $C > 0$ such that for all $\lambda \in \mathrm{S}_\theta$, $x_0 \in \overline{\Omega}$, $0 < r < \min\{s_0/4, r_0/(4\iota), r_0/2, 1/(16M\sqrt{d})\}$ and $u \in \mathrm{W}^{1,2}_D(\Omega)$ satisfying*

$$\lambda \int_\Omega u\overline{v} \, \mathrm{d}x + \mathfrak{t}[u, v] = 0$$

for all $v \in \mathrm{W}^{1,2}_D(\Omega)$ with $\operatorname{supp}(v) \subset \overline{B(x_0, 2r)}$ it follows that

$$\Big(\frac{1}{r^d}\int_{\Omega(x_0,r)} \big\{|\lambda u| + |\lambda|^{\frac12}|\nabla u|\big\}^2 \, \mathrm{d}x\Big)^{\frac12} \leq C\Big(\frac{1}{r^d}\int_{\Omega(x_0,8M^2\sqrt{d}r)} \big\{|\lambda u| + |\lambda|^{\frac12}|\nabla u|\big\}^{2_*} \, \mathrm{d}x\Big)^{\frac{1}{2_*}},$$

where $2_* := \frac{2d}{d+2}$.

Proof. If $B(x_0, 2r) \subset \Omega$, then Lemma 4.3 with $c := -u_{B(x_0,2r)}$ gives

$$\begin{aligned}|\lambda|\int_{B(x_0,r)} |u|^2 \, \mathrm{d}x &+ \int_{B(x_0,r)} |\nabla u|^2 \, \mathrm{d}x \\ &\leq C\Big\{|\lambda||u_{B(x_0,2r)}|\int_{B(x_0,2r)} |u| \, \mathrm{d}x + \frac{1}{r^2}\int_{B(x_0,2r)} |u - u_{B(x_0,2r)}|^2 \, \mathrm{d}x\Big\}\end{aligned}$$

for a suitable constant $C > 0$. The first term on the right-hand side is controlled by Hölder's inequality and the second term is controlled by means of a Sobolev–Poincaré inequality. This altogether yields

$$\begin{aligned}|\lambda|\int_{B(x_0,r)} |u|^2 \, \mathrm{d}x &+ \int_{B(x_0,r)} |\nabla u|^2 \, \mathrm{d}x \\ &\leq C' r^d\Big\{|\lambda|\Big(\frac{1}{r^d}\int_{B(x_0,2r)} |u|^{2_*} \, \mathrm{d}x\Big)^{\frac{2}{2_*}} + \Big(\frac{1}{r^d}\int_{B(x_0,2r)} |\nabla u|^{2_*} \, \mathrm{d}x\Big)^{\frac{2}{2_*}}\Big\}.\end{aligned}$$

Now, multiplying by $|\lambda|$, dividing by r^d, and taking the square root of the inequality leads us to

$$\begin{aligned}&\Big(\frac{1}{r^d}\int_{B(x_0,r)} |\lambda u|^2 + \big(|\lambda|^{\frac12}|\nabla u|\big)^2 \, \mathrm{d}x\Big)^{\frac12} \\ &\qquad\leq C\Big\{\Big(\frac{1}{r^d}\int_{B(x_0,2r)} |\lambda u|^{2_*} \, \mathrm{d}x\Big)^{\frac{2}{2_*}} + \Big(\frac{1}{r^d}\int_{B(x_0,2r)} \big(|\lambda|^{\frac12}|\nabla u|\big)^{2_*} \, \mathrm{d}x\Big)^{\frac{2}{2_*}}\Big\}^{\frac12}.\end{aligned}$$

Finally, the equivalence of norms in $\mathbb{R}^2$, in particular the estimates

$$(a+b)^2 \leq 2(a^2+b^2), \quad a^{\frac{2}{2_*}} + b^{\frac{2}{2_*}} \leq (a+b)^{\frac{2}{2_*}} \quad \text{and} \quad a^{2_*} + b^{2_*} \leq (a+b)^{2_*}$$

for all $a, b \in [0, \infty)$ yield the desired estimate for interior balls.

If $x_0 \in \partial\Omega$ and $B(x_0, 2r) \cap D = \emptyset$, then we do almost the same, but we take $c := -(Eu)_{B(x_0,2r)}$, where E denotes the local reflection operator from Remark 2.3. In this case, Lemma 4.3 gives

$$\begin{aligned} &|\lambda| \int_{\Omega(x_0,r)} |u|^2 \, \mathrm{d}x + \int_{\Omega(x_0,r)} |\nabla u|^2 \, \mathrm{d}x \\ &\quad \leq C \Big\{ |\lambda| |(Eu)_{B(x_0,2r)}| \int_{\Omega(x_0,2r)} |u| \, \mathrm{d}x + \frac{1}{r^2} \int_{B(x_0,2r)} |Eu - (Eu)_{B(x_0,2r)}|^2 \, \mathrm{d}x \Big\} \\ &\quad \leq C r^d \Big\{ |\lambda| |(Eu)_{B(x_0,2r)}| \frac{1}{r^d} \int_{\Omega(x_0,2r)} |u| \, \mathrm{d}x + \Big(\frac{1}{r^d} \int_{B(x_0,2r)} |\nabla (Eu)|^{2_*} \, \mathrm{d}x \Big)^{\frac{2}{2_*}} \Big\}. \end{aligned}$$

The estimates given in Remark 2.3 imply

$$\Big(\frac{1}{r^d} \int_{B(x_0,2r)} |\nabla (Eu)|^{2_*} \, \mathrm{d}x \Big)^{\frac{2}{2_*}} \leq C' \Big(\frac{1}{r^d} \int_{\Omega(x_0, 2M^2\sqrt{d} r)} |\nabla u|^{2_*} \, \mathrm{d}x \Big)^{\frac{2}{2_*}},$$

where $C' > 0$ only depends on d and M and

$$|(Eu)_{B(x_0,2r)}| \leq \frac{C''}{r^d} \int_{\Omega(x_0, 2M^2\sqrt{d} r)} |u| \, \mathrm{d}x.$$

Using once again the Hölder inequality we arrive at

$$\begin{aligned} &|\lambda| \int_{\Omega(x_0,r)} |u|^2 \, \mathrm{d}x + \int_{\Omega(x_0,r)} |\nabla u|^2 \, \mathrm{d}x \\ &\quad \leq C''' r^d \Big\{ |\lambda| \Big(\frac{1}{r^d} \int_{\Omega(x_0, 2M^2\sqrt{d} r)} |u|^{2_*} \, \mathrm{d}x \Big)^{\frac{2}{2_*}} + \Big(\frac{1}{r^d} \int_{\Omega(x_0, 2M^2\sqrt{d} r)} |\nabla u|^{2_*} \, \mathrm{d}x \Big)^{\frac{2}{2_*}} \Big\}. \end{aligned}$$

Multiplying again by $|\lambda|$, dividing by r^d, taking the square root and employing the equivalence of norms in $\mathbb{R}^2$ as above establishes the desired inequality.

If $x_0 \in \partial\Omega$ and $B(x_0, 2r) \cap D \neq \emptyset$, then Lemma 4.3 implies

$$|\lambda| \int_{\Omega(x_0,r)} |u|^2 \, \mathrm{d}x + \int_{\Omega(x_0,r)} |\nabla u|^2 \, \mathrm{d}x \leq \frac{C}{r^2} \int_{\Omega(x_0,2r)} |u|^2 \, \mathrm{d}x.$$

Now Lemma 2.4 guarantees that

$$\frac{1}{r^2} \int_{\Omega(x_0,2r)} |u|^2 \, \mathrm{d}x \leq C r^d \Big\{ \Big(\frac{1}{r^d} \int_{\Omega(x_0, 2M^2\sqrt{d} r)} |\nabla u|^{2_*} \, \mathrm{d}x \Big)^{\frac{1}{2_*}} + \frac{1}{r} \Big(\frac{1}{r^d} \int_{\Omega(x_0,2r)} |u|^{2_*} \, \mathrm{d}x \Big)^{\frac{1}{2_*}} \Big\}^2.$$

An application of Lemma 2.7 with $p = 2_*$ yields the desired inequality. □

5 From weak reverse Hölder estimates to L^p-estimates

In this section we provide the proofs of Theorems 3.1 and 3.4. The proofs fundamentally base on the following L^p-extrapolation theorem of Shen [48, Thm. 3.3] which was initially proved on bounded Lipschitz domains and generalised in [49, Thm. 4.1] to general bounded measurable sets.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^d$ be bounded and Lebesgue-measurable, let $k, m \in \mathbb{N}$, $\mathcal{M} > 0$, and let $T \in \mathcal{L}(\mathrm{L}^2(\Omega;\mathbb{C}^k), \mathrm{L}^2(\Omega;\mathbb{C}^m))$ with $\|T\|_{\mathcal{L}(\mathrm{L}^2(\Omega;\mathbb{C}^k),\mathrm{L}^2(\Omega;\mathbb{C}^m))} \leq \mathcal{M}$. Further let $p > 2$, $R_0 > 0$, $\alpha_2 > \alpha_1 > 1$ and $\mathcal{C} > 0$. Suppose that*

$$\Big(\frac{1}{r^d}\int_{\Omega(x_0,r)} |Tf|^p \,\mathrm{d}x\Big)^{1/p} \leq \mathcal{C}\Big(\frac{1}{r^d}\int_{\Omega(x_0,\alpha_1 r)} |Tf|^2 \,\mathrm{d}x\Big)^{1/2}$$

for all $0 < r < R_0$ and x_0 that either satisfy $x_0 \in \partial\Omega$ or $B(x_0, \alpha_2 r) \subset \Omega$ and for all compactly supported $f \in \mathrm{L}^\infty(\Omega;\mathbb{C}^k)$ with $f = 0$ on $\Omega(x_0, \alpha_2 r)$.

Then T restricts for all $2 < q < p$ to a bounded operator in $\mathcal{L}(\mathrm{L}^q(\Omega;\mathbb{C}^k), \mathrm{L}^q(\Omega;\mathbb{C}^m))$ with an operator norm that is bounded by a constant depending on d, p, q, α_1, α_2, $\mathcal{C}$, $\mathcal{M}$, R_0 and $\operatorname{diam}(\Omega)$.

Now we are in the position to give a proof of Theorem 3.1.

Proof of Theorem 3.1 and Theorem 3.4(a). By the Lax–Milgram lemma, there are $\theta_1 \in (\pi/2, \pi)$ and $\mathcal{M} > 0$, depending only on $c_\bullet$ and $c^\bullet$ such that $\mathrm{S}_{\theta_1} \subset \rho(-A_2)$ and

$$\|\lambda(\lambda + A_2)^{-1} f\|_{\mathrm{L}^2(\Omega)} \leq \mathcal{M}\|f\|_{\mathrm{L}^2(\Omega)} \qquad (f \in \mathrm{L}^2(\Omega),\ \lambda \in \mathrm{S}_{\theta_1}).$$

Similarly, it follows that

$$|\lambda|^{\frac{1}{2}}\|(\lambda + A_2)^{-1} \operatorname{div} g\|_{\mathrm{L}^2(\Omega)} \leq \mathcal{M}\|g\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \qquad (g \in \mathrm{L}^2(\Omega;\mathbb{C}^d),\ \lambda \in \mathrm{S}_{\theta_1}). \tag{23}$$

Here the operator $(\lambda + A_2)^{-1} \operatorname{div}$ is understood as the solution operator to the equation

$$\lambda \int_\Omega u\overline{v} \,\mathrm{d}x + \mathfrak{t}[u, v] = -\int_\Omega \langle g, \nabla v\rangle \,\mathrm{d}x \qquad (v \in \mathrm{W}^{1,2}_D(\Omega)).$$

Let $2 < p < p_0(\mu)$. Furthermore, let $\theta_2 \in (\pi/2, \pi)$ denote the angle obtained in Theorem 4.2. Define $\theta := \min\{\theta_1, \theta_2\}$. Let $0 < c < 1$ be the constant (which only depends on d, p and M) as in Theorem 4.2. Let $\mathcal{C}$ be as in Theorem 4.2. Note that $\mathcal{C}$ depends only on d, p, M, $c_\bullet$, $c^\bullet$ and γ_0. For all $\lambda \in \mathrm{S}_\theta$ define

$$T_\lambda \colon \mathrm{L}^2(\Omega;\mathbb{C}^{1+d}) \to \mathrm{L}^2(\Omega;\mathbb{C}^{1+1}), \quad T_\lambda(f, g) = \begin{pmatrix} \lambda(\lambda + A_2)^{-1} f \\ |\lambda|^{\frac{1}{2}}(\lambda + A_2)^{-1} \operatorname{div} g \end{pmatrix}.$$

Let $0 < r < cM/4 =: R_0$ and $x_0 \in \overline{\Omega}$. Suppose either $x_0 \in \partial\Omega$ or $B(x_0, (3/c)r) \subset \Omega$. Let $(f, g) \in \mathrm{L}^\infty(\Omega;\mathbb{C}^{1+d})$ and suppose that (f, g) vanishes on $\Omega(x_0, (3/c)r)$. Then

$$\Big(\frac{1}{r^d}\int_{\Omega(x_0,r)} |T_\lambda(f, g)|^{\frac{pd}{d-2}} \,\mathrm{d}x\Big)^{\frac{d-2}{pd}} \leq \mathcal{C}\Big(\frac{1}{r^d}\int_{\Omega(x_0,(2/c)r)} |T_\lambda(f, g)|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}},$$

by Theorem 4.2. Taking $\alpha_1 := 2/c$ and $\alpha_2 := 3/c$ it follows from Theorem 5.1 that T_λ restricts to a bounded operator on $\mathrm{L}^q(\Omega)$ for all $q \in (2, pd/(d-2))$. Moreover, the operator norm is bounded by a constant that depends only on d, p, q, M, c, $\mathcal{C}$, $\mathcal{M}$, γ_0 and $\operatorname{diam}(\Omega)$. This implies that there exists a constant $C > 0$ such that

$$|\lambda|\|(\lambda + A_2)^{-1} f\|_{\mathrm{L}^q(\Omega)} \leq C\|f\|_{\mathrm{L}^q(\Omega)} \qquad (f \in \mathrm{L}^q(\Omega),\ \lambda \in \mathrm{S}_\theta)$$

and

$$|\lambda|^{\frac{1}{2}}\|(\lambda + A_2)^{-1} \operatorname{div} g\|_{\mathrm{L}^q(\Omega)} \leq C\|g\|_{\mathrm{L}^q(\Omega;\mathbb{C}^d)} \qquad (g \in \mathrm{L}^q(\Omega;\mathbb{C}^d),\ \lambda \in \mathrm{S}_\theta).$$

Since A_q is the part of A_2 in $\mathrm{L}^q(\Omega)$ this immediately yields $\mathrm{S}_\theta \subset \rho(-A_q)$ and $(\lambda + A_2)^{-1} f = (\lambda + A_q)^{-1} f$ for all $f \in \mathrm{L}^q(\Omega)$ and $\lambda \in \mathrm{S}_\theta$. Consequently, A_q is a sectorial operator in $\mathrm{L}^q(\Omega)$ and by [34, Prop. 2.1.1(h)] it is densely defined. Now Lemma 2.12 yields that $(A_2^*)_{q'}$ is well-defined and by duality it is sectorial of the same angle. The operator A_2^* is the divergence form operator that is associated to the matrix μ^*. Finally, since $(\mu^*)^* = \mu$ and because $p_0(\mu) = p_0(\mu^*)$, by [9, Cor. 5.17], we find by the argumentation above with μ replaced by μ^* that $A_{q'}$ is well-defined and sectorial. Moreover, the gradient estimate in Theorem 3.4(a) follows by duality as well. Since $2 < p < p_0(\mu)$ is arbitrary this concludes the proof. □

The proof of Theorem 3.4(b) and (c) is similar.

Proof of Theorem 3.4(b) and (c). Dualising (23) implies the estimate

$$|\lambda|^{\frac{1}{2}} \|\nabla(\lambda + A_2)^{-1} f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \leq \mathcal{M} \|f\|_{\mathrm{L}^2(\Omega)} \qquad (f \in \mathrm{L}^2(\Omega), \lambda \in \mathrm{S}_{\theta_1}).$$

Moreover, the Lax–Milgram lemma implies the validity of

$$\|\nabla(\lambda + A_2)^{-1} \operatorname{div} g\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \leq \mathcal{M} \|g\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \qquad (g \in \mathrm{L}^2(\Omega;\mathbb{C}^d), \lambda \in \mathrm{S}_{\theta_1}).$$

Let $\theta_2 \in (\pi/2, \pi)$ denote the angle as in Lemma 4.3 and define $\theta := \min\{\theta_1, \theta_2\}$. Define the operator

$$S_\lambda \colon \mathrm{L}^2(\Omega;\mathbb{C}^{1+d}) \to \mathrm{L}^2(\Omega;\mathbb{C}^{1+d+1+d}), \quad S_\lambda(f,g) = \begin{pmatrix} \lambda(\lambda + A_2)^{-1} f \\ |\lambda|^{\frac{1}{2}} \nabla(\lambda + A_2)^{-1} f \\ |\lambda|^{\frac{1}{2}} (\lambda + A_2)^{-1} \operatorname{div} g \\ \nabla(\lambda + A_2)^{-1} \operatorname{div} g \end{pmatrix},$$

which defines a uniformly bounded family of operators on L^2.

Let $0 < r < \min\{s_0/4, r_0/(4\iota), r_0, 1/(16M\sqrt{d})\}$ and let $x_0 \in \overline{\Omega}$ satisfy either $x_0 \in \partial\Omega$ or $B(x_0, 16M^2\sqrt{d}r) \subset \Omega$. Let $(f,g) \in \mathrm{L}^\infty(\Omega;\mathbb{C}^{1+d})$ vanish on $\Omega(x_0, 16M^2\sqrt{d}r)$. Then by Lemma 4.4 there exists a constant $C > 0$ such that the weak reverse Hölder estimate

$$\Big(\frac{1}{r^d} \int_{\Omega(x_0,r)} |S_\lambda(f,g)|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}} \leq C \Big(\frac{1}{r^d} \int_{\Omega(x_0, 8M^2\sqrt{d}r)} |S_\lambda(f,g)|^{\frac{2d}{d+2}} \, \mathrm{d}x\Big)^{\frac{d+2}{2d}}$$

is valid. By [49, Lem. 4.2] the weak reverse Hölder estimate is even valid for all $x_0 \in \mathbb{R}^d$ with a different constant C (independent of x_0 and r). Consequently, the self-improving property of weak reverse Hölder estimates, see e.g., [31, Thm. 6.38], establishes the existence of an $\varepsilon > 0$ such that the weak reverse Hölder estimate

$$\Big(\frac{1}{r^d} \int_{\Omega(x_0,r)} |S_\lambda(f,g)|^p \, \mathrm{d}x\Big)^{\frac{1}{p}} \leq C \Big(\frac{1}{r^d} \int_{\Omega(x_0, 8M^2\sqrt{d}r)} |S_\lambda(f,g)|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}}$$

is valid for all $2 < p < 2 + \varepsilon$. It follows that S_λ restricts to a uniformly bounded family of operators from $\mathrm{L}^q(\Omega;\mathbb{C}^{1+d})$ into $\mathrm{L}^q(\Omega;\mathbb{C}^{1+d+1+d})$ for all $2 < q < 2 + \varepsilon$. Noticing that the boundedness of S_λ implies the boundedness of the operators in each row proves the theorem. □

6 Perturbation of real-valued matrices

This section is devoted to the perturbation theory of elliptic operators with complex coefficients. First, we record the following lemma that shows that p-ellipticity is stable under small perturbations of the coefficients.

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, where $d \geq 3$. Let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ be subject to Assumption 2.11. Let $\nu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ and suppose that $\|\nu\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))} < c_\bullet$. Then $\mu + \nu$ is subject to Assumption 2.11. Moreover,*

$$\Delta_p(\mu+\nu) \geq \Delta_p(\mu) - 2\max\Big(\frac{1}{p}, \frac{1}{p'}\Big)\|\nu\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))}$$

for all $p \in (1,\infty)$. In particular, if μ is p-elliptic and $2\max(\frac{1}{p}, \frac{1}{p'})\|\nu\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))} < \Delta_p(\mu)$, then $\mu + \nu$ is p-elliptic.

Proof. The first part is easy.

Let $\alpha, \beta \in \mathbb{R}^d$. Write $\xi = \alpha + \mathrm{i}\beta$. Let $x \in \Omega$. Then $|\mathcal{J}_p(\xi)|^2 = \frac{4}{(p')^2}|\alpha|^2 + \frac{4}{p^2}|\beta|^2 \leq 4\max(\frac{1}{p^2}, \frac{1}{(p')^2})|\xi|^2$. So

$$|\langle \nu(x)\xi, \mathcal{J}_p(\xi)\rangle| \leq \|\nu\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))}|\xi||\mathcal{J}_p(\xi)| \leq 2\max\Big(\frac{1}{p}, \frac{1}{p'}\Big)\|\nu\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))}|\xi|^2$$

and the lemma follows. □

Next, we prove Theorem 3.8. Recall that Lemma 2.15 states that μ is p-elliptic for all $p \in (1,\infty)$ if and only if $\mathrm{Im}(\mu)$ = 0. Thus, Theorem 3.8 *cannot* be concluded by perturbing the p-ellipticity as in Lemma 6.1 and then by applying Theorem 3.1.Instead, the proof is based on the Gaussian kernel estimates obtained in [29, Thm. 7.5], which were a consequence of De Giorgi estimates for the operator A_2. The De Giorgi estimates allow to have a (complex) perturbation. This was shown first for operators on $\mathbb{R}^d$ by Auscher [2, Thm. 4.4]. We modify the proofs in [2] and [29] for our situation. In the proof we need the next well-known lemma of Campanato, which allows for the perturbation of the De Giorgi estimates, cf. [31, Lemma 5.12] or [30, Lemma III.2.1].

Lemma 6.2. *For all $c, \alpha, \beta > 0$ with $\alpha > \beta$ there exist $\varepsilon > 0$ and $\tilde{c} > 0$ (depending only on c, α and β) such that the following is valid.*

Let $B \geq 0$, $R_0 > 0$, and let $\Psi\colon (0, R_0] \to [0,\infty)$ be an increasing function with the property that

$$\Psi(r) \leq c\Big(\Big(\frac{r}{R}\Big)^\alpha + \varepsilon\Big)\Psi(R) + BR^\beta$$

for all $r, R \in \mathbb{R}$ with $0 < r \leq R \leq R_0$. Then

$$\Psi(r) \leq \tilde{c}\Big(\Big(\frac{r}{R}\Big)^\beta \Psi(R) + Br^\beta\Big)$$

for all $0 < r \leq R \leq R_0$.

In the forthcoming proofs it will be important to not emphasise the set D, where functions in $\mathrm{W}^{1,2}_D(\Omega)$ vanish, but to emphasise the complementary set $N = \partial\Omega \setminus D$. Thus, in order to simplify the notation, we introduce

$$\widehat{\mathrm{W}}^{1,2}_N(\Omega) := \mathrm{W}^{1,2}_{\partial\Omega\setminus N}(\Omega).$$

Moreover, for all $x \in \mathbb{R}^d$ and $r > 0$ we shall write in the following $N(x,r) = N \cap B(x,r)$ and $Q(x,r) = Q \cap B(x,r)$, where Q will be $(-1,1)^{d-1} \times (0,1)$. Recall that we already write $\Omega(x,r)$ for $\Omega \cap B(x,r)$.

Definition 6.3. Let $\Omega \subset \mathbb{R}^d$ be open, N a relatively open subset of $\partial\Omega$ and $\Delta \subset \partial\Omega \setminus N$ be closed. Let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ subject to Assumption 2.11 and let A_2 be the associated operator. Let $\kappa_0 \in (0,1)$, $c_{DG} > 0$ and $\Upsilon \subset \overline{\Omega}$ be a set. Then we say that A_2 satisfies (κ_0, c_{DG})-*De Giorgi estimates* on Υ for functions vanishing on Δ and Neumann boundary conditions on N if

$$\int_{\Omega(x,r)} |\nabla u|^2 \, \mathrm{d}x \le c_{DG} \Big(\frac{r}{R}\Big)^{d-2+2\kappa_0} \int_{\Omega(x,R)} |\nabla u|^2 \, \mathrm{d}x$$

for all $x \in \Upsilon$, $0 < r \le R \le 1$ and $u \in \mathrm{W}^{1,2}_\Delta(\Omega)$ satisfying

$$\int_{\Omega(x,R)} \langle \mu \nabla u, \nabla v \rangle \, \mathrm{d}x = 0$$

for all $v \in \widehat{\mathrm{W}}^{1,2}_{N(x,R)}(\Omega(x,R))$.

We start with a perturbation of [29, Prop. 5.3]. As we work in the following only in the L^2 scale, we drop the index $p = 2$ at the operator. Furthermore, we adopt the notation A^μ to indicate that A^μ is the divergence-form operator with coefficients μ.

Proposition 6.4. *Let $\Omega \subset \mathbb{R}^d$ be open, $D \subset \partial\Omega$ be closed and $N := \partial\Omega \setminus D$ subject to Assumption N. Let $x_0 \in \overline{N} \cap D$ and let $\Phi := \Phi_{x_0}$ be the bi-Lipschitz homeomorphism of Assumption N with corresponding neighbourhood $U := U_{x_0}$. In addition, suppose that Assumption P is valid and let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{R}^{d\times d})$ be subject to Assumption 2.11.*

Then there exists an $\varepsilon > 0$ such that for all (complex valued)$\nu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ satisfying $\|\nu\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))} < \varepsilon$ the operator $A^{\mu+\nu}$ is m-sectorial. Moreover, if $A^{\mu+\nu}_\Phi$ denotes the operator in $(-1,1)^{d-1} \times (0,1)$ obtained from $A^{\mu+\nu}$ under the transformation Φ, there are $\kappa_0 \in (0,1)$ and $c_{DG} > 0$ such that the operator $A^{\mu+\nu}_\Phi$ satisfies (κ_0, c_{DG})-De Giorgi estimates on $(-\frac{1}{2}, \frac{1}{2})^{d-1} \times (0, \frac{1}{2})$ for functions vanishing on $\overline{\Phi(U \cap D)}$ and Neumann boundary conditions on $\Phi(U \cap N)$. The constants ε, κ_0 and c_{DG} can be chosen to depend only on d, M, $c_\bullet$, $c^\bullet$, c_0 and c_1.

Proof. By [29, Prop. 5.3] there exist $\kappa_0 \in (0,1)$ and $c_{DG} > 0$ such that the transformed operator A^μ_Φ satisfies (κ_0, c_{DG})-De Giorgi estimates on $(-\frac{1}{2}, \frac{1}{2})^{d-1} \times (0, \frac{1}{2})$ for functions vanishing on $\overline{\Phi(U \cap D)}$ and Neumann boundary conditions on $\Phi(U \cap N)$. Set $c = 4c_{DG}$, $\alpha = d-2+2\kappa_0$ and $\beta = d-2+\kappa_0$. Let $\varepsilon > 0$ and $\tilde{c} > 0$ be as in Lemma 6.2. Define

$$\tilde{\varepsilon} = \frac{c_\bullet}{2} \wedge \frac{\sqrt{\varepsilon} c_\bullet}{2(d! M^{d+2})^2} \Big(\frac{4c_{DG}}{2 + 4c_{DG}}\Big)^{1/2}.$$

Then

$$\frac{2 + 4c_{DG}}{4c_{DG}} (d! M^{d+2})^4 \frac{4\tilde{\varepsilon}^2}{c_\bullet^2} \le \varepsilon$$

and $\tilde{\varepsilon} \in (0, c_\bullet)$.

Let $\nu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ with $\|\nu\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))} < \tilde{\varepsilon}$. Let μ^Φ and $(\mu+\nu)^\Phi$ be the coefficient functions on $(-1,1)^{d-1} \times (0,1)$ obtained from μ and $\mu+\nu$ under the transformation Φ. Define $Q = (-1,1)^{d-1} \times$

$(0,1)$, $D_Q = \overline{\Phi(U \cap D)}$ and $N_Q = \Phi(U \cap N)$. Let $u \in \mathrm{W}^{1,2}_{D_Q}(Q)$ and $x \in \frac{1}{2}Q$. Fix $R_0 \in (0,1]$. Suppose that

$$\int_{Q(x,R_0)} \langle (\mu+\nu)^\Phi \nabla u, \nabla v\rangle \, \mathrm{d}x = 0$$

for all $v \in \widehat{\mathrm{W}}^{1,2}_{N_Q(x,R)}(Q(x,R_0))$. Let $0 < r \leq R \leq R_0$. By [29, Lem. 6.1(b)] and the Lax–Milgram lemma there exists a unique $\tilde v \in \widehat{\mathrm{W}}^{1,2}_{N_Q(x,R)}(Q(x,R))$ such that

$$\int_{Q(x,R)} \langle \mu^\Phi \nabla \tilde v, \nabla \varphi\rangle \, \mathrm{d}x = \int_{Q(x,R)} \langle \mu^\Phi \nabla u, \nabla \varphi\rangle \, \mathrm{d}x \tag{24}$$

for all $\varphi \in \widehat{\mathrm{W}}^{1,2}_{N_Q(x,R)}(Q(x,R))$. Define $v\colon Q \to \mathbb{C}$ by

$$v(y) = \begin{cases} \tilde v(y), & \text{if } y \in Q(x,R), \\ 0, & \text{if } y \in Q \setminus Q(x,R). \end{cases}$$

Then $v \in \widehat{\mathrm{W}}^{1,2}_{N_Q}(Q) \subset \mathrm{W}^{1,2}_{D_Q}(Q)$ by [29, Lem. 6.4]. Set $w = u - \tilde v$. Then $w \in \mathrm{W}^{1,2}_{D_Q}(Q)$. Moreover,

$$\int_{Q(x,R)} \langle \mu^\Phi \nabla w, \nabla \varphi\rangle \, \mathrm{d}x = 0$$

for all $\varphi \in \widehat{\mathrm{W}}^{1,2}_{N_Q(x,R)}(Q(x,R))$ by (24). The De Giorgi inequalities applied to the function w imply

$$\begin{aligned}
\int_{Q(x,r)} |\nabla u|^2 \, \mathrm{d}x &\leq 2\int_{Q(x,r)} |\nabla w|^2 \, \mathrm{d}x + 2\int_{Q(x,r)} |\nabla \tilde v|^2 \, \mathrm{d}x \\
&\leq 2c_{DG}\Big(\frac{r}{R}\Big)^{d-2+2\kappa_0} \int_{Q(x,R)} |\nabla w|^2 \, \mathrm{d}x + 2\int_{Q(x,r)} |\nabla \tilde v|^2 \, \mathrm{d}x \\
&\leq 4c_{DG}\Big(\frac{r}{R}\Big)^{d-2+2\kappa_0} \int_{Q(x,R)} |\nabla u|^2 \, \mathrm{d}x + (2+4c_{DG})\int_{Q(x,R)} |\nabla \tilde v|^2 \, \mathrm{d}x.
\end{aligned}$$

Choose $\varphi = \tilde v$ in (24). Then

$$\int_{Q(x,R)} \langle \mu^\Phi \nabla \tilde v, \nabla \tilde v\rangle \, \mathrm{d}x = \int_{Q(x,R)} \langle \mu^\Phi \nabla u, \nabla \tilde v\rangle \, \mathrm{d}x = \int_{Q(x,R)} \langle (\mu^\Phi - (\mu+\nu)^\Phi) \nabla u, \nabla \tilde v\rangle \, \mathrm{d}x.$$

Hence the estimate $c_\bullet - \tilde\varepsilon \geq \frac{c_\bullet}{2}$, ellipticity on Q (see [29] Prop. 4.3(b)) and the Cauchy–Schwarz inequality give

$$(d!M^{d+2})^{-1}\frac{c_\bullet}{2}\int_{Q(x,R)} |\nabla \tilde v|^2 \, \mathrm{d}x \leq d!M^{d+2}\tilde\varepsilon\Big(\int_{Q(x,R)} |\nabla \tilde u|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_{Q(x,R)} |\nabla \tilde v|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}}.$$

Therefore

$$\int_{Q(x,R)} |\nabla \tilde v|^2 \, \mathrm{d}x \leq (d!M^{d+2})^4 \frac{4\tilde\varepsilon^2}{c_\bullet^2}\int_{Q(x,R)} |\nabla \tilde u|^2 \, \mathrm{d}x$$

and

$$\begin{aligned}
\int_{Q(x,r)} |\nabla u|^2 \, \mathrm{d}x &\leq 4c_{DG}\Big(\frac{r}{R}\Big)^{d-2+2\kappa_0}\int_{Q(x,R)} |\nabla u|^2 \, \mathrm{d}x \\
&\qquad + (2+4c_{DG})(d!M^{d+2})^4\frac{4\tilde\varepsilon^2}{c_\bullet^2}\int_{Q(x,R)} |\nabla \tilde u|^2 \, \mathrm{d}x \\
&\leq 4c_{DG}\Big(\Big(\frac{r}{R}\Big)^{d-2+2\kappa_0} + \varepsilon\Big)\int_{Q(x,R)} |\nabla u|^2 \, \mathrm{d}x.
\end{aligned}$$

Hence by Lemma 6.2 one concludes that

$$\int_{Q(x,r)} |\nabla u|^2 \, \mathrm{d}x \le \tilde{c}\Big(\frac{r}{R}\Big)^{d-2+\kappa_0} \int_{Q(x,R)} |\nabla u|^2 \, \mathrm{d}x$$

for all $0 < r \le R \le R_0$. The proposition follows by choosing $R = R_0$. □

The next proposition gives perturbed De Giorgi estimates near the Neumann part of the boundary, but away from the Dirichlet part of the boundary.

Proposition 6.5. *Let $\Omega \subset \mathbb{R}^d$ be open, $D \subset \partial\Omega$ be closed and $N = \partial\Omega \setminus D$ subject to Assumption N. Let $x_0 \in N$ and let Φ_{x_0} be the bi-Lipschitz homeomorphism of Assumption N whose corresponding set U_{x_0} satisfies $U_{x_0} \cap \partial\Omega \subset N$. Moreover, let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{R}^{d\times d})$ be subject to Assumption 2.11.*

Then there exists an $\varepsilon > 0$ such that for all (complex valued) $\nu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ satisfying $\|\nu\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))} < \varepsilon$ the operator $A^{\mu+\nu}$ is m-sectorial. Moreover, if $A^{\mu+\nu}_\Phi$ denotes the operator in $(-1,1)^{d-1} \times (0,1)$ obtained from $A^{\mu+\nu}$ under the transformation Φ, there are $\kappa_0 \in (0,1)$ and $c_{DG} > 0$ such that $A^{\mu+\nu}_\Phi$ satisfies (κ_0, c_{DG})-De Giorgi estimates on $(-\frac{1}{2},\frac{1}{2})^{d-1} \times (0,\frac{1}{2})$ for functions vanishing on $\emptyset$ and Neumann boundary conditions on $(-1,1)^{d-1} \times \{0\}$. The constants ε, κ_0 and c_{DG} can be chosen to depend only on d, M, $c_\bullet$ and $c^\bullet$.

Proof. The proof is similar to the proof of Proposition 6.4, using [29, Lem. 5.1] instead of [29, Prop. 5.3]. □

Finally we need a perturbed version of [29, Prop. 2.1].

Proposition 6.6. *Let $\Omega \subset \mathbb{R}^d$ be open, $D \subset \partial\Omega$ closed, and set $N = \partial\Omega \setminus D$. Let $\Upsilon \subset \overline{\Omega}$ be a subset. Suppose there exist $\alpha, \zeta > 0$ such that $\operatorname{dist}(N, \Upsilon) \ge \zeta$ and $|B(x,r) \setminus \Omega| \ge \alpha r^d$ for all $r \in (0,1]$ and $x \in \partial\Omega$ with $\operatorname{dist}(x, \Upsilon) < \zeta$.*

Then there exists an $\varepsilon > 0$ such that for all (complex valued) $\nu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ satisfying $\|\nu\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))} < \varepsilon$ the operator $A^{\mu+\nu}$ is m-sectorial. Moreover there are $\kappa_0 \in (0,1)$ and $c_{DG} > 0$ such that the operator $A^{\mu+\nu}$ satisfies (κ_0, c_{DG})-De Giorgi estimates on Υ for functions vanishing on D and Neumann boundary conditions on $\emptyset$. The constants ε, κ_0 and c_{DG} can be chosen to depend only on d, ζ, α, $c_\bullet$ and $c^\bullet$.

Proof. The proof is similar to the proof of Proposition 6.4, but restrict to the case $R_0 \le \zeta$. Of course this time one uses [29, Prop. 2.1] instead of [29, Prop. 5.3]. This gives the De Giorgi estimates for $0 < r \le R \le \zeta$. The case $R \in (\zeta, 1]$ follows easily from the case $R = \zeta$. □

Proof of Theorem 3.8. The proof is similar to the proof of [29, Thm. 7.5], with the obvious changes to use Propositions 6.4, 6.5 and 6.6. One deduces the Gaussian kernel bounds of Theorem 3.8. These imply that the semigroup on $\mathrm{L}^2(\Omega)$ extends consistently to $\mathrm{L}^p(\Omega)$ for all $p \in [1,\infty)$. □

Finally, we present the proof of Corollary 3.9.

Proof of Corollary 3.9. Let $\varepsilon > 0$ be the ε from Theorem 3.8 that belongs to *real valued* matrices with ellipticity constants $c_\bullet/2$ and $2c^\bullet$. Choose $p_c > 2$ such that

$$\frac{(p_c-1)^{1/2}}{p_c-2} c^\bullet < \min\Big\{\varepsilon, c^\bullet, \frac{c_\bullet}{2}\Big\}.$$

Now let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ be subject to Assumption 2.11 and suppose that $p_0(\mu) > p_c$. Then Lemma 2.15 gives

$$\|\mathrm{Im}(\mu)\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))} \leq \frac{(p_0(\mu)-1)^{1/2}}{p_0(\mu)-2} c^\bullet \leq \frac{(p_c-1)^{1/2}}{p_c-2} c^\bullet < \varepsilon.$$

Therefore Theorem 3.8 is applicable with μ replaced by $\mathrm{Re}(\mu)$ and $\nu = \mathrm{i}\mathrm{Im}(\mu)$, and the corollary follows. □

7 Regularity for the induced operators on the $\mathrm{W}_D^{-1,q}$ scale

In Corollary 3.3 we investigated regularity properties for the divergence operators in the L^p scale. In view of parabolic and general evolution equations this is the most commonly used one, but in the treatment of real world problems there are at least two effects that make this choice inadequate. These are, on the one hand, inhomogeneous Neumann boundary conditions and, on the other hand, reaction terms which live on lower dimensional manifolds. For instance, the latter is of eminent importance when treating the semiconductor equations where it is quite common to consider generation/recombination mechanisms that are situated on surfaces. See [47, Section 4.2] and [21, Section 3] for a detailed discussion from the mathematical viewpoint. If the above-mentioned phenomena occur, the adequate spaces for the treatment of semilinear and quasilinear parabolic equations are often spaces from the $\mathrm{W}_D^{-1,q}$ scale or duals of Bessel potential spaces, see the detailed discussion in [35, Section 6].

For this purpose we will deduce parabolic regularity results on the $\mathrm{W}_D^{-1,q}$ scale from the results received above. Following the philosophy of [35] (see also [4, Section 11] for more advanced ideas) we will 'transport' properties for the divergence operators $-\nabla \cdot \mu \nabla$, formerly obtained on the L^p scale, into the $\mathrm{W}_D^{-1,q}$ scale by the knowledge of the square root isomorphism

$$\big(-\nabla \cdot \mu\nabla + 1\big)^{-\frac{1}{2}} \colon \mathrm{W}_D^{-1,q}(\Omega) \to \mathrm{L}^q(\Omega). \tag{25}$$

The isomorphism property of (25) is deduced in [4] for real coefficient functions μ, while in his pioneering paper [25] Egert succeeded to prove the isomorphy (25) for complex coefficient functions (and even for systems) as long as the corresponding semigroup is well-behaved on $\mathrm{L}^p(\Omega)$.

In order to go into details, we quote and apply the results from [25] that are relevant for our purposes.

Definition 7.1. Let $\Omega \subset \mathbb{R}^d$ be open and let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ be subject to Assumption 2.11. Let $J(\mu)$ be the interval of all numbers $p \in (1,\infty)$ for which the semigroup generated by $-A_2$ on $\mathrm{L}^2(\Omega)$ extrapolates consistently to a bounded C_0-semigroup $S^{(p)}$ on the space $\mathrm{L}^p(\Omega)$, that is $\sup_{t\in(0,\infty)} \|S_t^{(p)}\|_{\mathcal{L}(\mathrm{L}^p(\Omega))} < \infty$.

The first result from [25] is a bounded holomorphic functional calculus, see [25, Thm. 1.3].

Theorem 7.2. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, where $d \geq 3$. Let $D \subset \partial\Omega$ be closed and set $N = \partial\Omega \setminus D$. Adopt Assumption N and let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ be subject to Assumption 2.11. Let ω denote the half angle of sectoriality for the operator A_2. Further, let $p_1 \in J(\mu)$ and $p \in (p_1, 2) \cup (2, p_1)$. Let $\zeta \in (\omega, \pi)$. Then there exists a $c > 0$ such that*

$$\|f(A_2)u\|_{\mathrm{L}^p(\Omega)} \leq c\|f\|_{\mathrm{L}^\infty(\mathrm{S}_\zeta)}\|u\|_{\mathrm{L}^p(\Omega)}, \quad (u \in \mathrm{L}^2(\Omega) \cap \mathrm{L}^p(\Omega))$$

for every bounded, holomorphic function f on S_ζ. The constant c can be chosen to depend only on d, M, $c_\bullet$, $c^\bullet$, ζ, p and $\sup_{t\in(0,\infty)} \|S_t^{(p_1)}\|_{\mathcal{L}(\mathrm{L}^{p_1}(\Omega))}$.

The second is a topological isomorphism.

Theorem 7.3. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, where $d \geq 3$. Let $D \subset \partial\Omega$ be closed and set $N = \partial\Omega \setminus D$. Suppose that Ω is a d-set and D is a $(d-1)$-set. Adopt Assumption N and let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ be subject to Assumption 2.11. Then for all $p \in (\frac{p_0(\mu)d}{d(p_0(\mu)-1)+2}, 2]$ the operator*

$$\big(A_p + 1\big)^{-\frac{1}{2}} \colon \mathrm{L}^p(\Omega) \to \mathrm{W}^{1,p}_D(\Omega) \tag{26}$$

is a topological isomorphism.

Proof. This follows from Corollary 3.2 together with [25, Thm. 1.2(i)]. □

In all what follows, for a given coefficient function μ and $p \in (\frac{p_0(\mu)d}{d(p_0(\mu)-1)+2}, \frac{p_0(\mu)d}{d-2})$ we denote by A^μ_p the operator corresponding to μ. Then $A^{(\mu^*)}_p$ denotes the operator which corresponds to the conjugate coefficient function μ^*. It is standard that $(A^{(\mu^*)}_2)^* = A^\mu_2$.

Noting that $p_0(\mu)$ and $p_0(\mu^*)$ coincide (see Section 2.2), one obtains the following corollary.

Corollary 7.4. *Adopt the notation and assumptions of Theorem 7.3 above. Then for every $p \in \big(\frac{p_0(\mu)d}{d(p_0(\mu)-1)+2}, 2\big]$ the adjoint map $\big((A^{(\mu^*)}_p + 1)^{-\frac{1}{2}}\big)'$ is a topological isomorphism from $\mathrm{W}^{-1,q}_D(\Omega)$ onto $\mathrm{L}^q(\Omega)$ which is consistent with $(A^\mu_2 + 1)^{-\frac{1}{2}}$.*

Definition 7.5. Adopt the notation and assumptions of Theorem 7.3. For all $q \in [2, \frac{p_0(\mu)d}{d-2})$ we denote by $E^\mu_q \colon \mathrm{W}^{-1,q}_D(\Omega) \to \mathrm{L}^q(\Omega)$ the adjoint map of

$$\big(A^{(\mu^*)}_p + 1\big)^{-\frac{1}{2}} \colon \mathrm{L}^p(\Omega) \to \mathrm{W}^{1,p}_D(\Omega).$$

We emphasise that $E^\mu_q : W^{-1,q}_D(\Omega) \to L^q(\Omega)$ is a topological isomorphism. Let $q \in [2, \frac{p_0(\mu)d}{d-2})$. Then $-A^\mu_q$ is the generator of a C_0-semigroup $S^{(q)}$ on $\mathrm{L}^q(\Omega)$ by Corollary 3.2. We use E^μ_q to transfer $S^{(q)}$ to a C_0-semigroup $T^{(q)}$ on $\mathrm{W}^{-1,q}_D(\Omega)$ defined by

$$T^{(q)}_t = (E^\mu_q)^{-1}\, S^{(q)}_t\, E^\mu_q.$$

We denote the generator of $T^{(q)}$ by $-B^\mu_q$. Next, define $\mathcal{A} \colon \mathrm{W}^{1,2}_D(\Omega) \to \mathrm{W}^{-1,2}_D(\Omega)$ by $(\mathcal{A}u)(v) = \mathfrak{t}[u, v]$ for all $u, v \in \mathrm{W}^{1,2}_D(\Omega)$, where $\mathfrak{t}$ was defined in (6). We consider $\mathcal{A}$ as a densily defined operator in $\mathrm{W}^{-1,2}_D(\Omega)$.

Lemma 7.6. *Adopt the notation and assumptions of Theorem 7.3. Let $q \in [2, \frac{p_0(\mu)d}{d-2})$. Then B^μ_q is the part of $\mathcal{A}$ in $\mathrm{W}^{-1,q}_D(\Omega)$, that is $\mathrm{dom}(B^\mu_q) = \{u \in \mathrm{W}^{-1,q}_D(\Omega) : \mathcal{A}u \in \mathrm{W}^{-1,q}_D(\Omega)\}$ and $B^\mu_q u = \mathcal{A}u$ for all $u \in \mathrm{dom}(B^\mu_q)$.*

Proof. This follows as in [20, Lem. 6.9(c)]. □

Theorem 7.7. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, where $d \geq 3$. Let $D \subset \partial\Omega$ be closed and set $N = \partial\Omega \setminus D$. Suppose that Ω is a d-set and D is a $(d-1)$-set. Adopt Assumption N and let $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ be subject to Assumption 2.11. Let ω denote the half angle of sectoriality for the operator A_2. Further let $q \in [2, \frac{p_0(\mu)d}{d-2})$. Then one has the following.*

(a) *For all $\zeta \in (\omega, \pi/2)$ the operator $B_q^\mu + 1$ admits a bounded holomorphic calculus on the sector S_ζ. In particular, $B_q^\mu + 1$ admits bounded imaginary powers on $\mathrm{W}_D^{-1,q}(\Omega)$ and*

$$\sup_{s \in [-1,1]} \|(B_q^\mu + 1)^{\mathrm{i}s}\|_{\mathcal{L}(\mathrm{W}_D^{-1,q}(\Omega))} < \infty.$$

(b) *The operator B_q^μ has maximal regularity on $\mathrm{W}_D^{-1,q}(\Omega)$.*

Proof. '(a)'. Since $[2, \frac{p_0(\mu)d}{d-2}) \subset J(\mu)$ by Corollary 3.2, it follows that the operator $A_q^\mu + 1$ has a bounded holomorphic calculus on $\mathrm{L}^q(\Omega)$ by Theorem 7.2. Since E_q^μ is a topological isomorphism the bounded holomorphic calculus transfers to $\mathrm{W}_D^{-1,q}(\Omega)$, with the same angle as for A_q^μ, is implied by [18, Prop. 2.11]. The boundedness of the purely imaginary powers follows from this, see [18, p. 25].

'(b)'. The operator $B_q^\mu + 1$ has maximal parabolic regularity by Statement (a) and [22]. Therefore also the operator B_q^μ has maximal parabolic regularity. □

Remark 7.8.

- It is known that maximal parabolic regularity is preserved under (real and complex) interpolation, see [35, Lemma 5.3]. Using the interpolation results from [7], this shows that the minus generator of the consistent C_0-semigroup on the (dual of) Bessel potential spaces with non-integer differentiability index between -1 and 0 also satisfies maximal parabolic regularity. This considerably generalizes the results in [35, Thm. 5.16 and Section 6] in view of complex coefficients and much more general admissible geometries for Ω and D.

- If one is only interested in maximal parabolic regularity, it is also possible to transport this property directly from $\mathrm{L}^q(\Omega)$ to $\mathrm{W}_D^{-1,q}(\Omega)$ by the square root isomorphism (25), see [36, Lem. 5.12].

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