Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Deep quench approximation and optimal control of general Cahn–Hilliard systems with fractional operators and double obstacle potentials

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submitted: December 13, 2018

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No. 2559
Berlin 2018

2010 *Mathematics Subject Classification.* 35K45, 35K90, 49K20, 49K27.

Key words and phrases. Fractional operators, Cahn–Hilliard systems, optimal control, double obstacles, necessary optimality conditions.

This research was supported by the Italian Ministry of Education, University and Research (MIUR): Dipartimenti di Eccellenza Program (2018–2022) – Dept. of Mathematics “F. Casorati”, University of Pavia. In addition, PC and CG gratefully acknowledge some other financial support from the MIUR-PRIN Grant 2015PA5MP7 “Calculus of Variations”, the GNAMPA (Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica) and the IMATI – C.N.R. Pavia.

Edited by
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Abstract

In the recent paper *"Well-posedness and regularity for a generalized fractional Cahn–Hilliard system"*, the same authors derived general well-posedness and regularity results for a rather general system of evolutionary operator equations having the structure of a Cahn–Hilliard system. The operators appearing in the system equations were fractional versions in the spectral sense of general linear operators A and B having compact resolvents and are densely defined, unbounded, selfadjoint, and monotone in a Hilbert space of functions defined in a smooth domain. The associated double-well potentials driving the phase separation process modeled by the Cahn–Hilliard system could be of a very general type that includes standard physically meaningful cases such as polynomial, logarithmic, and double obstacle nonlinearities. In the subsequent paper *"Optimal distributed control of a generalized fractional Cahn–Hilliard system"* (Appl. Math. Optim. (2018), https://doi.org/10.1007/s00245-018-9540-7) by the same authors, an analysis of distributed optimal control problems was performed for such evolutionary systems, where only the differentiable case of certain polynomial and logarithmic double-well potentials could be admitted. Results concerning existence of optimizers and first-order necessary optimality conditions were derived, where more restrictive conditions on the operators A and B had to be assumed in order to be able to show differentiability properties for the associated control-to-state operator. In the present paper, we complement these results by studying a distributed control problem for such evolutionary systems in the case of nondifferentiable nonlinearities of double obstacle type. For such nonlinearities, it is well known that the standard constraint qualifications cannot be applied to construct appropriate Lagrange multipliers. To overcome this difficulty, we follow here the so-called "deep quench" method. This technique, in which the nondifferentiable double obstacle nonlinearity is approximated by differentiable logarithmic nonlinearities, was first developed by P. Colli, M.H. Farshbaf-Shaker and J. Sprekels in the paper *"A deep quench approach to the optimal control of an Allen–Cahn equation with dynamic boundary conditions and double obstacles"* (Appl. Math. Optim. **71** (2015), pp. 1-24) and has proved to be a powerful tool in a number of optimal control problems with double obstacle potentials in the framework of systems of Cahn–Hilliard type. We first give a general convergence analysis of the deep quench approximation that includes an error estimate and then demonstrate that its use leads in the double obstacle case to appropriate first-order necessary optimality conditions in terms of a variational inequality and the associated adjoint state system.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ denote an open, bounded, and connected set with smooth boundary Γ and outward normal derivative $\partial_{\mathbf{n}}$, let $T > 0$ be a final time, and let $H := L^2(\Omega)$ denote the Hilbert space of square-integrable real-valued functions defined on Ω, endowed with the standard inner product $(\cdot, \cdot)$

DOI 10.20347/WIAS.PREPRINT.2559 Berlin 2018

and norm $\|\cdot\|$, respectively. We denote $Q_t := \Omega \times (0,t)$ for $0 < t < T$ and $Q := \Omega \times (0,T)$. We investigate in this paper the approximation and optimal control of an abstract system of evolutionary variational (in)equalities. More precisely, the variational state system has the following form: we look for functions (μ, y) such that

$$y \in H^1(0,T;V_A^{-r}) \cap L^\infty(0,T;V_B^\sigma) \quad \text{and} \quad \tau\partial_t y \in L^2(0,T;H) \tag{1.1}$$
$$\mu \in L^2(0,T;V_A^r), \tag{1.2}$$
$$f_1(y) \in L^1(Q), \tag{1.3}$$

and satisfying

$$\langle \partial_t y(t), v\rangle_{A,r} + (A^r\mu(t), A^r v) = 0 \quad \text{for every } v \in V_A^r \text{ and a.e. } t \in (0,T), \tag{1.4}$$
$$(\tau\partial_t y(t), y(t)-v) + (B^\sigma y(t), B^\sigma(y(t)-v)) + \int_\Omega f_1(y(t))$$
$$+ (f_2'(y(t)) - u(t), y(t)-v) \le (\mu(t), y(t)-v) + \int_\Omega f_1(v)$$
$$\text{for every } v \in V_B^\sigma \text{ and a.e. } t \in (0,T), \tag{1.5}$$
$$y(0) = y_0 \quad \text{in } \Omega. \tag{1.6}$$

Here, it is understood that $\int_\Omega f_1(v) = +\infty$ whenever $f_1(v) \notin L^1(\Omega)$. The precise meaning of the involved quantities and spaces will be given below. Notice that (1.4)–(1.6) is a generalized version of the evolutionary system

$$\partial_t y + A^{2r}\mu = 0 \quad \text{in } Q, \tag{1.7}$$
$$\tau\partial_t y + B^{2\sigma} y + \partial f_1(y) + f_2'(y) \ni \mu + u \quad \text{in } Q, \tag{1.8}$$
$$y(0) = y_0 \quad \text{in } \Omega. \tag{1.9}$$

Here, $\tau \ge 0$ is a constant, $f_2 : \mathbb{R} \to \mathbb{R}$ is a smooth function, and $f_1 : \mathbb{R} \to [0,+\infty]$ denotes a proper, convex, and lower semicontinuous function with $f_1(0) = 0$, whose effective domain $D(f_1)$ is a closed interval in $\mathbb{R}$ (possibly $\mathbb{R}$ itself) and which is smooth in the interior of $D(f_1)$. In (1.8), ∂f_1 denotes the subdifferential of f_1, which is a multivalued operator, in general, so that the inclusion replaces the equality. The linear operators A^{2r}, and $B^{2\sigma}$, with $r > 0$ and $\sigma > 0$, denote fractional powers (in the spectral sense) of operators A and B. We will give a proper definition of such operators in the next section. Throughout this paper, we generally assume:

(A1) $A : D(A) \subset H \to H$ and $B : D(B) \subset H \to H$ are unbounded, monotone, and selfadjoint linear operators with compact resolvents.

This assumption implies that there are sequences $\{\lambda_j\}$ and $\{\lambda_j'\}$ of eigenvalues and orthonormal sequences $\{e_j\}$ and $\{e_j'\}$ of corresponding eigenvectors, that is,

$$Ae_j = \lambda_j e_j, \quad Be_j' = \lambda_j' e_j', \quad \text{and} \quad (e_i, e_j) = (e_i', e_j') = \delta_{ij}, \quad \text{for } i,j = 1,2,\dots, \tag{1.10}$$

with δ_{ij} denoting the Kronecker index, such that

$$0 \le \lambda_1 \le \lambda_2 \le \dots, \quad \text{and} \quad 0 \le \lambda_1' \le \lambda_2' \le \dots, \quad \text{with} \quad \lim_{j\to\infty} \lambda_j = \lim_{j\to\infty} \lambda_j' = +\infty, \tag{1.11}$$
$$\{e_j\} \text{ and } \{e_j'\} \text{ are complete systems in } H. \tag{1.12}$$

The state system (1.7)–(1.9) (and thus also (1.4)–(1.6)) can be seen as a generalization of the famous Cahn–Hilliard system which models a phase separation process taking place in the container Ω. In this case, one typically has $A^{2r} = B^{2\sigma} = -\Delta$ with zero Neumann or Dirichlet boundary conditions, and the unknown functions y and μ stand for the *order parameter* (usually a scaled density of one of the involved phases) and the *chemical potential* associated with the phase transition, respectively. Moreover, $f := f_1 + f_2$ is a double-well potential. Typical cases are the *classical regular potential*, the *logarithmic potential*, and the *double obstacle potential*, which (in this order) are given by

$$f_{\rm reg}(v) := \frac{1}{4}\,(v^2-1)^2\,, \quad v \in \mathbb{R}, \tag{1.13}$$

$$f_{\log}(v) := \begin{cases} (1+v)\ln(1+v) + (1-v)\ln(1-v) - c_1 v^2 & \text{for } v \in (-1,1) \\ 2\ln(2) - c_1 & \text{for } v \in \{-1,1\} \\ +\infty & \text{for } v \notin [-1,1] \end{cases} \tag{1.14}$$

$$f_{\rm obs}(v) := \begin{cases} -c_1\, v^2 & \text{if } |v| \le 1 \\ +\infty & \text{otherwise} \end{cases} \tag{1.15}$$

Here the constant $c_1 > 0$ is such that the above potentials are nonconvex.

Recently, in [24, Thm. 2.6 and 2.8], it was shown that the system (1.4)–(1.6) admits a solution (μ, y) satisfying (1.1)–(1.3), where the admissible nonlinearities include all of the three cases (1.13)–(1.15). In the analysis, it turned out that the first eigenvalue λ_1 of A plays an important role. Indeed, the main assumption for the operators A, B besides **(A1)** was the following:

(A2) Either

(i) $\lambda_1 > 0$

or

(ii) $0 = \lambda_1 < \lambda_2$, and e_1 is a constant and belongs to the domain of B^σ.

The existence proof in [24] was based on Moreau–Yosida approximation, which is generally applicable to all of the three cases (1.13)–(1.15). It turned out that the second component y of the solutions (μ, y) is always uniquely determined, while this is not necessarily so for the chemical potential μ (for cases in which also μ is unique, see [24, Rem. 4.1] and [25, Rem. 3.4]).

In this paper, we focus on the case when $f = f_{\rm obs}$, that is, when $f_1 = I_{[-1,1]}$ is the indicator function of the interval $[-1,1]$, given by $I_{[-1,1]}(v) = 0$ if $v \in [-1,1]$ and $I_{[-1,1]}(v) = +\infty$ otherwise. In this case, any solution (μ, y) of (1.4)–(1.6) must satisfy $\int_Q I_{[-1,1]}(y) < +\infty$, which entails that $y \in [-1,1]$ almost everywhere in Q and thus $\int_\Omega f_1(y(t)) = 0$ for almost every $t \in (0,T)$ in (1.5). While the question of well-posedness was settled in [24, Thm. 2.6 and 2.8] for $f_1 = I_{[-1,1]}$, the matter of optimal control is still open. Indeed, the optimal control theory recently developed in [25] applies to certain classes of differentiable potentials only. In this paper, we aim at extending this theory to the case $f = f_{\rm obs}$. More precisely, we investigate the following optimal control problem:

$(\mathcal{CP}_0)$ Minimize the tracking-type cost functional

$$\mathcal{J}(y,u) := \frac{\beta_1}{2}\,\|y(T) - y_\Omega\|^2 + \frac{\beta_2}{2}\int_0^T \|y(t) - y_Q(t)\|^2\,dt + \frac{\beta_3}{2}\int_0^T \|u(t)\|^2\,dt \tag{1.16}$$

over the admissible set

$$\mathcal{U}_{\rm ad} := \left\{ u \in H^1(0,T;L^2(\Omega)) : \ |u| \le \rho_1 \text{ a.e. in } Q, \ \ \|u\|_{H^1(0,T;L^2(\Omega))} \le \rho_2 \right\}, \tag{1.17}$$

subject to (1.4)–(1.6) with $f_1 = I_{[-1,1]}$. Here, $\rho_1 > 0$ and $\rho_2 > 0$ are such that $\mathcal{U}_{\rm ad} \ne \emptyset$, β_i, $i = 1,2,3$, are nonnegative but not all zero, and the given target functions satisfy $y_\Omega \in L^2(\Omega)$ and

$y_Q \in L^2(Q)$. Note that ($\mathcal{CP}_0$) is well defined, since the component y of the solutions to the state system is uniquely determined.

The main difficulty inherent in ($\mathcal{CP}_0$) is the nondifferentiability of the nonlinearity $I_{[-1,1]}$, which entails that standard constraint qualifications from optimal control theory are violated, so that suitable Lagrange multipliers cannot easily be constructed. In such situations, the so-called "deep quench" approximation has proved to be a useful tool in a number of cases in the framework of Cahn–Hilliard systems (see, e.g., [13, 15, 20, 23, 26]). In all of these works, the starting point was that the optimal control problem (we will later denote this problem by ($\mathcal{CP}_\alpha$)) had been successfully treated (by proving the Fréchet differentiability of the control-to-state operator and establishing first-order necessary optimality conditions in terms of a variational inequality and the adjoint state system) for the case when in the state system (1.4)–(1.6) the nonlinearity $f_1 = I_{[-1,1]}$ is for $\alpha > 0$ replaced by $f_1 = h^\alpha := \varphi(\alpha)h$, with the functions

(A3) $\varphi \in C^1[0,+\infty)$ is strictly increasing and satisfies $\lim_{\alpha\searrow 0} \varphi(\alpha) = 0;$ (1.18)

$$h(v) = \begin{cases} (1+v)\ln(1+v) + (1-v)\ln(1-v), & v \in (-1,1) \\ 2\ln(2), & v \in \{-1,1\} \\ +\infty, & v \notin [-1,1] \end{cases} . \tag{1.19}$$

We obviously have that

$$0 \le h^{\alpha_1}(v) \le h^{\alpha_2}(v) \quad \forall\, v \in \mathbb{R}, \quad \text{if } 0 < \alpha_1 < \alpha_2, \tag{1.20}$$

$$\lim_{\alpha\searrow 0} h^\alpha(v) = I_{[-1,1]}(v) \quad \forall\, v \in \mathbb{R}. \tag{1.21}$$

In addition, $h'(v) = \ln\left(\frac{1+v}{1-v}\right)$ and $h''(v) = \frac{2}{1-v^2} > 0$ for $v \in (-1,1)$, and thus, in particular,

$$\lim_{\alpha\searrow 0} \varphi(\alpha)h'(v) = 0 \quad \text{for } -1 < v < 1, \tag{1.22}$$

$$\lim_{\alpha\searrow 0}\Big(\varphi(\alpha) \lim_{v\searrow -1} h'(v)\Big) = -\infty, \quad \lim_{\alpha\searrow 0}\Big(\varphi(\alpha) \lim_{v\nearrow +1} h'(v)\Big) = +\infty. \tag{1.23}$$

We may therefore regard the graphs of the single-valued functions

$$(h^\alpha)'(v) = \varphi(\alpha)h'(v), \quad \text{for} \quad v \in (-1,1) \quad \text{and} \quad \alpha > 0, \tag{1.24}$$

as approximations to the graph of the multi-valued subdifferential $\partial I_{[-1,1]}$. Now the well-posedness results of [24, 25] apply, yielding a solution pair (μ^α, y^α) for every $\alpha > 0$, where the component y^α is uniquely determined. It is a natural question whether we have $y^\alpha \to y$ as $\alpha \searrow 0$ in a suitable topology. Below (cf. Theorem 3.5), we will show that this is actually true; in Corollary 3.6, we will show that in a very special case with some global constant $K_2 > 0$ a quantitative error estimate of the form

$$\|y^\alpha - y\|_{C^0([0,T];L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le K_2\, |\alpha|^{1/2} \tag{1.25}$$

is valid. Also, owing to the construction, the approximate functions y^α automatically attain values in the domain of $I_{[-1,1]}$; that is, we have $\|y^\alpha\|_{L^\infty(Q)} \le 1$ for all $\alpha > 0$.

As far as the optimal control problem is concerned, the general strategy is then to derive uniform (with respect to $\alpha \in (0,1]$) a priori estimates for the state and adjoint state variables of an "adapted" version of ($\mathcal{CP}_\alpha$) that are sufficiently strong as to permit a passage to the limit as $\alpha \searrow 0$ in order to derive meaningful first-order necessary optimality conditions also for ($\mathcal{CP}_0$). We can follow this strategy in this paper, since in [25] a corresponding theory for ($\mathcal{CP}_\alpha$) with the logarithmic potential (1.14) has been developed.

However, while the approximation results for the solutions of the state system hold true under essentially the same assumptions as those imposed in [24] for the well-posedness results, it seems impossible to prove that the control-to-state operator is Fréchet differentiable between suitable Banach spaces without having at disposal suitable uniform $L^\infty(Q)$ bounds for both the state component y and the functions $f^{(i)}(y)$, for $i = 1, 2, 3$. In the case of the logarithmic potentials h^α, which we intend to use for the deep quench approximation, this means that we need to separate y^α away from the critical arguments ± 1. Unfortunately, this postulate has the unpleasant consequence that Fréchet differentiability (and thus satisfactory first-order necessary optimality conditions) can only be established under rather restrictive conditions on the operators A and B. A particular case in which our analysis will work is given if $A = B = -\Delta$ with zero Neumann boundary condition, $\sigma = 1/2$, and $r \geq 3/8$.

Let us add a few remarks on the existing literature. One can find numerous contributions on viscous/nonviscous, local/nonlocal, convective/nonconvective Cahn–Hilliard systems for the classical (non-fractional) case $A = B = -\Delta$, $2r = 2\sigma = 1$, where various types of boundary conditions (e.g., Dirichlet, Neumann, dynamic) and different assumptions on the nonlinearity were considered. We refer the interested reader to the recent paper [21] for a selection of associated references. Some papers also address the coupled Cahn–Hilliard/Navier–Stokes system (see, e.g, [28], [29], and the references given therein).

The literature on optimal control problems for non-fractional Cahn–Hilliard systems is still rather scarce. The case of Dirichlet and/or Neumann boundary conditions for various types of such systems were the subject of, e.g., the works [16, 18–20, 27, 46, 49, 50], while the case of dynamic boundary conditions was studied in [13–15, 17, 19, 22, 23, 26, 32]. The optimal control of convective Cahn–Hilliard systems was addressed in [22, 23, 43, 47, 48], while the papers [11, 12, 30, 31, 37–42] were concerned with coupled Cahn–Hilliard/Navier–Stokes systems.

There are only a few contributions to the theory of Cahn–Hilliard systems involving fractional operators. In the connection of well-posedness and regularity results, we refer to [1,2] for the case of the fractional negative Laplacian with zero Dirichlet boundary conditions; general operators other than the negative Laplacian have apparently only been studied in [24,33–35]. As of now, aspects of optimal control have been scarcely dealt with even for simpler linear evolutionary systems involving fractional operators; for such systems, some identification problems were addressed in the recent contributions [36, 45], while for optimal control problems for such cases we refer to [6] (for the stationary (elliptic) case, see also [3–5, 7–9]). However, to the authors' best knowledge, the present paper appears to be the first contribution that addresses optimal control problems for Cahn–Hilliard systems with general fractional order operators and potentials of double obstacle type.

The paper is organized as follows: the subsequent Section 2 brings some auxiliary functional analytic material on fractional order operators, while in Section 3 we establish some general convergence results for the deep quench approximation of the state system (1.4)–(1.6). In particular, an error estimate is proved. In Section 4, we investigate the relations between the solutions to the optimal control problems ($\mathcal{P}_0$) and the solutions to the corresponding optimal control problems for the deep quench approximations. In the final Section 5, we then employ the results from [25] to establish the first-order necessary optimality conditions for ($\mathcal{P}_0$).

Throughout the paper, we denote for a general Banach space X other than $H = L^2(\Omega)$ by $\|\cdot\|_X$ and X^* its norm and dual space, respectively; the dual pairing between elements of X^* and X is denoted by $\langle \cdot, \cdot \rangle_X$.

2 Fractional powers and auxiliary results

In this section, we collect some auxiliary material concerning functional analytic notions. To this end, we generally assume that the conditions **(A1)** and **(A2)** are satisfied. At this point, some remarks on the assumption **(A2)** are in order.

Remark 2.1. The condition $\lambda_1 > 0$ is satisfied for many standard elliptic operators of second or higher order with zero Dirichlet boundary conditions (however, also zero mixed boundary conditions could be considered, with proper definitions of the domains of the operators); typical cases are the (negative) Laplacian $A = -\Delta$ with the domain $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ or the bi-harmonic operator $A = \Delta^2$ with the domain $D(\Delta^2) = H^4(\Omega) \cap H_0^2(\Omega)$. On the other hand, we have $0 = \lambda_1 < \lambda_2$ and $e_1 \equiv \mathrm{const.}$ for important problems with zero Neumann boundary conditions; typical examples are $A = -\Delta$ with the domain $D(-\Delta) = \{v \in H^2(\Omega) : \ \partial_{\mathbf{n}} v = 0 \text{ on } \Gamma\}$ and $A = \Delta^2$ with the domain $D(\Delta^2) = \{v \in H^4(\Omega) : \ \partial_{\mathbf{n}} v = \partial_{\mathbf{n}} \Delta v = 0 \text{ on } \Gamma\}$. We also point out that A and B can be completely unrelated if $\lambda_1 > 0$, while in the other case the constant functions have to belong to $D(B^\sigma)$. The latter holds true if $B = -\Delta$ with the domain $D(-\Delta) = \{v \in H^2(\Omega) : \partial_{\mathbf{n}} v = 0 \text{ on } \Gamma\}$, while in the Dirichlet case $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ no nontrivial constant functions are contained in $D(B)$; however, if $0 < \sigma < 1/4$, then $D(B^\sigma)$ coincides with the usual Sobolev–Slobodeckij space $H^{2\sigma}(\Omega)$ and thus contains all constant functions.

Using the facts summarized in (1.10)–(1.12), we can define the powers of A and B for an arbitrary positive real exponent. For the first operator, we have

$$V_A^r := D(A^r) = \Big\{ v \in H : \ \sum_{j=1}^{\infty} |\lambda_j^r (v, e_j)|^2 < +\infty \Big\} \quad \text{and} \tag{2.1}$$

$$A^r v = \sum_{j=1}^{\infty} \lambda_j^r (v, e_j) e_j \quad \text{for } v \in V_A^r, \tag{2.2}$$

the series being convergent in the strong topology of H, due to the properties (2.1) of the coefficients. In principle, we can endow V_A^r with the (graph) norm and inner product

$$\|v\|_{gr,A,r}^2 := (v, v)_{gr,A,r} \quad \text{and} \quad (v, w)_{gr,A,r} := (v, w) + (A^r v, A^r w) \quad \text{for } v, w \in V_A^r. \tag{2.3}$$

This makes V_A^r a Hilbert space. However, we can choose any equivalent Hilbert norm. Indeed, in view of assumption **(A2)**, it is more convenient to work with the Hilbert norm

$$\|v\|_{A,r}^2 := \begin{cases} \|A^r v\|^2 = \displaystyle\sum_{j=1}^{\infty} |\lambda_j^r (v, e_j)|^2 & \text{if } \lambda_1 > 0, \\ |(v, e_1)|^2 + \|A^r v\|^2 = |(v, e_1)|^2 + \displaystyle\sum_{j=2}^{\infty} |\lambda_j^r (v, e_j)|^2 & \text{if } \lambda_1 = 0. \end{cases} \tag{2.4}$$

In [24, Prop. 3.1] it has been shown that this norm is equivalent to the graph norm defined in (2.3), and we always will work with the norm (2.4) instead of with (2.3). We also use the corresponding inner product in V_A^r given by

$$(v, w)_{A,r} = (A^r v, A^r w) \quad \text{or} \quad (v, w)_{A,r} = (v, e_1)(w, e_1) + (A^r v, A^r w),$$
$$\text{depending on whether } \lambda_1 > 0 \text{ or } \lambda_1 = 0, \quad \text{for } v, w \in V_A^r. \tag{2.5}$$

Remark 2.2. Observe that in the case $\lambda_1 = 0$ the constant value of e_1 equals one of the numbers $\pm|\Omega|^{-1/2}$, where $|\Omega|$ is the volume of Ω. It follows for every $v \in H$ that the first term $(v, e_1)e_1$ of the Fourier series of v is the constant function whose value is the mean value of v, which is defined by

$$\operatorname{mean}(v) := \frac{1}{|\Omega|} \int_\Omega v \,. \tag{2.6}$$

In the same way as for A, starting from (1.10)–(1.12) for B, we can define the power B^σ of B for every $\sigma > 0$, where for V_B^σ we choose the graph norm. We therefore set

$$\begin{aligned} &V_B^\sigma := D(B^\sigma), \quad \text{with the norm } \|\cdot\|_{B,\sigma} \text{ associated to the inner product} \\ &(v,w)_{B,\sigma} := (v,w) + (B^\sigma v, B^\sigma w) \quad \text{for } v,w \in V_B^\sigma. \end{aligned} \tag{2.7}$$

To resume our preparations, we observe that if r_i and σ_i are arbitrary positive exponents, then it is easily seen that we have the "Green type" formulas

$$(A^{r_1+r_2} v, w) = (A^{r_1} v, A^{r_2} w) \quad \text{for every } v \in V_A^{r_1+r_2} \text{ and } w \in V_A^{r_2}, \tag{2.8}$$

$$(B^{\sigma_1+\sigma_2} v, w) = (B^{\sigma_1} v, B^{\sigma_2} w) \quad \text{for every } v \in V_B^{\sigma_1+\sigma_2} \text{ and } w \in V_B^{\sigma_2}. \tag{2.9}$$

The next step is the introduction of some spaces with negative exponents. We set

$$V_A^{-r} := (V_A^r)^* \quad \text{for } r > 0, \tag{2.10}$$

and endow V_A^{-r} with the dual norm $\|\cdot\|_{A,-r}$ of $\|\cdot\|_{A,r}$. We use the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_{A,r}$ for the duality pairing between V_A^{-r} and V_A^r and identify H with a subspace of V_A^{-r} in the usual way, i.e., such that $\langle v, w\rangle_{A,r} = (v,w)$ for every $v \in H$ and $w \in V_A^r$. Likewise, we set

$$V_B^{-\sigma} := (V_B^\sigma)^* \quad \text{for } \sigma > 0. \tag{2.11}$$

As V_B^σ is dense in H, we have the analogous embedding

$$H \subset V_B^{-\sigma}. \tag{2.12}$$

Observe that the following embedding results are valid:

The embeddings $V_A^{r_2} \subset V_A^{r_1} \subset H$ are dense and compact for $0 < r_1 < r_2$. (2.13)

The embeddings $H \subset V_A^{-r_1} \subset V_A^{-r_2}$ are dense and compact for $0 < r_1 < r_2$. (2.14)

The embeddings $V_B^{\sigma_2} \subset V_B^{\sigma_1} \subset H$ are dense and compact for $0 < \sigma_1 < \sigma_2$. (2.15)

At this point, we introduce the Riesz isomorphism $\mathcal{R}_r : V_A^r \to V_A^{-r}$ associated with the inner product (2.5), which is given by

$$\langle \mathcal{R}_r v, w\rangle_{A,r} = (v,w)_{A,r} \quad \text{for every } v,w \in V_A^r. \tag{2.16}$$

Moreover, we set

$$\begin{aligned} &V_0^r := V_A^r \quad \text{and} \quad V_0^{-r} := V_A^{-r} \quad \text{if } \lambda_1 > 0, \\ &V_0^r := \{v \in V_A^r : \operatorname{mean}(v) = 0\} \quad \text{and} \quad V_0^{-r} := \{v \in V_A^{-r} : \langle v, 1\rangle_{A,r} = 0\} \quad \text{if } \lambda_1 = 0\,. \end{aligned} \tag{2.17}$$

According to [24, Prop. 3.2], $\mathcal{R}_r$ maps V_0^r onto V_0^{-r} and extends to V_0^r the restriction of A^{2r} to V_0^{2r}. In view of this result, it is reasonable to use a proper notation for the restrictions of $\mathcal{R}_r$ and $\mathcal{R}_r^{-1}$ to the subspaces V_0^r and V_0^{-r}, respectively. We set

$$A_0^{2r} := (\mathcal{R}_r)_{|V_0^r} \quad \text{and} \quad A_0^{-2r} := (\mathcal{R}_r^{-1})_{|V_0^{-r}}\,, \tag{2.18}$$

where the index 0 has no meaning if $\lambda_1 > 0$ (since then $V_0^{\pm r} = V_A^{\pm r}$), while it reflects the zero mean value condition in the case $\lambda_1 = 0$. We thus have

$$A_0^{2r} \in \mathcal{L}(V_0^r, V_0^{-r}), \quad A_0^{-2r} \in \mathcal{L}(V_0^{-r}, V_0^r) \quad \text{and} \quad A_0^{-2r} = (A_0^{2r})^{-1}\,, \tag{2.19}$$

$$\langle A_0^{2r} v, w\rangle_{A,r} = (v,w)_{A,r} = (A^r v, A^r w) \quad \text{for every } v \in V_0^r \text{ and } w \in V_A^r\,, \tag{2.20}$$

$$\langle f, A_0^{-2r} f\rangle_{A,r} = \|A_0^{-2r} f\|_{A,r}^2 = \|f\|_{A,-r}^2 \quad \text{for every } f \in V_0^{-r}. \tag{2.21}$$

3 Deep quench approximation of the state system

In this section, we state our general assumptions and discuss the deep quench approximation of the state system (1.4)–(1.6). Besides **(A1)–(A3)**, we generally assume for the structure and the data of the state system:

(A4) $r > 0$, $\sigma > 0$, and $\tau \ge 0$ are fixed real numbers.

(A5) $f_2 \in C^3(\mathbb{R})$, and f_2' is Lipschitz continuous on $\mathbb{R}$ with Lipschitz constant $L > 0$.

(A6) $y_0 \in V_B^\sigma$, and $-1 < \inf \mathrm{ess}_{x\in\Omega}\, y_0(x), \quad \sup \mathrm{ess}_{x\in\Omega}\, y_0(x) < +1.$

(A7) $u \in \mathcal{X} := H^1(0,T;L^2(\Omega)) \cap L^\infty(Q).$

We draw a few consequences from **(A6)**. Namely, the mean value of y_0 belongs to the interior of both $D(\partial I_{[-1,1]})$ and $D((h^\alpha)')$, for all $\alpha > 0$. Moreover, we have $I_{[-1,1]}(y_0) \in L^1(\Omega)$ and $h(y_0) \in L^1(\Omega)$, and $h'(y_0)$ belongs to $L^2(\Omega)$. Thus, the conditions [24, (2.27), (2.28)] on y_0 for the application of [24, Thm. 2.6] are satisfied, where we note that

$$\|h^\alpha(y_0)\|_{L^1(\Omega)} + \|(h^\alpha)'(y_0)\|_{L^2(\Omega)} \le \widehat{c} \quad \forall\, \alpha \in (0,1], \tag{3.1}$$

with some constant $\widehat{c} > 0$ which is independent of $\alpha \in (0,1]$.

We now consider the state system (1.4)–(1.6) for the cases $f_1 = I_{[-1,1]}$ and $f_1 = h^\alpha$ ($\alpha \in (0,1]$), respectively. By virtue of [24, Thm. 2.6], there exist solution pairs (μ, y) and (μ^α, y^α), respectively, which enjoy the properties (1.1)–(1.3), and the (uniquely determined) second components satisfy

$$-1 \le y \le 1 \text{ a.e. in } Q, \quad -1 \le y^\alpha \le 1 \text{ a.e. in } Q. \tag{3.2}$$

We are now going to investigate the behavior of the family $\{(\mu^\alpha, y^\alpha)\}_{\alpha>0}$ of deep quench approximations for $\alpha \searrow 0$. We begin our analysis with the derivation of general a priori estimates.

Theorem 3.1. *Suppose that the general assumptions* **(A1)–(A7)** *are fulfilled, and assume that* (μ^α, y^α) *are solution pairs to the problem* (1.4)–(1.6) *with* $f_1 = h^\alpha$ *for* $\alpha \in (0,1]$ *as established in [24, Thm. 2.6]. Then there exists a constant* $K_1 > 0$*, which only depends on the data of the system*

(1.4)–(1.6), *such that*

$$\|\mu^\alpha\|_{L^2(0,T;V_A^r)} + \|y^\alpha\|_{H^1(0,T;V_A^{-r})\cap L^\infty(0,T;V_B^\sigma)\cap L^\infty(Q)} + \|\varphi(\alpha)h(y^\alpha)\|_{L^\infty(0,T;L^1(\Omega))}$$
$$+ \|\tau^{1/2}\partial_t y^\alpha\|_{L^2(0,T;H)} \le K_1 \quad \forall\,\alpha\in(0,1]. \tag{3.3}$$

If, in addition,

$$\tau > 0 \quad \textit{and} \quad y_0 \in V_B^{2\sigma}, \tag{3.4}$$

then we have the additional bounds

$$\|y^\alpha\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V_B^\sigma)} + \|\mu^\alpha\|_{L^\infty(0,T;V_A^{2r})}$$
$$+ \int_0^T\!\!\int_\Omega \varphi(\alpha)h''(y^\alpha)\,|\partial_t y^\alpha|^2 \le K_1 \quad \forall\,\alpha\in(0,1]. \tag{3.5}$$

Proof. To establish the validity of (3.3), we have to follow the lines of the proof of [24, Thm. 2.6]. The method of proof of [24, Thm. 2.6], specified to our situation where the convex part of the nonlinearity is given by h^α, was the following:

STEP 1: Replace in (1.5) the function $f_1 = h^\alpha$ by its Moreau–Yosida approximation h^α_λ, where $\lambda > 0$.

STEP 2: Approximate the resulting system of variational inequalities (which on the level of Moreau–Yosida approximations become variational equalities) via time discretization.

STEP 3: Show unique solvability for the discrete system and derive a priori estimates for the discrete approximations.

STEP 4: Take the time step-size to zero in the time-discrete system to establish unique solvability of the system governing the Moreau–Yosida approximations.

STEP 5: Take the limit as $\lambda \searrow 0$ to obtain the solvability of the system (1.4)–(1.6) for $f_1 = h^\alpha$.

STEP 6: Show the uniqueness of the second solution component y^α.

Now, a closer inspection reveals that in our case all of the bounds established in the a priori estimates performed in STEP 3 are uniform with respect to $\alpha\in(0,1]$, and due to the semicontinuity properties of norms, they persist under the limit processes as the step-size of the time discretization and the Moreau–Yosida parameter λ approach zero. The validity of the estimate (3.3) is thus a consequence of the estimate [24, Eq. (6.1)].

To offer to the reader a little flavor of the argument, we give a formal derivation of a part of (3.3) (which becomes rigorous on the level of the time-discrete approximation). To this end, let us assume that $\partial_t y^\alpha \in L^2(0,T;V_B^\sigma)$ (which is satisfied under the assumption (3.4)) and that the variational inequality (1.5) with $f_1 = h^\alpha$ is equivalent to the variational equation

$$(\tau\partial_t y^\alpha(t), v) + (B^\sigma y^\alpha(t), B^\sigma v) + ((h^\alpha)'(y^\alpha(t)) + f_2'(y^\alpha(t)), v) = (\mu^\alpha(t) + u(t), v)$$
$$\text{for every } v\in V_B^\sigma \text{ and a.e. } t\in(0,T). \tag{3.6}$$

The latter is certainly satisfied on the level of the Moreau–Yosida approximations to the deep quench approximations (μ^α, y^α). We then insert $v = \mu^\alpha(t)$ in (1.4) (written for $(\mu,y) = (\mu^\alpha,y^\alpha)$) and $v = \partial_t y^\alpha(t)$ in (3.6), add the resulting equations, and integrate with respect to time over $[0,t]$, where $t\in(0,T]$ is arbitrary. It then follows after an obvious cancellation of terms that

$$\tau\int_0^t\!\!\int_\Omega |\partial_t y^\alpha|^2 + \frac12\|B^\sigma y^\alpha(t)\|^2 + \int_0^t \|A^r\mu^\alpha(s)\|^2\,ds + \int_\Omega h^\alpha(y^\alpha(t))$$
$$= \frac12\|B^\sigma y_0\|^2 + \int_\Omega (h^\alpha(y_0) - f_2(y^\alpha(t)) + f_2(y_0)) + \int_0^t\!\!\int_\Omega u\,y^\alpha.$$

Now we recall (3.1) and the fact that $y_0 \in V_B^\sigma$ (cf. **(A6)**). We thus can infer from (3.2), **(A5)**, and **(A7)**, that all of the terms on the right-hand side are bounded independently of $\alpha \in (0,1]$ by a constant that depends in a continuous and monotone way on $\|u\|_{L^1(Q)}$. But this means that

$$\|A^r\mu^\alpha\|_{L^2(0,T;H)} + \|y^\alpha\|_{L^\infty(0,T;V_B^\sigma)\cap L^\infty(Q)} + \|h^\alpha(y^\alpha)\|_{L^\infty(0,T;L^1(\Omega))} \\ + \|\tau^{1/2}\partial_t y^\alpha\|_{L^2(0,T;H)} \le C,$$

where $C > 0$ is independent of $\alpha \in (0,1]$. This is already a part of the asserted bound (3.3). Now, if $\lambda_1 > 0$, then (2.4) and the above estimate immediately entail that $\{\mu^\alpha\}_{\alpha\in(0,1]}$ is bounded in $L^2(0,T;V_A^r)$, and comparison in (1.7) yields a uniform bound for $\{\partial_t y^\alpha\}_{\alpha\in(0,1]}$ in $L^2(0,T;V_A^{-r})$, which then shows that (3.3) is valid. In the case when $\lambda_1 = 0$, the boundednes of $\{\mu^\alpha\}_{\alpha\in(0,1]}$ in $L^2(0,T;V_A^r)$ is shown by proving that the mean values of $\{\mu^\alpha(t)\}_{\alpha\in(0,1]}$ are uniformly bounded in $L^2(0,T)$. For this argument, we refer the reader to the proof of [24, Thm. 2.6].

Assume now that also the condition (3.4) is fulfilled. In order to prove the bounds (3.5), we follow the proof of [24, Thm. 2.8], which again uses the time-discrete approximation scheme for the system governing the Moreau–Yosida approximations mentioned above in describing STEP 3 in the proof of [24, Thm. 2.6]. At this point, we recall the estimate (3.1). With this estimate in mind, it turns out that all of the estimates performed in the proof of [24, Thm. 2.8] on the discrete approximations yield bounds that do not depend on $\alpha \in (0,1]$ and persist under the limit processes of taking the time step-size and the Moreau–Yosida parameter λ to zero. Since (3.5) exactly reflects the bounds established there, the assertion is proved.

For the reader's convenience, we again provide a formal sketch of the argument. To this end, we formally differentiate (3.6) with respect to t, obtaining the identity

$$(\tau\partial_{tt}^2 y^\alpha, v) + (B^\sigma\partial_t y^\alpha, B^\sigma v) + (\varphi(\alpha)h''(y^\alpha)\partial_t y^\alpha + f_2''(y^\alpha)\partial_t y^\alpha, v) = (\partial_t\mu^\alpha + \partial_t u, v) \\ \text{for every } v \in V_B^\sigma \text{ and a.e. in } (0,T). \tag{3.7}$$

Then we formally test (1.4) by $v = \partial_t\mu^\alpha$ and (3.7) by $v = \partial_t y^\alpha$, and add the resulting identities. After an obvious cancellation of terms, we arrive at

$$\frac{\tau}{2}\|\partial_t y^\alpha(t)\|^2 + \frac12\|A^r\mu^\alpha(t)\|^2 + \int_0^t\!\!\int_\Omega |B^\sigma\partial_t y^\alpha|^2 + \int_0^t \varphi(\alpha)h''(y^\alpha)|\partial_t y^\alpha|^2 \\ = \frac{\tau}{2}\|\partial_t y^\alpha(0)\|^2 + \frac12\|A^r\mu^\alpha(0)\|^2 - \int_0^t f_2''(y^\alpha)|\partial_t y^\alpha|^2 + \int_0^t\!\!\int_\Omega \partial_t u\,\partial_t y^\alpha, \tag{3.8}$$

where the last summand on the left-hand side is nonnegative and the last two terms on the right-hand side can be estimated by an expression of the form

$$C_1\int_0^t\!\!\int_\Omega \left(|\partial_t u|^2 + |\partial_t y^\alpha|^2\right),$$

where $C_1 > 0$ is independent of $\alpha \in (0,1]$. We thus are left to estimate the initial value terms. To this end, we formally write (1.4) and (3.6) for $t = 0$, obtaining the identities

$$\langle\partial_t y^\alpha(0), v\rangle_{A,r} + (A^r\mu^\alpha(0), A^r v) = 0 \quad \forall\, v \in V_A^r, \tag{3.9}$$

$$(\tau\partial_t y^\alpha(0), v) + (B^{2\sigma}y_0 + (h^\alpha)'(y_0) + f_2'(y_0), v) = (\mu^\alpha(0) + u(0), v) \quad \forall\, v \in V_B^\sigma. \tag{3.10}$$

Now observe that, by virtue of (3.4), (3.1), and **(A5)**, the sum $B^{2\sigma}y_0 + (h^\alpha)'(y_0) + f_2'(y_0)$ is bounded in $L^2(\Omega)$, uniformly with respect to $\alpha \in (0,1]$. Hence, if we (formally) test (3.9) by $\mu^\alpha(0)$ and (3.10) by $\partial_t y^\alpha(0)$, add the resulting identities, and apply Young's inequality (note that we have $\tau > 0$ by assumption (3.4)), then we arrive at an estimate of the form

$$\|A^r \mu^\alpha(0)\|^2 + \frac{\tau}{2}\,\|\partial_t y^\alpha(0)\|^2 \le C_2\,\tau^{-1}(1 + \|u(0)\|^2),$$

where $C_2 > 0$ is independent of $\alpha \in (0,1]$. We may then combine this estimate with (3.8) to conclude from Gronwall's lemma that

$$\|\partial_t y^\alpha\|_{L^\infty(0,T;H)\cap L^2(0,T;V_B^\sigma)} + \|A^r\mu^\alpha\|_{L^\infty(0,T;H)} + \int_0^T\!\!\int_\Omega \varphi(\alpha)h''(y^\alpha)\,|\partial_t y^\alpha|^2 \le C_3, \tag{3.11}$$

where $C_3 > 0$ is independent of $\alpha \in (0,1]$. With this, the first and third summands on the left of (3.5) are uniformly bounded, which then, by comparison in (1.7), also holds true for $\|A^{2r}\mu^\alpha\|_{L^\infty(0,T;H)}$. Hence, (3.5) is proved if $\lambda_1 > 0$. In the case $\lambda_1 = 0$, it is necessary to derive a uniform $L^\infty(0,T)$ bound for the mean values of $\{\mu^\alpha(t)\}_{\alpha\in(0,1]}$. About this, we again refer the reader to the proof of [24, Thm. 2.8]. □

Remark 3.2. A closer inspection of the a priori estimates for the time-discretized systems mentioned above reveals that the constant K_1 depends in a monotone and continuous way on the norm $\|u\|_{\mathcal{X}}$. Hence, for any bounded subset $\mathcal{U}$ of $\mathcal{X}$ (in particular, for $\mathcal{U} = \mathcal{U}_{\rm ad}$) it follows that there is a constant, which is again denoted by K_1, such that the estimates (3.3) and (3.5), respectively, hold true whenever u is an arbitrary element of $\mathcal{U}$.

Next, we show the convergence of the deep quench approximations. Before formulating the result, we notice that the following control-to-state operators are well defined on the space $\mathcal{X}$:

$$\mathcal{S}_0 : \mathcal{X} \ni u \mapsto \mathcal{S}_0(u) := y, \tag{3.12}$$
$$\mathcal{S}_\alpha : \mathcal{X} \ni u \mapsto \mathcal{S}_\alpha(u) := y^\alpha, \tag{3.13}$$

where (μ, y) and (μ^α, y^α) denote solutions to the systems (1.4)–(1.6) for $f_1 = I_{[-1,1]}$ and $f_1 = h^\alpha$, $\alpha \in (0,1]$, respectively, as established in [24, Thm. 2.6]. We have the following result.

Theorem 3.3. *Suppose that the assumptions* **(A1)**–**(A7)** *are fulfilled, and let sequences* $\{\alpha_n\} \subset (0,1]$ *and* $\{u^{\alpha_n}\} \subset \mathcal{X}$ *be given such that* $\alpha_n \searrow 0$ *and* $u^{\alpha_n} \to u$ *weakly-star in* $\mathcal{X}$ *as* $n \to \infty$ *for some* $u \in \mathcal{X}$. *Moreover, let* $(\mu^{\alpha_n}, y^{\alpha_n})$ *be solutions to* (1.4)–(1.6) *for* $f_1 = h^{\alpha_n}$ *and* $u = u_n$, $n \in \mathbb{N}$, *as established in [24, Thm. 2.6]. Then there are a solution* (μ, y) *with* $y = \mathcal{S}_0(u)$ *to the problem* (1.4)–(1.6) *with* $f_1 = I_{[-1,1]}$ *and a subsequence* $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ *of* $\{\alpha_n\}$ *such that, as* $k \to \infty$,

$$\mu^{\alpha_{n_k}} \to \mu \quad \textit{weakly in } L^2(0,T;V_A^r), \tag{3.14}$$
$$y^{\alpha_{n_k}} \to y \quad \textit{weakly-star in } H^1(0,T;V_A^{-r}) \cap L^\infty(0,T;V_B^\sigma)$$
$$\textit{and strongly in } C^0([0,T];H), \tag{3.15}$$
$$\partial_t y^{\alpha_{n_k}} \to \partial_t y \quad \textit{weakly in } L^2(0,T;H) \quad \textit{if } \tau > 0. \tag{3.16}$$

Moreover, if (3.4) *is fulfilled, then the above solution* (μ, y) *also satisfies*

$$\mu^{\alpha_{n_k}} \to \mu \quad \textit{weakly-star in } L^\infty(0,T;V_A^{2r}), \tag{3.17}$$
$$y^{\alpha_{n_k}} \to y \quad \textit{weakly-star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V_B^\sigma). \tag{3.18}$$

Proof. The sequence $\{u^{\alpha_n}\}$ converges weakly-star in $\mathcal{X}$ and thus forms a bounded subset of $\mathcal{X}$. According to Remark 3.2, the bounds (3.3) and (3.5) (the latter if (3.4) is satisfied) apply, where the constant K_1 is independent of n. Therefore, there are limits $(\bar\mu, \bar y)$ and a subsequence of $\{(\mu^{\alpha_n}, y^{\alpha_n})\}$, which is for convenience again indexed by n, such that, as $n \to \infty$,

$$\mu^{\alpha_n} \to \bar\mu \quad \text{weakly in } L^2(0,T;V_A^r), \tag{3.19}$$
$$y^{\alpha_n} \to \bar y \quad \text{weakly-star in } H^1(0,T;V_A^{-r}) \cap L^\infty(0,T;V_B^\sigma), \tag{3.20}$$
$$y^{\alpha_n} \to \bar y \quad \text{strongly in } C^0([0,T];H) \text{ and pointwise a.e. in } Q, \tag{3.21}$$
$$\partial_t y^{\alpha_n} \to \partial_t \bar y \quad \text{weakly in } L^2(0,T;H) \quad \text{if } \tau > 0, \tag{3.22}$$

and, if (3.4) is satisfied,

$$\mu^{\alpha_n} \to \bar\mu \quad \text{weakly-star in } L^\infty(0,T;V_A^{2r}), \tag{3.23}$$
$$y^{\alpha_n} \to \bar y \quad \text{weakly-star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V_B^\sigma). \tag{3.24}$$

Notice that the strong convergence result in (3.21) follows from [44, Sect. 8, Cor. 4], since, by (2.15), V_B^σ is compactly emdedded in H; we thus may without loss of generality assume that $y^{\alpha_n} \to \bar y$ pointwise a.e. in Q. Since, by virtue of (3.2), $-1 \le y^{\alpha_n} \le +1$ a.e. in Q, we infer that $-1 \le \bar y \le 1$ a.e. in Q, and thus $I_{[-1,1]}(\bar y) \in L^1(Q)$ with

$$\int_0^T \int_\Omega I_{[-1,1]}(\bar y) = 0.$$

It remains to show that $(\bar\mu, \bar y)$ is a solution to (1.4)–(1.6) in the sense of [24, Thm. 2.6] for $f_1 = I_{[-1,1]}$ and control u. To this end, we pass to the limit as $n \to \infty$ in the system (1.4)–(1.6), written for $f_1 = h^{\alpha_n}$ and $u = u_n$, for $n \in \mathbb{N}$. We immediately see that $\bar y(0) = y_0$ and that (1.4) holds true for $(\bar\mu, \bar y)$. Also, the Lipschitz continuity of f_2' and (3.21) imply that $f_2'(y^{\alpha_n}) \to f_2'(\bar y)$ strongly in $C^0([0,T];H)$. Now, recall that $B^\sigma y^{\alpha_n} \to B^\sigma \bar y$ weakly in $L^2(0,T;H)$, by virtue of (3.20). We thus have, by lower semicontinuity,

$$\begin{aligned}
&\int_0^T (B^\sigma \bar y(t), B^\sigma(\bar y(t) - v(t)))\, dt \\
&\le \liminf_{n\to\infty} \int_0^T \big(B^\sigma y^{\alpha_n}(t), B^\sigma y^{\alpha_n}(t)\big)\, dt - \lim_{n\to\infty} \int_0^T \big(B^\sigma y^{\alpha_n}(t), B^\sigma v(t)\big)\, dt \\
&= \liminf_{n\to\infty} \int_0^T \big(B^\sigma y^{\alpha_n}(t), B^\sigma(y^{\alpha_n}(t) - v(t))\big)\, dt
\end{aligned}$$

for every $v \in L^2(0,T;V_B^\sigma)$. In conclusion, owing to (1.21) as well, we have that

$$\begin{aligned}
&\int_Q I_{[-1,1]}(\bar y) + \int_0^T \big(B^\sigma \bar y(t), B^\sigma(\bar y(t) - v(t))\big)\, dt = \int_0^T \big(B^\sigma \bar y(t), B^\sigma(\bar y(t) - v(t))\big)\, dt \\
&\le \liminf_{n\to\infty} \int_Q h^{\alpha_n}(y^{\alpha_n}) + \liminf_{n\to\infty} \int_0^T \big(B^\sigma y^{\alpha_n}(t), B^\sigma(y^{\alpha_n}(t) - v(t))\big)\, dt \\
&\le \liminf_{n\to\infty} \Big(\int_Q h^{\alpha_n}(y^{\alpha_n}) + \int_0^T \big(B^\sigma y^{\alpha_n}(t), B^\sigma(y^{\alpha_n}(t) - v(t))\big)\, dt\Big) \\
&\le \lim_{n\to\infty} \Big(\int_0^T \big(-\tau\partial_t y^{\alpha_n}(t) - f_2'(y^{\alpha_n}(t)) + u(t) + \mu^{\alpha_n}(t), y^{\alpha_n}(t) - v(t)\big)\, dt + \int_Q h^{\alpha_n}(v)\Big) \\
&= \int_0^T \big(-\tau\partial_t \bar y(t) - f_2'(\bar y(t)) + u(t) + \bar\mu(t), \bar y(t) - v(t)\big)\, dt + \int_Q I_{[-1,1]}(v),
\end{aligned}$$

for all $v \in L^2(0,T;V_B^\sigma)$. Thus the time-integrated version of (1.5), with time-dependent test functions, holds true. Since this version is equivalent to (1.5), we see that $(\bar\mu, \bar y)$ is indeed a solution in the sense of [24, Thm. 2.6] to (1.4)–(1.6) for $f_1 = I_{[-1,1]}$. The assertion is thus proved. □

Remark 3.4. According to [24, Thm. 2.6], the second solution component y and the expression $A^r\mu$ are uniquely determined. This entails that $\bar y = \mathcal{S}_0(u)$ and that the convergence properties (3.20), (3.22) and (3.24) are valid for the entire sequence $\{\alpha_n\}$ and not only for a subsequence. In addition, we can infer from (3.19) that $A^r\mu^{\alpha_n} \to A^r\bar\mu$ weakly in $L^2(0,T;H)$ as $n \to \infty$. If $\lambda_1 > 0$, then even μ^{α_n} converges to $\bar\mu$ weakly in $L^2(0,T;V_A^r)$.

In the following theorem, we prove a quantitative estimate that yields information on the order of convergence as $\alpha \searrow 0$ in a very special (but important) situation. To this end, we need further assumptions that will also be needed in the derivation of first-order necessary optimality conditions in Section 5.

Theorem 3.5. *Suppose that in addition to* **(A1)–(A6)** *the following assumptions are fulfilled:*

(A8) *The condition* (3.4) *is satisfied.*

(A9) $B = -\Delta$ *with the domain* $D(B) = \{v \in H^2(\Omega) : \partial_{\mathbf{n}} v = 0$ *on* $\Gamma\}$, $\sigma = \frac{1}{2}$, *and* $V_A^{2r} \subset L^\infty(\Omega)$.

Moreover, assume that $u^{\alpha_1}, u^{\alpha_2} \in \mathcal{X}$ *are given, where* $0 < \alpha_1 < \alpha_2 < 1$, *and that* $(\mu^{\alpha_i}, y^{\alpha_i})$ *are solutions to* (1.4)–(1.6) *for* $f_1 = h^{\alpha_i}$ *and* $u = u_i$ *in the sense of [24, Thm. 2.6], for* $i = 1, 2$. *Then there is a constant* $K_2 > 0$, *which depends only on the data of the problem, such that it holds, for all* $t \in (0,T]$,

$$\|y^{\alpha_1} - y^{\alpha_2}\|_{C^0([0,t];L^2(\Omega))\cap L^2(0,t;H^1(\Omega))} + \left\| \int_0^{\bullet} A^r(\mu^{\alpha_1} - \mu^{\alpha_2})(s)ds \right\|_{C^0([0,t];L^2(\Omega))}$$
$$\le K_2 \left(|\alpha_1 - \alpha_2|^{1/2} + \|u^{\alpha_1} - u^{\alpha_2}\|_{L^2(0,t;H)} \right). \tag{3.25}$$

Proof. We first observe that in [25, Example 1] it has been shown that a uniform separation property is valid for the solutions to (1.4)–(1.6) with $f_1 = h^\alpha$ under the assumptions **(A1)–(A9)**; that is, there are constants $r_*, r^* \in (-1,1)$ (depending on α) such that

$$r_* \le y^\alpha \le r^* \quad \text{a.e. in } Q. \tag{3.26}$$

Moreover, we have $V_{-\Delta}^{1/2} = H^1(\Omega)$, and thus we can infer from [25, Remarks 3.4, 3.5 and 3.6] that for any $\alpha > 0$ the solution (μ^α, y^α) to (1.4)–(1.6) in the sense of [24, Thm. 2.6] for $f_1 = h^\alpha$ is in fact uniquely determined and satisfies the variational equality (which in this special case turns out to be equivalent to (1.5))

$$(\tau\partial_t y^\alpha(t), v) + (\nabla y^\alpha(t), \nabla v) + ((h^\alpha)'(y^\alpha(t)), v) + (f_2'(y^\alpha(t)), v) = (\mu^\alpha(t) + u(t), v)$$
$$\text{for a.e. } t \in (0,T) \text{ and every } v \in H^1(\Omega). \tag{3.27}$$

Now, let $u := u^{\alpha_1} - u^{\alpha_2}$, $\mu := \mu^{\alpha_1} - \mu^{\alpha_2}$, and $y := y^{\alpha_1} - y^{\alpha_2}$. Then, taking the difference in (1.4) for the two different cases $\alpha = \alpha_1$, $\alpha = \alpha_2$, and integrating the resulting equality over $[0,t]$ with respect to time, where $t \in (0,T]$, we obtain the identity

$$\langle y(t), v\rangle_{A,r} + \left(A^r \textstyle\int_0^t \mu(s)ds, A^r v \right) = 0 \quad \text{for all } t \in (0,T] \text{ and } v \in V_A^r.$$

Testing this identity by $v = \mu(t)$, and noting that $\langle y(t), \mu(t)\rangle_{A,r} = (y(t), \mu(t))$ for almost every $t \in (0,T)$, we thus obtain that

$$\int_0^t\!\!\int_\Omega y\mu \,=\, -\int_0^t \Big(A^r\mu(s), \int_0^s A^r\mu(\rho)d\rho\Big)ds \,=\, -\frac{1}{2}\Big\|\int_0^t A^r\mu(s)ds\Big\|^2 . \tag{3.28}$$

Next, we insert $v = -y$ in the variational equality (3.27) for $\alpha = \alpha_2$, and $v = y$ in (3.27) for $\alpha = \alpha_1$. Summation of the resulting identities then yields the equality

$$\begin{aligned}&\frac{\tau}{2}\,\|y(t)\|^2 \,+\, \int_0^t \|\nabla y(s)\|^2\,ds \,+\, \int_0^t\!\!\int_\Omega \varphi(\alpha_1)(h'(y^{\alpha_1}) - h'(y^{\alpha_2}))(y^{\alpha_1} - y^{\alpha_2})\\ &= -\int_0^t\!\!\int_\Omega (\varphi(\alpha_1) - \varphi(\alpha_2))h'(y^{\alpha_2})(y^{\alpha_1} - y^{\alpha_2}) \,-\, \int_0^t\!\!\int_\Omega (f_2'(y^{\alpha_1}) - f_2'(y^{\alpha_2}))\,y\\ &\quad+\, \int_0^t\!\!\int_\Omega y\mu \,+\, \int_0^t\!\!\int_\Omega u\,y. \end{aligned} \tag{3.29}$$

Owing to the monotonicity of h', the third summand on the left-hand side of (3.29) is nonnegative. Moreover, $h'(y^{\alpha_2})(y^{\alpha_1} - y^{\alpha_2}) \le h(y^{\alpha_1}) - h(y^{\alpha_2})$ almost everywhere in Q, since $h \in C^1(-1,1)$ is convex, and $\varphi(\alpha_1) < \varphi(\alpha_2)$. So the first summand on the right-hand side of (3.29), which we denote by I, satisfies

$$I \,\le\, (\varphi(\alpha_2) - \varphi(\alpha_1)) \int_0^t\!\!\int_\Omega (|h(y^{\alpha_1})| \,+\, |h(y^{\alpha_2})|) \,\le\, C_1\,(\alpha_2 - \alpha_1)\,, \tag{3.30}$$

with $C_1 := 4\ln(2)\,|\Omega|\,T\,\|\varphi'\|_{C^0([0,1])}$, where $|\Omega|$ denotes the volume of Ω. Therefore, adding (3.28) and (3.29), and using the Lipschitz continuity of f_2', we obtain from Young's inequality an estimate of the form

$$\begin{aligned}&\frac{\tau}{2}\,\|y(t)\|^2 \,+\, \int_0^t \|\nabla y(s)\|^2\,ds \,+\, \frac{1}{2}\Big\|\int_0^t A^r\mu(s)ds\Big\|^2\\ &\le\, C_1\,|\alpha_1 - \alpha_2| \,+\, (L+1)\int_0^t\!\!\int_\Omega |y|^2 \,+\, \frac{1}{4}\int_0^t\!\!\int_\Omega |u|^2\,, \end{aligned} \tag{3.31}$$

and (3.25) follows from Gronwall's lemma. □

Corollary 3.6. *Suppose that* **(A1)–(A9)** *are fulfilled and that* $\alpha \in (0,1]$. *Moreover, let* $y = \mathcal{S}_0(u)$ *and* $y^\alpha = \mathcal{S}_\alpha(u)$. *Then*

$$\|y^\alpha - y\|_{C^0([0,t];L^2(\Omega))\cap L^2(0,t;H^1(\Omega))} \,\le\, K_2\,|\alpha|^{1/2}\,. \tag{3.32}$$

Proof. We apply (3.25) with $\alpha_1 = \alpha$, $\alpha_2 = \alpha_n$, where $\alpha_n \searrow 0$, and $u^{\alpha_1} = u^{\alpha_2} = u$, which with $y^{\alpha_n} = \mathcal{S}_{\alpha_n}(u)$ yields the estimate

$$\begin{aligned}&\|y^\alpha - y^{\alpha_n}\|_{C^0([0,t];L^2(\Omega))\cap L^2(0,t;H^1(\Omega))} \,+\, \Big\|\int_0^{\bullet} A^r(\mu^\alpha - \mu^{\alpha_n})(s)ds\Big\|_{C^0([0,t];L^2(\Omega))}\\ &\le\, K_2\,|\alpha - \alpha_n|^{1/2}\,.\end{aligned}$$

The assertion now follows from (3.15) in Theorem 3.2 by taking the limit as $n \to \infty$, invoking Remark 3.4 and the semicontinuity of norms. □

4 Existence and approximation of optimal controls

Beginning with this section, we investigate the optimal control problem ($\mathcal{CP}_0$) of minimizing the cost functional (1.16) over the admissible set $\mathcal{U}_{\rm ad}$ subject to state system (1.4)–(1.6) where $f_1 = I_{[-1,1]}$. In comparison with ($\mathcal{CP}_0$), we consider for $\alpha > 0$ the following control problem:

($\mathcal{CP}_\alpha$) Minimize $\mathcal{J}(y,u)$ for $u \in \mathcal{U}_{\rm ad}$, subject to the condition that $y = \mathcal{S}_\alpha(u)$ for some solution (μ, y) to the state system (1.4)–(1.6) with $f_1 = h^\alpha$, in the sense of [24, Thm. 2.6].

We expect that the minimizers of ($\mathcal{CP}_\alpha$) are for $\alpha \searrow 0$ related to minimizers of ($\mathcal{CP}_0$). Prior to giving an affirmative answer to this conjecture, we first show an existence result for ($\mathcal{CP}_\alpha$).

Proposition 4.1. *Suppose that* **(A1)**–**(A6)** *are satisfied. Then* $(\mathcal{CP}_\alpha)$ *has for every* $\alpha > 0$ *a solution.*

Proof. Let $\alpha > 0$ be fixed, and assume that a minimizing sequence $\{(y_n, u_n)\}$ for ($\mathcal{CP}_\alpha$) is given, where $y_n = \mathcal{S}_\alpha(u_n)$ for some solution pair (μ_n, y_n) to the state system with $u = u_n \in \mathcal{U}_{\rm ad}$ and $f_1 = h^\alpha$, for $n \in \mathbb{N}$. Then it holds for every $n \in \mathbb{N}$ that

$$\langle \partial_t y_n(t), v\rangle_{A,r} + (A^r \mu_n(t), A^r v) = 0 \quad \text{for a.e. } t \in (0,T) \text{ and every } v \in V_A^r, \tag{4.1}$$

$$\begin{aligned}&(\tau \partial_t y_n(t), y_n(t) - v) + (B^\sigma y_n(t), B^\sigma(y_n(t) - v)) + h^\alpha(y_n(t))\\ &\quad \le (\mu_n(t) + u_n(t) - f_2'(y_n(t)), y_n(t) - v) + h^\alpha(v)\\ &\text{for a.e. } t \in (0,T) \text{ and every } v \in V_B^\sigma,\end{aligned} \tag{4.2}$$

$$y_n(0) = y_0. \tag{4.3}$$

Taking estimate (3.3) into account, we may without loss of generality assume that there are $\bar u \in \mathcal{U}_{\rm ad}$ and $(\bar\mu, \bar y)$ such that

$$u_n \to \bar u \quad \text{weakly-star in } \mathcal{X}, \tag{4.4}$$

$$\mu_n \to \bar\mu \quad \text{weakly in } L^2(0,T;V_A^r), \tag{4.5}$$

$$\begin{aligned}y_n \to \bar y \quad &\text{weakly-star in } H^1(0,T;V_A^{-r}) \cap L^\infty(0,T;V_B^\sigma),\\ &\text{strongly in } C^0([0,T];H), \text{ and pointwise a.e. in } Q.\end{aligned} \tag{4.6}$$

Then also $f_2'(y_n) \to f_2'(\bar y)$ strongly in $C^0([0,T];H)$. Moreover, it holds

$$\int_0^T\!\!\int_\Omega h^\alpha(y_n) \le K_1 \quad \text{for every } n \in \mathbb{N}. \tag{4.7}$$

Therefore, we have $y_n \in [-1,1]$ almost everywhere in Q, and since h^α is continuous in $[-1,1]$, it follows that $h^\alpha(y_n) \to h^\alpha(\bar y)$ pointwise almost everywhere in Q. Lebesgue's dominated convergence theorem then yields that

$$\int_0^T\!\!\int_\Omega h^\alpha(y_n) \to \int_0^T\!\!\int_\Omega h^\alpha(\bar y)\,.$$

In addition, by lower semicontinuity, we have that

$$\int_0^T (B^\sigma \bar y(t)), B^\sigma \bar y(t))\,dt \le \liminf_{n\to\infty} \int_0^T (B^\sigma y_n(t), B^\sigma y_n(t))\,dt\,.$$

Combining the convergence results shown above, we obtain by passage to the limit as $n \to \infty$ that

$$\int_0^T \langle \partial_t \bar{y}(t), v(t) \rangle_{A,r}\, dt \,+\, \int_0^T (A^r \bar{\mu}(t), A^r v(t))\, dt \,=\, 0 \quad \forall\, v \in L^2(0,T;V_A^r), \tag{4.8}$$

$$\int_0^T (\tau \partial_t \bar{y}(t), \bar{y}(t) - v(t))\, dt \,+\, \int_0^T (B^\sigma \bar{y}(t), B^\sigma(\bar{y}(t) - v(t)))\, dt \,+\, \int_0^T\!\!\int_\Omega h^\alpha(\bar{y})$$
$$\le \int_0^T (\bar{\mu}(t) + \bar{u}(t) - f_2'(\bar{y}(t)), \bar{y}(t) - v(t))\, dt \,+\, \int_0^T\!\!\int_\Omega h^\alpha(v)$$
for every $v \in L^2(0,T;V_B^\sigma)$, (4.9)

$$\bar{y}(0) = y_0. \tag{4.10}$$

Apparently, (4.8)–(4.9) is just the time-integrated version of (1.4)–(1.5) for $u = \bar{u}$ and $f_1 = h^\alpha$, written with time-dependent test functions, which is equivalent to (1.4)–(1.5). Hence, $(\bar{\mu}, \bar{y})$ solves (1.4)–(1.5) for $u = \bar{u}$ and $f_1 = h^\alpha$ in the sense of [24, Thm. 2.6]. In particular, we have $\bar{y} = \mathcal{S}_\alpha(\bar{u})$. But this means that $(\bar{y}, \bar{u})$ is admissible for ($\mathcal{CP}_\alpha$). From the semicontinuity properties of the cost functional (1.16) it then follows that $(\bar{y}, \bar{u})$ is an optimal pair, which concludes the proof of the assertion. □

Proposition 4.2. *Let* **(A1)–(A6)** *be fulfilled, and suppose that sequences* $\{\alpha_n\} \subset (0,1]$ *and* $\{u_n\} \subset \mathcal{U}_{\rm ad}$ *are given such that* $\alpha_n \searrow 0$ *and* $u_n \to u$ *weakly-star in* $\mathcal{X}$ *for some* $u \in \mathcal{U}_{\rm ad}$. *Then, with the solution operators defined in* (3.12) *and* (3.13)*,*

$$\mathcal{J}(\mathcal{S}_0(u), u) \;\le\; \liminf_{n\to\infty}\, \mathcal{J}(\mathcal{S}_{\alpha_n}(u^{\alpha_n}), u^{\alpha_n}), \tag{4.11}$$

$$\mathcal{J}(\mathcal{S}_0(v), v) \;=\; \lim_{n\to\infty}\, \mathcal{J}(\mathcal{S}_{\alpha_n}(v), v) \quad \forall\, v \in \mathcal{U}_{\rm ad}. \tag{4.12}$$

Proof. Under the given assumptions, we may apply (3.15) in Theorem 3.3 and Remark 3.4 to infer that $\mathcal{S}_{\alpha_n}(u^{\alpha_n}) \to \mathcal{S}_0(u)$ strongly in $C^0([0,T];H)$. The validity of (4.11) is then a direct consequence of the weak and weak-star sequential semicontinuity properties of the cost functional (1.16). Now suppose that $v \in \mathcal{U}_{\rm ad}$ is arbitrarily chosen, and put $y^{\alpha_n} := \mathcal{S}_{\alpha_n}(v)$ for all $n \in \mathbb{N}$. Then, again by Theorem 3.3 and Remark 3.4, $y^{\alpha_n} \to \mathcal{S}_0(v)$ strongly in $C^0([0,T];H)$. Next, observe that the first two summands of the cost functional are obviously continuous with respect to the strong topology of $C^0([0,T];H)$, which then shows the validity of (4.12). □

We are now in a position to prove the existence of minimizers for the problem ($\mathcal{CP}_0$). We have the following result.

Corollary 4.3. *Under the assumptions of Proposition 4.2, the optimal control problem* ($\mathcal{CP}_0$) *has at least one solution.*

Proof. Pick an arbitrary sequence $\{\alpha_n\} \subset (0,1]$ such that $\alpha_n \searrow 0$ as $n \to \infty$. By virtue of Proposition 4.1, the optimal control problem ($\mathcal{CP}_{\alpha_n}$) has for every $n \in \mathbb{N}$ a solution $(y^{\alpha_n}, u^{\alpha_n})$, where $y^{\alpha_n} = \mathcal{S}_{\alpha_n}(u^{\alpha_n})$ for a solution $(\mu^{\alpha_n}, y^{\alpha_n})$ to the corresponding state system. Since $\mathcal{U}_{\rm ad}$ is bounded in $\mathcal{X}$, we may without loss of generality assume that $u^{\alpha_n} \to u$ weakly-star in $\mathcal{X}$ for some $u \in \mathcal{U}_{\rm ad}$. At this point, we apply Theorem 3.3 to the present situation. We then infer that the convergence results (3.14) and (3.15) hold true for some subsequence $\{\alpha_{n_k}\}$ with a pair (μ, y) satisfying $y = \mathcal{S}_0(u)$. Invoking the optimality of $(y^{\alpha_n}, u^{\alpha_n})$ for ($\mathcal{CP}_{\alpha_n}$), we then find for every $v \in \mathcal{U}_{\rm ad}$ the chain of (in)equalities

$$\mathcal{J}(y,u) \;=\; \mathcal{J}(\mathcal{S}_0(u), u) \;\le\; \liminf_{k\to\infty}\, \mathcal{J}(\mathcal{S}_{\alpha_{n_k}}(u^{\alpha_{n_k}}), u^{\alpha_{n_k}})$$
$$\le\; \liminf_{k\to\infty}\, \mathcal{J}(\mathcal{S}_{\alpha_{n_k}}(v), v) \;\le\; \mathcal{J}(\mathcal{S}_0(v), v), \tag{4.13}$$

which yields that (y, u) is an optimal pair for $(\mathcal{CP}_0)$. The assertion is thus proved. $\square$

Theorem 3.3 and the proof of Corollary 4.3 indicate that optimal controls of $(\mathcal{CP}_\alpha)$ are "close" to optimal controls of $(\mathcal{CP}_0)$. However, they do not yield any information on whether every optimal control of $(\mathcal{CP}_0)$ can be approximated in this way. In fact, such a global result cannot be expected to hold true. However, a local answer can be given. For this purpose, we employ a trick introduced in [10]. To this end, let $\bar{u} \in \mathcal{U}_{\rm ad}$ be an optimal control for $(\mathcal{CP}_0)$ with the associated state $(\bar{\mu}, \bar{y})$ where $\bar{y} = \mathcal{S}_0(\bar{u})$. We associate with this optimal control the *adapted cost functional*

$$\widetilde{\mathcal{J}}(y, u) := \mathcal{J}(y, u) \, + \, \frac{1}{2} \, \|u - \bar{u}\|^2_{L^2(Q)} \tag{4.14}$$

and a corresponding *adapted optimal control problem* for $\alpha > 0$, namely:

$(\widetilde{\mathcal{CP}}_\alpha)$ Minimize $\widetilde{\mathcal{J}}(y, u)$ for $u \in \mathcal{U}_{\rm ad}$, subject to the condition that $y = \mathcal{S}_\alpha(u)$ for some solution (μ, y) to the state system (1.4)–(1.6) with $f_1 = h^\alpha$ in the sense of [24, Thm. 2.6].

With essentially the same proof as that of Proposition 4.1 (which needs no repetition here), we can show the following result.

Lemma 4.4. *Suppose that the assumptions* **(A1)–(A6)** *are fulfilled. Then the optimal control problem* $(\widetilde{\mathcal{CP}}_\alpha)$ *has for every* $\alpha > 0$ *at least one solution.*

We are now in the position to give a partial answer to the question raised above. We have the following result.

Theorem 4.5. *Let the assumptions* **(A1)–(A6)** *be fulfilled, suppose that* $\bar{u} \in \mathcal{U}_{\rm ad}$ *is an arbitrary optimal control of* $(\mathcal{CP}_0)$ *with associated state* $(\bar{\mu}, \bar{y})$ *where* $\bar{y} = \mathcal{S}_0(\bar{u})$*, and let* $\{\alpha_n\} \subset (0, 1]$ *be any sequence such that* $\alpha_n \searrow 0$ *as* $n \to \infty$*. Then there exist a subsequence* $\{\alpha_{n_k}\}$ *of* $\{\alpha_n\}$*, and, for every* $k \in \mathbb{N}$*, an optimal control* $u^{\alpha_{n_k}} \in \mathcal{U}_{\rm ad}$ *of the adapted problem* $(\widetilde{\mathcal{CP}}_{\alpha_{n_k}})$ *with associated state* $(\mu^{\alpha_{n_k}}, y^{\alpha_{n_k}})$*, where* $y^{\alpha_{n_k}} = \mathcal{S}_{\alpha_{n_k}}(u^{\alpha_{n_k}})$*, such that, as* $k \to \infty$*,*

$$u^{\alpha_{n_k}} \to \bar{u} \quad \text{strongly in } L^2(Q), \tag{4.15}$$

and such that the property (3.15) *is satisfied with* y *replaced by* $\bar{y}$*. Moreover, we have*

$$\lim_{k \to \infty} \widetilde{\mathcal{J}}(y^{\alpha_{n_k}}, u^{\alpha_{n_k}}) \, = \, \mathcal{J}(\bar{y}, \bar{u}) \, . \tag{4.16}$$

Proof. Let $\alpha_n \searrow 0$ as $n \to \infty$. For any $n \in \mathbb{N}$, we pick an optimal control $u^{\alpha_n} \in \mathcal{U}_{\rm ad}$ for the adapted problem $(\widetilde{\mathcal{CP}}_{\alpha_n})$ and denote by $(\mu^{\alpha_n}, y^{\alpha_n})$, where $y^{\alpha_n} = \mathcal{S}_{\alpha_n}(u^{\alpha_n})$, an associated solution to (1.4)–(1.6) with $f_1 = h^{\alpha_n}$ and $u = u^{\alpha_n}$. By the boundedness of $\mathcal{U}_{\rm ad}$ in $\mathcal{X}$, there is some subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that

$$u^{\alpha_{n_k}} \to u \quad \text{weakly-star in } \mathcal{X} \quad \text{as } k \to \infty, \tag{4.17}$$

with some $u \in \mathcal{U}_{\rm ad}$. Thanks to Theorem 3.3, the convergence properties (3.14)–(3.15) hold true with some pair (μ, y) satisfying $y = \mathcal{S}_0(u)$. In particular, the pair (y, u) is admissible for $(\mathcal{CP}_0)$.

We now aim to prove that $u = \bar{u}$. Once this is shown, it follows from the uniqueness of the second solution component to the state system (1.4)–(1.6) that also $y = \bar{y}$, which implies that (3.15) holds true with y replaced by $\bar{y}$.

Now observe that, owing to the weak sequential lower semicontinuity of $\widetilde{\mathcal{J}}$, and in view of the optimality property of $(\bar{y}, \bar{u})$ for problem $(\mathcal{CP}_0)$,

$$\liminf_{k\to\infty} \widetilde{\mathcal{J}}(y^{\alpha_{n_k}}, u^{\alpha_{n_k}}) \ge \mathcal{J}(y,u) + \frac{1}{2}\|u-\bar{u}\|^2_{L^2(Q)}$$
$$\ge \mathcal{J}(\bar{y},\bar{u}) + \frac{1}{2}\|u-\bar{u}\|^2_{L^2(Q)} . \tag{4.18}$$

On the other hand, the optimality property of $(y^{\alpha_{n_k}}, u^{\alpha_{n_k}})$ for problem $(\widetilde{\mathcal{CP}}_{\alpha_{n_k}})$ yields that for any $k \in \mathbb{N}$ we have

$$\widetilde{\mathcal{J}}(y^{\alpha_{n_k}}, u^{\alpha_{n_k}}) = \widetilde{\mathcal{J}}(\mathcal{S}_{\alpha_{n_k}}(u^{\alpha_{n_k}}), u^{\alpha_{n_k}}) \le \widetilde{\mathcal{J}}(\mathcal{S}_{\alpha_{n_k}}(\bar{u}), \bar{u}) , \tag{4.19}$$

whence, taking the limit superior as $k \to \infty$ on both sides and invoking (4.12) in Proposition 4.2,

$$\limsup_{k\to\infty} \widetilde{\mathcal{J}}(y^{\alpha_{n_k}}, u^{\alpha_{n_k}}) \le \widetilde{\mathcal{J}}(\mathcal{S}_0(\bar{u}), \bar{u}) = \widetilde{\mathcal{J}}(\bar{y}, \bar{u}) = \mathcal{J}(\bar{y}, \bar{u}) . \tag{4.20}$$

Combining (4.18) with (4.20), we have thus shown that $\frac{1}{2}\|u-\bar{u}\|^2_{L^2(Q)} = 0$, so that $u = \bar{u}$ and thus also $y = \bar{y}$. Moreover, (4.18) and (4.20) also imply that

$$\mathcal{J}(\bar{y},\bar{u}) = \widetilde{\mathcal{J}}(\bar{y},\bar{u}) = \liminf_{k\to\infty} \widetilde{\mathcal{J}}(y^{\alpha_{n_k}}, u^{\alpha_{n_k}})$$
$$= \limsup_{k\to\infty} \widetilde{\mathcal{J}}(y^{\alpha_{n_k}}, u^{\alpha_{n_k}}) = \lim_{k\to\infty} \widetilde{\mathcal{J}}(y^{\alpha_{n_k}}, u^{\alpha_{n_k}}) , \tag{4.21}$$

which proves (4.16) and, at the same time, also (4.15). This concludes the proof of the assertion. □

5 Adjoint system and first-order optimality conditions

In this section, we aim at deriving first-order necessary optimality conditions for the optimal control problem ($\mathcal{CP}_0$) using the deep quench approximation. Throughout the section, we assume that $\bar{u} \in \mathcal{U}_{\rm ad}$ is an optimal control of ($\mathcal{CP}_0$) with associated state $(\bar{\mu}, \bar{y})$, with $\bar{y} = \mathcal{S}_0(\bar{u})$. The derivation will be achieved by a passage to the limit as $\alpha \searrow 0$ in the first-order optimality conditions for the adapted optimal control problems ($\widetilde{\mathcal{CP}}_\alpha$) that can be derived as in [25] with only minor and obvious changes. This approach will not be possible in full generality. In fact, we have to assume that, besides **(A1)–(A7)**, the assumptions **(A8)–(A9)** from Theorem 3.5 are fulfilled.

Remark 5.1. Observe that **(A8)** yields the validity of the stronger regularity properties (3.5) from Theorem 3.1. Also, **(A9)** implies that the constant functions belong to $V_{-\Delta}^{1/2} = H^1(\Omega)$, so that **(A2)** is automatically fulfilled. In addition, since $H^1(\Omega) \cap L^\infty(\Omega)$ is dense in $H^1(\Omega)$ and $H^1(\Omega)$ is continuously embedded in $L^4(\Omega)$, the conditions [25, **(A8)** and **(A9)**] are satisfied.

Remark 5.2. The condition that $V_A^{2r} \subset L^\infty(\Omega)$ is, for instance, satisfied if $A = -\Delta$ with zero Dirichlet or Neumann boundary condition and $r > \frac{3}{8}$. Indeed, we have in this case that $V_A^{2r} \subset H^{4r}(\Omega) \subset L^\infty(\Omega)$, since $4r > \frac{3}{2}$. Likewise, if $A = \Delta^2$ with domain $D(A) \subset H^4(\Omega)$, then $V_A^{2r} \subset L^\infty(\Omega)$ provided that $r > \frac{3}{16}$. In this sense, while the improvement obtained in the following results over previously known results for the classical case $A = B = -\Delta$, $r = \sigma = \frac{1}{2}$, is not too large, the results are entirely new for other operators A; in fact, to our best knowledge, they constitute the first ever first-order necessary optimality conditions for Cahn–Hilliard type systems with fractional operators and nondifferentiable nonlinearities of double obstacle type.

As already mentioned in the proof of Theorem 3.5, it follows under the assumptions **(A1)–(A9)** that also the solution component μ^α of the solutions (μ^α, y^α) to (1.4)–(1.6) in the sense of [24, Thm. 2.6] for $f_1 = h^\alpha$ is uniquely determined, so that a corresponding solution operator

$$\widetilde{\mathcal{S}}_\alpha = (\widetilde{\mathcal{S}}^1_\alpha, \widetilde{\mathcal{S}}^2_\alpha) : u \ni \mathcal{U}_{\rm ad} \mapsto \widetilde{\mathcal{S}}_\alpha(u) = (\widetilde{\mathcal{S}}^1_\alpha(u), \widetilde{\mathcal{S}}^2_\alpha(u)) := (\mu^\alpha, y^\alpha)$$

is well defined. Clearly, we have $\widetilde{\mathcal{S}}^2_\alpha = \mathcal{S}_\alpha$. Moreover, (μ^α, y^α) satisfies the variational equality (3.27), which in this situation is equivalent to (1.5). In addition, a uniform separation property is satisfied; indeed, thanks to **(A6)**, for every $\alpha > 0$ and every bounded set $\mathcal{U} \subset \mathcal{X}$, there exist constants $r_*(\alpha), r^*(\alpha) \in (-1, 1)$, which depend only on $\mathcal{U}$, such that the following holds true: whenever $(\mu^\alpha, y^\alpha) = \widetilde{\mathcal{S}}_\alpha(u)$ for some $u \in \mathcal{U}$, then

$$r_*(\alpha) \le y^\alpha \le r^*(\alpha) \quad \text{a.e. in } Q, \quad r_*(\alpha) \le y_0 \le r^*(\alpha) \quad \text{a.e. in } \Omega. \tag{5.1}$$

In particular, the condition [25, **(GB)**], which was crucial for the analysis carried out in [25], is fulfilled for the potentials $f_1 = h^\alpha$, $\alpha > 0$, and we may take advantage of the results derived there.

Remark 5.3. Owing to the separation property (5.1), there is, for every $\alpha > 0$ and every bounded $\mathcal{U} \subset \mathcal{X}$, some constant $K_\alpha > 0$, which depends only on $\mathcal{U}$, such that

$$\max_{0\le i\le 3} \|(h^\alpha)^{(i)}(y^\alpha)\|_{L^\infty(Q)} \le K_\alpha \quad \text{whenever } y^\alpha = \mathcal{S}_\alpha(u) \text{ for some } u \in \mathcal{U}. \tag{5.2}$$

Now we have $V_A^{2r} \subset L^\infty(\Omega)$ and thus, by (3.5), $\mu^\alpha \in L^\infty(Q)$. Since also $\partial_t y^\alpha \in L^\infty(0,T;H)$, comparison in (3.27) shows that then $y^\alpha \in L^\infty(0,T;H^2(\Omega))$, which means that the state equations (1.4), (1.5) for $f_1 = h^\alpha$ are even satisfied in the strong sense, that is, we have

$$\partial_t y^\alpha + A^{2r}\mu^\alpha = 0 \quad \text{a.e. in } Q, \tag{5.3}$$

$$\tau\partial_t y^\alpha - \Delta y^\alpha + (h^\alpha)'(y^\alpha) + f_2'(y^\alpha) = \mu^\alpha + u \quad \text{a.e. in } Q. \tag{5.4}$$

At this point, we observe that that the state systems associated with $(\mathcal{CP}_\alpha)$ and $(\widetilde{\mathcal{CP}}_\alpha)$ are exactly the same. Hence, if $\bar u^\alpha \in \mathcal{U}_{\rm ad}$ is an optimal control of $(\widetilde{\mathcal{CP}}_\alpha)$ with associated state $(\bar\mu^\alpha, \bar y^\alpha) = \widetilde{\mathcal{S}}_\alpha(\bar u^\alpha)$ for some $\alpha > 0$, then $(\bar\mu^\alpha, \bar y^\alpha)$ satisfies the global bounds (3.3), (3.5), (5.2), as well as the separation property (5.1), and the state equations hold true in the form (5.3), (5.4). Moreover, introducing for $\alpha > 0$ the abbreviating notation

$$g_1^\alpha := \beta_1(\bar y^\alpha(T) - y_\Omega), \quad g_2^\alpha := \beta_2(\bar y^\alpha - y_Q), \quad \psi_1^\alpha := f_2''(\bar y^\alpha), \quad \psi_2^\alpha := \varphi(\alpha) h''(\bar y^\alpha), \tag{5.5}$$

we observe that (3.3), (3.5), (3.2), and **(A5)** imply the global bound

$$\|g_1^\alpha\|_{L^2(\Omega)} + \|g_2^\alpha\|_{L^2(Q)} + \|\psi_1^\alpha\|_{L^\infty(Q)} \le C_1 \quad \forall\, \alpha \in (0,1], \tag{5.6}$$

where, here and in the following, C_i, $i \in \mathbb{N}$, denote positive constants that may depend on the data of the state system but not on $\alpha \in (0,1]$. Observe that a corresponding bound for ψ_2^α cannot be expected: indeed, it may well happen that the separation constants $r_*(\alpha)$ and/or $r^*(\alpha)$ introduced in (5.1) approach ± 1 as $\alpha \searrow 0$, so that $\psi_2^\alpha = \frac{2\varphi(\alpha)}{1-(\bar y^\alpha)^2}$ may become unbounded as $\alpha \searrow 0$.

Next, we consider the adjoint system associated with the adapted optimal control problem $(\widetilde{\mathcal{CP}}_\alpha)$. According to [25, Sect. 5], it has the following form:

$$(A^r p^\alpha(t), A^r v) - (q^\alpha(t), v) = 0 \quad \text{for a.e. } t \in (0,T) \text{ and every } v \in V_A^r, \tag{5.7}$$

$$\langle -\partial_t(p^\alpha + \tau q^\alpha)(t), v\rangle + (\nabla q^\alpha(t), \nabla v) + ((\psi_1^\alpha(t) + \psi_2^\alpha(t))\, q^\alpha(t), v)$$
$$= (g_2^\alpha(t), v) \quad \text{for a.e. } t \in (0,T) \text{ and every } v \in H^1(\Omega), \tag{5.8}$$

$$(p^\alpha + \tau q^\alpha)(T) = g_1^\alpha \quad \text{in } \Omega. \tag{5.9}$$

Here, for the sake of simplicity, we have denoted by $\langle \cdot, \cdot \rangle$ the dual pairing between $H^1(\Omega)^*$ and $H^1(\Omega)$.

The system (5.7)–(5.9) is a special case of the type of systems that has been analyzed in [25, Sect. 5]. We briefly summarize some of the results established there (cf., [25, Prop. 5.2, Lem. 5.3, Lem. 5.4, Rem. 5.7, Thm. 5.8]), where we have to distinguish the following cases:

CASE 1: $\lambda_1 > 0$.

In this case, the system (5.7)–(5.9) admits a unique solution (p^α, q^α) satisfying

$$p^\alpha \in L^2(0,T;V_A^{2r}), \tag{5.10}$$
$$q^\alpha \in L^2(0,T;H^1(\Omega)), \tag{5.11}$$
$$p^\alpha + \tau q^\alpha \in H^1(0,T;H^1(\Omega)^*). \tag{5.12}$$

Notice that (5.12) implies that $p + \tau q \in C^0([0,T];H^1(\Omega)^*)$, so that the endpoint condition (5.9) is meaningful. Now observe that the operator $A^{2r} \in \mathcal{L}(V_A^{2r}, H)$ is for $\lambda_1 > 0$ a topological isomorphism, and with $A^{-2r} := (A^{2r})^{-1} : H \to V_A^{2r}$ the variational equation (5.7) takes the simple form $p^\alpha = A^{-2r} q^\alpha$. Inserting this in (5.8) and (5.9), we obtain that

$$\langle -\partial_t \big((A^{-2r} + \tau I) q^\alpha \big)(t), v \rangle + \int_\Omega \nabla q^\alpha(t) \cdot \nabla v + (\psi_1^\alpha(t)\, q^\alpha(t), v)$$
$$+ (\psi_2^\alpha(t) q^\alpha(t), v) = (g_2^\alpha(t), v) \quad \text{for a.e. } t \in (0,T) \text{ and all } v \in H^1(\Omega), \tag{5.13}$$
$$(A^{-2r} + \tau I) q^\alpha(T) = g_1^\alpha \qquad \text{a.e. in } \Omega, \tag{5.14}$$

where I denotes the identity operator in H. Moreover, since also the linear operator $A^{-2r} + \tau I \in \mathcal{L}(H,H)$ is obviously a topological isomorphism, (5.14) can be equivalently written as

$$q^\alpha(T) = (A^{-2r} + \tau I)^{-1} g_1^\alpha, \tag{5.15}$$

which gives $q^\alpha(T)$ a proper meaning as well.

We now derive an estimate for the adjoint variables that is uniform in $\alpha > 0$. Testing (5.13) by $q^\alpha(t)$ and integrating with respect to time over $[t,T]$, where $t \in [0,T)$, we then conclude the equation

$$\int_t^T \langle -\partial_t \big((A^{-2r} + \tau I) q^\alpha \big)(\rho), q^\alpha(\rho) \rangle \, d\rho + \int_t^T \|\nabla q^\alpha(\rho)\|^2 \, d\rho + \int_t^T \int_\Omega \psi_2^\alpha \, |q^\alpha|^2$$
$$= \int_t^T \int_\Omega \big(-\psi_1^\alpha q^\alpha + g_2^\alpha \big) \, q^\alpha \,, \tag{5.16}$$

where the last term on the left-hand side is nonnegative and, owing to (5.6), the right-hand side is bounded by an expression of the form

$$C_2 + C_3 \int_t^T \int_\Omega |q^\alpha|^2 \,. \tag{5.17}$$

Now observe that, by definition (2.2), and since $\lambda_1 > 0$, it holds for every $v \in H$ that

$$(A^{-2r} + \tau I)^{1/2} v = \sum_{j=1}^\infty \left(\lambda_j^{-2r} + \tau \right)^{1/2} (v, e_j) e_j, \tag{5.18}$$

and we have the estimates

$$\left\|(A^{-2r}+\tau I)^{1/2}v\right\|^2 \,=\, \sum_{j=1}^{\infty}\left(\lambda_j^{-2r}+\tau\right)|(v,e_j)|^2 \,\ge\, \tau\,\|v\|^2, \tag{5.19}$$

$$\left\|(A^{-2r}+\tau I)^{1/2}v\right\|^2 \,=\, \sum_{j=1}^{\infty}\left(\lambda_j^{-2r}+\tau\right)|(v,e_j)|^2 \,\le\, (\lambda_1^{-2r}+\tau)\,\|v\|^2, \tag{5.20}$$

$$\left\|(A^{-2r}+\tau I)^{-1}v\right\|^2 \,=\, \sum_{j=1}^{\infty}(\lambda_j^{-2r}+\tau)^{-2}|(v,e_j)|^2 \,\le\, \tau^{-2}\,\|v\|^2. \tag{5.21}$$

Moreover, it is easily verified that

$$-\left\langle \partial_t(A^{-2r}+\tau I)q^\alpha(t),q^\alpha(t)\right\rangle \,=\, -\frac{1}{2}\,\frac{d}{dt}\left\|(A^{-2r}+\tau I)^{1/2}q^\alpha(t)\right\|^2. \tag{5.22}$$

Therefore, by virtue of (5.15), the first term on the left-hand side of (5.16) is equal to

$$\frac{1}{2}\left\|(A^{-2r}+\tau I)^{1/2}q^\alpha(t)\right\|^2 \,-\, \frac{1}{2}\left\|(A^{-2r}+\tau I)^{1/2}(A^{-2r}+\tau I)^{-1}g_1^\alpha\right\|^2, \tag{5.23}$$

which, by (5.6) and (5.19)–(5.21), is bounded from below by $\frac{\tau}{2}\,\|q^\alpha(t)\|^2 - C_4$, with some global constant $C_4 > 0$. At this point, we invoke Gronwall's lemma, taken backward in time, as well as the fact that $p = A^{-2r}q$, to conclude that

$$\|p^\alpha\|_{L^\infty(0,T;V_A^{2r})} \,+\, \|q^\alpha\|_{L^\infty(0,T;H)\cap L^2(0,T;H^1(\Omega))} \,\le\, C_5 \quad \forall\,\alpha\in(0,1]. \tag{5.24}$$

CASE 2: $\lambda_1 = 0$.

This case is considerably more difficult to handle. To motivate this, we denote by $\mathbf{1}$ both the functions that are identically equal to 1 in either Ω or Q. Then, by **(A2)**(ii), $A^r\mathbf{1} = 0$, and insertion of $v = \mathbf{1}$ in (5.7) yields that

$$\text{mean}\,(q^\alpha(t)) = 0 \quad \text{for a.e. } t\in(0,T). \tag{5.25}$$

At this point, and also for later use, we recall an integration-by-parts formula that was proved in [22, Lem. 4.5]: if $(\mathcal{V},\mathcal{H},\mathcal{V}^*)$ is a Hilbert triple and

$$w \in H^1(0,T;\mathcal{H})\cap L^2(0,T;\mathcal{V}) \quad \text{and} \quad z\in H^1(0,T;\mathcal{V}^*)\cap L^2(0,T;\mathcal{H}), \tag{5.26}$$

then the function $t\mapsto (w(t),z(t))_{\mathcal{H}}$ is absolutely continuous, and for every $t_1,t_2\in[0,T]$ it holds the formula

$$\int_{t_1}^{t_2}\left[(\partial_t w(t),z(t))_{\mathcal{H}} \,+\, \langle\partial_t z(t),w(t)\rangle_{\mathcal{V}}\right]dt \,=\, (w(t_2),z(t_2))_{\mathcal{H}} - (w(t_1),z(t_1))_{\mathcal{H}}, \tag{5.27}$$

where $(\cdot,\cdot)_{\mathcal{H}}$ denotes the inner product in $\mathcal{H}$.

We now insert $v = \mathbf{1}$ in (5.8) and integrate the resulting identity with respect to time over $[t,T]$. Using (5.27) formally (this will later be justified by the regularity properties of the involved functions), we then obtain for every $t\in[0,T]$ the representation formula

$$\text{mean}\,(p^\alpha(t)+\tau q^\alpha(t)) \,=\, \text{mean}\,(g_1^\alpha) \,+\, \frac{1}{|\Omega|}\int_t^T\!\!\int_\Omega (g_2^\alpha - \psi_1^\alpha\,q^\alpha - \psi_2^\alpha\,q^\alpha), \tag{5.28}$$

where the left-hand side equals mean $(p^\alpha(t))$ for almost every $t \in (0,T)$ by (5.25). In view of this identity, we cannot expect the bound (5.24) to hold also in this case: indeed, due to the presence of the term $-\int_t^T\int_\Omega \psi_2^\alpha\, q^\alpha$ on the right-hand side of (5.28), we cannot hope to be able to control the mean value of p^α independently of $\alpha > 0$.

Nevertheless, a proper solution to (5.7)–(5.9) exists also in this case. To this end, we eliminate mean (p^α) from the problem, following a strategy introduced in [13] and [19]. We put

$$H_0 := \{v \in H : \text{mean}(v) = |\Omega|^{-1}(v, \mathbf{1}) = 0\}. \tag{5.29}$$

Then $H = H_0 \oplus \text{span}\{\mathbf{1}\}$, and we have (cf. (2.17)) that $V_0^r = V_A^r \cap H_0$ for $\lambda_1 = 0$. Moreover, the linear operator $A_0^{2r} = A^{2r}_{|V_0^{2r}}$ is a topological isomorphism from V_0^{2r} onto H_0, where we have the representation formulas

$$A_0^{2r} v \,=\, A^{2r} v \,=\, \sum_{j=2}^{\infty} \lambda_j^{2r}(v, e_j)e_j \quad \forall\, v \in V_0^{2r}, \tag{5.30}$$

$$A_0^{-2r} v \,:=\, (A_0^{2r})^{-1} v \,=\, \sum_{j=2}^{\infty} \lambda_j^{-2r}(v, e_j)e_j \quad \forall\, v \in H_0. \tag{5.31}$$

Moreover, with

$$H^{1,0}(\Omega) := H^1(\Omega) \cap H_0, \tag{5.32}$$

we have (cf. [25, Sect. 5]) that $(H^{1,0}(\Omega), H_0, (H^{1,0}(\Omega))^*)$ is a Hilbert triple with dense, continuous, and compact embeddings.

Now observe that $A^r(\text{mean}\,(p^\alpha(t))\mathbf{1}) = \text{mean}\,(p^\alpha(t))\, A^r\mathbf{1} = 0$, and thus (5.7) becomes

$$\begin{aligned} &(A^r(p^\alpha(t) - \text{mean}(p^\alpha(t))\mathbf{1}), A^r v) \,=\, (q^\alpha(t), v) \\ &\text{for a.e. } t \in (0,T) \text{ and every } v \in V_A^r. \end{aligned} \tag{5.33}$$

Since $p^\alpha(t) - \text{mean}(p^\alpha(t))\mathbf{1} \in H_0$ for almost every $t \in (0,T)$, this is equivalent to

$$A_0^{2r}(p^\alpha - \text{mean}(p^\alpha)\mathbf{1}) = q^\alpha \quad \text{and} \quad p^\alpha - \text{mean}(p^\alpha)\mathbf{1} \,=\, A_0^{-2r} q^\alpha. \tag{5.34}$$

At this point, we are able to state the existence result for the system (5.7)–(5.9) in the case $\lambda_1 = 0$ by adapting the results established in [25, Sect. 5] to the present situation. We then can infer that there exists a unique solution (p^α, q^α) such that

$$A_0^{-2r} q^\alpha \in L^\infty(0,T; V_0^{2r}), \tag{5.35}$$

$$q^\alpha \in L^\infty(0,T; H_0) \cap L^2(0,T; H^{1,0}(\Omega)), \tag{5.36}$$

$$(A_0^{-2r} + \tau I) q^\alpha \in H^1(0,T; (H^{1,0}(\Omega))^*), \tag{5.37}$$

as well as

$$\text{mean}(p^\alpha + \tau q^\alpha) \text{ satisfies (5.28) for every } t \in [0,T], \tag{5.38}$$

$$p^\alpha - \text{mean}(p^\alpha)\mathbf{1} = A_0^{-2r} q^\alpha, \tag{5.39}$$

$$\begin{aligned} &\big\langle -\partial_t\big(A_0^{-2r} + \tau I\big) q^\alpha(t), v\big\rangle_{H^{1,0}(\Omega)} + \int_\Omega \nabla q^\alpha(t) \cdot \nabla v \,+\, \big((\psi_1^\alpha(t) + \psi_2^\alpha(t))\, q^\alpha(t), v\big) \\ &\quad = \,(g_2^\alpha(t), v) \quad \text{for a.e. } t \in (0,T) \text{ and every } v \in H^{1,0}(\Omega), \end{aligned} \tag{5.40}$$

$$\big\langle (A_0^{-2r} + \tau I) q^\alpha(T), v\big\rangle_{H^{1,0}(\Omega)} \,=\, \big(g_1^\alpha - \text{mean}(g_1^\alpha)\mathbf{1}, v\big) \quad \text{for all } v \in H^{1,0}(\Omega). \tag{5.41}$$

Notice that, by (5.37), we have $(A_0^{-2r}+\tau I)q^\alpha \in C^0([0,T];(H^{1,0}(\Omega))^*)$, which gives the endpoint condition (5.41) a proper meaning: indeed, (5.41) means that $(A_0^{-2r}+\tau I)q^\alpha(T) = g_1^\alpha - \text{mean}(g_1^\alpha)\mathbf{1}$ in $(H^{1,0}(\Omega))^*$, where the right-hand side belongs to H_0. Now observe that the operator

$$(A_0^{-2r}+\tau I)v \;=\; \sum_{j=2}^{\infty}(\lambda_j^{-2r}+\tau)(v,e_j)e_j \quad \forall\, v\in H_0 \tag{5.42}$$

is a topological isomorphism from H_0 into itself with the inverse

$$(A_0^{-2r}+\tau I)^{-1}v \;=\; \sum_{j=2}^{\infty}(\lambda_j^{-2r}+\tau)^{-1}(v,e_j)e_j \quad \forall\, v\in H_0. \tag{5.43}$$

Hence, also $q^\alpha(T) = (A_0^{-2r}+\tau I)^{-1}(g_1^\alpha - \text{mean}(g_1^\alpha)\mathbf{1})$ has a proper meaning as an element of H_0. Next, we consider the mapping

$$(A_0^{-2r}+\tau I)^{1/2}v \;=\; \sum_{j=2}^{\infty}(\lambda_j^{-2r}+\tau)^{1/2}(v,e_j)e_j \quad \forall\, v\in H_0. \tag{5.44}$$

It is readily seen that the estimates (5.19)–(5.21) have the analogues

$$\|(A_0^{-2r}+\tau I)^{1/2}v\|^2 \;\ge\; \tau\,\|v\|^2 \quad \forall\, v\in H_0, \tag{5.45}$$

$$\|(A_0^{-2r}+\tau I)^{1/2}v\|^2 \;\le\; (\lambda_2^{-2r}+\tau)\,\|v\|^2 \quad \forall\, v\in H_0, \tag{5.46}$$

$$\|(A_0^{-2r}+\tau I)^{-1}v\|^2 \;\le\; \tau^{-2}\,\|v\|^2 \quad \forall\, v\in H_0. \tag{5.47}$$

Now observe that for a.e. $t\in(0,T)$ it holds that

$$-\,\langle (A_0^{-2r}+\tau I)\,q^\alpha(t), q^\alpha(t)\rangle_{H^{1,0}(\Omega)} \;=\; -\,\frac{1}{2}\,\frac{d}{dt}\left\|(A_0^{-2r}+\tau I)^{1/2}q^\alpha(t)\right\|^2. \tag{5.48}$$

At this point, we insert $v = q^\alpha(t)\in H^{1,0}(\Omega)$ in (5.40) and integrate over $[t,T]$, where $t\in[0,T)$, to recover the identity (5.16), only that in the first term the expression A^{-2r} and the dual pairing between $H^1(\Omega)^*$ and $H^1(\Omega)$ are replaced by A_0^{-2r} and the dual pairing between $(H^{1,0}(\Omega))^*$ and $H^{1,0}(\Omega)$, respectively. Again, the third summand on the left-hand side is nonnegative, and the right-hand side is bounded by the expression (5.17). Moreover, the first summand on the left-hand side, which we denote by $I_1^\alpha(t)$, can by (5.46) and (5.47) be estimated as follows:

$$I_1^\alpha(t) \;=\; \frac{1}{2}\left\|(A_0^{-2r}+\tau I)^{1/2}q^\alpha(t)\right\|^2 \;-\; \frac{1}{2}\left\|(A_0^{-2r}+\tau I)^{1/2}q^\alpha(T)\right\|^2 \;\ge\; \frac{\tau}{2}\,\|q^\alpha(t)\|^2 - \frac{1}{2}\,C_6. \tag{5.49}$$

At this point, we can again employ Gronwall's lemma to conclude the estimate

$$\|p^\alpha - \text{mean}(p^\alpha)\mathbf{1}\|_{L^\infty(0,T;V_0^{2r})} \;+\; \|q^\alpha\|_{L^\infty(0,T;H_0)\cap L^2(0,T;H^{1,0}(\Omega))} \;\le\; C_7 \quad \forall\,\alpha\in(0,1], \tag{5.50}$$

which is the sought analogue of (5.24).

In the following, we complement (5.24) and (5.50) by further estimates. We treat the two cases $\lambda_1>0$ and $\lambda_1=0$ simultaneously, where it is understood that the spaces V_0^r and the operators A_0^r are defined as in (2.17) and (2.18), respectively. We now introduce the space

$$\mathcal{Z} := \begin{cases} \{v\in H^1(0,T;H)\cap L^2(0,T;H^1(\Omega)) : v(0)=0\} & \text{if } \lambda_1>0 \\ \{v\in H^1(0,T;H_0)\cap L^2(0,T;H^1(\Omega)) : v(0)=0\} & \text{if } \lambda_1=0 \end{cases}, \tag{5.51}$$

which is a Hilbert space when endowed with its natural inner product and norm. Moreover, setting

$$G = H \quad \text{for } \lambda_1 > 0 \quad \text{and} \quad G = H_0 \quad \text{for } \lambda_1 = 0, \tag{5.52}$$

we see that the embedding $\mathcal{Z} \subset C^0([0,T];G)$ is continuous. Furthermore, we also have the dense and continuous embeddings $\mathcal{Z} \subset L^2(0,T;G) \subset \mathcal{Z}^*$, where it is understood that

$$\langle v, z\rangle_{\mathcal{Z}} = \int_0^T (v(t), z(t))\, dt \quad \text{for all } z \in \mathcal{Z} \text{ and } v \in L^2(0,T;G). \tag{5.53}$$

In order to avoid to have to distinguish between the two cases, we employ in the following the same notation $\langle\cdot,\cdot\rangle$ for the dual pairings $\langle\cdot,\cdot\rangle_{H^1(\Omega)}$ and $\langle\cdot,\cdot\rangle_{H^{1,0}(\Omega)}$, where the former corresponds to the case $\lambda_1 > 0$ and the latter to the case $\lambda_1 = 0$.

At this point, by recalling (5.10)–(5.12) for $\lambda_1 > 0$ and (5.35)–(5.37) for $\lambda_1 = 0$, we may employ the integration-by-parts formula (5.27) with $z = A_0^{-2r} q^\alpha + \tau q^\alpha$ to conclude that for every $v \in \mathcal{Z}$ it holds that

$$\begin{aligned}
\langle -\partial_t(p^\alpha + \tau q^\alpha), v\rangle_{\mathcal{Z}} \,&=\, -\int_0^T \langle \partial_t (A_0^{-2r} q^\alpha(t) + \tau q^\alpha(t), v(t)\rangle\, dt \\
&= \int_0^T (\partial_t v(t), (A_0^{-2r} + \tau I) q^\alpha(t))\, dt \,-\, (g_1^\alpha, v(T)) \\
&\le \|\partial_t v\|_{L^2(0,T;H)}\, \|(A_0^{-2r} + \tau I) q^\alpha\|_{L^2(0,T;H)} \,+\, \|g_1^\alpha\|_H\, \|v(T)\|_H \\
&\le C_7\, \|v\|_{\mathcal{Z}},
\end{aligned} \tag{5.54}$$

which implies that

$$\|\partial_t (p^\alpha + \tau q^\alpha)\|_{\mathcal{Z}^*} \,\le\, C_7 \quad \forall\, \alpha \in (0,1]. \tag{5.55}$$

Now observe that for any $v \in \mathcal{Z}$ it holds that

$$\begin{aligned}
&\int_0^T (\nabla q^\alpha(t), \nabla v(t))\, dt \,+ \int_0^T (\psi_1^\alpha(t) q^\alpha(t), v(t))\, dt \,- \int_0^T (g_2^\alpha(t), v(t))\, dt \\
&\le \|q^\alpha\|_{L^2(0,T;H^1(\Omega))}\, \|v\|_{\mathcal{Z}} \,+\, C_8\, \|q^\alpha\|_{L^2(0,T;H)}\, \|v\|_{L^2(0,T;H)} \,+\, C_9\, \|v\|_{L^2(0,T;H)} \\
&\le C_{10}\, \|v\|_{\mathcal{Z}},
\end{aligned}$$

and it follows from comparison in (5.8) that, with $\Lambda^\alpha := \psi_2^\alpha q^\alpha = \varphi(\alpha) h''(\bar{y}^\alpha) q^\alpha$,

$$\|\Lambda^\alpha\|_{\mathcal{Z}^*} \,\le\, C_{11} \quad \forall\, \alpha \in (0,1]. \tag{5.56}$$

At this point, we choose any sequence $\{\alpha_n\}$ such that $\alpha_n \searrow 0$. We infer from Theorem 3.3 and Theorem 4.5 that, at least for a subsequence which is again indexed by n,

$$\bar{u}^{\alpha_n} \to \bar{u} \quad \text{strongly in } L^2(Q), \tag{5.57}$$

$$\bar{y}^{\alpha_n} \to \bar{y} \quad \text{weakly-star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;H^1(\Omega)). \tag{5.58}$$

By virtue of [44, Sect. 8, Cor. 4], we may also assume that

$$\bar{y}^{\alpha_n} \to \bar{y} \quad \text{strongly in } C^0([0,T];L^p(\Omega)) \quad \forall\, p \in [1,6), \tag{5.59}$$

which entails, in particular, that

$$f_2''(\bar y^{\alpha_n}) \to f_2''(\bar y) \quad \text{strongly in } C^0([0,T];L^p(\Omega)) \quad \forall\, p \in [1,6), \tag{5.60}$$

$$g_1^{\alpha_n} \to \beta_1(\bar y(T) - y_\Omega) \quad \text{strongly in } H, \tag{5.61}$$

$$g_2^{\alpha_n} \to \beta_2(\bar y - y_Q) \quad \text{strongly in } L^2(Q). \tag{5.62}$$

Moreover, by virtue of the estimates (5.24), (5.50), (5.55), and (5.56), there are limits $\zeta, \bar q, \Lambda$ such that, at least for another subsequence which is again indexed by n,

$$\partial_t(A_0^{-2r} + \tau I) q^{\alpha_n} \to \zeta \quad \text{weakly in } \mathcal{Z}^*, \tag{5.63}$$

$$q^{\alpha_n} \to \bar q \quad \text{weakly-star in } L^\infty(0,T;G) \cap L^2(0,T;H^1(\Omega)), \tag{5.64}$$

$$A_0^{-2r} q^{\alpha_n} \to A_0^{-2r} \bar q \quad \text{weakly-star in } L^\infty(0,T;V_0^{2r}), \tag{5.65}$$

$$\Lambda^{\alpha_n} \to \Lambda \quad \text{weakly in } \mathcal{Z}^*. \tag{5.66}$$

The limit $\zeta \in \mathcal{Z}^*$ is readily identified. Indeed, by formula (5.27) we have, for every $v \in \mathcal{Z}$,

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \langle \partial_t (A_0^{-2r} + \tau I) q^{\alpha_n}(t), v(t) \rangle \, dt \\
&= \lim_{n\to\infty} \Big[- \int_0^T \big(\partial_t v(t), (A_0^{-2r} + \tau I) q^{\alpha_n}(t) \big) \, dt + \big(g_1^{\alpha_n}, v(T) \big) \Big] \\
&= - \int_0^T \big(\partial_t v(t), (A_0^{-2r} + \tau I) \bar q(t) \big) \, dt \, + \, \big(\beta_1(\bar y(T) - y_\Omega), v(T) \big) \, =: \, \langle \zeta, v \rangle_{\mathcal{Z}}.
\end{aligned} \tag{5.67}$$

Moreover, by combining the strong convergence (5.60) with (5.64), it is easily checked that

$$f_2''(\bar y^{\alpha_n}) \, q^{\alpha_n} \to f_2''(\bar y) \, \bar q \quad \text{weakly in } L^2(Q). \tag{5.68}$$

At this point, we recall that

$$\langle -\partial_t (p^\alpha + \tau q^\alpha)(t), v(t) \rangle \, = \, \langle -\partial_t (A_0^{-2r} + \tau I) q^\alpha(t), v(t) \rangle \quad \text{for a.e. } t \in (0,T) \text{ and } v \in \mathcal{Z}.$$

We now write the adjoint system (5.7)–(5.8) for $\alpha = \alpha_n$, insert $v = v(t)$ for an arbitrary $v \in \mathcal{Z}$, integrate the resulting identity with respect to time over $[0,T]$, and pass to the limit as $n \to \infty$. It then results the following equation:

$$\begin{aligned}
\langle \Lambda, v \rangle_{\mathcal{Z}} \, = \, & - \int_0^T \big(\partial_t v(t), (A_0^{-2r} + \tau I) \bar q(t) \big) \, dt \, + \beta_1(\bar y(T) - y_\Omega, v(T)) \\
& - \int_0^T \int_\Omega \nabla \bar q \cdot \nabla v \, + \int_0^T \int_\Omega \big(\beta_2(\bar y - y_Q) - f_2''(\bar y) \bar q \big) \, v \quad \forall\, v \in \mathcal{Z}.
\end{aligned} \tag{5.69}$$

Finally, we need to identify the variational inequality relating the optimal control to the adjoint variables. In this regard, we can infer, with the same argument as in the proof of [25, Thm. 5.9], that the optimal control $\bar u^{\alpha_n}$ satisfies the variational inequality

$$\int_0^T \int_\Omega (q^{\alpha_n} + \beta_3 \bar u^{\alpha_n} + \bar u^{\alpha_n} - \bar u)(v - \bar u^{\alpha_n}) \, \geq \, 0 \quad \forall\, v \in \mathcal{U}_{\rm ad}. \tag{5.70}$$

Taking the limit as $n \to \infty$ in (5.70), and using (5.57) and (5.64), we arrive at the necessary optimality condition

$$\int_0^T \int_\Omega (\bar q \, + \, \beta_3 \, \bar u)(v - \bar u) \, \geq \, 0 \quad \forall\, v \in \mathcal{U}_{\rm ad}. \tag{5.71}$$

From the above considerations, we can conclude the following first-order necessary optimality conditions for the optimal control problem ($\mathcal{CP}_0$):

Theorem 5.4. *Suppose that the conditions* **(A1)–(A6)**, **(A8)**, **(A9)** *are satisfied, and let* $\bar{u} \in \mathcal{U}_{\rm ad}$ *be an optimal control for* $(\mathcal{CP}_0)$ *with associated state* $(\bar{\mu}, \bar{y})$ *where* $\bar{y} = \mathcal{S}_0(\bar{u})$*. Then there exist* $(\bar{q}, \Lambda)$ *such that the following statements hold true:*

(i) *We have the regularity properties*

$$\bar{q} \in L^\infty(0,T;G) \cap L^2(0,T;H^1(\Omega)), \quad \Lambda \in \mathcal{Z}^*. \tag{5.72}$$

(ii) *The adjoint equation* (5.69) *is fulfilled.*

(iii) *The necessary optimality condition* (5.71) *is satisfied.*

Remark 5.5. From (5.71) we infer that, in the case $\beta_3 > 0$, $\bar{u}$ is nothing but the $L^2(Q)$-orthogonal projection of $-\beta_3^{-1}\, q$ onto $\mathcal{U}_{\rm ad}$.

Remark 5.6. Unfortunately, we are unable to derive any complementarity slackness conditions for the Lagrange multiplier Λ. Indeed, while it is easily seen that

$$\liminf_{n\to\infty} \int_0^T\!\!\int_\Omega \Lambda^{\alpha_n}\, q^{\alpha_n} \;=\; \liminf_{n\to\infty} \int_0^T\!\!\int_\Omega \frac{2\varphi(\alpha_n)}{1-(\bar{y}^{\alpha_n})^2}\, |q^{\alpha_n}|^2 \;\geq\; 0 \quad \forall\, n \in \mathbb{N},$$

the convergence properties (5.58) and (5.64) do not suffice to conclude that $\langle \Lambda, \bar{q} \rangle_{\mathcal{Z}} \;\geq\; 0$.

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