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Abstract

This paper is concerned with Kolmogorov's two-equation model for free turbulence in $\mathbb{R}^3$ involving the mean velocity $\boldsymbol{u}$, the pressure p, an average frequency $\omega > 0$, and a mean turbulent kinetic energy k. We first discuss scaling laws for a slightly more general two-equation models to highlight the special role of the model devised by Kolmogorov in 1942. The main part of the paper consists in proving the existence of weak solutions of Kolmogorov's two-equation model under space-periodic boundary conditions in cubes $\Omega = (\,]\,0, l\,[\,)^3$ with $l > 0$. For this we provide new a priori estimates and invoke existence result for pseudo-monotone operators.

1 Introduction

In 1942, A.N. Kolmogorov (see [Kol42] and [Spa91, pp. 214–216] for an English translation) postulated the following system of PDEs as a model for the isotropic homogeneous turbulent motion of an incompressible fluid in $\mathbb{R}^3 \times [\,0, +\infty\,[$:

$$\operatorname{div} \boldsymbol{u} = 0\,, \tag{1.1a}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nu_0 \operatorname{div}\Big(\frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u})\Big) - \nabla p + \boldsymbol{f}, \tag{1.1b}$$

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u}\cdot\nabla\omega = \nu_1 \operatorname{div}\Big(\frac{k}{\omega}\nabla\omega\Big) - \alpha_1\omega^2, \tag{1.1c}$$

$$\frac{\partial k}{\partial t} + \boldsymbol{u}\cdot\nabla k = \nu_2 \operatorname{div}\Big(\frac{k}{\omega}\nabla k\Big) + \nu_0\frac{k}{\omega}\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 - \alpha_2 k\omega. \tag{1.1d}$$

Throughout the paper, bold letters denote functions with values in $\mathbb{R}^3$ or $\mathbb{R}^9$ as well as normed spaces of such functions. Here, the unknowns have the following physical meaning:

- $\boldsymbol{u}$ is the velocity of the mean flow,
- p is the average of the pressure,
- ω is the average of the frequency associated with the turbulent kinetic energy,
- k is the mean turbulent kinetic energy.

The velocity field $\boldsymbol{v}$ of the fluid motion is given by $\boldsymbol{v} = \boldsymbol{u}+\tilde{\boldsymbol{u}}$, where $\tilde{\boldsymbol{u}}$ denotes the turbulent fluctuation velocity, such that the scalar k is the temporal average $\overline{\frac{1}{2}\,|\tilde{\boldsymbol{u}}|^2}$. Further,

$\nu_0,\ \nu_1,\ \nu_2 > 0$ are dimensionless and constant closure coefficients;

$\alpha_2,\ \alpha_1 > 0$ are dimensionless and constant closure coefficients;

$\boldsymbol{f}$ is the given external force,

$\boldsymbol{D}(\boldsymbol{u}) = \dfrac{1}{2}\big(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top\big)$ is the mean strain-rate tensor.

The function $\nu_0\frac{k}{\omega}$ denotes the kinematic eddy viscosity, while $\nu_1\frac{k}{\omega}$ and $\nu_2\frac{k}{\omega}$ denote the corresponding diffusion constants for the scalars ω and k. We refer to Section 2 for the scaling properties of this specific choice by Kolmogorov. Since the numerical values of ν_1 and ν_2 are not relevant for the theory of weak solutions for (1.1) we are going to develop below, we assume them to be equal to 1. A detailed discussion of the numerical values of closure coefficients and their role in turbulence modeling can be found, e.g., in [Bau13] and [Wil06, Chap. 4.3.1]. However, we keep the coefficient ν_0 to emphasize that the viscous dissipation generated by the viscous term in (1.1a) is feeding into the mean turbulent kinetic energy, see the second last term in (1.1d). Hence, for sufficiently smooth solutions we have the formal energy relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\Big(\frac{1}{2}|\boldsymbol{u}|^2 + k\Big)\,\mathrm{d}x = \int_{\mathbb{R}^3}\Big(\boldsymbol{f}\cdot\boldsymbol{u} - \alpha_2\omega k\Big)\,\mathrm{d}x,$$

where the first term on the right-hand side gives the power of the external forces, while the second term is Kolmogorov's way of modeling dissipative losses, e.g. through thermal radiation.

From the two closure coefficients α_1 and α_2 in the last terms of (1.1c) and (1.1d), respectively, only the ratio α_2/α_1 is relevant, as α_1 can be set to 1 by suitable rescalings, see Section 2. Note that for $\boldsymbol{f} \equiv 0$ there are explicit spatially constant solutions

$$\boldsymbol{u} \equiv \boldsymbol{u}_\circ, \quad \omega(t) = \frac{\omega_\circ}{1+\alpha_1\omega_\circ t}, \quad k(t) = \frac{k_\circ}{(1+\omega_\circ\alpha_1 t)^{\alpha_2/\alpha_1}}, \tag{1.2}$$

i.e. the mean turbulent kinetic energy decays like $t^{-\alpha_2/\alpha_1}$, if there is no feeding through macroscopic viscous dissipation. Under the assumptions $\omega(0,x) \geq \omega_* > 0$ and $k(0,x) \geq k_* > 0$ for all $x \in \mathbb{R}^3$ we will show that in the general case the solutions satisfy similar lower bounds, see Section 5.2.

System (1.1) is an outgrowth of A.N. Kolmogorov's theory of turbulence published in a series of papers in 1941. Comprehensive presentations of this theory can be found, e.g., in [Fri04] and [MoY07, Vol. I, Chap. 6.1, 6.2; Vol. II, Chap. 8] (see also the article [Tik91, pp. 488–503]). The function $L = \frac{k^{1/2}}{\omega}$ ("external length scale" or "size of largest eddies") plays an important role for the study of the energy spectrum of the turbulence (see [LaL91, Chap. 33], [Wil06, Chap. 8.1]). A review of the work of A.N. Kolmogorov and the Russian school of turbulence can be found in [Yag94]. This paper contains also some remarks about a possibly "missing source term" in (1.1c) (cf. [Spa91, p. 212]).

A profound discussion of the mathematical background of Obukhoff–Kolmogorov's spectral theory of turbulence (K41-functions, bounds for the energy spectrum for low and high frequencies) is given in [Vig10].

In place of $\mathbb{R}^3 \times [\,0,+\infty\,[$, in the present paper we study system (1.1) in the space-time cylinder $Q = \Omega\times\,]\,0,T\,[$, where $\Omega = \big(\,]\,0,l\,[\,\big)^3$ with $0 < l < +\infty$ and $0 < T < +\infty$

arbitrary, but fixed. To implement periodic boundary conditions we interpret Ω as a torus by identifying the opposite sides. If $\partial\Omega$ denotes the boundary of the cube $\Omega \subset \mathbb{R}^3$ we set

$$\Gamma_i = \partial\Omega \cap \{x_i = 0\}, \quad \Gamma_{i+3} = \partial\Omega \cap \{x_{i+3} = l\} \quad \text{for } i = 1,2,3,$$

and complement (1.1) with periodic boundary conditions and initial conditions as follows:

$$\left.\begin{array}{l} \boldsymbol{u}\big|_{\Gamma_i\times\,]0,T[} = \boldsymbol{u}\big|_{\Gamma_{i+3}\times]0,T[}, \text{ analogously for } p,\omega,k, \\ \boldsymbol{D}(\boldsymbol{u})\big|_{\Gamma_i\times\,]0,T[} = \boldsymbol{D}(\boldsymbol{u})\big|_{\Gamma_{i+3}\times\,]0,T[}, \text{ analogously for } \nabla\omega, \nabla k \\ (i=1,2,3) \end{array}\right\} \tag{1.3a}$$

$$\boldsymbol{u} = \boldsymbol{u}_0, \quad \omega = \omega_0, \quad k = k_0 \text{ in } \Omega\times\{0\}. \tag{1.3b}$$

Initial/boundary-value problem (1.1) and (1.3) characterizes a turbulent motion of an incompressible fluid in Q that evolves from $\{\boldsymbol{u}_0, \omega_0, k_0\}$ at time $t = 0$.

On physical grounds, the size l of the underlying cube Ω should be greater than certain quantities of the turbulent motion. A detailed discussion of this aspect is given in [Dav04, pp. 25–26, 424–435] (cf. also item 2° below). This is one of the main reasons why we consider a cube Ω of side length l and periodic boundary conditions which provides an analysis that is completely independent of l. In particular, we can choose l much bigger than the "external length scale" $L = \frac{k^{1/2}}{\omega}$.

In [BuM16], the authors study system (1.1) in $\Omega\times\,]0,T[$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain with $C^{1,1}$-boundary $\partial\Omega$, mixed boundary conditions for ω and k, the condition $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ and a condition for the normal traction of the tensor $-p\boldsymbol{I} + \nu_0 \frac{k}{\omega} \boldsymbol{D}(\boldsymbol{u})$ on $\partial\Omega\times\,]0,T[$. Under these boundary conditions, system (1.1) characterizes a wall-bounded turbulent motion, i.e., turbulence is generated at the Dirichlet part of the boundary. The authors complete this boundary value problem by the initial conditions (1.3b) and prove the existence of a weak solution by combining a truncation method and the Galerkin approximation.

The emphasis of our present paper is quite different from that of [BuM16], as we are interested in free turbulence which is independent of the boundary and thus has to obey certain scaling invariances (see Section 2). This can only be understood by using periodic boundary conditions and assuming that the cube size l is much larger than the structures under consideration. However, it is well-known that the study of Navier-Stokes equations with periodic boundary conditions is more delicate due to mean-flow effects and several choices of the Helmholtz decomposition (see, e.g., [ChI94, IoM91, KaW97] for more details).

Our proof of the existence of weak solutions of (1.1) and (1.3), which has been already sketched in [MiN15], is entirely independent of the discussion in [BuM16]. More specifically, the basic aspects of our paper are:

1° In Section 3 we introduce the notion of weak solution $\{\boldsymbol{u}, \omega, k\}$ with defect measure μ for (1.1) and (1.3) . This notion leads to a balance law for $\int_\Omega k(x,\cdot)\,\mathrm{d}x$ and gives a connection between the energy equality for $\frac{1}{2}\int_\Omega \big|\boldsymbol{u}(x,\cdot)\big|^2\,\mathrm{d}x$ and the vanishing of μ (cf. the Proposition, and Corollary 4.5 in Section 4).

2° In Section 4 we present our existence theorem for weak solutions $\{\boldsymbol{u}, \omega, k\}$ with defect measure μ. Based on comparison arguments with the solution in (1.2) this solution satisfies the inequality

$$L = \frac{k^{1/2}}{\omega} \geq c\,(1+t)^{1-\alpha_2/(2\alpha_1)} \quad \text{for all } t \in [0,T\,],$$

where α_2 and α_1 are from (1.1c) and (1.1d), and where $c = \text{const} > 0$ neither depends on l nor on T (cf. Corollary 4.4 in Section 4). If $\alpha_2/\alpha_1 \leq 10/7$, then L *grows at least as* $t^{2/7}$ (cf. A.N. Kolmogorov [Kol42]).

3° The proof of our existence theorem is given in Section 5. It is based on the existence of an approximate solution $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ (without defect measure) of (1.1) and (1.3), establishing a-priori estimates independently of ε and then carrying out the limit passage $\varepsilon \to 0$. The existence of the approximate solutions is obtained by applying an abstract existence results for evolutionary equations with pseudo-monotone operators from [Rou13, Thm. 8.9], see Appendix A for the details.

4° Our approach is easily adaptable to more general domains with suitable boundary conditions, and to the full-space $\mathbb{R}^d$ with general $d \in \mathbb{N}$. However, for notational convenience and physical relevance we restrict ourselves to $d = 3$ and the spatially periodic case.

In subsequent work we will investigate similarity solutions that are induced by the scaling laws discussed in Section 2. The most challenging question will be the derivation of suitable solution concepts that allow the turbulent kinetic energy k to vanish on parts of the domain. This would allow us to study the predictions of the Kolmogorov model (1.1) in the way turbulent regions invade non-turbulent regions.

2 Scaling laws

We consider the free turbulent motion of an incompressible fluid in $\mathbb{R}^3 \times \,]\,0,+\infty\,[$ which is governed by the following system of PDEs (note that $\boldsymbol{f} \equiv 0$):

$$\operatorname{div} \boldsymbol{u} = 0, \tag{2.1a}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \operatorname{div}\big(d_1(\omega,k)\boldsymbol{D}(\boldsymbol{u})\big) - \nabla p, \tag{2.1b}$$

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u}\cdot\nabla\omega = \operatorname{div}\big(d_2(\omega,k)\nabla\omega\big) - g_2(\omega,k)\omega, \tag{2.1c}$$

$$\frac{\partial k}{\partial t} + \boldsymbol{u}\cdot\nabla k = \operatorname{div}\big(d_3(\omega,k)\nabla k\big) + d_1(\omega,k)\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 - g_3(\omega,k)k, \tag{2.1d}$$

where $\boldsymbol{u}$, p, ω and k are the unknowns, and

$$d_i : \big(\,]\,0,+\infty\,[\,\big)^2 \longrightarrow \,]\,0,+\infty\,[\qquad (i = 1,2,3),$$
$$g_m : \big(\,]\,0,+\infty\,[\,\big)^2 \longrightarrow \,]\,0,+\infty\,[\qquad (m = 2,3)$$

are given coefficients. The coefficient $d_1(\omega,k)$ represents a "generalized" viscosity of the fluid. System (2.1) obviously includes Kolmogorov's two-equation model (1.1) with

$$d_1(\omega,k) = \nu_0\frac{k}{\omega}, \quad d_2(\omega,k) = \nu_1\frac{k}{\omega}, \quad d_3(\omega,k) = \nu_2\frac{k}{\omega},$$

$$g_2(\omega,k) = \alpha_1\omega, \quad g_3(\omega,k) = \alpha_2\omega.$$

We want to show that these choices are special, because they give a richer structure of scaling invariances than more general functions. We refer to

Let $\{\boldsymbol{u}, \omega, k\}$ be a classical solution of (2.1) that has a suitable decay for $|x| \to \infty$ such that the following integrals over $\mathbb{R}^3$ exist. We multiply (2.1b) by $\boldsymbol{u}$, integrate by parts over $\mathbb{R}^3$, integrate (2.1d) over $\mathbb{R}^3$, and add the equations obtained. This gives the energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\Big(\frac{1}{2}|\boldsymbol{u}|^2 + k\Big)\,\mathrm{d}x = -\int_{\mathbb{R}^3} g_3(\omega,k)k\,\mathrm{d}x, \quad t\in\,]\,0,+\infty\,] \tag{2.2}$$

(cf. Section 4, Corollary 4.5).

We are now studying the invariance of $\{\boldsymbol{u}, \omega, k\}$ under the scaling

$$\partial_t \mapsto \alpha\partial_t, \quad \partial_{x_j} \mapsto \beta\partial_{x_j}, \quad \boldsymbol{u} \mapsto \gamma\boldsymbol{u}, \quad \omega \mapsto \rho\omega, \quad k \mapsto \sigma k, \tag{2.3}$$

where $(\alpha,\beta,\gamma,\rho,\sigma) \in \big(\,]\,0,+\infty\,[\,\big)^5$. Here, the pressure p is omitted, for it can be always suitably scaled. In addition to the well-known scaling laws for the Navier-Stokes equations, the scaling (2.3) have to leave invariant the coefficients $d_i(\omega,k)$ and $g_m(\omega,t)$ for $i=1,2,3$ and $m=2,3$, too.

To this end, we consider the following conditions for the family of parameters $(\alpha,\beta,\gamma,\rho,\sigma)$ and the coefficients d_i and g_m:

$$\alpha = \beta\gamma, \qquad \sigma = \gamma^2, \tag{2.4}$$

$$\forall\,\omega,k>0: \begin{cases} \beta^2 d_i(\rho\omega,\sigma k) = \alpha d_i(\omega,k), & i=1,2,3,\\ g_m(\rho\omega,\sigma k) = \alpha g_m(\omega,k), & m=2,3. \end{cases} \tag{2.5}$$

The first condition in (2.4) implies the invariance of the convective derivative $\partial_t + \boldsymbol{u}\cdot\nabla$ under (2.3), while the second condition implies that $|\boldsymbol{u}|^2$ and k have the same scaling property which is necessary for the conservation law (2.2) to hold. It is now easy to see that system (2.1) is invariant under the scaling laws (2.3) if the conditions (2.4) and (2.5) hold.

In order to relate the present discussion to Kolmogorov's two-equation model (1.1) we make an "ansatz" for the parameter β as well as for the coefficients d_i and g_m. For $(\gamma,\rho),(\omega,k) \in \big(\,]\,0,+\infty\,[\,\big)^2$ define

$$\beta = \rho^A\gamma^{1-2B} \tag{2.6}$$

$$d_i(\omega,k) = D_i\omega^{-A}k^B, \quad g_m(\omega,k) = G_m\omega^A k^{1-B}, \tag{2.7}$$

where D_i, G_m ($i=1,2,3$; $m=2,3$) and A, B are arbitrary positive constants. Condition (2.6) is equivalent to

$$\frac{\beta}{\gamma}\,\rho^{-A}\gamma^{2B} = 1 \quad \text{ resp. } \quad \frac{1}{\beta\gamma}\,\rho^A\gamma^{2(1-B)} = 1.$$

Observing (2.4), it is readily seen that d_i and g_m as in (2.7) obey the scaling conditions (2.5) for all choices of D_i, G_m, A, and B.

Finally, let $A=B=1$ in (2.6) and (2.7), i.e. g_m does not depend on k. Then we obtain

$$d_i(\omega,k) = D_i\frac{k}{\omega}, \quad g_m(\omega,k) = G_m\omega \quad (i=1,2,3;\ m=2,3).$$

Hence, Kolmogorov's two-equation model of turbulence, which is obtained for $D_i = \nu_{i-1}$, $G_2 = \alpha_1$, and $G_3 = \alpha_2$, is invariant under the scaling (2.3) with the two-parameter family

$$(\rho,\gamma) \mapsto (\alpha,\beta,\gamma,\rho,\sigma) = \Big(\rho, \frac{\rho}{\gamma}, \gamma, \rho, \gamma^2\Big). \tag{2.8}$$

3 Definition of weak solutions

We begin with introducing notations that will be used throughout the paper.

Let X denote any real normed space with norm $|\cdot|_X$, and let $\langle x^*, x\rangle_X$ denote the dual pairing of $x^* \in X^*$ and $x \in X$. By $L^p(0,T;X)$ $(1 \le p \le +\infty)$ we denote the vector space of all equivalence classes of Bochner measurable mappings $u : [\,0,T\,] \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \Big(\int_0^T \big|u(t)\big|_X^p \,\mathrm{d}t \Big)^{1/p} & \text{if } \ 1 \le p < +\infty, \\ \operatorname*{ess\,sup}_{t\in[0,T\,]} \big|u(t)\big|_X & \text{if } \ p = +\infty \end{cases}$$

is finite (see, e.g., [Bou65, Chap. III, §3, Chap. IV, §3], [Bre73, App.] and [Dro01] for details). Let $\Omega \subseteq \mathbb{R}^N$ $(N \ge 2)$ be any open set, and let $Q = \Omega \times]\,0,T\,[$ $(0 < T < +\infty)$. For $1 \le p < +\infty$ and $u \in L^p(Q)$ define

$$[u](t)(\cdot) = u(\cdot, t) \quad \text{for a.a. } \ t \in [\,0,T\,].$$

By Fubini's theorem, the function $t \mapsto \int_\Omega \big|u(x,t)\big|^p \mathrm{d}x$ is in $L^1(0,T)$ and there holds

$$\int_0^T \big\|[u](t)\big\|_{L^p(\Omega)}^p \,\mathrm{d}t = \int_Q \big|u(x,t)\big|^p \,\mathrm{d}x\,\mathrm{d}t.$$

An elementary argument shows that the mapping $u \mapsto [u]$ is a linear isometry of $L^p(Q)$ onto $L^p\big(0,T;L^p(\Omega)\big)$. Therefore, these spaces will be identified in what follows. By $W^{1,p}(\Omega)$ we denote the usual Sobolev space, and we set $\boldsymbol{W}^{1,p}(\Omega) = \big(W^{1,p}(\Omega)\big)^N$.

Unless otherwise stated, from now on let $\Omega = \big(\,]0,l\,[\,\big)^3$ denote the cube introduced in Section 1. We define

$$\begin{aligned}
&W^{1,p}_{\mathrm{per,div}}(\Omega) = \big\{ u \in W^{1,p}(\Omega);\ u\big|_{\Gamma_i} = u\big|_{\Gamma_{i+3}} \ (i=1,2,3) \big\},\\
&\boldsymbol{W}^{1,p}_{\mathrm{per,div}}(\Omega) = \big\{ \boldsymbol{u} \in \boldsymbol{W}^{1,p}_{\mathrm{per}}(\Omega);\ \operatorname{div} \boldsymbol{u} = 0 \text{ a.e. in } \Omega \big\},\\
&C^1_{\mathrm{per},T}(\bar{Q}) = \big\{ \varphi \in C^1(\bar{Q});\ \varphi\big|_{\Gamma_i \times]\,0,T\,[} = \varphi\big|_{\Gamma_{i+3} \times]\,0,T\,[}\\
&\qquad\qquad\qquad (i=1,2,3),\ \varphi(x,T) = 0 \ \ \forall\, x \in \Omega \big\},\\
&\boldsymbol{C}^1_{\mathrm{per},T,\mathrm{div}}(\bar{Q}) = \big\{ \boldsymbol{v} \in \boldsymbol{C}^1_{\mathrm{per},T}(\bar{Q});\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } Q \big\}.
\end{aligned}$$

We emphasize that the test functions in $C^1_{\mathrm{per},T}(\bar{Q})$ vanish at $t = T$. Finally, by $\mathcal{M}(\bar{Q})$ we denote the vector space of all non-negative, bounded Radon measures on the σ-algebra of Borel sets $\subseteq \bar{Q}$.

To simplify the notation we subsequently set $\alpha_1 = 1$ and $\nu_2 = 1$, which can always be achieved by exploiting the scaling (2.8). We further set $\nu_1 = 1$, but keep the constant $\nu_0 > 0$ to emphasize that the source term in the equation (1.1d) for the turbulent energy k arises from the dissipation in the momentum equation (1.1b) for $\boldsymbol{u}$.

Definition 3.1. *Let $\boldsymbol{f} \in \boldsymbol{L}^1(Q)$, $\boldsymbol{u}_0 \in \boldsymbol{L}^1(\Omega)$ and $\omega_0, k_0 \in L^1(\Omega)$ such that $\omega_0, k_0 \ge 0$ a.e. in Ω. A triple of measurable functions $\{\boldsymbol{u}, \omega, k\}$ in Q is called* weak solution of (1.1) and (1.3) with defect measure *$\mu \in \mathcal{M}(\bar{Q})$, if*

$$\omega > 0, \ \ \frac{k}{\omega} \ge \textit{const} > 0 \ \ \textit{a.e. in } Q, \tag{3.1}$$

$$\left.\begin{aligned} &\boldsymbol{u} \in L^\infty\big(0,T;\boldsymbol{L}^2(\Omega)\big) \cap L^2\big(0,T;\boldsymbol{W}^{1,2}_{\mathrm{per,div}}(\Omega)\big),\\ &\omega \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;W^{1,2}_{\mathrm{per}}(\Omega)\big),\\ &k \in L^\infty\big(0,T;L^1(\Omega)\big) \cap L^{15/14}\big(0,T;W^{1,15/14}_{\mathrm{per}}(\Omega)\big), \end{aligned}\right\} \tag{3.2}$$

$$\int_Q \frac{k}{\omega}\big(\big(1+\big|\boldsymbol{D}(\boldsymbol{u})\big|\big)\big|\boldsymbol{D}(\boldsymbol{u})\big| + |\nabla\omega| + |\nabla k|\big)\,\mathrm{d}x\,\mathrm{d}t < +\infty, \tag{3.3}$$

and the following weak equations holds:

$$\left.\begin{aligned} &-\int_Q \boldsymbol{u}\cdot\frac{\partial\boldsymbol{v}}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_Q (\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t + \nu_0\int_Q \frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_\Omega \boldsymbol{u}_0(x)\cdot\boldsymbol{v}(x,0)\,\mathrm{d}x + \int_Q \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{C}^1_{\mathrm{per},T,\mathrm{div}}(\bar{Q}), \end{aligned}\right\} \tag{3.4}$$

$$\left.\begin{aligned} &-\int_Q \omega\frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_Q \omega\boldsymbol{u}\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_Q \frac{k}{\omega}\nabla\omega\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_\Omega \omega_0(x)\varphi(x,0)\,\mathrm{d}x - \int_Q \omega^2\varphi\,\mathrm{d}x\,\mathrm{d}t \quad \textit{for all } \varphi \in C^1_{\mathrm{per},T}(\bar{Q}), \end{aligned}\right\} \tag{3.5}$$

$$\left.\begin{aligned} &-\int_Q k\frac{\partial z}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_Q k\boldsymbol{u}\cdot\nabla z\,\mathrm{d}x\,\mathrm{d}t + \int_Q \frac{k}{\omega}\nabla k\cdot\nabla z\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_\Omega k_0(x)z(x,0)\,\mathrm{d}x + \int_Q \Big(\nu_0\frac{k}{\omega}\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 - \alpha_2 k\omega\Big)z\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_{\bar{Q}} z\,\mathrm{d}\mu \quad \textit{for all } z \in C^1_{\mathrm{per},T}(\bar{Q}). \end{aligned}\right\} \tag{3.6}$$

It is easy to see that all integrals in (3.4)–(3.6) are well-defined. It suffices to consider the integrals with integrands $k\boldsymbol{u}\cdot\nabla z$ and $\frac{k}{\omega}\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 z$ in (3.6). Firstly, it is well-known that condition (3.2) on $\boldsymbol{u}$ implies $\boldsymbol{u} \in \boldsymbol{L}^{10/3}(Q)$ (combine Hölder's inequality and Sobolev's embedding theorem). Analogously, the condition (3.2) on k implies $k \in L^{10/7}(Q)$ (take $N = 3$, $\theta = 3/4$, $(p_1,p_2) = (1,\frac{15}{14})$, and $(s_1,s_2) = (\infty,\frac{15}{14})$ in Lemma 4.2(B) below). Hence, $k\boldsymbol{u} \in \boldsymbol{L}^1(Q)$. Secondly, $\frac{k}{\omega}\,\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 \in L^1(Q)$ by virtue of (3.3).

Remark 3.2. *Every sufficiently regular classical solution* $\{\boldsymbol{u},\omega,k\}$ *of* (1.1) *and* (1.3) *satisfies the variational identities* (3.4), (3.5) *and* (3.6) *with defect measure* $\mu = 0$. *To verify this, we multiply* (1.1b), (1.1c) *and* (1.1d) *by the test functions* $\boldsymbol{v}$, φ *and* z, *respectively, and integrate by parts over the cube* Ω *and then over the interval* $[\,0,T\,]$.

Remark 3.3. *The condition* $k/\omega \ge$ *const* > 0 *is crucial for our existence theory, in particular for obtaining the regularities for* $\{\boldsymbol{u},\omega,k\}$ *stated in* (3.2). *It would be desirable to develop an existence theory without this condition, because this would allow us to study how the support of* k, *which is may be called the turbulent region, invades the non-turbulent region where* $k \equiv 0$.

Remark 3.4. *From* (3.6) *it follows that*

$$\begin{aligned} &-\int_Q k\frac{\partial z}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_Q k\boldsymbol{u}\cdot\nabla z\,\mathrm{d}x\,\mathrm{d}t + \int_Q \frac{k}{\omega}\nabla\omega\cdot\nabla z\,\mathrm{d}x\,\mathrm{d}t\\ &\ge \int_\Omega k_0(x)z(x,0)\,\mathrm{d}x + \int_Q \Big(\nu_0\frac{k}{\omega}\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 - \alpha_2 k\omega\Big)z\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.7}$$

for all $z \in C^1_{\mathrm{per},T}(\bar{Q})$ *with* $z \geq 0$ *in* Q. *Choosing* $z \equiv 1$ *we find*

$$\int_\Omega k(x,T)\,\mathrm{d}x \geq \int_\Omega k_0(x)\,\mathrm{d}x + \int_Q \Big(\nu_0 \frac{k}{\omega}\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 - \alpha_2 k\omega\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

A suitably adapted version of inequality (3.7) has been obtained in [BuM16, p. 10, eqn. (1.55)]. Our concept of weak solution of (1.1) and (1.3) with defect measure provides a more precise statement than this inequality and corresponds to the well-known energy equality for weak solutions of the Navier-Stokes equations (cf. (3.11) and Corollary 4.5 below).

More specifically, the variational identity in (3.6) gives the following result about possible jump discontinuities of the function $t \mapsto \int_\Omega k(x,t)\,\mathrm{d}x$.

Proposition 3.5. *Let* $\{\boldsymbol{u},\omega,k\}$ *be a weak solution of* (1.1) and (1.3) *with defect measure* μ. *Then, we have the integral relations*

$$\int_\Omega \omega(x,t)\,\mathrm{d}x + \int_0^t \int_\Omega \omega^2\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega \omega_0(x)\,\mathrm{d}x \quad \textit{for all } t \in [0,T\,], \tag{3.8}$$

$$\left.\begin{aligned} \int_\Omega k(x,t)\,\mathrm{d}x = \int_\Omega k_0(x)\,\mathrm{d}x + \int_0^t\!\!\int_\Omega \Big(\nu_0 \frac{k}{\omega}\,\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 - \alpha_2 k\omega\Big)\,\mathrm{d}x\,\mathrm{d}s \\ +\mu\big(\bar\Omega \times [0,t\,]\big) \quad \textit{for a.a. } t \in [0,T\,], \end{aligned}\right\} \tag{3.9}$$

$$\lim_{t\to 0} \int_\Omega k(x,t)\,\mathrm{d}x = \int_\Omega k_0(x)\,\mathrm{d}x + \mu\big(\bar\Omega \times \{0\}\big), \tag{3.10}$$

$$\left.\begin{aligned} \int_\Omega k(x,t)\,\mathrm{d}x = \int_\Omega k(x,s)\,\mathrm{d}x + \int_s^t\!\!\int_\Omega \Big(\nu_0 \frac{k}{\omega}\,\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 - \alpha_2 k\omega\Big)\,\mathrm{d}x\,\mathrm{d}\tau \\ +\mu\big(\bar\Omega \times\,]\,s,t\,]\big) \quad \textit{for a.a. } s,t \in [0,T\,] \textit{ with } s<t. \end{aligned}\right\} \tag{3.11}$$

Proof. It suffices to prove (3.9). The same reasoning gives (3.8), and (3.10) and (3.11) follow from (3.9). For $t \in\,]\,0,T\,[$ and $m > \frac{1}{T-t}$ ($m \in \mathbb{N}$) we define

$$\eta_m(\tau) = \begin{cases} 1 & \text{if } \ 0 \leq \tau \leq t, \\ m\left(t + \frac{1}{m} - \tau\right) & \text{if } \ t < \tau < t + \frac{1}{m}, \\ 0 & \text{if } \ t + \frac{1}{m} \leq \tau < T. \end{cases}$$

Taking $z(x,\tau) = 1 \cdot \eta_m(\tau)$, $(x,t) \in Q$ in (3.6), we arrive at

$$m\int_t^{t+1/m}\!\!\int_\Omega k(x,\tau)\,\,\mathrm{d}x\,\mathrm{d}\tau \tag{3.12}$$

$$\begin{aligned} = \int_\Omega k_0(x)\,\mathrm{d}x + \int_0^{t+1/m}\!\!\int_\Omega \Big(\nu_0 \frac{k}{\omega}\,\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 - \alpha_2 k\omega\Big)\eta_m\,\,\mathrm{d}x\,\mathrm{d}\tau \\ + \mu\big(\bar\Omega \times [\,0,t\,]\big) + m\int_{\bar\Omega\times]t,t+\frac{1}{m}[} \Big(t + \frac{1}{m} - \tau\Big)\,\,\mathrm{d}\mu. \end{aligned} \tag{3.13}$$

Observing that

$$m\int_{\bar\Omega\times]t,t+\frac{1}{m}[}\Big(t+\frac{1}{m}-\tau\Big)\,\mathrm{d}\mu \le \mu\Big(\bar\Omega\times\,]\,t,t+\frac{1}{m}\,[\,\Big)\longrightarrow 0\ \text{ as }\ m\to\infty,$$

the limit passage $m\to\infty$ in (3.13) gives (3.9) for every Lebesgue point $t\in[0,T\,]$ of the function $t\mapsto\int_\Omega k(x,t)\,\mathrm{d}x$. $\square$

4 An existence theorem for weak solutions

We define the function spaces

$$\begin{aligned} C^\infty_{\mathrm{per}}(\Omega) &= \big\{u|_\Omega\ ;\ u\in C^\infty(\mathbb{R}^3), u \text{ is } l\text{-periodic} \\ &\qquad\qquad\qquad \text{in the directions } \boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\big\},\\ \boldsymbol{C}^\infty_{\mathrm{per,div}}(\Omega) &= \big\{\boldsymbol{u}\in\boldsymbol{C}^\infty_{\mathrm{per}}(\Omega)\ ;\ \operatorname{div}\boldsymbol{u}=0\ \text{ in }\ \Omega\big\}.\end{aligned}$$

We impose the following conditions upon the right-hand side in (1.1b) and the initial data in (1.3b):

$$\left.\begin{array}{l} \boldsymbol{f}\in\boldsymbol{L}^2(Q);\ \boldsymbol{u}_0\in\overline{\boldsymbol{C}^\infty_{\mathrm{per,div}}(\Omega)}^{\|\cdot\|_{\boldsymbol{L}^2(\Omega)}},\ \omega_0\in L^\infty(\Omega),\ k_0\in L^1(\Omega),\\ \textit{there exist positive } \omega_*,\ \omega^* \textit{ such that } \omega_*\le\omega_0(x)\le\omega^* \textit{ for a.a. } x\in\Omega,\\ \textit{there exist positive } k_* \textit{ such that } k_0(x)\ge k_* \textit{ for a.a. } x\in\Omega. \end{array}\right\}\tag{4.1}$$

The following theorem is the main result of our paper.

Theorem 4.1 (Main existence result)**.** *Assume* (4.1) *and* $\alpha_2=\mathrm{const}>0$ *(cf.* (1.1d)*). Then there exists a triple of measurable functions* $\{\boldsymbol{u},\omega,k\}$ *in* Q *and a measure* $\mu\in\mathcal{M}(\bar Q)$ *such that*

$$\frac{\omega_*}{1+t\omega_*}\le\omega(x,t)\le\frac{\omega^*}{1+t\omega^*}\ \textit{and}\ \frac{k_*}{(1+t\omega^*)^{\alpha_2}}\le k(x,t)\ \textit{ for a.a. } (x,t)\in Q;\tag{4.2}$$

$$\left.\begin{array}{l} \boldsymbol{u}\in C_w\big([\,0,T\,];\boldsymbol{L}^2(\Omega)\big)\cap L^2(0,T;\boldsymbol{W}^{1,2}_{\mathrm{per,div}}(\Omega)),\\ \omega\in C_w\big([\,0,T\,];L^2(\Omega)\big)\cap L^2\big(0,T;W^{1,2}_{\mathrm{per}}(\Omega)\big),\\ k\in L^\infty\big(0,T;L^1(\Omega)\big)\cap\bigcap\limits_{1\le p<2}L^p\big(0,T;W^{1,p}_{\mathrm{per}}(\Omega)\big); \end{array}\right\}\tag{4.3}$$

$$\int_Q k\,\big(\big|\boldsymbol{D}(\boldsymbol{u})\big|^2+|\nabla\omega|^2\big)\,\mathrm{d}x\,\mathrm{d}t<+\infty,\tag{4.4}$$

$$\left.\begin{array}{l} \boldsymbol{u}':=\dfrac{\partial}{\partial t}\boldsymbol{u}\in\bigcap_{\sigma>16/5}L^{4/3}\big(0,T;\big(\boldsymbol{W}^{1,\sigma}_{\mathrm{per,div}}(\Omega)\big)^*\big),\\ \omega':=\dfrac{\partial}{\partial t}\omega\in\bigcap_{\sigma>16/5}L^{4/3}\big(0,T;\big(W^{1,\sigma}_{\mathrm{per}}(\Omega)\big)^*\big). \end{array}\right\}\tag{4.5}$$

The triple $\{\boldsymbol{u},k,\omega\}$ *is a weak solution of* (1.1) *and* (1.3) *in the sense of Definition 3.1. In particular, for all* $\sigma>16/5$ *we have that*

$$\left.\begin{array}{l} \displaystyle\int_0^T\big\langle\boldsymbol{u}'(t),\boldsymbol{v}(t)\big\rangle_{W^{1,\sigma}_{\mathrm{per,div}}}\mathrm{d}t+\int_Q\Big(-(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla\boldsymbol{v}+\nu_0\frac{k}{\omega}\,\boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{v})\Big)\,\mathrm{d}x\,\mathrm{d}t\\ \displaystyle=\int_Q\boldsymbol{f}\cdot\boldsymbol{v}\,\,\mathrm{d}x\,\mathrm{d}t\qquad \textit{for all } \boldsymbol{v}\in L^\sigma\big(0,T;\boldsymbol{W}^{1,\sigma}_{\mathrm{per,div}}(\Omega)\big)\ \textit{with}\ \boldsymbol{v}(\cdot,T)=0; \end{array}\right\}\tag{4.6}$$

$$\left.\begin{aligned}&\int_0^T \big\langle \omega'(t),\varphi(t)\big\rangle_{W^{1,\sigma}_{\mathrm{per}}}\,\mathrm{d}t-\int_Q \omega\boldsymbol{u}\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_Q \frac{k}{\omega}\,\nabla\omega\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\\&=-\int_Q \omega^2\varphi\,\mathrm{d}x\,\mathrm{d}t\quad \textit{for all } \varphi\in L^\sigma\big(0,T;W^{1,\sigma}_{\mathrm{per}}(\Omega)\big)\ \textit{with } \varphi(\cdot,T)=0;\end{aligned}\right\}\tag{4.7}$$

that (3.6) *holds; and that*

$$\boldsymbol{u}(0)=\boldsymbol{u}_0\ \textit{in } \boldsymbol{L}^2(\Omega)\quad \textit{and}\quad \omega(0)=\omega_0\ \textit{in } L^2(\Omega);\tag{4.8}$$

$$\left.\begin{aligned}&\frac12\int_\Omega \big|\boldsymbol{u}(x,t)\big|^2\,\mathrm{d}x+\nu_0\int_0^t\!\!\int_\Omega \frac{k}{\omega}\,\big|\boldsymbol{D}(\boldsymbol{u})\big|^2\,\mathrm{d}x\,\mathrm{d}s\\&\le \frac12\int_\Omega \big|\boldsymbol{u}_0(x)\big|^2+\int_0^t\!\!\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}s\end{aligned}\right\}\ \textit{for a.a. } t\in[0,T];\tag{4.9}$$

$$\left.\begin{aligned}&\int_\Omega\Big(\frac12\big|\boldsymbol{u}(x,t)\big|^2+k(x,t)\Big)\,\mathrm{d}x+\alpha_2\int_0^t\!\!\int_\Omega k\omega\,\mathrm{d}x\,\mathrm{d}s\\&\le\int_\Omega\Big(\frac12\big|\boldsymbol{u}_0(x)\big|^2+k_0(x)\Big)\,\mathrm{d}x+\int_0^t\!\!\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}s\end{aligned}\right\}\ \textit{for a.a. } t\in[0,T].\tag{4.10}$$

Of course, in (4.6) and (4.7) it suffices to consider $\sigma=\frac{16}{5}+\eta$ for an arbitrarily small $\eta>0$. The derivatives $\boldsymbol{u}'$ and ω' in (4.5) are understood in the sense of distributions from $]\,0,T\,[$ into $\big(\boldsymbol{W}^{1,\sigma}_{\mathrm{per,div}}(\Omega)\big)^*$ and $\big(W^{1,\sigma}_{\mathrm{per}}(\Omega)\big)^*$, respectively. Here we have used the continuous and dense embeddings (see, e.g., [Bre73, App.], [Dro01, pp. 54–56] for details)

$$W^{1,2}_{\mathrm{per}}(\Omega)\subset L^2(\Omega)\subset\big(W^{1,\sigma}_{\mathrm{per}}(\Omega)\big)^*\quad\text{for } \sigma\ge\frac65.$$

To see that $\{\boldsymbol{u},\omega,k\}$ together with the measure μ in the above theorem are a weak solution of (1.1) and (1.3) in the sense of the Definition 3.1, it suffices to note that (3.4) and (3.5) follow from (4.6) and (4.7), respectively, by integration by parts of the first integrals on the left-hand sides.

Before starting the proof it is instructive to check that the above estimates (4.2) to (4.5) are enough to show that all terms in (4.6) to (3.6) are well defined. For this, we first recall the classical Gagliardo-Nirenberg estimate and then provide an anisotropic version that is adjusted to the parabolic problems on $Q=[0,T]\times\Omega$, we use the short-hand notations

$$L^s(L^p):=L^s(0,T;L^p(\Omega))\quad\text{and}\quad J_\theta(a,b):=a^{1-\theta}\big(a{+}b\big)^\theta.$$

Lemma 4.2 (Gagliardo-Nirenberg estimates)**.** *For $N\in\mathbb{N}$ consider a bounded Lipschitz domain $\Omega\subset\mathbb{R}^N$.*
(A) (Classical isotropic version) Assume $1\le p_1<p<\infty$, $p_2\in\,]1,N[$ and $\theta\in\,]0,1[$ such that

$$\frac1p=(1{-}\theta)\,\frac1{p_1}+\theta\,\big(\frac1{p_2}-\frac1N\big).\tag{4.11}$$

Then, there exists a constant $C>0$ such that for all $\psi\in W^{1,p_2}(\Sigma)$ we have

$$\|\psi\|_{L^p(\Omega)}\le C\,J_\theta\big(\|\psi\|_{L^{p_1}(\Omega)},\|\nabla\psi\|_{L^{p_2}(\Omega)}\big).\tag{4.12}$$

(B) (Anisotropic version) Consider p, p_1, p_2, and θ as in (A) and s, s_1, and s_2 satisfying

$$1\le s_2\le s\le s_1\quad \textit{and}\ \frac1s=(1{-}\theta)\,\frac1{s_1}+\theta\,\frac1{s_2}.\tag{4.13}$$

Then, there exists $C^* > 0$ such that for all $\varphi \in L^{s_2}(0,T;W^{1,p_2}(\Omega))$ we have

$$\|\varphi\|_{L^s(L^p)} \le C^* J_\theta\Big(\|\varphi\|_{L^{s_1}(L^{p_1})}, \|\nabla\varphi\|_{L^{s_2}(L^{p_2})}\Big). \tag{4.14}$$

Proof. Part (A) is well-known, see e.g. [Rou13, Thm. 1.24].

To establish Part (B) we apply Part (A) for $\psi = \varphi(t)$ a.a. $t \in [0,T]$. Thus, we obtain (abbreviating $\|\psi\|_p := \|\psi\|_{L^p(\Omega)}$)

$$\begin{aligned}
\|\varphi\|^s_{L^s(L^p)} &= \int_0^T \|\varphi(t)\|^s_p \,\mathrm{d}t \overset{(4.12)}{\le} C_1 \int_0^T \|\varphi(t)\|^{(1-\theta)s}_{p_1}\big(\|\varphi(t)\|_{p_1} + \|\nabla\varphi(t)\|_{p_2}\big)^{\theta s}\,\mathrm{d}t \\
&\overset{\text{Hölder+(4.13)}}{\le} C_1 \big\| \|\varphi\|_{p_1}\big\|^{(1-\theta)s}_{L^{s_1}(0,T)} \Big\| \|\varphi\|_{p_1} + \|\nabla\varphi\|_{p_2}\Big\|^{\theta s}_{L^{s_2}(0,T)} \\
&\overset{s_1 \ge s_2}{\le} C_1 \|\varphi\|^{(1-\theta)s}_{L^{s_1}(L^{p_1})}\big(T^{1/s_2 - 1/s_1}\|\varphi\|_{L^{s_1}(L^{p_1})} + \|\nabla\varphi\|_{L^{s_2}(L^{p_2})}\big)^{\theta s} \\
&\le C_2\Big(J_\theta\big(\|\varphi\|_{L^{s_1}(L^{p_1})}, \|\nabla\varphi\|_{L^{s_2}(L^{p_2})}\big)\Big)^s,
\end{aligned}$$

which is the desired estimate. □

Remark 4.3 (Well-definedness of nonlinear terms)**.** *We first show that the second integral on the left-hand side of the variational identity in* (4.6) *are well-defined. For the integral of $(\boldsymbol{u}\otimes\boldsymbol{u}){:}\nabla\boldsymbol{v}$ we see that* (4.3) *allows us to use Lemma 4.2 with $N = 3$, $(s_1, p_1) = (\infty, 2)$ and $(s_2, p_2) = (2, 2)$. With $\theta = 3/4$ part (A) gives*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \le C\Big(\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}\|^{1/4}_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{u}\|^{3/4}_{\boldsymbol{L}^2(\Omega)}\Big), \tag{4.15}$$

whereas part (B) leads to $\boldsymbol{u} \in L^{8/3}(0,T;\boldsymbol{L}^4(\Omega))$, which implies

$$\boldsymbol{u}\otimes\boldsymbol{u} \in L^{4/3}(0,T;L^2(\Omega)). \tag{4.16}$$

With $\sigma > 16/5 > 2$ we have $\nabla\boldsymbol{v} \in L^2(0,T;L^2(\Omega))$ and $\int_Q(\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t$ is well defined. Using $\theta = 3/5$ in Lemma 4.2(B) we obtain $s = p = 10/3$ and hence conclude

$$\|\boldsymbol{u}\|_{L^{10/3}(Q)} \le C_2 J_{3/5}\Big(\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{L}^2)}, \|\nabla\boldsymbol{u}\|_{L^2(L^2)}\Big). \tag{4.17}$$

For the integral of $\frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u}){:}\boldsymbol{D}(\boldsymbol{v})$ we use $\omega \ge \omega_*/(1+T\omega_*) > 0$ from (4.2), *$k^{1/2}\boldsymbol{D}(\boldsymbol{u}) \in L^2(Q)$ from* (4.4). *Using* (4.3) *we can apply Lemma 4.2(B) to k with $N = 3$, $(s_1, p_1) = (\infty, 1)$, and $s_2 = p_2 \in [1, 2[$. Choosing $\theta = 3/4$ we obtain $s = p = 4p_2/3$, such that k lies in $L^{4p_2/3}(0,T;L^{4p_2/3}(\Omega)) = L^{4p_2/3}(Q)$. As $p_2 \in [1,2[$ is arbitrary, we have $k^{1/2} \in L^q(Q)$ for all $q \in [1, 16/3[$. By Hölder's inequality we arrive at*

$$k\boldsymbol{D}(\boldsymbol{u}) = k^{1/2}\, k^{1/2}\boldsymbol{D}(\boldsymbol{u}) \in L^{\overline{p}}(Q) \quad \text{for all } \overline{p} \in [1, 16/11[. \tag{4.18}$$

Using $\boldsymbol{D}(\boldsymbol{v}) \in L^\sigma(0,T;L^\sigma(\Omega)) = L^\sigma(Q)$ with $\tau > 16/5$ we see that there is always a $\overline{p} \in [1, 16/11[$ such that $\frac{1}{\sigma} + \frac{1}{\overline{p}} \le 1$. Hence we conclude

$$\int_Q \Big|\frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u}){:}\boldsymbol{D}(\boldsymbol{v})\Big|\,\mathrm{d}x\,\mathrm{d}t \le C\|k\boldsymbol{D}(\boldsymbol{u})\|_{L^{\overline{p}}(Q)}\|\boldsymbol{D}(\boldsymbol{v})\|_{L^\sigma(Q)} < \infty.$$

Thus, by a routine argument, (4.16) *and* (4.18) *lead to the existence of the distributional derivative $\boldsymbol{u}'$ as in* (4.5), *see also Sections 5.4–5.6.*

An analogous reasoning applies to the second and the third integral on the left-hand side of the variational identity in (4.7)*.*

Finally, combining $\boldsymbol{u} \in \boldsymbol{L}^2(Q)$ *and* $\nabla k \in L^p(Q)$ *for all* $p \in [1,2[$ *(see* (4.3)*) and* $k \in L^{4p/3}(Q)$ *from above, Hölder's inequality gives we find*

$$k\boldsymbol{u} \in \boldsymbol{L}^q(Q) \text{ and } k\nabla k \in \boldsymbol{L}^q(Q) \quad \text{for all } q \in [1, 8/7[,$$

i.e., the second and third integral on the left-hand side in (3.6) *are well defined.*

The estimates (4.2), which will be derived by using suitable comparison arguments, allow us to deduce the following result (based on the choice $\alpha_1 = 1$).

Corollary 4.4. *For a.a.* $(x,t) \in Q$*, we have the following estimates:*

$$L(x,t) = \frac{k(x,t)^{1/2}}{\omega(x,t)} \geq \frac{k_*^{1/2}}{\omega^*}\,(1 + t\omega^*)^{1-\alpha_2/2}, \tag{4.19}$$

$$\frac{1}{\omega^*} + t \leq \frac{1}{\omega(x,t)} \leq \frac{1}{\omega_*} + t. \tag{4.20}$$

Kolmogorov claimed in [Kol42] that $L = L(x,t)$ "... grows in proportion of $t^{2/7}$ ..." (see also [Spa91, p. 215], [Tik91, p. 329]). Clearly, from (4.19) with $\alpha_2 = 10/7$ it follows

$$L(x,t) \geq \frac{k_*^{1/2}}{\omega^*}(1 + t\omega^*)^{2/7} \quad \text{for a.a.} \quad (x,t) \in \Omega\times\,]\,t_0, T\,[\,.$$

Of course, Kolmogorov's claim is compatible with our lower estimate for any choice $\alpha_2 \geq 10/7$. However, it cannot be true for $\alpha_2 \in\,]0, 10/7[$.

Recalling the energy balance (3.9) we easily obtain the following result.

Corollary 4.5 (Energy equalities and defect measure)**.** *Let* $\{u, \omega, k\}$ *and* μ *be as in the theorem and assume that equality holds in* (4.10)*, i.e.,*

$$\begin{aligned}\int_\Omega \Big(\frac{1}{2}\big|\boldsymbol{u}(x,t)\big|^2 + k(x,t)\Big)\,\mathrm{d}x + \alpha_2 \int_0^t\!\!\int_\Omega k\omega\,\mathrm{d}x\,\mathrm{d}s \\ = \int_\Omega \Big(\frac{1}{2}\big|\boldsymbol{u}_0(x)\big|^2 + k_0(x)\Big)\,\mathrm{d}x + \int_0^t\!\!\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

for a.a. $t \in [0, T\,]$*. Then the following two statements are equivalent:*

(i) $\mu = 0$*;*

(ii) $$\begin{aligned}\frac{1}{2}\int_\Omega \big|\boldsymbol{u}(x,t)\big|^2\,\mathrm{d}x + \nu_0 \int_0^t\!\!\int_\Omega \frac{k}{\omega}\big|\boldsymbol{D}(\boldsymbol{u})\big|^2\,\mathrm{d}x\,\mathrm{d}s \\ = \frac{1}{2}\int_\Omega \big|\boldsymbol{u}_0(x)\big|^2\,\mathrm{d}x + \int_0^t\!\!\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}s \quad \textit{for a.a.} \quad t \in [\,0, T\,].\end{aligned}$$

This result shows that inequalities (4.9), (4.10) and the defect measure μ in (3.6) are related to the deep problem of proving an energy equality for weak solutions of the Navier-Stokes equations. A similar result for the case of Navier-Stokes equations with temperature dependent viscosities has been obtained in [Nau08]. Defect measures also appear in a natural way in the context of weak solutions of other types of nonlinear PDEs (see, e.g., [AlV02, Har06, LLZ95]).

5 Proof of the existence theorem

The proof of the main Theorem 4.1 proceeds in several steps. First we regularize the problem by adding small higher-order dissipation terms of r-Laplacian type and small coercivity-generating lower order terms. A general result for pseudo-monotone operators, which is detailed in Appendix A, then provides approximate solutions $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$. In Section 5.2 we provide ε-independent upper and lower bounds for ω_ε and k_ε by comparison arguments. In Section 5.3 we complement the standard energy estimates by improved integral estimates for k_ε that allow us to pass to the limit $\varepsilon \searrow 0$ in Section 5.5.

5.1 Defining suitable approximate solutions $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$

Let be ω_*, ω^* and k_* as in (4.1). We introduce the comparison functions

$$\underline{\omega}(t) = \frac{\omega_*}{1+t\omega_*}, \quad \overline{\omega}(t) = \frac{\omega^*}{1+t\omega^*}, \quad \kappa(t) = \frac{k_*}{(1+t\omega^*)^{\alpha_2}} \quad \text{for } t \in [0,T], \tag{5.1}$$

which will be the desired bounds for ω_ε and k_ε in Q. Subsequently we will use the notion

$$\xi^+ := \max\{\xi, 0\} \geq 0 \quad \text{and} \quad \xi^- = \min\{\xi, 0\} \leq 0$$

for the positive and negative parts of real numbers or real-valued functions.

We choose a fixed number $r \in\,]3,\infty[$ and consider for all small $\varepsilon > 0$ the following *r-Laplacian approximation* of (1.1), where we add the coercivity-generating terms $\varepsilon\big(\underline{\omega}(t)\big)^{r-1}$ and $\varepsilon\big(\kappa(t)\big)^{r-1}$ to the right-hand sides of (1.1c) and (1.1d), respectively:

$$\operatorname{div} \boldsymbol{u} = 0, \tag{5.2a}$$

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= \nu_0 \operatorname{div}\Big(\frac{k^+}{\varepsilon+\omega^+}\,\boldsymbol{D}(\boldsymbol{u})\Big) - \nabla p + \boldsymbol{f} \\ &\quad + \varepsilon\Big(\operatorname{div}\big(\big|\boldsymbol{D}(\boldsymbol{u})\big|^{r-2}\boldsymbol{D}(\boldsymbol{u})\big) - |\boldsymbol{u}|^{r-2}\boldsymbol{u}\Big), \end{aligned} \tag{5.2b}$$

$$\begin{aligned} \frac{\partial \omega}{\partial t} + \boldsymbol{u}\cdot\nabla\omega &= \operatorname{div}\Big(\frac{k^+}{\varepsilon+\omega^+}\,\nabla\omega\Big) - \omega^+\omega \\ &\quad + \varepsilon\Big(\operatorname{div}\big(|\nabla\omega|^{r-2}\nabla\omega\big) - |\omega|^{r-2}\omega\Big) + \varepsilon\big(\underline{\omega}(t)\big)^{r-1}, \end{aligned} \tag{5.2c}$$

$$\begin{aligned} \frac{\partial k}{\partial t} + \boldsymbol{u}\cdot\nabla k &= \operatorname{div}\Big(\frac{k^+}{\varepsilon+\omega^+}\,\nabla k\Big) + \nu_0\frac{k^+}{\varepsilon+\omega^+ +\varepsilon k^+}\,\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 - \alpha_2 k\omega^+ \\ &\quad + \varepsilon\Big(\operatorname{div}\big(|\nabla k|^{r-2}\nabla k\big) - |k|^{r-2}k\Big) + \varepsilon\big(\kappa(t)\big)^{r-1}. \end{aligned} \tag{5.2d}$$

We consider system (5.2) with initial data $\{\boldsymbol{u}_{0,\varepsilon}, \omega_{0,\varepsilon}, k_{0,\varepsilon}\}$ satisfying

$$\{\boldsymbol{u}_{0,\varepsilon}, \omega_{0,\varepsilon}, k_{0,\varepsilon}\} \in \boldsymbol{W}^{1,r}_{\mathrm{per,div}}(\Omega) \times W^{1,r}_{\mathrm{per}}(\Omega) \times W^{1,r}_{\mathrm{per}}(\Omega), \tag{5.3a}$$

$$\omega_* \leq \omega_{0,\varepsilon}(x) \leq \omega^* \ \text{ and } \ k_{0,\varepsilon}(x) \geq k_* \quad \text{a.e. in } \Omega, \tag{5.3b}$$

$$\left.\begin{aligned} &\boldsymbol{u}_{0,\varepsilon} \longrightarrow \boldsymbol{u}_0 \text{ in } \boldsymbol{L}^2(\Omega), \quad \omega_{0,\varepsilon} \longrightarrow \omega_0 \text{ a.e. in } \Omega, \\ &k_{0,\varepsilon} \longrightarrow k_0 \ \text{ in } \ L^1(\Omega) \ \text{ for } \varepsilon \to 0. \end{aligned}\right\} \tag{5.3c}$$

The existence of a sequence $\{\boldsymbol{u}_{0,\varepsilon}\}_{\varepsilon>0}$ which satisfies (5.3a) follows immediately from the condition on $\boldsymbol{u}_0$ in (4.1), whereas the existence of sequences $\{\omega_{0,\varepsilon}\}_{\varepsilon>0}$ and $\{k_{0,\varepsilon}\}_{\varepsilon>0}$

satisfying (5.3) can be derived by routine argument from the conditions on ω_0 and k_0 in (4.1).

The following lemma states the existence of weak solutions of (5.2) under the periodic boundary conditions (1.3a) and initial data (5.3). This result, which we derive in Appendix A by a direct application of existence results for pseudo-monotone evolutionary problems (see Theorem A.1), forms the starting point for our discussion in Subsections 5.2–5.6.

Proposition 5.1 (Existence of approximate solutions). *Let $\{\boldsymbol{u}_{0,\varepsilon}, \omega_{0,\varepsilon}, k_{0,\varepsilon}\}_{\varepsilon>0}$ be as in (5.3), $r > 3$, and $\boldsymbol{f} \in \boldsymbol{L}^2(Q)$. Then, for every $\varepsilon > 0$ there exists a triple $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ such that*

$$\boldsymbol{u}_\varepsilon \in C\big([0,T]; \boldsymbol{L}^2(\Omega)\big) \cap L^r\big(0,T; \boldsymbol{W}^{1,r}_{\mathrm{per,div}}(\Omega)\big), \tag{5.4a}$$

$$\omega_\varepsilon, k_\varepsilon \in C\big([0,T]; L^2(\Omega)\big) \cap L^r\big(0,T; W^{1,r}_{\mathrm{per}}(\Omega)\big), \tag{5.4b}$$

$$\boldsymbol{u}'_\varepsilon \in L^{r'}\big(0,T; \big(\boldsymbol{W}^{1,r}_{\mathrm{per,div}}(\Omega)\big)^*\big), \quad \omega'_\varepsilon,\, k'_\varepsilon \in L^{r'}\big(0,T; \big(W^{1,r}_{\mathrm{per}}(\Omega)\big)^*\big), \tag{5.4c}$$

and

$$\left.\begin{aligned}
&\int_0^T \big\langle \boldsymbol{u}'_\varepsilon(t), \boldsymbol{v}(t)\big\rangle_{W^{1,r}_{\mathrm{per,div}}} \mathrm{d}t + \int_Q \sum_{i=1}^3 u_{\varepsilon,i}(\partial_i \boldsymbol{u}_\varepsilon)\cdot \boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \nu_0 \int_Q \frac{k_\varepsilon^+}{\varepsilon + \omega_\varepsilon^+} \boldsymbol{D}(\boldsymbol{u}_\varepsilon) : \boldsymbol{D}(\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \varepsilon \int_Q \big(\big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^{r-2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon) : \boldsymbol{D}(\boldsymbol{v}) + |\boldsymbol{u}_\varepsilon|^{r-2}\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v}\big)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_Q \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t \qquad \textit{for all } \boldsymbol{v} \in L^r\big(0,T; \boldsymbol{W}^{1,r}_{\mathrm{per,div}}(\Omega)\big),
\end{aligned}\right\} \tag{5.5a}$$

$$\left.\begin{aligned}
&\int_0^T \big\langle \omega'_\varepsilon(t), \varphi(t)\big\rangle_{W^{1,r}_{\mathrm{per}}} \mathrm{d}t + \int_Q \varphi\boldsymbol{u}_\varepsilon\cdot\nabla\omega_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_Q \frac{k_\varepsilon^+}{\varepsilon+\omega_\varepsilon^+}\nabla\omega_\varepsilon\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_Q \omega_\varepsilon^+\omega_\varepsilon\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \varepsilon\int_Q \big(|\nabla\omega_\varepsilon|^{r-2}\nabla\omega_\varepsilon\cdot\nabla\varphi + |\omega_\varepsilon|^{r-2}\omega_\varepsilon\varphi\big)\,\mathrm{d}x\,\mathrm{d}t\\
&= \varepsilon\int_Q \big(\underline{\omega}(t)\big)^{r-1}\varphi\,\mathrm{d}x\,\mathrm{d}t \qquad \textit{for all } \varphi \in L^r\big(0,T; W^{1,r}_{\mathrm{per}}(\Omega)\big),
\end{aligned}\right\} \tag{5.5b}$$

$$\left.\begin{aligned}
&\int_0^T \big\langle k'_\varepsilon(t), z(t)\big\rangle_{W^{1,r}_{\mathrm{per}}} \mathrm{d}t + \int_Q z\boldsymbol{u}_\varepsilon\cdot\nabla k_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_Q \frac{k_\varepsilon^+}{\varepsilon{+}\omega_\varepsilon^+}\nabla k_\varepsilon{\cdot}\nabla z\,\mathrm{d}x\,\mathrm{d}t - \nu_0\int_Q \frac{k_\varepsilon^+}{\varepsilon{+}\omega_\varepsilon^+{+}\varepsilon k_\varepsilon^+}\big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^2 z\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \alpha_2\int_Q k_\varepsilon\omega_\varepsilon^+ z\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_Q \big(|\nabla k_\varepsilon|^{r-2}\nabla k_\varepsilon{\cdot}\nabla z{+}|k_\varepsilon|^{r-2}k_\varepsilon z\big)\,\mathrm{d}x\,\mathrm{d}t\\
&= \varepsilon\int_Q \big(\kappa(t)\big)^{r-1} z\,\mathrm{d}x\,\mathrm{d}t \qquad \textit{for all } z \in L^r\big(0,T; W^{1,r}_{\mathrm{per}}(\Omega)\big),
\end{aligned}\right\} \tag{5.5c}$$

$$\boldsymbol{u}_\varepsilon(0) = \boldsymbol{u}_{0,\varepsilon}, \quad \omega_\varepsilon(0) = \omega_{0,\varepsilon}, \quad k_\varepsilon(0) = k_{0,\varepsilon}. \tag{5.6}$$

The proof of Proposition 5.1 is the content of Appendix A. Observing the separability of $\boldsymbol{W}^{1,r}_{\mathrm{per,div}}(\Omega)$ and $W^{1,r}_{\mathrm{per}}(\Omega)$ and using (5.4), a routine argument yields that the system (5.5) is equivalent to the following conditions for a.a. $t \in [0,T]$:

$$\left.\begin{aligned}
&\big\langle \boldsymbol{u}'_\varepsilon(t), \boldsymbol{w}\big\rangle_{W^{1,r}_{\mathrm{per,div}}} + \int_\Omega\Big(\big(\boldsymbol{u}_\varepsilon(t)\cdot\nabla\boldsymbol{u}(t)\big)\cdot\boldsymbol{w} + \nu_0\frac{k^+_\varepsilon(t)}{\varepsilon+\omega^+_\varepsilon(t)}\,\boldsymbol{D}\big(\boldsymbol{u}_\varepsilon(t)\big):\boldsymbol{D}(\boldsymbol{w})\Big)\,\mathrm{d}x\\
&\quad+\varepsilon\int_\Omega\big(\big|\boldsymbol{D}\big(\boldsymbol{u}_\varepsilon(t)\big)\big|^{r-2}\boldsymbol{D}\big(\boldsymbol{u}_\varepsilon(t)\big):\boldsymbol{D}(\boldsymbol{w}) + \big|\boldsymbol{u}_\varepsilon(t)\big|^{r-2}\boldsymbol{u}_\varepsilon(t)\cdot\boldsymbol{w}\big)\,\mathrm{d}x\\
&=\int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{w}\,\mathrm{d}x \qquad\qquad \text{for all } \boldsymbol{w}\in\boldsymbol{W}^{1,r}_{\mathrm{per,div}}(\Omega),
\end{aligned}\right\}\tag{5.7a}$$

$$\left.\begin{aligned}
&\big\langle \omega'_\varepsilon(t), \psi\big\rangle_{W^{1,r}_{\mathrm{per}}} + \int_\Omega\Big(\psi\boldsymbol{u}_\varepsilon(t)\cdot\nabla\omega_\varepsilon(t) + \frac{k^+_\varepsilon(t)}{\varepsilon+\omega^+_\varepsilon(t)}\,\nabla\omega_\varepsilon(t)\cdot\nabla\psi\Big)\,\mathrm{d}x\\
&\quad+\int_\Omega\Big(\omega^+_\varepsilon(t)\omega_\varepsilon(t)\psi + \varepsilon\big(\big|\nabla\omega_\varepsilon(t)\big|^{r-2}\nabla\omega_\varepsilon(t)\cdot\nabla\psi + \big|\omega_\varepsilon(t)\big|^{r-2}\omega_\varepsilon(t)\psi\big)\Big)\,\mathrm{d}x\\
&=\varepsilon\big(\underline{\omega}(t)\big)^{r-1}\int_\Omega\psi\,\mathrm{d}x \qquad\qquad \text{for all } \psi\in W^{1,r}_{\mathrm{per}}(\Omega),
\end{aligned}\right\}\tag{5.7b}$$

$$\left.\begin{aligned}
&\big\langle k'_\varepsilon(t), z\big\rangle_{W^{1,r}_{\mathrm{per}}} + \int_\Omega\Big(z\boldsymbol{u}_\varepsilon(t)\cdot\nabla k_\varepsilon(t) + \frac{k^+_\varepsilon(t)}{\varepsilon+\omega^+_\varepsilon(t)}\,\nabla k_\varepsilon(t)\cdot\nabla z\Big)\,\mathrm{d}x\\
&\quad-\nu_0\int_\Omega\frac{k^+_\varepsilon(t)}{\varepsilon+\omega^+_\varepsilon(t)+\varepsilon k^+_\varepsilon(t)}\,\big|\boldsymbol{D}\big(\boldsymbol{u}_\varepsilon(t)\big)\big|^2 z\,\mathrm{d}x + \alpha_2\int_\Omega k_\varepsilon(t)\omega^+_\varepsilon(t)z\,\mathrm{d}x\\
&\quad+\varepsilon\int_\Omega\big(\big|\nabla k_\varepsilon(t)\big|^{r-2}\nabla k_\varepsilon(t)\cdot\nabla z + \big|k_\varepsilon(t)\big|^{r-2}k_\varepsilon(t)z\big)\,\mathrm{d}x\\
&=\varepsilon\big(\kappa(t)\big)^{r-1}\int_\Omega z\,\mathrm{d}x \qquad\qquad \text{for all } z\in W^{1,r}_{\mathrm{per}}(\Omega)
\end{aligned}\right\}\tag{5.7c}$$

We notice that the set $\mathcal{N}\subset[\,0,T\,]$ of measure zero of those t where (5.7) fails, does not depend on $(\boldsymbol{w},\psi,z)$. More specifically, if $\varepsilon=\varepsilon_m>0$ with $\lim_{m\to\infty}\varepsilon_m=0$, then $\mathcal{N}$ can be chosen independently of m.

The variational identities in (5.7) are the point of departure for the proof of a series of the a priori estimates for $\{\boldsymbol{u}_\varepsilon,\omega_\varepsilon,k_\varepsilon\}$ we are going to derive in Subsections 5.2–5.4.

5.2 Upper and lower bounds for $\{\omega_\varepsilon, k_\varepsilon\}$

Let $\underline{\omega}$, $\overline{\omega}$ and κ be as in (5.1) and $r>3$ as chosen in Section 5.1. The following result provides pointwise upper and lower bounds that are obtained via classical comparison arguments for weak solutions of the scalar parabolic equations for ω and k, cf. (1.1c) and (1.1d), respectively.

Lemma 5.2. *Let be $\{\boldsymbol{u}_\varepsilon,\omega_\varepsilon,k_\varepsilon\}$ a triple according to Proposition 5.1 with $r>3$. Then,*

$$\underline{\omega}(t)\le\omega_\varepsilon(x,t)\le\overline{\omega}(t)\quad\text{ and }\quad\kappa(t)\le k_\varepsilon(x,t)\tag{5.8}$$

for a.a. $(x,t)\in Q$ and for all $\varepsilon>0$.

Proof. For notational simplicity, we set $\boldsymbol{u}\equiv\boldsymbol{u}_\varepsilon$, $\omega\equiv\omega_\varepsilon$ and $k\equiv k_\varepsilon$ within this proof.

Step 1: $\omega\ge\underline{\omega}$. The function $\psi=\big(\omega(\cdot,t)-\underline{\omega}(t)\big)^-$ is an admissible test function for (5.7b). Since $\underline{\omega}(t)$ does not depend on x we have $\frac{1}{2}\nabla(\psi^2)=\psi\nabla\omega$ and $\nabla\omega\cdot\nabla\psi=|\nabla\psi|^2\ge$

0. Using $\underline{\omega} > 0$ and the monotonicity of $\omega \mapsto |\omega|^{r-2}\omega$ we arrive at

$$\begin{aligned}\big\langle \omega'(t), \big(\omega(t) - \underline{\omega}(t)\big)^{-}\big\rangle_{W^{1,r}_{\mathrm{per}}} + \int_\Omega \omega^2\big(\omega - \underline{\omega}(t)\big)^{-}\,\mathrm{d}x \\ \leq \varepsilon \int_\Omega \Big(\big(\underline{\omega}(t)\big)^{r-1} - |\omega|^{r-2}\omega\Big)\big(\omega - \underline{\omega}(t)\big)^{-}\,\mathrm{d}x \qquad \leq 0 \end{aligned} \tag{5.9}$$

for a.a. $t \in [0,T]$. By construction we have $\underline{\omega}'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\underline{\omega}(t) = -\big(\underline{\omega}(t)\big)^2$. Identifying $\underline{\omega}$ with a function in $C^1\big([0,T]; W^{1,r}_{\mathrm{per}}(\Omega)\big)$ the estimate (5.9) leads to

$$\big\langle \omega'(t) - \underline{\omega}'(t), \big(\omega(t){-}\underline{\omega}(t)\big)^{-}\big\rangle_{W^{1,r}_{\mathrm{per}}} \leq -\int_\Omega \Big(\omega^2 - \big(\underline{\omega}(t)\big)^2\Big)\big(\omega - \underline{\omega}(t)\big)^{-}\,\mathrm{d}x \leq 0.$$

By (5.1) and (5.3b), we have $\omega(x,0) - \underline{\omega}(0) \geq 0$, which means $\psi(x,0) = 0$ for a.a. $x \in \Omega$. Using a slight modification of [Lio69, pp. 290–291] we find

$$\int_\Omega \frac{1}{2}\psi(t)^2\,\mathrm{d}x = \int_\Omega \frac{1}{2}\psi(0)^2\,\mathrm{d}x + \int_0^t \langle \psi', \psi\rangle_{W^{1,r}_{\mathrm{per}}}\,\mathrm{d}t = 0 + \int_0^t \langle \omega'{-}\underline{\omega}', (\omega{-}\underline{\omega})^{-}\rangle_{W^{1,r}_{\mathrm{per}}}\,\mathrm{d}t \leq 0.$$

Hence, we conclude $\psi(t) = 0$ for all t, which means that

$$\omega(x,t) \geq \underline{\omega}(t) \quad \text{ for a.a. } \ (x,t) \in Q. \tag{5.10}$$

Step 2: $\omega \leq \overline{\omega}$. Next, we insert $\psi = \big(\omega(\cdot,t) - \overline{\omega}(t)\big)^{+}$ into (5.7c) and argue as in Step 1 to find

$$\big\langle \omega', (\omega{-}\overline{\omega})^{+}\big\rangle_{W^{1,r}_{\mathrm{per}}} + \int_\Omega \omega^2\big(\omega - \overline{\omega}\big)^{+}\,\mathrm{d}x \leq \varepsilon\int_\Omega \big((\underline{\omega})^{r-1}{-}\omega^{r-1}\big)\big(\omega{-}\overline{\omega}\big)^{+}\,\mathrm{d}x \leq 0.$$

For the last estimate we used $\omega \geq \underline{\omega}$, which was obtained in Step 1. Hence, as above,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\psi(t)^2\,\mathrm{d}x = \big\langle \omega'(t) - \dot{\overline{\omega}}(t), \big(\omega(t) - \overline{\omega}(t)\big)^{+}\big\rangle_{W^{1,r}_{\mathrm{per}}} \leq -\int_\Omega \big(\omega^2{-}\overline{\omega}^2\big)\big(\omega{-}\overline{\omega}\big)^{+}\,\mathrm{d}x \leq 0$$

for a.a. $t \in [0,T]$. Again by (5.1) and (5.3b), we have $\psi(0) = 0$ a.e. in Ω and conclude

$$\omega(x,t) \leq \overline{\omega}(t) \quad \text{ for a.a. } (x,t) \in Q. \tag{5.11}$$

Step 3: $k \geq \kappa$. We first insert $z = k^{-}(\cdot,t)$ into (5.7c) and find $k \geq 0$ a.e. in Q. Next, we insert the test function $z(x,t) = \big(k(x,t) - \kappa(t)\big)^{-}$ and obtain as above

$$\big\langle k'(t), \big(k(t) - \kappa(t)\big)^{-}\big\rangle_{W^{1,r}_{\mathrm{per}}} + \alpha_2\int_\Omega k(t)\omega(t)\big(k{-}\kappa(t)\big)^{-}\,\mathrm{d}x \leq 0$$

for a.a. $t \in [0,T]$. By construction κ satisfies $\kappa'(t) = -\alpha_2\kappa(t)\overline{\omega}(t)$ for all $t \in [0,T]$. It follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\Big(\big(k(t){-}\kappa(t)\big)^{-}\Big)^2\,\mathrm{d}x = \big\langle k'(t) - \dot{\kappa}(t), \big(k(t) - \kappa(t)\big)^{-}\big\rangle_{W^{1,r}_{\mathrm{per}}} \\ \leq -\alpha_2\int_\Omega \big(k(t)\omega(t) - \kappa(t)\overline{\omega}(t)\big)\big(k(t) - \kappa(t)\big)^{-}\,\mathrm{d}x \ \leq \ 0.\end{aligned}$$

To see the last inequality, we use $\omega \leq \overline{\omega}$ a.e. in Q from Step 2, which gives $k(x,t)\omega(x,t) \leq \kappa(t)\overline{\omega}(t)$ for a.a. x of the set $\big\{x \in \Omega\,;\, k(x,t) \leq \kappa(t)\big\}$. Since $k(x,0) \geq \kappa(0)$ for a.a. $x \in \Omega$ by (5.1) and (5.3b) we obtain, as above,

$$k(x,t) \geq \kappa(t) \quad \text{ for a.a. } \ (x,t) \in Q.$$

Altogether the upper and lower bounds in (5.8) are established. □

5.3 Energy estimates for $(\boldsymbol{u}_\varepsilon, \omega_\varepsilon)$ and improved estimates for k_ε

For the subsequent estimates we fix the data

$$\mathfrak{D} = \{T,\ \boldsymbol{f},\ \omega_*,\ \omega^*,\ k_*,\ r\}$$

and will indicate constants that only depend on $\mathfrak{D}$ by $C_\mathfrak{D}$. However, depending on the context the constants $C_\mathfrak{D}$ may be different. We also define the constant

$$\beta_* = \frac{k_*}{(1+\omega^*)(1+T\omega^*)^{\alpha_2}},$$

which according to Lemma 5.2b is a lower bound for $k_\varepsilon/(\varepsilon+\omega_\varepsilon)$. This will allows us to derive the standard estimates for $\boldsymbol{u}_\varepsilon$ and ω_ε.

Lemma 5.3. *There exists a constant $C_\mathfrak{D} > 0$ such for all $\varepsilon \in \,]0,1]$ and all solutions $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ as in Proposition 5.1 we have the estimates*

$$\left.\begin{aligned} \|\boldsymbol{u}_\varepsilon\|^2_{L^\infty(\boldsymbol{L}^2)} + \int_Q \big(\beta_* + \frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\big)\big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon \int_Q \big(\big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^r + |\boldsymbol{u}_\varepsilon|^r\big)\,\mathrm{d}x\,\mathrm{d}t& \\ \le C_\mathfrak{D}\big(\|\boldsymbol{u}_{0,\varepsilon}\|^2_{\boldsymbol{L}^2} + \|\boldsymbol{f}\|^2_{\boldsymbol{L}^2}\big),& \end{aligned}\right\} \tag{5.12a}$$

$$\left.\begin{aligned} \|\omega_\varepsilon\|^2_{L^\infty(L^2)} + \int_Q \Big(\beta_* + \frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\Big)|\nabla\omega_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon \int_Q \big(|\nabla\omega_\varepsilon|^r + \omega_\varepsilon^r\big)\,\mathrm{d}x\,\mathrm{d}t& \\ \le C_\mathfrak{D}\big(1 + \|\omega_{0,\varepsilon}\|^2_{L^2}\big).& \end{aligned}\right\} \tag{5.12b}$$

Proof. We insert the test functions $\boldsymbol{w} = \boldsymbol{u}_\varepsilon$ and $\psi = \omega_\varepsilon$ in (5.7a) and (5.7b), respectively. Integrating over $[0,t]$ and using $\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon} \ge \beta_*$ a.e. in Q (cf. (5.8)), the desired estimates (5.12) are readily obtained by the aid of Gronwall's lemma. ☐

By (5.3) the approximative initial conditions satisfy $\sup_{0<\varepsilon\le1}\big(\|\boldsymbol{u}_{0,\varepsilon}\|_{\boldsymbol{L}^2} + \|\omega_{0,\varepsilon}\|_{L^2}\big) < +\infty$. Therefore all terms on the left hand sides of (5.12) are bounded independently of $\varepsilon \in \,]0,1]$.

Of course, one obtains a trivial bound for k_ε in $L^\infty(0,T;L^1(\Omega))$ by testing (5.7c) with $z \equiv 1$. We include this result in the following non-trivial estimate that implies uniform higher integrability of k_ε as well as suitable bounds for ∇k_ε. For this we test (5.7c) by $z = 1 - (1+k_\varepsilon)^{-\delta}$ for $\delta \in \,]0,1[$, which is a well-known technique for treating diffusion equations with an L^1 right-hand side, see e.g. [Rak91, BD*97].

Proposition 5.4. *For $\mathfrak{D}$, $p \in [1,2[$, and $\delta \in \,]0,1[$, there exists $C^{p,\delta}_\mathfrak{D} > 0$ such that for all $\varepsilon \in \,]0,1]$ and all $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ as in Proposition 5.1, we have the estimate*

$$\left.\begin{aligned} \|k_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \int_Q \Big(k_\varepsilon^{4p/3} + |\nabla k_\varepsilon|^p + \frac{|\nabla k_\varepsilon|^2}{(1+k_\varepsilon)^\delta}\Big)\,\mathrm{d}x\,\mathrm{d}t& \\ + \varepsilon \int_Q \Big(\frac{|\nabla k_\varepsilon|^r}{(1+k_\varepsilon)^{1+\delta}} + k_\varepsilon^{r-1}\Big)\,\mathrm{d}x\,\mathrm{d}t& \\ \le C^{p,\delta}_\mathfrak{D}\big(1 + \|\boldsymbol{u}_{0,\varepsilon}\|^2_{\boldsymbol{L}^2(\Omega)} + \|k_{0,\varepsilon}\|_{L^1(\Omega)}\big).& \end{aligned}\right\} \tag{5.13}$$

Proof. Step 1: For $0 < \delta < 1$ we define $\Phi : [0,\infty[\, \to [0,\infty[$ via

$$\Phi(\tau) = \tau + \frac{1}{1-\delta}\big(1 - (1+\tau)^{1-\delta}\big), \quad 0 \le \tau < +\infty.$$

Hence, Φ is convex and satisfies, for all $\tau \geq 0$, the estimates

$$\frac{\tau}{2} - \frac{2}{1-\delta} \leq \Phi(\tau) \leq \tau, \quad \Phi'(\tau) = 1 - \frac{1}{(1+\tau)^\delta} \in [0,1], \quad \Phi''(\tau) = \frac{\delta}{(1+\tau)^{1+\delta}}. \tag{5.14}$$

From [Rak92, pp. 360–361; cf. also pp. 365–366] (with $W^{1,p}_{\mathrm{per}}(\Omega)$ in place of $W^{1,p}_0(\Omega)$) we have the chain rule

$$\int_0^t \big\langle k_\varepsilon'(s), \Phi'\big(k_\varepsilon(s)\big)\big\rangle_{W^{1,r}_{\mathrm{per}}} \,\mathrm{d}s = \int_\Omega \Phi\big(k_\varepsilon(x,t)\big)\,\mathrm{d}x - \int_\Omega \Phi\big(k_{0,\varepsilon}(x)\big)\,\mathrm{d}x$$

for all $t \in [0,T]$. When inserting $z = \Phi'\big(k_\varepsilon(\cdot,t)\big)$ into (5.7c) we obtain

$$\int_\Omega \Phi'\big(k_\varepsilon(\cdot,t)\big)\boldsymbol{u}_\varepsilon(t)\cdot\nabla k_\varepsilon(t)\,\mathrm{d}x = \int_\Omega \boldsymbol{u}_\varepsilon(t)\cdot\nabla\big(\Phi(k_\varepsilon(\cdot,t)\big)\,\mathrm{d}x = 0 \quad \text{for a.a. } t \in [0,T],$$

where we used $\operatorname{div}\boldsymbol{u}_\varepsilon = 0$. With this we obtain (recall $\nu_0 = 1 = \alpha_2$)

$$\begin{aligned}
&\int_\Omega \Phi\big(k_\varepsilon(x,t)\big)\,\mathrm{d}x + \delta\int_0^t\!\!\int_\Omega \frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\,\frac{|\nabla k_\varepsilon|^2}{(1+k_\varepsilon)^{1+\delta}}\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t\!\!\int_\Omega \Big(-\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon+\varepsilon k_\varepsilon}\,\big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^2 + k_\varepsilon\omega_\varepsilon\Big)\Big(1 - \frac{1}{(1+k_\varepsilon)^\delta}\Big)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \varepsilon\int_0^t\!\!\int_\Omega \Big(\delta\,\frac{|\nabla k_\varepsilon|^r}{(1+k_\varepsilon)^{1+\delta}} + k_\varepsilon^{r-1}\big(1 - \frac{1}{(1+k_\varepsilon)^\delta}\big)\Big)\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_\Omega \Phi\big(k_{0,\varepsilon}(x)\big)\,\mathrm{d}x + \varepsilon\int_0^t\!\!\int_\Omega \big(\kappa(s)\big)^{r-1}\Big(1 - \frac{1}{(1+k_\varepsilon)^\delta}\Big)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

for all $t \in [0,T]$. By (5.12a), (5.14), and $k_\varepsilon/\big((\varepsilon+\omega_\varepsilon)(1+k_\varepsilon)\big) \geq 1/(1+\overline{\omega}(T)) > 0$ we find

$$\begin{aligned}
&\|k_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \delta\int_Q \frac{|\nabla k_\varepsilon|^2}{(1+k_\varepsilon)^\delta}\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\delta\int_Q \frac{|\nabla k_\varepsilon|^r}{(1+k_\varepsilon)^{1+\delta}}\,\mathrm{d}x\,\mathrm{d}s + \varepsilon\int_Q k_\varepsilon^{r-1}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq c\Big(\frac{1}{1-\delta} + \|\boldsymbol{u}_{0,\varepsilon}\|^2_{\boldsymbol{L}^2} + \|k_{0,\varepsilon}\|_{L^1} + \|\boldsymbol{f}\|^2_{\boldsymbol{L}^2} + k_*^{r-1}\Big),
\end{aligned} \tag{5.15}$$

where the constant c is independent of δ and ε. Thus, we have estimated all the term on the left-hand side of (5.13) except for the second and third.

Step 2: To estimate ∇k_ε we choose $p \in]1,2[$ and $\delta = (2-p)/p \in]0,1[$. With Hölder's inequality we find

$$\begin{aligned}
\int_Q |\nabla k_\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}t &= \int_Q \frac{|\nabla k_\varepsilon|^p}{(1+k_\varepsilon)^{p\delta/2}}\,(1+k_\varepsilon)^{p\delta/2}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \Big(\int_Q \frac{|\nabla k_\varepsilon|^2}{(1+k_\varepsilon)^\delta}\,\mathrm{d}x\,\mathrm{d}t\Big)^{p/2}\Big(\int_Q (1+k_\varepsilon)^{\delta p/(2-p)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{(2-p)/2} \\
&\leq \frac{1}{\delta^{p/2}}\Big(\delta\int_Q \frac{|\nabla k_\varepsilon|^2}{(1+k_\varepsilon)^\delta}\,\mathrm{d}x\,\mathrm{d}t\Big)^{p/2}\,T\Big(|\Omega| + \|k_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))}\Big).
\end{aligned}$$

Using (5.15) this provides the estimate for the third term on the left-hand side of (5.13).

Step 3: To show higher integrability of k_ε we simply use the Gagliardo–Nirenberg interpolation from Lemma 4.2 for $z \in W^{1,p}(\Omega)$ with $\Omega \subset \mathbb{R}^3$ where $p \in [1,2[$ as in Step 2:

$$\|z\|_{L^{4p/3}(\Omega)} \leq C_{\mathrm{GN}}\|z\|^{1/4}_{L^1(\Omega)}\big(\|z\|_{L^1(\Omega)} + \|z\|_{L^p(\Omega)}\big)^{3/4},$$

Applying this to $z = k_\varepsilon(t)$, taking the power $4p/3$, and integrating $t \in [0,T]$ we obtain

$$\int_Q |k_\varepsilon|^{4p/3}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \|k_\varepsilon(t)\|^{4p/3}_{L^{4p/3}(\Omega)}\,\mathrm{d}t \le C_{\mathrm{GN}}^{4p/3}\int_0^T K_\varepsilon^{p/3}\big(K_\varepsilon + \|\nabla k_\varepsilon(t)\|_{L^p(\Omega)}\big)^p\,\mathrm{d}t,$$

where $K_\varepsilon := \|k_\varepsilon(\cdot)\|_{L^\infty(L^1(\Omega))} \le C < \infty$ by Step 1. Hence, together with Step 2 the second term on the left-hand side of (5.13)is uniformly bounded by the right-hand side of (5.13).

In summary, the desired a priori estimate (5.13) is established. □

5.4 Estimates for $\{\boldsymbol{u}_\varepsilon', \omega_\varepsilon', k_\varepsilon'\}$

We now provide a priori estimates on the time derivative. To obtain estimates that are independent of $\varepsilon \in\,]0,1]$ we recall $r \ge 3$ and will use $\sigma > r$ and estimate in the dual space of $W^{1,\sigma}(\Omega)$. While for $\boldsymbol{u}_\varepsilon'$ and ω_ε' we obtain estimates in spaces $L^q\big(0,T;((W^{1,\sigma}(\Omega))^*\big)$, the time derivate k_ε' can only be estimated only for $q=1$ because of the source term $\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon+\varepsilon k_\varepsilon}|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|^2$, for which the only ε-independent a priori estimate is in $L^1(Q) = L^1(0,T;L^1(\Omega))$. This problem will result in the occurrence of the defect measure μ. The estimates for $\boldsymbol{u}_\varepsilon'$ and ω_ε' will work for arbitrary $r \ge 3$, however, for the estimate of k_ε' we need to restrict r to the small interval $[3, 11/3[$. Here the upper bound $r < 11/3$ seems to be critical for $N = 3$, while $2 < r < 3$ might still be considered.

Proposition 5.5. *Let $\mathfrak{D}$ be fixed.*
(A) For all $r \ge 3$ (implying $r' = r/(r{-}1) \le 3/2$) and $\sigma > r$ there exists a constant C_1 such that for all $0 < \varepsilon \le 1$ the solutions $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ of Proposition 5.1 satisfy the estimates

$$\|\boldsymbol{u}_\varepsilon'\|_{L^{r'}(0,T;(\boldsymbol{W}^{1,\sigma}_{\mathrm{per,div}}(\Omega))^*)} + \|\omega_\varepsilon'\|_{L^{r'}(0,T;(W^{1,\sigma}_{\mathrm{per}}(\Omega))^*)} \le C_1. \tag{5.16}$$

(B) For all $r \in [3, 11/3[$ and $\sigma > 8r/(11{-}3r)$ there exists a constant C_2 such that for all $0 < \varepsilon \le 1$ the solutions $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ of Proposition 5.1 satisfy

$$\|k_\varepsilon'\|_{L^1(0,T;(W^{1,\sigma}_{\mathrm{per}})^*)} \le C_2. \tag{5.17}$$

Proof. Step 1. Estimate for $\boldsymbol{u}_\varepsilon'$: For $\boldsymbol{w} \in \boldsymbol{W}^{1,\sigma}_{\mathrm{per,div}}(\Omega)$, we write (5.7a) in the form

$$\begin{aligned}
&\big\langle \boldsymbol{u}_\varepsilon'(t), \boldsymbol{w}\big\rangle_{\boldsymbol{W}^{1,\sigma}_{\mathrm{per,div}}} = \big\langle \boldsymbol{u}_\varepsilon'(t), \boldsymbol{w}\big\rangle_{\boldsymbol{W}^{1,r}_{\mathrm{per,div}}} \\
&= \int_\Omega \big(\boldsymbol{u}_\varepsilon(t)\otimes\boldsymbol{u}_\varepsilon(t)\big){:}\nabla\boldsymbol{w}\,\mathrm{d}x - \nu_0 \int_\Omega \frac{k_\varepsilon(t)}{\varepsilon+\omega_\varepsilon(t)}\,\boldsymbol{D}\big(\boldsymbol{u}_\varepsilon(t)\big){:}\boldsymbol{D}(\boldsymbol{w})\,\mathrm{d}x \\
&\quad - \varepsilon\int_\Omega \Big(\big|\boldsymbol{D}\big(\boldsymbol{u}_\varepsilon(t)\big)\big|^{r-2}\boldsymbol{D}\big(\boldsymbol{u}_\varepsilon(t)\big){:}\boldsymbol{D}(\boldsymbol{w}) + \big|\boldsymbol{u}_\varepsilon(t)\big|^{r-2}\boldsymbol{u}_\varepsilon(t)\cdot\boldsymbol{w}\Big)\,\mathrm{d}x + \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{w}\,\mathrm{d}x \\
&= \sum_{m=1}^4 I_{\varepsilon,m}(t) \quad \text{ for a.a. } t \in [0,T].
\end{aligned} \tag{5.18}$$

The aim is to show $|I_{\varepsilon,m}(t)| \le f_{\varepsilon,m}(t)\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,\sigma}(\Omega)}$ with $f_{\varepsilon,m}$ bounded in $L^{\overline{q}_m}(0,T)$ for some $\overline{q}_m \ge r/(r{-}1)$. For this, we proceed as in Remark 4.3, but use now that $\boldsymbol{w} \in \boldsymbol{W}^{1,\sigma}_{\mathrm{per,div}}(\Omega)$ is fixed.

For $I_{\varepsilon,1}$ we use $\nabla\boldsymbol{w} \in \boldsymbol{L}^\sigma(\Omega)$ and need to bound $|\boldsymbol{u}_\varepsilon\otimes\boldsymbol{u}_\varepsilon| \le |\boldsymbol{u}_\varepsilon|^2$ in $L^{\sigma'}(\Omega)$, which means $\boldsymbol{u}_\varepsilon \in \boldsymbol{L}^p(\Omega)$ with $p = 2\sigma/(\sigma{-}1)$. For this we use the bounds (5.12a) for $\boldsymbol{u}_\varepsilon$, which allow us to apply Lemma 4.2(B) with $(s_1,p_1) = (\infty,2)$, $(s_2,p_2) = (2,2)$, $N = 3$, and $\theta = 3/(2\sigma) < 1/2$. This provides the desired $p = 2\sigma/(\sigma{-}1)$ and $\overline{q}_1 = s = 4\sigma/3$.

To estimate $I_{\varepsilon,2}$ we use $\varepsilon+\omega_\varepsilon(t,x)\geq \underline{\omega}(T)>0$ and need to bound

$$|k_\varepsilon \boldsymbol{D}(\boldsymbol{u}_\varepsilon)| = k_\varepsilon^{1/2}\ |k_\varepsilon^{1/2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon)| \quad \text{ in } L^{\overline{q}_2}(0,T;L^{\sigma'}(\Omega)).$$

By (5.12a) we have a uniform bound for $|k_\varepsilon^{1/2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|$ in $L^2(Q)=L^2(0,T;L^2(\Omega))$. Moreover, (5.13) provides uniform bounds for $\|k_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))}$ and for $\|\nabla k_\varepsilon\|_{\boldsymbol{L}^p(Q)}$ with $p\in[1,2[$. Hence, restricting to $\overline{q}_2\in[1,2]$ we proceed as follows:

$$\|k_\varepsilon \boldsymbol{D}(\boldsymbol{u}_\varepsilon)\|^{\overline{q}_2}_{L^{\overline{q}_2}(0,T;L^{\sigma'}(\Omega))} \leq \int_0^T \big(\|k_\varepsilon^{1/2}\|_{L^{2\sigma/(\sigma-2)}}\|k_\varepsilon^{1/2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\|_{L^2}\big)^{\overline{q}_2}\,\mathrm{d}t$$
$$\leq \int_0^T \|k_\varepsilon\|^{\overline{q}_2/2}_{L^{\sigma/(\sigma-2)}}\|k_\varepsilon^{1/2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\|^{\overline{q}_2}_{L^2}\,\mathrm{d}t \overset{\text{Hölder}}{\leq} \Big(\int_0^T \|k_\varepsilon\|^{\overline{q}_2/(2-\overline{q}_2)}_{L^{\sigma/(\sigma-2)}}\,\mathrm{d}t\Big)^{(2-\overline{q}_2)/2}\Big(\int_Q k_\varepsilon|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|^2\,\mathrm{d}t\Big)^{\overline{q}_2/2}.$$

The second term in the last product is already uniformly bounded. To estimate the first term we apply Lemma 4.2(B) with $(s_1,p_1)=(\infty,1)$, $s_2=p_2\in[1,2[$, $N=3$, and $\theta=6p_2/((4p_2{-}3)\sigma)\in\,]0,1[$, where we use $\sigma>r\geq 3$ such that p_2 can be chosen close to 2. From the interpolation condition (4.13) we obtain the range of possible $\overline{q}_2$ via

$$\frac{2}{\overline{q}_2}-1=\frac{2{-}\overline{q}_2}{\overline{q}_2}=\frac{1}{s}=(1{-}\theta)\frac{1}{s_1}+\theta\frac{1}{s_2}=0+\theta\frac{1}{p_2}=\frac{6}{(4p_2{-}3)\sigma}.$$

Thus, we are able to choose all $\overline{q}_2\in[1,10\sigma/(5\sigma{+}6)[$ by adjusting p_2 suitably. As $\sigma>r\geq 3$ we see that $\overline{q}_2=3/2$ is always admissible.

Using $\sigma\geq r\geq 3$ and Hölder's inequality, we obtain

$$\big|I_{\varepsilon,3}(t)\big|\leq f_{\varepsilon,3}(t)\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,\sigma}} \quad \text{ with } f_{\varepsilon,3}(t)=C\varepsilon\big\|\boldsymbol{u}_\varepsilon(t)\big\|^{r-1}_{\boldsymbol{W}^{1,r}}.$$

By the uniform bound (5.12a) we obtain $\|f_{\varepsilon,3}\|_{L^{r'}(0,T)}\leq C_*\varepsilon^{1/(r-1)}$ with a constant C_* independent of ε. Thus, we can choose $\overline{q}_3=r'=r/(r{-}1)\leq 3/2$.

With $|I_{\varepsilon,4}(t)|\leq\|\boldsymbol{f}(t)\|_{L^2}\|\boldsymbol{w}(t)\|_{\boldsymbol{L}^2}\leq C\|\boldsymbol{f}(t)\|_{L^2}\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,\sigma}}$ and $\boldsymbol{f}\in\boldsymbol{L}^2(Q)=L^2(0,T;\boldsymbol{L}^2(\Omega))$ we obtain $\overline{q}_4=2$, and conclude that in all cases we have $\overline{q}_m\geq r'=r/(r{-}1)$ and the first part of (5.16) is established.

Step 2. Estimate for ω_ε': We proceed as in Step 1 by writing (5.7b) in the form

$$\big\langle \omega_\varepsilon'(t),\psi\big\rangle_{W^{1,\sigma}}=\sum_{m=1}^5 J_{\varepsilon,m}(t) \quad \text{ with } |J_{\varepsilon,m}(t)|\leq g_{\varepsilon,m}(t)\|\psi\|_{W^{1,\sigma}},$$

where $g_{\varepsilon,m}$ has to be bounded in $L^{\widetilde{q}_m}(0,T)$ for suitable $\widetilde{q}_m\geq r'=r/(r{-}1)$. Exploiting Lemma 5.2, namely $0<\underline{\omega}(T)\leq\omega_\varepsilon(x,t)\leq\overline{\omega}(0)=\omega^*$ and (5.12b) and proceeding as in Step 1 we easily find $\widetilde{q}_1=\widetilde{q}_3=\widetilde{q}_5=\infty$, $\widetilde{q}_2=10\sigma/(5\sigma{+}6)\geq 3/2$, and $\widetilde{q}_4=r'\leq 3/2$. Thus, the second part of (5.16), and hence all of (5.16), is established.

Step 3. Estimate for k_ε': We again write

$$\begin{aligned}\big\langle k_\varepsilon'(t),z\big\rangle =& -\int_\Omega z\boldsymbol{u}_\varepsilon(t)\cdot\nabla k_\varepsilon(t)\,\mathrm{d}x-\int_\Omega\frac{k_\varepsilon(t)}{\varepsilon+\omega_\varepsilon(t)}\,\nabla k_\varepsilon(t)\cdot\nabla z\,\mathrm{d}x \\ &+\nu_0\int_\Omega\frac{k_\varepsilon(t)}{\varepsilon+\omega_\varepsilon(t)+\varepsilon k_\varepsilon(t)}\,\big|\boldsymbol{D}\big(\boldsymbol{u}_\varepsilon(t)\big)\big|^2 z\,\mathrm{d}x-\alpha_2\int_\Omega k_\varepsilon(t)\omega_\varepsilon(t)z\,\mathrm{d}x \\ &-\varepsilon\int_\Omega\big(\big|\nabla k_\varepsilon(t)\big|^{r-2}\nabla k_\varepsilon(t)\cdot\nabla z+\big|k_\varepsilon(t)\big|^{r-2}k_\varepsilon(t)z\big)\,\mathrm{d}x+\varepsilon\big(\kappa(t)\big)^{r-1}\int_\Omega z\,\mathrm{d}x \\ =:&\sum_{m=1}^7 K_{\varepsilon,m}(t)\end{aligned} \tag{5.19}$$

and have to show that $K_{\varepsilon,m}(t) \leq h_{\varepsilon,m}(t)\|z\|_{W^{1,\sigma}}$, where all $h_{\varepsilon,m}$ are bounded in $L^1(0,T)$ independently of $\varepsilon \in]0,1[$ and $m = 1,\dots,7$.

Before starting the estimates we note that the condition $r \in [3,11/3[$ and $\sigma > 8r/(11{-}3r)$ implies $\sigma > 12$, which will be useful below.

For $m = 1$ we integrate by parts using $\operatorname{div} \boldsymbol{u}_\varepsilon = 0$ and obtain

$$|K_{\varepsilon,1}(t)| = \Big|\int_\Omega k_\varepsilon \boldsymbol{u}_\varepsilon{\cdot}\nabla z \,\mathrm{d}x\Big| \leq h_{\varepsilon,1}(t)\|z\|_{W^{1,\sigma}} \quad \text{with } h_{\varepsilon,1}(t) = \|k_\varepsilon \boldsymbol{u}_\varepsilon\|_{L^{\sigma'}}.$$

Using (5.12a) for $\boldsymbol{u}_\varepsilon$ and applying Lemma 4.2 with $(s_1,p_1) = (\infty,2)$, $(s_2,p_2) = (2,2)$, $N = 3$, and $\theta = 3/5$ we find $(s,p) = (10/3,10/3)$ which means that $\boldsymbol{u}_\varepsilon$ is uniformly bounded in $L^{10/3}(Q)$. Using the uniform bound (5.13) for k_ε in $L^q(Q)$ for all $q \in [1,8/3[$ we can use $\frac{1}{q} + \frac{3}{10} \leq 1\sigma' < 1$ as $\sigma > 40/13$ and obtain

$$\int_0^T h_{\varepsilon,1}(t)\,\mathrm{d}t \leq \int_0^T C\|k_\varepsilon(t)\|_{L^q(\Omega)}\|\boldsymbol{u}_\varepsilon(t)\|_{L^{10/3}(\Omega)}\,\mathrm{d}t \leq C_T\|k_\varepsilon\|_{L^q(Q)}\|\boldsymbol{u}_\varepsilon\|_{L^{10/3}(Q)} \leq C_{T,1}.$$

For $m = 2$ we again use (5.13) and $\sigma > 8$. Choosing $p \in [1,2[$ with $3/(4p){+}1/p{+}1/\sigma \leq 1$ Hölder's inequality gives

$$\int_0^T |K_{\varepsilon,2}(t)|\,\mathrm{d}t \leq \int_0^T \|k_\varepsilon\|_{L^{4p/3}}\|\nabla k_\varepsilon\|_{L^p}\|\nabla z\|_{L^\sigma}\,\mathrm{d}t \leq C_{T,2}\|k_\varepsilon\|_{L^{4p/3}(Q)}\|\nabla k_\varepsilon\|_{L^p(Q)}\|z\|_{W^{1,\sigma}}.$$

The case $m = 3$ follows easily as $\|z\|_{L^\infty(\Omega)} \leq C\|z\|_{W^{1,\sigma}}$ because $\sigma > N$. Together with the simple energy estimate (5.12a) (uniform boundedness of the dissipation) we obtain

$$\int_0^T |K_{\varepsilon,3}(t)|\,\mathrm{d}t \leq C\int_Q \frac{k_\varepsilon}{\varepsilon{+}\omega_\varepsilon}|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t\|z\|_{L^\infty} \leq C_3\|z\|_{W^{1,\sigma}}.$$

The case $m = 4$ is also trivial, since $|K_{\varepsilon,4}(t)| \leq C\|k_\varepsilon(t)\|\omega^*\|z\|_{L^\infty}$.

The most difficult term is $K_{\varepsilon,5}$ because we do not have an a priori bound on $\varepsilon|\nabla k_\varepsilon|^r$. We adapt the method developed in Step 2 of the proof of Proposition 5.4. Using

$$|K_{\varepsilon,5}(t)| \leq h_{\varepsilon,5}(t)\|z\|_{W^{1,\sigma}} \qquad \text{with } h_{\varepsilon,5}(t) = \varepsilon\big\||\nabla k_\varepsilon(t)|^{r-1}\big\|_{L^{\sigma'}}$$

we proceed as follows:

$$\begin{aligned}\int_0^T h_{\varepsilon,5}\,\mathrm{d}t &= \varepsilon\int_0^T \|\nabla k_\varepsilon(t)\|^{r-1}_{L^{(r-1)\sigma'}}\,\mathrm{d}t \leq \varepsilon T^{1/\sigma}\|\nabla k_\varepsilon\|^{r-1}_{L^{(r-1)\sigma'}(Q)}\\ &\leq \varepsilon T^{1/\sigma}\Big(\int_Q \frac{|\nabla k_\varepsilon|^{(r-1)\sigma'}}{(1{+}k_\varepsilon)^\rho}(1{+}k_\varepsilon)^\rho\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/\sigma'}\end{aligned}$$

for a $\rho > 0$ to be chosen appropriately. Applying Hölder's inequality with $p = r'/\sigma' > 1$ and using $\varepsilon = \varepsilon^{1/r}\varepsilon^{1/(p\sigma')}$ we continue

$$\leq \varepsilon^{1/r}T^{1/\sigma}\Big(\int_Q \frac{\varepsilon|\nabla k_\varepsilon|^r}{(1{+}k_\varepsilon)^{p\rho}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/(p\sigma')}\Big(\int_Q (1{+}k_\varepsilon)^{p'\rho}\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/(p'\sigma')}.$$

According to (5.13) both integral terms are uniformly bounded if we can choose ρ such that $p\rho \in]1,2]$ and $p'\rho < 8/3$. Writing $\kappa = 1/p$ this means $\kappa < \rho < \min\{2\kappa, 8(1{-}\kappa)/3\}$, which has solutions ρ if and only if $\kappa \in]0,8/11[$, i.e. we need $p = r'/\sigma' > 11/8$ which in term can only be possible if $r' > 11/8$ or $r < 11/3$. Then, $p = r'/\sigma' > 11/8$ is equivalent to $\sigma > 8r/(11-3r)$. This explains the restriction for r and σ in (5.17) and provides the L^1 bound $\int_0^T |K_{\varepsilon,5}(t)|\,\mathrm{d}t \le \varepsilon^{1/r} C_{r,\sigma}\|z\|_{W^{1,\sigma}}$.

The estimate of $K_{\varepsilon,6}$ follows easily from (5.13) using $r-1 \in [2, 8/3[$, which implies $\|k_\varepsilon\|_{L^{r-1}(Q)} \le C$ and thus

$$\int_0^T |K_{\varepsilon,6}(t)|\,\mathrm{d}t \le \int_0^T \varepsilon \|k_\varepsilon\|^{r-1}_{L^{r-1}}\,\mathrm{d}t\,\|z\|_{L^\infty} \le \varepsilon C\|z\|_{W^{1,\sigma}}.$$

The case of $K_{\varepsilon,7}$ is trivial.

For later use in the limit passage $\varepsilon \to 0$ we note that

$$\int_0^T \big(|K_{\varepsilon,5}(t)| + |K_{\varepsilon,6}(t)| + |K_{\varepsilon,7}(t)|\big)\,\mathrm{d}t \;\le\; \varepsilon^{1/r} C_{r,\sigma}\|z\|_{W^{1,\sigma}}. \tag{5.20}$$

Hence, the a priori estimate (5.17) for k_ε' is established. □

5.5 Convergent subsequences

After having derived a series of a priori estimates we are now able to choose weakly converging subsequences for $\varepsilon \to 0$. Of course the major step is to identify the limits of the nonlinear terms. For simplicity we now choose one fixed $r_* \in [3, 11/3[$ and a $\sigma_* > 12$, which implies that Part (A) and (B) of Proposition 5.5 can be applied. From (5.8), (5.12), (5.13), (5.16), and (5.17) we obtain a limit triple $\{\boldsymbol{u}, \omega, k\}$ with the properties

$$\left.\begin{array}{l}
\underline{\omega} \le \omega \le \overline{\omega} \text{ a.e. on } Q,\\
\boldsymbol{u} \in L^2(0,T;\boldsymbol{W}^{1,2}(\Omega)) \cap L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \cap W^{1,r_*'}\big(0,T;(\boldsymbol{W}^{1,\sigma_*}_{\mathrm{per,div}}(\Omega))^*\big),\\
\omega \in L^\infty(Q) \cap L^2(0,T;W^{1,2}(\Omega)) \cap W^{1,r_*'}\big(0,T;(W^{1,\sigma_*}_{\mathrm{per}}(\Omega))^*\big),\\
k \in L^\infty(0,T;L^1(\Omega)) \cap L^{4p/3}(Q) \cap L^p(0,T;W^{1,p}_{\mathrm{per}}(\Omega)) \cap \mathrm{BV}(0,T;(W^{1,\sigma_*}_{\mathrm{per}}(\Omega))^*)
\end{array}\right\} \tag{5.21}$$

for all $p \in [1, 2[$, such that along a suitable subsequence (not relabeled) we have

$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u} \text{ in } L^2\big(0,T;\boldsymbol{W}^{1,2}_{\mathrm{per,div}}(\Omega)\big) \text{ and weakly}^* \text{ in } L^\infty\big(0,T;\boldsymbol{L}^2(\Omega)\big), \tag{5.22a}$$

$$\boldsymbol{u}_\varepsilon' \rightharpoonup \boldsymbol{u}' \text{ in } L^{r_*'}\big(0,T;(W^{1,\sigma_*}_{\mathrm{per,div}}(\Omega))^*\big), \tag{5.22b}$$

$$\omega_\varepsilon \rightharpoonup \omega \text{ in } L^2\big(0,T;W^{1,2}_{\mathrm{per}}(\Omega)\big) \text{ and weakly}^* \text{ in } L^\infty(Q), \tag{5.22c}$$

$$\omega_\varepsilon' \rightharpoonup \omega' \text{ in } L^{r_*'}\big(0,T;(W^{1,\sigma_*}_{\mathrm{per}}(\Omega))^*\big), \tag{5.22d}$$

$$k_\varepsilon \rightharpoonup k \text{ in } L^p\big(0,T;W^{1,p}_{\mathrm{per}}(\Omega)\big) \text{ and in } L^{4p/3}(Q) \text{ for all } p \in [1,2[. \tag{5.22e}$$

These weak convergences imply the corresponding properties of the limits $\boldsymbol{u}$ and ω in (5.21). Moreover, $\|k\|_{L^\infty(0,T;L^1(\Omega))} \le C < \infty$ follows from (5.13) and (5.22e) by a routine argument. As in [BaP12, Sec. 1.3.2] the space $BV(0,T;X)$, where X is a Banach space, denotes all functions $g : [0,T] \to X$ such that $\mathrm{Var}_X(g,[a,b]) := \sup \sum_{i=1}^N \|g(t_i) - g(t_{i-1})\|_X < \infty$ where the supremum is taken over all finite partitions $a \le t_0 < t_1 < \cdots < t_N \le b$. Clearly, (5.17) implies $\mathrm{Var}_{(W^{1,\sigma}_{\mathrm{per}})^*}(k_\varepsilon,[0,T]) = \|k_\varepsilon'\|_{L^1(0,T;(W^{1,\sigma}_{\mathrm{per}})^*)} \le C_2$. Since for all partitions we have

$$\sum_{i=1}^N \|k(t_i) - k(t_{i-1})\|_{(W^{1,\sigma}_{\mathrm{per}})^*} \le \liminf_{\varepsilon\to 0} \sum_{i=1}^N \|k_\varepsilon(t_i) - k_\varepsilon(t_{i-1})\|_{(W^{1,\sigma}_{\mathrm{per}})^*} \le C_2,$$

which provides $\|k\|_{\mathrm{BV}(0,T;(W^{1,\sigma_*}_{\mathrm{per}}(\Omega))^*)} \leq C_2\infty$ as stated at the end of (5.21).

We next apply the Aubin-Lions-Simon lemma (see [Sim87, Cor. 4, p. 85], [Lio69, Th. 5.1, p. 58], or [Rou13, Lem. 7.7]) to obtain strong convergence. By taking a further subsequence (not relabeled) Vitali's theorem implies the pointwise convergence almost everywhere.

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad \text{in } \boldsymbol{L}^s(Q) \ \text{ for all } s \in [1, 10/3[\ \text{ and a.e. in } Q, \tag{5.23a}$$
$$\omega_\varepsilon \to \omega \quad \text{in } L^p(Q) \text{ for all } p > 1 \ \text{ and a.e. in } Q, \tag{5.23b}$$
$$k_\varepsilon \to k \quad \text{in } L^q(Q) \text{ for all } q \in [1, 8/3[\ \text{ and a.e. in } Q, \tag{5.23c}$$

To obtain the results in (5.23b) and (5.23c) we first derive strong convergence for $s = p = q = 2$ and then use the boundedness of the sequence for higher s, p, and q to obtain strong convergence for intermediate values by Riesz interpolation (use (4.17) for $\boldsymbol{u}_\varepsilon$).

We are now ready to consider also the limits of the nonlinear terms. We first treat the diffusive terms.

Lemma 5.6. *Along the chosen subsequences for $\varepsilon \to 0$ we have the convergences*

$$\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\boldsymbol{D}(\boldsymbol{u}_\varepsilon) \rightharpoonup \frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u}) \ \textit{and} \ \frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\nabla\omega_\varepsilon \rightharpoonup \frac{k}{\omega}\nabla\omega \ \textit{in } \boldsymbol{L}^s(Q) \ \textit{for all } s \in [1, 16/11[, \tag{5.24a}$$
$$\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\nabla k_\varepsilon \rightharpoonup \frac{k}{\omega}\nabla k \qquad \textit{in } \boldsymbol{L}^\sigma(Q) \ \textit{for all } \sigma \in [1, 8/7[. \tag{5.24b}$$

Proof. We first recall the weak convergences of the gradients $\boldsymbol{D}(\boldsymbol{u}_\varepsilon)$, $\nabla\omega_\varepsilon$, and ∇k_ε in $L^p(Q)$ for all $p \in [1, 2[$, see (5.22). Next we establish the strong convergence

$$\Big(\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\Big)^{1/2} \to \Big(\frac{k}{\omega}\Big)^{1/2} \quad \text{in } L^q(Q) \text{ for all } q \in [1, 16/3[. \tag{5.25}$$

To see this we use the explicit estimate

$$\begin{aligned}\Big\|\Big(\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\Big)^{1/2} - \Big(\frac{k}{\omega}\Big)^{1/2}\Big\|_{L^q(Q)} &\leq \Big\|\Big(\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\Big)^{1/2} - \Big(\frac{k}{\varepsilon+\omega_\varepsilon}\Big)^{1/2}\Big\|_{L^q(Q)} + \Big\|\Big(\frac{k}{\varepsilon+\omega_\varepsilon}\Big)^{1/2} - \Big(\frac{k}{\omega}\Big)^{1/2}\Big\|_{L^q(Q)} \\ &\leq \frac{\|k_\varepsilon - k\|^{1/2}_{L^{q/2}(Q)}}{(1+\underline{\omega}(T))^{1/2}} + \frac{\big\|(\varepsilon+\omega_\varepsilon-\omega)\,k^{1/2}\big\|_{L^q(Q)}}{2(1+\underline{\omega}(T))^{3/2}}.\end{aligned}$$

Clearly, the first term on the right-hand side tends to 0 using (5.23c) and $q/2 < 8/3$. For the second term we can still choose $\widetilde{q} \in\,]q, 16/3[$ and $\widetilde{p} \gg 1$ such that $1/q = 1/\widetilde{q} + 1/\widetilde{p}$. Then, Hölder's inequality, $k^{1/2} \in L^{\widetilde{q}}(Q)$, and (5.23b) for $p = \widetilde{p}$ yield the convergence to 0. Hence, the convergence (5.25) is established.

Now using the weak convergences $\boldsymbol{D}(\boldsymbol{u}_\varepsilon) \rightharpoonup \boldsymbol{D}(\boldsymbol{u})$ and $\nabla\omega_\varepsilon \rightharpoonup \nabla\omega$, and $\nabla k_\varepsilon \rightharpoonup \nabla k$ in $L^p(Q)$ for $p \in [1, 2[$ and (5.25) we obtain the weak convergences

$$\big(\tfrac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\big)^{1/2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon) \rightharpoonup \big(\tfrac{k}{\omega}\big)^{1/2}\boldsymbol{D}(\boldsymbol{u}), \quad \big(\tfrac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\big)^{1/2}\nabla\omega_\varepsilon \rightharpoonup \big(\tfrac{k}{\omega}\big)^{1/2}\nabla\omega, \quad \big(\tfrac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\big)^{1/2}\nabla k_\varepsilon \rightharpoonup \big(\tfrac{k}{\omega}\big)^{1/2}\nabla k$$

in $L^q(Q)$ for all $q \in [1, 16/11[$.

However, by the standard a priori estimates (5.12) we see that the first two sequences are bounded in $L^2(Q)$ and hence converge weakly in $L^2(Q)$ as well. The convergence of the third term cannot be improved, because we don't have appropriate a priori bounds.

Multiplying once again by $\big(k_\varepsilon/(\varepsilon+\omega_\varepsilon)\big)^{1/2}$, which converges strongly according to (5.25), we obtain the results in (5.24). $\square$

5.6 Limit passage $\varepsilon \to 0$ and appearance of the defect measure

In this subsection we finalize the proof of Theorem 4.1.

Using the convergences derived above it is now straight forward to perform the limit passage $\varepsilon \to 0$ in the equation for $\boldsymbol{u}_\varepsilon$ and ω_ε. In the energy equation for k_ε we have to be a little more careful to show the occurrence of the defect measure μ.

In the Steps 1 to 3 the limit $\varepsilon \to 0$ will be done with test functions with high integrability $\overline{s}$ in $t \in [0,T]$ taking values in the Sobolev $W^{1,\overline{r}}(\Omega)$ with large $\overline{r}$. This choice will be independent of the chosen r_* in the regularization terms. After the artificial r_* has disappeared in the limit, in Step 4 we discuss which minimal $\overline{s}$ and $\overline{r}$ can be chosen in the weak form.

Step 1. Limit in the momentum balance for $\boldsymbol{u}_\varepsilon$; from (5.5a) to (4.6):

We consider a fixed test function $\boldsymbol{v} \in L^{\overline{s}}\big(0,T;\boldsymbol{W}^{1,\overline{r}}_{\mathrm{per,div}}(\Omega))^*\big)$ with $\overline{s} = 4$ and $\overline{r} \geq s_* > 12$ and discuss the convergence of the five terms on the left-hand side of (5.5a) individually.

The first term is linear in $\boldsymbol{u}_\varepsilon'$ and converges because of (5.22b). The second term can be rewritten as $\int_\Omega \big(\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon\big) : \nabla \boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}t$ and converges by (5.23a).

For the third term we use the nonlinear convergences from Lemma 5.6, cf. the first in (5.24b). Finally, the fourth and fifth terms converge to 0 by the estimate $\int_0^T |I_{\varepsilon,3}(t)|\,\mathrm{d}t \leq C_* \varepsilon^{1/(r_*-1)} \|\boldsymbol{D}(\boldsymbol{v})\|_{L^{r_*}(\boldsymbol{L}^{\sigma_*})} \leq C\varepsilon^{1/(r_*-1)}\|\boldsymbol{v}\|_{L^{\overline{s}}(\boldsymbol{W}^{1,\overline{r}})}$, see Step 1 of the proof of Proposition 5.5.

Thus, (4.6) is established for test functions $v \in L^{\overline{s}}\big(0,T;\boldsymbol{W}^{1,\overline{r}}_{\mathrm{per,div}}(\Omega))^*\big)$.

Step 2. Limit for ω_ε, from (5.5b) to (4.7):

This case works similar as Step 1.

Step 3. Limit in the energy equation for k_ε, from (5.5c) to (3.6):

For this limit passage we choose a test function $z \in C^1_{\mathrm{per},T}(\overline{Q})$, because we want to take the limit of the dissipation which is bounded only in $L^1(Q)$.

The first term of the left-hand side in(5.5c) is integrated by parts in time to obtain

$$\int_0^T \big\langle k_\varepsilon'(t), z(t)\big\rangle_{W^{1,r}_{\mathrm{per}}} \mathrm{d}t = \int_\Omega k_{0,\varepsilon} z(\cdot,0)\,\mathrm{d}x - \int_Q k_\varepsilon z' \,\mathrm{d}x\,\mathrm{d}t \;\to\; \int_\Omega \int_\Omega k_0 z(\cdot,0)\,\mathrm{d}x - \int_Q k z' \,\mathrm{d}x\,\mathrm{d}t$$

by (5.3c) and and (5.22e). For the second term we use (5.23) and conclude

$$\int_Q z\boldsymbol{u}_\varepsilon \cdot \nabla k_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = -\int_Q k_\varepsilon \nabla \boldsymbol{u}_\varepsilon \cdot \nabla z \,\mathrm{d}\,\mathrm{d}t \;\to\; -\int_Q k\boldsymbol{u}\cdot\nabla z \,\mathrm{d}x\,\mathrm{d}t.$$

For the third term Lemma 5.6 can be exploited (cf. (5.24b)) to find

$$\int_Q \frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon} \nabla k_\varepsilon \cdot \nabla z \,\mathrm{d}x\,\mathrm{d}t \;\to\; \int_Q \frac{k}{\omega} \nabla k \cdot \nabla z \,\mathrm{d}x\,\mathrm{d}t.$$

We return to the fourth term at the end and continue with the fifth term. Using (5.23) and $\omega_\varepsilon^+ = \omega_\varepsilon \geq \underline{\omega}(\cdot) > 0$ we easily find $\int_Q k_\varepsilon \omega_\varepsilon^+ z \,\mathrm{d}x\,\mathrm{d}t \to \int_Q k\omega z \,\mathrm{d}x\,\mathrm{d}t$.

The sixth and seventh term on the left-hand side and the single term on the right-hand side converge to 0, which was establish in Step 3 of the proof of Proposition 5.5, see (5.20).

For the fourth term, it remains to prove the *appearance of the defect measure* $\mu \in \mathcal{M}(\overline{Q})$ such that

$$\int_Q \frac{\nu_0 k_\varepsilon}{\varepsilon{+}\omega_\varepsilon{+}\varepsilon k_\varepsilon} \big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^2 \phi \,\mathrm{d}x\,\mathrm{d}t \;\longrightarrow\; \int_Q \frac{\nu_0 k}{\omega} \big|\boldsymbol{D}(\boldsymbol{u})\big|^2 \phi \,\mathrm{d}x\,\mathrm{d}t + \int_{\overline{Q}} \phi \,\mathrm{d}\mu \;\text{ for all } \phi \in C(\overline{Q}). \tag{5.26}$$

Indeed, by the positivity of the integrand and the a priori estimate (5.12a) we can apply Riesz' Representation Theorem for linear continuous functionals on $C(\overline{Q})$. Hence, there exist $\widehat{\mu} \in \mathcal{M}(\overline{Q}) = \big(C(\overline{Q})\big)^*$ such that

$$\int_Q \frac{\nu_0 k_\varepsilon}{\varepsilon{+}\omega_\varepsilon{+}\varepsilon k_\varepsilon} \big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^2 \phi \,\mathrm{d}x\,\mathrm{d}t \;\to\; \int_{\overline{Q}} \phi \,\mathrm{d}\widehat{\mu} \;\text{ for all } \phi \in C(\overline{Q}).$$

As in Lemma 5.6 we can show that $\big(\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon+\varepsilon k_\varepsilon}\big)^{1/2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon)$ converges weakly to $(k/\omega)^{1/2}\boldsymbol{D}(\boldsymbol{u})$ in $\boldsymbol{L}^2(Q)$. Of course, this weak convergence remains true if we multiply by a continuous function $\psi \in C(\overline{Q})$. Thus, the lower semi-continuity of the L^2 norm yields

$$\int_Q \psi^2 \,\mathrm{d}\widehat{\mu} = \lim_{\varepsilon\to 0} \int_Q \frac{\nu_0 k_\varepsilon}{\varepsilon + \omega_\varepsilon + \varepsilon k_\varepsilon} \big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^2 \psi^2 \,\mathrm{d}x\,\mathrm{d}t \geq \int_Q \frac{\nu_0 k}{\omega} \big|\boldsymbol{D}(\boldsymbol{u})\big|^2 \psi^2 \,\mathrm{d}x\,\mathrm{d}t$$

for all $\psi \in C(\overline{Q})$. Thus, the linear functional $\phi \mapsto \int_Q \phi \,\mathrm{d}\widehat{\mu} - \int_Q \frac{\nu_0 k}{\omega} \big|\boldsymbol{D}(\boldsymbol{u})\big|^2 \phi \,\mathrm{d}x\,\mathrm{d}t$ is non-negative and defines the desired defect measure $\mu \in \mathcal{M}(\overline{Q})$, and

$$\int_{\overline{Q}} \phi \,\mathrm{d}\widehat{\mu} = \int_Q \frac{\nu_0 k}{\omega} \big|\boldsymbol{D}(\boldsymbol{u})\big|^2 \phi \,\mathrm{d}x\,\mathrm{d}t + \int_{\overline{Q}} \phi \,\mathrm{d}\mu \quad \text{for all } \phi \in C(\overline{Q}),$$

which gives the desired convergence (5.26).

Step 4. More test functions:

After having passed to the limit $\varepsilon \to 0$ the regularization terms involving the exponent r have disappeared. From the a priori estimates (5.21) for $\{\boldsymbol{u}, \omega, k\}$ we know that $\boldsymbol{u}{\otimes}\boldsymbol{u} \in L^{5/3}(Q)$ and $\frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u}) \in L^q(Q)$ for all $q \in [1, 16/11[$. Thus, by density we can extend the set of test function $\boldsymbol{v}$ in (4.5) can be chosen in $L^{\overline{s}}\big(0,T; W^{1,\overline{r}}_{\mathrm{per,div}}(\Omega)\big)$ for any $\overline{s} > 16/5$ and $\overline{r} > 16/5$. This proves (4.6) and (4.7) for the full set of test functions.

Moreover, we find $\boldsymbol{u}' \in L^q\big((W^{1,q'}_{\mathrm{per,div}}(\Omega))^*\big)$ for all $q \in [1, 16/11[$, which proves (4.5).

Step 5. Further statements:

To derive (4.4) we define the functional $\mathcal{J} : (k, \boldsymbol{u}, \omega) \mapsto \int_Q k\,(|\boldsymbol{D}(\boldsymbol{u})|^2{+}|\nabla\omega|^2)\,\mathrm{d}x\,\mathrm{d}t$ and use the a priori estimate $\mathcal{J}(k_\varepsilon, \boldsymbol{u}_\varepsilon, \omega_\varepsilon) \leq C$, which follows from (5.12) since $\omega_\varepsilon \geq \underline{\omega}(T) > 0$. The functional is convex in $\boldsymbol{u}$ and ω, hence it is lower semicontinuous with respect to strong convergence in k (see (5.23c)) and weak convergence for $(\boldsymbol{u}, \omega)$ (see (5.22a) and (5.22c)), so that

$$\mathcal{J}(k, \boldsymbol{u}, \omega) \leq \liminf_{\varepsilon\to 0} \mathcal{J}(k_\varepsilon, \boldsymbol{u}_\varepsilon, \omega_\varepsilon) \leq C,$$

which is the desired estimate (4.4).

The limit passage $\varepsilon \to 0$ in the pointwise a priori estimates (5.8) leads immediately to the pointwise estimates (4.2) for ω and k.

By (5.22b) and (5.22d) the functions $\boldsymbol{u}_\varepsilon(\cdot)$ and ω_ε are uniformly bounded with respect to $\varepsilon \in\,]0,1]$ in $W^{1,r_*}\big(0,T; (W^{1,\sigma_*}(\Omega))^*\big) \subset C^{1/r_*}\big([0,T]; (W^{1,\sigma_*}(\Omega))^*\big)$. Thus, we have uniform convergence and obtain $(\boldsymbol{u}, \omega) \in C^{1/r_*}\big([0,T]; (\boldsymbol{W}^{1,\sigma_*}(\Omega))^* {\times} (W^{1,\sigma_*}(\Omega))^*\big)$. Together with the essential boundedness of $(\boldsymbol{u}, \omega)$ in $L^2(\Omega){\times}L^2(\Omega)$ this implies

$$(\boldsymbol{u}, \omega) \in C_w([0,T]; \boldsymbol{L}^2(\Omega){\times}L^2(\Omega)).$$

Hence (4.3) is established. Moreover, with (5.3c) and the uniform convergence we deduce the initial conditions (4.8), i.e. $\boldsymbol{u}(\cdot,0)=\boldsymbol{u}_0$ and $\omega(\cdot,0)=\omega_0$. To obtain inequality (4.9), we insert $\boldsymbol{w}=\boldsymbol{u}_\varepsilon(t)$ into (5.7a), integrate over the interval $[0,t]$ and let tend $\varepsilon\to 0$.

Finally, we insert $z=1$ into (5.7c), integrate over $[0,t]$ and add this identity to the one just obtained for $\boldsymbol{u}_\varepsilon$. Using $\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}-\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon+\varepsilon k_\varepsilon}\geq 0$ we can drop the dissipation term involving $|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|^2$, and the limit passage $\varepsilon\to 0$ yields (5.8).

With this, the proof of our main existence result in Theorem 4.1 is complete.

A Appendix. Existence of approximate solutions

We now provide the proof of Proposition 5.1, which will be obtained as an application of a general existence result of evolutionary equations of pseudo-monotone type.

We consider a separable reflexive Banach space $\boldsymbol{V}$ that is continuously and densely embedded in a Hilbert space $\boldsymbol{H}$ such that $\boldsymbol{V}\subset\boldsymbol{H}\approx\boldsymbol{H}^*\subset\boldsymbol{V}^*$. For $U\in\boldsymbol{V}$ and $\Xi\in\boldsymbol{V}^*$ we denote the dual pairing by $\langle\Xi,U\rangle$. Our operator $A:\boldsymbol{V}\to\boldsymbol{V}^*$ is assumed to satisfy the following conditions depending on $p>1$:

$$\text{p-boundedness:}\qquad \exists C_1>0:\ \|A(U)\|_{\boldsymbol{V}^*}\leq C_1\big(1+\|U\|_{\boldsymbol{V}}^{p-1}\big)\ \text{ for all } U\in\boldsymbol{V}; \tag{A.1a}$$

$$\text{p-coercivity:}\qquad \exists C_2>0:\ \langle A(U),U\rangle\geq\frac{1}{C_2}\|U\|_{\boldsymbol{V}}^p-C_2\qquad \text{for all } U\in\boldsymbol{V}; \tag{A.1b}$$

$$\text{pseudo-monotonicity:}\ \left\{\begin{array}{l}\text{if } U_m\rightharpoonup U \text{ in } \boldsymbol{V} \text{ and } \limsup\limits_{m\to\infty}\langle A(U_m),U_m{-}U\rangle\leq 0,\ \text{then}\\ \langle A(U),U{-}V\rangle\leq\liminf\limits_{m\to\infty}\langle A(U_m),U_m{-}V\rangle \text{ for all } V\in\boldsymbol{V}.\end{array}\right\} \tag{A.1c}$$

Under these conditions the following existence result is available.

Theorem A.1 (see e.g. [Rou13, Thm. 8.9])**.** *Let $\boldsymbol{V}$ and $\boldsymbol{H}$ be as above and let the operator $A:\boldsymbol{V}\to\boldsymbol{V}^*$ satisfy the assumptions* (A.1) *with $p>1$. Then, for all $T>0$, all $u_0\in\boldsymbol{H}$, and all $g\in L^{p'}([0,T];\boldsymbol{V}^*)$ there exists a solution $u\in L^p(0,T;\boldsymbol{V})\cap C([0,T];\boldsymbol{H})\cap W^{1,p'}(0,T;\boldsymbol{V}^*)$ of the Cauchy problem*

$$u'(t)+A(u(t))=f(t)\quad \textit{in } \boldsymbol{V}^* \textit{ for a.a. } t\in[0,T]\qquad \textit{and}\qquad u(0)=u_0. \tag{A.2}$$

To apply this result we choose $p=r>3$, $U=(\boldsymbol{u},\omega,k)$,

$$\boldsymbol{H}=\boldsymbol{L}^2_{\mathrm{per,div}}(\Omega)\times L^2(\Omega)\times L^2(\Omega),\quad \text{and}\quad \boldsymbol{V}=\boldsymbol{W}^{1,r}_{\mathrm{per,div}}(\Omega)\times W^{1,r}_{\mathrm{per}}(\Omega)\times W^{1,r}_{\mathrm{per}}(\Omega).$$

The operator A is defined to make the approximate system (5.5) equivalent to the abstract Cauchy problem (A.2). We recall that $\varepsilon>0$ is fixed in Proposition 5.1, so we do not keep track of the dependence on ε. With $V = (\boldsymbol{v}, \varphi, w)$ we define $A : V \to V^*$ by

$$\begin{aligned}
&\langle A(U), V\rangle = I(U, V) \\
&:= \int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v} + \int_\Omega \frac{k^+}{\varepsilon+\omega^+}\,\boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) \\
&\quad + \int_\Omega \varphi\boldsymbol{u}\cdot\nabla\omega + \int_\Omega \frac{k^+}{\varepsilon+\omega^+}\,\nabla\omega\cdot\nabla\varphi + \int_\Omega \omega^+\omega\varphi \\
&\quad + \int_\Omega w\boldsymbol{u}\cdot\nabla k + \int_\Omega \frac{k^+}{\varepsilon+\omega^+}\,\nabla k\cdot\nabla w - \int_\Omega \frac{k^+}{\varepsilon+\omega^+ +\varepsilon k^+}\,\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 w + \int_\Omega k^+\omega^+ w \\
&\quad + \varepsilon\int_\Omega \Big(\big|\boldsymbol{D}(\boldsymbol{u})\big|^{r-2}\boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) + |\boldsymbol{u}|^{r-2}\boldsymbol{u}\cdot\boldsymbol{v} \\
&\qquad\qquad + |\nabla\omega|^{r-2}\nabla\omega\cdot\nabla\varphi + |\omega|^{r-2}\omega\varphi + |\nabla k|^{r-2}\nabla k\cdot\nabla w + |k|^{r-2}kw\Big).
\end{aligned} \tag{A.3}$$

For the rest of this appendix we continue to omit the measure symbol " $\mathrm{d}x$" for integration over Ω. Moreover we have set $\alpha_2 = \nu_0 = 1$ for notational simplicity, because these numerical constant have no influence on the analysis.

Proof of Proposition 5.1. It remains to establish the conditions (A.1) on the operator A.

Step 1. r-boundedness (A.1a): Using $r > 3$ and Hölder's inequality, it is easily seen that all integrals in the definition of $I(U, V)$ are well defined. In particular, we find a constant $c_1 > 0$ such that

$$\big|I(U,V)\big| \le c_1\big(\|U\|_{\boldsymbol{V}}^2 + \|U\|_{\boldsymbol{V}}^{r-1}\big)\|V\|_{\boldsymbol{V}} \quad \text{for all } U, V \in \boldsymbol{V}. \tag{A.4}$$

But this implies (A.1a) because of $r \ge 3$.

Step 2. r-coercivity (A.1b): For estimating $\langle A(U), U\rangle = I(U, U)$ from below we see that all convective terms disappear because of $\operatorname{div}\boldsymbol{u} = 0$. After dropping the three non-negative terms arising from the dissipation terms involving $k^+/(\varepsilon+\omega^+)$ we find, for all $U \in \boldsymbol{V}$,

$$\langle A(U), U\rangle = I(U,U) \ge \varepsilon\big\|(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{u}, \nabla\omega, \omega, \nabla k, k)\big\|_{L^r(\Omega)}^r - \int_\Omega \frac{k^+}{\varepsilon+\omega^+ +\varepsilon k^+}\,\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 k. \tag{A.5}$$

By Hölder's and Young's inequality and $r \gneqq 3$ we find $c_2 > 0$ such that

$$\int_\Omega \frac{k^+}{\varepsilon+\omega^+ +\varepsilon k^+}\,\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 k \le \frac{1}{\varepsilon}\int_\Omega \big|\boldsymbol{D}(\boldsymbol{u})\big|^2 k \le \frac{\varepsilon}{2}\int_\Omega \big|\boldsymbol{D}(\boldsymbol{u})\big|^r + \frac{\varepsilon}{2}\int_\Omega |k|^r + c_2,$$

where the constant c_2 depends on $\varepsilon > 0$, $r > 3$, and $\operatorname{vol}(\Omega)$. Inserting this into (A.5) and using Korn's inequality in $\boldsymbol{W}^{1,r}(\Omega)$ we have established (A.1b) for $p = r$.

Step 3. Strong convergence: In the remaining two steps we consider a sequence $U_m = (\boldsymbol{u}_m, \omega_m, k_m)$ satisfying the assumptions in condition (A.1c), namely

$$\text{(a)}\ \ U_m \rightharpoonup U \ \text{ in } \boldsymbol{V} \qquad\qquad \text{(b)}\ \ \limsup_{m\to\infty}\langle A(U_m), U_m{-}U\rangle \le 0. \tag{A.6}$$

In this step we first show that this implies the strong convergence $U_m \to U$ in $\boldsymbol{V}$, and in Step 4 we deduce the liminf estimate for (A.1c).

Combining parts (a) and (b) of (A.6) we immediately obtain

$$\limsup_{m\to\infty} \big\langle A(U_m) - A(V)\,,\, U_m - V\big\rangle \leq 0 \quad \text{for all } V \in \boldsymbol{V}. \tag{A.7}$$

We decompose these duality products into ten separate integrals, namely

$$\begin{aligned}
&\big\langle A(U_m) - A(U), U_m - U\big\rangle = \sum_{j=1}^{10} K_{j,m} \qquad\qquad (A.8)\\
&:= \int_\Omega \big[\boldsymbol{u}_m\cdot\nabla\boldsymbol{u}_m - \boldsymbol{u}\cdot\nabla\boldsymbol{u}\big]\cdot(\boldsymbol{u}_m - \boldsymbol{u}) + \int_\Omega \Big[\frac{k_m^+}{\varepsilon+\omega_m^+}\,\boldsymbol{D}(\boldsymbol{u}_m) - \frac{k^+}{\varepsilon+\omega^+}\,\boldsymbol{D}(\boldsymbol{u})\Big] : \boldsymbol{D}(\boldsymbol{u}_m - \boldsymbol{u})\\
&\quad + \int_\Omega (\boldsymbol{u}_m\cdot\nabla\omega_m - \boldsymbol{u}\cdot\nabla\omega)\,(\omega_m - \omega) + \int_\Omega \Big[\frac{k_m^+}{\varepsilon+\omega_m^+}\,\nabla\omega_m - \frac{k^+}{\varepsilon+\omega^+}\,\nabla\omega\Big]\cdot\nabla(\omega_m - \omega)\\
&\quad + \int_\Omega (\omega_m^+\omega_m - \omega^+\omega)(\omega_m - \omega) + \int_\Omega (\boldsymbol{u}_m\cdot\nabla k_m - \boldsymbol{u}\cdot\nabla k)(k_m - k)\\
&\quad + \int_\Omega \Big[\frac{k_m^+}{\varepsilon+\omega_m^+}\,\nabla k_m - \frac{k^+}{\varepsilon+\omega^+}\,\nabla k\Big]\cdot\nabla(k_m - k) + \int_\Omega (k_m\omega_m^+ - k\omega^+)(k_m - k)\\
&\quad - \int_\Omega \Big(\frac{k_m^+}{\varepsilon+\omega_m^+ + \varepsilon k_m^+}\,\big|\boldsymbol{D}(\boldsymbol{u}_m)\big|^2 - \frac{k^+}{\varepsilon+\omega^+ + \varepsilon k^+}\,\big|\boldsymbol{D}(\boldsymbol{u})\big|^2\Big)(k_m - k)\\
&\quad + \int_\Omega \varepsilon\Big(\big(\Phi_r(\boldsymbol{D}(\boldsymbol{u}_m)) - \Phi_r(\boldsymbol{D}(\boldsymbol{u}))\big):\boldsymbol{D}(u_m - \boldsymbol{u}) + \big(\Phi_r(\boldsymbol{u}_m) - \Phi_r(\boldsymbol{u})\big)\cdot(\boldsymbol{u}_m - \boldsymbol{u})\\
&\qquad + \big(\Phi_r(\nabla\omega_m) - \Phi_r(\nabla\omega)\big)\cdot\nabla(\omega_m - \omega) + \big(\Phi_r(\omega_m) - \Phi_r(\omega)\big)(\omega_m - \omega)\\
&\qquad + \big(\Phi_r(\nabla k_m) - \Phi_r(\nabla k)\big)\cdot\nabla(k_m - k) + \big(\Phi_r(k_m) - \Phi_r(k)\big)\,(k_m - k)\Big),
\end{aligned}$$

where $\Phi_r(\boldsymbol{\xi}) := |\boldsymbol{\xi}|^{r-2}\boldsymbol{\xi}$. The last term $K_{10,m}$ can be used to control $U_m - U$ in the norm of $\boldsymbol{V}$ by using the estimate

$$\big(\Phi_r(\boldsymbol{\xi}) - \Phi_r(\boldsymbol{\eta})\big)\cdot(\boldsymbol{\xi} - \boldsymbol{\eta}) \geq 2^{2-r}\big|\boldsymbol{\xi} - \boldsymbol{\eta}\big|^r \quad \text{for all } \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N,$$

see [Lin06] for the derivation of the exact constant. In particular, we find

$$K_{10,m} \geq \varepsilon 2^{2-r}\big\|U_m - U\big\|_{\boldsymbol{V}}^r, \tag{A.9}$$

and the strong convergence $U_m \to U$ follows if we show $\limsup_{m\to\infty} K_{10,m} \leq 0$.

By (A.7) we control the limsup of $\sum_1^{10} K_{j,m}$ and hence obtain

$$\begin{aligned}
\limsup_{m\to\infty} K_{10,m} &= \limsup_{m\to\infty}\Big(\sum_{j=1}^{10} K_{j,m} - \sum_{l=1}^{9} K_{l,m}\Big)\\
&\leq \limsup_{m\to\infty}\sum_{j=1}^{10} K_{j,m} - \liminf_{m\to\infty}\sum_{l=1}^{9} K_{l,m} \overset{\text{(A.7)}}{\leq} 0 \; - \; \sum_{l=1}^{9}\liminf_{m\to\infty} K_{l,m}.
\end{aligned}$$

Thus, it suffices to establish $\liminf_{m\to\infty} K_{l,m} \geq 0$ for all $l \in \{1, ..., 9\}$. To do so, we only use $U_m \rightharpoonup U$ (i.e. (A.6a)), which by $r > 3$ and the compact embedding $W^{1,r}(\Omega) \Subset C^0(\overline{\Omega})$ implies

$$\boldsymbol{u}_m \to \boldsymbol{u}, \quad \omega_m \to \omega, \quad k_m \to k \quad \text{uniformly in } \Omega. \tag{A.10}$$

For treating $K_{1,m}$ we use integration by parts and $\operatorname{div}\boldsymbol{u}_m = \operatorname{div}\boldsymbol{u} = 0$ to find

$$K_{1,m} = \int_\Omega \big(\operatorname{div}(\boldsymbol{u}_m\otimes\boldsymbol{u}_m) : \nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}_m\big) \;\to\; \int_\Omega \big(\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}\big) \;=\; 0,$$

because of the uniform convergence $\boldsymbol{u}_m \to \boldsymbol{u}$.

Similarly, the other convective terms $K_{3,m}$ and $K_{6.m}$ convergence to 0, as only on factor converges weakly.

For the second term $K_{2,m}$ we again use the uniform convergence in the decomposition

$$K_{2,m} = \int_\Omega \Big(\frac{k_m^+}{\varepsilon{+}\omega_m^+} - \frac{k^+}{\varepsilon{+}\omega^+}\Big)\,\boldsymbol{D}(\boldsymbol{u}_m) : \boldsymbol{D}(\boldsymbol{u}_m{-}\boldsymbol{u}) + \int_\Omega \frac{k^+}{\varepsilon{+}\omega^+}\,\boldsymbol{D}(\boldsymbol{u}_m{-}\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{u}_m{-}\boldsymbol{u}).$$

The first integral converges to 0 as the two terms involving $\boldsymbol{D}$ are bounded in $\boldsymbol{L}^r(\Omega) \subset \boldsymbol{L}^2(\Omega)$ while the prefactor converges to 0 uniformly. The second integral is non-negative, hence $\liminf_{m\to\infty} K_{2,m} \geq 0$ follows. Analogously, the $\liminf\limits_{m\to\infty}$ of $K_{4,m}$ and $K_{7,m}$ is non-negative.

By uniform convergence of the integrands we easily obtain $K_{5,m} \to 0$ and $K_{8,m} \to 0$.

In $K_{9,m}$ the integrand is a product of a function bounded uniformly in $L^{r/2}(\Omega)$ and $k_m{-}k$, which converges uniformly to 0; hence $K_{9,m} \to 0$ as well.

This finishes the proof of Step 3 guaranteeing $U_m \to U$ in $\boldsymbol{V}$.

Step 4. A is pseudo-monotone: For the sequence U_m satisfying (A.6) we have to show

$$\langle A(U), U{-}V\rangle \leq \liminf_{m\to\infty}\langle A(U_m), U_m{-}V\rangle \text{ for all } V = (\boldsymbol{v},\varphi,w) \in \boldsymbol{V} \tag{A.11}$$

Again we split the duality-product term into ten parts and treat the parts separately, where we are now able to use the strong convergence $U_m \to U$:

$$\begin{aligned}
\big\langle A(U_m), U_m - V\big\rangle &= \sum_{j=1}^{10} G_{j,m} \qquad\qquad (\text{A.12})\\
&=: \int_\Omega \boldsymbol{u}_m\cdot\nabla\boldsymbol{u}_m\cdot(\boldsymbol{u}_m{-}\boldsymbol{v}) + \int_\Omega \frac{k_m^+}{\varepsilon{+}\omega_m^+}\,\boldsymbol{D}(\boldsymbol{u}_m) : \boldsymbol{D}(\boldsymbol{u}_m{-}\boldsymbol{v})\\
&\quad + \int_\Omega \boldsymbol{u}_m\cdot\nabla\omega_m\,(\omega_m{-}\varphi) + \int_\Omega \frac{k_m^+}{\varepsilon{+}\omega_m^+}\,\nabla\omega_m\cdot\nabla(\omega_m{-}\varphi)\\
&\quad + \int_\Omega \omega_m^+\omega_m(\omega_m{-}\varphi) + \int_\Omega \boldsymbol{u}_m\cdot\nabla k_m\,(k_m{-}w) + \int_\Omega \frac{k_m^+}{\varepsilon{+}\omega_m^+}\,\nabla k_m\cdot\nabla(k_m{-}w)\\
&\quad + \int_\Omega k_m\omega_m^+(k_m{-}w) - \int_\Omega \frac{k_m^+}{\varepsilon{+}\omega_m^+{+}\varepsilon k_m^+}\,\big|\boldsymbol{D}(\boldsymbol{u}_m)\big|^2(k_m{-}w)\\
&\quad + \int_\Omega \varepsilon\Big(\Phi_r(\boldsymbol{D}(\boldsymbol{u}_m)) : \boldsymbol{D}(\boldsymbol{u}_m{-}\boldsymbol{v}) + \Phi_r(\boldsymbol{u}_m)\cdot(\boldsymbol{u}_m{-}\boldsymbol{v}) + \Phi_r(\nabla\omega_m)\cdot\nabla(\omega_m{-}\varphi)\\
&\qquad\qquad + \Phi_r(\omega_m)(\omega_m{-}\varphi) + \Phi_r(\nabla k_m)\cdot\nabla(k_m{-}w) + \Phi_r(k_m)(k_m{-}w)\Big).
\end{aligned}$$

Using the uniform convergence of U_m (see (A.10)) and the strong convergence in $L^r(\Omega)$ of the derivatives ∇U_m it is straight forward to see that the integrals $G_{j,m}$ for $j \in \{1,...,9\}$ converge to their respective limits. For $G_{10,m}$ we can use the estimate

$$\big|\Phi_r(\boldsymbol{\xi}) - \Phi_r(\boldsymbol{\eta})\big| \leq 3r\big(|\boldsymbol{\xi}| + |\boldsymbol{\eta}|\big)^{r-1}\big|\boldsymbol{\xi} - \boldsymbol{\eta}\big| \quad \text{for all } \boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^N,$$

see [Bou65, exerc. 10.a, p. 257]. Thus, we conclude that (A.11) holds, even with equality.

Hence, all the assumptions in (A.1) are established, Theorem A.1 is applicable, and the proof of Proposition 5.1 is complete. $\square$

Remark A.2. *An alternative proof for Proposition 5.1 is given in the first draft [MiN18] of the present work. That proof is based on the method of elliptic regularization of abstract evolution equations, cf. [Lio69, Ch. 3, Thm. 1.2].*

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