RANDOM WALK FOR ELLIPTIC EQUATIONS AND BOUNDARY LAYER

G.N. MILSTEIN

18. July 1996

1991 *Mathematics Subject Classification.* 35J25, 60J15, 65N99.

Key words and phrases. Boundary value problem, weak methods of numerical integration of SDE, random walk, boundary layer.

Abstract. We consider the Dirichlet problem for equations of elliptic type in a domain G with a boundary ∂G. A probabilistic representation of solutions to the problem is connected with a system of stochastic differential equations (SDE). Unlike usual approximation of SDE when a time-discretization is exploited, here a space-discretization is recommended. We construct weak approximations for which an estimate of their errors contains derivatives of the required solution to the Dirichlet problem only of lower order. In particular, it is important for problems with a boundary layer. We simulate a Markov chain in G on the basis of a one-step approximation using variable step in the space. The chain should be stopped entering a sufficiently small neighborhood of the boundary ∂G. We estimate the average number of steps before stopping and state some convergence theorems.

1. **Introduction**

Consider the Dirichlet problem for an equation of elliptic type

$$Lu(x) + g(x) :=$$

$$\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x^i} + c(x)u + g(x) = 0, \;\; x \in G \tag{1.1}$$

$$u\mid_{\partial G} = \varphi(x) \tag{1.2}$$

The following conditions are assumed to be satisfied :

(i) G is open bounded set with twice continuously differentiable boundary ∂G;

(ii) the coefficients $a^{ij}(x),\ b^i(x),\ c(x),\ g(x)$ belong to the class $\mathbf{C}^2(\bar{G})$, $c(x) \leq 0$, $\varphi \in \mathbf{C}^4(\partial G)$;

(iii) $a^{ij} = a^{ji}$, and the matrix $a(x) = \{a^{ij}(x)\}$ satisfies a strict ellipticity condition, i.e., a constant $a > 0$ exists such that for any $x \in \bar{G},\ y \in \mathbf{R}^n$ the following inequality

$$\sum_{i,j=1}^{n} a^{ij}(x)y^i y^j \geq a^2 \sum_{i=1}^{n} {y^i}^2 \tag{1.3}$$

holds.

The conditions (i)-(iii) ensure the existence of the unique solution $u(x)$ of the problem (1.1)–(1.2) belonging to the class $\mathbf{C}^4(\bar{G})$ [14].

Let $\sigma(x)$ be a matrix (for instance, a lower triangular matrix) that is obtained from the following equality

$$a(x) = \sigma(x)\sigma^{\top}(x)$$

The solution to the problem (1.1)–(1.2) has various probabilistic representations:

$$u(x) = \mathbf{E}\int_0^{\tau} g(X_x(t))Y_{x,1}(t)dt + \mathbf{E}\varphi(X_x(\tau))Y_{x,1}(\tau) \tag{1.4}$$

where $X_x(t),\ Y_{x,1}(t)$ is the solution of the Cauchy problem for the system of stochastic differential equations (SDE)

$$dX = b(X)dt - \sigma(X)h(X)dt + \sigma(X)dw(t), \;\; X(0) = x \tag{1.5}$$

$$dY = c(X)Y dt + h^{\top}(X)Y dw(t), \;\; Y(0) = 1 \tag{1.6}$$

In (1.4)–(1.6) $b(x) = (b^1(x), ..., b^n(x))^{\top}$, $h(x) = (h^1(x), ..., h^n(x))^{\top}$, $h^i(x)$, $i = 1, ..., n$, are arbitrary functions belonging to the class $\mathbf{C}^2(\bar{G})$, $w(t) = (w^1(t), ..., w^n(t))^{\top}$ is a standard Wiener process, which is defined on a probabilistic space $(\Omega, \mathcal{F}, \mathbf{P})$ and which is measurable with respect to the flow $\mathcal{F}_t,\ t \geq 0$, Y is a scalar, τ is a first passage time of the path $X_x(t)$ to the boundary ∂G. The usual representation (see [1]) can be seen in (1.4)–(1.6) if $h = 0$, other are rest on Girsanov's theorem. Let us apply the representation (1.4)–(1.6) under

$$h(x) = \sigma^{-1}(x)b(x)$$

Then it has the form

$$u(x) = \mathbf{E}(\varphi(X_x(\tau))Y_{x,1}(\tau) + Z_{x,1,0}(\tau)) \tag{1.7}$$

where $X_x(t)$, $Y_{x,1}(t)$, $Z_{x,1,0}(t)$ is the solution to the system

$$dX = \sigma(X)dw(t) \tag{1.8}$$

$$dY = c(X)Y dt + (\sigma^{-1}(X)b(X))^\top Y dw(t) \tag{1.9}$$

$$dZ = g(X)Y dt \tag{1.10}$$

with the initial data $X(0) = x$, $Y(0) = 1$, $Z(0) = 0$. Denote the solution of the system (1.8)–(1.10) with the initial data $X(0) = x$, $Y(0) = y$, $Z(0) = z$ by $X_x(t)$, $Y_{x,y}(t)$, $Z_{x,y,z}(t)$. We always set $y > 0$ for definiteness.

Introduce the function

$$v(x, y, z) = \mathbf{E}(\varphi(X_x(\tau))Y_{x,y}(\tau) + Z_{x,y,z}(\tau)) \tag{1.11}$$

Clearly

$$v(x, y, z) = u(x)y + z \tag{1.12}$$

Only the last section (Section 7) is devoted to the general problem (1.1)–(1.2). The most effective results can be obtained in the case of constant coefficients at higher derivatives in (1.1). For simplicity, here (see Sections 3-6) we consider the more special case when $a^{ij}(x) = \delta_{ij}a^2$, where $a > 0$ and δ_{ij} is the Kronecker delta. In this case the equations (1.8) and (1.9) acquire the folowing form

$$dX = adw(t) \tag{1.13}$$

$$dY = c(X)Y dt + \frac{1}{a}b^\top(X)Y dw(t) \tag{1.14}$$

The simplicity of the equation (1.13) allows to simulate its solution exactly.

Let Γ_δ ($\Gamma_{\alpha r}$) be the interior of a δ-neighborhood (of an αr-neighborhood) of the boundary ∂G belonging to $\bar{G}$. Let $\alpha \geq a$ and $\delta < \alpha r/2$. Usually r is taken sufficiently small and $\delta = O(r^q)$, $q > 1$. Introduce in $\mathbf{R}^n$ balls U_ρ and $U(x)$, $x \in G\backslash\Gamma_\delta : U_\rho$ is the open ball of radius ρ with centre at the origin; $U(x)$ for $x \in G\backslash\Gamma_{\alpha r}$ is the open ball of radius ar and $U(x)$ for $x \in \Gamma_{\alpha r}\backslash\Gamma_\delta$ is the open tangent ball of radius $\rho(x, \partial G)$ with centre at x.

Consider the following random walk over small spheres which starts at $x \in G\backslash\Gamma_\delta$. For definiteness let $x \in G\backslash\Gamma_{\alpha r}$. We set $X_0 = x$. Let ϑ_1 be the first passage time of the Wiener process $w(t)$ to the sphere ∂U_r; we set $X_1 = X_0 + aw(\vartheta_1)$. Clearly, X_1 has the uniform distribution on $\partial U(X_1)$. If $X_1 \in G\backslash\Gamma_{\alpha r}$, we search $\vartheta_1 + \vartheta_2$ which is the first passage time of the process $w(t) - w(\vartheta_1)$, $t \geq \vartheta_1$, to the same sphere ∂U_r and we set $X_2 = X_1 + a(w(\vartheta_1 + \vartheta_2) - w(\vartheta_1))$. If $X_1 \in \Gamma_{\alpha r}\backslash\Gamma_\delta$, we turn to a walk over boundary of tangent ball $U(X_1)$: we search $\vartheta_1 + \vartheta_2$ which is the first passage time of the process $w(t) - w(\vartheta_1)$, $t \geq \vartheta_1$, to the sphere $\partial U_{\frac{1}{a}\rho(X_1,\partial G)}$ and we set $X_2 = X_1 + a(w(\vartheta_1 + \vartheta_2) - w(\vartheta_1))$ as before. If $X_2 \in G\backslash\Gamma_{\alpha r}$, we turn again to the walk over a sphere of radius ar with centre at X_2, and if $X_2 \in \Gamma_{\alpha r}\backslash\Gamma_\delta$, we continue the walk over tangent sphere and so on. At each k-th step it is a random walk over surface

$\partial U(X_{k-1})$. Clearly, X_k has the uniform distribution on $\partial U(X_{k-1})$. Let $\nu = \nu_x$ be the first number at which $X_\nu \in \Gamma_\delta$. Let us set $\vartheta_\kappa = 0$ for $k > \nu$ and $X_k = X_\nu$ for $k \geq \nu$. So, we obtain a random walk

$$
\begin{aligned}
&X_0 = x \\
&X_1 = X_0 + aw(\vartheta_1) \\
&\ldots\ldots\ldots\ldots\ldots \\
&X_k = X_{k-1} + a(w(\vartheta_1 + \ldots + \vartheta_k) - w(\vartheta_1 + \ldots + \vartheta_{k-1})),\ k = 1, \ldots, \nu \\
&X_k = X_\nu,\ k \geq \nu
\end{aligned}
$$

which stops at a random step ν. It is a Markov chain.

Let $\mathcal{B}_k = \sigma(X_0, X_1, \ldots, X_k)$, $k = 1, 2, \ldots$, be the sequence of σ-algebras generated by the random walk $X_0, X_1, \ldots, X_k, \ldots$.

Presuppose that a method of approximation of the system (1.13)–(1.14), (1.10) is done and the sequences $Y_0, Y_1, \ldots, Y_k, \ldots$, $Z_0, Z_1, \ldots, Z_k, \ldots$ which approximate $Y_{x,y}(\vartheta_1 + \ldots + \vartheta_k)$, $Z_{x,y,z}(\vartheta_1 + \ldots + \vartheta_k)$, $k = 1, 2, \ldots$, correspondingly are constructed such that Y_k, Z_k are $\mathcal{B}_k$-measurable and they are stopped at the random step ν. Let $\bar{X}_\nu$ be the point of the boundary ∂G closest to X_ν. Put $\bar{Y}_\nu = Y_\nu$, $\bar{Z}_\nu = Z_\nu$. We are interested in the difference

$$\mathcal{R} = \mathbf{E}v(\bar{X}_\nu, \bar{Y}_\nu, \bar{Z}_\nu) - \mathbf{E}v(X_x(\tau), Y_{x,y}(\tau), Z_{x,y,z}(\tau)) = \mathbf{E}v(\bar{X}_\nu, \bar{Y}_\nu, \bar{Z}_\nu) - v(x, y, z)$$

since $\mathbf{E}v(\bar{X}_\nu, \bar{Y}_\nu, \bar{Z}_\nu) = \mathbf{E}(\varphi(\bar{X}_\nu)\bar{Y}_\nu + \bar{Z}_\nu) = \mathbf{E}(\varphi(\bar{X}_\nu)Y_\nu + Z_\nu)$ is taken as an approximation of $v(x, y, z) = u(x)y + z$.

We have

$$v(\bar{X}_\nu, \bar{Y}_\nu, \bar{Z}_\nu) - v(x, y, z) = (v(\bar{X}_\nu, \bar{Y}_\nu, \bar{Z}_\nu) - v(X_\nu, Y_\nu, Z_\nu))$$

$$+(v(X_\nu, Y_\nu, Z_\nu) - v(X_{\nu-1}, Y_{\nu-1}, Z_{\nu-1})) + \ldots + (v(X_1, Y_1, Z_1) - v(x, y, z))$$

$$= (u(\bar{X}_\nu) - u(X_\nu))Y_\nu + \sum_{k=1}^{\infty}(v(X_k, Y_k, Z_k) - v(X_{k-1}, Y_{k-1}, Z_{k-1}))\chi_{\nu \geq k}$$

and consequently

$$\mathcal{R} = \mathbf{E}(u(\bar{X}_\nu) - u(X_\nu))Y_\nu +$$

$$\sum_{k=1}^{\infty} \mathbf{E}(v(X_k, Y_k, Z_k) - v(X_{k-1}, Y_{k-1}, Z_{k-1}))\chi_{\nu \geq k} \tag{1.15}$$

An evaluation of $\mathcal{R}$ depends on a bound of the first term and on a one-step approximation which gives bounds for the summands in right-hand side of (1.15). Our aim is to find such one-step approximations that do not use the simulation of ϑ_k (it is a fairly difficult problem) and error of which can be bounded without using any derivatives or at least without using high derivatives of the solution $u(x)$ to the input problem (1.1)–(1.2). The latter is very important for problems with a small parameter at higher derivatives because a boundary layer arises in such a situation, and the higher derivatives of the solution u the larger values they take. Such approximations are based on simulation of some conditional mathematical expectations like as

$$\xi^i = \mathbf{E}(\int_0^\vartheta w^i(s)ds/w(\vartheta)),\ \xi^{ij} = \mathbf{E}(\int_0^\vartheta w^i(s)dw^j(s)/w(\vartheta))$$

Section 2 is devoted to some auxiliary lemmas and to simulation of ξ^i, ξ^{ij}. Various one-step approximations are constructed in Section 3. The bound of the first term in (1.15) essentially depends on δ. The average number of steps $\mathbf{E}\nu$ also depends on δ. A choice of δ is connected with exactness of a one-step approximation. As usual $\delta = O(r^k)$ if the order of one-step approximation is equal to $O(r^{k+2})$. Theorems on the average number of steps $\mathbf{E}\nu$ and other results relevant for evaluation of the sum in (1.15) are obtained in Section 4. The convergence theorems are proved in Section 5. In the case of a small parameter at the second derivatives (this case is treated in Section 6), the system (1.8)–(1.10) becomes a system with a small noise and we construct some specific methods for its approximate integration. Another way in this case rests on the fact that in the almost whole domain G with the exception of a narrow boundary layer the solution to the Dirichlet problem can be found sufficiently precise and simply by analytical tools (this part of the solution is known as external expansion). Basing on this, we propose a method of random walk in the narrow layer for searching the remaining part of the solution (known as interior expansion). The effectiveness of this analytic-numerical method is achieved because of small average number of steps for the random paths in the greatly narrow domain. In the last section (Section 7) we consider two methods for the general problem (1.1)–(1.2). In contrast to the case of constant σ in (1.8) we cannot obtain the exact random walk X_k now. In the first method, which is the essentially modified variant of the method from [11], we solve (1.8) approximately by freezing its coefficients at every step at the point X_{k-1}. The next point X_k is found by a random walk over the boundary of a small ellipsoid. The second method is remarkable in the respect that the corresponding random walk terminates on ∂G. Therefore, we do not require the neighborhood Γ_δ of the boundary ∂G, and the part $\mathbf{E}(u(\bar{X}_\nu) - u(X_\nu))Y_\nu$ of the error $\mathcal{R}$ disappears. The methods represented in Section 7 are similar to the methods developed for the boundary value problems for the equations of parabolic type [9].

The contents of the present paper are connected with weak approximations for SDE [7], [17], [15] (see also [8], [5]). Unlike these works where a time-discretization is exploited, here a space-discretization is used, which is necessary to solve boundary value problems. Moreover, here we pay a special attention to numerical analysis of the boundary layer which arises in the case of a small diffusion. Other approaches to probabilistic methods of solving boundary value problems for differential equations with partial derivatives are discussed, for instance, in [2], [6], [16].

2. Conditional expectation of Ito's integrals connected with Wiener's process in the ball

Here both a probabilistic representation and an explicit form of solution will be exploited for Dirichlet's problem in the ball $U_r = \{x = (x^1, ..., x^n) : |x|^2 = {x^1}^2 + ... + {x^n}^2 \le r^2\}$:

$$\frac{1}{2}\Delta u + g(x) = 0, \ \ |x| < r \tag{2.1}$$

$$u \mid_{|x|=r} = \varphi(x) \tag{2.2}$$

In (2.1)–(2.2) $g(x) \in \mathbf{C}^1(|x| \le r)$, $\varphi(x) \in \mathbf{C}(|x| = r)$.

The probabilistic representation for the solution to the problem (2.1)–(2.2) has the form

$$u(x) = \mathbf{E}\varphi(x + w(\vartheta_x)) + \mathbf{E}\int_0^{\vartheta_x} g(x + w(s))ds \tag{2.3}$$

where $w(t) = (w^1(t), ..., w^n(t))$ is an n-dimensional standard Wiener process and ϑ_x is the first passage time of the process $x + w(t)$ to the sphere ∂U_r.

The explicit formula for the solution has the following form [14]

$$u(x) = \int_{|\xi|=r} P_r(x,\xi)\varphi(\xi)dS_\xi + \int_{|\xi|<r} G_r(x,\xi)g(\xi)d\xi \tag{2.4}$$

where P_r is the Poisson kernel:

$$P_r(x,\xi) = \frac{r^2 - |x|^2}{\sigma_n r|x-\xi|^n} \tag{2.5}$$

and G_r is the Green function which for $n = 2$ is equal to

$$G_r(x,\xi) = \frac{1}{2\pi} \ln \frac{|x| \cdot |(r/|x|)^2 x - \xi|}{r|x-\xi|}, \ n = 2 \tag{2.6}$$

and for $n > 2$ is equal to

$$G_r(x,\xi) = \frac{1}{(n-2)\sigma_n} \cdot (\frac{1}{|x-\xi|^{n-2}} - \frac{(r/|x|)^{n-2}}{|(r/|x|)^2 x - \xi|^{n-2}}), \ n > 2 \tag{2.7}$$

In (2.5), (2.7) σ_n is area of the unit sphere in $\mathbf{R}^n : \sigma_n = 2\pi^{n/2}/\Gamma(n/2)$. Remember that $\sigma_n r^{n-1}$ is area of the sphere ∂U_r and $\sigma_n r^n/n$ is volume of the ball U_r.

Proceeding to simulation of the conditional expectation $\mathbf{E}(\int_0^\vartheta w^i(s)ds/w(\vartheta))$ where $\vartheta = \vartheta_0$ is the first passage time of the Wiener process $w(t)$ to the sphere ∂U_r let us assume that

$$\mathbf{E}(\int_0^\vartheta w^i(s)ds/w(\vartheta)) = \alpha w^i(\vartheta), \ i = 1, ..., n \tag{2.8}$$

If (2.8) is true then the constant α can be found from the condition

$$\mathbf{E}(\int_0^\vartheta w^i(s)ds - \alpha w^i(\vartheta))^2 \longrightarrow \min_\alpha$$

i.e.,

$$\alpha = \frac{\mathbf{E}w^i(\vartheta)\int_0^\vartheta w^i(s)ds}{\mathbf{E}w^{i^2}(\vartheta)} \tag{2.9}$$

Lemma 2.1. *For every $i = 1, ..., n$ the following formulae hold:*

$$\mathbf{E}w^{i^2}(\vartheta) = \frac{r^2}{n} \tag{2.10}$$

$$\mathbf{E}w^i(\vartheta)\int_0^\vartheta w^i(s)ds = \mathbf{E}\int_0^\vartheta w^{i^2}(s)ds = \frac{r^4}{2n(n+2)} \tag{2.11}$$

and consequently α from (2.9) is equal to

$$\alpha = \frac{r^2}{2(n+2)} \tag{2.12}$$

Proof. The relation (2.10) is evident due to the identity $w^{1^2}(\vartheta) + ... + w^{n^2}(\vartheta) = r^2$. Further from Ito's formula

$$dw^i(t)\int_0^t w^i(s)ds = \int_0^t w^i(s)ds \cdot dw^i(t) + w^{i^2}(t)dt$$

and therefore

$$\mathbf{E}w^i(\vartheta)\int_0^\vartheta w^i(s)ds = \mathbf{E}\int_0^\vartheta w^{i^2}(s)ds$$

It is not difficult to verify that the function $u = r^4 - |x|^4$ is a solution to the problem

$$\frac{1}{2}\Delta u + 2(n+2)|x|^2 = 0, \; u_{||x|=r} = 0$$

Therefore (see (2.3))

$$u(0) = r^4 = 2(n+2)\mathbf{E}\int_0^\vartheta \sum_{k=1}^n w^{k^2}(s)ds = 2n(n+2)\mathbf{E}\int_0^\vartheta w^{i^2}(s)ds \tag{2.13}$$

that gives (2.11). Lemma 2.1 is proved.

It turns out that the hypothesis (2.8) is true.

Theorem 2.1. *For every $i = 1, ..., n$ the following equality holds:*

$$\mathbf{E}(\int_0^\vartheta w^i(s)ds / w(\vartheta)) = \frac{r^2}{2(n+2)} w^i(\vartheta) \tag{2.14}$$

Proof. The equality (2.14) will be proved if we prove the following relation

$$\mathbf{E}(\varphi(w(\vartheta)) \cdot \int_0^\vartheta w^i(s)ds) = \frac{r^2}{2(n+2)}\mathbf{E}(\varphi(w(\vartheta)) \cdot w^i(\vartheta)) \tag{2.15}$$

for a sufficiently large class of functions φ. We shall prove (2.15) for all $\varphi(x) \in \mathbf{C}(|x| = r)$. Let us extend the function $\varphi(x) \in \mathbf{C}(|x| = r)$ to a function $\varphi(x) \in \mathbf{C}(|x| \le r)$ as the harmonic function in the open ball $U_r \backslash \partial U_r$, i.e.,

$$\frac{1}{2}\Delta\varphi = 0, \; |x| < r \tag{2.16}$$

Hence due to (2.4) and (2.5)

$$\varphi(\xi) = \int_{\partial U_r} \frac{r^2 - |\xi|^2}{\sigma_n r |\xi - \eta|^n} \varphi(\eta) dS_\eta, \; |\xi| < r \tag{2.17}$$

As $w(\vartheta)$ has the uniform distribution on the sphere ∂U_r, we have

$$\mathbf{E}(\varphi(w(\vartheta)) \cdot w^i(\vartheta)) = \frac{1}{\sigma_n r^{n-1}} \int_{\partial U_r} \varphi(\eta)\eta^i dS_\eta \tag{2.18}$$

Thanks to Ito's formula

$$d(\varphi(w(t)) \cdot \int_0^t w^i(s)ds) = \sum_{k=1}^n \frac{\partial\varphi}{\partial x^k}(w(t)) \cdot \int_0^t w^i(s)ds \cdot dw^k(t)$$

$$+\varphi(w(t)) \cdot w^i(t) + \frac{1}{2}\sum_{k=1}^n \frac{\partial^2\varphi}{\partial x^{k^2}}(w(t)) \cdot \int_0^t w^i(s)ds \cdot dt$$

Taking into account (2.16) we obtain from here

$$\mathbf{E}(\varphi(w(\vartheta)) \cdot \int_0^\vartheta w^i(s)ds) = \mathbf{E}(\int_0^\vartheta \varphi(w(s)) \cdot w^i(s)ds) \tag{2.19}$$

Thus from (2.3)

$$\mathbf{E}(\varphi(w(\vartheta)) \cdot \int_0^\vartheta w^i(s)ds) = u(0) \tag{2.20}$$

where $u(x)$ is the solution to the problem

$$\frac{1}{2}\Delta u + \varphi(x) \cdot x^i = 0, \; |x| < r; \; u_{||x|=r} = 0 \tag{2.21}$$

Using now (2.4) and (2.7) for $x = 0$ we obtain in the case $n > 2$

$$u(0) = \int_{|\xi|<r} \frac{1}{(n-2)\sigma_n} \cdot (\frac{1}{|\xi|^{n-2}} - \frac{1}{r^{n-2}}) \cdot \varphi(\xi) \cdot \xi^i d\xi$$

Substituting $\varphi(\xi)$ from (2.17) and using Fubini's theorem we can write

$$u(0) = \frac{1}{(n-2)\sigma_n^2} \int_{|\eta|=r} [\int_{|\xi|<r} (\frac{1}{|\xi|^{n-2}} - \frac{1}{r^{n-2}}) \cdot \frac{r^2 - |\xi|^2}{r|\xi - \eta|^n} \cdot \xi^i d\xi] \cdot \varphi(\eta) dS_\eta \tag{2.22}$$

Let us calculate the integral over $|\xi| < r$ in (2.22). For definiteness take $i = 1$ and let $\eta = (\eta^1, ..., \eta^n)$, ${\eta^1}^2 + ... + {\eta^n}^2 = r^2$. If $\eta^1 = 0$ then the integral over $|\xi| < r$ is obviously equal to zero. Let $\eta^1 \neq 0$. Introduce the vector $\eta_0 = (r, 0, ..., 0)$ and consider the orthogonal transformation $T = \{t_{ij}\}$, $i, j = 1, ..., n$, such that

$$T\eta = \eta_0 \tag{2.23}$$

It follows from (2.23) that the vectors $(t_{k1}, ..., t_{kn})$, $k = 2, ..., n$, are orthogonal to the vector η and consequently the vector $(t_{11}, ..., t_{1n})$ is collinear with the vector η, i.e.,

$$t_{11} = \frac{\eta^1}{r},\ t_{12} = \frac{\eta^2}{r}, ...,\ t_{1n} = \frac{\eta^n}{r}$$

Let us change variables in the integral over $|\xi| < r$ according to the formula

$$\xi = T^{-1}\zeta$$

Note that

$$\xi^1 = t_{11}\zeta^1 + t_{21}\zeta^2 + ... + t_{n1}\zeta^n = \frac{\eta^1}{r}\zeta^1 + t_{21}\zeta^2 + ... + t_{n1}\zeta^n$$

As T is orthogonal and all the components of η_0 beginning from the second component are equal to zero, we obtain

$$\int_{|\xi|<r} (\frac{1}{|\xi|^{n-2}} - \frac{1}{r^{n-2}}) \cdot \frac{r^2 - |\xi|^2}{r|\xi - \eta|^n} \cdot \xi^1 d\xi$$

$$= \int_{|\zeta|<r} (\frac{1}{|\zeta|^{n-2}} - \frac{1}{r^{n-2}}) \cdot \frac{r^2 - |\zeta|^2}{r|\zeta - \eta_0|^n} \cdot (\frac{\eta^1}{r}\zeta^1 + t_{21}\zeta^2 + ... + t_{n1}\zeta^n) d\zeta = C_n \eta^1 \tag{2.24}$$

where

$$C_n = \frac{1}{r} \int_{|\zeta|<r} (\frac{1}{|\zeta|^{n-2}} - \frac{1}{r^{n-2}}) \cdot \frac{r^2 - |\zeta|^2}{r|\zeta - \eta_0|^n} \cdot \zeta^1 d\zeta$$

To calculate C_n put in (2.20), (2.22) $i = 1$ and $\varphi(x) = x^1$ ($\varphi(x) = x^1$ is evidently harmonic). For such a function from (2.20) and (2.11) we have

$$u(0) = \mathbf{E} \int_0^\vartheta {w^1}^2(s) ds = \frac{r^4}{2n(n+2)}$$

But from (2.22) and (2.24)

$$u(0) = \frac{1}{(n-2)\sigma_n^2} \int_{|\eta|=r} C_n {\eta^1}^2 dS_\eta$$

and evidently

$$\int_{|\eta|=r} {\eta^1}^2 dS_\eta = \frac{\sigma_n r^{n-1}}{n}$$

Therefore

$$C_n = \frac{n-2}{2(n+2)} \cdot \frac{\sigma_n}{r^{n-3}}$$

Now for $\varphi(x) \in \mathbf{C}(|x| = r)$ from (2.20), (2.22) and (2.18)

$$\mathbf{E}(\varphi(w(\vartheta)) \cdot \int_0^\vartheta w^i(s)ds) = u(0) =$$

$$\frac{1}{(n-2)\sigma_n^2} \cdot \frac{n-2}{2(n+2)} \cdot \frac{\sigma_n}{r^{n-3}} \int_{|\eta|=r} \eta^i \varphi(\eta) dS_\eta = \frac{r^2}{2(n+2)} \cdot \mathbf{E}(\varphi(w(\vartheta)) \cdot w^i(\vartheta))$$

Thus, the theorem is proved for $n > 2$. The case $n = 2$ can be considered quite analogously. Consider finally the case $n = 1$. For even functions φ the relation

$$\mathbf{E}(\varphi(w(\vartheta)) \cdot \int_0^\vartheta w(s)ds) = \frac{r^2}{6}\mathbf{E}(\varphi(w(\vartheta)) \cdot w(\vartheta)) \tag{2.25}$$

is evidently fulfilled because both sides of (2.25) are equal to zero. Let φ be odd and $\varphi(-r) = -\varphi(r) = c$. Then one can take the function $\varphi(x) = \frac{c}{r}x$ as a function φ in both sides of (2.25) and obtain (2.25) as a consequence of (2.11). Theorem 2.1 is proved in full.

Lemma 2.2. *Let φ be harmonic in U_r and $\varphi \in \mathbf{C}^3(|x| \leq r)$. Then*

$$\mathbf{E}\int_0^\vartheta \frac{\partial \varphi}{\partial x^i}(w(s)) \cdot w^j(s)ds =$$

$$\mathbf{E}\int_0^\vartheta \frac{\partial \varphi}{\partial x^j}(w(s)) \cdot w^i(s)ds = \frac{r^2}{2(n+2)}\mathbf{E}\int_0^\vartheta \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(w(s))ds \tag{2.26}$$

Proof. Let ψ be a harmonic function and $\psi \in \mathbf{C}^2(|x| \leq r)$. Due to harmonicity of ψ we have from Ito's formula that

$$d\psi(w(t)) \cdot w^i(t) = \sum_{k=1}^n \frac{\partial \psi}{\partial x^k}(w(t)) \cdot w^i(t) dw^k(t)$$

$$+\psi(w(t))dw^i(t) + \frac{\partial \psi}{\partial x^i}(w(t))dt$$

and hence

$$\mathbf{E}(\psi(w(\vartheta)) \cdot w^i(\vartheta)) = \mathbf{E}\int_0^\vartheta \frac{\partial \psi}{\partial x^i}(w(s))ds \tag{2.27}$$

Using (2.19), (2.15) and (2.27) we obtain

$$\mathbf{E}\int_0^\vartheta \psi(w(s)) \cdot w^i(s)ds = \frac{r^2}{2(n+2)}\mathbf{E}\int_0^\vartheta \frac{\partial \psi}{\partial x^i}(w(s))ds \tag{2.28}$$

But the function $\psi(x) = \frac{\partial \varphi}{\partial x^j}(x) \in \mathbf{C}^2(|x| \leq r)$ is harmonic. Substituting it in (2.28) we arrive at (2.26). Lemma 2.2 is proved.

Theorem 2.2. *For every $i, j = 1, ..., n$ the following formulae hold:*

$$\mathbf{E}(\int_0^\vartheta w^i(s)dw^i(s)/w(\vartheta)) = \frac{1}{2}w^{i^2}(\vartheta) - \frac{r^2}{2n} \tag{2.29}$$

$$\mathbf{E}(\int_0^{\vartheta} w^i(s)dw^j(s)/w(\vartheta)) = \frac{1}{2}w^i(\vartheta)w^j(\vartheta),\ i \neq j \tag{2.30}$$

Proof. The equality (2.29) is obvious since ϑ does not depend on $w(\vartheta)$ and $\mathbf{E}\vartheta = r^2/n$. For (2.30) it is sufficient to prove

$$\mathbf{E}(\varphi(w(\vartheta)) \cdot \int_0^{\vartheta} w^i(s)dw^j(s)) = \frac{1}{2}\mathbf{E}(\varphi(w(\vartheta)) \cdot w^i(\vartheta)w^j(\vartheta)) \tag{2.31}$$

for any φ which is the trace of harmonic $\varphi \in \mathbf{C}^3(|x| \leq r)$ on ∂U_r.

We have

$$d(\varphi(w(t)) \cdot \int_0^t w^i(s)dw^j(s)) = \sum_{k=1}^{n} \frac{\partial\varphi}{\partial x^k}(w(t)) \cdot \int_0^t w^i(s)dw^j(s) \cdot dw^k(t)$$

$$+\varphi(w(t)) \cdot w^i(t)dw^j(t) + \frac{\partial\varphi}{\partial x^j}(w(t)) \cdot w^i(t)dt$$

From here and from Lemma 2.2

$$\mathbf{E}(\varphi(w(\vartheta)) \cdot \int_0^{\vartheta} w^i(s)dw^j(s)) = \mathbf{E}\int_0^{\vartheta} \frac{\partial\varphi}{\partial x^j}(w(s)) \cdot w^i(s)ds$$

$$= \mathbf{E}\int_0^{\vartheta} \frac{\partial\varphi}{\partial x^i}(w(s)) \cdot w^j(s)ds = \mathbf{E}(\varphi(w(\vartheta)) \cdot \int_0^{\vartheta} w^j(s)dw^i(s)) \tag{2.32}$$

But

$$\int_0^{\vartheta} w^j(s)dw^i(s) = w^i(\vartheta)w^j(\vartheta) - \int_0^{\vartheta} w^i(s)dw^j(s) \tag{2.33}$$

The relation (2.31) follows from (2.32) and (2.33). Theorem 2.2 is proved.

Introduce the functions

$$h_m(x) = \mathbf{E}\vartheta_x^m,\ m = 1, 2, ...$$

where $x \in U_r$, ϑ_x is the first passage time of the process $x + w(t)$ to the sphere ∂U_r.

As it follows from one of Dynkin's theorems (see [1], Theorem 13.17), the function $h_m(x)$ is the only solution to the following Dirichlet problem

$$\frac{1}{2}\Delta h_1 + 1 = 0,\ h_1 \mid_{\partial U_r} = 0$$

$$\frac{1}{2}\Delta h_m + mh_{m-1}(x) = 0,\ h_m \mid_{\partial U_r} = 0,\ m = 2, 3, ... \tag{2.34}$$

The solution of the problem is obviously a function of the variable $\chi = (x, x)^{1/2} = |x|,\ 0 \leq \chi \leq r$. We denote this function as $q_m(\chi)$. We easily obtain the following boundary value problem for $n > 1$ (we recall that n is a dimension of the Wiener process $w(t)$)

$$\frac{1}{2}q_1^{''} + \frac{n-1}{2\chi}q_1^{'} + 1 = 0,\ q_1(0) < \infty,\ q_1(r) = 0$$

$$\frac{1}{2}q_m^{''} + \frac{n-1}{2\chi}q_m^{'} + mq_{m-1}(\chi) = 0,\ q_m(0) < \infty,\ q_m(r) = 0 \tag{2.35}$$

We mark that if $n = 1$ then (2.34) can be rewritten in the form

$$\frac{1}{2}h_m^{''} + mh_{m-1}(x) = 0,\ h_m(-r) = h_m(r) = 0$$

The equations (2.35) are solvable by quadratures. One can also find the required solution in the form

$$q_m(\chi) = \alpha_0\chi^{2m} + \alpha_1\chi^{2(m-1)}r^2 + \alpha_2\chi^{2(m-2)}r^4 + ... + \alpha_m r^{2m}$$

By such a way we can sequentially obtain

$$h_1(x) = \frac{r^2 - |x|^2}{n}$$

$$h_2(x) = \frac{|x|^4}{n(n+2)} - \frac{2r^2|x|^2}{n^2} + \frac{(n+4)r^4}{n^2(n+2)}$$

and so on.

In particular

$$\mathbf{E}\vartheta = \frac{r^2}{n},\ \mathbf{E}\vartheta^2 = \frac{n+4}{n^2(n+2)}r^4,\ \mathbf{D}\vartheta = \frac{2}{n^2(n+2)}r^4 \tag{2.36}$$

But with growth of m such formulae become complicated. For example,

$$\mathbf{E}\vartheta^3 = \frac{n^2+12n+48}{n^3(n+2)(n+4)}r^6$$

Therefore, it is useful to obtain some simple bounds for $h_m(x)$.

Lemma 2.3. *The following bounds*

$$\frac{1}{n^m}(r^2 - |x|^2)^m \le h_m(x) \le \frac{m!}{n^m}r^{2m-2}(r^2 - |x|^2),\ m = 1, 2, ... \tag{2.37}$$

hold. Consequently

$$\frac{1}{n^m}r^{2m} \le \mathbf{E}\vartheta^m \le \frac{m!}{n^m}r^{2m} \tag{2.38}$$

and for $\lambda < n/r^2$

$$\mathbf{E}\exp(\lambda\vartheta) \le \frac{n}{n - \lambda r^2} \tag{2.39}$$

Proof. The inequalities (2.37) are true for $m = 1$ because $h_1(x) = (r^2 - |x|^2)/n$. Let the right part of (2.37) be true for the number m. Consider the function $\bar{h}_{m+1}(x)$ satisfying the equation

$$\frac{1}{2}\Delta\bar{h}_{m+1} + (m+1)\frac{m!}{n^m}r^{2m-2}(r^2 - |x|^2) = 0,\ \bar{h}_{m+1}\mid_{\partial U_r} = 0 \tag{2.40}$$

The function $h_{m+1}(x)$ satisfies the following equation (see (2.34))

$$\frac{1}{2}\Delta h_{m+1} + (m+1)h_m(x) = 0,\ h_{m+1}\mid_{\partial U_r} = 0 \tag{2.41}$$

Due to the inductive hypothesis we get from (2.40) and (2.41) that

$$h_{m+1}(x) \le \bar{h}_{m+1}(x) \tag{2.42}$$

Consider now

$$\tilde{h}_{m+1}(x) = \frac{(m+1)!}{n^{m+1}}r^{2m}(r^2 - |x|^2)$$

We obtain directly

$$\frac{1}{2}\Delta\tilde{h}_{m+1} + \frac{(m+1)!}{n^m}r^{2m} = 0,\ \tilde{h}_{m+1}\mid_{\partial U_r} = 0 \tag{2.43}$$

But

$$\frac{(m+1)!}{n^m} r^{2m} \geq \frac{(m+1)!}{n^m} r^{2m-2}(r^2 - |x|^2),\ 0 \leq |x| \leq r$$

Hence it follows from (2.40), (2.43) and (2.42) that

$$h_{m+1}(x) \leq \bar{h}_{m+1}(x) \leq \tilde{h}_{m+1}(x)$$

The right side of (2.37) is proved.

Now prove the left part of the inequality (2.37). Introduce the function

$$\bar{h}_m(x) = \frac{1}{n^m}(r^2 - |x|^2)^m$$

Due to the inductive hypothesis $\bar{h}_m(x) \leq h_m(x)$. For $\bar{h}_{m+1}(x)$ we obtain directly

$$\frac{1}{2}\Delta \bar{h}_{m+1} = -\frac{m+1}{n^m}(r^2 - |x|^2)^m + \frac{4m(m+1)}{n^{m+1}}(r^2 - |x|^2)^{m-1}|x|^2 \geq$$

$$-(m+1)\bar{h}_m(x) \geq -(m+1)h_m(x),\ \bar{h}_{m+1}\ |_{\partial U_r} = 0 \tag{2.44}$$

Comparing (2.41) with (2.44) we get

$$\bar{h}_{m+1}(x) \leq h_{m+1}(x)$$

The inequalities (2.37) imply (2.38) and (2.39) easily. Lemma 2.3 is proved.

Lemma 2.4. *Let $\varphi(t)$ be an $\mathcal{F}_t$-measurable process with continuous sample functions. Let*

$$\mathbf{E}\int_0^{\vartheta} \varphi^2(s)ds < \infty \tag{2.45}$$

Then

$$\mathbf{E}(\max_{0\leq t\leq \vartheta} |\int_0^t \varphi(s)dw_i(s)|)^{2m} \leq$$

$$K(\mathbf{E}\max_{0\leq s\leq \vartheta}|\varphi(s)|^{4m})^{1/2}\cdot r^{2m},\ i = 1, ..., n;\ m = 1, 2, ... \tag{2.46}$$

Proof. Let τ be the first passage time of the process

$$\Phi_i(t) = \int_0^t \chi_{\vartheta \geq s}\varphi(s)dw_i(s) = \int_0^{\vartheta} \varphi(s)dw_i(s)$$

to the endpoints of the interval $(-R, R)$. Introduce

$$Z(t) = \int_0^t \chi_{\tau\wedge\vartheta\geq s}\varphi(s)dw_i(s) = \int_0^{\tau\wedge\vartheta} \varphi(s)dw_i(s)$$

Of course, $Z(t)$ depends on i and R. Clearly $|Z(t)| \leq R$.

We have

$$dZ^{2m}(s) = 2mZ^{2m-1}(s)\chi_{\tau\wedge\vartheta\geq s}\varphi(s)dw_i(s) + m(2m-1)Z^{2m-2}(s)\chi_{\tau\wedge\vartheta\geq s}\varphi^2(s)ds \tag{2.47}$$

Due to the boundedness of $Z(t)$ and the condition (2.45)

$$\mathbf{E}\int_0^t Z^{4m-2}(s)\chi_{\tau\wedge\vartheta\geq s}\varphi^2(s)ds < \infty$$

Hence from (2.47)

$$\mathbf{E}|Z(t)|^{2m} = \mathbf{E}\int_0^t m(2m-1)|Z(s)|^{2m-2}\chi_{\tau\wedge\vartheta\geq s}\varphi^2(s)ds \leq$$

$$m(2m-1)\mathbf{E}(\max_{0\le s\le t}|Z(s)|^{2m-2}\cdot\max_{0\le s\le\vartheta}\varphi^2(s)\cdot\vartheta) \tag{2.48}$$

By applying the Hölder inequality with $p=\dfrac{2m}{2m-2}$ (see such a reception, for instance, in [3]) we get

$$\mathbf{E}|Z(t)|^{2m}\le m(2m-1)(\mathbf{E}\max_{0\le s\le t}|Z(s)|^{2m})^{\frac{2m-2}{2m}}\cdot(\mathbf{E}(\max_{0\le s\le\vartheta}|\varphi(s)|^{2m}\cdot\vartheta^m))^{\frac{1}{m}} \tag{2.49}$$

As $Z(t)$ is a martingale, we can use the Doob inequality

$$\mathbf{E}\max_{0\le s\le t}|Z(s)|^{2m}\le(\frac{2m}{2m-1})^{2m}\mathbf{E}|Z(t)|^{2m}$$

From here and (2.49) and then from the Cauchy-Bunyakovskii inequality and Lemma 2.3 we have

$$\mathbf{E}\max_{0\le s\le t}|Z(s)|^{2m}\le K\mathbf{E}(\max_{0\le s\le\vartheta}|\varphi(s)|^{2m}\cdot\vartheta^m)\le$$

$$K(\mathbf{E}\max_{0\le s\le\vartheta}|\varphi(s)|^{4m})^{1/2}\cdot(\mathbf{E}\vartheta^{2m})^{1/2}\le K(\mathbf{E}\max_{0\le s\le\vartheta}|\varphi(s)|^{4m})^{1/2}\cdot r^{2m}$$

As the right side of this inequality does not depend on t and R, we can direct them to infinity and obtain the inequality (2.46). Lemma 2.4 is proved.

3. One-step approximations

Let for definiteness $x\in G\backslash\Gamma_{\alpha r}$ (see Introduction). Then $U(x)$ is a ball of radius ar with centre at x. Let ϑ be the first passage-time of the Wiener process $w(t)$ to the sphere ∂U_r. Then $X_x(\vartheta)=x+aw(\vartheta)\in\partial U(x)$ and ϑ is the first passage time of the solution $X_x(t)$ of the equation (1.13) to the sphere $\partial U(x)$. Consider the solution $X_x(t)$, $Y_{x,y}(t)$, $Z_{x,y,z}(t)$ of the system (1.13)–(1.14), (1.10) at the time ϑ : $X_x(\vartheta)$, $Y_{x,y}(\vartheta)$, $Z_{x,y,z}(\vartheta)$. Clearly, $X_1=X_x(\vartheta)$ has the uniform distribution on $\partial U(x)$ and it can be simulated exactly. Our aim is to construct approximation Y_1, Z_1 for $Y_{x,y}(\vartheta)$, $Z_{x,y,z}(\vartheta)$ so that the difference

$$d=\mathbf{E}(v(X_1,Y_1,Z_1)-v(X_x(\vartheta),Y_{x,y}(\vartheta),Z_{x,y,z}(\vartheta)))$$

$$=\mathbf{E}(u(X_1)Y_1+Z_1-u(X_x(\vartheta))Y_{x,y}(\vartheta)-Z_{x,y,z}(\vartheta))$$

$$=\mathbf{E}u(X_x(\vartheta))(Y_1-Y_{x,y}(\vartheta))+\mathbf{E}(Z_1-Z_{x,y,z}(\vartheta)) \tag{3.1}$$

should be small.

Repeatedly applying Ito's formula like Wagner-Platen expansion [18], [8], [5] we can obtain the following formula

$$Y_{x,y}(\vartheta)=y+\frac{1}{a}y\sum_{i=1}^n b^i(x)w^i(\vartheta)+c(x)y\vartheta+\frac{1}{a^2}y\sum_{i=1}^n\sum_{j=1}^n b^i(x)b^j(x)\int_0^\vartheta w^j(t)dw^i(t)$$

$$+y\sum_{i=1}^n\sum_{j=1}^n\frac{\partial b^i}{\partial x^j}(x)\int_0^\vartheta w^j(t)dw^i(t)+\rho_{11}+\rho_{12}+\rho_{13} \tag{3.2}$$

where

$$\rho_{11}=a\sum_{i=1}^n\int_0^\vartheta\int_0^t\frac{\partial c}{\partial x^i}(X_x(s))Y_{x,y}(s)dw^i(s)dt$$

$$+\frac{1}{a}\sum_{i=1}^{n}\int_0^{\vartheta}\int_0^{t} c(X_x(s))b^i(X_x(s))Y_{x,y}(s)dw^i(s)dt$$

$$+\frac{1}{a}\sum_{i=1}^{n}\int_0^{\vartheta}\int_0^{t} c(X_x(s))b^i(X_x(s))Y_{x,y}(s)dsdw^i(t)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^{\vartheta}\int_0^{t}(\frac{a}{2}\frac{\partial^2 b^i}{\partial x^{j^2}}(X_x(s)) + \frac{1}{a}\frac{\partial b^i}{\partial x^j}(X_x(s))b^j(X_x(s)))Y_{x,y}(s)dsdw^i(t) \tag{3.3}$$

$$\rho_{12} = \int_0^{\vartheta}\int_0^{t}(c^2(X_x(s)) + \frac{a^2}{2}\sum_{i=1}^{n}\frac{\partial^2 c}{\partial x^{i^2}}(X_x(s)))Y_{x,y}(s)dsdt$$

$$+\sum_{i=1}^{n}\int_0^{\vartheta}\int_0^{t}\frac{\partial c}{\partial x^i}(X_x(s))b^i(X_x(s))Y_{x,y}(s)dsdt \tag{3.4}$$

and ρ_{13} contains a sum of integrals like

$$I_{i_1,i_2,i_3} = \int_0^{\vartheta}\int_0^{t}\int_0^{s} f^{i_1 i_2 i_3}(X_x(s_1))Y_{x,y}(s_1)dw^{i_1}(s_1)dw^{i_2}(s)dw^{i_3}(t)$$

$$i_1 = 0, 1, ..., n;\ \ i_2 \neq 0;\ \ i_3 \neq 0 \tag{3.5}$$

where $f^{i_1 i_2 i_3}$ is a finite sum of products and any product has not more than three factors of the form b^i, $\partial b^i/\partial x^j$, $\partial^2 b^i/\partial x^j \partial x^k$, $\partial^3 b^i/\partial x^j \partial x^k \partial x^l$, and c. Underline that $\rho_{13} = 0$ if $b = 0$.

For Z we have

$$Z_{x,y,z}(\vartheta) = z + g(x)y\vartheta + \rho_{21} + \rho_{22} \tag{3.6}$$

where

$$\rho_{21} = a\sum_{i=1}^{n}\int_0^{\vartheta}\int_0^{t}\frac{\partial g}{\partial x^i}(X_x(s))Y_{x,y}(s)dw^i(s)dt$$

$$+\frac{1}{a}\sum_{i=1}^{n}\int_0^{\vartheta}\int_0^{t} g(X_x(s))b^i(X_x(s))Y_{x.y}(s)dw^i(s)dt \tag{3.7}$$

and

$$\rho_{22} = \int_0^{\vartheta}\int_0^{t}(g(X_x(s))c(X_x(s)) + \frac{a^2}{2}\sum_{i=1}^{n}\frac{\partial^2 g}{\partial x^{i^2}}(X_x(s)))Y_{x,y}(s)dsdt$$

$$+\sum_{i=1}^{n}\int_0^{\vartheta}\int_0^{t}\frac{\partial g}{\partial x^i}(X_x(s))b^i(X_x(s))Y_{x,y}(s)dsdt \tag{3.8}$$

Let us put

$$Y_1 = y + \frac{1}{a}y\sum_{i=1}^{n}b^i(x)w^i(\vartheta) + c(x)y\frac{r^2}{n} + \frac{1}{2a^2}y\sum_{i=1}^{n}\sum_{j=1}^{n}b^i(x)b^j(x)w^i(\vartheta)w^j(\vartheta)$$

$$-\frac{1}{2a^2}y\frac{r^2}{n}\sum_{i=1}^{n}b^{i^2}(x) + \frac{1}{2}y\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial b^i}{\partial x^j}(x)w^i(\vartheta)w^j(\vartheta) - \frac{1}{2}y\frac{r^2}{n}\sum_{i=1}^{n}\frac{\partial b^i}{\partial x^i}(x) \tag{3.9}$$

$$Z_1 = z + g(x)y\frac{r^2}{n} \tag{3.10}$$

We note that $Y_1 > 0$ for sufficiently small r as it has been supposed $y > 0$.

We have

$$d = \mathbf{E}(u(X_x(\vartheta))\mathbf{E}(Y_1 - Y_{x,y}(\vartheta) \mid w(\vartheta))) + \mathbf{E}(Z_1 - Z_{x,y,z}(\vartheta)) \tag{3.11}$$

Due to the relation $\mathbf{E}(\vartheta \mid w(\vartheta)) = \mathbf{E}\vartheta = r^2/n$, Theorem 2.2, formulae (3.2) and (3.6) we have from here that

$$d = -\mathbf{E}u(X_x(\vartheta)\mathbf{E}(\rho_{11} + \rho_{12} + \rho_{13} \mid w(\vartheta)) - \mathbf{E}(\rho_{21} + \rho_{22}) =$$

$$-\mathbf{E}u(X_x(\vartheta)(\rho_{11} + \rho_{12} + \rho_{13}) - \mathbf{E}(\rho_{21} + \rho_{22}) \tag{3.12}$$

Introduce the following integrals

$$I_{i_1}(t; f, r) = \int_0^{t\wedge\vartheta} f(X_x(s))Y_{x,y}(s)dw_{i_1}(s) = \int_0^t \chi_{\vartheta\geq s} f(X_x(s))Y_{x,y}(s)dw_{i_1}(s)$$

$$I_{i_1,\dots,i_k}(t; f, r) = \int_0^{t\wedge\vartheta}\int_0^{t_1}\dots\int_0^{t_{k-1}} f(X_x(t_k))Y_{x,y}(t_k)dw_{i_1}(t_k)\dots dw_{i_k}(t_1),\ k > 1 \tag{3.13}$$

where the indices $i_1, \dots, i_k$ take values in the set $\{0, 1, \dots, n\}$, and where $dw_0(t)$ is understood to mean dt.

We set $I_{i_1,\dots,i_k}(f, r) := I_{i_1,\dots,i_k}(\vartheta; f, r)$.

The following lemma will be used below.

Lemma 3.1. *Let r be sufficiently small, ϑ be the first passage time of $w(t)$ to the sphere ∂U_r, and f be a continuous function defined in $\bar{U}_{r_0}$, $r \leq r_0$. Then for all sufficiently small r the integral $I_{i_1,\dots,i_k}(f, r)$ satisfies the inequality*

$$(\mathbf{E}|I_{i_1,\dots,i_k}(f, r)|^{2m})^{1/2m} \leq Kyr^{\sum_{j=1}^k(1+\delta_{i_j})},\ m = 1, 2, \dots \tag{3.14}$$

where K is a constant depending on k and m, and

$$\delta_{i_j} = \begin{cases} 1, & i_j = 0 \\ 0, & i_j \neq 0 \end{cases}$$

i.e., the degree of smallness of the integral $I_{i_1,\dots,i_k}(f, r)$ with respect to r can be guided by the following rule: dt contributes two to the order of smallness, and $dw_i(t)$, $i = 1, \dots, n$, contributes one.

Furthermore, if at least one index i_j, $j = 1, \dots, k$, is not equal to zero then

$$\mathbf{E}I_{i_1,\dots,i_k}(f, r) = 0,\ \sum_{j=1}^k i_j^2 \neq 0 \tag{3.15}$$

Proof. The last assertion of this lemma is obvious. We shall prove (3.14) by induction on k. And we shall prove more. Namely, we prove for any $m = 1, 2, \dots$ that

$$(\mathbf{E}\max_{0\leq t\leq\vartheta}|I_{i_1,\dots,i_k}(t; f, r)|^{2m})^{1/2m} \leq Kyr^{\sum_{j=1}^k(1+\delta_{i_j})},\ m = 1, 2, \dots \tag{3.16}$$

We note that various constants in the proof are given the same letter K.

Let $k = 1$. Let $|f(x)| \leq K$ for $x \in \bar{U}_{r_0}$. If $i_1 = 0$ (i.e., $dw_{i_1}(t) = dt$) then

$$\mathbf{E}\max_{0\leq t\leq\vartheta}|I_0(t; f, r)|^{2m} \leq K\mathbf{E}(\vartheta^{2m}\max_{0\leq t\leq\vartheta}Y_{x,y}^{2m}(t)) \leq$$

$$K(\mathbf{E}\vartheta^{4m})^{1/2}(\mathbf{E}\max_{0\leq t\leq\vartheta}Y_{x,y}^{4m}(t))^{1/2} \tag{3.17}$$

If $i_1 \neq 0$ then according to Lemma 2.4

$$\mathbf{E} \max_{0\le t\le \vartheta} |I_{i_1}(t; f, r)|^{2m} \le K r^{2m} (\mathbf{E} \max_{0\le t\le \vartheta} Y_{x,y}^{4m}(t))^{1/2} \tag{3.18}$$

Let us prove that $\mathbf{E}\max_{0\le t\le\vartheta} Y_{x,y}^{4m}(t) < \infty$.

Since $c(x) \le 0$, then

$$0 < Y_{x,y}(t) \le \bar{Y}_{x,y}(t)$$

where $\bar{Y}_{x,y}(t)$ is a positive martingale satisfying the following equation

$$d\bar{Y} = \frac{1}{a} b^\top(X)\bar{Y} dw(t)$$

We have

$$d\bar{Y}^{4m} = \frac{4m}{a} b^\top(X)\bar{Y}^{4m} dw(t) + \frac{4m(4m-1)}{2}\frac{1}{a^2}|b(X)|^2 \bar{Y}^{4m} dt$$

From here

$$\mathbf{E} \max_{0\le t\le\vartheta} Y_{x,y}^{4m}(t) \le \mathbf{E} \max_{0\le t\le\vartheta} \bar{Y}_{x,y}^{4m}(t) \le (\frac{4m}{4m-1})^{4m} \mathbf{E}\bar{Y}_{x,y}^{4m}(\vartheta) =$$

$$(\frac{4m}{4m-1})^{4m} \mathbf{E} \exp\left\{ \frac{2m(4m-1)}{a^2} \int_0^\vartheta |b(X_x(s))|^2 ds \right\} \le K\mathbf{E}\exp\{B\vartheta\} \tag{3.19}$$

where B is a constant.

Now (3.16) for $k = 1$ and for all sufficiently small r follows from (3.17)–(3.19) and from Lemma 2.3 (see (2.39)).

Due to inductive hypothesis and (2.38) and under $dw_{i_{k+1}}(t) = dt$ we have (underline that the inequality (3.16) under given k is true for all m and, in particular, for $2m$)

$$\mathbf{E} \max_{0\le t\le\vartheta} |I_{i_1,\dots,i_k,i_{k+1}}(t; f, r)|^{2m} = \mathbf{E} \max_{0\le t\le\vartheta} | \int_0^t \chi_{\vartheta\ge s} I_{i_1,\dots,i_k}(s; f, r) ds|^{2m} \le$$

$$\mathbf{E}(\max_{0\le t\le\vartheta} |I_{i_1,\dots,i_k}(t; f, r)|^{2m} \cdot \vartheta^{2m}) \le (\mathbf{E} \max_{0\le t\le\vartheta} |I_{i_1,\dots,i_k}(t; f, r)|^{4m})^{1/2} \cdot (\mathbf{E}\vartheta^{4m})^{1/2} \le$$

$$(Kyr^{\sum_{j=1}^k (1+\delta_{i_j})})^{2m} \cdot r^{4m}$$

From here (as $i_{k+1} = 1$)

$$(\mathbf{E} \max_{0\le t\le\vartheta} |I_{i_1,\dots,i_k,i_{k+1}}(t; f, r)|^{2m})^{1/2m} \le Kyr^{\sum_{j=1}^k (1+\delta_{i_j})} \cdot r^2 = Kyr^{\sum_{j=1}^{k+1} (1+\delta_{i_j})}$$

i.e., the inequality (3.16) is proved for $i_{k+1} = 1$.

Now let $i_{k+1} = i \neq 0$. Then due to Lemma 2.4

$$\mathbf{E} \max_{0\le t\le\vartheta} |I_{i_1,\dots,i_k,i_{k+1}}(t; f, r)|^{2m} = \mathbf{E} \max_{0\le t\le\vartheta} | \int_0^t \chi_{\vartheta\ge s} I_{i_1,\dots,i_k}(s; f, r) dw_{i_{k+1}}(s)|^{2m} \le$$

$$K(\mathbf{E} \max_{0\le t\le\vartheta} |I_{i_1,\dots,i_k}(t; f, r)|^{4m})^{1/2} \cdot (\mathbf{E}\vartheta^{2m})^{1/2} \le (Kyr^{\sum_{j=1}^k (1+\delta_{i_j})})^{2m} \cdot r^{2m}$$

which is equivalent to (3.16). Lemma 3.1 is proved in full.

Let us return to (3.12). According to the mean value theorem

$$u(X_x(\vartheta)) = u(x) + a\sum_{k=1}^n \frac{\partial u}{\partial x^k}(\xi) w^k(\vartheta) \tag{3.20}$$

where ξ is a point between x and $X_x(\vartheta)$. Let

$$M_0(x) = \max_{\xi \in \bar{U}(x)} |u(\xi)|, \ M_1(x) = \max_{\xi \in \bar{U}(x), \, 1 \le i \le n} |\frac{\partial u}{\partial x^i}(\xi)|$$

We have (see (3.15))

$$\mathbf{E}u(x)(\rho_{11} + \rho_{13}) = 0, \ \mathbf{E}\rho_{21} = 0$$

Due to Lemma 3.1 and the Cauchy-Bunyakovskii inequality

$$|d| \le M_0(x)|\mathbf{E}\rho_{12}| + aM_1(x) \sum_{k=1}^{n} \mathbf{E}|w^k(\vartheta)(\rho_{11} + \rho_{13})| + \mathbf{E}|\rho_{22}|$$

$$\le (K_0 M_0(x) + K_1 M_1(x) + K_2) y r^4 \tag{3.21}$$

where $K_0, \ K_1, \ K_2$ are constants depending only on a, b, c, and g.

So, we obtain the following theorem.

Theorem 3.1. *The one-step error $d = d(x, y, r)$ of the approximation $X_1 = x + aw(\vartheta)$ and (3.9)–(3.10) has the form (3.21), i.e., the degree of smallness of this approximation with respect to r is equal to 4.*

Consider another approximation

$$Y_1 = y + \frac{1}{a} y \sum_{i=1}^{n} b^i(x) w^i(\vartheta) + c(x) y \frac{r^2}{n} \tag{3.22}$$

$$Z_1 = z + g(x) y \frac{r^2}{n} \tag{3.23}$$

Now from (3.11) instead of (3.12) we obtain (again using Theorem 2.2)

$$d = -\mathbf{E}u(X_x(\vartheta))\bar{Y} - \mathbf{E}u(X_x(\vartheta))(\rho_{11} + \rho_{12} + \rho_{13}) - \mathbf{E}(\rho_{21} + \rho_{22}) \tag{3.24}$$

where

$$\bar{Y} = \frac{1}{2a^2} y \sum_{i=1}^{n} \sum_{j=1}^{n} b^i(x) b^j(x) w^i(\vartheta) w^j(\vartheta) - \frac{1}{2a^2} y \frac{r^2}{n} \sum_{i=1}^{n} b^{i^2}(x) +$$

$$\frac{1}{2} y \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial b^i}{\partial x^j}(x) w^i(\vartheta) w^j(\vartheta) - \frac{1}{2} y \frac{r^2}{n} \sum_{i=1}^{n} \frac{\partial b^i}{\partial x^i}(x)$$

The last two terms in (3.24) can be bounded like (3.21). For evaluating of the first term let us write down

$$u(X_x(\vartheta)) = u(x) + a \sum_{k=1}^{n} \frac{\partial u}{\partial x^k}(x) w^k(\vartheta) + \frac{1}{2} a^2 \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x^k \partial x^j}(\xi) w^k(\vartheta) w^j(\vartheta) \tag{3.25}$$

and denote

$$M_2(x) = \max_{\xi \in \bar{U}(x), \, 1 \le i,j \le n} |\frac{\partial^2 u}{\partial x^i \partial x^j}(\xi)|$$

Since

$$\mathbf{E}(u(x) + a \sum_{k=1}^{n} \frac{\partial u}{\partial x^k}(x) w^k(\vartheta))\bar{Y} = 0$$

we have obtained the following theorem.

Theorem 3.2. *The one-step error $d = d(x, y, r)$ of the approximation $X_1 = x + aw(\vartheta)$ and (3.22)–(3.23) has the form*

$$|d| \leq (K_0M_0(x) + K_1M_1(x) + K_2M_2(x) + K_3)yr^4 \tag{3.26}$$

Remark 3.1. We note that the degree of smallness of both approximations (3.9)–(3.10) and (3.22)–(3.23) is equal to 4. But the bound (3.21) does not depend on second derivatives of the function u.

Consider the case $b^i(x) = 0,\ i = 1, ..., n$. In this case

$$Y_{x,y}(\vartheta) = y + c(x)y\vartheta +$$

$$a\sum_{i=1}^{n}\int_0^{\vartheta}\int_0^{t}\frac{\partial c}{\partial x^i}(X_x(s))Y_{x,y}(s)dw^i(s)dt + \int_0^{\vartheta}\int_0^{t}c_1(X_x(s))Y_{x,y}(s)dsdt =$$

$$y + c(x)y\vartheta + a\sum_{i=1}^{n}\frac{\partial c}{\partial x^i}(x)y\int_0^{\vartheta}w^i(t)dt + c_1(x)y\frac{\vartheta^2}{2} + \rho_{11} + \rho_{12} + \rho_{13} \tag{3.27}$$

where

$$c_1(x) = c^2(x) + \frac{a^2}{2}\sum_{i=1}^{n}\frac{\partial^2 c}{\partial x^{i2}}(x)$$

$$\rho_{11} = a^2\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^{\vartheta}\int_0^{t}\int_0^{s}\frac{\partial^2 c}{\partial x^i\partial x^j}(X_x(s_1))Y_{x,y}(s_1)dw^j(s_1)dw^i(s)dt$$

$$\rho_{12} =$$

$$a\sum_{i=1}^{n}\int_0^{\vartheta}\int_0^{t}\int_0^{s}(\frac{\partial c}{\partial x^i}(X_x(s_1))c(X_x(s_1)) + \frac{1}{2}a^2\sum_{j=1}^{n}\frac{\partial^3 c}{\partial x^i\partial x^{j2}}(X_x(s_1)))Y_{x,y}(s_1)ds_1dw^i(s)dt$$

$$+a\sum_{j=1}^{n}\int_0^{\vartheta}\int_0^{t}\int_0^{s}\frac{\partial}{\partial x^j}c_1(X_x(s_1))\cdot Y_{x,y}(s_1)dw^j(s_1)dsdt$$

$$\rho_{13} = \int_0^{\vartheta}\int_0^{t}\int_0^{s}(c_1(X_x(s_1))c(X_x(s_1)) + \frac{1}{2}a^2\sum_{j=1}^{n}\frac{\partial^2 c_1}{\partial x^{j2}}(X_x(s_1)))\cdot Y_{x,y}(s_1)ds_1dsdt$$

For Z we have

$$Z_{x,y,z}(\vartheta) = z + g(x)y\vartheta +$$

$$a\sum_{i=1}^{n}\int_0^{\vartheta}\int_0^{t}\frac{\partial g}{\partial x^i}(X_x(s))Y_{x,y}(s)dw^i(s)dt + \int_0^{\vartheta}\int_0^{t}g_1(X_x(s))Y_{x,y}(s)dsdt$$

$$= z + g(x)y\vartheta + a\sum_{i=1}^{n}\frac{\partial g}{\partial x^i}(x)y\int_0^{\vartheta}w^i(t)dt + g_1(x)y\frac{\vartheta^2}{2} + \rho_{21} + \rho_{22} + \rho_{23} \tag{3.28}$$

where

$$g_1(x) = g(x)c(x) + \frac{a^2}{2}\sum_{i=1}^{n}\frac{\partial^2 g}{\partial x^{i2}}(x)$$

and ρ_{21}, ρ_{22}, ρ_{23} are similar to ρ_{11}, ρ_{12}, ρ_{13} (we do not write them down here).

We note due to Lemma 3.1 that

$$\mathbf{E}\rho_{11} = \mathbf{E}\rho_{12} = \mathbf{E}\rho_{21} = \mathbf{E}\rho_{22} = 0$$

$$\mathbf{E}w^k(\vartheta)\rho_{11} = 0, k = 1, ..., n \tag{3.29}$$

and that for ρ_{11}, ρ_{21} the degree of smallness with respect to r is equal to 4, for ρ_{12}, ρ_{22} it is equal to 5, for ρ_{13}, ρ_{23} it is equal to 6.

Consider the following one-step approximation (see Theorem 2.1 and (2.36))

$$Y_1 = y + c(x)y\frac{r^2}{n} + a\sum_{i=1}^{n}\frac{\partial c}{\partial x^i}(x)y\cdot\frac{r^2}{2(n+2)}w^i(\vartheta) + \frac{1}{2}c_1(x)y\cdot\frac{4+n}{n^2(2+n)}r^4 \tag{3.30}$$

$$Z_1 = z + g(x)y\frac{r^2}{n} + \frac{1}{2}g_1(x)y\cdot\frac{4+n}{n^2(2+n)}r^4 \tag{3.31}$$

Theorem 3.3. *The one-step error $d = d(x,y,r)$ of the approximation $X_1 = x + aw(\vartheta)$ and (3.30)–(3.31) has the form*

$$|d| \le (K_0M_0(x) + a^2K_1M_1(x) + a^4K_2M_2(x) + K_3)yr^6 \tag{3.32}$$

where K_0, K_1, K_2, K_3 depend only on c and g.

Proof. Due to Theorem 2.1, formula (2.36) for $\mathbf{E}\vartheta^2$, and (3.29) we have

$$d = \mathbf{E}(u(X_x(\vartheta))\mathbf{E}(Y_1 - Y_{x,y}(\vartheta) \mid w(\vartheta)) + \mathbf{E}(Z_1 - Z_{x,y,z}(\vartheta)) =$$

$$-\mathbf{E}u(X_x(\vartheta))(\rho_{11} + \rho_{12} + \rho_{13}) - \mathbf{E}\rho_{23}$$

Using expansions (3.20) and (3.25) we obtain

$$d = -\mathbf{E}\frac{1}{2}a^2\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x^k \partial x^j}(\xi)w^k(\vartheta)w^j(\vartheta)\rho_{11} -$$

$$\mathbf{E}a\sum_{k=1}^{n}\frac{\partial u}{\partial x^k}(\xi_1)w^k(\vartheta)\rho_{12} - \mathbf{E}u(X_x(\vartheta))\rho_{13} - \mathbf{E}\rho_{23}$$

Finally, the Cauchy-Bunyakovskii inequality and Lemma 3.1 imply (3.32).
Theorem 3.3 is proved.

It is not difficult to obtain the following result.

Theorem 3.4. *The one-step error $d = d(x,y,r)$ of the approximation*

$$X_1 = x + aw(\vartheta),\ Y_1 = y + c(x)y\frac{r^2}{n},\ Z_1 = z + g(x)y\frac{r^2}{n} \tag{3.33}$$

has the form

$$|d| \le (K_0M_0(x) + a^2K_1M_1(x) + K_2)yr^4 \tag{3.34}$$

We note that the bound (3.34) does not contain second derivatives in contrast to the case $b \neq 0$.

4. **The average number of steps**

Consider the question about average characteristics of ν. In connection with the homogeneous Markov chain X_k we introduce a one-step transition function

$$P(x, B) = \mathbf{P}(X_1 \in B \mid X_0 = x)$$

where B is a Borel set belonging to $\bar{G}$. If $x \in G \backslash \Gamma_{\alpha r}$ then $P(x, B)$ is concentrated on the surface $\partial U(x)$ of radius ar, if $x \in \Gamma_{\alpha r} \backslash \Gamma_\delta$ then $P(x, B)$ is concentrated on the surface $\partial U(x)$ of radius $\rho(x, \partial G)$, and if $x \in \Gamma_\delta$ then $P(x, B)$ is concentrated at the point x.

Define an operation P acting on functions $v(x)$, $x \in \bar{G}$, by formula

$$Pv(x) = \int_{\bar{G}} P(x, dy) v(y) = Ev(X_1), \ X_0 = x$$

and an operator

$$Av(x) = Pv(x) - v(x)$$

which is called by generator of the chain.

The generator gives an average increment of function v on the trajectory of the considering chain per step.

Consider a boundary value problem in $\bar{G}$

$$Pv(x) - v(x) = -f(x), \ x \in G \backslash \Gamma_\delta \tag{4.1}$$

$$v(x) = 0, \ x \in \Gamma_\delta \tag{4.2}$$

which is connected with the chain X_k.

In (4.1) $f(x)$ is a continuous function defined on the compact $G \backslash \Gamma_\delta : f \in \mathbf{C}(G \backslash \Gamma_\delta)$. It is not difficult to prove that there exists the only solution to the problem (4.1)–(4.2) which is a continuous function on $G \backslash \Gamma_\delta$. This solution is known (see [19]) to be the following function

$$v(x) = \mathbf{E} \sum_{k=0}^{\nu_x - 1} f(X_k), \ X_0 = x \tag{4.3}$$

where ν_x relates to the chain starting at x.

If $f \equiv 1$ then

$$v(x) = \mathbf{E}\nu_x$$

Further, if $v(x)$ is the solution of the boundary value problem (4.1)–(4.2) with the function $f(x)$ satisfying in $G \backslash \Gamma_\delta$ the inequality

$$f(x) \geq 1$$

then thanks to (4.3) we have

$$\mathbf{E}\nu_x \leq v(x) \tag{4.4}$$

Consider the function

$$V_1(x) = \begin{cases} A^2 + (h, x) - x^2, & x \in G \backslash \Gamma_\delta \\ 0, & x \in \Gamma_\delta \end{cases}$$

where the constant A^2 and the vector h are such that for all $x \in \bar{G}$ the inequality

$$A^2 + (h, x) - x^2 \geq 0$$

is fulfilled. As in [10] we obtain the following result.

Lemma 4.1. *The inequalities*

$$PV_1(x) - V_1(x) \le -a^2r^2, \; x \in G\backslash\Gamma_{\alpha r} \tag{4.5}$$

$$PV_1(x) - V_1(x) \le 0, \; x \in \Gamma_{\alpha r}\backslash\Gamma_\delta \tag{4.6}$$

hold.

Proof. Let $x \in G\backslash\Gamma_{\alpha r}$ and let $U(x)$ do not intersect with Γ_δ. The measure $P(x, B)$ concentrates on $\partial U(x)$ and due to the inclusion $U(x) \in G\backslash\Gamma_\delta$ the function $V_1(y)$ on $\partial U(x)$ is equal to $A^2 + (h, y) - y^2$. Let dS be an area element of the surface ∂U_r and let S be an area of this surface (remember U_r is a sphere with centre at the origin). We have

$$PV_1(x) = EV_1(X_1) = EV_1(x + aw(\vartheta)) =$$

$$\frac{1}{S}\int_{\partial U_r}(A^2 + (h, x + az) - (x + az)^2)dS =$$

$$A^2 + (h, x) - x^2 - \frac{1}{S}\int_{\partial U_r}(-h + 2x, az)dS - \frac{1}{S}\int_{\partial U_r} a^2z^2dS \tag{4.7}$$

Clearly

$$\int_{\partial U_r}(h + 2x, az)dS = 0, \; \frac{1}{S}\int_{\partial U_r} a^2z^2dS = a^2r^2$$

and the equality (4.7) implies

$$PV_1(x) - V_1(x) = -a^2r^2$$

Let now $x \in G\backslash\Gamma_{\alpha r}$ but the part of $U(x)$ can belong to Γ_δ. Introduce temporarily a function $\bar{V}_1(y)$ which is equal to $A^2 + (h, y) - y^2$ on the all surface $\partial U(x)$. Therefore, as in (4.7) we obtain

$$P\bar{V}_1(x) = A^2 + (h, x) - x^2 - a^2r^2$$

Since $V_1(y) \le \bar{V}_1(y)$ on $\partial U(x)$ we have $PV_1(x) \le P\bar{V}_1(x)$ and consequently the inequality (4.5) is proved for all $x \in G\backslash\Gamma_{\alpha r}$. By the same way it can be proved that for $x \in \Gamma_{\alpha r}\backslash\Gamma_\delta$ the inequality $PV_1(x) - V_1(x) \le -\rho^2(x, \partial G)$ is fulfilled. Lemma 4.1 is proved.

Now introduce the function

$$V_2(x) = \begin{cases} \ln\dfrac{\alpha r}{\delta} + 1, & x \in G\backslash\Gamma_{\alpha r} \\ \ln\dfrac{\rho(x)}{\delta} + 1, & x \in \Gamma_{\alpha r}\backslash\Gamma_\delta \\ 0, & x \in \Gamma_\delta \end{cases}$$

where $\rho(x) = \rho(x, \partial G)$.

Lemma 4.2. *If $r > 0$ is sufficiently small then the inequalities*

$$PV_2(x) - V_2(x) \le 0, \; x \in G\backslash\Gamma_{\alpha r} \tag{4.8}$$

$$PV_2(x) - V_2(x) \le -C_n, \; x \in \Gamma_{\alpha r}\backslash\Gamma_\delta \tag{4.9}$$

hold. Here C_n does not depend on x. If the set G is convex, the assumption of smallness of r can be omitted.

Proof. As $V_2(x) \leq \ln\frac{\alpha r}{\delta}+1$ for all $x \in G$ then $PV_2(x) \leq \ln\frac{\alpha r}{\delta}+1$ for all x too. Consequently, the inequality (4.8) is proved. Now let $x \in \Gamma_{\alpha r}\backslash\Gamma_\delta$ and $\rho(x) = \rho(x,\partial G) = \rho$. At the beginning we consider the case $\delta < \rho \leq \frac{\alpha r}{2}$. In the case $U(x) \subset \Gamma_{\alpha r}$ we have

$$PV_2(x) = \frac{1}{S}\int_{\partial U(x)} V_2(y)dS$$

Let $S(h)$, $0 \leq h \leq 2\rho$, be an area of spherical segment of height h. We have

$$S(h) = \frac{S}{\sqrt{\pi}} \cdot \frac{\Gamma(n/2)}{\Gamma((n-1)/2)} \cdot \frac{1}{\rho^{n-2}} \int_0^{\sqrt{2\rho h-h^2}} \frac{\eta^{n-2}}{\sqrt{\rho^2-\eta^2}}d\eta \;,\; 0 \leq h \leq \rho$$

$$S(h) = S - S(2\rho - h),\;\; \rho \leq h \leq 2\rho$$

where S is an area of sphere of radius ρ :

$$S = \frac{2\pi^{n/2}}{\Gamma(n/2)}\rho^{n-1}$$

In the case of convexity G we have

$$PV_2(x) \leq \frac{1}{S}\int_\delta^{2\rho}(\ln\frac{h}{\delta}+1)S'(h)dh \tag{4.10}$$

From here

$$PV_2(x) - V_2(x) \leq 1 + \ln 2 + \ln\frac{\rho}{\delta} - \frac{S(\delta)}{S} - \frac{1}{S}\int_\delta^{2\rho}\frac{S(h)}{h}dh - (\ln\frac{\rho}{\delta}+1)$$

$$= -\frac{S(\delta)}{S} + \frac{1}{S}\int_0^\delta \frac{S(h)}{2\rho-h}dh - \frac{1}{S}\int_\delta^\rho \frac{2(\rho-h)}{h(2\rho-h)}S(h)dh \tag{4.11}$$

For $h \leq \rho$ let us bound $\frac{S(h)}{S}$ from below. We have

$$\frac{S(h)}{S} = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(n/2)}{\Gamma((n-1)/2)} \cdot \frac{1}{\rho^{n-2}} \int_0^{\sqrt{2\rho h-h^2}} \frac{\eta^{n-2}}{\sqrt{\rho^2-\eta^2}}d\eta$$

$$\geq \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(n/2)}{\Gamma((n-1)/2)} \cdot \frac{1}{\rho^{n-2}} \int_0^{\sqrt{2\rho h-h^2}} \frac{\eta^{n-2}}{\rho}d\eta = A_n \cdot \frac{1}{\rho^{n-1}}(2\rho h-h^2)^{(n-1)/2} \tag{4.12}$$

where

$$A_n = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(n/2)}{\Gamma((n-1)/2)} \cdot \frac{1}{n-1}$$

Using the inequality $2\rho h - h^2 \geq \rho h$ under $h \leq \rho$ and continuing (4.12) we obtain

$$\frac{S(h)}{S} \geq A_n \cdot (\frac{h}{\rho})^{(n-1)/2} \tag{4.13}$$

As $\delta < \rho$ we can use (4.13) and obtain

$$-\frac{S(\delta)}{S} + \frac{1}{S}\int_0^\delta \frac{S(h)}{2\rho-h}dh < -\frac{S(\delta)}{S} + \frac{S(\delta)}{S}\int_0^\delta \frac{dh}{2\rho-h} = \frac{S(\delta)}{S}(-1+\ln\frac{2\rho}{2\rho-\delta})$$

$$\leq \frac{S(\delta)}{S}(-1+\ln 2) \leq (-1+\ln 2)\cdot A_n \cdot (\frac{\delta}{\rho})^{(n-1)/2} \tag{4.14}$$

Using (4.12) find an upper bound for the third term in the right-hand part of (4.11)

$$-\frac{1}{S}\int_{\delta}^{\rho}\frac{2(\rho-h)}{2\rho h-h^2}S(h)dh\leq$$

$$-A_n\cdot\frac{1}{\rho^{n-1}}\int_{\delta}^{\rho}2(\rho-h)(2\rho h-h^2)^{(n-3)/2}dh\leq -A_n\cdot\frac{1}{\rho^{n-1}}\int_{\delta}^{\rho}2(\rho-h)(\rho h)^{(n-3)/2}dh=$$

$$-A_n(\frac{8}{n^2-1}-\frac{4}{n-1}(\frac{\delta}{\rho})^{(n-1)/2}+\frac{4}{n+1}(\frac{\delta}{\rho})^{(n+1)/2}) \tag{4.15}$$

The relations (4.11), (4.14) and (4.15) imply

$$PV_2(x)-V_2(x)\leq -A_n(\frac{8}{n^2-1}+(1-\ln 2-\frac{4}{n-1})(\frac{\delta}{\rho})^{(n-1)/2}+\frac{4}{n+1}(\frac{\delta}{\rho})^{(n+1)/2}) \tag{4.16}$$

Remember that (4.16) is proved for x which satisfies the inequality $\delta<\rho=\rho(x,\partial G)\leq \frac{\alpha r}{2}$. Examine now $x\in\Gamma_{\alpha r}\backslash\Gamma_\delta$ with $\rho(x,\partial G)>\frac{\alpha r}{2}$. Introduce temporarily a function $\bar{V}_2(y)$ which is equal to $\bar{V}_2(y)=\ln\frac{\rho(y)}{\delta}+1$ on the all surface $\partial U(x)$. Clearly, the inequality (4.16) is fulfilled for the function $\bar{V}_2(y)$ too. But $\bar{V}_2(y)\geq V_2(y)$ on $\partial U(x)$. Consequently $PV_2(x)\leq P\bar{V}_2(x)$. As $\bar{V}_2(x)=V_2(x)$ the inequality (4.16) is proved for all $x\in\Gamma_{\alpha r}\backslash\Gamma_\delta$. It is not difficult to find for any $n=2,3,...$ a constant $C_n>0$ such that under $\rho=\rho(x)>\delta$

$$A_n(\frac{8}{n^2-1}+(1-\ln 2-\frac{4}{n-1})(\frac{\delta}{\rho})^{(n-1)/2}+\frac{4}{n+1}(\frac{\delta}{\rho})^{(n+1)/2})\geq C_n \tag{4.17}$$

If G is not necessarily convex but r is sufficiently small, we can use another inequality instead of (4.10). The new inequality is distinguished from (4.10) only by presence of a small term in the right hand side. It is easy to see that the term is $O(r^2)$. As a result we obtain (4.17) with a new constant C_n which differs from the old one by a quantity of $O(r^2)$ and, consequently, new C_n will be positive again. Lemma 4.2 is proved.

Remark 4.1. We do not aim for the highest precision and the bound (4.17) is fairly rough. For example, under $n=3$ the area $S(h)$ is equal to $Sh/2\rho$, the integral in (4.10) is equal to $\ln 2\rho/\delta$ and it holds

$$PV_2(x)-V_2(x)\leq\ln 2-1,\ x\in\Gamma_{\alpha r}\backslash\Gamma_\delta,\ n=3$$

Theorem 4.1. *If $r>0$ is sufficiently small then there exist constants B and C such that for any x*

$$E\nu_x\leq\frac{B}{a^2r^2}+C\ln\frac{\alpha r}{\delta} \tag{4.18}$$

If $\delta=O(r^p)$, $p>1$, then

$$E\nu_x\leq\frac{B+1}{a^2r^2} \tag{4.19}$$

If G is convex and $r\geq d/2$ where d is a diameter of G then the random walk is realized over touching spheres and

$$E\nu_x\leq C\ln\frac{d}{2\delta} \tag{4.20}$$

Proof. The inequalities (4.18)–(4.20) simply flow out from Lemma 4.1, Lemma 4.2, and (4.4) if in the capacity of $v(x)$ we take

$$v(x) = \frac{V_1(x)}{a^2r^2} + \frac{V_2(x)}{C_n} \tag{4.21}$$

Theorem 4.1 is proved.

Remark 4.2. Let us emphasize that the number p does not play any essential role for the upper bound of average number of steps $\mathbf{E}\nu$.

Lemma 4.3 (see [19]). *Let* $q(x) > 0,\ q \in \mathbf{C}(G\backslash\Gamma_\delta),\ f(x) \geq 0,\ f \in \mathbf{C}(G\backslash\Gamma_\delta),\ f(x) = 0$ *for* $x \in \Gamma_\delta$. *Let* $z(x)$ *be a solution to the boundary value problem*

$$q(x)Pz(x) - z(x) = -f(x),\ x \in G\backslash\Gamma_\delta \tag{4.22}$$

$$z(x) = 0,\ x \in \Gamma_\delta \tag{4.23}$$

Then for $x \in G\backslash\Gamma_\delta$

$$z(x) = f(x) + \mathbf{E}\sum_{k=1}^{\nu_x - 1} f(X_k)\Pi_{i=0}^{k-1}q(X_i) \tag{4.24}$$

Proof. We have for $x \in G\backslash\Gamma_\delta$

$$z(x) = f(x) + q(x)Pz(x) = f(x) + q(x)\mathbf{E}z(X_1) =$$

$$f(x) + q(x)\mathbf{E}(f(X_1) + q(X_1)Pz(X_1)) =$$

$$f(x) + q(x)\mathbf{E}(\chi_{\nu_x>1}f(X_1)) + q(x)\mathbf{E}(\chi_{\nu_x>1}q(X_1)\mathbf{E}(z(X_2)/X_1)) =$$

$$f(x) + q(x)\mathbf{E}(\chi_{\nu_x>1}f(X_1)) + q(x)\mathbf{E}(\chi_{\nu_x>2}q(X_1)z(X_2)) =$$

$$f(x) + q(x)\mathbf{E}(\chi_{\nu_x>1}f(X_1)) + q(x)\mathbf{E}(\chi_{\nu_x>2}q(X_1)f(X_2)) +$$

$$q(x)\mathbf{E}(\chi_{\nu_x>3}q(X_1)q(X_2)z(X_3)) = ... = f(x) + q(x)\mathbf{E}(\chi_{\nu_x>1}f(X_1)) + ... +$$

$$q(x)\mathbf{E}(\chi_{\nu_x>N}q(X_1)...q(X_{N-1})f(X_N)) + q(x)\mathbf{E}(\chi_{\nu_x>N+1}q(X_1)...q(X_N)z(X_{N+1}))$$

Turning N to infinity we obtain (4.24). Lemma 4.3 is proved.

Corollary. Let the conditions of Lemma 4.3 be fulfilled. If $q = const > 1,\ f(x) = 1$ for $x \in G\backslash\Gamma_\delta$, then

$$z(x) = \mathbf{E}(1 + q + q^2 + ... + q^{\nu_x - 1}) = \frac{1}{q-1}(\mathbf{E}q^{\nu_x} - 1)$$

If $q = const > 1,\ f(x) \geq c$ for $x \in G\backslash\Gamma_\delta$, then

$$\mathbf{E}q^{\nu_x} < \infty,\ \mathbf{E}(1 + q + ... + q^{\nu_x - 1}) \leq \frac{1}{c}z(x)$$

Lemma 4.4. *Let* $\delta = O(r^p),\ p > 1$. *Then there exists a constant* $\beta > 0$ *and a constant* $K > 0$ *such that for all sufficiently small* r

$$\mathbf{E}\sum_{k=0}^{\nu_x - 1}(1 + \beta r^2)^k = O(\frac{1}{r^2}) \tag{4.25}$$

$$\mathbf{E}(1+\beta r^2)^{\nu_x} < K \tag{4.26}$$

$$\mathbf{P}(\nu_x \geq k) \leq K(1-\beta r^2)^k \tag{4.27}$$

Proof. For the function $v(x)$ from (4.21) we have

$$(1+\beta r^2)Pv - (1+\beta r^2)v \leq -(1+\beta r^2),\ x \in G\backslash\Gamma_\delta$$

or

$$(1+\beta r^2)Pv - v \leq \beta r^2 v - (1+\beta r^2),\ x \in G\backslash\Gamma_\delta$$

$$v = 0,\ x \in \Gamma_\delta$$

Thus, the function $v(x)$ is a solution to the problem (4.22)–(4.23) with $q(x) = 1+\beta r^2$ and with $f(x)$ that satisfies the following inequality

$$f(x) \geq 1+\beta r^2 - \beta r^2 v = 1+\beta r^2 - \frac{\beta}{a^2}V_1(x) - \frac{\beta r^2}{C_n}V_2(x)$$

Clearly, $f(x) \geq 1/2$ for sufficiently small β and r. By Corollary to Lemma 4.3

$$\mathbf{E}\sum_{k=0}^{\nu_x-1}(1+\beta r^2)^k \leq 2v(x)$$

and, consequently, (4.25) is proved. The relation (4.25) implies (4.26) easily. The relation (4.27) is obtained from (4.26) with the help of the Chebyshev inequality. Lemma 4.4 is proved.

5. Convergence theorems

Here we construct a number of algorithms for the Dirichlet problem

$$\frac{1}{2}a^2\Delta u + \sum_{i=1}^n b^i(x)\frac{\partial u}{\partial x^i} + c(x)u + g(x) = 0,\ x \in G, \tag{5.1}$$

$$u\mid_{\partial G} = \varphi(x) \tag{5.2}$$

which are based on the one-step approximations obtained in Section 3.

The domain G and the coefficients $b^i(x)$, $c(x)$, $g(x)$ and the function $\varphi(x)$ in (5.1)–(5.2) are supposed to satisfy the conditions (i)–(ii) (see Introduction). We remember that Γ_δ is the interior of a δ-neighborhood of the boundary ∂G belonging to $\bar{G}$. Let $U \in \mathbf{R}^n$ be an open ball of radius $\mathbf{1}$ with centre at the origin and with the boundary ∂U. Let ξ be a point uniformly distributed on the sphere ∂U and $\xi_1,\ \xi_2, ...$ be such independent random points.

Basing on the one-step approximation (3.22)–(3.23) we construct the following algorithm. For definiteness, let $x \in G\backslash\Gamma_{ar}$ where r is sufficiently small. We set $X_0 = x,\ r_1 = r$ and

$$X_1 = X_0 + ar_1\xi_1$$

If $X_k \in G\backslash\Gamma_{ar}$, we set $r_{k+1} = r$. If $X_k \in \Gamma_{ar}\backslash\Gamma_{r^2}$, we set $r_{k+1} = \dfrac{1}{a}\rho(X_k, \partial G)$. And in both cases

$$X_{k+1} = X_k + ar_{k+1}\xi_{k+1}$$

Let $\nu = \nu_x$ be the first number at which $X_\nu \in \Gamma_{r^2}$. Then we set $X_k = X_\nu$ for $k \geq \nu$, i.e., our algorithm is stopped at a random step ν. We note that $r_k \leq r$ for all k. Now we can write for $k < \nu$:

$$X_{k+1} = X_k + a r_{k+1} \xi_{k+1} \ , \ X_0 = x \tag{5.3}$$

$$Y_{k+1} = Y_k \cdot (1 + \frac{r_{k+1}}{a} \sum_{i=1}^{n} b^i(X_k) \xi^i_{k+1} + c(X_k) \frac{r^2_{k+1}}{n}) \ , \ Y_0 = 1 \tag{5.4}$$

$$Z_{k+1} = Z_k + Y_k g(X_k) \frac{r^2_{k+1}}{n} \ , \ Z_0 = 0 \tag{5.5}$$

We note that if r is sufficiently small then all Y_k are positive: $Y_k > 0$.

After obtaining $X_\nu, \ Y_\nu, \ Z_\nu$ we find the point $\bar{X}_\nu \in \partial G$ which is the closest to X_ν and set $\bar{Y}_\nu = Y_\nu, \ \bar{Z}_\nu = Z_\nu$. Then we calculate

$$v(\bar{X}_\nu, \bar{Y}_\nu, \bar{Z}_\nu) = u(\bar{X}_\nu)\bar{Y}_\nu + \bar{Z}_\nu = \varphi(\bar{X}_\nu) Y_\nu + Z_\nu$$

The desired solution to the problem (5.1)–(5.2) is approximately equal to

$$u(x) \cong \mathbf{E}(\varphi(\bar{X}_\nu) Y_\nu + Z_\nu) \cong \frac{1}{N} \sum_{m=1}^{n} (\varphi(\bar{X}^{(m)}_\nu) Y^{(m)}_\nu + Z^{(m)}_\nu) \tag{5.6}$$

where $\bar{X}^{(m)}_\nu, \ Y^{(m)}_\nu, \ Z^{(m)}_\nu, \ m = 1, ..., N$, are independent realizations of the algorithm (5.3)–(5.5). The first approximate equality in (5.6) involves an error brought about by replacing $X_x(\tau), \ Y_{x,1}(\tau), \ Z_{x,1,0}(\tau)$ by $\bar{X}_\nu, \ Y_\nu, \ Z_\nu$; in the second approximate equality the error comes from the Monte-Carlo method. The first error is estimated by $O(r^2)$ (see Theorem 5.1 below) and the second one by $O(1/\sqrt{N})$.

Construct an algorithm basing on the one-step approximation (3.9)–(3.10):

$$X_{k+1} = X_k + a r_{k+1} \xi_{k+1} \ , \ X_0 = x \tag{5.7}$$

$$Y_{k+1} = Y_k \cdot (1 + \frac{r_{k+1}}{a} \sum_{i=1}^{n} b^i(X_k) \xi^i_{k+1} + c(X_k) \frac{r^2_{k+1}}{n} + \gamma(X_k, r_{k+1}, \xi_{k+1})) \ , \ Y_0 = 1 \tag{5.8}$$

$$Z_{k+1} = Z_k + Y_k g(X_k) \frac{r^2_{k+1}}{n} \ , \ Z_0 = 0 \tag{5.9}$$

where

$$\gamma(x, r, \xi) = \frac{r^2}{2a^2} \sum_{i=1}^{n} \sum_{j=1}^{n} b^i(x) b^j(x) \xi^i \xi^j -$$

$$\frac{r^2}{2a^2 n} \sum_{i=1}^{n} b^{i^2}(x) + \frac{r^2}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial b^i}{\partial x^j}(x) \xi^i \xi^j - \frac{r^2}{2n} \sum_{i=1}^{n} \frac{\partial b^i}{\partial x^i}(x)$$

Let us note that

$$|\gamma(x, r, \xi)| = O(r^2)$$

where O is uniform with respect to $x \in G, \ \xi \in \partial U$ and

$$\mathbf{E}(\gamma(X_k, r_{k+1}, \xi_{k+1}) / \mathcal{B}_k) = 0 \tag{5.10}$$

For the Dirichlet problem

$$\frac{1}{2} a^2 \Delta u + c(x) u + g(x) = 0, \ x \in G \tag{5.11}$$

$$u \mid_{\partial G} = \varphi(x) \tag{5.12}$$

we can also suggest two algorithms. One of them is based on the one-step approximation (3.33). We write down another one which is based on the one-step approximation (3.30)–(3.31). Now we choose Γ_{r^4} as Γ_δ and set $r_{k+1} = \frac{1}{a}\rho(X_k, \partial G)$ if $X_k \in \Gamma_{ar} \backslash \Gamma_{r^4}$. We obtain

$$X_{k+1} = X_k + ar_{k+1}\xi_{k+1} \ , \ X_0 = x \tag{5.13}$$

$$Y_{k+1} = Y_k \cdot (1 + c(X_k)\frac{r_{k+1}^2}{n} + \frac{a}{2(n+2)}\sum_{i=1}^{n}\frac{\partial c}{\partial x^i}(X_k)r_{k+1}^3\xi_{k+1}^i) +$$

$$Y_k \cdot \frac{4+n}{2n^2(2+n)}c_1(X_k)r_{k+1}^4 \ , \ Y_0 = 1 \tag{5.14}$$

$$Z_{k+1} = Z_k + Y_k \cdot (g(X_k)\frac{r_{k+1}^2}{n} + \frac{4+n}{2n^2(2+n)}g_1(X_k)r_{k+1}^4) \ , \ Z_0 = 0 \tag{5.15}$$

We note that by Theorem 4.1 the average number of steps for all the methods presented here is $O(\frac{1}{r^2})$.

Proceeding to convergence theorems let us use the relation (1.15)

$$|\mathbf{E}(\varphi(\bar{X}_\nu)Y_\nu - Z_\nu) - u(x)| = |\mathcal{R}| \leq |\mathbf{E}(u(\bar{X}_\nu) - u(X_\nu))Y_\nu| + \sum_{k=1}^{\infty}|\tilde{d}_k|$$

where

$$\tilde{d}_k = \mathbf{E}\chi_{\nu>k-1}(v(X_k, Y_k, Z_k) - v(X_{k-1}, Y_{k-1}, Z_{k-1}))$$

We note that $\nu = \nu_x$ here and below.

Clearly

$$u(X_{k-1})Y_{k-1} - Z_{k-1} = v(X_{k-1}, Y_{k-1}, Z_{k-1}) =$$

$$\mathbf{E}(v(X_{X_{k-1}}(\vartheta_k), Y_{X_{k-1},Y_{k-1}}(\vartheta_k), Z_{X_{k-1},Y_{k-1},Z_{k-1}}(\vartheta_k))/\mathcal{B}_{k-1})$$

Therefore

$$\tilde{d}_k = \mathbf{E}\chi_{\nu>k-1}d_k$$

where

$$d_k = \mathbf{E}(v(X_k, Y_k, Z_k) - v(X_{X_{k-1}}(\vartheta_k), Y_{X_{k-1},Y_{k-1}}(\vartheta_k), Z_{X_{k-1},Y_{k-1},Z_{k-1}}(\vartheta_k))/\mathcal{B}_{k-1}) \tag{5.16}$$

is a one-step error for the point $(X_{k-1}, Y_{k-1}, Z_{k-1})$. Thus

$$|\mathcal{R}| \leq |\mathbf{E}(u(\bar{X}_\nu) - u(X_\nu))Y_\nu| + \sum_{k=1}^{\infty}|\mathbf{E}\chi_{\nu>k-1}d_k| \tag{5.17}$$

Theorem 5.1. *Let* $c(x) \leq -c_0 < 0$. *Then both the method* (5.3)–(5.5) *and the method* (5.7)–(5.9) *have the second order of convergence with respect to* r, *i.e., for all sufficiently small* r

$$|\mathbf{E}(\varphi(\bar{X}_\nu)Y_\nu - Z_\nu) - u(x)| \leq Kr^2 \tag{5.18}$$

In addition, the constant K *for the method* (5.3)–(5.5) *depends on the first and second derivatives of the required solution* $u(x)$ *while this constant for the method* (5.7)–(5.9) *depends only on first derivatives.*

Proof. Let us restrict ourselves to the proof of the method (5.7)–(5.9). Due to Theorem 3.1 the one-step error d_k from (5.16) satisfies the following inequality

$$|d_k| \le KY_{k-1}r^4 \tag{5.19}$$

where (see (3.21))

$$K_0M_0(x) + K_1M_1(x) + K_2 \le K$$

for all $x \in \bar{G}$.

As $X_\nu \in \Gamma_{r^2}$ we have

$$|u(\bar{X}_\nu) - u(X_\nu)| \le Kr^2$$

Therefore

$$|\mathcal{R}| \le Kr^2\mathbf{E}Y_\nu + Kr^4\sum_{k=0}^{\infty}\mathbf{E}\chi_{\nu>k}Y_k \tag{5.20}$$

Clearly, K does not depend on the second derivatives of $u(x)$ here.

We have for $k > 0$ (see (5.10)

$$\mathbf{E}\chi_{\nu>k}Y_k \le \mathbf{E}\chi_{\nu>k-1}Y_k =$$

$$\mathbf{E}(\chi_{\nu>k-1}Y_{k-1}\mathbf{E}(1+\frac{r_k}{a}\sum_{i=1}^{n}b^i(X_{k-1})\xi_k^i + c(X_{k-1})\frac{r_k^2}{2} + \gamma(X_{k-1},r_k,\xi_k)/\mathcal{B}_{k-1})) =$$

$$\mathbf{E}(\chi_{\nu>k-1}Y_{k-1}(1+c(X_{k-1})\frac{r_k^2}{2})) \le (1-\frac{c_0}{2}r^2)\cdot\mathbf{E}\chi_{\nu>k-1}Y_{k-1} \le$$

$$(1-\frac{c_0}{2}r^2)^2\cdot\mathbf{E}\chi_{\nu>k-2}Y_{k-2} \le ... \le (1-\frac{c_0}{2}r^2)^k$$

From here

$$Kr^4\sum_{k=0}^{\infty}\mathbf{E}\chi_{\nu>k}Y_k \le Kr^2 \tag{5.21}$$

(remember that often various constants in this paper are given the same letter K).

Further

$$\mathbf{E}Y_\nu = \sum_{k=1}^{\infty}\mathbf{E}\chi_{\nu=k}Y_k = \sum_{k=1}^{\infty}(\mathbf{E}\chi_{\nu>k-1}Y_k - \mathbf{E}\chi_{\nu>k}Y_k) = \mathbf{E}\chi_{\nu>0}Y_1 + \sum_{k=1}^{\infty}\mathbf{E}\chi_{\nu>k}(Y_{k+1}-Y_k)$$

But

$$\mathbf{E}\chi_{\nu>k}(Y_{k+1}-Y_k) = \mathbf{E}\chi_{\nu>k}Y_k(\frac{r_{k+1}}{a}\sum_{i=1}^{n}b^i(X_k)\xi_{k+1}^i + c(X_k)\frac{r_{k+1}^2}{2} + \gamma(X_k,r_{k+1},\xi_{k+1})) =$$

$$\mathbf{E}(\chi_{\nu>k}Y_k\mathbf{E}(\frac{r_{k+1}}{a}\sum_{i=1}^{n}b^i(X_k)\xi_{k+1}^i + c(X_k)\frac{r_{k+1}^2}{2} + \gamma(X_k,r_{k+1},\xi_{k+1})/\mathcal{B}_k)) =$$

$$\mathbf{E}(\chi_{\nu>k}Y_k\cdot c(X_k)\frac{r_{k+1}^2}{2}) < 0$$

Hence

$$\mathbf{E}Y_\nu \le \mathbf{E}\chi_{\nu>0}Y_1 \le \mathbf{E}Y_1 < 1 \tag{5.22}$$

The relations (5.20), (5.21) and (5.22) imply (5.18). Theorem 5.1 is proved.

Theorem 5.2. *Let $c(x) \le 0$. Then the method* (5.13)–(5.15) *has the fourth order of convergence with respect to r :*

$$|\mathbf{E}(\varphi(\bar{X}_\nu)Y_\nu - Z_\nu) - u(x)| \le Kr^4 \tag{5.23}$$

The constant K depends on the first and second derivatives of $u(x)$.

Proof. Due to Theorem 3.3 the one-step error d_k from (5.17) satisfies the following inequality

$$|d_k| \le KY_{k-1}r^6$$

As $X_\nu \in \Gamma_{r^4}$ we have

$$|u(\bar{X}_\nu) - u(X_\nu)| \le Kr^4$$

where K depends on the first derivatives of $u(x)$.

Therefore

$$|\mathcal{R}| \le Kr^4\mathbf{E}Y_\nu + Kr^6\sum_{k=0}^{\infty}\mathbf{E}\chi_{\nu>k}Y_k \tag{5.24}$$

with K satisfying Theorem 5.1.

It follows from (5.14) that for anyhow small β_0 there exists $r_0 > 0$ such that for all $r \le r_0$

$$Y_k \le Y_{k-1}(1+\beta_0 r^2) \le ... \le (1+\beta_0 r^2)^k$$

Therefore

$$\mathbf{E}\chi_{\nu>k}Y_k \le (1+\beta_0 r^2)^k\mathbf{E}\chi_{\nu>k}$$

Let $\beta_0 < \beta$ for β from Lemma 4.4. Then for all sufficiently small r

$$\mathbf{E}\chi_{\nu>k}Y_k \le (1-(\beta-\beta_0)r^2)^k \tag{5.25}$$

In the same way as in the previous theorem one can prove that

$$\mathbf{E}Y_\nu \le 1 \tag{5.26}$$

The relations (5.24), (5.25) and (5.26) imply (5.23). Theorem 5.2 is proved.

Remark 5.1. The more simple method based on the one-step approximation (3.33) has the second order of convergence with a constant K depending on the first derivatives of $u(x)$.

Remark 5.2. We have considered a number of methods in the case $a^{ij}(x) = a^2\delta_{ij}$ where δ_{ij} is the Kronecker delta. But all the results can be carried over to the case of constant coefficients a^{ij}. Let us construct an algorithm analogous to (5.7)–(5.9) for definiteness. In this case we have to integrate the system (1.8)–(1.10) with a constant matrix σ.

Clearly, together with (1.3) the following inequality

$$a^2\sum_{i=1}^{n} {y^i}^2 \le \sum_{i,j=1}^{n} a^{ij}y^iy^j \le \bar{a}^2\sum_{i=1}^{n} {y^i}^2 \tag{5.27}$$

holds for any $y \in \mathbf{R}^n$ and a constant $\bar{a} > 0$.

Let $X_0 = x$. If $X_k \in G\backslash\Gamma_{\bar{a}r}$, we set $r_{k+1} = r$, and if $X_k \in \Gamma_{\bar{a}r}\backslash\Gamma_{r^2}$, we search a number r_{k+1} such that the ellipsoid $(\sigma^{-1}(X-X_k), \sigma^{-1}(X-X_k)) = r_{k+1}^2$ touches ∂G. In both cases we set

$$X_{k+1} = X_k + \sigma r_{k+1}\xi_{k+1}$$

Let $\nu = \nu_x$ be the first number at which $X_\nu \in \Gamma_{r^2}$. Then we set $X_k = X_\nu$ for $k \geq \nu$, i.e., our algorithm is stopped at a random step ν. We note that $r_k \leq \frac{\bar{a}}{a} r$ for all k. Underline that the distinction, consisting in a random walk over small ellipsoids instead of a random walk over small spheres, is not essential.

Proceeding to an integration of (1.8)–(1.10) we get (remember that $\sigma^{-1}b(x) = h(x)$)

$$Y_{x,y}(\vartheta) \cong y + y\sum_{i=1}^{n} h^i(x)w^i(\vartheta) + c(x)y\vartheta + y\sum_{i=1}^{n}\sum_{j=1}^{n} h^i(x)h^j(x)\int_0^{\vartheta} w^j(t)dw^i(t) +$$

$$y\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial h^i}{\partial x^j}(x)\int_0^{\vartheta} (\sigma w(t))^j dw^i(t) \tag{5.28}$$

$$Z_{x,y,z}(\vartheta) \cong z + g(x)y\vartheta \tag{5.29}$$

instead of (3.2) and (3.6).

Then we construct the one-step approximation Y_1, Z_1 similar to (3.9), (3.10), i.e., we substitute $\frac{r^2}{n}$ instead of ϑ, $\frac{1}{2}w^j(\vartheta)w^i(\vartheta)$, $i \neq j$, instead of $\int_0^\vartheta w^j(t)dw^i(t)$ and $\frac{1}{2}w^{i^2}(\vartheta) - \frac{r^2}{2n}$ instead of $\int_0^\vartheta w^i(t)dw^i(t)$ in (5.28) and (5.29) (see Theorem 2.2 and formula (2.36)). In addition we take into account that

$$\int_0^{\vartheta} (\sigma w(t))^j dw^i(t) = \sum_{k=1}^{n} \sigma^{jk}\int_0^{\vartheta} w^k(t)dw^i(t)$$

Having the one-step approximation we can easily obtain an algorithm similar to (5.7)–(5.9) substituting $r_{k+1}\xi^i_{k+1}$ instead of $w^i(\vartheta)$ on $(k+1)$-st step.

6. Boundary layer

Proceeding to the numerical investigation of a boundary layer let us consider the following model problem:

$$\frac{1}{2}\varepsilon^2\Delta u + c(x)u = g(x),\ x \in U_R \tag{6.1}$$

$$u\,|_{\partial U_R} = 0 \tag{6.2}$$

where $\varepsilon \ll 1$, $U_R \in \mathbf{R}^n$ is an open ball of radius R with center at the origin, $c(x)$ and $g(x)$ belong to $\mathbf{C}^\infty(\bar{U}_R)$ and $c(x) \leq -c_0 < 0,\ x \in \bar{U}_R$.

A solution $u(x,\varepsilon)$ to this problem has a fluent alteration everywhere in U_R with the exception of a small neighborhood of ∂U_R which is called boundary layer and which is narrowed with decreasing ε. The solution $u(x,\varepsilon)$ varies sharply in the boundary layer. It is well known (see [4] and references therein) that the width of the boundary layer for the problem (6.1)–(6.2) is evaluated by $l\varepsilon$ (l is a number), i.e., boundary layer has a form $\Gamma_{l\varepsilon}$. Moreover, it is known that

$$|u(x,\varepsilon)| \leq K,\ |\frac{\partial u}{\partial x^i}(x,\varepsilon)| \leq K,\ |\frac{\partial^2 u}{\partial x^i \partial x^j}(x,\varepsilon)| \leq K,\ x \in U_R\backslash\Gamma_{l\varepsilon},$$

$$|u(x,\varepsilon)| \leq K,\ |\frac{\partial u}{\partial x^i}(x,\varepsilon)| \leq \frac{K}{\varepsilon},\ |\frac{\partial^2 u}{\partial x^i \partial x^j}(x,\varepsilon)| \leq \frac{K}{\varepsilon^2},\ x \in \Gamma_{l\varepsilon} \tag{6.3}$$

An analytical approach to this problem consists in construction of an external asymptotic expansion $V(x,\varepsilon)$ and of an interior asymptotic expansion $W(x,\varepsilon)$. They describe the solution in $U_R \backslash \Gamma_{l\varepsilon}$ and in $\Gamma_{l\varepsilon}$ correspondingly. The external expansion has a form

$$V(x,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^{2k} v_k(x)$$

where

$$v_0(x) = \frac{g(x)}{c(x)},\ v_k(x) = -\frac{\Delta v_{k-1}(x)}{c(x)},\ k \geq 1$$

The function $V(x,\varepsilon)$ is an asymptotic solution in $U_R \backslash \Gamma_{l\varepsilon}$, i.e., the function

$$V_m(x,\varepsilon) = \sum_{k=0}^{m} \varepsilon^{2k} v_k(x) \tag{6.4}$$

is distinguished from the solution in $U_R \backslash \Gamma_{l\varepsilon}$ by $O(\varepsilon^{2m+2})$.

The interior expansion $W(x,\varepsilon)$ is necessary for compensation of a discrepancy in the boundary conditions. It turned out that outside of the boundary layer $W(x,\varepsilon) = O(\varepsilon^N)$, $\varepsilon \to 0$, for any N. The sum $V+W$ is an asymptotic solution of the problem (6.1)–(6.2). The interior expansion is constructed in a more complicated way and it is not brought here.

It should be noted that the problem (6.1)–(6.2) is one of the simplest ones in the theory of boundary layer. If, for instance, the condition $c(x) \leq -c_0 < 0,\ x \in \bar{U}_R$, is violated so that the function $c(x)$ may take zero values then analytical investigation of a corresponding problem becomes exceedingly intricate. Therefore, a numerical approach to problems with a small or with an intermediate parameter at higher derivatives is actual. But it should not be supposed that one can use general numerical methods (for example, the methods from Section 5) without taking into account the smallness of the parameter at higher derivatives. Principal difficulties lie in the fact that the average number of steps evaluated as $O(\frac{1}{\varepsilon^2 r^2})$ by Theorem 4.1 is big, and derivatives of the solution in the boundary layer are great. Let us analyze these and some other difficulties for the problem (6.1)–(6.2).

As before we consider a random walk over spheres with radius εr in $U_R \backslash \Gamma_{\varepsilon r}$ (as we have ε instead of a now) and over tangent to ∂U_R spheres in $\Gamma_{\varepsilon r} \backslash \Gamma_\delta$ where δ is sufficiently small (in any case $\delta < \varepsilon r/2$).

Now it is convenient to present the error $\mathcal{R}$ (see (5.17)) in the following form

$$|\mathcal{R}| \leq |\mathbf{E}(u(\bar{X}_\nu) - u(X_\nu))Y_\nu| + \sum_{k=1}^{\infty} |\mathbf{E}\chi_{\nu>k-1} d_k| \leq$$

$$|\mathbf{E}(u(\bar{X}_\nu) - u(X_\nu))Y_\nu| + \sum_{k=1}^{\infty} |\mathbf{E}\chi_{\Gamma_{l\varepsilon}\backslash\Gamma_\delta}(X_{k-1}) d_k| + \sum_{k=1}^{\infty} |\mathbf{E}\chi_{U_R\backslash\Gamma_{l\varepsilon}}(X_{k-1}) d_k| \tag{6.5}$$

because

$$\chi_{\nu>k-1} = \chi_{\Gamma_{l\varepsilon}\backslash\Gamma_\delta}(X_{k-1}) + \chi_{U_R\backslash\Gamma_{l\varepsilon}}(X_{k-1})$$

Let the one-step error d_k be bounded by $\delta_0(r,\varepsilon)Y_{k-1}$ in the part $\Gamma_{l\varepsilon} \backslash \Gamma_\delta$ of the boundary layer $\Gamma_{l\varepsilon}$ and by $\delta_1(r,\varepsilon)Y_{k-1}$ outside of the boundary layer, i.e., in $U_R \backslash \Gamma_{l\varepsilon}$. We note that the method (5.3)–(5.5) under $b(x) = 0$ and the method (5.13)–(5.15) have $Y_k \leq 1$ for sufficiently small r if $c(x) \leq -c_0 < 0$. For similar methods we obtain from (6.5) that

$$|\mathcal{R}| \leq |\mathbf{E}(u(\bar{X}_\nu) - u(X_\nu))Y_\nu| + \delta_0(r,\varepsilon)\mathbf{E}\nu_0 + \delta_1(r,\varepsilon)\mathbf{E}\nu_1 \tag{6.6}$$

where ν_0 and ν_1 are random numbers of steps inside and outside of the boundary layer correspondingly. Clearly, ν_0 and ν_1 depend on x. Due to Theorem 4.1 we have $\mathbf{E}\nu_1 \le \frac{K}{\varepsilon^2 r^2}$. Fortunately, due to the stated below lemma $\mathbf{E}\nu_0 \le \frac{K}{r^2}$.

Lemma 6.1. *There exists a constant $K > 0$ such that for any $x \in U_R \backslash \Gamma_\delta$ and sufficiently small both ε and r*

$$\mathbf{E}\nu_0 \le \frac{K}{r^2} \tag{6.7}$$

Proof. We have

$$\mathbf{E}\nu_0 = \mathbf{E}\sum_{k=1}^{\infty} \chi_{\Gamma_{l\varepsilon}\backslash\Gamma_\delta}(X_{k-1})$$

Consider the following function

$$v(x) = \begin{cases} 3l^2\varepsilon^2, & 0 \le |x| \le R - l\varepsilon \\ (R-|x|)(|x| - (R - 4l\varepsilon)), & R - l\varepsilon \le |x| \le R - \delta \\ 0, & R - \delta \le |x| \le R \end{cases}$$

Clearly $v \in \mathbf{C}(G\backslash\Gamma_\delta)$ and

$$|v(x)| \le 4l^2\varepsilon^2, \; x \in G\backslash\Gamma_\delta$$

Evaluate $Pv(x) - v(x)$ for x belonging to the intersection of the boundary layer with $U_R\backslash\Gamma_\delta$, *i.e.*, $x \in \Gamma_{l\varepsilon}\backslash\Gamma_\delta$. At first let $x \in \Gamma_{l\varepsilon}\backslash\Gamma_\delta$ be such that $U_{\varepsilon r}(x) \in \Gamma_{l\varepsilon}\backslash\Gamma_\delta$. Then

$$Pv(x) = \mathbf{E}v(X_1) = \mathbf{E}v(x + \varepsilon w(\vartheta)) =$$

$$\frac{1}{S}\int_{\partial U_r} (R - |x+\varepsilon z|)(|x + \varepsilon z| - (R - 4l\varepsilon))dS =$$

$$-R^2 + 4lR\varepsilon - |x|^2 - \varepsilon^2 r^2 + \frac{1}{S}(2R - 4l\varepsilon)\int_{\partial U_r} |x + \varepsilon z| dS \tag{6.8}$$

Due to the Taylor formula we have

$$|x + \varepsilon z| = |x| + \frac{(x,z)}{|x|}\varepsilon + \frac{1}{2}(\frac{|z|^2}{|x|} - \frac{(x,z)^2}{|x|^3})\varepsilon^2 + O((r\varepsilon)^3) \tag{6.9}$$

Since

$$\frac{1}{S}\int_{\partial U_r} (z^1)^2 dS = \ldots = \frac{1}{S}\int_{\partial U_r} (z^n)^2 dS = \frac{1}{nS}\int_{\partial U_r} \sum_{i=1}^{n} (z^i)^2 dS = \frac{r^2}{n}$$

we get

$$\frac{1}{S}\int_{\partial U_r} (x,z)^2 dS = \frac{1}{S}\sum_{i=1}^{n}\int_{\partial U_r} (x^i z^i)^2 dS = \frac{r^2}{n}|x|^2 \tag{6.10}$$

From (6.9) and (6.10)

$$\frac{1}{S}(2R - 4l\varepsilon)\int_{\partial U_r} |x+\varepsilon z| dS = (2R - 4l\varepsilon)(|x| + \frac{1}{2}\frac{r^2\varepsilon^2}{|x|} - \frac{1}{2}\frac{r^2\varepsilon^2}{n|x|}) + O((r\varepsilon)^3) \le$$

$$(2R - 4l\varepsilon)|x| + r^2\varepsilon^2(1 - \frac{1}{n}) + O((r\varepsilon)^3)$$

Therefore, from (6.8) under sufficiently small $r\varepsilon$ we obtain

$$Pv(x) - v(x) \le -\frac{1}{2n}\varepsilon^2 r^2$$

This inequality can be proved for all $x \in \Gamma_{l\varepsilon} \backslash \Gamma_\delta$ by the same way as Lemma 4.1 has been proved.

If $0 \le |x| \le R - l\varepsilon - r\varepsilon$ then $Pv(x) - v(x) = 0$ because $v(x + \varepsilon z) = 3l^2\varepsilon^2 = const$ under $z \in \partial U_r$. Finally, if $R - l\varepsilon - r\varepsilon \le |x| \le R - l\varepsilon$, we prove that $Pv(x) - v(x) \le 0$ as in Lemma 4.1 introducing the function $\bar{v}(y) = 3l^2\varepsilon^2 \ge v(y)$.

Thus, we obtain for the function $V(x) = \dfrac{2n}{\varepsilon^2 r^2} v$ that

$$PV(x) - V(x) \le -\chi_{\Gamma_{l\varepsilon} \backslash \Gamma_\delta}(x)$$

Consequently

$$\mathbf{E}\nu_0 \le V(x) \le \frac{2n}{\varepsilon^2 r^2} 4l^2\varepsilon^2 = \frac{K}{r^2}$$

Lemma 6.1 is proved.

Let us return to the inequality (6.6). The first term in the right side of (6.6) is bounded by $\dfrac{K\delta}{\varepsilon}$ according to (6.3) (to the point let us note that for the problem (6.1)–(6.2) we need not seek $\bar{X}_\nu$ as $u(\bar{X}_\nu) = 0$). If we choose $\delta = O(r^p)$, then the first term can be done sufficiently small. At the same time due to Theorem 4.1 the average number of steps depends on p insignificantly and as before it is evaluated by $O(\dfrac{1}{\varepsilon^2 r^2})$. The factor $\mathbf{E}\nu_0 = O(\dfrac{1}{r^2})$ in the second term (Lemma 6.1) is comparatively not big and the other factor $\delta_0(r, \varepsilon)$ depends on behavior of the solution in the boundary layer and it may take big values. But the methods from the previous section do not contain in their errors any too higher order derivatives of the solution and therefore the second term can also be done small. The third term in (6.6) has the very big factor $\mathbf{E}\nu_1 = O(\dfrac{1}{\varepsilon^2 r^2})$ and consequently this term can be decreased only by means of $\delta_1(r, \varepsilon)$. Thus, the principal problem is contained in construction of a sufficiently precise and effective one-step approximation in the larger domain $U_R \backslash \Gamma_{l\varepsilon}$. Let us take into consideration that the system (1.8)–(1.10) for the problem (6.1)–(6.2) is a system with small noise:

$$dX = \varepsilon dw(t) \tag{6.11}$$

$$dY = c(X)Y dt \tag{6.12}$$

$$dZ = g(X)Y dt \tag{6.13}$$

In [12], [13] some specific methods for systems with small noise are constructed. The errors of those methods have not a traditional form $O(h^q)$ (here h is a step with respect to time) but are estimated by $O(h^p + \varepsilon^k h^q)$, $q < p$. Time-step order of such a method is equal to q which is comparatively low and thanks to this fact one may reach a certain efficiency. Moreover, according to large p and the factor ε^k at h^q the method error becomes sufficiently small, and the method reaches high exactness. These ideas can be carried over to the approximation under space-discretization as well. We shall construct an efficient one-step approximation in the main domain $U_R \backslash \Gamma_{l\varepsilon}$ with an error of the form $O(r^{2p} + \varepsilon^k r^{2q})$. We remember that the solution $u(x, \varepsilon)$ has a fluent alteration in $U_R \backslash \Gamma_{l\varepsilon}$.

At the beginning let us analyze a method based on the one-step approximation (3.33). According to (6.3) $M_0(x)$ and $M_1(x)$, $x \in \Gamma_{l\varepsilon}$, in (3.34) are bounded by K and K/ε correspondingly. Hence $\delta_0(r, \varepsilon) \le Kr^4$ (of course, we have to take ε instead of a in (3.34)) and due to Lemma 6.1 the second term in (6.6) has the acceptable bound

$O(r^2)$. Clearly, the third term has the following bound $\delta_1(r,\varepsilon)\mathbf{E}\nu_1 \le Kr^4 \cdot \dfrac{K}{\varepsilon^2 r^2} \le \dfrac{Kr^2}{\varepsilon^2}$ and we have to choose too small r to obtain an acceptable accuracy. This circumstance leads in turn to increasing the average number of steps.

For the method (5.13)–(5.15) we get analogously: $\delta_0(r,\varepsilon)\mathbf{E}\nu_0 \le Kr^4$, $\delta_1(r,\varepsilon) \le Kr^6$, $\delta_1(r,\varepsilon)\mathbf{E}\nu_1 \le \dfrac{Kr^4}{\varepsilon^2}$. But this method can be simplified without an essential loss of accuracy. To this aim consider the following method:

$$X_{k+1} = X_k + \varepsilon r_{k+1}\xi_{k+1}\ ,\ X_0 = x \tag{6.14}$$

$$Y_{k+1} = Y_k \cdot (1 + c(X_k)\frac{r_{k+1}^2}{n} + \frac{\varepsilon}{2(n+2)}\sum_{i=1}^{n}\frac{\partial c}{\partial x^i}(X_k)r_{k+1}^3\xi_{k+1}^i)+$$

$$Y_k \cdot \frac{4+n}{2n^2(2+n)}c^2(X_k)r_{k+1}^4\ ,\ Y_0 = 1 \tag{6.15}$$

$$Z_{k+1} = Z_k + Y_k \cdot (g(X_k)\frac{r_{k+1}^2}{n} + \frac{4+n}{2n^2(2+n)}c(X_k)g(X_k)r_{k+1}^4)\ ,\ Z_0 = 0 \tag{6.16}$$

Here we choose Γ_{r^3} as Γ_δ and set: $r_{k+1} = r$ if $X_k \in U_R\backslash\Gamma_{\varepsilon r}$; $r_{k+1} = \dfrac{1}{\varepsilon}(R - |X_k|)$ if $X_k \in \Gamma_{\varepsilon r}\backslash\Gamma_\delta$.

This method does not require calculation of the second derivatives $\dfrac{\partial^2 c}{\partial x^{i^2}}$ and $\dfrac{\partial^2 g}{\partial x^{i^2}}$ at every step in contrast to the method (5.13)–(5.15). Analogously to Theorem 3.3 one can prove that

$$|d| \le K(M_0(x) + \varepsilon^2 M_1(x))yr^6 + K\varepsilon^4 M_2(x)yr^4 + K(\varepsilon^2 r^4 + r^6)y$$

Therefore (see (6.3)) for both $\delta_0(r,\varepsilon)$ and $\delta_1(r,\varepsilon)$ we have

$$|\delta_i(r,\varepsilon)| \le K(\varepsilon^2 r^4 + r^6),\ i = 0,1 \tag{6.17}$$

The error (6.17) is only of the fourth order with respect to r (due to this fact the method (6.14)–(6.16) is fairly simple) but at the same time it is sufficiently small due to the factor ε^2. Using (6.17) with regard to $\delta = r^3$ it is not difficult to obtain the following result (we remark that now the first term in (6.6) is $O(\dfrac{r^3}{\varepsilon}) \le Kr^2 + K\dfrac{r^4}{\varepsilon^2}$).

Theorem 6.1. *Let $\delta = r^3$. The error of the method (6.14)–(6.16) is estimated by*

$$|\mathcal{R}| \le Kr^2 + K\frac{r^4}{\varepsilon^2} \tag{6.18}$$

and the average number of steps for this method is equal to $O(\dfrac{1}{\varepsilon^2 r^2})$.

Let us emphasize that the big average number of steps leads to the extraordinary computational expenses. At the same time we can find the solution of the problem (6.1)–(6.2) in $U_R\backslash\Gamma_{l\varepsilon}$ with great accuracy according to (6.4). We use this fact and construct below an analytic-numerical method.

We set

$$u(x,\varepsilon) \cong V_m(x,\varepsilon),\ x \in U_R\backslash\Gamma_{l\varepsilon}$$

and instead of (6.1)–(6.2) we introduce

$$\frac{1}{2}\varepsilon^2\Delta u + c(x)u = g(x),\ R - l\varepsilon < |x| < R \tag{6.19}$$

$$u\mid_{|x|=R-l\varepsilon}=V_m(x,\varepsilon),\ u\mid_{|x|=R}=0 \tag{6.20}$$

Consider the following random walk defined by $r<\max(\varepsilon,l\varepsilon)$ and $\delta\ll r$ in the layer $R-l\varepsilon\le|x|\le R$: if $R-l\varepsilon\le|X_k|<R-l\varepsilon+\delta$ or $R-\delta<|X_k|\le R$, then $X_{k+1}=X_k$; if $R-l\varepsilon+\delta\le|X_k|<R-l\varepsilon+\varepsilon r$ or $R-\varepsilon r<|X_k|\le R-\delta$, then r_{k+1} is equal to $\frac{1}{\varepsilon}(|X_k|-(R-l\varepsilon))$ or $\frac{1}{\varepsilon}(R-|X_k|)$ correspondingly; if $R-l\varepsilon+\varepsilon r\le|X_k|\le R-\varepsilon r$, then $r_{k+1}=r$. In the second and third cases we put

$$X_{k+1}=X_k+\varepsilon r_{k+1}\xi_{k+1} \tag{6.21}$$

Lemma 6.2. *The average number of steps for the random walk* (6.21) *is estimated by* $O(\frac{1}{r^2})$.

Proof. This lemma can be proved in just the same way as Lemma 6.1 by introducing the function

$$v(x)=\begin{cases}0, & R-l\varepsilon\le|x|<R-l\varepsilon+\delta\\ (R-|x|)(|x|-(R-l\varepsilon)), & R-l\varepsilon+\delta\le|x|\le R-\delta\\ 0, & R-\delta<|x|\le R\end{cases}$$

It is not difficult to prove the following theorem.

Theorem 6.2. *Let* $\delta=r^5$. *Then the error of the method* (6.14)–(6.16) *for the problem* (6.19)–(6.20) *is estimated by*

$$|\mathcal{R}|\le K(\varepsilon^2r^2+r^4)+K\frac{r^5}{\varepsilon} \tag{6.22}$$

and the average number of steps is equal to $O(\frac{1}{r^2})$.

It is clear that the error for the original problem (6.1)–(6.2) is more than (6.22) about $O(\varepsilon^{2m+2})$. We see the proposed analytic-numerical method be greatly effective: it is more exact (compare the errors (6.22) and (6.18)) and it has the lesser average number of steps.

Remark 6.1. Undoubtedly, many results obtained for the model problem (6.1)–(6.2) here can be used for more general problems. In particular, they can be carried over to the problem (5.1)–(5.2) under $a=\varepsilon,\ b^i(x)=0,\ c(x)\le-c_0<0$ without any essential change.

7. General problem

We offer two methods for the general problem (1.1)–(1.2) here. As in the case of constant coefficients a^{ij} (see Remark 5.2) we can write the inequality (5.27). Now a^{ij} depends on x and (5.27) holds for any $x\in\bar G,\ y\in\mathbf{R}^n$. For constructing a random walk in $\bar G$ we use the system (1.8) with frozen coefficients and choose $\delta=r^2$. Let $X_0=x$. If $X_k\in G\backslash\Gamma_{\bar a r}$, we set $r_{k+1}=r$, and if $X_k\in\Gamma_{\bar a r}\backslash\Gamma_{r^2}$, we search a number r_{k+1} such that the ellipsoid $(\sigma^{-1}(X_k)(X-X_k),\sigma^{-1}(X_k)(X-X_k))=r_{k+1}^2$ touches ∂G. In both cases we set

$$X_{k+1}=X_k+\sigma(X_k)r_{k+1}\xi_{k+1} \tag{7.1}$$

Let $\nu=\nu_x$ be the first number at which $X_\nu\in\Gamma_{r^2}$. Then we set $X_k=X_\nu$ for $k\ge\nu$, i.e., the random walk is stopped at a random step ν.

Consider the following one-step approximation of the solution to the system (1.8)–(1.10):

$$X_1=x+\sigma(x)w(\vartheta) \tag{7.2}$$

$$Y_1 = y + yc(x)\frac{r^2}{n} + yh^\top(x)w(\vartheta) \tag{7.3}$$

$$Z_1 = z + yg(x)\frac{r^2}{n} \tag{7.4}$$

where $w(\vartheta)$ has the uniform distribution on the sphere ∂U_r and r is such that the ellipsoid $(\sigma^{-1}(x)(X-x), \sigma^{-1}(x)(X-x)) = r^2$ belongs to $\bar{G}$.

Let $u(x)$ be a solution to the problem (1.1)–(1.2) and let $v(x,y,z) = u(x)y + z$. In connection with (1.15) let us evaluate

$$\mathbf{E}v(X_1, Y_1, Z_1) - v(x,y,z) = \mathbf{E}(u(x + \sigma(x)w(\vartheta)) \cdot (y + yc(x)\frac{r^2}{n} + yh^\top(x)w(\vartheta)))+$$

$$z + yg(x)\frac{r^2}{n} - (u(x)y + z) \tag{7.5}$$

We have

$$u(x + \sigma(x)w(\vartheta)) = u(x) + \sum_{i=1}^{n} \frac{\partial u}{\partial x^i}(x)(\sigma(x)w(\vartheta))^i +$$

$$\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x^i \partial x^j}(x)(\sigma(x)w(\vartheta))^i \cdot (\sigma(x)w(\vartheta))^j +$$

$$\frac{1}{6}\sum_{i,j,m=1}^{n} \frac{\partial^3 u}{\partial x^i \partial x^j \partial x^m}(x)(\sigma(x)w(\vartheta))^i \cdot (\sigma(x)w(\vartheta))^j \cdot (\sigma(x)w(\vartheta))^m + \rho \tag{7.6}$$

In (7.6) ρ evidently satisfies the following inequality

$$|\rho| \le KM_4 r^4 \tag{7.7}$$

where M_4 is an upper bound for the fourth partial derivatives of the solution $u(x)$ in $\bar{G}$.

Let us write several relations which are necessary for our calculations:

$$\mathbf{E}w^i(\vartheta) = 0,\ \mathbf{E}w^i(\vartheta)w^j(\vartheta) = \delta_{ij}\frac{r^2}{n},\ \mathbf{E}w^i(\vartheta)w^j(\vartheta)w^m(\vartheta) = 0 \tag{7.8}$$

$$\frac{1}{2}\mathbf{E}\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x^i \partial x^j}(x)(\sigma(x)w(\vartheta))^i \cdot (\sigma(x)w(\vartheta))^j =$$

$$\frac{1}{2}\mathbf{E}\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x^i \partial x^j}(x)\sum_{k=1}^{n}\sigma^{ik}(x)w^k(\vartheta) \cdot \sum_{m=1}^{n}\sigma^{jm}(x)w^m(\vartheta) =$$

$$\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x^i \partial x^j}(x)\sum_{k=1}^{n}\sigma^{im}(x)\sigma^{jm}(x)\frac{r^2}{n} = \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j}(x) \cdot \frac{r^2}{n} \tag{7.9}$$

$$\mathbf{E}\sum_{i=1}^{n} \frac{\partial u}{\partial x^i}(x)(\sigma(x)w(\vartheta))^i \cdot h^\top(x)w(\vartheta) =$$

$$\mathbf{E}\sum_{i=1}^{n} \frac{\partial u}{\partial x^i}(x)\sum_{k=1}^{n}\sigma^{ik}(x)w^k(\vartheta) \cdot \sum_{m=1}^{n}h^m(x)w^m(\vartheta) =$$

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x^i}(x)\sum_{m=1}^{n}\sigma^{im}(x)h^m(x) \cdot \frac{r^2}{n} = \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x^i}(x) \cdot \frac{r^2}{n} \tag{7.10}$$

Using the relations (7.6)–(7.10) and the fact that $u(x)$ is a solution to the equation (1.1) we easily get from (7.5):

$$\mathbf{E}v(X_1, Y_1, Z_1) - v(x, y, z) =$$

$$y(\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x^i} + c(x)u + g(x)) \cdot \frac{r^2}{n} + y\rho_1 = y\rho_1$$

where ρ_1 satisfies

$$|\rho_1| \leq K(M_2 + M_3 + M_4)r^4$$

We have obtained the following lemma.

Lemma 7.1. *The degree of smallness of the one-step approximation* (7.2)–(7.4) *with respect to r is equal to 4:*

$$|\mathbf{E}v(X_1, Y_1, Z_1) - v(x, y, z)| \leq Kyr^4$$

where K depends on derivatives of $u(x)$ up to the fourth order.

Basing on the random walk (7.1) and on the one-step approximation (7.2)–(7.4) we can construct the corresponding algorithm by the same way as it has been done in Section 5. The average number of steps for this algorithm is equal to $O(\frac{1}{r^2})$. If $c(x) \leq -c_0 < 0$ then this algorithm has the second order of convergence with respect to r, i.e., the relation (5.18) is fulfilled. This assertion can be proved without any change in comparison with Theorem 5.1. But the constant K in the considered method depends on the higher derivatives of $u(x)$ than, for instance, in the method (5.3)–(5.5).

Let us turn to the second method. Its random walk is constructed by the following way.

Let η be a vector with coordinates η^i, $i = 1, ..., n$, that are mutually independent random variables taking values $\pm\frac{1}{\sqrt{n}}$ with probability $\frac{1}{2}$. Clearly, if $x \in G\backslash\Gamma_{\bar{a}r}$, then $x + \sigma(x)\eta \in \bar{G}$. For $x \in G$ we set $X_0 = x$. If $X_k \in G\backslash\Gamma_{\bar{a}r}$, we set $r_{k+1} = r$, and if $X_k \in \Gamma_{\bar{a}r}\backslash\partial G$, we search a minimal number r_{k+1} such that one of points from the set $\{X : X = X_k + \sigma(X_k)r_{k+1}\eta\}$ belongs to ∂G. In both cases we set

$$X_{k+1} = X_k + \sigma(X_k)r_{k+1}\eta_{k+1} \tag{7.11}$$

In the second case the point X_{k+1} with probability $\frac{1}{2^n}$ falls on ∂G.

Let $\nu = \nu_x$ be the first number at which $X_\nu \in \partial G$. Then we set $X_k = X_\nu$ for $k \geq \nu$, i.e., the random walk is stopped at a random step ν. The obtained random walk gets a finite number of values at every step (it is equal to 2^n) in contrast to the previous walk and it does not require any neighborhood Γ_δ of the boundary ∂G. Due to this fact we need not seek the point $\bar{X}_\nu$ and the first term in (1.15) for the second method is lacking.

A one-step approximation in the second method is of the form

$$X_1 = x + \sigma(x)r\eta \tag{7.12}$$

$$Y_1 = y + yc(x)\frac{r^2}{n} + yh^\top(x)r\eta \tag{7.13}$$

$$Z_1 = z + yg(x)\frac{r^2}{n} \tag{7.14}$$

Now we have

$$u(x+\sigma(x)r\eta) = u(x) + r\sum_{i=1}^{n}\frac{\partial u}{\partial x^i}(x)(\sigma(x)\eta)^i +$$

$$\frac{1}{2}r^2\sum_{i,j=1}^{n}\frac{\partial^2 u}{\partial x^i\partial x^j}(x)(\sigma(x)\eta)^i\cdot(\sigma(x)\eta)^j +$$

$$\frac{1}{6}r^3\sum_{i,j,m=1}^{n}\frac{\partial^3 u}{\partial x^i\partial x^j\partial x^m}(x)(\sigma(x)\eta)^i\cdot(\sigma(x)\eta)^j\cdot(\sigma(x)\eta)^m + \rho \tag{7.15}$$

where ρ satisfies the equality (7.7) again.

Instead of (7.8)–(7.10) we get

$$\mathbf{E}\eta^i = 0,\ \mathbf{E}\eta^i\eta^j = \delta_{ij}\frac{1}{n},\ \mathbf{E}\eta^i\eta^j\eta^m = 0 \tag{7.16}$$

$$\frac{1}{2}\mathbf{E}\sum_{i,j=1}^{n}\frac{\partial^2 u}{\partial x^i\partial x^j}(x)(\sigma(x)\eta)^i\cdot(\sigma(x)\eta)^j = \frac{1}{2n}\sum_{i,j=1}^{n}a^{ij}(x)\frac{\partial^2 u}{\partial x^i\partial x^j}(x) \tag{7.17}$$

$$\mathbf{E}\sum_{i=1}^{n}\frac{\partial u}{\partial x^i}(x)(\sigma(x)\eta)^i\cdot h^\top(x)\eta = \frac{1}{n}\sum_{i=1}^{n}b^i(x)\frac{\partial u}{\partial x^i}(x) \tag{7.18}$$

Using (7.15)–(7.18) we can obtain the same results as for the first method: the method based on the random walk (7.11) and on the one-step approximation (7.12)–(7.14) has the second order of convergence with respect to r and its average number of steps is equal to $O(\frac{1}{r^2})$.

References

[1] E.B. Dynkin. Markov Processes (engl. transl.). Springer, Berlin, 1965.

[2] S.M. Ermakov, V.V. Nekrutkin, A.S. Spirin. Random Processes for Solving the Equations of Mathematical Physics. Nauka, Moscow, 1984.

[3] I.I. Gichman, A.V. Skorochod. Stochastic Differential Equations. Naukova Dumka, Kiev, 1968.

[4] A.M. Ilyin. Matching of Asymptotic Expansions of Solutions of Boundary Value Problems. Nauka, Moscow, 1989.

[5] P.E. Kloeden, E. Platen. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1992.

[6] H.J. Kushner. Probability Methods for Approximations in Stochastic Control and for Elliptic Equations. Academic Press, New York, 1977.

[7] G.N. Milstein. A method of second-order accuracy integration of stochastic differential equations. Theory Prob. Appl. 23(1978), 396-401.

[8] G.N. Milstein. Numerical Integration of Stochastic Differential Equations (engl. transl.). Kluwer Academic Publishers, 1995.

[9] G.N. Milstein. Solving the first boundary value problem of parabolic type by numerical integration of stochastic differential equations. Theory Prob. Appl. 40(1995), 657-665.

[10] G.N. Milstein. The simulation of phase trajectories of a diffusion process in a bounded domain. Stochastics and Stochastic Reports, 1996 (accepted)

[11] G.N. Milstein and N.F. Rybkina. An algorithm for random walks over small ellipsoids for solving the general Dirichlet problem. Comput. Maths Math. Phys. 33(1993), No. 5, 631-647.

[12] G.N. Milstein and M.V. Tret'yakov. Mean-square numerical methods for stochastic differential equations with small noises. SIAM J. on Scientific Computing, 1997 (accepted)

[13] G.N. Milstein and M.V. Tret'yakov. Numerical methods in the weak sense for stochastic differential equations with small noise. SIAM J. on Numerical Analysis, 1997 (accepted)

[14] C. Miranda. Partial Differential Equations of Elliptic Type. Springer, 1970.

[15] E. Pardoux and D. Talay. Discretization and simulation of stochastic differential equations. Acta Appl. Math. 3(1985), 23-47.

[16] K.K. Sabelfeld. Monte-Carlo methods for boundary problems. Nauka, Novosibirsk, 1989.

[17] D. Talay. Efficient numerical schemes for the approximation of expectations of functionals of the solution of a stochastic differential equation, and applications. Lecture Notes Control Inform. Sci. 61(1984), 294-313.

[18] W. Wagner and E. Platen. Approximation of Ito integral equations. Preprint ZIMM, Akad. der Wiss. der DDR, Berlin, 1978.

[19] A.D. Wentsell. A Course in the Theory of Random Processes. Nauka, Moscow, 1975.