1 Introduction

In this paper, we present the optimal control of a system of two coupled nonlinear partial differential equations. This system describes the dynamics of first order martensitic phase transitions occurring in a thin rod of a shape memory alloy (SMA) which is fixed on one side and pushed and pulled on the other side in the course of time by an elongation m. This so–called deformation–driven experiment, and related ones, are performed by I.Müller and his co–workers [8,9], for instance.

We have chosen the Landau–Ginzburg model developed by Falk to describe this experiment. A large number of papers is dealing with the general derivation of this model [e.g. 2,5,6,7,14]; we omit the details, here. For details concerning the application of the model to this experiment, the determination of physical parameters and numerical simulations, we refer the reader to [3,4].

Summarizing, we have the following system ($\Omega := (0,l)$, $\Omega_T := \Omega \times (0,T)$):

$$\rho\, u_{tt} - (\gamma\,(\theta-\theta_1)\,u_x - \beta\, u_x^3 + \alpha\, u_x^5)_x + \delta\, u_{xxxx} = 0, \quad \text{in} \quad \Omega_T, \tag{1.1a}$$

$$c_e\,\theta_t - \kappa\,\theta_{xx} - \gamma\,\theta\, u_x\, u_{xt} = g(x,t), \quad \text{in} \quad \Omega_T, \tag{1.1b}$$

$$u(0,t) = u_{xx}(0,t) = u_{xx}(l,t) = 0, \quad u(l,t) = m(t), \quad \forall t \in [0,T], \tag{1.1c}$$

$$\theta_x(0,t) = 0, \quad -\kappa\,\theta_x(l,t) = \bar{\kappa}\,(\theta(l,t) - \theta_\Gamma(t)), \quad \forall t \in [0,T], \tag{1.1d}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \forall x \in \overline{\Omega}, \tag{1.1e}$$

$$\theta(x,0) = \theta_0(x), \quad \forall x \in \overline{\Omega}. \tag{1.1f}$$

The equations (1.1a) and (1.1b) represent the balance laws of momentum and energy, respectively. The physical meanings of the involved quantities are: ρ – constant mass density, u – displacement in the direction of the rod, θ – absolute temperature, u_x – strain in the direction of the rod, c_e – specific heat, κ – positive constant heat conductivity, g – density of heat sources or sinks, l – length of the rod (which is normalized to unity: $l := 1$), $\bar{\kappa}$ – positive constant heat exchange coefficient, θ_Γ – temperature of the surrounding medium. The couple stress leads to the Ginzburg–term $\delta \cdot u_{xxxx}$, the linearized strain is $\varepsilon = u_x$, and α, β, γ, and δ are material constants to be determined for each specimen. The boundary condition for u at $x = 1$ reflects the pulling and pushing of the rod in the course of time by a prescribed elongation m. The other boundary condition for the momentum balance has been taken in analogy to [16]. The boundary condition for the energy balance models a heat exchange with the surrounding temperature at $x = 1$ using Newton's law. For the mathematical analysis, as well as for the numerical approximation, the system is transformed by $\tilde{u}(x,t) := u(x,t) - x \cdot m(t)$. Then we deal with homogeneous boundary conditions. An additional term $\rho \cdot x \cdot \ddot{m}(t)$ appears only on the left hand side of the momentum balance. Furthermore, we normalize all physical constants to 1, except for θ_1 which is set to 0, and we set $F(\varepsilon) := -\frac{\varepsilon^4}{4} + \frac{\varepsilon}{6}^6$. We obtain the following system:

$$
\begin{aligned}
&\tilde{u}_{tt} + x\,\ddot{m}(t) - \Big(\tilde{\theta}\,(\tilde{u}_x + m(t)) + F'(\tilde{u}_x + m(t))\Big)_x + \tilde{u}_{xxxx} = 0, \quad \text{in} \quad \Omega_T, && \text{(1.2a)}\\
&\tilde{\theta}_t - \tilde{\theta}\,(\tilde{u}_x + m(t))\,(\tilde{u}_x + m(t))_t - \tilde{\theta}_{xx} = g(x,t), \quad \text{in} \quad \Omega_T, && \text{(1.2b)}\\
&\tilde{u}(0,t) = \tilde{u}(1,t) = 0 = \tilde{u}_{xx}(0,t) = \tilde{u}_{xx}(1,t), \quad \forall t \in [0,T], && \text{(1.2c)}\\
&\tilde{\theta}_x(0,t) = 0, \qquad -\tilde{\theta}_x(1,t) = \tilde{\theta}(1,t) - \theta_\Gamma(t), \quad \forall t \in [0,T], && \text{(1.2d)}\\
&\tilde{u}(x,0) = u(x,0) - x\,m(0) = u_0(x), && \\
&\tilde{u}_t(x,0) = u_t(x,0) - x\,\dot{m}(0) = u_1(x), \quad \tilde{\theta}(x,0) = \theta_0(x), \quad \forall x \in \overline{\Omega}. && \text{(1.2e)}
\end{aligned}
$$

Here, we have assumed that $m(0) = 0 = \dot{m}(0)$. We impose the following compatibility conditions (For simplicity, the tilde is omitted from now on.).

$$
\begin{aligned}
(H1) \qquad & u_0(0) = u_0(1) = 0; \quad u_1(0) = u_1(1) = 0; && \text{(1.3a)}\\
& u_0''(0) = u_0''(1) = 0; \quad u_1''(0) = u_1''(1) = 0; && \text{(1.3b)}\\
& u_0''''(0) = 0; \quad u_0''''(1) = -\ddot{m}(0) - (\theta_0(1) - \theta_\Gamma(0))\,u_0'(1); && \text{(1.3c)}\\
& \theta_0'(0) = 0; \quad -\theta_0'(1) = \theta_0(1) - \theta_\Gamma(0). && \text{(1.3d)}
\end{aligned}
$$

Some control problems concerning applications of or experiments on SMA, respectively, have been studied in recent years: dynamical shape control problems of a thin rod in [11], the optimal control of phase transitions in a load–driven experiment in [1,2], and these control problems with state constraints in [12,13]. We will make use of some of these results to deal with the actual case of the optimal control of phase transitions in a deformation–driven experiment. We will consider a weak formulation of the system (1.2) which is introduced in (3.5). The aim is to achieve, possibly isothermically, a prescribed distribution of the phases. Therefore, it is natural to consider a cost functional involving the order parameter $\varepsilon = u_x$ and θ (or rather, the stress $\sigma = \gamma(\theta - \theta_1)u_x - \beta u_x^3 + \alpha u_x^5$), as well as the natural control variables m, θ_Γ, and g:

$$J(u,\theta;m,g,\theta_\Gamma) = \int_0^T \int_\Omega \Phi_1(u_x(x,t),\theta(x,t))\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \Big(\Phi_2(\dddot{m}\,(t)) + \Phi_4(\theta_\Gamma(t))\Big)\,\mathrm{d}t + \int_0^T \int_\Omega \Phi_3(g(x,t))\,\mathrm{d}x\,\mathrm{d}t\,, \tag{1.4}$$

where Φ_1, $\Phi_3 \in C^2(\mathbb{R}^2)$, Φ_2, $\Phi_4 \in C^1(\mathbb{R})$, and Φ_2, Φ_3 and Φ_4 are convex in their arguments. For instance, one could investigate the case when

$$J(u,\theta;m,g,\theta_\Gamma) = \alpha_1\Big(\|\,\sigma-\overline{\sigma}\,\|^2_{L^2(\Omega_T)} + \|\,\theta-\overline{\theta}\,\|^2_{L^2(\Omega_T)}\Big) + \alpha_2\|\,\dddot{m}\,\|^2_{L^2(0,T)} + \alpha_3\|\,g\,\|^2_{L^2(\Omega_T)} + \alpha_4\|\,\theta_\Gamma\,\|^2_{L^2(0,T)}, \tag{1.5}$$

where α_i, $i=1,\ldots,4$, are non–negative constants, and where $\overline{\sigma}$ and $\overline{\theta}$ denote the desired stress and temperature distribution during the evolution of the process, respectively.

The following problem is considered.

(CP) Minimize $J(u,\theta;m,g,\theta_\Gamma)$, subject to (3.5) and $(m,g,\theta_\Gamma)\in U_{ad}$.

Here, U_{ad} denotes the set of admissible controls which is assumed to be a non–empty, bounded, convex and closed subset of

$$M := M_m \times M_g \times M_{\theta_\Gamma}\,, \tag{1.6}$$

where

$$\begin{aligned} M_m &:= \Big\{\ m\in H^3(0,T)\ \Big|\ m(0)=0,\ \ \dot{m}(0)=0\ \Big\},\\ M_g &:= \Big\{\ g\in L^2(0,T;L^2(\Omega))\ \Big|\ g(x,t)\geq 0\ \text{ on }\ \overline{\Omega_T}\ \Big\},\\ M_{\theta_\Gamma} &:= \Big\{\ \theta_\Gamma\in H^1(0,T)\ \Big|\ \theta_\Gamma(t)>0\ \text{ on }\ [0,T]\ \Big\}. \end{aligned} \tag{1.7}$$

The existence of at least one solution to **(CP)** can be proved in the same way as theorem 3.1 in [15] (see [3]).

In Section 2 the existence of a local classical solution to the system (1.2) is shown. We prove global existence of a classical and, under weaker assumptions on the data, of a weak solution, respectively, as well as the corresponding uniqueness in Section 3. Section 4 finally presents the differentiability of the observation operator and the necessary conditions of optimality.

2 Local existence

In this section, we sketch the proof of the existence for a local classical solution. We define the following spaces.

$$X_{1,\tau} := L^2(0,\tau;H^5(\Omega)) \cap H^2(0,\tau;H^1(\Omega)), \qquad X_{2,\tau} := H^{4,2}(\Omega_\tau), \quad \text{and} \tag{2.1}$$

$$X_\tau \quad := \quad X_{1,\tau} \quad \times \quad X_{2,\tau}. \tag{2.2}$$

In the sequel, $\|.\|$ denotes the norm of $L^2(\Omega)$. Let

$$\begin{aligned}
K_\tau := \Big\{ \quad & (u,\theta) \in X_\tau \quad \Big| \\
& u(x,0) = u_0(x),\, u_t(x,0) = u_1(x),\, \theta(x,0) = \theta_0(x), \quad \forall x \in \overline{\Omega}, \\
& u(0,t) = u(1,t) = 0 = u_{xx}(0,t) = u_{xx}(1,t), \quad \forall t \in [0,\tau], \\
& \theta_x(0,t) = 0,\, -\theta_x(1,t) = \theta(1,t) - \theta_\Gamma(t), \quad \forall t \in [0,\tau], \\
& \max_{0\le t\le\tau} \left(\|u_t(t)\|^2 + \|u_{xx}(t)\|^2 + \|u_x(t)\|^2_{L^\infty(\Omega)} \right) \le M_1, \\
& \max_{0\le t\le\tau} \|\theta(t)\|^2 + \int_0^\tau \|\theta_x(t)\|^2 \,\mathrm{d}t \le M_2, \\
& \max_{0\le t\le\tau} \left(\|u_{xt}(t)\|^2 + \|u_{xxx}(t)\|^2 + \|u_{xx}(t)\|^2_{L^\infty(\Omega)} \right) \le M_3, \\
& \max_{0\le t\le\tau} \left(\|\theta_x(t)\|^2 + \theta^2(1,t) \right) + \int_0^\tau \|\theta_t(t)\|^2 \,\mathrm{d}t \le M_4, \\
& \max_{0\le t\le\tau} \|\theta(t)\|^2_{L^\infty(\Omega)} \le M_5, \\
& \max_{0\le t\le\tau} \left(\|u_{tt}(t)\|^2 + \|u_{xxt}(t)\|^2 + \|u_{xt}(t)\|^2_{L^\infty(\Omega)} \right) \le M_6, \\
& \max_{0\le t\le\tau} \left(\|\theta_{xx}(t)\|^2 + \|\theta_x(t)\|^2_{L^\infty(\Omega)} \right) + \int_0^\tau \left(\|\theta_{xt}(t)\|^2 + \theta_t^2(1,t) \right) \mathrm{d}t \le M_7, \\
& \max_{0\le t\le\tau} \left(\|u_{xxxx}(t)\|^2 + \|\theta_t(t)\|^2 + \|u_{xxx}(t)\|^2_{L^\infty(\Omega)} \right) \\
& \qquad\quad + \int_0^\tau \|\theta_t(t)\|^2_{L^\infty(\Omega)} \,\mathrm{d}t \le M_8, \\
& \int_0^\tau \left(\|u_{xtt}(t)\|^2 + \|u_{xxxt}(t)\|^2 \right) \mathrm{d}t \le M_9, \\
& \max_{0\le t\le\tau} \left(\|\theta_{xt}(t)\|^2 + \theta_t^2(1,t) \right) + \int_0^\tau \left(\|\theta_{tt}(t)\|^2 + \|\theta_{xxt}(t)\|^2 \right) \mathrm{d}t \le M_{10}, \\
& \max_{0\le t\le\tau} \left(\|\theta_{xxx}(t)\|^2 + \|\theta_t(t)\|^2_{L^\infty(\Omega)} + \|\theta_{xx}(t)\|^2_{L^\infty(\Omega)} \right) \\
& \qquad\quad + \int_0^\tau \left(\|u_{xxxxx}(t)\|^2 + \|\theta_{xxxx}(t)\|^2 \right) \mathrm{d}t \le M_{11} \quad \Big\},
\end{aligned} \tag{2.3}$$

where M_i, $i = 1, \ldots, 11$, denote positive constants which have to be constructed.

Let $\tau \in (0,T]$. We consider the operator

$$\mathcal{T} : K_\tau \subseteq X_\tau \to X_\tau, \qquad (\hat{u},\hat{\theta}) \mapsto \mathcal{T}[\hat{u},\hat{\theta}] = (u,\theta), \tag{2.4}$$

where (u,θ) solves the linear initial–boundary value problem

$$u_{tt} + u_{xxxx} \;=\; f_1, \quad \text{in} \quad \Omega_\tau, \tag{2.5a}$$

$$\begin{aligned}
\theta_t - \theta_{xx} &= f_2, \quad \text{in} \quad \Omega_\tau, & (2.5b)\\
u(0,t) = u(1,t) &= 0 = u_{xx}(0,t) = u_{xx}(1,t), \quad \forall t \in [0,\tau], & (2.5c)\\
\theta_x(0,t) = 0, \quad -\theta_x(1,t) &= \theta(1,t) - \theta_\Gamma(t), \qquad \forall t \in [0,\tau], & (2.5d)\\
u(x,0) = u_0(x), \quad u_t(x,0) &= u_1(x), \quad \theta(x,0) = \theta_0(x), \quad \forall x \in \overline{\Omega}. & (2.5e)
\end{aligned}$$

The right hand side is given by

$$f_1 := -x\ddot{m}(t) + \left(\hat{\theta}\,(\hat{u}_x + m(t)) + F'(\hat{u}_x + m(t))\right)_x, \tag{2.6}$$

$$f_2 := g(x,t) + \hat{\theta}\,(\hat{u}_x + m(t))\,(\hat{u}_x + m(t))_t. \tag{2.7}$$

The proof of the following result can be found in [3].

Lemma 2.1 *Let $\tau > 0$, and suppose that $u_0 \in H^5(\Omega), u_1 \in H^4(\Omega), \theta_0 \in H^3(\Omega), m \in H^4(0,\tau), g \in L^2(0,\tau; H^2(\Omega)) \cap H^1(0,\tau; H^1(\Omega)), \theta_\Gamma \in H^2(0,\tau)$, and that the compatibility conditions (H1) are satisfied. Therefore,*

$$f_1 \in H^{2,1}(\Omega_\tau) \cap H^1(0,\tau; H^1(\Omega)) \quad \textit{and} \quad f_2 \in H^{2,1}(\Omega_\tau). \tag{2.8}$$

Then the linear problem (2.5) has a unique solution (u,θ) satisfying

$$u \in L^2(0,\tau; H^5(\Omega)) \cap H^2(0,\tau; H^1(\Omega)), \qquad \theta \in H^{4,2}(\Omega_\tau). \tag{2.9}$$

Next, we show the existence of a local solution to the system (1.2).

Theorem 2.1 *Let the compatibility conditions (H1) be satisfied, $u_0 \in H^5(\Omega), u_1 \in H^4(\Omega), \theta_0 \in H^3(\Omega), m \in H^4(0,T), g \in L^2(0,T; H^2(\Omega)) \cap H^1(0,T; H^1(\Omega))$, and $\theta_\Gamma \in H^2(0,T)$. Then there exists some $\tau > 0$, depending only on $u_0, u_1, \theta_0, m, g, \theta_\Gamma$, and T, such that the system (1.2) has a solution (u,θ) on Ω_τ satisfying*

$$u \in L^2(0,\tau; H^5(\Omega)) \cap H^2(0,\tau; H^1(\Omega)), \qquad \theta \in H^{4,2}(\Omega_\tau). \tag{2.10}$$

Proof. In order to prove this theorem we apply Tikhonov's fixed point theorem which can be found, for example, as Corollary 9.7 in Chapter 9 in [17]. It claims: If X is a reflexive and separable Banach space, M a nonempty, closed, bounded, and convex subset of X, and $\mathcal{T}$ a weakly sequentially continuous operator on M with

$$\mathcal{T} : M \subseteq X \to M, \tag{2.11}$$

then $\mathcal{T}$ has a fixed point in M.

To apply the fixed point theorem to our problem, we have to show that

(I) the operator $\mathcal{T}$ maps K_τ into itself,

(II) the operator $\mathcal{T}$ is weakly sequentially continuous on K_τ, that is

$$\text{if} \quad \{(\hat{u}_n, \hat{\theta}_n)\} \subset K_\tau \quad \text{and} \quad (\hat{u}_n, \hat{\theta}_n) \ \to \ (\hat{u}, \hat{\theta}) \quad \text{weakly in} \quad X_\tau, \\ \text{then} \quad \mathcal{T}[\hat{u}_n, \hat{\theta}_n] \ \to \ \mathcal{T}[\hat{u}, \hat{\theta}] \quad \text{weakly in} \quad X_\tau. \tag{2.12}$$

We start showing (I). This takes 8 steps of which we will only show the first two. Again, for further details we refer the reader to [3].

In order to show that the operator $\mathcal{T}$ maps K_τ into itself for sufficiently small $\tau > 0$, suitable constants M_i, $i = 1, \ldots, 11$, are constructed. We define

$$\rho := \max_{|\xi| \le \sqrt{M_1} + \max\limits_{0 \le t \le \tau} |m(t)|} \Big(|\xi| + |F(\xi)| + |F'(\xi)| + |F''(\xi)| + |F'''(\xi)| + |F''''(\xi)| \Big), \tag{2.13}$$

where $\tau \in (0, T)$ is arbitrary and $t \in (0, \tau]$. With $C_i, \hat{C}_i, i \in \mathbb{N}$, positive constants are denoted, the former depending only on $u_0, u_1, \theta_0, m, g, \theta_\Gamma$, and T. We will make use of the fact that for functions $v \in L^2(0, \tau; H^1(\Omega)) \cap C([0, \tau]; L^2(\Omega))$ it holds

$$\|v(t)\|^2_{L^\infty(\Omega)} \le \hat{C}_1 \Big(\|v_x(t)\|^2 + \|v(t)\|^2 \Big), \quad \text{a.e. in} \quad (0, \tau), \tag{2.14}$$

and

$$\int_0^\tau \|v(s)\|^2_{L^\infty(\Omega)} \, \mathrm{d}s \ \le \hat{C}_3 \sqrt{\tau} \Big(\max_{0 \le t \le \tau} \|v(t)\|^2 + \int_0^\tau \|v_x(s)\|^2 \, \mathrm{d}s \Big), \tag{2.15}$$

where the inequalities of Young, Hölder and Nirenberg have been used.
Step 1. Multiplying (2.5a) by u_t and integrating over Ω_t, we arrive at

$$\frac{1}{2} \Big(\|u_t(t)\|^2 + \|u_{xx}(t)\|^2 \Big) \le C_1 + \int_0^t \int_\Omega f_1 \, u_t \, \mathrm{d}x \, \mathrm{d}s \ \le C_1 + \max_{0 \le s \le t} \|u_t(s)\| \int_0^t \|f_1(s)\| \, \mathrm{d}s \, . \tag{2.16}$$

Hölder's inequality gives

$$\begin{aligned} &\int_0^t \|f_1(s)\| \, \mathrm{d}s \\ &\le \ \int_0^t \Big(\| - x \, \ddot{m}(s) \| + \|\hat{\theta}_x \, (\hat{u}_x + m(s))\| + \|\hat{\theta} \, \hat{u}_{xx}\| + \|F''(\hat{u}_x + m(s)) \, \hat{u}_{xx}\| \Big) \, \mathrm{d}s \\ &\le \ C_2 \Big(\sqrt{t} + \rho \int_0^t \|\hat{\theta}_x(s)\| \, \mathrm{d}s \ + \rho \int_0^t \|\hat{u}_{xx}(s)\| \, \mathrm{d}s \ + \int_0^t \|\hat{\theta}(s)\|_{L^\infty(\Omega)} \cdot \|\hat{u}_{xx}(s)\| \, \mathrm{d}s \Big) \\ &\le \ C_3 \sqrt{t} \Big(1 + \rho \sqrt{M_2} + \rho \sqrt{M_1} + \sqrt{M_1} \Big(\int_0^t \|\hat{\theta}(s)\|^2_{L^\infty(\Omega)} \, \mathrm{d}s \Big)^{\frac{1}{2}} \Big). \end{aligned} \tag{2.17}$$

Then, (2.15) leads to

$$\int_0^t \|f_1(s)\| \, \mathrm{d}s \ \le C_4 \sqrt{t} \Big(1 + \rho \sqrt{M_2} + \rho \sqrt{M_1} + \sqrt{M_1 M_2} \Big). \tag{2.18}$$

Now, from Schwarz's inequality and the boundary conditions, we infer that

$$\|u_x(t)\|_{L^\infty(\Omega)} \le \|u_{xx}(t)\|. \tag{2.19}$$

Taking the maximum over $t \in [0,\tau]$ on both sides of (2.16), Young's inequality yields

$$\max_{0\le t\le \tau}\Big(\|u_t(t)\|^2 + \|u_{xx}(t)\|^2 + \|u_x(t)\|^2_{L^\infty(\Omega)}\Big) \le C_5\Big(1+\tau\Big(\rho^2 M_2 + \rho^2 M_1 + M_1 M_2\Big)\Big). \tag{2.20}$$

Step 2. Testing (2.5b) with θ, one obtains, again by the help of Young's inequality,

$$\frac{1}{2}\|\theta(t)\|^2 + \int_0^t \|\theta_x(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \theta^2(1,s)\,\mathrm{d}s \le C_6 + \frac{1}{2}\int_0^t \theta_\Gamma^2(s)\,\mathrm{d}s + \int_0^t\int_\Omega f_2\,\theta\,\mathrm{d}x\,\mathrm{d}s\,. \tag{2.21}$$

It holds

$$\int_0^t\int_\Omega |\,g\,\theta\,|\,\mathrm{d}x\,\mathrm{d}s \le \int_0^t \|g(s)\|\cdot\|\theta(s)\|\,\mathrm{d}s \le C_7\sqrt{t}\max_{0\le s\le t}\|\theta(s)\| \le \frac{1}{10}\max_{0\le s\le t}\|\theta(s)\|^2 + C_8\,t. \tag{2.22}$$

Furthermore,

$$\begin{aligned} \Big|\int_0^t\int_\Omega \theta\,\hat\theta\,(\hat u_x + m(s))\,\dot m(s)\,\mathrm{d}x\,\mathrm{d}s\Big| &\le C_9\rho\int_0^t \|\theta(s)\|\cdot\|\hat\theta(s)\|\,\mathrm{d}s \\ &\le C_{10}\rho\sqrt{t\,M_2}\max_{0\le s\le t}\|\theta(s)\| \le \frac{1}{10}\max_{0\le s\le t}\|\theta(s)\|^2 + C_{11}\rho^2 M_2\,t, \end{aligned} \tag{2.23}$$

and, using partial integration, we have for the other term of f_2

$$\begin{aligned} \Big|\int_0^t\int_\Omega \theta\,\hat\theta\,(\hat u_x + m(s))\,\hat u_{xt}\,\mathrm{d}x\,\mathrm{d}s\Big| &\le \Big|\int_0^t\int_\Omega \theta\,\hat\theta\,\hat u_{xx}\,\hat u_t\,\mathrm{d}x\,\mathrm{d}s\Big| \qquad (2.24)\\ &+ \Big|\int_0^t\int_\Omega \theta_x\,\hat\theta\,(\hat u_x + m(s))\,\hat u_t\,\mathrm{d}x\,\mathrm{d}s\Big| + \Big|\int_0^t\int_\Omega \theta\,\hat\theta_x\,(\hat u_x + m(s))\,\hat u_t\,\mathrm{d}x\,\mathrm{d}s\Big| =: I_1 + I_2 + I_3. \end{aligned}$$

By virtue of (2.15) and owing to the inequalities of Hölder and Young, one has

$$\begin{aligned} |I_1| &\le \int_0^t \|\theta(s)\|_{L^\infty(\Omega)}\cdot\|\hat\theta(s)\|_{L^\infty(\Omega)}\cdot\|\hat u_{xx}(s)\|\cdot\|\hat u_t(s)\|\,\mathrm{d}s \\ &\le M_1\Big(\int_0^t \|\theta(s)\|^2_{L^\infty(\Omega)}\,\mathrm{d}s\Big)^{\frac{1}{2}}\cdot\Big(\int_0^t \|\hat\theta(s)\|^2_{L^\infty(\Omega)}\,\mathrm{d}s\Big)^{\frac{1}{2}} \\ &\le M_1 C_{12}\sqrt{t\,M_2}\Big(\max_{0\le s\le t}\|\theta(s)\| + \Big(\int_0^t \|\theta_x(s)\|^2\,\mathrm{d}s\Big)^{\frac{1}{2}}\Big) \\ &\le \frac{1}{10}\max_{0\le s\le t}\|\theta(s)\|^2 + \frac{1}{4}\int_0^t \|\theta_x(s)\|^2\,\mathrm{d}s + C_{13}M_1^2 M_2\,t, \qquad (2.25)\\ |I_2| &\le \frac{1}{4}\int_0^t \|\theta_x(s)\|^2\,\mathrm{d}s + C_{14}\rho^2\int_0^t \|\hat\theta(s)\|^2_{L^\infty(\Omega)}\cdot\|\hat u_t(s)\|^2\,\mathrm{d}s \\ &\le \frac{1}{4}\int_0^t \|\theta_x(s)\|^2\,\mathrm{d}s + C_{15}\rho^2 M_1 M_2\sqrt{t}, \qquad (2.26)\\ |I_3| &\le \rho\int_0^t \|\theta(s)\|_{L^\infty(\Omega)}\cdot\|\hat\theta_x(s)\|\cdot\|\hat u_t(s)\|\,\mathrm{d}s \\ &\le \rho\sqrt{M_1 M_2}\Big(\int_0^t \|\theta(s)\|^2_{L^\infty(\Omega)}\,\mathrm{d}s\Big)^{\frac{1}{2}} \\ &\le C_{16}\sqrt{M_1 M_2}\,\rho\,t^{\frac{1}{4}}\Big(\max_{0\le s\le t}\|\theta(s)\| + \Big(\int_0^t \|\theta_x(s)\|^2\,\mathrm{d}s\Big)^{\frac{1}{2}}\Big) \\ &\le \frac{1}{10}\max_{0\le s\le t}\|\theta(s)\|^2 + \frac{1}{4}\int_0^t \|\theta_x(s)\|^2\,\mathrm{d}s + C_{17}\,\rho^2 M_1 M_2\sqrt{t}. \qquad (2.27) \end{aligned}$$

Taking the maximum over $t \in [0,\tau]$, we arrive at

$$\max_{0\le t\le\tau} \|\theta(t)\|^2 + \int_0^\tau \|\theta_x(s)\|^2 \,\mathrm{d}s + \int_0^\tau \theta^2(1,s)\,\mathrm{d}s \le C_{18}\Big(1+\sqrt{\tau}\Big(1+\rho^2 M_2 + M_1^2 M_2 + \rho^2 M_1 M_2\Big)\Big). \tag{2.28}$$

Now, we define:

$$M_1 := 2\cdot C_5, \qquad M_2 := 2\cdot C_{18}. \tag{2.29}$$

In this way ρ is fixed, and we deduce from (2.20) and (2.28) that there exists a $\tau_1 > 0$ with

$$\max_{0\le t\le\tau_1} \left(\|u_t(t)\|^2 + \|u_{xx}(t)\|^2 + \|u_x(t)\|^2_{L^\infty(\Omega)}\right) \le M_1, \tag{2.30}$$

$$\max_{0\le t\le\tau_1} \|\theta(t)\|^2 + \int_0^\tau \|\theta_x(s)\|^2 \,\mathrm{d}s + \int_0^\tau \theta^2(1,s)\,\mathrm{d}s \le M_2. \tag{2.31}$$

Continuing in a similar manner, it can be demonstrated that there exist some sufficiently small $\tau > 0$ such that $\mathcal{T}$ maps K_τ into itself with the values for the M_i given in (2.3).

We now show (II). The proof for the energy balance being analogous, we only work it out for the momentum balance.

Let $(\hat{u}_n, \hat{\theta}_n) \in K_\tau, n \in \mathbb{N}$, with $(\hat{u}_n, \hat{\theta}_n) \to (\hat{u}, \hat{\theta})$, weakly in X_τ. We have to show that

$$(u_n,\theta_n) := \mathcal{T}[\hat{u}_n,\hat{\theta}_n] \to \mathcal{T}[\hat{u},\hat{\theta}] =: (u,\theta), \quad \text{weakly} \quad \text{in} \quad X_\tau. \tag{2.32}$$

Since $\{\hat{\theta}_n\}$ converges weakly in $H^{4,2}(\Omega_\tau)$ to $\hat{\theta}$, $\{\hat{\theta}_{n,x}\}$ converges weakly in $H^{2,1}(\Omega_\tau)$ to $\hat{\theta}_x$, too; and since the embedding $H^{2,1}(\Omega_\tau) \hookrightarrow C(\overline{\Omega_\tau})$ is compact, $\{\hat{\theta}_{n,x}\}$ converges uniformly on $\overline{\Omega_\tau}$ to $\hat{\theta}_x$. Furthermore, Proposition 2.3, Chapter 4 in [10], states $H^{4,2}(\Omega_\tau) \hookrightarrow H^1(0,\tau;H^2(\Omega))$ in the sense of a continuous embedding. Thus, for any sequence $\{\hat{u}_n\} \subset H^{4,2}(\Omega_\tau)$, we have $\{\hat{u}_{n,xx}\} \subset H^{2,1}(\Omega_\tau)$ which therefore converges weakly in $H^{2,1}(\Omega_\tau)$ to $\hat{u}_{xx}$. As shown above, it follows that $\{\hat{u}_{n,xx}\}$ converges uniformly on $\overline{\Omega_\tau}$ to $\hat{u}_{xx}$. So, we find that

$$f_{1,n}(x,t) := -x\,\ddot{m}(t) + \hat{\theta}_n\,\hat{u}_{n,xx} + \hat{\theta}_{n,x}\,(\hat{u}_{n,x} + m(t)) + F''(\hat{u}_{n,x} + m(t))\,\hat{u}_{n,xx} \tag{2.33}$$

converges uniformly on $\overline{\Omega_\tau}$ to f_1. Since $(u_n,\theta_n) \in K_\tau$, $\{u_n\}$ is bounded in $X_{1,\tau}$, there exists a weakly convergent subsequence $\{u_{\tilde{n}}\}$ in $X_{1,\tau}$. Thus, there is a $\tilde{u} \in X_{1,\tau}$ such that $\{u_{\tilde{n},tt}\}$ and $\{u_{\tilde{n},xxxx}\}$ converge weakly in $L^2(0,\tau;H^1(\Omega))$ to $\tilde{u}_{tt}$ and $\tilde{u}_{xxxx}$, respectively. Then, for all $\phi \in L^2(0,\tau;L^2(\Omega))$, it holds

$$\begin{aligned}
\int_0^\tau\int_\Omega \tilde{u}_{tt}\,\phi\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega \tilde{u}_{xxxx}\,\phi\,\mathrm{d}x\,\mathrm{d}t &= \lim_{\tilde{n}\to\infty}\int_0^\tau\int_\Omega (u_{\tilde{n},tt} + u_{\tilde{n},xxxx})\,\phi\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{\tilde{n}\to\infty}\int_0^\tau\int_\Omega \Big(-x\,\ddot{m}(t) + \hat{\theta}_{\tilde{n}}\,\hat{u}_{\tilde{n},xx} + \hat{\theta}_{\tilde{n},x}\,(\hat{u}_{\tilde{n},x} + m(t)) + F''(\hat{u}_{\tilde{n},x} + m(t))\,\hat{u}_{\tilde{n},xx}\Big)\,\phi\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau\int_\Omega \Big(-x\,\ddot{m}(t) + \hat{\theta}\,\hat{u}_{xx} + \hat{\theta}_x\,(\hat{u}_x + m(t)) + F''(\hat{u}_x + m(t))\,\hat{u}_{xx}\Big)\,\phi\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned} \tag{2.34}$$

Thus,

$$\tilde{u}_{tt} + \tilde{u}_{xxxx} = -x\,\ddot{m}(t) + \hat{\theta}\,\hat{u}_{xx} + \hat{\theta}_x\,(\hat{u}_x + m(t)) + F''(\hat{u}_x + m(t))\,\hat{u}_{xx}, \quad \text{a.e.} \quad \text{in} \quad \Omega_\tau. \tag{2.35}$$

Uniform convergence yields

$$\tilde{u}(0,t) = \tilde{u}_{xx}(0,t) = 0 = \tilde{u}(1,t) = \tilde{u}_{xx}(1,t), \quad 0 \le t \le \tau, \quad \text{and} \quad \tilde{u}(x,0) = u_0(x), \quad x \in \overline{\Omega}.$$

Owing to the compact embedding $H^1(0,\tau;H^1(\Omega)) \hookrightarrow C(\overline{\Omega_\tau})$, we infer $u_{\tilde{n},t} \to \tilde{u}_t$, uniformly on $\overline{\Omega_\tau}$, i.e. $\tilde{u}_t(x,0) = u_1(x), x \in \overline{\Omega}$. We have $\tilde{u} = u$ because of the uniqueness of the linear problem (2.1). Since the limit does not depend on the choice of the subsequence, the whole sequence $\{u_n\}$ converges weakly in $X_{1,\tau}$ to u. The operator $\mathcal{T}$ is weakly sequentially continuous. □

Remark 2.1 A consequence of the proof is that

$$u_{xtt} \in L^\infty(0,\tau;L^2(\Omega)) \quad \text{and} \quad u_{xxxxx} \in L^\infty(0,\tau;L^2(\Omega)), \tag{2.36}$$

whence the Hölder continuity of the functions u_{tt}, u_{xxxx}, u_{xxt}, θ_t, and θ_{xx} on $\overline{\Omega_T}$ can be shown in the same way as in [18].

3 Global existence

3.1 Classical solution

In this subsection, we prove the existence of a global classical solution. The following assumptions are needed.

$$\begin{aligned}
(H2) \quad & m \in H^4_{loc}(0,\infty); \quad g \in L^2_{loc}(0,\infty;H^2(\Omega)) \cap H^1_{loc}(0,\infty;H^1(\Omega)); \\
& g(x,t) \ge 0 \quad \text{on} \quad \overline{\Omega} \times [0,\infty); \quad \theta_\Gamma \in H^2_{loc}(0,\infty); \quad \theta_\Gamma(t) > 0 \quad \text{on} \quad [0,\infty). \qquad (3.1)\\
(H3) \quad & u_0 \in H^5_E(\Omega) := \{u \in H^5(\Omega) \,\big|\, u(0) = u''(0) = u(1) = u''(1) = 0\}; \\
& u_1 \in H^4_E(\Omega) := \{u \in H^4(\Omega) \,\big|\, u(0) = u''(0) = u(1) = u''(1) = 0\}; \\
& \theta_0 \in H^3(\Omega); \quad \theta_0(x) > 0 \quad \text{on} \quad \overline{\Omega}. \qquad (3.2)
\end{aligned}$$

We define

$$\tilde{X}_{1,T} := X_{1,T} \cap W^{2,\infty}(0,T;H^1(\Omega)) \cap W^{1,\infty}(0,T;H^3(\Omega)) \cap L^\infty(0,T;H^5(\Omega)), \tag{3.3}$$

and

$$\tilde{X}_T := \tilde{X}_{1,T} \times X_{2,T}. \tag{3.4}$$

There holds

Theorem 3.1 *Suppose that (H1)–(H3) are satisfied. Then the system (1.2) has a classical solution* (u, θ) *on* $\overline{\Omega} \times [0, \infty)$ *with* $\theta(x,t) > 0$ *on* $\overline{\Omega} \times [0, \infty)$. *Furthermore, we have* $(u, \theta) \in \tilde{X}_T$ *for any* $T > 0$.

Proof. Since the proof is almost identical with the proof of theorem 2.1 in [16], except for the first a priori estimate, we omit it here. It has already been mentioned in [19] that the proof of global existence for the system describing load–driven experiments can be carried over to other boundary conditions. The first a priori estimate is identical with the first one given in the proof of the next theorem for a weak solution, so we refer to it. □

3.2 Weak solution

In order to deal with less assumptions for the initial, boundary and compatibility conditions, we investigate a weak formulation. The following weak formulation of the system (1.2) is considered.

$$\int_0^T < u_{tt}(s), \phi(s) >_{H^{-1} \times H_0^1} \, \mathrm{d}s \, + \int_0^T \int_\Omega x \, \ddot{m}(s) \, \phi \, \mathrm{d}x \, \mathrm{d}s \, + \int_0^T \int_\Omega \Big(\theta \, (u_x + m(s)) \\ + F(u_x + m(s)) \Big) \, \phi_x \, \mathrm{d}x \, \mathrm{d}s \, - \int_0^T \int_\Omega u_{xxx} \, \phi_x \, \mathrm{d}x \, \mathrm{d}s \, = 0, \quad \forall \phi \in L^2(0,T; H_0^1(\Omega)), \tag{3.5a}$$

$$\theta_t - \theta \, (u_x + m(t)) \, (u_{xt} + \dot{m}(t)) - \theta_{xx} = g, \quad \text{a.e. in} \quad \Omega_T, \tag{3.5b}$$

$$u(0,t) = u(1,t) = 0, \quad \forall t \in [0,T], \quad u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad \text{a.e. in} \quad (0,T), \\ \theta_x(0,t) = 0, \quad -\theta_x(1,t) = \theta(1,t) - \theta_\Gamma(t), \quad \text{a.e. in} \quad (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \quad \forall x \in \overline{\Omega}. \tag{3.5c}$$

Instead of $(H1)$ through $(H3)$, we impose the following assumptions:

$$(H4) \qquad m \in H^3(0,T); \quad g \in L^2(0,T; L^2(\Omega)); \quad g(x,t) \geq 0 \quad \text{on} \quad \overline{\Omega} \times [0,T]; \\ \theta_\Gamma \in H^1(0,T); \quad \theta_\Gamma(t) > 0 \quad \text{on} \quad [0,T]. \tag{3.6}$$

$$(H5) \qquad u_0 \in H_E^3(\Omega) := \{ u \in H^3(\Omega) \,\big|\, u(0) = u''(0) = u(1) = u''(1) = 0 \}; \\ u_1 \in H_0^1(\Omega); \quad \theta_0 \in H^1(\Omega); \quad \theta_0(x) > 0 \quad \text{on} \quad \overline{\Omega}. \tag{3.7}$$

There holds

Theorem 3.2 *Suppose that (H4) and (H5) are satisfied. Then the system (3.5) has a solution* (u, θ) *on* $\overline{\Omega} \times [0,T]$ *satisfying*

$$u \ \in \ X_{3,T} := W^{2,\infty}(0,T; H^{-1}(\Omega)) \cap W^{1,\infty}(0,T; H^1(\Omega)) \cap L^\infty(0,T; H_E^3(\Omega)) \quad \textit{and} \\ \theta \ \in \ X_{4,T} := H^{2,1}(\Omega_T) \cap L^\infty(0,T; H^1(\Omega)), \tag{3.8}$$

for any $T > 0$.

Proof. In order to invoke the existence of a solution, we introduce sequences of smooth approximations of the initial and boundary data, respectively. Let $\{u_0^n\}$, $\{u_1^n\}$, $\{\theta_0^n\}$, $\{m^n\}$, $\{g^n\}$, and $\{\theta_\Gamma^n\}$ satisfy the assumptions of theorem 3.1 and

$$\begin{aligned} u_0^n \rightharpoonup u_0 \quad \text{in} \quad H_E^3(\Omega), \quad u_1^n \rightharpoonup u_1 \quad \text{in} \quad H_0^1(\Omega), \quad \theta_0^n \rightharpoonup \theta_0 \quad \text{in} \quad H^1(\Omega), \\ m^n \rightharpoonup m \quad \text{in} \quad H^3(0,T), \quad g^n \rightharpoonup g \quad \text{in} \quad L^2(0,T;L^2(\Omega)), \quad \text{and} \\ \theta_\Gamma^n \rightharpoonup \theta_\Gamma \quad \text{in} \quad H^1(0,T), \quad n \to \infty, \quad \text{for} \quad \text{some} \quad T > 0. \end{aligned} \tag{3.9}$$

Such sequences can be constructed in formulating appropriate boundary value problems so that they comply with the compatibility conditions (1.3). The corresponding solutions are denoted by $\{(u^n, \theta^n)\}$. We infer from the maximum principle for parabolic equations that $\theta^n(x,t) > 0$ on $\overline{\Omega_T}$. Taking the initial and boundary data given in (3.9), we will derive a priori estimates that do not depend on n.

The necessary a priori estimates will be given in the following four lemmas. In the sequel, the index 'n' is omitted for simplicity if no confusion will arise. Furthermore, C_i, $C_{\delta,i}$, $\hat{C}_i$, $i \in \mathbb{N}$, and C, respectively, denote positive constants which may depend on T, but not on n.

Lemma 3.1 *It holds*

$$\sup_{t\in(0,T)} \Big(\|\theta(t)\|_{L^1(\Omega)} + \|u_t(t)\|^2 + \|u_{xx}(t)\|^2 + \|u_x(t) + m(t)\|_{L^6(\Omega)}^6 + \|u_x(t)\|_{L^\infty(\Omega)}^2 \Big) \le C. \tag{3.10}$$

Proof. We proceed in two steps. First, testing (1.2a) with u yields

$$\begin{aligned} -\int_0^t \int_\Omega u_t^2 \,\mathrm{d}x\,\mathrm{d}s + \int_\Omega u_t\, u \Big|_0^t \mathrm{d}x + \int_0^t \int_\Omega x\, \ddot{m}(s)\, u \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega u_{xx}^2 \,\mathrm{d}x\,\mathrm{d}s \\ + \int_0^t \int_\Omega u_x \Big(\theta\,(u_x + m(s)) + F'(u_x + m(s)) \Big) \,\mathrm{d}x\,\mathrm{d}s = 0, \end{aligned} \tag{3.11}$$

and therefore

$$\begin{aligned} \int_0^t \int_\Omega \Big(\theta\, u_x^2 + u_{xx}^2 + F'(u_x + m(s))\,(u_x + m(s)) \Big) \,\mathrm{d}x\,\mathrm{d}s \le \int_0^t \int_\Omega \Big(\,|\, F'(u_x + m(s))\, m(s)\,| \\ + \;|\,\theta\, u_x\, m(s)\,| + |\, x\, \ddot{m}(s)\, u\,| + u_t^2 \Big) \,\mathrm{d}x\,\mathrm{d}s - \int_\Omega u_t\, u \Big|_0^t \mathrm{d}x\,. \end{aligned} \tag{3.12}$$

Since $u(0,t) = 0 = u(1,t)$, we conclude from (2.19) that $\|u_x(t)\|^2 \le \|u_{xx}(t)\|^2$. Thus,

$$F'(u_x + m(s))\,(u_x + m(s)) \;\ge\; C_1 (u_x + m(s))^6 - C_2, \tag{3.13}$$

$$\begin{aligned} \Big| F'(u_x + m(s))\, m(s) \Big| \;&\le\; \|m\|_{H^1(0,t)} \,(C_3 |u_x + m(s)|^6 + C_4), \\ \int_0^t \int_\Omega |\,\theta\, u_x\, m(s)\,| \,\mathrm{d}x\,\mathrm{d}s \;&\le\; \frac{1}{2} \int_0^t \int_\Omega \theta\, u_x^2 \,\mathrm{d}x\,\mathrm{d}s + C_5 \|m\|_{H^1(0,t)}^2 \int_0^t \int_\Omega \theta \,\mathrm{d}x\,\mathrm{d}s\,, \end{aligned} \tag{3.14}$$

$$\begin{aligned} \int_0^t \int_\Omega |\, x\, \ddot{m}(s)\, u\,| \,\mathrm{d}x\,\mathrm{d}s \;&\le\; \frac{1}{2}\|m\|_{H^2(0,t)}^2 + \frac{1}{2} \int_0^t \int_\Omega u^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\le\; \frac{1}{2}\|m\|_{H^2(0,t)}^2 + \frac{1}{2} \int_0^t \|u_x(s)\|^2 \,\mathrm{d}s \\ &\le\; \frac{1}{2}\|m\|_{H^2(0,t)}^2 + \frac{1}{2} \int_0^t \|u_{xx}(s)\|^2 \,\mathrm{d}s\,, \end{aligned} \tag{3.15}$$

and

$$
\begin{aligned}
-\int_\Omega u_t\, u\Big|_0^t \,\mathrm{d}x \;&\le\; \frac{1}{2}(\|u_0\|^2+\|u_1\|^2)+\delta_1\int_\Omega u_t^2(t)\,\mathrm{d}x \;+C_{1,\delta}\int_\Omega u^2(t)\,\mathrm{d}x\\
&\le\; \frac{1}{2}(\|u_0\|^2+\|u_1\|^2)+\delta_1\int_\Omega u_t^2(t)\,\mathrm{d}x \;+C_{1,\delta}\,t\int_0^t\int_\Omega u_t^2(x,s)\,\mathrm{d}x\,\mathrm{d}s\,, \qquad (3.16)
\end{aligned}
$$

where we have made use of the inequalities of Young, Poincaré and Hölder. Altogether, we end up with

$$
\begin{aligned}
\frac{1}{2}\int_0^t\int_\Omega(\theta\, u_x^2+u_{xx}^2)\,\mathrm{d}x\,\mathrm{d}s \;&\le\; C_6\Big(1+\int_0^t\int_\Omega u_t^2\,\mathrm{d}x\,\mathrm{d}s\;+\|m\|_{H^1(0,t)}\int_0^t\int_\Omega(u_x+m(s))^6\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\|m\|_{H^1(0,t)}^2\int_0^t\int_\Omega\theta\,\mathrm{d}x\,\mathrm{d}s\;+\|m\|_{H^1(0,t)}+\|m\|_{H^2(0,t)}^2\\
&\quad+\|u_0\|^2+\|u_1\|^2\Big)+\delta_1\int_\Omega u_t^2(t)\,\mathrm{d}x\,. \qquad (3.17)
\end{aligned}
$$

Similarly, one has

$$
\int_0^t\int_\Omega|\,\theta\, u_x\,\dot m(s)\,|\,\mathrm{d}x\,\mathrm{d}s\;\le\frac{1}{2}\int_0^t\int_\Omega\theta\, u_x^2\,\mathrm{d}x\,\mathrm{d}s\;+C_7\|m\|_{H^2(0,t)}^2\int_0^t\int_\Omega\theta\,\mathrm{d}x\,\mathrm{d}s\,, \qquad (3.18)
$$

and therefore

$$
\begin{aligned}
\int_0^t\int_\Omega|\,\theta\, u_x\,\dot m(s)\,|\,\mathrm{d}x\,\mathrm{d}s\;&\le\; C_8\Big(1+\int_0^t\int_\Omega u_t^2\,\mathrm{d}x\,\mathrm{d}s\;+\|m\|_{H^1(0,t)}\int_0^t\int_\Omega(u_x+m(s))^6\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\|m\|_{H^2(0,t)}^2\int_0^t\int_\Omega\theta\,\mathrm{d}x\,\mathrm{d}s\;+\|m\|_{H^2(0,t)}^2\\
&\quad+\|u_0\|^2+\|u_1\|^2\Big)+\delta_1\int_\Omega u_t^2(t)\,\mathrm{d}x\,. \qquad (3.19)
\end{aligned}
$$

Secondly, we test (1.2a) with u_t and add equation (1.2b) which has been integrated over time and space. Partial integration yields

$$
\begin{aligned}
&\frac{1}{2}\|u_t(t)\|^2+\int_0^t\int_\Omega x\,\ddot m(s)\,u_t\,\mathrm{d}x\,\mathrm{d}s\;+\int_0^t\int_\Omega u_{xt}\,\theta\,(u_x+m(s))\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad+\int_0^t\int_\Omega F'(u_x+m(s))\,u_{xt}\,\mathrm{d}x\,\mathrm{d}s\;+\int_0^t\int_\Omega u_{xx}\,u_{xxt}\,\mathrm{d}x\,\mathrm{d}s\;+\int_0^t\int_\Omega\theta_t\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad-\int_0^t\int_\Omega\theta\,(u_x+m(s))\,u_{xt}\,\mathrm{d}x\,\mathrm{d}s\;-\int_0^t\int_\Omega\theta\,(u_x+m(s))\,\dot m(s)\,\mathrm{d}x\,\mathrm{d}s\;-\int_0^t\int_\Omega\theta_{xx}\,\mathrm{d}x\,\mathrm{d}s\\
&\le\;\int_0^t\int_\Omega|\,g\,|\,\mathrm{d}x\,\mathrm{d}s\;+\frac{1}{2}\|u_1\|^2. \qquad (3.20)
\end{aligned}
$$

The third and the seventh term on the left hand side cancel. Involving the boundary conditions for θ, we find

$$
\begin{aligned}
&\frac{1}{2}\|u_t(t)\|^2+\int_0^t\int_\Omega x\,\ddot m(s)\,u_t\,\mathrm{d}x\,\mathrm{d}s\;+\int_0^t\int_\Omega F'(u_x+m(s))\,u_{xt}\,\mathrm{d}x\,\mathrm{d}s\;+\frac{1}{2}\|u_{xx}(t)\|^2\\
&\qquad+\int_\Omega\theta(x,t)\,\mathrm{d}x\;-\int_0^t\int_\Omega\theta\,(u_x+m(s))\,\dot m(s)\,\mathrm{d}x\,\mathrm{d}s\;+\int_0^t\theta(1,s)\,\mathrm{d}s\\
&=:\;\frac{1}{2}\|u_t(t)\|^2+I_1+I_2+\frac{1}{2}\|u_{xx}(t)\|^2+\int_\Omega\theta(x,t)\,\mathrm{d}x\;-(I_3+I_4)+\int_0^t\theta(1,s)\,\mathrm{d}s\\
&\le\;\frac{1}{2}\|u_0\|_{H^2(0,t)}^2+\int_\Omega\theta(x,0)\,\mathrm{d}x\;+\int_0^t\theta_\Gamma(s)\,\mathrm{d}s\,. \qquad (3.21)
\end{aligned}
$$

Recall that θ remains positive for all times. To deal with I_3, we use (3.19). Furthermore, one has

$$
\begin{aligned}
|I_1| &\le \frac{1}{2}\int_0^t\int_\Omega x^2\,\ddot{m}^2(s)\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_0^t\int_\Omega u_t^2\,\mathrm{d}x\,\mathrm{d}s \le C_9\|m\|^2_{H^2(0,t)} + \frac{1}{2}\int_0^t\int_\Omega u_t^2\,\mathrm{d}x\,\mathrm{d}s\,;\\
|I_4| &\le C_{10}\|m\|^2_{H^2(0,t)}\int_0^t\int_\Omega \theta\,\mathrm{d}x\,\mathrm{d}s\,. \qquad (3.22)
\end{aligned}
$$

Partial integration and Young's inequality lead to

$$
\begin{aligned}
I_2 &= \int_0^t \frac{\partial}{\partial s}\int_\Omega F(u_x+m(s))\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega F'(u_x+m(s))\,\dot{m}(s)\,\mathrm{d}x\,\mathrm{d}s\\
&\ge \int_\Omega F(u_x+m(t))\,\mathrm{d}x - C_{11}(\|u_0\|^6_{L^6(\Omega)} + |m(0)|^6) - C_{12}\\
&\quad -\int_0^t\int_\Omega |\,F'(u_x+m(s))\,|\cdot|\,\dot{m}(s)\,|\,\mathrm{d}x\,\mathrm{d}s\\
&\ge C_{13}\int_\Omega (u_x+m(t))^6\,\mathrm{d}x - C_{11}\|u_0\|^6_{L^6(\Omega)} - C_{14}\\
&\quad -C_{15}\|m\|_{H^2(0,t)}\Big(\int_0^t\int_\Omega (u_x+m(s))^6\,\mathrm{d}x\,\mathrm{d}s - 1\Big). \qquad (3.23)
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
&\frac{1}{2}\Big(\|u_t(t)\|^2 + \|u_{xx}(t)\|^2\Big) + C_{13}\int_\Omega (u_x+m(t))^6\,\mathrm{d}x + \int_\Omega \theta(x,t)\,\mathrm{d}x + \int_0^t \theta(1,s)\,\mathrm{d}s\\
&\le C_{16}\Big(1 + \int_0^t\int_\Omega u_t^2\,\mathrm{d}x\,\mathrm{d}s + \|m\|_{H^2(0,t)}\int_0^t\int_\Omega (u_x+m(s))^6\,\mathrm{d}x\,\mathrm{d}s + \|m\|^2_{H^2(0,t)}\int_0^t\int_\Omega \theta\,\mathrm{d}x\,\mathrm{d}s\\
&\quad +\|u_0\|^2_{H^2(0,t)} + \|u_0\|^6_{L^6(\Omega)} + \|u_1\|^2 + \|\theta_0\|^2\\
&\quad +\|g\|^2_{L^2(\Omega_t)} + \|m\|^2_{H^2(0,t)} + \|m\|_{H^2(0,t)} + \|\theta_\Gamma\|^2_{L^2(0,t)}\Big) + \int_0^t\int_\Omega |\,\theta\,u_x\,\dot{m}(s)\,|\,\mathrm{d}x\,\mathrm{d}s\,. \qquad (3.24)
\end{aligned}
$$

Invoking the first step (3.19), one concludes

$$
\begin{aligned}
&\frac{1}{2}\Big(\|u_t(t)\|^2 + \|u_{xx}(t)\|^2\Big) + C_{13}\int_\Omega (u_x+m(t))^6\,\mathrm{d}x + \int_\Omega \theta(x,t)\,\mathrm{d}x + \int_0^t \theta(1,s)\,\mathrm{d}s\\
&\le C_{17}\Big(1 + \int_0^t\int_\Omega u_t^2\,\mathrm{d}x\,\mathrm{d}s + \|m\|_{H^2(0,t)}\int_0^t\int_\Omega (u_x+m(s))^6\,\mathrm{d}x\,\mathrm{d}s + \|m\|^2_{H^2(0,t)}\int_0^t\int_\Omega \theta\,\mathrm{d}x\,\mathrm{d}s\\
&\quad +\|u_0\|^2_{H^2(0,t)} + \|u_0\|^6_{L^6(\Omega)} + \|u_1\|^2 + \|\theta_0\|^2\\
&\quad +\|g\|^2_{L^2(\Omega_t)} + \|m\|^2_{H^2(0,t)} + \|\theta_\Gamma\|^2_{L^2(0,t)}\Big) + \delta_1\|u_t(t)\|^2. \qquad (3.25)
\end{aligned}
$$

Recall that the norms on the right hand side are bounded. Now, with suitably chosen δ_1, we apply Gronwall's lemma and take the supremum over $[0,T]$. We arrive at

$$
\sup_{t\in(0,T)}\Big(\|\theta(t)\|_{L^1(\Omega)} + \|u_t(t)\|^2 + \|u_{xx}(t)\|^2 + \|u_x+m(t)\|^6_{L^6(\Omega)}\Big) \le C_{18}. \qquad (3.26)
$$

Invoking (2.19) completes the proof. □

Lemma 3.2 *It holds*

$$\sup_{t\in(0,T)} \|\theta(t)\|^2 + \int_0^T \Big(\|\theta_x(t)\|^2 + \theta^2(1,t) + \|\theta(t)\|^2_{L^\infty(\Omega)}\Big)\,\mathrm{d}t \;\le C. \tag{3.27}$$

Proof. We remark that, in contrast to the foregoing lemma, the bounds for the norms of the initial and boundary data, respectively, are immediately included into the constants C_i. Furthermore, the proofs of this lemma and the following one are almost identical with the proofs of lemma 2.5 and 2.6 of [16], respectively. We will give the proof to this lemma in order to show that it works for our purposes as well, but the next proof then will be omitted.

Testing (1.2b) with θ, one obtains

$$\begin{aligned}\frac{1}{2}\|\theta(t)\|^2 + \int_0^t \|\theta_x(s)\|^2\,\mathrm{d}s \;+ \frac{1}{2}\int_0^t \theta^2(1,s)\,\mathrm{d}s \;- \int_0^t\int_\Omega \theta^2\,(u_x + m(s))\,u_{xt}\,\mathrm{d}x\,\mathrm{d}s \\ - \int_0^t\int_\Omega \theta^2 (u_x + m(s))\dot{m}(s)\,\mathrm{d}x\,\mathrm{d}s \;\le C_1 + \int_0^t \|\theta(s)\|^2\,\mathrm{d}s\,.\end{aligned} \tag{3.28}$$

It is

$$\begin{aligned}\Big|\int_0^t\int_\Omega \theta^2\,(u_x + m(t))\,u_{xt}\,\mathrm{d}x\,\mathrm{d}s\Big| = \Big|\int_0^t\int_\Omega \Big(\theta^2\,(u_x + m(s))\,u_t\Big)_x\,\mathrm{d}x\,\mathrm{d}s \\ - \int_0^t\int_\Omega 2\,\theta\,\theta_x\,(u_x + m(s))\,u_t\,\mathrm{d}x\,\mathrm{d}s \;- \int_0^t\int_\Omega \theta^2\,u_{xx}\,u_t\,\mathrm{d}x\,\mathrm{d}s\Big| =: I_1 + I_2.\end{aligned} \tag{3.29}$$

The first term on the right hand side vanishes due to the boundary conditions. With (3.10), it holds

$$\begin{aligned}\Big|I_1\Big| \;&\le\; \delta_1 \int_0^t\int_\Omega \theta_x^2\,\mathrm{d}x\,\mathrm{d}s \;+ C_{1,\delta}\int_0^t\int_\Omega \theta^2\,u_t^2\,(u_x + m(s))^2\,\mathrm{d}x\,\mathrm{d}s \\ &\le\; \delta_1 \int_0^t\int_\Omega \theta_x^2\,\mathrm{d}x\,\mathrm{d}s \;+ C_2\,C_{1,\delta}\int_0^t \|\theta(s)\|^2_{L^\infty(\Omega)}\,\mathrm{d}s\,.\end{aligned} \tag{3.30}$$

Nirenberg's inequality yields

$$\|\theta(t)\|_{L^\infty(\Omega)} \le \hat{C}_1\|\theta_x(t)\|^{\frac{2}{3}} \cdot \|\theta(t)\|^{\frac{1}{3}}_{L^1(\Omega)} + \hat{C}_2\|\theta(t)\|_{L^1(\Omega)}, \tag{3.31}$$

and Young's inequality gives

$$\begin{aligned}\|\theta(t)\|^2_{L^\infty(\Omega)} \;&\le\; \Big(\hat{C}_3\|\theta_x(t)\|^{\frac{2}{3}} + \hat{C}_4\Big)^2 \le 2\,\hat{C}_3^2\,\|\theta_x(t)\|^{\frac{4}{3}} + 2\,\hat{C}_4^2 \\ &\le\; \delta_2\,\hat{C}_5\|\theta_x(t)\|^2 + C_{2,\delta}.\end{aligned} \tag{3.32}$$

Altogether, we have for I_1

$$\begin{aligned}|I_1| \;&\le\; \delta_1\int_0^t\int_\Omega \theta_x^2\,\mathrm{d}x\,\mathrm{d}s \;+ C_{1,\delta}\int_0^t \Big(\delta_2\,C_3\|\theta_x(s)\|^2 + C_{2,\delta}\,C_2\Big)\,\mathrm{d}s \\ &\le\; \Big(\delta_1 + (C_{1,\delta}\,C_3\,\delta_2)\Big)\int_0^t\int_\Omega \theta_x^2\,\mathrm{d}x\,\mathrm{d}s \;+ C_{1,\delta}\,C_{2,\delta}\,C_4.\end{aligned} \tag{3.33}$$

Analogously, the second integral can be estimated by

$$\begin{aligned} |I_2| &\leq C_5 \int_0^t \|\theta(s)\|^2_{L^\infty(\Omega)} \cdot \|u_{xx}(s)\| \cdot \|u_t(s)\| \, \mathrm{d}s \\ &\leq C_6 \int_0^t \|\theta(s)\|^2_{L^\infty(\Omega)} \, \mathrm{d}s \leq C_7 \, \delta_2 \int_0^t \int_\Omega \theta_x^2 \, \mathrm{d}x \, \mathrm{d}s + C_{2,\delta} \, C_8. \end{aligned} \tag{3.34}$$

Furthermore,

$$\int_0^t \int_\Omega |\, \theta^2 \, (u_x + m(s)) \, \dot{m}(s)) \, | \, \mathrm{d}x \, \mathrm{d}s \leq C_9 \int_0^t \int_\Omega \theta^2 \, \mathrm{d}x \, \mathrm{d}s \, . \tag{3.35}$$

We arrive at

$$\begin{aligned} &\frac{1}{2}\|\theta(t)\|^2 + \left(1 - \delta_1 - C_{1,\delta} \, C_3 \, \delta_2 - C_7 \, \delta_2\right) \int_0^t \int_\Omega \theta_x^2 \, \mathrm{d}x \, \mathrm{d}s + \frac{1}{2} \int_0^t \theta^2(1,s) \, \mathrm{d}s \\ &\quad \leq C_1 + C_{10} \int_0^t \int_\Omega \theta^2 \, \mathrm{d}s + C_{1,\delta} \, C_{2,\delta} \, C_4 + C_{2,\delta} \, C_8. \end{aligned} \tag{3.36}$$

Again, applying Gronwall's lemma, choosing δ_1 and δ_2 appropriately, and taking the supremum on both sides, we arrive at the statement (3.27) except for the last estimate. Due to (3.32), also

$$\int_0^T \|\theta(t)\|^2_{L^\infty(\Omega)} \, \mathrm{d}t \leq C \tag{3.37}$$

holds. □

Lemma 3.3 *It holds*

$$\sup_{t\in(0,T)} \left(\|u_{xt}(t)\|^2 + \|u_{xxx}(t)\|^2 + \|\theta_x(t)\|^2 + \theta^2(1,t)\right) + \int_0^T \left(\|\theta_t(t)\|^2 + \|\theta_{xx}(t)\|^2\right) \mathrm{d}t \leq C. \tag{3.38}$$

Proof. See [16], lemma 2.6. □

Lemma 3.4 *It holds*

$$\sup_{t\in(0,T)} \|u_{tt}(t)\|^2_{H^{-1}(\Omega)} \leq C. \tag{3.39}$$

Proof. From equation (3.5a), we have

$$\begin{aligned} < u_{tt}(t), \phi(t) >_{H^{-1}\times H_0^1} &= \int_\Omega x \, \ddot{m}(t) \, \phi \, \mathrm{d}x + \int_\Omega \Big(\theta \, (u_x + m(t)) \\ &+ F(u_x + m(t))\Big) \, \phi_x \, \mathrm{d}x - \int_\Omega u_{xxx} \, \phi_x \, \mathrm{d}x = 0, \quad \forall \phi \in L^2(0,T; H_0^1(\Omega)). \end{aligned} \tag{3.40}$$

Therefore, taking into account the estimates of the foregoing lemmas,

$$< u_{tt}(t), \phi(t) >_{H^{-1}\times H_0^1} \leq C \cdot \|\phi\|_{H_0^1(\Omega)}, \quad \forall \phi \in L^2(0,T; H_0^1(\Omega)) \quad \text{a.e. in} \quad (0,T), \tag{3.41}$$

and

$$\|u_{tt}(t)\|_{H^{-1}(\Omega)} \leq C \quad \text{a.e. in} \quad (0,T), \tag{3.42}$$

whence the assertion follows. □

Now, from lemma 3.1 to 3.4 it follows that, possibly for a subsequence which is again denoted by $\{(u^n, \theta^n)\}$, there exist functions (u, θ) satisfying

$$u^n \to u, \quad \text{weakly–star} \quad \text{in} \quad X_{3,T}, \qquad \theta^n \to \theta, \quad \text{weakly–star} \quad \text{in} \quad X_{4,T}. \tag{3.43}$$

We will show that these convergences are sufficient to arrive at the weak solution (3.5).

Since the embedding $H^{2,1}(\Omega_T) \hookrightarrow C(\overline{\Omega_T})$ is compact, $\{\theta^n\}$ converges uniformly to θ on $\overline{\Omega_T}$. In addition, $\{u_x^n\} \subset W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^2(\Omega)) \subset H^{2,1}(\Omega_T)$, and therefore the same argument applies to $\{u_x^n\}$. Thus, $\forall \phi \in L^2(0,T;H_0^1(\Omega))$,

$$\begin{aligned} \lim_{n\to\infty} \quad & \int_0^T \int_\Omega \Big((u_{tt}^n - u_{tt})\,\phi - (u_{xxx}^n - u_{xxx})\,\phi_x \\ & +(\theta^n\,(u_x^n + m^n) - \theta\,(u_x + m))\,\phi_x + (F(u_x^n + m^n) - F(u_x + m))\,\phi_x \Big) \;\; \mathrm{d}x\,\mathrm{d}t \\ = \quad & \lim_{n\to\infty} \int_0^T \int_\Omega \Big((\theta^n - \theta)\,(u_x^n + m^n)\,\phi_x + \theta\,(u_x^n + m^n - u_x - m)\,\phi_x \Big) \;\; \mathrm{d}x\,\mathrm{d}t \;=\; 0, \end{aligned} \tag{3.44}$$

since F is a continuous function and $\{m^n\}$ converges uniformly to m. Similarly, as $\{u_{xt}^n\} \subset L^\infty(0,T;L^2(\Omega))$ and $\dot{m}^n \to \dot{m}$ uniformly,

$$\begin{aligned} \lim_{n\to\infty} \quad & \int_0^T \int_\Omega \Big((\theta_t^n - \theta_t)\,\phi - (\theta_{xx}^n - \theta_{xx})\,\phi \\ & -(\theta^n\,(u_x^n + m^n)\,(u_{xt}^n + \dot{m}^n) - \theta\,(u_x + m)\,(u_{xt} + \dot{m}))\,\phi \Big) \;\; \mathrm{d}x\,\mathrm{d}t \\ = \quad & \lim_{n\to\infty} \int_0^T \int_\Omega \Big((\theta^n - \theta)\,(u_x^n + m^n)\,(u_{xt}^n + \dot{m}^n) + \theta\,(u_x^n + m^n - u_x - m)\,(u_{xt}^n + \dot{m}^n) \\ & +\theta\,(u_x + m)\,(u_{xt}^n + \dot{m}^n - u_{xt} - \dot{m}) \Big)\,\phi \;\; \mathrm{d}x\,\mathrm{d}t \;=\; 0. \end{aligned} \tag{3.45}$$

Due to the regularity of the solutions $\{(u^n, \theta^n)\}$, no problem arises concerning the boundary data. The theorem is completely proved. □

3.3 Uniqueness

We have uniqueness of the global classical and of the weak solution, respectively. The proof of uniqueness for the weak solution is almost identical with the proofs of theorem 2.3 in [15] and of theorem 2.2 in [1], respectively. These papers deal with the problem of load–driven experiments for which existence has been shown in [16], as mentioned above. So, we only state the result of the obtained stability result, implying uniqueness, and omit the proof. In the next section, on the control problem, we will make use of this stability result.

Lemma 3.5 *Let the assumptions of theorem 3.2 be satisfied. We denote by $u^{(i)}, \theta^{(i)}$, $i = 1, 2$, weak solutions to the problem (1.2) in the sense of theorem 3.2, respectively, and by $m^{(i)}, g^{(i)}$*

and $\theta_\Gamma^{(i)}$ the corresponding boundary data, and set $u := u^{(1)} - u^{(2)}, \theta := \theta^{(1)} - \theta^{(2)}, m := m^{(1)} - m^{(2)}, g := g^{(1)} - g^{(2)}$ and $\theta_\Gamma := \theta_\Gamma^{(1)} - \theta_\Gamma^{(2)}$. Then it holds

$$
\begin{aligned}
&\sup_{t\in(0,T)} \left(\|u_t(t)\|^2 + \|u_{xx}(t)\|^2 + \|\theta(t)\|^2 + \|u_x(t)\|^2_{L^\infty(\Omega)}\right) \\
&\qquad + \int_0^T \left(\|\theta_x(t)\|^2 + \theta^2(1,t) + \|\theta(t)\|^2_{L^\infty(\Omega)}\right) \mathrm{d}t \; + \\
&\sup_{t\in(0,T)} \left(\|u_{xt}(t)\|^2 + \|u_{xxx}(t)\|^2 + \|\theta_x(t)\|^2 + \theta^2(1,t) + \|u_t(t)\|^2_{L^\infty(\Omega)}\right. \\
&\qquad \left. + \|u_{xx}(t)\|^2_{L^\infty(\Omega)} + \|u_{tt}(t)\|^2_{H^{-1}(\Omega)} + \|\theta(t)\|^2_{L^\infty(\Omega)}\right) + \int_0^T \left(\|\theta_t(t)\|^2 + \|\theta_{xx}(t)\|^2\right) \mathrm{d}t \\
&\leq \;\; C \cdot \left(\|m\|^2_{H^3(0,T)} + \|g\|^2_{L^2(\Omega_T)} + \|\theta_\Gamma\|^2_{H^1(0,T)}\right). \qquad (3.46)
\end{aligned}
$$

Uniqueness for the global classical solution can be shown in the same way as in the proof of lemma 3.5. In fact, one has to derive the same estimates for u and θ as for the global existence which has been shown in [3]. So, this proof is omitted, too.

4 The control problem

4.1 Differentiability of the observation operator

In order to derive the necessary conditions of optimality, we have to show the Fréchet differentiability of the observation operator S. The operator maps each (m, g, θ_Γ) in the control set to the corresponding unique solution (u, θ) of (3.5). The control space is defined as

$$Z = H^3(0,T) \times L^2(0,T; L^2(\Omega)) \times H^1(0,T), \qquad (4.1)$$

therefore $M \subset Z$. For $(m, g, \theta_\Gamma) \in U_{ad}$ we define

$$
\begin{aligned}
K^\pm(m, g, \theta_\Gamma) \;\; := \;\; \Big\{ \; (h,k,l) \in Z \;\Big|\; & (m \pm \lambda h, \; g \pm \lambda k, \; \theta_\Gamma \pm \lambda l) \in U_{ad} \\
& \forall\, \lambda \in [0, \overline{\lambda}], \quad \overline{\lambda} > 0 \; \Big\}. \qquad (4.2)
\end{aligned}
$$

There holds:

Theorem 4.1 *Let the assumptions of theorem 3.2 be satisfied and $(m, g, \theta_\Gamma) \in U_{ad}$. Then S has a directional derivative*

$$(\phi, \psi) = D_{(h,k,l)} S(m, g, \theta_\Gamma) \qquad (4.3)$$

at (m, g, θ_Γ) in the direction (h, k, l) for all $(h,k,l) \in K^+(m, g, \theta_\Gamma)$. Moreover, with $(u, \theta) = S(m, g, \theta_\Gamma)$ and $\varepsilon = u_x + m(t)$, $(\phi, \psi) \in X_{3,T} \times X_{4,T}$ solves the linear initial boundary value problem

$$\int_0^T < \phi_{tt}(s), \xi(s) >_{H^{-1}\times H_0^1} \,\mathrm{d}s - \int_0^T \int_\Omega \phi_{xxx}\, \xi_x \,\mathrm{d}x\,\mathrm{d}s = - \int_0^T \int_\Omega x\, \ddot{h}(s)\, \xi \,\mathrm{d}x\,\mathrm{d}s \tag{4.4a}$$

$$- \int_0^T \int_\Omega \Big(\varepsilon\, \psi + (\theta + F''(\varepsilon))\, (\phi_x + h(s))\Big)\, \xi_x \,\mathrm{d}x\,\mathrm{d}s\,, \quad \forall \xi \in L^2(0,T; H_0^1(\Omega)), \tag{4.4b}$$

$$\psi_t - \psi_{xx} = k + \theta\, \varepsilon_t\, (\phi_x + h(t)) + \varepsilon\, \varepsilon_t\, \psi + \theta\, \varepsilon\, (\phi_{xt} + \dot{h}(t)), \quad \textit{a.e.} \quad \textit{in} \quad \Omega_T, \tag{4.4c}$$

$$\phi(x,0) = \phi_t(x,0) = 0 = \psi(x,0), \qquad \forall x \in \overline{\Omega}, \tag{4.4d}$$

$$\phi(0,t) = \phi(1,t) = 0, \quad \forall t \in [0,T], \quad \phi_{xx}(0,t) = \phi_{xx}(1,t) = 0, \quad \textit{a.e.} \quad \textit{in} \quad (0,T),$$

$$\psi_x(0,t) = 0, \quad -\psi_x(1,t) = \psi(1,t) - l(t), \qquad \textit{a.e.} \quad \textit{in} \quad (0,T). \tag{4.4e}$$

A corresponding result holds for the directional derivative $D_{(-h,-k,-l)}S(m,g,\theta_\Gamma)$ *of* S *at* (m,g,θ_Γ) *in the direction* $(-h,-k,-l)$ *if* $(h,k,l) \in K^-(m,g,\theta_\Gamma)$.

Proof. It follows from the standard theory of linear partial differential equations that (4.4) has a unique solution $(\phi,\psi) \in X_{3,T} \times X_{4,T}$. Now, let $(h,k,l) \in K^+(m,g,\theta_\Gamma)$ and $\overline{\lambda} > 0$ such that $(m + \lambda h, g + \lambda k, \theta_\Gamma + \lambda l) \in U_{ad}$, whenever $0 \le \lambda \le \overline{\lambda}$. We define

$$(u^\lambda, \theta^\lambda) := S(m + \lambda h,\, g + \lambda k,\, \theta_\Gamma + \lambda l), \qquad \varepsilon^\lambda := u_x^\lambda + m(t) + \lambda h(t),$$

$$p^\lambda := u^\lambda - u - \lambda \phi, \qquad q^\lambda := \theta^\lambda - \theta - \lambda \psi. \tag{4.5}$$

To prove the theorem we have to show that

$$\|(p^\lambda, q^\lambda)\|_{X_{3,T}\times X_{4,T}} = o(\lambda) \quad \text{as} \quad \lambda \quad \to \quad 0+\,. \tag{4.6}$$

For this purpose we need some preparation. Setting

$$G(\varepsilon,\theta) := \theta\, \varepsilon + F'(\varepsilon), \tag{4.7}$$

we obtain the following system which is solved by (p^λ, q^λ):

$$\int_0^T < p_{tt}^\lambda(s), \varphi(s) >_{H^{-1}\times H_0^1} \,\mathrm{d}s - \int_0^t \int_\Omega p_{xxx}^\lambda\, \varphi_x \,\mathrm{d}x\,\mathrm{d}s = - \int_0^t \int_\Omega \Big(G(\varepsilon^\lambda, \theta^\lambda) - G(\varepsilon,\theta)$$

$$-\lambda\, (G_\varepsilon(\varepsilon,\theta)\, (\phi_x + h(t)) + G_\theta(\varepsilon,\theta)\, \psi)\Big)\, \varphi_x \,\mathrm{d}x\,\mathrm{d}s\,, \qquad \forall \varphi \in L^2(0,T; H_0^1(\Omega)), \tag{4.8a}$$

$$q_t^\lambda - q_{xx}^\lambda = \theta^\lambda\, \varepsilon^\lambda\, \varepsilon_t^\lambda - \theta\, \varepsilon\, \varepsilon_t - \lambda\, \Big(\theta\, \varepsilon_t\, (\phi_x + h(t)) + \varepsilon\, \varepsilon_t\, \psi$$

$$+\theta\, \varepsilon\, (\phi_{xt} + \dot{h}(t))\Big), \qquad \text{a.e.} \quad \text{in} \quad \Omega_T, \tag{4.8b}$$

$$p^\lambda(x,0) = p_t^\lambda(x,0) = 0 = q^\lambda(x,0), \qquad x \in \overline{\Omega}, \tag{4.8c}$$

$$p^\lambda(0,t) = p^\lambda(1,t) = 0, \quad \forall t \in [0,T], \quad p_{xx}^\lambda(0,t) = p_{xx}^\lambda(1,t) = 0, \quad \text{a.e.} \quad \text{in} \quad (0,T),$$

$$q_x^\lambda(0,t) = 0, \quad -q_x^\lambda(1,t) = q^\lambda(1,t), \qquad \text{a.e.} \quad \text{in} \quad (0,T). \tag{4.8d}$$

Taylor's theorem leads to

$$\begin{aligned} G(\varepsilon^\lambda,\theta^\lambda) &= G(\varepsilon,\theta)+G_\varepsilon(\varepsilon,\theta)\,(\varepsilon^\lambda-\varepsilon)+G_\theta(\varepsilon,\theta)\,(\theta^\lambda-\theta) \\ &\quad +G_1(\varepsilon^\lambda,\varepsilon)\,(\varepsilon^\lambda-\varepsilon)^2+(\varepsilon^\lambda-\varepsilon)\,(\theta^\lambda-\theta), \end{aligned} \tag{4.9}$$

with $G_1(\varepsilon^\lambda,\varepsilon)=\Big(10\varepsilon^3-3\varepsilon+(10\varepsilon^2-1)(\varepsilon^\lambda-\varepsilon)+5\varepsilon(\varepsilon^\lambda-\varepsilon)^2+(\varepsilon^\lambda-\varepsilon)^3\Big)$. Since $\varepsilon^\lambda-\varepsilon=p_x^\lambda+\lambda(\phi_x+h(t))$ and $\theta^\lambda-\theta=q^\lambda+\lambda\psi$, we have

$$\begin{aligned} A &:= \frac{\partial}{\partial x}\Big(G(\varepsilon^\lambda,\theta^\lambda)-G(\varepsilon,\theta)-\lambda\,(G_\varepsilon(\varepsilon,\theta)\,(\phi_x+h(t))+G_\theta(\varepsilon,\theta)\,\psi)\Big) \\ &= \frac{\partial}{\partial x}\Big(G_\varepsilon(\varepsilon,\theta)\,p_x^\lambda+G_\theta(\varepsilon,\theta)\,q^\lambda+G_1(\varepsilon^\lambda,\varepsilon)\,(\varepsilon^\lambda-\varepsilon)^2+(\varepsilon^\lambda-\varepsilon)\,(\theta^\lambda-\theta)\Big) \\ &=: \frac{\partial}{\partial x}\tilde{A}=G_\varepsilon(\varepsilon,\theta)\,p_{xx}^\lambda+G_\theta(\varepsilon,\theta)\,q_x^\lambda+G_{\varepsilon\theta}(\varepsilon,\theta)\,\theta_x\,p_x^\lambda \\ &\quad +G_{\varepsilon\theta}(\varepsilon,\theta)\,\varepsilon_x\,q^\lambda+G_{\varepsilon\varepsilon}(\varepsilon,\theta)\,\varepsilon_x\,p_x^\lambda+G_1(\varepsilon^\lambda,\varepsilon)\,2\,(\varepsilon^\lambda-\varepsilon)(\varepsilon_x^\lambda-\varepsilon_x) \\ &\quad +(\varepsilon_x^\lambda-\varepsilon_x)\,(\theta^\lambda-\theta)+(\varepsilon^\lambda-\varepsilon)\,(\theta_x^\lambda-\theta_x)+\Big(G_{1,\varepsilon^\lambda}(\varepsilon^\lambda,\varepsilon)\,\varepsilon_x^\lambda+G_{1,\varepsilon}(\varepsilon^\lambda,\varepsilon)\,\varepsilon_x\Big)\,(\varepsilon^\lambda-\varepsilon)^2. \end{aligned} \tag{4.10}$$

Taylor's theorem for the right hand side of equation (4.8b) yields

$$\begin{aligned} B &:= \varepsilon\,\varepsilon_t\,q^\lambda+\theta\,\varepsilon_t\,p_x^\lambda+\theta\,\varepsilon\,p_{xt}^\lambda+\Big((\theta^\lambda-\theta)\,\varepsilon_t\,(\varepsilon^\lambda-\varepsilon) \\ &\quad +(\theta^\lambda-\theta)\,\varepsilon\,(\varepsilon_t^\lambda-\varepsilon_t)+\theta\,(\varepsilon^\lambda-\varepsilon)\,(\varepsilon_t^\lambda-\varepsilon_t)\Big). \end{aligned} \tag{4.11}$$

We now prove assertion (4.6). We divide the proof into three lemmas. In the following, $C_i, i\in\mathbb{N}$, denote suitably chosen constants.

Lemma 4.1 *It holds*

$$\begin{aligned} &\sup_{t\in(0,T)}\Big(\|p^\lambda(t)\|^2+\|p_t^\lambda(t)\|^2+\|p_{xx}^\lambda(t)\|^2+\|q^\lambda(t)\|^2+\|p_x^\lambda(t)\|^2_{L^\infty(\Omega)}\Big) \\ &\qquad +\int_0^T\Big(\|q_x^\lambda(t)\|^2+q^\lambda(1,t)^2\Big)\,\mathrm{d}t\;=O(\lambda^4). \end{aligned} \tag{4.12}$$

Proof. Testing (4.8a) with p_t^λ, partial integration, Hölder's and Young's inequalities and the corresponding stability result (see lemma 3.5) lead to

$$\begin{aligned} &\frac{1}{2}\Big(\|p_t^\lambda(t)\|^2+\|p_{xx}^\lambda(t)\|^2\Big)=\int_0^t\int_\Omega A\,p_t^\lambda\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\le\; C_1\int_0^t\Big(\|p_t^\lambda(s)\|^2+\|p_x^\lambda(s)\|^2+\|p_{xx}^\lambda(s)\|^2+\|q^\lambda(s)\|^2+\delta_1\|q_x^\lambda(s)\|^2\Big)\,\mathrm{d}s \\ &\qquad +C_2\int_0^t\int_\Omega\Big(|\varepsilon^\lambda-\varepsilon|^4+|\varepsilon_x^\lambda-\varepsilon_x|^4+|\theta^\lambda-\theta|^4\Big)\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad +\int_0^t\int_\Omega(\varepsilon^\lambda-\varepsilon)\,(\theta_x^\lambda-\theta_x)\,p_t^\lambda\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\le\; C_3\int_0^t\Big(\|p_t^\lambda(s)\|^2+\|p_x^\lambda(s)\|^2+\|p_{xx}^\lambda(s)\|^2+\|q^\lambda(s)\|^2+\delta_1\|q_x^\lambda(s)\|^2\Big)\,\mathrm{d}s \\ &\qquad +C_2\int_0^t\int_\Omega\Big(|\varepsilon^\lambda-\varepsilon|^4+|\varepsilon_x^\lambda-\varepsilon_x|^4+|\theta^\lambda-\theta|^4\Big)\,\mathrm{d}x\,\mathrm{d}s\;+C_4\lambda^4 \\ &\quad\le\; C_3\int_0^t\Big(\|p_t^\lambda(s)\|^2+\|p_x^\lambda(s)\|^2+\|p_{xx}^\lambda(s)\|^2+\|q^\lambda(s)\|^2+\delta_1\|q_x^\lambda(s)\|^2\Big)\,\mathrm{d}s\;+C_5\lambda^4. \end{aligned} \tag{4.13}$$

Next, we test (4.8b) with q^λ. Again, partial integration and Young's inequality give

$$
\begin{aligned}
&\frac{1}{2}\|q^\lambda(t)\|^2 + \int_0^t \|q_x^\lambda(s)\|^2\,\mathrm{d}s \;-\int_0^t q_x^\lambda(s)\; q^\lambda(s)\,\Big|_0^1\,\mathrm{d}s \\
&\quad\le\; \int_0^t\int_\Omega q^\lambda\Big(\theta\,\varepsilon\,p_{xt}^\lambda + q^\lambda\,\varepsilon\,\varepsilon_t + \theta\,\varepsilon_t\,p_x^\lambda\Big)\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \frac{1}{6}\int_0^t \|q^\lambda(s)\|^2\,\mathrm{d}s \;+ C_6\int_0^t\int_\Omega \Big(|\varepsilon^\lambda-\varepsilon|^4 + |\theta^\lambda-\theta|^4\Big)\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \int_0^t\int_\Omega \Big((\theta^\lambda-\theta)\,\varepsilon\,(\varepsilon_t^\lambda-\varepsilon_t) + \theta\,(\varepsilon^\lambda-\varepsilon)\,(\varepsilon_t^\lambda-\varepsilon_t)\Big)\,q^\lambda\,\mathrm{d}x\,\mathrm{d}s\,. \qquad (4.14)
\end{aligned}
$$

With the help of our stability result we have that the last terms are bounded by $C_7\lambda^4 + \frac{1}{3}\int_0^t \|q^\lambda(s)\|^2\,\mathrm{d}s$. Moreover,

$$
\begin{aligned}
\Big|\int_0^t\int_\Omega q^\lambda\Big(q^\lambda\,\varepsilon\,\varepsilon_t + \theta\,\varepsilon_t\,p_x^\lambda\Big)\,\mathrm{d}x\,\mathrm{d}s\Big| &\le C_8\int_0^t \Big(\|q^\lambda(s)\|^2 + \delta_2\|q_x^\lambda(s)\|^2\Big)\,\mathrm{d}s \\
&\quad + \delta_3 C_9 \sup_{t\in(0,T)} \|p_{xx}^\lambda(t)\|^2, \qquad (4.15)
\end{aligned}
$$

where we have made use of (2.14) and (2.19) for $p^\lambda(t)$. The first term on the right hand side of (4.14) yields

$$
\begin{aligned}
\int_0^t\int_\Omega q^\lambda\,\theta\,\varepsilon\,p_{xt}^\lambda\,\mathrm{d}x\,\mathrm{d}s \;&=\; \int_0^t\int_\Omega \frac{d}{dx}\Big(q^\lambda\,\theta\,\varepsilon\,p_t^\lambda\Big)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad -\int_0^t\int_\Omega \Big(q_x^\lambda\,\theta\,\varepsilon\,p_t^\lambda + q^\lambda\,\theta_x\,\varepsilon\,p_t^\lambda + q^\lambda\,\theta\,\varepsilon_x\,p_t^\lambda\Big)\,\mathrm{d}x\,\mathrm{d}s\,. \qquad (4.16)
\end{aligned}
$$

The first term on the right hand side vanishes due to the boundary conditions. Again, invoking (2.14), Hölder's and Young's inequalities, we have

$$
\begin{aligned}
&\frac{1}{2}\Big(\|q^\lambda(t)\|^2 + \int_0^t \|q_x^\lambda(s)\|^2\,\mathrm{d}s\Big) + \int_0^t q^\lambda(1,s)^2\,\mathrm{d}s \;\le C_7\lambda^4 \qquad (4.17) \\
&\quad + C_{10}\int_0^t \Big(\|p_t^\lambda(s)\|^2 + \|q^\lambda(s)\|^2 + (\delta_2+\delta_4+\delta_5)\,\|q_x^\lambda(s)\|^2\Big)\,\mathrm{d}s \;+ \delta_3 C_9 \sup_{t\in(0,T)} \|p_{xx}^\lambda(t)\|^2.
\end{aligned}
$$

Altogether, we conclude from (4.13) and (4.17):

$$
\begin{aligned}
&\|p_t^\lambda(t)\|^2 + \|p_{xx}^\lambda(t)\|^2 + \|q^\lambda(t)\|^2 + \int_0^t \Big(\|q_x^\lambda(s)\|^2 + q^\lambda(1,s)^2\Big)\,\mathrm{d}s \\
&\quad\le\; C_{11}\lambda^4 + C_{12}\int_0^t \Big(\|p_t^\lambda(s)\|^2 + \|p_x^\lambda(s)\|^2 + \|p_{xx}^\lambda(s)\|^2 + \|q^\lambda(s)\|^2 \\
&\qquad + (\delta_1+\delta_2+\delta_4+\delta_5)\,\|q_x^\lambda(s)\|^2\Big)\,\mathrm{d}s \;+ \delta_3 C_9 \sup_{t\in(0,T)} \|p_{xx}^\lambda(t)\|^2. \qquad (4.18)
\end{aligned}
$$

Taking into account Poincaré's inequality, i.e. $\|p^\lambda(t)\| \le \|p_x^\lambda(t)\|$, (2.19) for $p^\lambda(t)$, chosing δ_i, $i=1,\dots,5$, suitably, applying Gronwall's lemma, and taking the supremum on both sides, the assertion is proved. □

Lemma 4.2 *It holds*

$$\sup_{t\in(0,T)} \Big(\|p^\lambda_{xt}(t)\|^2 + \|p^\lambda_{xxx}(t)\|^2 + \|q^\lambda_x(t)\|^2 + q^\lambda(1,t)^2 + \|p^\lambda_t(t)\|^2_{L^\infty(\Omega)} + \|p^\lambda_{xx}(t)\|^2_{L^\infty(\Omega)}$$
$$+\|q^\lambda(t)\|^2_{L^\infty(\Omega)}\Big) + \int_0^T \Big(\|q^\lambda_t(t)\|^2 + \|q^\lambda_{xx}(t)\|^2\Big)\,\mathrm{d}t \;=\; O(\lambda^4). \tag{4.19}$$

Proof. Testing (4.8b) with q^λ_t, partial integration yields

$$\int_0^t \|q^\lambda_t(s)\|^2\,\mathrm{d}s \;+ \int_0^t\int_\Omega q^\lambda_{xt}\,q^\lambda_x\,\mathrm{d}x\,\mathrm{d}s \;- \int_0^t q^\lambda_x\,q^\lambda_t\Big|_0^1\mathrm{d}s \;\le \delta_1\int_0^t \|q^\lambda_t(s)\|^2\,\mathrm{d}s \;+ C_1\lambda^4, \tag{4.20}$$

where we proceeded as in the foregoing lemma. Considering the initial and boundary conditions, we obtain

$$(1-\delta_1)\int_0^t \|q^\lambda_t(s)\|^2\,\mathrm{d}s \;+ \frac{1}{2}\|q^\lambda_x(t)\|^2 + \frac{1}{2}\,q^\lambda(1,t)^2 \le C_1\lambda^4. \tag{4.21}$$

Next, we test (4.8a) with $-p^\lambda_{xxt}$. Using the initial and boundary conditions, we arrive at

$$\begin{aligned}\frac{1}{2}\Big(\|p^\lambda_{xt}(t)\|^2 + \|p^\lambda_{xxx}(t)\|^2\Big) \;&=\; -\int_0^t\int_\Omega A\,p^\lambda_{xxt}\,\mathrm{d}x\,\mathrm{d}s\\ &=\; -\int_0^t\int_\Omega \tilde{A}_t\,p^\lambda_{xxx}\,\mathrm{d}x\,\mathrm{d}s \;+ \int_\Omega \tilde{A}\,p^\lambda_{xxx}(t)\,\mathrm{d}x\,. \end{aligned}\tag{4.22}$$

The first integral can be estimated by

$$\begin{aligned}&\Big|\int_0^t\int_\Omega \Big(G_\varepsilon(\varepsilon,\theta)\,p^\lambda_{xt} + G_\theta(\varepsilon,\theta)\,q^\lambda_t + G_{\varepsilon\theta}(\varepsilon,\theta)\,\theta_t\,p^\lambda_x + G_{\varepsilon\theta}(\varepsilon,\theta)\,\varepsilon_t\,q^\lambda + G_{\varepsilon\varepsilon}(\varepsilon,\theta)\,\varepsilon_t\,p^\lambda_x\\ &\quad +G_1(\varepsilon^\lambda,\varepsilon)\,2\,(\varepsilon^\lambda-\varepsilon)\,(\varepsilon^\lambda_t-\varepsilon_t) + (\varepsilon^\lambda_t-\varepsilon_t)\,(\theta^\lambda-\theta)\\ &\quad +(\varepsilon^\lambda-\varepsilon)\,(\theta^\lambda_t-\theta_t) + \Big(G_{1,\varepsilon^\lambda}(\varepsilon^\lambda,\varepsilon)\,\varepsilon^\lambda_t + G_{1,\varepsilon}(\varepsilon^\lambda,\varepsilon)\,\varepsilon_t\Big)\,(\varepsilon^\lambda-\varepsilon)^2\Big)\,p^\lambda_{xxx}\,\mathrm{d}x\,\mathrm{d}s\,\Big|\\ &\quad\le\; C_2\int_0^t \Big(\|p^\lambda_{xt}(s)\|^2 + \|p^\lambda_x(s)\|^2 + \|p^\lambda_{xxx}(s)\|^2 + \|q^\lambda(s)\|^2 + \delta_2\|q^\lambda_t(s)\|^2\Big)\,\mathrm{d}s \;+ C_3\lambda^4\\ &\quad\le\; C_4\int_0^t \Big(\|p^\lambda_{xt}(s)\|^2 + \|p^\lambda_{xxx}(s)\|^2 + \delta_2\|q^\lambda_t(s)\|^2\Big)\,\mathrm{d}s \;+ C_5\lambda^4, \end{aligned}\tag{4.23}$$

where we have made use of the foregoing lemma and the stability result. Also, that is why the second integral is bounded by

$$\begin{aligned}&\delta_3\|p^\lambda_{xxx}(t)\|^2 + C_6\Big(\|p^\lambda_x(t)\|^2 + \|q^\lambda(t)\|^2\Big) + C_7\int_\Omega \Big(|\varepsilon^\lambda-\varepsilon|^4 + |\theta^\lambda-\theta|^4\Big)\,\mathrm{d}x\\ &\quad\le\; \delta_3\|p^\lambda_{xxx}(t)\|^2 + C_8\lambda^4. \end{aligned}\tag{4.24}$$

Both inequalities lead to

$$\begin{aligned}&\frac{1}{2}\|p^\lambda_{xt}(t)\|^2 + (\frac{1}{2}-\delta_3)\|p^\lambda_{xxx}(t)\|^2 + (\frac{1}{2}-\delta_1-\delta_2)\int_0^t \|q^\lambda_t(s)\|^2\,\mathrm{d}s \;+ \frac{1}{2}\|q^\lambda_x(t)\|^2 + \frac{1}{2}\,q^\lambda(1,t)^2\\ &\quad\le\; C_9\lambda^4 + C_4\int_0^t \Big(\|p^\lambda_{xt}(s)\|^2 + \|p^\lambda_{xxx}(s)\|^2\Big)\,\mathrm{d}s\,. \end{aligned}\tag{4.25}$$

Now, a suitable choice of δ_i, $i = 1,2,3$, Gronwall's lemma and taking the supremum on both sides, invoking (2.14) and lemma 4.1 as usual, the lemma is proved except for the term $\int_0^T \|q_{xx}^\lambda(t)\|^2 \, \mathrm{d}t$. Looking into equations (4.8b) and (4.11), respectively, one easily sees that it is also bounded by $O(\lambda^4)$. □

Lemma 4.3 *It holds*

$$\sup_{t\in(0,T)} \|p_{tt}^\lambda(t)\|_{H^{-1}(\Omega)}^2 \le O(\lambda^4). \tag{4.26}$$

Proof. This works analogously to lemma 3.4. With the help of equations (4.8a), (4.10) and the foregoing lemma, one finds

$$\sup_{t\in(0,T)} \|p_{tt}^\lambda(t)\|_{H^{-1}(\Omega)} \le O(\lambda^2) \quad \text{a.e.} \quad \text{in} \quad (0,T), \tag{4.27}$$

whence the assertions follows. □

Thus, theorem 4.1 is completely proved. □

Remark 4.1 The result of theorem 4.1 is much stronger than the corresponding result of theorem 2.3 in [1]. In fact, here we have shown the differentiability of S as mapping into the solution space $X_{3,T} \times X_{4,T}$, while in [1] only the differentiability into the Banach space

$$\begin{aligned} B \;\; = \;\; & W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;\overset{\circ}{H}_1(\Omega)\cap H^2(\Omega)) \\ \times \;\; & L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \end{aligned} \tag{4.28}$$

has been proved. Since $X_{4,T}$ is continuously imbedded in $C(\overline{\Omega_T})$, this means that also pointwise constraints on the temperature θ could be included in the control problem. This was not possible in [12] and [13] where only pointwise constraints on the displacement u and the strain ε, respectively, could be admitted. Note that pointwise constraints for θ are very realistic for the particular experimental setup discussed here, where θ is kept close to a prescribed (constant) temperature $\overline{\theta}$.

4.2 Necessary conditions of optimality

We introduce the adjoint system. Let (p,q) be the adjoint variables to $(u^*,\theta^*) \in X_{3,T} \times X_{4,T}$, we have

$$\begin{aligned} &\int_0^T \int_\Omega \xi_t \, p_t \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega \xi(t) \, p_t(t) \, \mathrm{d}x - \int_0^T \int_\Omega \xi_{xx} \, p_{xx} \, \mathrm{d}x \, \mathrm{d}t = - \int_t^T \int_\Omega \xi \, \frac{\partial}{\partial x}\Big((\theta^* \\ &\quad + F''(\varepsilon^*)) \, p_x + q_t \, \theta^* \, \varepsilon^* + q \, \theta_t^* \, \varepsilon^* + D_1\Phi_1(u_x^*, \theta^*)\Big) \, \mathrm{d}x \, \mathrm{d}s \,, \quad \text{in} \quad \Omega_T, \\ &\quad \forall \xi \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;\overset{\circ}{H}_1(\Omega)\cap H^2(\Omega)), \quad 0 \le t \le T, \end{aligned} \tag{4.29a}$$

$$q_t + q_{xx} - p_x\,\varepsilon^* + q\,\varepsilon^*\,\varepsilon_t^* = D_2\Phi_1(u_x^*, \theta^*), \quad \text{a.e.} \quad \text{in} \quad \Omega_T, \tag{4.29b}$$

$$p(x,T) = p_t(x,T) = 0 = q(x,T), \quad \text{for} \quad x \in \overline{\Omega}, \tag{4.29c}$$

$$p(0,t) = p(1,t) = 0 = p_{xx}(0,t) = p_{xx}(1,t),$$

$$q_x(0,t) = 0 = q_x(1,t) + q(1,t), \quad \text{for} \quad 0 \le t \le T, \quad \varepsilon^* = u_x^* + m^*(t). \tag{4.29d}$$

The following theorem can be proved as theorem 3.1 in [1].

Theorem 4.2 *Suppose that $(u^*, \theta^*) \in X_{3,T} \times X_{4,T}$ and $\Phi_1 \in C^2(\mathbb{R}^2)$. Then there exists a pair (p^*, q^*) such that $p^* \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; \overset{\circ}{H}_1(\Omega) \cap H^2(\Omega))$ and $q^* \in H^1(0,T; H^1(\Omega)) \cap L^2(0,T; H^3(\Omega))$, which solve the adjoint system (4.29).*

The following theorem is the main result of this section.

Theorem 4.3 *Let $(u^*, \theta^*; m^*, g^*, \theta_\Gamma^*)$ denote any solution of the control problem* **(CP)**. *Then there exist functions (p^*, q^*) as in theorem 4.2 which solve the following variational inequality:*

$$\begin{aligned}
&\int_0^T \int_\Omega \left\{ \ \ddot{h}(t)\, p^*\, x - \dot{h}(t)\, q^*\, \theta^*\, \varepsilon^* + h(t) \left(p_x^*\, (\theta^* + F''(\varepsilon^*)) - q^*\, \theta^*\, \varepsilon_t^* \right) \ \right\} \, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \left\{ \ \Phi_2'(\dddot{m}^*(t))\, h(t) \ \right\} \, \mathrm{d}t + \int_0^T \int_\Omega \left\{ \ \Phi_3'(g^*) - q^* \ \right\} \, k \ \, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \left\{ \ \Phi_4'(\theta_\Gamma^*(t)) - q^*(1,t) \ \right\} \, l(t) \ \, \mathrm{d}t \quad \ge \quad 0,
\end{aligned} \tag{4.30}$$

$\forall (h,k,l) \in K^+(m^*, g^*, \theta_\Gamma^*)$. For $(h,k,l) \in K^-(m^*, g^*, \theta_\Gamma^*)$ we obtain the reverse inequality.

Proof. Let $(u^*, \theta^*; m^*, g^*, \theta_\Gamma^*)$ solve **(CP)**. Then, for sufficiently small $\lambda > 0$, one has

$$J(u^\lambda, \theta^\lambda; m^\lambda, g^\lambda, \theta_\Gamma^\lambda) \ge J(u^*, \theta^*; m^*, g^*, \theta_\Gamma^*), \quad \forall (h,k,l) \in K^+(m^*, g^*, \theta_\Gamma^*). \tag{4.31}$$

Theorem 4.1 then yields

$$(u^\lambda, \theta^\lambda) = (u^*, \theta^*) + \lambda(\phi^*, \psi^*) + o(\lambda), \tag{4.32}$$

where (ϕ^*, ψ^*) solves (4.4). Taking the limit $\lambda \to 0+$, invoking (1.4) and (4.31), one arrives at

$$\begin{aligned}
&\int_0^T \int_\Omega \left(D_1\Phi_1(u_x^*, \theta^*)\, \phi_x^* + D_2\Phi_1(u_x^*, \theta^*)\, \psi^* \right) \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \Phi_2'(\dddot{m}^*(t))\, h(t)\, \mathrm{d}t + \int_0^T \int_\Omega \Phi_3'(g^*)\, k\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \Phi_4'(\theta_\Gamma^*(t))\, l(t)\, \mathrm{d}t \quad \ge \quad 0, \quad \forall (h,k,l) \in K^+(m^*, g^*, \theta_\Gamma^*).
\end{aligned} \tag{4.33}$$

Denoting by (p^*, q^*) the adjoint variables, we have

$$\int_0^T < \phi_{tt}^*, p^* >_{H^{-1}\times H_0^1} \,\mathrm{d}t - \int_0^T \int_\Omega \phi_{xxx}^* \, p_x^* \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_\Omega x \, \ddot{h}(t) \, p^* \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_\Omega p^* \frac{\partial}{\partial x}\Big(\varepsilon^* \, \psi^*$$
$$+ \quad (\theta^* + F''(\varepsilon^*)) \, (\phi_x^* + h(t))\Big) \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_\Omega q^* \Big(\psi_t^* - \psi_{xx}^* - k - \Big(\theta^* \, \varepsilon_t^* \, (\phi_x^* + h(t))$$
$$+ \quad \varepsilon^* \, \varepsilon_t^* \, \psi^* + \theta^* \; \varepsilon^* \, (\phi_{xt}^* + \dot{h}(t))\Big)\Big) \,\mathrm{d}x \,\mathrm{d}t \quad = \quad 0, \quad \varepsilon_x^* = u_x^* + m^*(t). \tag{4.34}$$

Partial integration leads to

$$-\int_0^T \int_\Omega \phi_t^* \, p_t^* \,\mathrm{d}x \,\mathrm{d}t - \int_\Omega \phi_t^*(t) \, p_t^*(t) \,\mathrm{d}x + \int_0^T \int_\Omega \phi_{xx}^* \, p_{xx} \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_\Omega \Big(p_x^* \, (\theta^*$$
$$+ \quad F''(\varepsilon^*)) \, (\phi_x^* + h(t)) + p^* \, x \, \ddot{h}(t) + p_x^* \, \varepsilon^* \, \psi^* + \psi^*(-q_t^* - q_{xx}^*) - q^* k - q^* \, \theta^* \, \varepsilon_t^* \, (\phi_x^* + h(t))$$
$$- \quad q^* \, \varepsilon^* \, \varepsilon_t^* \, \psi^* + (q^* \, \theta^* \, \varepsilon^*)_x \, \phi_t^* - q^* \, \theta^* \, \varepsilon^* \, \dot{h}(t)\Big) \,\mathrm{d}x \,\mathrm{d}t - \int_0^T q^*(1,t) \, l(t) \,\mathrm{d}t \quad = \quad 0. \tag{4.35}$$

It follows

$$-\int_0^T \int_\Omega \phi_t^* \, p_t^* \,\mathrm{d}x \,\mathrm{d}t - \int_\Omega \phi_t^*(t) \, p_t^*(t) \,\mathrm{d}x + \int_0^T \int_\Omega \phi_{xx}^* \, p_{xx} \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_\Omega \phi^* \Big(\frac{\partial}{\partial x}\Big(p_x^* \, (\theta^*$$
$$+ \quad F''(\varepsilon^*)) - q^* \, \theta^* \, \varepsilon_t^* + (q^* \, \theta^* \, \varepsilon^*)_t\Big)\Big) \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_\Omega \psi^* \Big(- q_t^* - q_{xx}^* + p_x^* \, \varepsilon^* - q^* \, \varepsilon^* \, \varepsilon_t^*\Big) \,\mathrm{d}x \,\mathrm{d}t$$
$$+ \quad \int_0^T \int_\Omega \Big(p_x^* \, (\theta^* + F''(\varepsilon^*)) \, h(t) + p^* \, x \, \ddot{h}(t) - q^* \, \theta^* \, \varepsilon_t^* \, h(t)$$
$$- \quad q^* k - q^* \, \theta^* \, \varepsilon^* \, \dot{h}(t)\Big) \,\mathrm{d}x \,\mathrm{d}t - \int_0^T q^*(1,t) \, l(t) \,\mathrm{d}t \quad = \quad 0, \tag{4.36}$$

and therefore

$$-\int_0^T \int_\Omega \phi_t^* \, p_t^* \,\mathrm{d}x \,\mathrm{d}t - \int_\Omega \phi_t^*(t) \, p_t^*(t) \,\mathrm{d}x + \int_0^T \int_\Omega \phi_{xx}^* \, p_{xx} \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_\Omega \phi^* \Big(\frac{\partial}{\partial x}\Big(p_x^* \, (\theta^*$$
$$+ \quad F''(\varepsilon^*)) + q_t^* \, \theta^* \, \varepsilon^* + q^* \, \theta_t^* \, \varepsilon^*\Big)\Big) \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_\Omega \psi^* \Big(q_t^* + q_{xx}^* - p_x^* \, \varepsilon^* + q^* \, \varepsilon^* \, \varepsilon_t^*\Big) \,\mathrm{d}x \,\mathrm{d}t$$
$$+ \quad \int_0^T \int_\Omega \Big(h(t) \, p_x^* \, (\theta^* + F''(\varepsilon^*)) + \ddot{h}(t) \, p^* \, x - q^* k$$
$$- \quad q^* \, \theta^* \, (h(t) \, \varepsilon_t^* + \dot{h}(t) \, \varepsilon^*)\Big) \,\mathrm{d}x \,\mathrm{d}t - \int_0^T q^*(1,t) \, l(t) \,\mathrm{d}t \quad = \quad 0. \tag{4.37}$$

With theorem 4.2, we obtain

$$\int_0^T \int_\Omega \phi^* \frac{\partial}{\partial x} D_1 \Phi_1(u_x^*, \theta^*) \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_\Omega \psi^* \, D_2 \Phi_1(u_x^*, \theta^*) \,\mathrm{d}x \,\mathrm{d}t$$
$$+ \quad \int_0^T \int_\Omega \Big(h(t) \, p_x^* \, (\theta^* + F''(\varepsilon^*)) + \ddot{h}(t) \, p^* \, x - q^* k$$
$$- \quad q^* \, \theta^* \, (h(t) \, \varepsilon_t^* + \dot{h}(t) \, \varepsilon^*)\Big) \,\mathrm{d}x \,\mathrm{d}t - \int_0^T q^*(1,t) \, l(t) \,\mathrm{d}t \quad = \quad 0. \tag{4.38}$$

Invoking equation (4.33) yields

$$\int_0^T \int_\Omega \Big(h(t) \, p_x^* \, (\theta^* + F''(\varepsilon^*)) + \ddot{h}(t) \, p^* \, x - q^* k$$

$$
- \quad q^* \, \theta^* \left(h(t) \, \varepsilon_t^* + \dot{h}(t) \, \varepsilon^* \right) \big) \, \mathrm{d}x \, \mathrm{d}t \; - \int_0^T q^*(1,t) \, l(t) \, \mathrm{d}t
$$

$$
+ \quad \int_0^T \Phi_2'(\ddot{m}^*(t)) \, h(t) \, \mathrm{d}t + \int_0^T \int_\Omega \Phi_3'(g^*) \, k \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \Phi_4'(\theta_\Gamma^*(t)) \, l(t) \, \mathrm{d}t \quad \geq \quad 0. \qquad (4.39)
$$

Hence, the variational inequality (4.30) follows. □

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