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Abstract

We study a general class of discrete p-Laplace operators in the random conductance model with long-range jumps and ergodic weights. Using a variational formulation of the problem, we show that under the assumption of bounded first moments and a suitable lower moment condition on the weights, the homogenized limit operator is a fractional p-Laplace operator.

Under strengthened lower moment conditions, we can apply our insights also to the spectral homogenization of the discrete Lapalace operator to the continuous fractional Laplace operator.

1 Introduction

In a recent work [9], the authors together with Slowik studied homogenization of a discrete Laplace operator on $\mathbb{Z}^d_\varepsilon := \varepsilon\mathbb{Z}^d$ with long range jumps of the form

$$\mathcal{L}_\varepsilon u(x) := \varepsilon^{-2} \sum_{y\in\mathbb{Z}^d_\varepsilon\backslash\{x\}} \omega_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \left(u(y) - u(x)\right) . \tag{1}$$

The operator was studied on a bounded domain under proper rescaling with Dirichlet boundary conditions. The coefficients $\omega_{x,y}$ being random and positive with $\omega_{x,y} = \omega_{y,x}$, the operator $\mathcal{L}_\varepsilon$ acts on functions $\mathbb{Z}^d_\varepsilon \to \mathbb{R}$, and the corresponding linear equation in [9] reads

$$\mathcal{L}_\varepsilon u(x) = f(x) , \qquad u(x) = 0 \text{ on } \mathbb{Z}^d_\varepsilon\backslash\boldsymbol{Q} . \tag{2}$$

The assumptions on $\omega_{x,y}$ imposed in [9] are ergodicity and stationarity in x, together with a first moment condition of the form

$$\mathbb{E}\left(\sum_{z\in\mathbb{Z}^d} \omega_{0,z} \left|z\right|^2\right) < \infty , \tag{3}$$

and a lower moment condition of the form

$$\exists \mathfrak{q} > \frac{d}{2} : \quad \mathbb{E}\left(\sum_{i=1}^{d} \omega_{0,e_i}^{-\mathfrak{q}} + \omega_{0,-e_i}^{-\mathfrak{q}}\right) < \infty , \tag{4}$$

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where e_i is the i-th unit vector in $\mathbb{Z}^d$. Under these assumptions, it could be shown that in the limit $\varepsilon \to 0$ the homogenized operator on $L^2(\boldsymbol{Q})$ is a second order elliptic operator $\nabla \cdot (A_{\mathrm{hom}}\nabla \bullet)$ on $\boldsymbol{Q}$ with Dirichlet boundary conditions, where $A_{\mathrm{hom}} \in \mathbb{R}^{d\times d}$ is symmetric and positive definite.

In the present work, we generalize the findings of [9] to the case of fractional Laplacians. More precisely, we study the operator

$$\mathcal{L}_\varepsilon u(x) := \varepsilon^d \sum_{y\in\varepsilon\mathbb{Z}^d\setminus\{x\}} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{u(y)-u(x)}{|x-y|^{d+2s}}\,, \tag{5}$$

where c is stationary ergodic on $\mathbb{Z}^{2d}$ and relates to ω through

$$c_{x,y} := \omega_{x,y}\,|x-y|^{d+2s}\,.$$

Note that the prefactor ε^d balances $|x-y|^d$ and ε^{-2} is changed into ε^{-2s}, respectively into $|x-y|^{2s}$.

One might expect in this case that the limit operator is no longer a second order elliptic operator but rather a nonlocal fractional operator of the form

$$(-\Delta)^{2s} u(x) := \mathrm{PV}\int_{\mathbb{R}^d} \frac{u(y)-u(x)}{|x-y|^{d+2s}}\,\mathrm{d}y\,,$$

where PV stands for *principle value* of the integral. We refer to [15] for a list of equivalent characterizations, among which the most common is the Fourier-symbol $|\xi|^{2s}$.

In this work, we study the above homogenization problem in a more general setting. Our focus lies on energy functionals which take the form

$$\mathscr{E}_{p,s,\varepsilon}(u) = \varepsilon^{2d} \sum\sum_{(x,y)\in\mathbb{Z}^{2d}_\varepsilon} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}} + \varepsilon^d \sum_{x\in\mathbb{Z}^d_\varepsilon} G(u(x)) - \sum_{x\in\mathbb{Z}^d_\varepsilon} u(x) f_\varepsilon(x)\,,$$

where we will study both the convergence behavior on the whole of $\mathbb{R}^d$ and on the restriction $u(x) = 0$ for $x \notin \boldsymbol{Q}$, where $\boldsymbol{Q} \subset \mathbb{R}^d$ is a bounded domain. The corresponding limit functional (in the sense of Γ-convergence) will turn out to be

$$\mathscr{E}_{p,s}(u) = \mathbb{E}(c) \iint_{\mathbb{R}^{2d}} \frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}} + \varepsilon^d \int_{\mathbb{R}^d} G(u(x))\mathrm{d}x - \int_{\mathbb{R}^d} u(x)f(x)\,.$$

Thus, in some sense, we will partially generalize previous work by Neukamm, Schäffner and Schlömerkemper[16] on the homogenization of discrete non-convex functionals with finite range. In case $V(\xi) = |\xi|^p$ this functional generates the fractional p-Laplace equation (see [11] and reference therein). In what follows, we will shortly recall the relation between the homogenization problem for the linear equation and the homogenization of convex functionals.

In order to understand our way to approach this problem, note that the weak formulation of (2) with $\mathcal{L}_\varepsilon$ given by (5) reads

$$\sum_{x\in\mathbb{Z}^d_\varepsilon} \varepsilon^d \sum_{y\in\mathbb{Z}^d_\varepsilon} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{u(y)-u(x)}{|x-y|^{d+2s}} \left(v(y)-v(x)\right) = \sum_{x\in\mathbb{Z}^d_\varepsilon} f(x)v(x) \tag{6}$$

and in a variational formulation, u is the minimizer of the energy potential

$$\mathscr{E}_{2,s,\varepsilon}(u)=\varepsilon^{2d}\frac{1}{2}\sum_{x\in\mathbb{Z}^d_\varepsilon}\sum_{y\in\mathbb{Z}^d_\varepsilon}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{\left(u(y)-u(x)\right)^2}{|x-y|^{d+2s}}-\varepsilon^d\sum_{x\in\mathbb{Z}^d_\varepsilon}u(x)f(x)\,.$$

We will also look at the constraint $u(x)=0$ on $\mathbb{Z}^d_\varepsilon\backslash\boldsymbol{Q}$.

In the continuum, a corresponding functional is known for the solutions of the fractional Laplace equation $(-\Delta)^{2s}u=f$ and on $\boldsymbol{Q}=\mathbb{R}^d$ it reads

$$\mathscr{E}_{2,s}(u)=\iint_{\mathbb{R}^{2d}}\frac{\left(u(x)-u(y)\right)^2}{|x-y|^{d+2s}}-\int_{\mathbb{R}^d}u(x)f(x)\,.$$

The minimizers of $\mathscr{E}_{2,s}$ lie in the space $W^{s,p}(\mathbb{R}^d)$, which we will introduce in Section 3.1. Hence, a Γ-convergence result for $\mathscr{E}_{2,s,\varepsilon}\xrightarrow{\Gamma}\mathscr{E}_{2,s}$ implies homogenization of (6) to

$$(-\Delta)^s u:=\mathrm{PV}\int_{\mathbb{R}^d}\frac{u(y)-u(x)}{|x-y|^{d+2s}}\,\mathrm{d}y=f\,,$$

see Section 2.3.

On bounded domains $\boldsymbol{Q}\subset\mathbb{R}^d$, there is a problem though. This problem is related to the choice of boundary conditions. It is known that the notion of boundary conditions in spaces $W^{s,p}(\boldsymbol{Q})$ does not make sense in case $s\leq\frac{1}{p}$ and hence the minimizers of $\mathscr{E}_{2,s}$ are not unique up to solutions of the homogeneous equation $(-\Delta)^{2s}u=0$.

We take this into account by studying two different types of functionals which are a bit more oriented at the definition of $W^{s,p}(\boldsymbol{Q})$-seminorms in [8]. They read

$$\mathscr{E}_{p,s,\varepsilon,\boldsymbol{Q}}(u)=\varepsilon^{2d}\sum\sum_{(x,y)\in\boldsymbol{Q}^\varepsilon\times\boldsymbol{Q}^\varepsilon}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}}+\varepsilon^d\sum_{x\in\boldsymbol{Q}^\varepsilon}G(u(x))-\sum_{x\in\boldsymbol{Q}^\varepsilon}u(x)f_\varepsilon(x)\,,$$

where $\boldsymbol{Q}^\varepsilon=\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon$.From the analytical point of view, it then makes sense to consider the restriction of $\mathscr{E}_{p,s,\varepsilon,\boldsymbol{Q}}$ to functions with zero boundary conditions and zero mean value conditions. In order to formulate discrete Dirichlet conditions, let

$$\partial\boldsymbol{Q}^\varepsilon=\left\{x\in\mathbb{Z}^d_\varepsilon\,:\,\partial\boldsymbol{Q}\cap\left(x+[-\varepsilon,\varepsilon]^d\right)\neq\emptyset\right\}\,. \tag{7}$$

In every of the above mentioned cases, the corresponding Γ-limit functional will turn out to be

$$\mathscr{E}_{p,s}(u)=\mathbb{E}(c)\iint_{\boldsymbol{Q}\times\boldsymbol{Q}}\frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}}+\varepsilon^d\int_{\boldsymbol{Q}}G(u(x))\mathrm{d}x-\int_{\boldsymbol{Q}}u(x)f(x)\,.$$

However, as we will see below, we even obtain a kind of Mosco convergence in suitable spaces $L^r(\boldsymbol{Q})$. Mosco convergence means that the lim inf-estimate can be obtained for weakly converging sequences while the recovery sequence can be constructud with respect to strong convergence.

Our convergence results rest upon a well-balanced interplay between p, s, c and d, which we formulate in the following condition on the coefficients:

Assumption 1. *We assume that the random variable c is ergodic in $\mathbb{Z}^d\times\mathbb{Z}^d$ with $\mathbb{E}(c)<\infty$ and given $s\in(0,1)$, $p>1$ we assume that there exists $\mathfrak{q}\in\left(\frac{d}{ps},+\infty\right]$ and $r\in(1,p)$ such that $\mathbb{E}(c^{-\mathfrak{q}})<\infty$ and $\mathfrak{q}\geq\frac{r}{p-r}>\frac{d}{ps}$.*

In the hypothetical case $s=1$ and $p=2$, the last assumption reduces to $\mathfrak{q}>\frac{d}{2}$. Hence Assumption 1 is in accordance with the assumptions in [9], which we recalled in (3)–(4).

In view of [9] one could get the idea that our setting corresponds to a relaxation of condition (3) to, say

$$\mathbb{E}\left(\sum_{z\in\mathbb{Z}^d}\omega_{0,z}\,|z|^{2s}\right)<\infty\,,\qquad s\in(0,1)\,. \tag{8}$$

However, our first moment condition is not equivalent with (8) but corresponds to (see Lemma 29)

$$\mathbb{E}\left(\sum_{z\in\mathbb{Z}^d}\omega_{0,z}\,|z|^{2s}\right)=\infty\,.$$

As discussed in Remark 36 below, the proof of our Lemma 35 suggests that (8) leads to a localization of $\mathcal{L}_\varepsilon$ in the limit $\varepsilon\to0$, indicating that the limit is different from a fractional, nonlocal operator. Recent results like in [16] suggest that in this case, the limit problem reduces to a "cell-problem", i.e. a linear problem for the first order corrector.

Remark 2. As we will see in Theorems 4–7, sequences u_ε with bounded $\mathscr{E}_{p,s,\varepsilon}(u_\varepsilon)$ or $\mathscr{E}_{\varepsilon,p,\boldsymbol{Q}}(u_\varepsilon)$ are bounded in $L^r(\mathbb{Z}^d_\varepsilon)$ or $L^r(\boldsymbol{Q}^\varepsilon)$ if $r\in[1,p^\star_\mathfrak{q})$ for

$$p^\star_\mathfrak{q}=\frac{dp\mathfrak{q}}{2d+d\mathfrak{q}-sp\mathfrak{q}}\,.$$

In particular, it turns out that $\mathfrak{q}>\frac{2d}{ps}$ is a sufficient condition to have boundedness of u_ε in $L^p(\boldsymbol{Q}^\varepsilon)$. In order to obtain suitable bounds on u_ε in $L^r(\boldsymbol{Q})$, we ask that V satisfies the following assumption.

The notation $p^\star_\mathfrak{q}$ is related to the fractional critical exponent $p^\star$ in the classical theory of fractional Sobolev spaces, which is introduced in Theorem 17. However, we will see that the random weights c will force us to lower the value of the classical $p^\star$ with decreasing $\mathfrak{q}$.

We finally introduce our assumptions on V. These assumptions are a natural generalization of the fractional p-Laplace potential and are also natural in the context of Sobolev spaces which we will use.

Assumption 3. *We assume that $V:\mathbb{R}^m\to\mathbb{R}$ is convex, lower semi continuous and there exists $\alpha,\beta,c>0$ such that*

$$\alpha|\xi|^p\leq V(\xi)\leq c+\beta|\xi|^p\,,$$
$$\xi\mapsto|\xi|^{-p}V(\xi)\qquad\textit{is continuous in }0\,.$$

The study of discrete elliptic operators has some history starting from works by Künnemann[14] and Kozlov[13]. The interest in this topic has been tremendous both from the physical point of view, e.g. as a model for Brownian motion (see [3, 5]), or from mathematical point of view when studying numerical scemes (see [3, 13] or [10] for a numerical application). The current research particularly focuses on higher order corrector estimates, see e.g. [2] and references therein. However, the above works where on finite range jumps while models for long-range jumps are not related to the above mentioned methods or applications. In particular, the situation studied in this paper leaves the realm of Brownian motion as it accounts for Levi-flights, such as are used to model the movement of bacteria.

The homogenization of the fractional Laplace operator seems to be only recent and rather unexplored. However, there are a few results in the literature: Most of them are focused on the periodic homogenization of the continuous fractional Laplace operator $(-\Delta)^s$, starting from a work by Piatnitskii and Zhizhina [18]. A first result on the stochastic homogenization of the (continuum) fractional Laplace operator with uniformly bounded c is given in [19]. We will not investigate the relation between [19] and the present work, but we expect that the methods developed below could help to generalize [19] to non-uniformly bounded coefficients with bounded moment conditions.

From the point of view of discrete operators, our work is of course related to our previous result [9] but also to a recent result by Chen, Kumagai and Wang [6]. These authors show homogenization of the discrete fractional Laplace on $\mathbb{Z}^d$ in case $d > 4 - 4s$ and under the assumption $\mathbb{E}(\omega^p) + \mathbb{E}(\omega^{-q}) < \infty$ where $p > C_{d,\alpha} > 1$ and $q > 2\frac{d+2}{2}$. Note that the authors of [6] also allow for percolation, which we exclude for simplicity. Hence these results are complementing each other.

The outline of the paper is as follows: In the next section we first provide Mosco convergence of $\mathscr{E}_{p,s,\varepsilon}$ and $\mathscr{E}_{p,s,\varepsilon,\boldsymbol{Q}}$ to $\mathscr{E}_{p,s}$ and $\mathscr{E}_{p,s,\boldsymbol{Q}}$ respectively. Recall that Mosco convergence is slightly stronger than weak or strong Γ-convergence. Based on these results, we formulate our homogenization results for the fractional Laplace operator, including also spectral homogenization in case $\mathfrak{q} > \frac{2d}{ps}$. In Section 3 we a provide basic knowledge on fractional Sobolev spaces and generalize these to the discrete setting. Lemma 30 in Section 3.4 can be considered as the heart of our homogenization results. Finally, in Section 4 we prove the main theorems from Section 2. For readability of Section 3, we shift some standard proofs to the appendix.

2 Main results

The discrete space, on which our functionals $\mathscr{E}_{p,s,\varepsilon}$ and $\mathscr{E}_{p,s,\varepsilon,\boldsymbol{Q}}$ are defined, are denoted

$$\mathcal{H}_\varepsilon := \left\{u : \mathbb{Z}^d_\varepsilon \to \mathbb{R}\right\} , \quad \text{resp.} \quad \mathcal{H}_\varepsilon(\boldsymbol{Q}) := \left\{u \in \mathcal{H}^\varepsilon \,:\, \forall x \notin \boldsymbol{Q} :\, u(x) = 0\right\} .$$

However, the limit functionals are defined on the measurable functions on $\mathbb{R}^d$ and in order to compare discrete solutions with continuous functions we introduce the operators $\mathcal{R}^*_\varepsilon$ through

$$\mathcal{R}^*_\varepsilon u(x) = u(x_i) \quad \text{if } x_i \in \mathbb{Z}^d_\varepsilon \text{ and } x \in x_i + \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)^d .$$

As observed in [9], the operator $\mathcal{R}^*_\varepsilon$ is the dual of the operator

$$\left(\mathcal{R}_\varepsilon u\right)(x) = \varepsilon^{-d} \int_{x_i + \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)^d} u(y)\mathrm{d}y \quad \text{if } x_i \in \mathbb{Z}^d_\varepsilon \text{ and } x \in x_i + \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)^d .$$

2.1 Homogenization of the global energy $\mathscr{E}_{p,s,\varepsilon}$

On bounded domains $\boldsymbol{Q} \subset \mathbb{R}^d$ we find the following convergence behavior of $\mathscr{E}_{p,s,\varepsilon}$.

Theorem 4. *Let $\boldsymbol{Q} \subset \mathbb{R}^d$ be a bounded domain. Let c, s, p, $\mathfrak{q}$ and V satisfy Assumptions 1 and 3, $G : \mathbb{R} \to \mathbb{R}$ be non-negative and convex with $G(\xi) \leq \alpha|\xi|^m$, $\alpha > 0$, $m < p^\star_{\mathfrak{q}}$, and let $f_\varepsilon \in \mathcal{H}_\varepsilon$ be such that $\mathcal{R}^*_\varepsilon f_\varepsilon \rightharpoonup f$ in $L^{r^*}(\boldsymbol{Q})$, where $\frac{1}{r^*} + \frac{1}{p^\star_{\mathfrak{q}}} < 1$. Then the sequence $\mathscr{E}_{p,s,\varepsilon}$ restricted to $\mathcal{H}_\varepsilon(\boldsymbol{Q})$ Mosco-converges* almost surely *to $\mathscr{E}_{p,s}$ in the following sense:*

1. *For $r = \frac{r^*}{r^*-1}$ there exists $C > 0$ such that*

$$\forall u_\varepsilon \in \mathcal{H}_\varepsilon(\boldsymbol{Q}) : \qquad \|u_\varepsilon\|_{L^r(\boldsymbol{Q}^\varepsilon)} \leq C\mathscr{E}_{p,s,\varepsilon}(u_\varepsilon) \quad \text{for all } \varepsilon > 0\,.$$

   For every sequence $u_\varepsilon \in \mathcal{H}_\varepsilon(\boldsymbol{Q})$ such that $\sup_\varepsilon \mathscr{E}_{p,s,\varepsilon}(u_\varepsilon) < \infty$ there exists $u \in W^{s,p}(\boldsymbol{Q})$, $u = 0$ on $\mathbb{R}^d \backslash \boldsymbol{Q}$, and a subsequence $\mathcal{R}^*_\varepsilon u_\varepsilon \to u$ pointwise a.e. with $\mathcal{R}^*_\varepsilon u_\varepsilon \to u$ strongly in $L^r(\boldsymbol{Q})$, and

$$\liminf_{\varepsilon\to 0} \mathscr{E}_{p,s,\varepsilon}(u_\varepsilon) \geq \mathscr{E}_{p,s}(u)\,,$$

2. *For every $u \in W^{s,p}(\boldsymbol{Q})$ and $r = \frac{r^*}{r^*-1}$ there exists a sequence $u_\varepsilon \in \mathcal{H}_\varepsilon$ such that $\mathcal{R}^*_\varepsilon u^\varepsilon \to u$ strongly in $L^r(\boldsymbol{Q})$ and*

$$\limsup_{\varepsilon\to 0} \mathscr{E}_{p,s,\varepsilon}(u_\varepsilon) = \mathscr{E}_{p,s}(u)\,. \tag{9}$$

Note that for $\mathfrak{q} > \frac{2d}{ps}$ we can choose $r = p$.

Theorem 5. *Let c, s, p, $\mathfrak{q}$ and V satisfy Assumptions 1 and 3, and let the sequence f_ε and the function $G : \mathbb{R} \to \mathbb{R}$ satisfy either one of the following conditions:*

1. *G is non-negative and convex and there exists a bounded $C^{0,1}$ domain $\boldsymbol{Q} \subset \mathbb{R}^d$ such that every f_ε has support in $\boldsymbol{Q}$. Furthermore $\mathcal{R}^*_\varepsilon f_\varepsilon \rightharpoonup f$ in $L^{r^*}(\boldsymbol{Q})$, where $\frac{1}{r^*} + \frac{1}{p^\star_{\mathfrak{q}}} < 1$.*

2. *$G(\xi) = \alpha|\xi|^r + \tilde{G}$, $\tilde{G}$ is non-negative and convex and $r, r^* > 1$ with $\frac{1}{r} + \frac{1}{r^*} = 1$. Furthermore, $\mathcal{R}^*_\varepsilon f_\varepsilon \rightharpoonup f$ in $L^{r^*}(\boldsymbol{Q})$.*

Then the sequence $\mathscr{E}_{p,s,\varepsilon}$ restricted to $\mathcal{H}_\varepsilon$ Mosco-converges to $\mathscr{E}_{p,s}$ in the following sense:

1. *For every sequence $u_\varepsilon \in \mathcal{H}_\varepsilon$ such that $\sup_\varepsilon \mathscr{E}_{p,s,\varepsilon}(u_\varepsilon) < \infty$ there exists $u \in W^{s,p}(\mathbb{R}^d)$, and a subsequence $\varepsilon' \to 0$ such that $\mathcal{R}^*_{\varepsilon'} u_{\varepsilon'} \to u$ pointwise almost everywhere and*

$$\liminf_{\varepsilon\to 0} \mathscr{E}_{p,s,\varepsilon}(u_\varepsilon) \geq \mathscr{E}_{p,s}(u)\,,$$

2. *For every $u \in W^{s,p}(\mathbb{R}^d)$ there exists a sequence $u_\varepsilon \in \mathcal{H}_\varepsilon$ such that $\mathcal{R}^*_\varepsilon u^\varepsilon \to u$ pointwise almost everywhere and*

$$\limsup_{\varepsilon\to 0} \mathscr{E}_{p,s,\varepsilon}(u_\varepsilon) = \mathscr{E}_{p,s}(u)\,. \tag{10}$$

2.2 Homogenization of the local energy $\mathcal{E}_{p,s,\varepsilon,\boldsymbol{Q}}$

The following two theorems deal with the homogenization of the functional $\mathcal{E}_{p,s,\varepsilon,\boldsymbol{Q}}$. In this work, we will study $\mathcal{E}_{p,s,\varepsilon,\boldsymbol{Q}}$ with boundary conditions $u_\varepsilon|_{\partial\boldsymbol{Q}^\varepsilon} \equiv 0$, mean value conditions or with suitable conditions on G. In a first step, we define the following spaces similar to the continuum case:

$$\mathcal{H}_{\varepsilon,0}(\boldsymbol{Q}) := \{u \in \mathcal{H}_\varepsilon(\boldsymbol{Q}) \,:\, \forall x \in \partial\boldsymbol{Q}^\varepsilon \; u(x) = 0\} \,,$$
$$\mathcal{H}_{\varepsilon,(0)}(\boldsymbol{Q}) := \left\{u \in \mathcal{H}_\varepsilon(\boldsymbol{Q}) \,:\, \sum_{x\in\boldsymbol{Q}^\varepsilon} u(x) = 0\right\} .$$

Theorem 6. *Let $\boldsymbol{Q} \subset \mathbb{R}^d$ be a bounded $C^{0,1}$-domain.Let c, s, p, $\mathfrak{q}$ and V satisfy Assumptions 1 and 3, $sp > 1$, $G : \mathbb{R} \to \mathbb{R}$ non-negative and convex with $G(\xi) \le \alpha|\xi|^m$,$\alpha > 0$, $m < p_{\mathfrak{q}}^\star$, and let $f_\varepsilon \in \mathcal{H}_\varepsilon$ be such that $\mathcal{R}_\varepsilon^* f_\varepsilon \rightharpoonup f$ in $L^{r^*}(\boldsymbol{Q})$, where $\frac{1}{r^*} + \frac{1}{p_{\mathfrak{q}}^\star} < 1$. Then the sequence $\mathcal{E}_{p,s,\varepsilon,\boldsymbol{Q}}$ restricted to $\mathcal{H}_{\varepsilon,0}(\boldsymbol{Q})$ Mosco-converges to $\mathcal{E}_{p,s}$ restricted to $W_0^{s,p}(\boldsymbol{Q})$ in the following sense:*

1. *For every sequence $u_\varepsilon \in \mathcal{H}_\varepsilon$ such that $\sup_\varepsilon \mathcal{E}_{p,s,\varepsilon}(u_\varepsilon) < \infty$ there exists $u \in W_0^{s,p}(\boldsymbol{Q})$, $u = 0$ on $\mathbb{R}^d\backslash\boldsymbol{Q}$, and a subsequence $\mathcal{R}_\varepsilon^* u_\varepsilon \to u$ pointwise a.e. with $\mathcal{R}_\varepsilon^* u_\varepsilon \to u$ strongly in $L^r(\boldsymbol{Q})$ for $r = \frac{r^*}{r^*-1}$ and*
$$\liminf_{\varepsilon\to0} \mathcal{E}_{p,s,\varepsilon,\boldsymbol{Q}}(u_\varepsilon) \ge \mathcal{E}_{p,s}(u) \,,$$

2. *For every $u \in W_0^{s,p}(\boldsymbol{Q})$ there exists a sequence $u_\varepsilon \in \mathcal{H}_\varepsilon$ such that $\mathcal{R}_\varepsilon^* u^\varepsilon \to u$ strongly in $L^r(\boldsymbol{Q})$ for $r = \frac{r^*}{r^*-1}$ and*
$$\limsup_{\varepsilon\to0} \mathcal{E}_{p,s,\varepsilon}(u_\varepsilon) = \mathcal{E}_{p,s}(u) \,. \tag{11}$$

If we do not consider zero Dirichlet boundary conditions, we have to find a suitable replacement that guarantees that the necessary (compact) embeddings hold. We use the concept of uniform extension domains introduced in Definition 15.

Theorem 7. *Let c, s, p, $\mathfrak{q}$ and V satisfy Assumptions 1 and 3, $G : \mathbb{R} \to \mathbb{R}$ non-negative and convex with $G(\xi) \le \alpha|\xi|^m$,$\alpha > 0$, $m < p_{\mathfrak{q}}^\star$, and let $f_\varepsilon \in \mathcal{H}_\varepsilon$ be such that $\mathcal{R}_\varepsilon^* f_\varepsilon \rightharpoonup f$ in $L^{r^*}(\boldsymbol{Q})$, where $\frac{1}{r^*} + \frac{1}{p_{\mathfrak{q}}^\star} < 1$. Then the sequence $\mathcal{E}_{p,s,\varepsilon,\boldsymbol{Q}}$ restricted to $\mathcal{H}_{\varepsilon,(0)}(\boldsymbol{Q})$ Mosco-converges to $\mathcal{E}_{p,s}$ restricted to $W_{(0)}^{s,p}(\boldsymbol{Q})$ in the following sense:*

1. *For every sequence $u_\varepsilon \in \mathcal{H}_\varepsilon$ such that $\sup_\varepsilon \mathcal{E}_{p,s,\varepsilon}(u_\varepsilon) < \infty$ there exists $u \in W^{s,p}(\boldsymbol{Q})$, $u = 0$ on $\mathbb{R}^d\backslash\boldsymbol{Q}$, and a subsequence $\mathcal{R}_\varepsilon^* u_\varepsilon \to u$ pointwise a.e. with $\mathcal{R}_\varepsilon^* u_\varepsilon \to u$ strongly in $L^r(\boldsymbol{Q})$ for $r = \frac{r^*}{r^*-1}$ and*
$$\liminf_{\varepsilon\to0} \mathcal{E}_{p,s,\varepsilon}(u_\varepsilon) \ge \mathcal{E}_{p,s}(u) \,,$$

2. *For every $u \in W^{s,p}(\boldsymbol{Q})$ there exists a sequence $u_\varepsilon \in \mathcal{H}_\varepsilon$ such that $\mathcal{R}_\varepsilon^* u_\varepsilon \to u$ strongly in $L^r(\boldsymbol{Q})$ for $r = \frac{r^*}{r^*-1}$ and*
$$\limsup_{\varepsilon\to0} \mathcal{E}_{p,s,\varepsilon}(u_\varepsilon) = \mathcal{E}_{p,s}(u) \,. \tag{12}$$

2.3 Application to the (spectral) homogenization of the fractional Laplace operator

It is a standard and well-known observation that Γ-convergence of $\mathscr{E}_{2,s,\varepsilon}$ for $V(\xi)=\frac{1}{2}|\xi|^2$ implies strong convergence of minimizers $\mathcal{R}_\varepsilon^* u_\varepsilon \to u$ in $L^2(\boldsymbol{Q})$ where u is the minimizer of the limiting functional $\mathscr{E}_{2,s}$. Hence, solutions of $\mathcal{L}_\varepsilon u_\varepsilon = f$ converge to solutions of the fractional equation

$$\int_{\mathbb{R}^d} \frac{u(y)-u(x)}{|x-y|^{d+2s}}\,\mathrm{d}y = f(x)\,.$$

We recall the proof in the context of the following result.

Theorem 8. *Let the assumptions of Theorem 4 hold with $p=2$. For every $\varepsilon>0$ there exists a unique solution $u_\varepsilon \in \mathcal{H}_\varepsilon(\boldsymbol{Q})$ such that for every $v\in\mathcal{H}_\varepsilon(\boldsymbol{Q})$ it holds*

$$\sum_{x\in\mathbb{Z}^d_\varepsilon} \varepsilon^{2d} \sum_{y\in\mathbb{Z}^d_\varepsilon} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{u_\varepsilon(y)-u_\varepsilon(x)}{|x-y|^{d+2s}}\left(v(y)-v(x)\right) = \sum_{x\in\mathbb{Z}^d_\varepsilon} f(x)v(x)\,,$$

and as $\varepsilon\to 0$ we find $\mathcal{R}_\varepsilon^* u_\varepsilon \to u$ strongly in $L^r(\boldsymbol{Q})$ and $u\in W^{s,2}(\mathbb{R}^d)$ is the unique solution to the equation

$$\forall v\in W^{s,2}(\mathbb{R}^d):\quad \mathbb{E}(c)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{u(y)-u(x)}{|x-y|^{d+2s}}\left(v(y)-v(x)\right)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^d} fv\,,$$

where $u=0$ outside of $\boldsymbol{Q}$.

Proof. Let u be the unique minimizer of $\mathscr{E}_{2,s}$ and let u_ε^* be a sequence such that $\mathcal{R}_\varepsilon^* u_\varepsilon^* \to u$ strongly in $L^r(\boldsymbol{Q})$ and (9) holds. Furthermore, let $u_\varepsilon\in\mathcal{H}_\varepsilon(\boldsymbol{Q})$ be the minimizer of $\mathscr{E}_{\varepsilon,2}$ and let $\tilde u = \lim_{\varepsilon\to0}\mathcal{R}_\varepsilon^* u_\varepsilon$ according to Theorem 4. Then

$$\mathscr{E}_{2,s}(u) = \lim_{\varepsilon\to0}\mathscr{E}_{2,s,\varepsilon}(u_\varepsilon^*) \geq \liminf_{\varepsilon\to0}\mathscr{E}_{2,s,\varepsilon}(u_\varepsilon) \geq \mathscr{E}_{2,s}(\tilde u) \geq \mathscr{E}_{2,s}(u)\,.$$

where we used in the last inequality that u is the minimizer of $\mathscr{E}_{2,s}$. Since the minimizer of $\mathscr{E}_{2,s}$ is unique, we obtain $\tilde u = u$ and the Theorem is proved. □

In a similar way, we prove the following theorems.

Theorem 9. *Let the assumptions of Theorem 5 hold with $p=2$. For every $\varepsilon>0$ there exists a unique solution $u_\varepsilon\in\mathcal{H}_\varepsilon$ such that for every $v\in\mathcal{H}_\varepsilon$ it holds*

$$\sum_{x\in\mathbb{Z}^d_\varepsilon} \varepsilon^{2d} \sum_{y\in\mathbb{Z}^d_\varepsilon} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{u_\varepsilon(y)-u_\varepsilon(x)}{|x-y|^{d+2s}}\left(v(y)-v(x)\right) = \sum_{x\in\mathbb{Z}^d_\varepsilon} f(x)v(x)\,,$$

and as $\varepsilon\to0$ we find $\mathcal{R}_\varepsilon^* u_\varepsilon \to u$ pointwise where $u\in W^{s,2}(\mathbb{R}^d)$ is the unique solution to the equation

$$\forall v\in W^{s,2}(\mathbb{R}^d):\quad \mathbb{E}(c)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{u(y)-u(x)}{|x-y|^{d+2s}}\left(v(y)-v(x)\right)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^d} fv\,.$$

Theorem 10. *Let the assumptions of Theorem 6 hold with $p = 2$. For every $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in \mathcal{H}_{\varepsilon,0}(\boldsymbol{Q})$ such that for every $v \in \mathcal{H}_{\varepsilon,0}(\boldsymbol{Q})$ it holds*

$$\sum_{x\in\boldsymbol{Q}^\varepsilon} \varepsilon^{2d} \sum_{y\in\boldsymbol{Q}^\varepsilon} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{u_\varepsilon(y)-u_\varepsilon(x)}{|x-y|^{d+2s}} \left(v(y)-v(x)\right) = \sum_{x\in\boldsymbol{Q}^\varepsilon} f(x)v(x)\,, \tag{13}$$

and as $\varepsilon \to 0$ we find $\mathcal{R}_\varepsilon^* u_\varepsilon \to u$ strongly in $L^r(\boldsymbol{Q})$ and $u \in W_0^{s,2}(\boldsymbol{Q})$ is the unique solution to the equation

$$\forall v \in W_0^{s,2}(\boldsymbol{Q}) : \quad \mathbb{E}(c) \int_{\boldsymbol{Q}} \int_{\boldsymbol{Q}} \frac{u(y)-u(x)}{|x-y|^{d+2s}} \left(v(y)-v(x)\right) \mathrm{d}x\mathrm{d}y = \int_{\boldsymbol{Q}} fv\,. \tag{14}$$

Theorem 11. *Let the assumptions of Theorem 7 hold with $p = 2$. For every $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in \mathcal{H}_{\varepsilon,(0)}(\boldsymbol{Q})$ such that for every $v \in \mathcal{H}_{\varepsilon,(0)}(\boldsymbol{Q})$ it holds*

$$\sum_{x\in\boldsymbol{Q}^\varepsilon} \varepsilon^{2d} \sum_{y\in\boldsymbol{Q}^\varepsilon} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{u_\varepsilon(y)-u_\varepsilon(x)}{|x-y|^{d+2s}} \left(v(y)-v(x)\right) = \sum_{x\in\boldsymbol{Q}^\varepsilon} f(x)v(x)\,,$$

and as $\varepsilon \to 0$ we find $\mathcal{R}_\varepsilon^* u_\varepsilon \to u$ strongly in $L^r(\boldsymbol{Q})$ and $u \in W_{(0)}^{s,2}(\boldsymbol{Q})$ is the unique solution to the equation

$$\forall v \in W_{(0)}^{s,2}(\boldsymbol{Q}) : \quad \mathbb{E}(c) \int_{\boldsymbol{Q}} \int_{\boldsymbol{Q}} \frac{u(y)-u(x)}{|x-y|^{d+2s}} \left(v(y)-v(x)\right) \mathrm{d}x\mathrm{d}y = \int_{\boldsymbol{Q}} fv\,.$$

We finally take a look on the topic of spectral homogenization. Theorem 31 together with Remark 2 and Theorem 6 shows that the operators $\mathcal{B}_c^\varepsilon : \mathcal{H}_\varepsilon(\boldsymbol{Q}) \to \mathcal{H}_{\varepsilon,0}(\boldsymbol{Q})$, where $\mathcal{B}_c^\varepsilon(f)$ solves (13), are uniformly compact with respect to the norm $L^p(\boldsymbol{Q}^\varepsilon)$. Furthermore, Theorem 6 yields that

$$\|\mathcal{R}_\varepsilon^* \mathcal{B}_c^\varepsilon f^\varepsilon - u\|_{L(\boldsymbol{Q})} \to 0 \quad \text{as } \varepsilon \to 0\,,$$

if $\mathcal{R}_\varepsilon^* f^\varepsilon \rightharpoonup f$ where u is the solution to (14). Furthermore, the solution operator $\mathcal{B}$ to (14) is compact by the compact embedding $W^{s,2}(\boldsymbol{Q}) \hookrightarrow L^2(\boldsymbol{Q})$. Hence, we obtain the following result from [12], Theorem 11.4 and 11.5 following the argumentation in Section 8 of [9].

Theorem 12. *Under the assumptions of Theorem 6 let μ_k^ε be the k-th eigenvalue (i.e. $\mu_1^\varepsilon \geq \mu_2^\varepsilon \geq \dots$) and ψ_k^ε the k-th eigenfunction of $\mathcal{B}_c^\varepsilon$. Furthermore, let μ_k be the k-th eigenvalue and ψ_k the k-th eigenfunction of $\mathcal{B}$. Then the following holds.*

- *Let $k \in \mathbb{N}$ and let ε_m be a null sequence. Then there $\mathbb{P}$-a.s. exists a family $\{\psi_j^0\}_{1\leq j\leq k}$ of eigenvectors of $\mathcal{B}$ and a subsequence still indexed by ε_m such that*

$$\left(\mathcal{R}_{\varepsilon_m}^* \psi_1^{\varepsilon_m}, \dots, \mathcal{R}_{\varepsilon_m}^* \psi_k^{\varepsilon_m}\right) \to \left(\psi_1^0, \dots, \psi_k^0\right) \quad \textit{strongly in } L^2(\boldsymbol{Q})\,.$$

- *If the multiplicity of μ_k is equal to s, i.e.*

$$\mu_{k-1} > \mu_k = \mu_{k+1} = \dots = \mu_{k+s} > \mu_{k+s+1}$$

then for $j = 1, \dots, s$ there $\mathbb{P}$-a.s. exists a sequence $\psi^\varepsilon \in \mathcal{H}^\varepsilon(\boldsymbol{Q})$ such that

$$\lim_{\varepsilon\to 0} \|\psi_{k+j} - \mathcal{R}_\varepsilon^* \psi^\varepsilon\|_{L^2(\boldsymbol{Q})} = 0$$

where ψ^ε is a linear combination of the eigenfunctions of the operator $\mathcal{B}_c^\varepsilon$ corresponding to $\mu_k^\varepsilon, \dots, \mu_{k+s}^\varepsilon$.

3 Preliminaries

We first fix some convenient notation for discrete integrals (i.e. higher dimensional sums) and function spaces. For $A \subset \mathbb{R}^d$ we write $|A|_\varepsilon := \varepsilon^d \sharp \left\{ A \cap \mathbb{Z}^d_\varepsilon \right\}$ and note that $|A|_\varepsilon \to |A|$ as $\varepsilon \to \infty$ for every open set $A \subset \mathbb{R}^d$. Moreover, for $A \subset \mathbb{Z}^d_\varepsilon$ and a function $f : A \to \mathbb{R}$ we define

$$\sum_{x\in A}\!{}_\varepsilon\, f(x) := \varepsilon^d \sum_{x\in A} f(x) \,.$$

Then, for every function $f \in C_c(\mathbb{R}^d)$ we find

$$\sum_{x\in\mathbb{Z}^d_\varepsilon}\!{}_\varepsilon\, f(x) \to \int_{\mathbb{R}^d} f \,.$$

Hence, $\sum_\varepsilon$ is a discrete equivalent of the integral $\int$.

3.1 Discrete and continuous Sobolev–Slobodeckij spaces

We introduce the Sobolev–Slobodeckij space $W^{s,p}(\mathbb{R}^d)$ as the closure of $C^\infty_c(\mathbb{R}^d)$ with respect to the norm

$$\|u\|^p_{s,p} := \|u\|^p_{L^p(\mathbb{R}^d)} + [u]^p_{s,p} \,, \quad \text{where} \quad [u]^p_{s,p} := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} \,\mathrm{d}x\,\mathrm{d}y$$

is the Gagliardo seminorm. This family of spaces is discussed in detail for example in [8, 21]. In general, they can be constructed as the interpolation of $W^{1,p}(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$, see e.g. [1, 21], but in this work, we follow the outline of [8]. We also consider Sobolev–Slobodeckij spaces $W^{s,p}(\boldsymbol{Q})$ on Lipschitz bounded domains $\boldsymbol{Q} \subset \mathbb{R}^d$. These are defined by the norm $\|u\|^p_{s,p,\boldsymbol{Q}} := \|u\|^p_{L^p(\boldsymbol{Q})} + [u]^p_{s,p,\boldsymbol{Q}}$, where the semi-norm $[u]^p_{s,p,\boldsymbol{Q}}$ is given through

$$[u]^p_{s,p,\boldsymbol{Q}} = \int_{\boldsymbol{Q}}\int_{\boldsymbol{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} \,\mathrm{d}x\,\mathrm{d}y \,.$$

As can be found for example in Theorem 5.4 of [8],

$$\text{the extention operator } W^{s,p}(\boldsymbol{Q}) \hookrightarrow W^{s,p}(\mathbb{R}^d) \text{ is continuous for every } s \in (0,1] \tag{15}$$

if $\partial\boldsymbol{Q}$ is bounded and of class $C^{0,1}$. Property (15) is called the $W^{s,p}$-extension property of domains $\boldsymbol{Q}$ and it is used to prove compactness of embeddings $W^{s,p}(\boldsymbol{Q}) \hookrightarrow W^{s',p}(\boldsymbol{Q})$ for $0 < s' < s < 1$ and $W^{s,p}(\boldsymbol{Q}) \hookrightarrow L^q(\boldsymbol{Q})$ for every $1 > s > 0$ and $\frac{s}{d} + \frac{1}{q} - \frac{1}{p} > 0$. If $\partial\boldsymbol{Q}$ is bounded and of class $C^{0,1}$ and $sp > 1$, it makes sense to consider

$$W^{s,p}_0(\boldsymbol{Q}) := \left\{ u \in W^{s,p}(\boldsymbol{Q}) \,:\, u|_{\partial\boldsymbol{Q}} \equiv 0 \right\} \,,$$

as in this case the trace is well defined.

Remark 13. In general, the space

$$W_0^{s,p}(\boldsymbol{Q}) := \big(L^p(\boldsymbol{Q}), W_0^{1,p}(\boldsymbol{Q})\big)_s \tag{16}$$

is the interpolate of $W_0^{1,p}(\boldsymbol{Q})$ and $L^p(\boldsymbol{Q})$ and hence the extension by 0 to $W_0^{s,p}(\boldsymbol{Q}) \hookrightarrow W^{s,p}(\boldsymbol{Q})$ is continuous and well defined (see [1, VII.7.17]). Interestingly, (16) is well defined also in case $sp \leq 1$ but on the whole $W^{s,p}(\boldsymbol{Q}) \hookrightarrow W^{s,p}(\mathbb{R}^d)$. Heuristically, this stems from the fact that $sp \leq 1$ implies that functions might have jumps across Lipschitz manifolds. Thus, we may formally identify $W_0^{s,p}(\boldsymbol{Q}) = W^{s,p}(\boldsymbol{Q})$ for $sp \leq 1$.

A further space we will use is

$$W_{(0)}^{s,p}(\boldsymbol{Q}) := \left\{ u \in W^{s,p}(\boldsymbol{Q}) \,:\, \int_{\boldsymbol{Q}} u = 0 \right\}.$$

On $\mathbb{R}^d$ we do not have compact embedding but it holds that $W^{s,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ continuously for every $q \in [p, p^\star]$, where $p^\star = dp/(d-sp)$ for $sp < d$. Furthermore, the set $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{s,p}(\mathbb{R}^d)$. We finally need the following approximation result.

Lemma 14. *Let $\eta \in C_c^\infty(B_1(0))$ with $\eta \geq 0$ and $\int \eta = 1$ and for $k \in \mathbb{N}$ denote $\eta_k(x) := \eta(kx)$. Denoting $f * \eta_k$ the convolution of a measurable function f and η_k we find for every $u \in W^{s,p}(\mathbb{R}^d)$ that*

$$\|u * \eta_k\|_{s,p} \leq \|u\|_{s,p} \qquad \textit{and} \qquad \lim_{k\to\infty} \|u * \eta_k - u\|_{s,p} = 0\,.$$

We shift the proof to the appendix, as it is standard.

In this work, we will need a discrete notion of Sobolev–Slobodeckij spaces and generalizations of the above embedding results to the discrete setting. To this aim, we consider the following normed subspaces of $\mathcal{H}_\varepsilon$. First, let $\boldsymbol{Q}^\varepsilon := \mathbb{Z}_\varepsilon^d \cap \boldsymbol{Q}$ for a bounded domain $\boldsymbol{Q} \subset \mathbb{R}^d$ to define

$$\|u\|_{L^p(\mathbb{Z}_\varepsilon^d)}^p := \sum_{x\in\mathbb{Z}_\varepsilon^d}\nolimits^{\varepsilon} |u(x)|^p \quad \text{and} \quad \|u\|_{L^p(\boldsymbol{Q}^\varepsilon)}^p := \sum_{x\in\boldsymbol{Q}^\varepsilon}\nolimits^{\varepsilon} |u(x)|^p\,,$$

and let $W^{s,p}(\mathbb{Z}_\varepsilon^d)$ be the closure of $C_c^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|u\|_{s,p,\varepsilon}^p := \|u\|_{L^p(\mathbb{Z}_\varepsilon^d)}^p + [u]_{s,p,\varepsilon}^p\,, \quad \text{where} \quad [u]_{s,p,\varepsilon}^p := \sum_{x\in\mathbb{Z}_\varepsilon^d}\nolimits^{\varepsilon}\sum_{y\in\mathbb{Z}_\varepsilon^d}\nolimits^{\varepsilon} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,.$$

When restricted to a bounded domain $\boldsymbol{Q} \subset \mathbb{R}^d$, we define $\|u\|_{s,p,\varepsilon,\boldsymbol{Q}}^p := \|u\|_{L^p(\boldsymbol{Q}^\varepsilon)}^p + [u]_{s,p,\varepsilon,\boldsymbol{Q}}^p$ the norm of the space $W^{s,p}(\boldsymbol{Q}^\varepsilon)$, where

$$[u]_{s,p,\varepsilon,\boldsymbol{Q}}^p := \sum_{x\in\boldsymbol{Q}^\varepsilon}\nolimits^{\varepsilon}\sum_{y\in\boldsymbol{Q}^\varepsilon}\nolimits^{\varepsilon} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,. \tag{17}$$

For some of the proofs below, we need a discrete version of the continuous extension property (15) which holds uniformly in ε. As announced in the introduction we formulate this condition in a definition.

Definition 15. A bounded domain $\boldsymbol{Q}\subset\mathbb{R}^d$ is called a *uniform extension domain* if there exists $C>0$ such that for every $\varepsilon>0$ there exists a linear extension operator $\mathcal{E}_\varepsilon\,:\,W^{s,p}(\boldsymbol{Q}^\varepsilon)\hookrightarrow W^{s,p}(\mathbb{Z}^d_\varepsilon)$ with $\|\mathcal{E}_\varepsilon\|\leq C$.

Remark 16. We may assume for a uniform extension domain $\boldsymbol{Q}$ that there exists a further bounded domain $\tilde{\boldsymbol{Q}}\supset\overline{\boldsymbol{Q}}$ and such that the extensions have compact support in $\tilde{\boldsymbol{Q}}$. We prove this in the appendix.

We will not go into details on this point but note that being a uniform extension domain is immediate for rectangular boxes $\boldsymbol{Q}=\prod_{i=1}^d(a_i,b_i)$, where $-\infty<a_i<b_i<+\infty$ for every $i=1,\dots d$. This can be checked by reflection at the boundaries. Furthermore, Theorem 20 suggests that every $C^{0,1}$ domain should be a uniform extension domain. However, the proof of such a statement is beyond the scope of this work.

In the following, we formulate the four most important results of this subsection. The proofs are technical and either standard (and hence shifted to the appendix) or will be presented in Section 3.2 below.

Theorem 17 (Discrete Sobolev inequality on $\mathbb{Z}^d_\varepsilon$)**.** *Let $s\in(0,1)$ and $p\in[1,\infty)$ be such that $sp<d$ and let $p^\star:=dp/(d-sp)$. Then, for every $q\in[p,p^\star]$, there exists a constant $C_{p,q}>0$ depending only on d, p, q and s such that for every $\varepsilon>0$ and every $u\in W^{s,p}(\mathbb{Z}^d_\varepsilon)$ it holds*

$$\|u\|_{L^q(\mathbb{Z}^d_\varepsilon)}\leq C_{p,q}\,\|u\|_{s,p,\varepsilon}\ . \tag{18}$$

The exponent $p^\star$ is called the *fractional critical exponent*. As a corollary, the last result extends to bounded domains.

Theorem 18. *Let $\boldsymbol{Q}\subset\mathbb{R}^d$ be a bounded uniform extension domain and let $s\in(0,1)$ and $p\in[1,\infty)$ be such that $sp<d$ and let $p^\star:=dp/(d-sp)$. Then, for every $q\in[p,p^\star]$, there exists a constant $C_{p,q}>0$ depending only on d, p, q, s and $\boldsymbol{Q}$ such that for every $\varepsilon>0$ and every $u\in W^{s,p}(\boldsymbol{Q}^\varepsilon)$ it holds*

$$\|u\|_{L^q(\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon)}\leq C_{p,q}\,\|u\|_{s,p,\varepsilon,\boldsymbol{Q}}\ .$$

Furthermore, we obtain the following compactness result on bounded domains.

Theorem 19. *Let $\boldsymbol{Q}\subset\mathbb{R}^d$ be a bounded uniform extension domain and let $s\in(0,1)$ and $p\in[1,\infty)$. Let $p^\star:=dp/(d-sp)$ if $sp<d$, and $p^\star=\infty$ else. For every $\varepsilon>0$ let $u_\varepsilon\in W^{1,p}(\boldsymbol{Q}^\varepsilon)$ such that $\sup_{\varepsilon>0}\|u_\varepsilon\|_{s,p,\varepsilon,\boldsymbol{Q}}<\infty$. Then, for every $q\in[p,p^\star)$ the family $(\mathcal{R}^*_\varepsilon u_\varepsilon)_{\varepsilon>0}$ is precompact in $L^q(\boldsymbol{Q}^\varepsilon)$.*

The proofs of Theorems 17 and 19 are very technical and mostly follow the outline of proofs from [8]. Hence, for better readability of the paper, we shift them to the appendix.

Finally, we turn to Poincaré-type inequalities on bounded domains with Dirichlet boundary conditions or zero mean value. We hence define the spaces

$$W^{s,p}_0(\boldsymbol{Q}^\varepsilon):=\left\{u\in W^{s,p}(\boldsymbol{Q}^\varepsilon)\,:\,u|_{\partial\boldsymbol{Q}^\varepsilon}\equiv0\right\}\,,$$
$$W^{s,p}_{(0)}(\boldsymbol{Q}^\varepsilon):=\left\{u\in W^{s,p}(\boldsymbol{Q}^\varepsilon)\,:\,\sum_{x\in\boldsymbol{Q}^\varepsilon}{}^{\varepsilon}u=0\right\}\,.$$

The corresponding embedding theorems are the following.

Theorem 20. *Let $\boldsymbol{Q}\subset\mathbb{R}^d$ be a bounded domain with $C^{0,1}$ boundary, let $p\in(1,\infty)$, $s\in(0,1)$. Identifying every function $u\in W^{s,p}_0(\boldsymbol{Q}^\varepsilon)$ with its extension by 0 outside $\boldsymbol{Q}^\varepsilon$, there exists $C>0$ independent from ε such that*

$$\forall u\in W^{s,p}_0(\boldsymbol{Q}^\varepsilon):\quad [u]_{s,p,\varepsilon}\leq C\,[u]_{s,p,\varepsilon,\boldsymbol{Q}}\,. \tag{19}$$

For every $q\in[p,p^\star]$, there exists a constant $C_{p,q}>0$ depending only on d, p, q, s and $\boldsymbol{Q}$ such that for every $\varepsilon>0$ it holds

$$\forall u\in W^{s,p}_0(\boldsymbol{Q}^\varepsilon):\quad \|u\|_{L^q(\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon)}\leq C_{p,q}\,[u]^p_{s,p,\varepsilon,\boldsymbol{Q}} \tag{20}$$

Finally, let $p^\star:=dp/(d-sp)$ if $sp<d$, and $p^\star=\infty$ else. For every $\varepsilon>0$ let $u_\varepsilon\in W^{1,p}(\boldsymbol{Q}^\varepsilon)$ such that $\sup_{\varepsilon>0}\lfloor u_\varepsilon\rfloor_{s,p,\varepsilon,\boldsymbol{Q}}<\infty$. Then, for every $q\in[p,p^\star)$ the family $(\mathcal{R}^*_\varepsilon u_\varepsilon)_{\varepsilon>0}$ is precompact in $L^q(\boldsymbol{Q}^\varepsilon)$.

Furthermore, we have a similar result in case $W^{s,p}_0(\boldsymbol{Q}^\varepsilon)$ is replaced by $W^{s,p}_{(0)}(\boldsymbol{Q}^\varepsilon)$.

Theorem 21. *Let $\boldsymbol{Q}\subset\mathbb{R}^d$ be a bounded uniform extension domain with $C^{0,1}$ boundary, let $p\in(1,\infty)$, $s\in(0,1)$. For every $q\in[p,p^\star]$, there exists a constant $C_{p,q}>0$ depending only on d, p, q, s and $\boldsymbol{Q}$ such that for every $\varepsilon>0$ it holds*

$$\forall u\in W^{s,p}_{(0)}(\boldsymbol{Q}^\varepsilon):\quad \|u\|_{L^q(\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon)}\leq C_{p,q}\,[u]^p_{s,p,\varepsilon,\boldsymbol{Q}} \tag{21}$$

Finally, let $p^\star:=dp/(d-sp)$ if $sp<d$, and $p^\star=\infty$ else. For every $\varepsilon>0$ let $u_\varepsilon\in W^{1,p}(\boldsymbol{Q}^\varepsilon)$ such that $\sup_{\varepsilon>0}\lfloor u_\varepsilon\rfloor_{s,p,\varepsilon,\boldsymbol{Q}}<\infty$. Then, for every $q\in[p,p^\star)$ the family $(\mathcal{R}^*_\varepsilon u_\varepsilon)_{\varepsilon>0}$ is precompact in $L^q(\boldsymbol{Q}^\varepsilon)$.

The proof of Theorems 20 and 21 is given in the following subsection. It will be based on the fact that $W^{s,p}(\mathbb{Z}^d_\varepsilon)$ embeds into $W^{s,p}(\mathbb{R}^d)$ via a finite element interpolation operator.

3.2 Proof of Theorems 20 and 21

We first study an interesting connection between $W^{s,p}(\mathbb{R}^d)$ and $W^{s,p}(\mathbb{Z}^d_\varepsilon)$. Let

$$\mathrm{P}:\ [0,1]\times\{0,1\}\to\mathbb{R}\,,\quad (x,\kappa)\mapsto\begin{cases}x & \text{if }\kappa=1\\ 1-x & \text{if }\kappa=0\end{cases},$$

we define for $x=(x_j)_{j=1\dots d}$ and $\kappa=(\kappa_j)_{j=1\dots d}\in\{0,1\}^d$ and $\varphi\in\mathcal{H}_\varepsilon$:

$$(\mathcal{Q}_\varepsilon\varphi)(x):=\sum_{\kappa\in\{0,1\}^d}\varphi\left(\varepsilon\left\lfloor\frac{x}{\varepsilon}\right\rfloor+\varepsilon\kappa\right)\prod_{j=1}^d\mathrm{P}\left(\left\{\frac{x_j}{\varepsilon}\right\},\kappa_j\right)\,,$$

the finite element interpolation of φ. Our first corollary on the operator $\mathcal{Q}_\varepsilon$ is the following.

Corollary 22. *Let $p\in[1,\infty)$. There exists a constant $C>0$ for every $\varphi\in\mathcal{H}_\varepsilon$*

$$C^{-1}\|\varphi\|_{L^p(\mathbb{Z}^d_\varepsilon)}\leq\|\mathcal{Q}_\varepsilon\varphi\|_{L^p(\mathbb{Z}^d_\varepsilon)}\leq C\,\|\varphi\|_{L^p(\mathbb{Z}^d_\varepsilon)}\,. \tag{22}$$

This corollary is straight forward to prove from the definition of $\mathcal{Q}_\varepsilon$. Moreover, we obtain the following natural property.

Lemma 23. *Let $p \in [1, \infty)$ and $s \in (0, 1)$. Then there exists $C > 0$ such that for every $\varepsilon > 0$*

$$\forall \varphi \in \mathcal{H}_\varepsilon : \quad [\mathcal{Q}_\varepsilon \varphi]_{s,p}^p \leq C\, [\varphi]_{s,p,\varepsilon}^p \tag{23}$$

Proof. For $\kappa \in \{0,1\}^d$ we write $\kappa^{i,0}$ and $\kappa^{i,1}$ for the vectors where the i-th entry of κ is replaced by 0 and 1 respectively. In order to reduce notation, we write

$$\left(\delta_i^\varepsilon \varphi\right)(x, \kappa) := \varphi\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon \kappa^{i,1}\right) - \varphi\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon \kappa^{i,0}\right)$$

and hence obtain

$$\partial_i \mathcal{Q}_\varepsilon \varphi = \frac{1}{\varepsilon} \sum_{\kappa \in \{0,1\}^d} \frac{1}{2} \left(\delta_i^\varepsilon \varphi\right)(x, \kappa) \prod_{j \neq i} \mathrm{P}\left(\left\{\frac{x_j}{\varepsilon}\right\}, \kappa_j\right) . \tag{24}$$

For every $x \in \mathbb{R}^d$ let $\lfloor x \rfloor_\varepsilon \in \mathbb{Z}_\varepsilon^d$ be the unique element such that $x \in \mathcal{C}_\varepsilon(x) := \lfloor x \rfloor_\varepsilon + [0, \varepsilon)^d$. We denote $\overline{x}_\varepsilon$ the center of $\mathcal{C}_\varepsilon(x)$ and define

$$A_\varepsilon(x) := \mathbb{Z}_\varepsilon^d \cap \left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + [-\varepsilon, \varepsilon]^d\right)$$

as well as $B_\varepsilon(x) = \overline{x}_\varepsilon + [-\frac{3}{2}\varepsilon, \frac{3}{2}\varepsilon]^d$ and $B_\varepsilon^\complement(x) := \mathbb{R}^d \setminus B_\varepsilon(x)$. We then find for $\tilde{\varphi} = \mathcal{Q}_\varepsilon \varphi$

$$[\tilde{\varphi}]_{s,p}^p \leq \sum_{z \in \mathbb{Z}_\varepsilon^d} \int_{z + (0,\varepsilon)^d} \mathrm{d}x \left(\int_{B_\varepsilon(x)} \frac{|\tilde{\varphi}(x) - \tilde{\varphi}(y)|^p}{|x - y|^{d+sp}} \mathrm{d}y + \int_{B_\varepsilon^\complement(x)} \frac{|\tilde{\varphi}(x) - \tilde{\varphi}(y)|^p}{|x - y|^{d+sp}} \mathrm{d}y \right) . \tag{25}$$

Now, observe that with (24) it holds

$$\begin{aligned}
\int_{B_\varepsilon(x)} \frac{|\tilde{\varphi}(x) - \tilde{\varphi}(y)|^p}{|x - y|^{d+sp}} \mathrm{d}y &\leq \|\nabla \tilde{\varphi}\|_{C^\infty(B_\varepsilon(x))}^p \int_{B_\varepsilon(x)} |x - y|^{p-sp-d} \,\mathrm{d}y \\
&\leq C \,\|\nabla \tilde{\varphi}\|_{C^\infty(B_\varepsilon(x))}^p \,\varepsilon^{p-sp} \\
&\leq C \varepsilon^{-sp} \sum_{z \in A_\varepsilon(x)} \sum_{i=1}^d \sum_{\tilde{\kappa} \in \{0,1\}^d} \varepsilon^d \frac{\left|\left(\delta_i^\varepsilon \varphi\right)(z, \kappa)\right|^p}{\varepsilon^d} \\
&\leq C \varepsilon^d \sum_{y, z \in B_\varepsilon(x)} \frac{|\varphi(z) - \varphi(y)|^p}{|z - y|^{d+ps}} ,
\end{aligned}$$

where C changes in each line but is independent from ε and φ. Furthermore, estimating $\frac{|\tilde{\varphi}(x) - \tilde{\varphi}(y)|^p}{|x-y|^{d+sp}}$ over each cell $\mathcal{C}_\varepsilon(y)$ it is easy to verify (see also the proof of Lemma 40 in the appendix) that we have

$$\int_{B_\varepsilon^\complement(x)} \frac{|\tilde{\varphi}(x) - \tilde{\varphi}(y)|^p}{|x - y|^{d+sp}} \mathrm{d}y \leq C \sum_{z \in \mathbb{Z}_\varepsilon^d \cap \left(\overline{x}_\varepsilon + (-\varepsilon, \varepsilon)^d\right)} \sum_{y \in \mathbb{Z}_\varepsilon^d \setminus \left(\overline{x}_\varepsilon + (-\varepsilon, \varepsilon)^d\right)} \varepsilon^d \frac{|varphi(z) - \varphi(y)|^p}{|z - y|^{d+sp}} .$$

Hence the term in brackets on the right hand side of (25) is independent from $x \in z+(0,\varepsilon)^d$ and we find

$$[u]_{s,p}^p \le C \sum_{x\in\mathbb{Z}_\varepsilon^d}\!\!{}^{\varepsilon} \left(\sum_{y\in\mathbb{Z}_\varepsilon^d\setminus\left(\overline{x}_\varepsilon+(-\varepsilon,\varepsilon)^d\right)} \varepsilon^d \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{d+sp}} + \varepsilon^d \sum_{y\in A_\varepsilon(x)} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{d+ps}} \right) .$$

Since C does not depend on ε or φ, this finally yields (23). □

Proof of Theorem 20. Let $u \in W_0^{s,p}(\boldsymbol{Q}^\varepsilon)$. Due to (22) and (23) we know that $\mathcal{Q}_\varepsilon u \in W_0^{s,p}(\boldsymbol{Q})$. We can now extend $v^\varepsilon := \mathcal{Q}_\varepsilon u$ to $\mathbb{R}^d$ by 0 and obtain $v^\varepsilon \in W^{s,p}(\mathbb{R}^d)$ with $\|v^\varepsilon\|_{s,p} \le C\|v^\varepsilon\|_{s,p,\boldsymbol{Q}}$, where $C>0$ depends on s, p and $\boldsymbol{Q}$. This follows from Remark 13.

We now show $\|u\|_{s,p,\varepsilon} \le C\|v^\varepsilon\|_{s,p}$. Since $\|v^\varepsilon\|_{s,p} \le C\|u\|_{s,p,\varepsilon,\boldsymbol{Q}}$ by Lemma 23, this in turn implies the Theorem by virtue of Theorems 17 and 19.

It only remains to show that

$$\sum_{x\in\mathbb{Z}_\varepsilon^d\setminus\boldsymbol{Q}}\!\!{}^{\varepsilon} \sum_{y\in\boldsymbol{Q}^\varepsilon\setminus\partial\boldsymbol{Q}^\varepsilon}\!\!{}^{\varepsilon} \frac{|u(y)|^p}{|x-y|^{d+ps}} < C\|v^\varepsilon\|_{s,p} . \tag{26}$$

To this reason, let $x\in\mathbb{Z}_\varepsilon^d\setminus\boldsymbol{Q}$ and $y\in\boldsymbol{Q}^\varepsilon\setminus\partial\boldsymbol{Q}^\varepsilon$. Then by definition of $\partial\boldsymbol{Q}^\varepsilon$ in (7) it holds $|x-y|\ge 2\varepsilon$. Let $\tilde{x}\in x+\left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]^d$, $\tilde{y}\in y+\left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]^d$. In order to provide an upper bound on $|\tilde{x}-\tilde{y}|$ in terms of $|x-y|$ assume that $x\in y+[-2\varepsilon,2\varepsilon]$. It then holds $\varepsilon\le|\tilde{x}-\tilde{y}|\le 3d^{\frac{1}{d}}\varepsilon$. Hence we can conclude that $3d^{\frac{1}{d}}|x-y|\ge|\tilde{x}-\tilde{y}|$. In case $x\notin y+[-2\varepsilon,2\varepsilon]$ the ratio between $|x-y|$ and $|\tilde{x}-\tilde{y}|$ becomes smaller. Furthermore, since $u\ge 0$, we have $\mathcal{Q}_\varepsilon u(\tilde{y})\ge 2^{-d}u(y)$ and

$$\begin{aligned}\int_{x+\left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]^d}\int_{y+\left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]^d} &\frac{|\mathcal{Q}_\varepsilon u(\tilde{y})|^p}{|\tilde{x}-\tilde{y}|^{d+ps}}\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} \\ &\ge 2^{-d}\left(3d^{\frac{1}{d}}\right)^{-d-ps}\int_{x+\left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]^d}\int_{y+\left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]^d}\frac{|u(y)|^p}{|x-y|^{d+ps}}\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}\,.\end{aligned}$$

Summing up the last inequality over x and y yields (26). □

Proof of Theorem 21. Let us first verify that (21) holds. Assume that (21) was wrong. Without loss of generality, we might assume that $q>p$. In particular, we use $\|u_{\varepsilon_k}\|_{L^q(\boldsymbol{Q}^{\varepsilon_k})}\le C\|u_{\varepsilon_k}\|_{L^p(\boldsymbol{Q}^{\varepsilon_k})}$. Then there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, $\varepsilon_k>0$, and a sequence of functions $u_{\varepsilon_k}\in\mathcal{H}_{\varepsilon_k,(0)}(\boldsymbol{Q})$ such that

$$\|u_{\varepsilon_k}\|_{L^q(\boldsymbol{Q}^{\varepsilon_k})} = 1 \ge k\,[u_{\varepsilon_k}]_{s,p,\varepsilon_k,\boldsymbol{Q}}^p ,$$

and we find $\mathcal{R}_{\varepsilon_k}^* u_{\varepsilon_k}\to u$ strongly in $L^q(\boldsymbol{Q})$ by Theorem 19. But then $u=0$ since $\mathcal{R}_{\varepsilon_k}^* u_{\varepsilon_k}\rightharpoonup 0$ weakly in $L^q(\boldsymbol{Q})$. This is a contradiction. The compactness follows from Theorem 19. □

3.3 Dynamical systems

Throughout this paper, we follow the setting of Papanicolaou and Varadhan [17] and make the following assumptions.

Assumption 24. *Let $D \in \mathbb{N}$ and let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a given family $(\tau_x)_{x\in\mathbb{Z}^D}$ of measurable bijective mappings $\tau_x : \Omega \mapsto \Omega$, having the properties of a* dynamical system *on $(\Omega, \mathscr{F}, \mathbb{P})$, i.e. they satisfy (i)-(iii):*

(i) $\tau_x \circ \tau_y = \tau_{x+y}$, $\tau_0 = id$ *(Group property)*

(ii) $\mathbb{P}(\tau_{-x}B) = \mathbb{P}(B) \quad \forall x \in \mathbb{Z}^D,\ B \in \mathscr{F}$ *(Measure preserving)*

(iii) $A:\ \mathbb{Z}^d \times \Omega \to \Omega \qquad (x, \omega) \mapsto \tau_x\omega$ *is measurable (Measurability of evaluation)*

Let the system $(\tau_x)_{x\in\mathbb{Z}^D}$ be ergodic i.e. for every $\mathscr{F}$-measurable set $B \subset \Omega$ holds

$$\left[\mathbb{P}\left((\tau_x(B) \cup B)\backslash(\tau_x(B) \cap B)\right) = 0\ \forall x \in \mathbb{Z}^d\right] \Rightarrow \left[\mathbb{P}(B) \in \{0, 1\}\right]. \tag{27}$$

Theorem 25 (Ergodic Theorem [7] Theorem 10.2.II and also [20])**.** *Let $(A_n)_{n\in\mathbb{N}}$ be a family of convex sets in $\mathbb{Z}^D$ such that $A_{n+1} \subset A_n$ and such that there exists a sequence r_n with $r_n \to \infty$ as $n \to \infty$ such that $B_{r_n}(0) \cap \mathbb{Z}^D \subseteq A_n$. If $(\omega_x)_{x\in\mathbb{Z}^D}$ is a stationary ergodic random variable with finite expectation, then almost surely*

$$\frac{1}{\#A_n} \sum_{x\in A_n} \omega_x \to \mathbb{E}(\omega)\,. \tag{28}$$

The last theorem has an important consequence for our work:

Lemma 26. *Let $(A_n)_{n\in\mathbb{N}}$ be a family of convex sets in $\mathbb{R}^D$ such that $A_{n+1} \subset A_n$ and such that there exists a sequence r_n with $r_n \to \infty$ as $n \to \infty$ such that $B_{r_n}(0) \subseteq A_n$. If $(c_x)_{x\in\mathbb{Z}^D}$ is a stationary ergodic random variable with finite expectation, then almost surely*

$$\mathfrak{c} := \sup_{\varepsilon,n} \frac{\varepsilon^D}{|A_n|} \sum_{x\in A_n\cap\mathbb{Z}^D_\varepsilon} c_{\frac{x}{\varepsilon}} < \infty\,. \tag{29}$$

Proof. Defining $\mathfrak{c}_{n,\varepsilon} := \frac{\varepsilon^D}{|A_n|} \sum_{x\in A_n\cap\mathbb{Z}^D_\varepsilon} c_{\frac{x}{\varepsilon}}$ we observe that Theorem (25) implies

$$\forall\varepsilon > 0\ :\ \mathfrak{c}_{n,\varepsilon} \to \mathbb{E}(\tilde{c}) \text{ as } n \to \infty\,, \qquad \text{and} \qquad \forall n\ :\ \mathfrak{c}_{n,\varepsilon} \to \mathbb{E}(c) \text{ as } \varepsilon \to 0\,. \tag{30}$$

Assume that (29) was wrong. Then there exists a sequence $(n_k, \varepsilon_k)_{n\in\mathbb{N}}$ such that $\mathfrak{c}_{n_k,\varepsilon_k} \to \infty$ as $k \to \infty$. If we assume n_k was bounded by N, then the second part of (30) implies existence of $C > 0$ such that

$$\sup_{\varepsilon_k,n_k} \mathfrak{c}_{n_k,\varepsilon_k} \leq \sup_{n\leq N} \sup_{\varepsilon} \mathfrak{c}_{n,\varepsilon} < C < \infty\,,$$

which is a contradiction to the assumption that (29) was wrong. Hence we can w.l.o.g. assume $n_k \uparrow \infty$.

By the same argument, we can assume $\varepsilon_k \downarrow 0$. But then, the Ergodic Theorem 28 implies $\mathfrak{c}_{n_k,\varepsilon_k} \to \mathbb{E}(\tilde{c})$, a contradiction with $\mathfrak{c}_{n_k,\varepsilon_k} \to \infty$. Hence (29) holds. □

A further important consequence is the following:

Lemma 27. *Let c be a random variable on $\mathbb{Z}^d\times\mathbb{Z}^d$ satisfying Assumption 1. Then for every bounded convex domain $\boldsymbol{Q}\subset\mathbb{R}^d$ and every $\alpha,\xi>0$ it holds*

$$\sup_{\varepsilon>0}\varepsilon^{2d}\sum_{x\in\boldsymbol{Q}^\varepsilon}\sum_{\substack{|x-y|<\xi\\ y\in\mathbb{Z}^d_\varepsilon}}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\left|x-y\right|^{-d+\alpha}<C\xi^\alpha\,,$$

where C only depends on $\boldsymbol{Q}$ and d.

Proof. We consider

$$\begin{aligned}\frac{\varepsilon^{2d}}{|\boldsymbol{Q}|}\sum_{x\in\boldsymbol{Q}^\varepsilon}\sum_{\substack{|x-y|<\xi\\ y\in\mathbb{Z}^d_\varepsilon}}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\left|x-y\right|^{-d+\alpha}&=\sum_{k=0}^\infty\frac{\varepsilon^{2d}}{|\boldsymbol{Q}|}\sum_{\substack{x\in\boldsymbol{Q}\\ x\in\mathbb{Z}^d_\varepsilon}}\sum_{\substack{\frac12\xi\le2^k|x-y|<\xi\\ y\in\mathbb{Z}^d_\varepsilon}}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\left|x-y\right|^{-d+\alpha}\\
&\le\sum_{k=0}^\infty\left(2^{-k}\xi\right)^\alpha\frac{\varepsilon^{2d}}{|\boldsymbol{Q}|}\sum_{\substack{x\in\boldsymbol{Q}\\ x\in\mathbb{Z}^d_\varepsilon}}\left(2^{-k-1}\xi\right)^{-d}\sum_{\substack{\frac12\xi\le2^k|x-y|<\xi\\ y\in\mathbb{Z}^d_\varepsilon}}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\\
&=\sum_{k=0}^\infty\left(2^{-k}\xi\right)^\alpha\frac{\varepsilon^{2d}}{|\boldsymbol{Q}|}\sum_{\substack{x\in2^k\boldsymbol{Q}\\ x\in\mathbb{Z}^d_{2^k\varepsilon}}}\left(2^{-k-1}\xi\right)^{-d}\sum_{\substack{\frac12\xi\le|x-y|<\xi\\ y\in\mathbb{Z}^d_{2^k\varepsilon}}}c_{\frac{x}{2^k\varepsilon},\frac{y}{2^k\varepsilon}}\\
&\le(2\xi)^{-d}\sum_{k=0}^\infty\left(2^{-k}\xi\right)^\alpha\frac{\left(2^k\varepsilon\right)^{2d}}{2^{kd}|\boldsymbol{Q}|}\sum_{\substack{x\in2^k\boldsymbol{Q}\\ x\in\mathbb{Z}^d_{2^k\varepsilon}}}\sum_{\substack{|x-y|<\xi\\ y\in\mathbb{Z}^d_{2^k\varepsilon}}}c_{\frac{x}{2^k\varepsilon},\frac{y}{2^k\varepsilon}}\,.\end{aligned}$$

Replacing $\delta:=2^k\varepsilon$, $\boldsymbol{Q}_k:=2^k\boldsymbol{Q}$, $\xi_k=(1+k)\xi$ and $\tilde{\boldsymbol{Q}}_k:=\{(x,y)\,:\,x\in\boldsymbol{Q}_k,\ |x-y|<\xi_k\}$ we obtain

$$\frac{\varepsilon^{2d}}{|\boldsymbol{Q}|}\sum_{x\in\boldsymbol{Q}^\varepsilon}\sum_{\substack{|x-y|<\xi\\ y\in\mathbb{Z}^d_\varepsilon}}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\left|x-y\right|^{-d+\alpha}\le2^{-d}\sum_{k=0}^\infty\left(2^{-k}\xi\right)^\alpha\frac{\delta^{2d}}{|\tilde{\boldsymbol{Q}}_k|}\left(1+k\right)^d\sum_{\substack{(x,y)\in\tilde{\boldsymbol{Q}}_k\\ (x,y)\in\mathbb{Z}^d_\delta\times\mathbb{Z}^d_\delta}}c_{\frac{x}{2^k\varepsilon},\frac{y}{2^k\varepsilon}}\,.$$

By Lemma 26 the sequence

$$\mathfrak{c}:=\sup_{\delta,k}\frac{\delta^{2d}}{|\tilde{\boldsymbol{Q}}_k|}\sum_{\substack{(x,y)\in\tilde{\boldsymbol{Q}}_k\\ (x,y)\in\mathbb{Z}^d_\delta\times\mathbb{Z}^d_\delta}}c_{\frac{x}{2^k\varepsilon},\frac{y}{2^k\varepsilon}}<\infty$$

is bounded. Hence we observe

$$\frac{\varepsilon^{2d}}{|\boldsymbol{Q}|}\sum_{x\in\boldsymbol{Q}^\varepsilon}\sum_{\substack{|x-y|<\xi\\ y\in\mathbb{Z}^d_\varepsilon}}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\left|x-y\right|^{-d+\alpha}\le\xi^\alpha2^{-d}\mathfrak{c}\sum_{k=0}^\infty2^{-k\alpha}\xi_k^d\,,$$

and since $\sum_{k=0}^\infty2^{-k\alpha}\xi_k^d$ is bounded, the lemma is proved. □

We will need to test the convergence (28) with a pointwise converging sequence of functions. The following necessary result by Flegel, Heida and Slowik is a generalization of [4, Theorem 3].

Theorem 28 (Extended ergodic Theorem [9, Theorem 5.2])**.** *Let $\boldsymbol{Q}\subset\mathbb{Z}^D$ be a convex set containing 0 and let f be a stationary random ergodic variable on $\mathbb{Z}^D$ with finite expectation. Furthermore, let $u_\varepsilon:\mathbb{Z}^D_\varepsilon\to\mathbb{R}$ a sequence of functions such that $\mathcal{R}_\varepsilon^* u_\varepsilon\to u$ pointwise a.e. and $\sup_\varepsilon\|u_\varepsilon\|_\infty<\infty$. Then*

$$\varepsilon^D\sum_{x\in\boldsymbol{Q}\cap\mathbb{Z}^D_\varepsilon} f(\frac{x}{\varepsilon})u_\varepsilon(x)\to\mathbb{E}(f)\int_{\boldsymbol{Q}}u(x)\mathrm{d}x\qquad \text{as } \varepsilon\to 0\,.$$

As a direct consequence of the above ergodic theorems we obtain the following result on our coefficients ω and c.

Lemma 29. *Let $0<\mathbb{E}(c)<\infty$ and let $c_{x,y}=\omega_{x,y-x}\left|x-y\right|^{d+ps}$. Then*

$$\mathbb{E}\left(\sum_{z\in\mathbb{Z}^d}\omega_{0,z}\left|z\right|^{ps}\right)=\infty\,.$$

Proof. For every $R>0$ and every $k<R$ we have

$$\begin{aligned}\frac{1}{R^d}\sum_{\substack{x\in\mathbb{Z}^d\\|x|\le R/2}}\sum_{\substack{z\in\mathbb{Z}^d\\|z|<R}}\omega_{x,z}|z|^{ps}&=\frac{1}{R^d}\sum_{\substack{x\in\mathbb{Z}^d\\|x|\le R/2}}\sum_{\substack{y\in\mathbb{Z}^d\\|y-x|\le R}}c_{x,y}\frac{1}{R^d}+\frac{1}{R^d}\sum_{\substack{x\in\mathbb{Z}^d\\|x|\le R/2}}\sum_{\substack{z\in\mathbb{Z}^d\\|z|\le R}}\left(1-\frac{|z|^d}{R^d}\right)\omega_{x,z}|z|^{ps}\\&>\frac{1}{R^{2d}}\sum_{\substack{x\in\mathbb{Z}^d\\|x|\le R/2}}\sum_{\substack{y\in\mathbb{Z}^d\\|y|\le R/2}}c_{x,y}+\frac{1}{R^d}\sum_{\substack{x\in\mathbb{Z}^d\\|x|\le R/2}}\sum_{\substack{z\in\mathbb{Z}^d\\|z|\le R\,k^{-1/d}}}\frac{k-1}{k}\omega_{x,z}|z|^{ps}\,.\end{aligned}$$

Hence, passing to the limit $R\to\infty$ on both sides we obtain

$$|S^{d-1}|\mathbb{E}\left(\sum_{z\in\mathbb{Z}^d}\omega_{0,z}\left|z\right|^{ps}\right)\ge|S^{d-1}|^2\mathbb{E}\left(c\right)+\frac{k-1}{k}|S^{d-1}|\mathbb{E}\left(\sum_{z\in\mathbb{Z}^d}\omega_{0,z}\left|z\right|^{ps}\right)\,,$$

where $|S^{d-1}|$ is the surface of the d-dimensional unit ball in $\mathbb{R}^d$. Since the last inequality holds for arbitrary $k\in\mathbb{N}$ we find $|S^{d-1}|\mathbb{E}\left(\sum_{z\in\mathbb{Z}^d}\omega_{0,z}\left|z\right|^{ps}\right)=\infty$.

□

3.4 Weighted discrete Sobolev–Slobodeckij spaces

This section is concerned with the (compact) embedding of discrete weighted Sobolev–Slobodeckij spaces into the discrete Sobolev–Slobodeckij spaces from Section 3.1. More precisely, the heart of this section (and of the whole article) is the inequality

$$\left(\sum_{x\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}\sum_{y\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}\frac{\left|u(x)-u(y)\right|^r}{\left|x-y\right|^{d+rs'}}\right)^{\frac{1}{r}}\le C\left(\sum_{x\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}\sum_{y\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{\left|u(x)-u(y)\right|^p}{\left|x-y\right|^{d+ps}}\right)^{\frac{1}{p}}\qquad(31)$$

for suitable $r>1$ and $s'\in(0,s)$, where C should depend on s, s', p and r but not on ε. Let us first establish some conditions on $c_{x,y}$, $s\in(0,1)$ and $p\in(1,\infty]$, under which we can expect existence of suitable r and s'. For simplicity of notation, we establish the following semi-norm corresponding to (17):

$$[u]_{s,p,\varepsilon,\boldsymbol{Q},c}:=\left(\sum_{x\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}\sum_{y\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{|u(x)-u(y)|^p}{|x-y|^{d+ps}}\right)^{\frac{1}{p}}.$$

We can use Hölder's inequality and observe that

$$\begin{aligned}\sum_{x\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}\sum_{y\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}\frac{|u(x)-u(y)|^r}{|x-y|^{d+rs'}}&=\sum_{x\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}\sum_{y\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}\frac{\left(c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\right)^{\frac{r}{p}}}{\left(c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\right)^{\frac{r}{p}}}\frac{|u(x)-u(y)|^r}{|x-y|^{(d+ps)\frac{r}{p}}}|x-y|^{d\frac{r}{p}-d+r(s-s')}\\&\leq[u]^r_{s,p,\varepsilon,\boldsymbol{Q},c}\left(\sum_{x\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}\sum_{y\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}\left(c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\right)^{-\frac{r}{p-r}}|x-y|^{-d\left(1-\frac{rp}{d(p-r)}(s-s')\right)}\right)^{\frac{p-r}{p}}.\end{aligned}\tag{32}$$

In order to obtain (31), it is necessary to show that the second factor on the right hand side of (32) is uniformly bounded in $\varepsilon>0$. We have to distinguish two cases.

In the first case, we assume that $1-\frac{rp}{d(p-r)}(s-s')\leq0$, which is equivalent to

$$\frac{r}{p-r}\geq\frac{d}{p(s-s')}\tag{33}$$

and can be fulfilled for a suitable $s'\in(0,s)$ if and only if $\frac{r}{p-r}>\frac{d}{ps}$. In this case, the factor $|x-y|^{-d\left(1-\frac{rp}{d(p-r)}(s-s')\right)}$ stays bounded since $\boldsymbol{Q}^\varepsilon$ is bounded. It follows that the right-hand side of (32) exists – provided that $\mathbb{E}(c^{-\frac{r}{p-r}})<\infty$.

In the second case, we assume that $1-\frac{rp}{d(p-r)}(s-s')>0$. Here, we choose a suitable $\mathfrak{q}$ and apply once more Hölder's inequality to obtain that the right-hand side of (32) is bounded by

$$[u]^r_{s,p,\varepsilon,\boldsymbol{Q},c}\left(\left(\sum_{x\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}\sum_{y\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}\left(c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\right)^{-\mathfrak{q}}\right)^{\frac{1}{\tilde q}}\left(\sum_{x\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}\sum_{y\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}|x-y|^{-d\left(1-\frac{rp}{d(p-r)}(s-s')\right)\frac{\tilde q}{\tilde q-1}}\right)^{\frac{\tilde q-1}{\tilde q}}\right)^{\frac{p-r}{p}},$$

where $\tilde q:=\mathfrak{q}\frac{p-r}{r}>1$. The limit $\varepsilon\to0$ of the right-hand side exists if and only if $\mathbb{E}(c^{-\mathfrak{q}})<\infty$ and

$$1>\left(1-\frac{rp}{d\,(p-r)}(s-s')\right)\frac{\tilde q}{\tilde q-1}\qquad\Leftrightarrow\qquad\mathfrak{q}>\frac{d}{p(s-s')}.$$

Hence, we infer the following Lemma.

Lemma 30. *Let $p\in(1,\infty)$ and let $s\in(0,1)$. If c satisfies Assumption 1 for some $\mathfrak{q}\in\left(\frac{d}{ps},+\infty\right]$, then for all $r>1$ such that $\mathfrak{q}>\frac{r}{p-r}$ there exists $s'\in(0,s)$ such that (31) holds uniformly in $\varepsilon>0$.*

Proof. Let us first assume that $\frac{r}{p-r} > \frac{d}{ps}$. It follows that there exists $s' \in (0,s)$ such that (33) is fulfilled and therefore the right-hand side of (32) stays bounded if $\mathbb{E}(c^{-\frac{r}{p-r}}) < \infty$. This, however, is clearly the case due to Assumption 1 and since $\mathfrak{q} > \frac{r}{p-r}$.

Let us now assume that $\frac{r}{p-r} \leq \frac{d}{ps}$. In this case, there exists $s' \in (0,s)$ such that $1 - \frac{rp}{d(p-r)}(s-s') > 0$ and therefore the claim of the lemma follows by the second case that we have considered above. □

Combined with Theorem 18, we obtain the following result as a consequence of Lemma 30.

Theorem 31. *Let $p \in (1,\infty)$, let $s \in (0,1)$ and let c satisfy Assumption 1 for some $\mathfrak{q} \in \left(\frac{d}{ps}, +\infty\right]$. If $\boldsymbol{Q} \subset \mathbb{R}^d$ is a uniform extesion domain, then for every $r^\star < p^\star_{\mathfrak{q}} = \frac{dp\mathfrak{q}}{2d+d\mathfrak{q}-sp\mathfrak{q}}$ there exists $C > 0$, which does not depend on ε, such that*

$$\|u\|_{L^{r^\star}(\boldsymbol{Q}^\varepsilon)} \leq C \left(\sum_{x\in\boldsymbol{Q}^\varepsilon}^{\varepsilon} \sum_{y\in\boldsymbol{Q}^\varepsilon}^{\varepsilon} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{|u(x)-u(y)|^p}{|x-y|^{d+ps}} \right)^{\frac{1}{p}} .$$

Moreover, for every sequence $u_\varepsilon \in W^{s,p}(\boldsymbol{Q}^\varepsilon, c)$ such that $\sup_{\varepsilon>0} \|u_\varepsilon\|_{s,p,\varepsilon,\boldsymbol{Q},c} < \infty$, the sequence $\mathcal{R}^*_\varepsilon u_\varepsilon$ is precompact in $L^{r^\star}(\boldsymbol{Q})$.

Finally, if $\boldsymbol{Q}$ is a bounded $C^{0,1}$-domain and $u_\varepsilon \in W^{s,p}_0(\boldsymbol{Q}^\varepsilon, c)$ such that $\sup_{\varepsilon>0} \|u_\varepsilon\|_{s,p,\varepsilon,\boldsymbol{Q},c} < \infty$, the sequence $\mathcal{R}^*_\varepsilon u_\varepsilon$ is precompact in $L^{r^\star}(\boldsymbol{Q})$.

Proof. Note that Theorem 18 and Lemma 30 imply that for every $r > 1$ such that $\mathfrak{q} > \frac{r}{p-r}$ and $s' \in (0,s)$ such that $\mathfrak{q} > \frac{d}{p(s-s')}$ and $r^\star := dr/(d-s'r)$ there exists a constant $C > 0$, which does not depend on ε, such that

$$\|u\|_{L^{r^\star}(\boldsymbol{Q}^\varepsilon)} \leq \left(\sum_{x\in\boldsymbol{Q}^\varepsilon}^{\varepsilon} \sum_{y\in\boldsymbol{Q}^\varepsilon}^{\varepsilon} \frac{|u(x)-u(y)|^r}{|x-y|^{d+rs'}} \right)^{\frac{1}{r}} \overset{(31)}{\leq} C \left(\sum_{x\in\boldsymbol{Q}^\varepsilon}^{\varepsilon} \sum_{y\in\boldsymbol{Q}^\varepsilon}^{\varepsilon} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{|u(x)-u(y)|^p}{|x-y|^{d+ps}} \right)^{\frac{1}{p}} ,$$

and the claimed compactness holds. It only remains to verify that $r^\star$ can take any value up to $dp\mathfrak{q}/(2d+d\mathfrak{q}-sp\mathfrak{q})$. Let us note the following equivalences

$$\mathfrak{q} > \frac{d}{p(s-s')} \quad \Leftrightarrow \quad s' < s - \frac{d}{p\mathfrak{q}} ,$$
$$\mathfrak{q} > \frac{r}{p-r} \quad \Leftrightarrow \quad r < \frac{p\mathfrak{q}}{1+\mathfrak{q}} ,$$

and that we can chose s' and r arbitrarily close to their upper bounds. Hence, we obtain from $r^\star = dr/(d-s'r)$ that

$$r^\star < \frac{dp\mathfrak{q}}{1+\mathfrak{q}} \left(d - \frac{p\mathfrak{q}}{1+\mathfrak{q}} \left(s - \frac{d}{p\mathfrak{q}} \right) \right)^{-1}$$

and $r^\star$ can take any value between 1 and the right-hand side. A short calculation shows that this is the claim.

□

4 Proof of Theorems 4 to 7

4.1 Auxiliary Lemmas

We recall the following useful Lemma.

Lemma 32. *Let $\boldsymbol{Q}\subset\mathbb{R}^d$ be a bounded domain and let $v:\mathbb{R}\to\mathbb{R}$ be non-negative and convex. Let $u_\varepsilon:\mathbb{Z}^d_\varepsilon\to\mathbb{R}$ a sequence of functions having support in $\boldsymbol{Q}^\varepsilon$ such that $\mathcal{R}^*_\varepsilon u_\varepsilon\to u$ pointwise a.e.. Then*

$$\liminf_{\varepsilon\to0}\varepsilon^d\sum_{x\in\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon}v(u_\varepsilon)\geq\int_{\boldsymbol{Q}}v(u(x))\mathrm{d}x\,.$$

The proof is standard. However, we recall it here as a preparation for the more involved proofs that will follow.

Proof. Without loss of generality, we may assume

$$\mathrm{E}_\infty:=\liminf_{\varepsilon\to0}\varepsilon^d\sum_{x\in\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon}v(u_\varepsilon)<+\infty\,.$$

For $M\in\mathbb{N}$ we denote $u_\varepsilon^M:=\max\{-M,\min\{u_\varepsilon,M\}\}$, the function u_ε cut to values in the interval $[-M,M]$. We then note that $\mathcal{R}^*_\varepsilon u^M_\varepsilon\to u^M$ pointwise a.e.. Using this insight and continuity , we obtain

$$\mathrm{E}_\infty\geq\liminf_{\varepsilon\to0}I_{M,\varepsilon}\,,\quad\text{where}\quad I_{M,\varepsilon}=\varepsilon^d\sum_{x\in\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon}v(u^M_\varepsilon)\,.$$

From continuity of v and the Lebesgue's dominated convergence theorem, we infer $I_{M,\varepsilon}\to\int_{\boldsymbol{Q}}v(u^M(x))\mathrm{d}x$ as $\varepsilon\to0$. Now, we infer from the Fatou lemma that

$$\int_{\boldsymbol{Q}}v(u(x))\mathrm{d}x=\int_{\boldsymbol{Q}}\liminf_{M\to\infty}v(u^M(x))\mathrm{d}x\leq E_\infty$$

and hence the lemma is proved. □

A related lemma is the following.

Lemma 33. *Let $\boldsymbol{Q}\subset\mathbb{R}^d$ be a bounded domain and let $v:\mathbb{R}\to\mathbb{R}$ be non-negative and convex such that for some $\alpha>0$ and $r'>1$ we have $v(\xi)\leq\alpha|\xi|^{r'}$. Let $u_\varepsilon:\mathbb{Z}^d_\varepsilon\to\mathbb{R}$ a sequence of functions having support in $\boldsymbol{Q}^\varepsilon$ such that for some $r\geq r'$ it holds $\sup_{\varepsilon>0}\|u_\varepsilon\|_{L^r(\boldsymbol{Q}^\varepsilon)}<\infty$ and $\mathcal{R}^*_\varepsilon u_\varepsilon\to u$ pointwise a.e.. Then*

$$\lim_{\varepsilon\to0}\varepsilon^d\sum_{x\in\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon}v(u_\varepsilon)=\int_{\boldsymbol{Q}}v(u(x))\mathrm{d}x\,.$$

Proof. We have for some positive constant C that

$$\limsup_{\varepsilon\to0}\varepsilon^d\sum_{x\in\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon}v(u_\varepsilon)<C\sup_{\varepsilon>0}\|u_\varepsilon\|^{r'}_{L^r(\boldsymbol{Q}^\varepsilon)}<+\infty\,.$$

Let $\delta > 0$. By Egorov's theorem there exists $\boldsymbol{Q}_\delta \subset \boldsymbol{Q}$ with $\boldsymbol{Q}_\delta^\complement = \boldsymbol{Q}\backslash\boldsymbol{Q}_\delta$, $|\boldsymbol{Q}_\delta| < \delta$ and such that $\mathcal{R}_\varepsilon^* u_\varepsilon \to u$ uniformly on $\boldsymbol{Q}_\delta^\complement$. Hence we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon^d \sum_{x\in\boldsymbol{Q}\cap\mathbb{Z}_\varepsilon^d} v(u_\varepsilon) - \int_{\boldsymbol{Q}} v(u(x))\mathrm{d}x &= \int_{\boldsymbol{Q}_\delta} \left(v(u(x)) - v(\mathcal{R}_\varepsilon^* u_\varepsilon(x))\right)\mathrm{d}x \\ &\le 2\sup_{\varepsilon>0}\int_{\boldsymbol{Q}_\delta} \alpha\left|\mathcal{R}_\varepsilon^* u_\varepsilon(x)\right|^{r'}\mathrm{d}x \\ &\le 2|\boldsymbol{Q}_\delta|^{\frac{r-r'}{r}}\alpha\sup_{\varepsilon>0}\|u_\varepsilon\|_{L^r(\boldsymbol{Q}^\varepsilon)}\end{aligned}$$

and hence the lemma is proved as δ becomes arbitrary small. □

Another important result connected with convex functions is the following.

Lemma 34. *Let $G : \mathbb{R} \to \mathbb{R}$ be non-negative and convex, let $u : \mathbb{R}^d \to \mathbb{R}$ be measurable and such that $\int_{\mathbb{R}^d} G(u(z))\,\mathrm{d}z < \infty$ and let $(\eta_k)_{k\in\mathbb{N}}$ be as in Lemma 14. Then*

$$\lim_{k\to\infty}\int_{\mathbb{R}^d} G(\eta_k * u) = \int_{\mathbb{R}^d} G(u)\ .$$

Proof. We note that $\eta_k(x-z)\mathrm{d}z$ induces a probability measure on $\mathbb{R}^d$ for every $k \in \mathbb{N}$ and every $x \in \mathbb{R}^d$. Hence we infer in a first step by Jensen's inequality

$$\begin{aligned}\int_{\mathbb{R}^d} G(\eta_k * u) &= \int_{\mathbb{R}^d} G\left(\int_{\mathbb{R}^d}\eta_k(x-z)u(z)\mathrm{d}z\right)\mathrm{d}x \\ &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\eta_k(x-z)G(u(z))\,\mathrm{d}z\mathrm{d}x \\ &\le \int_{\mathbb{R}^d} G(u(z))\,\mathrm{d}z\,.\end{aligned}$$

On the other hand, Fatou's Lemma yields

$$\int_{\mathbb{R}^d} G(u(z))\,\mathrm{d}z \le \liminf_{k\to\infty}\int_{\mathbb{R}^d} G\left(\left(u*\eta_k\right)(z)\right)\mathrm{d}z\,.$$

□

The following lemma is new to our knowledge. It is the basis for the proofs of our main results.

Lemma 35. *Let $\boldsymbol{Q} \subset \mathbb{R}^d$ be a bounded domain and let c, s, p, $\mathfrak{q}$ and $V : \mathbb{R} \to \mathbb{R}$ satisfy Assumptions 1 and 3. Furthermore, let $u_\varepsilon : \mathbb{Z}_\varepsilon^d \to \mathbb{R}$ a sequence of functions having support in $\boldsymbol{Q}^\varepsilon$ such that $\mathcal{R}_\varepsilon^* u_\varepsilon \to u$ pointwise a.e. and $\sup_\varepsilon \|u_\varepsilon\|_\infty < \infty$. Then*

$$\liminf_{\varepsilon\to 0}\varepsilon^{2d}\sum\sum_{(x,y)\in\mathbb{Z}_\varepsilon^{2d}} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}} \ge \mathbb{E}(c)\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}}\mathrm{d}x\mathrm{d}y\,. \tag{34}$$

$$\liminf_{\varepsilon\to 0}\varepsilon^{2d}\sum\sum_{(x,y)\in\boldsymbol{Q}^\varepsilon\times\boldsymbol{Q}^\varepsilon} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}} \ge \mathbb{E}(c)\iint_{\boldsymbol{Q}\times\boldsymbol{Q}}\frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}}\mathrm{d}x\mathrm{d}y\,. \tag{35}$$

Furthermore, if

$$\sup_{\varepsilon>0}\sup_{x,y\in\mathbb{Z}^d_\varepsilon}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|}{|x-y|}=:C_L<\infty \tag{36}$$

$$\textit{resp.}\qquad \sup_{\varepsilon>0}\sup_{x,y\in\boldsymbol{Q}^\varepsilon}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|}{|x-y|}=:C_L<\infty\,, \tag{37}$$

and $u\in C^1_c(\mathbb{R}^d)$*, resp.* $u\in W^{1,\infty}(\boldsymbol{Q})$ *then we have*

$$\lim_{\varepsilon\to0}\varepsilon^{2d}\sum\sum_{(x,y)\in\mathbb{Z}^{2d}_\varepsilon}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}=\mathbb{E}(c)\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}}\mathrm{d}x\mathrm{d}y\,. \tag{38}$$

$$\lim_{\varepsilon\to0}\varepsilon^{2d}\sum\sum_{(x,y)\in\boldsymbol{Q}^\varepsilon\times\boldsymbol{Q}^\varepsilon}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}=\mathbb{E}(c)\iint_{\boldsymbol{Q}\times\boldsymbol{Q}}\frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}}\mathrm{d}x\mathrm{d}y\,. \tag{39}$$

Proof. We only prove (34) and (38) and shortly discuss how to generalize the calculations to (35) and (39). Without loss of generality, we assume that the lim inf of the left hand side of (34) is bounded. For each $0<\xi<R<\infty$ the sum

$$\varepsilon^{2d}\sum\sum_{(x,y)\in\mathbb{Z}^{2d}_\varepsilon}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}$$

can be split into the three sums over $(x,y)\in\mathbb{Z}^d_\varepsilon\times\mathbb{Z}^d_\varepsilon$ such that either $\{|x-y|<\xi\}$, $\{\xi\le|x-y|<R\}$ or $\{|x-y|\ge R\}$. We denote the corresponding sums by I^ε_ξ, $I^\varepsilon_{\xi,R}$ and I^ε_R. In what follows, we prove in three steps that

$$I^\varepsilon_{\xi,R}=\varepsilon^{2d}\sum\sum_{\substack{(x,y)\in\mathbb{Z}^{2d}_\varepsilon\\ \xi\le|x-y|<R}}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}\to\mathbb{E}(c)\iint_{\xi\le|x-y|<R}\frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}}\,, \tag{40}$$

$$I^\varepsilon_R=\varepsilon^{2d}\sum\sum_{\substack{(x,y)\in\mathbb{Z}^{2d}_\varepsilon\\ |x-y|\ge R}}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}\to O(R^{-ps})\,, \tag{41}$$

$$I^\varepsilon_\xi=\varepsilon^{2d}\sum\sum_{\substack{(x,y)\in\mathbb{Z}^{2d}_\varepsilon\\ |x-y|<\xi}}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}\to O(\xi^{p-ps})\,, \tag{42}$$

where we show (42) only in case (36) holds. Without loss of generality, we will thereby assume that $R>2\mathrm{diam}(\boldsymbol{Q})$. In what follows, we prove (40)-(42) in 3 steps. This provides (38) on observing that

$$\iint_{|x-y|<\xi}\frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}}\le\|\nabla u\|_\infty\iint_{\substack{(x,y)\in2\boldsymbol{Q}\times2\boldsymbol{Q}\\ |x-y|<\xi}}\frac{|x-y|^p}{|x-y|^{d+ps}}=O(\xi^{p-ps})$$

and

$$\limsup_{\varepsilon\to0}\varepsilon^{2d}\sum\sum_{(x,y)\in\mathbb{Z}^{2d}_\varepsilon}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}\leq\limsup_{\varepsilon\to0}\left(I^\varepsilon_{\xi,R}+I^\varepsilon_\xi+I^\varepsilon_R\right)\,.$$

Inequality (34) can be proved on noting that

$$\begin{aligned}V_\infty:=\liminf_{\varepsilon\to0}\varepsilon^{2d}\sum\sum_{(x,y)\in\mathbb{Z}^{2d}_\varepsilon}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}\\ \geq\sup_{\xi,R}\ \mathbb{E}(c)\iint_{\xi\leq|x-y|<R}\frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}}\end{aligned}$$

and applying the Beppo Levi monotone convergence theorem as $\xi\to0$ and $R\to\infty$.

We note that (35) and (39) can be proved in the same way using some slight modification. In particular, we replace $V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)$ by

$$\tilde V\left(x,y,u_\varepsilon(x)-u_\varepsilon(y)\right):=\chi_{\boldsymbol{Q}}(x)\chi_{\boldsymbol{Q}}(y)V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)$$

and study

$$I^\varepsilon_{\xi,R}=\varepsilon^{2d}\sum\sum_{\substack{(x,y)\in B^\varepsilon\times B^\varepsilon\\ \xi\leq|x-y|}}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(x,y,u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}\to\mathbb{E}(c)\iint_{\substack{(x,y)\in B\times B\\ \xi\leq|x-y|}}\frac{V\left(x,y,u(x)-u(y)\right)}{|x-y|^{d+ps}}\,,$$

$$I^\varepsilon_{\xi}=\varepsilon^{2d}\sum\sum_{\substack{(x,y)\in B^\varepsilon\times B^\varepsilon\\ |x-y|<\xi}}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(x,y,u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}\to O(\xi^{p-ps})\,,$$

where $B=B(0)$ is an open ball around 0 that contains $\boldsymbol{Q}$ and $B^\varepsilon:=B\cap\mathbb{Z}^d_\varepsilon$.

Step 1: We consider the lower semi-continuous extension $g_\varepsilon(x,y):=\frac{V(u_\varepsilon(x)-u_\varepsilon(y))}{\max\left\{|x-y|^{d+2s},\xi^{d+2s}\right\}}$ to the set $|x-y|<R$. Moreover, since $R>2\mathrm{diam}(\boldsymbol{Q})$, it holds $V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)=0$ if $|x|\geq2R$ or $|y|\geq2R$. Hence, the support of g_ε is a compact convex subset of $|x-y|<R$ and we infer from Theorem 28 that

$$\sum\sum_{\substack{(x,y)\in\mathbb{Z}^{2d}_\varepsilon\\ |x-y|<R}}c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}g_\varepsilon(x,y)=\mathbb{E}(c)\iint_{|x-y|<R}g(x,y)\,,$$

where $g(x,y):=\frac{V(u(x)-u(y))}{\max\left\{|x-y|^{d+2s},\xi^{d+2s}\right\}}$. Since the same arguments hold on the set $|x-y|<\xi$, the limit (40) holds.

Step 2: Due to our assumption on R, we find $|x-y|>R$ implies that at most one of the points x,y lies in $\boldsymbol{Q}$. Since u vanishes outside $\boldsymbol{Q}$, we obtain by a symmetrization

$$I^\varepsilon_R\leq\left\|V(u_\varepsilon(\cdot))\right\|_\infty\varepsilon^{2d}\sum\sum_{\substack{R<|x-y|\\ x\in\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon}}\left(c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}+c_{\frac{y}{\varepsilon},\frac{x}{\varepsilon}}\right)\frac{1}{|x-y|^{d+ps}}\,.$$

For simplicity of notation, we write $\tilde{c}_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} := c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} + c_{\frac{y}{\varepsilon},\frac{x}{\varepsilon}}$ which is an ergodic variable with $\mathbb{E}(\tilde{c}) = 2\mathbb{E}(c)$. We denote $\mathcal{R}^\varepsilon_{k,R} := \left\{z \in \mathbb{Z}^d_\varepsilon \,|\, 2^k R < |z| \leq 2^{k+1}R\right\}$ and reformulate I^ε_R as

$$I^\varepsilon_R = \varepsilon^{2d} \sum_{\substack{R<|x-y| \\ x\in\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon}}\sum \tilde{c}_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{1}{|x-y|^{d+ps}} = \varepsilon^{2d} \sum_{x\in\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon} \sum_{k=0}^\infty \sum_{y\in\mathcal{R}^\varepsilon_{k,R}+x} \tilde{c}_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{1}{|x-y|^{d+ps}} ,$$

which we estimate as

$$\begin{aligned} I^\varepsilon_R &\leq \sum_{k=0}^\infty \left(2^k R\right)^{-ps} \varepsilon^{2d} \sum_{x\in(1+k)\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon} \left(2^k R\right)^{-d} \sum_{|x-y|<2^{k+1}R} \tilde{c}_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \\ &= |\boldsymbol{Q}| \sum_{k=0}^\infty \left(2^k R\right)^{-ps} (1+k)^d \varepsilon^{2d} \left|\mathbb{B}_{\boldsymbol{Q},k,R}\right|^{-1} \sum_{(x,y)\in\mathbb{B}^\varepsilon_{\boldsymbol{Q},k,R}} \tilde{c}_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} , \end{aligned}$$

where $\mathbb{B}_{\boldsymbol{Q},k,R} = \left\{(x,y) \in \mathbb{R}^d \,|\, x \in (1+k)\boldsymbol{Q}\,,\, |x-y| \leq 2^{k+1}R\right\}$ and $\mathbb{B}^\varepsilon_{\boldsymbol{Q},k,R} := \mathbb{B}_{\boldsymbol{Q},k,R} \cap \mathbb{Z}^d_\varepsilon$. Defining $\mathfrak{c}_{R,k,\varepsilon} := \varepsilon^{2d} \left|\mathbb{B}_{\boldsymbol{Q},k,R}\right|^{-1} \sum_{(x,y)\in\mathbb{B}^\varepsilon_{\boldsymbol{Q},k,R}} \tilde{c}_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}$ we infer from Lemma 26 an estimate $\mathfrak{c}_R := \sup_{\varepsilon,k} \mathfrak{c}_{R,k,\varepsilon} < \infty$ and boundedness of

$$I^\varepsilon_R \leq \mathfrak{c}_R R^{-ps/2} \frac{|\boldsymbol{Q}|\,\mathbb{E}(\tilde{c})}{1-2^{-ps/2}} \sup_k \left(2^{-kps/2}(1+k)^d\right) .$$

Step 3: Now let $\sup_{\varepsilon>0}\sup_{x,y\in\mathbb{Z}^d_\varepsilon} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|}{|x-y|} =: C_L < \infty$. In order to treat the remaining term I^ε_ξ, note that Assumption 3 implies uniform boundedness and lower semi-continuity of the function $\widetilde{V}_\varepsilon(x,y) := \frac{V(u_\varepsilon(x)-u_\varepsilon(y))}{|u_\varepsilon(x)-u_\varepsilon(y)|^p}$. Furthermore, if either $\mathrm{dist}(x,\boldsymbol{Q}) > \xi$ or $\mathrm{dist}(y,\boldsymbol{Q}) > \xi$ then $|x-y| < \xi$ implies $\widetilde{V}_\varepsilon(x,y) = 0$. Hence we obtain

$$\begin{aligned} I^\varepsilon_\xi &\leq \left\|\widetilde{V}\right\|_\infty \varepsilon^{2d} \sum_{\substack{x\in(2\boldsymbol{Q})\cap\mathbb{Z}^d_\varepsilon \\ |x-y|<\xi}}\sum \tilde{c}_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^p}{|x-y|^{d+ps}} \\ &\leq C_L \left\|\widetilde{V}\right\|_\infty \varepsilon^{2d} \sum_{\substack{x\in(2\boldsymbol{Q})\cap\mathbb{Z}^d_\varepsilon \\ |x-y|<\xi}}\sum \tilde{c}_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} |x-y|^{-d+p(1-s)} , \end{aligned}$$

and (42) follows from Lemma 27. □

Remark 36. Having a look on the proof of Lemma 35 and assuming (8) we obtain the following estimates using the relation $c_{x,y} := \omega_{x,y-x}\,|x-y|^{d+2s}$:

$$\begin{aligned} \varepsilon^{2d} \sum_{\substack{|x-y|>\xi \\ x\in\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon}}\sum \tilde{c}_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{1}{|x-y|^{d+ps}} &= \varepsilon^d \sum_{x\in\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon} \sum_{\substack{z\in\mathbb{Z}^d_\varepsilon \\ |z|>\xi/\varepsilon}} \omega_{\frac{x}{\varepsilon},z} \left|\frac{z}{\varepsilon}\right|^{ps} \frac{1}{|z|^{ps}} \\ &\leq \varepsilon^d \sum_{x\in\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon} \frac{1}{\xi^{ps}} \sum_{\substack{z\in\mathbb{Z}^d_\varepsilon \\ |z|>\xi/\varepsilon}} \omega_{\frac{x}{\varepsilon},z} |z|^{ps} \end{aligned}$$

and the last expression converges to 0 as $\varepsilon\to0$. On the other side, we obtain

$$\varepsilon^{2d}\sum_{\substack{|x-y|\leq\xi\\x\in\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon}}\sum\tilde{c}_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{1}{|x-y|^{d+ps}}\geq\varepsilon^d\sum_{x\in\boldsymbol{Q}\cap\mathbb{Z}^d_\varepsilon}\frac{1}{\xi^{ps}}\sum_{\substack{z\in\mathbb{Z}^d_\varepsilon\\|z|\leq\xi/\varepsilon}}\omega_{\frac{x}{\varepsilon},z}\,|z|^{ps}$$

and hence the left expression is bounded from below and away from 0. This suggests that the "natural" guess (8) for a discrete fractional Laplacian leads to a local operator in the limit $\varepsilon\to0$.

4.2 Proof of Theorems 4 and 6–7

We will only prove Theorem 4. Theorems 6–7 can be proved in the same way replacing Theorem 19 by the Embedding Theorems 20 and 21.

Proof of Part 1. Since $\mathcal{R}^*_\varepsilon f_\varepsilon\rightharpoonup f$ weakly in $L^{r^*}(\boldsymbol{Q})$, we find $\sup_\varepsilon\|f_\varepsilon\|_{L^{r^*}(\boldsymbol{Q}_\varepsilon)}<\infty$. Thus, we find from the scaled young inequality for every $\delta>0$ some C_δ such that

$$\left|\sum_{x\in\boldsymbol{Q}^\varepsilon}^{\varepsilon}f_\varepsilon(x)u_\varepsilon(x)\right|\leq\|u\|_{L^r(\boldsymbol{Q}^\varepsilon)}\|f_\varepsilon\|_{L^{r^*}(\boldsymbol{Q}_\varepsilon)}\leq\delta\,\|u\|^p_{L^r(\boldsymbol{Q}^\varepsilon)}+C_\delta\,\|f_\varepsilon\|^{p^*}_{L^{r^*}(\boldsymbol{Q}_\varepsilon)},$$

where $\frac{1}{p}+\frac{1}{p^*}=1$ and $r=\frac{r^*}{r^*-1}$. Since $r<p^\star_{\mathfrak{q}}$, we find from Theorem 31 that

$$\begin{aligned}\|u\|^p_{L^r(\boldsymbol{Q}^\varepsilon)}&\leq\sup_\varepsilon\left(\varepsilon^{2d}\sum_{(x,y)\in\mathbb{Z}^{2d}_\varepsilon}\sum c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}\right)+\varepsilon^d\sum_{x\in\mathbb{Z}^d_\varepsilon}G(u(x))\\&\leq\sup_\varepsilon\mathscr{E}_{p,s,\varepsilon}(u_\varepsilon)+\delta\,\|u\|^p_{L^r(\boldsymbol{Q}^\varepsilon)}+C_\delta\,\|f_\varepsilon\|^{p^*}_{L^{r^*}(\boldsymbol{Q}_\varepsilon)}\end{aligned}$$

implying (for suitable choice of δ) boundedness of

$$\varepsilon^{2d}\sum_{(x,y)\in\mathbb{Z}^{2d}_\varepsilon}\sum c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}+\varepsilon^d\sum_{x\in\mathbb{Z}^d_\varepsilon}G(u(x))\,.$$

In particular, we obtain that

$$\mathrm{E}_\infty:=\liminf_{\varepsilon\to0}\varepsilon^{2d}\sum_{(x,y)\in\mathbb{Z}^{2d}_\varepsilon}\sum c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}<+\infty$$

is bounded.

From Assumption 1 and Theorems 19 and 31 it follows that $\sup_{\varepsilon>0}\|\mathcal{R}^*_\varepsilon u_\varepsilon\|_{L^r}<\infty$ and the existence of $u\in L^r(\boldsymbol{Q})$ such that $\mathcal{R}^*_\varepsilon u_\varepsilon\to u$ strongly in $L^r(\boldsymbol{Q})$ and pointwise a.e. along a subsequence $\varepsilon'\to0$. Furthermore, for $M\in\mathbb{N}$ we denote $u^M_\varepsilon:=\max\{-M,\min\{u_\varepsilon,M\}\}$, the function u_ε cut to values in the interval $[-M,M]$. We then note that $\mathcal{R}_\varepsilon^* u_\varepsilon^M \to u^M$ strongly in $L^r(\boldsymbol{Q})$ and pointwise a.e.. Using this insight, we obtain using Lemma 35 that

$$\begin{aligned}\mathrm{E}_\infty &\geq \liminf_{\varepsilon\to 0} \sum\sum_{(x,y)\in\mathbb{Z}_\varepsilon^{2d}} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{V\left(u_\varepsilon^M(x)-u_\varepsilon^M(y)\right)}{|x-y|^{d+ps}} \\ &\geq \mathbb{E}(c) \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{V\left(u^M(x)-u^M(y)\right)}{|x-y|^{d+ps}} \mathrm{d}x\,\mathrm{d}y\end{aligned}$$

Since the above considerations hold for every M, we apply Fatous Lemma (resp. the monotone convergence theorem by Beppo-Levi) and find

$$\mathbb{E}(c)\iint_{\mathbb{R}^{2d}} \frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}} \mathrm{d}x\,\mathrm{d}y \leq \liminf_{M\to\infty} \mathbb{E}(c) \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{V\left(u^M(x)-u^M(y)\right)}{|x-y|^{d+ps}} \mathrm{d}x\,\mathrm{d}y \leq \mathrm{E}_\infty\,.$$

Moreover, we have from Lemma 32 that

$$\liminf_{\varepsilon\to 0} \varepsilon^d \sum_{x\in\mathbb{Z}_\varepsilon^d} G(u_\varepsilon(x)) - \sum_{x\in\mathbb{Z}_\varepsilon^d} u_\varepsilon(x) f_\varepsilon(x) \geq \varepsilon^d \int_{\mathbb{R}^d} G(u(x))\mathrm{d}x - \int_{\mathbb{R}^d} u(x) f(x)\,.$$

Proof of Part 2

We first consider $u \in C_c^1(\boldsymbol{Q})$. In this case, we set $u_\varepsilon(x) = u(x)$ for $x \in \mathbb{Z}_\varepsilon^d$. From Lemma 35 we infer

$$\lim_{\varepsilon\to 0} \varepsilon^{2d} \sum\sum_{(x,y)\in\mathbb{Z}_\varepsilon^{2d}} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} \frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}} = \mathbb{E}(c) \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}} \mathrm{d}x\,\mathrm{d}y\,.$$

Now, let $\mathscr{E}_{p,s}(u) < \infty$ with $u(x) = 0$ outside of $\boldsymbol{Q}$, set $\varepsilon_0 = 1$. By Assumption 3, we find $u \in W^{s,p}(\mathbb{R}^d)$ and in particular, there exists a sequence $u_k \in C_c^1(\boldsymbol{Q})$ such that $u_k \to u$ in $W^{s,p}(\mathbb{R}^d)$. Moreover, since $m < p_{\mathfrak{q}}^\star \leq p^\star$ the Lemma 33 yields $\int_{\mathbb{R}^d} G(u_k) \to \int_{\mathbb{R}^d} G(u)$ and hence

$$\lim_{k\to\infty} \mathscr{E}_{p,s}(u_k) = \mathscr{E}_{p,s}(u)\,.$$

From the above calculation, there exists $\varepsilon_k > 0$ such that for all $\varepsilon < \varepsilon_k$, $|\mathscr{E}_{p,s,\varepsilon}(u_k) - \mathscr{E}_{p,s}(u_k)| < |\mathscr{E}_{p,s}(u) - \mathscr{E}(\eta_k * u)|$ and in total

$$|\mathscr{E}_{p,s,\varepsilon}(u_k) - \mathscr{E}_{p,s}(u)| < 2\,|\mathscr{E}_{p,s}(u) - \mathscr{E}(\eta_k * u)|\,.$$

Setting $u^\varepsilon := u_k$ for all $\varepsilon \in [\varepsilon_{k+1}, \varepsilon_k)$, (9) holds.

4.3 Proof of Theorem 5

The proof mostly follows the lines of Section 4.2. However, as there are a few modifications due to the non-boundedness of the domain, we provide the full proof for completeness.

Proof of Part 1. Since $f \in C_c(\mathbb{R}^d)$, we chose some bounded domain $\boldsymbol{Q}$ such that f has its support in $\boldsymbol{Q}$. From here, we may follow the lines of Section 4.2 to obtain boundedness of

$$\varepsilon^{2d}\sum\sum_{(x,y)\in\mathbb{Z}_\varepsilon^{2d}} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}+\varepsilon^d\sum_{x\in\mathbb{Z}_\varepsilon^d}G(u(x))\,.$$

In particular, we obtain that

$$\mathrm{E}_\infty := \liminf_{\varepsilon\to 0}\varepsilon^{2d}\sum\sum_{(x,y)\in\mathbb{Z}_\varepsilon^{2d}} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}<+\infty$$

is bounded.

Now, let $m\in\mathbb{N}$ and consider $B_m := \left\{x\in\mathbb{R}^d\,:\,|x|<m\right\}$. From Assumption 1 and Theorem 31 it follows that $\sup_{\varepsilon>0}\left\|\mathcal{R}_\varepsilon^* u_\varepsilon\right\|_{L^r(B_m)}<\infty$ and the existence of $u_m\in L^r(B_m)$ and a subsequence ε_m such that $\mathcal{R}_{\varepsilon_m}^* u_{\varepsilon_m}\to u_m$ as $\varepsilon_m\to 0$ strongly in $L^r(B_m)$ and pointwise a.e. in B_m. Furthermore, for $M\in\mathbb{N}$ we denote $u_\varepsilon^M := \max\left\{-M,\min\left\{u_\varepsilon,M\right\}\right\}$ and obtain using Lemma 35 that

$$\begin{aligned}\mathrm{E}_\infty &\geq \liminf_{\varepsilon\to 0}\sum\sum_{(x,y)\in B_m^\varepsilon\times B_m^\varepsilon} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_{\varepsilon_m}^M(x)-u_{\varepsilon_m}^M(y)\right)}{|x-y|^{d+ps}}\\ &\geq \mathbb{E}(c)\iint_{B_m\times B_m}\frac{V\left(u_m^M(x)-u_m^M(y)\right)}{|x-y|^{d+ps}}\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

Since the above considerations hold for every M, we apply Fatous Lemma (resp. the monotone convergence theorem by Beppo-Levi) and find

$$\mathbb{E}(c)\iint_{B_m\times B_m}\frac{V\left(u_m(x)-u_m(y)\right)}{|x-y|^{d+ps}}\mathrm{d}x\,\mathrm{d}y\leq\liminf_{M\to\infty}\mathbb{E}(c)\iint_{B_m\times B_m}\frac{V\left(u_m^M(x)-u_m^M(y)\right)}{|x-y|^{d+ps}}\mathrm{d}x\,\mathrm{d}y\leq\mathrm{E}_\infty\,.$$

Using a Cantor argument, we infer the existence of a measurable $u:\mathbb{R}^d\to\mathbb{R}$ such that $\mathcal{R}_{\varepsilon'}^* u_{\varepsilon'}\to u$ pointwise a.e.. along a subsequence $\varepsilon'\to 0$ and the Fatou Lemma yields

$$\mathbb{E}(c)\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}}\mathrm{d}x\,\mathrm{d}y\leq\liminf_{m\to\infty}\mathbb{E}(c)\iint_{B_m\times B_m}\frac{V\left(u_m(x)-u_m(y)\right)}{|x-y|^{d+ps}}\mathrm{d}x\,\mathrm{d}y\leq\mathrm{E}_\infty\,.$$

Moreover, we have from Lemma 32 that

$$\liminf_{\varepsilon\to 0}\varepsilon^d\sum_{x\in\mathbb{Z}_\varepsilon^d}G(u_\varepsilon(x))-\sum_{x\in\mathbb{Z}_\varepsilon^d}u_\varepsilon(x)f_\varepsilon(x)\geq\varepsilon^d\int_{\mathbb{R}^d}G(u(x))\mathrm{d}x-\int_{\mathbb{R}^d}u(x)f(x)\,.$$

Proof of Part 2

We first consider $u\in C_c^1(\boldsymbol{Q})$. In this case, we set $u_\varepsilon(x)=u(x)$ for $x\in\mathbb{Z}_\varepsilon^d$. From Lemma 35 we infer

$$\lim_{\varepsilon\to 0}\varepsilon^{2d}\sum\sum_{(x,y)\in\mathbb{Z}_\varepsilon^{2d}} c_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}}\frac{V\left(u_\varepsilon(x)-u_\varepsilon(y)\right)}{|x-y|^{d+ps}}=\mathbb{E}(c)\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{V\left(u(x)-u(y)\right)}{|x-y|^{d+ps}}\mathrm{d}x\,\mathrm{d}y\,.$$

Now, let $\mathscr{E}_{p,s}(u) < \infty$ with $u(x) = 0$ outside of $\boldsymbol{Q}$, set $\varepsilon_0 = 1$. By Assumption 3, we find $u \in W^{s,p}(\mathbb{R}^d)$ and in particular, by Lemma 14 we infer $u_k := \eta_k * u \to u$ in $W^{s,p}(\mathbb{R}^d)$. Moreover, Lemma 34 yields $\int_{\mathbb{R}^d} G(\eta_k * u) \to \int_{\mathbb{R}^d} G(u)$ and hence

$$\lim_{k\to\infty} \mathscr{E}_{p,s}(u_k) = \mathscr{E}_{p,s}(u)\,.$$

From the above calculation, there exists $\varepsilon_k > 0$ such that for all $\varepsilon < \varepsilon_k$, $|\mathscr{E}_{p,s,\varepsilon}(u_k) - \mathscr{E}_{p,s}(u_k)| < |\mathscr{E}_{p,s}(u) - \mathscr{E}(\eta_k * u)|$ and in total

$$|\mathscr{E}_{p,s,\varepsilon}(u_k) - \mathscr{E}_{p,s}(u)| < 2\,|\mathscr{E}_{p,s}(u) - \mathscr{E}_{p,s}(\eta_k * u)|\;.$$

Setting $u^\varepsilon := u_k$ for all $\varepsilon \in [\varepsilon_{k+1}, \varepsilon_k)$, (9) holds.

A Proofs of Auxiliary results

A.1 Proof of Lemma 14

Lemma 37. *Let $u \in W^{s,p}(\mathbb{R}^d)$. Then*

$$\lim_{h\to 0} \|u(\cdot) - u(\cdot - h)\|_{s,p} \to 0\,. \tag{43}$$

Proof. It is well known that the

$$\lim_{h\to 0} \|u(\cdot) - u(\cdot - h)\|_{L^p(\mathbb{R}^d)} \to 0$$

and it only remains to show

$$\lim_{h\to 0} [u(\cdot) - u(\cdot - h)]_{s,p} \to 0\,.$$

Suppose $u \in C_c^\infty(\mathbb{R}^d)$ and let B be a ball that contains the support of u. We write $u_h(x) := u(x-h)$ as well as $f(x,y) = u(x) - u(y)$ and similarly $f_h(x,y)$. Since, for small h, $f(x,y) = f_h(x,y) = 0$ if both $x, y \notin 2B$, we observe that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x,y) - f_h(x,y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad = \int_{2B}\int_{2B} \frac{|f(x,y) - f_h(x,y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y + 2\int_{2B}\int_{\mathbb{R}^d\setminus 2B} \frac{|f(x,y) - f_h(x,y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad \le 2\int_{2B}\int_{\mathbb{R}^d} \frac{|u(x) - u_h(x) - u(y) + u_h(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

For every $\delta > 0$ the right-hand side can be split into an integral over

$$A_\delta := \{(x,y)\;:\; x \in 2B,\; |x-y| < \xi\}$$

and the complement. We find

$$[u(\cdot)-u(\cdot-h)]_{s,p}\leq 2^{p+1}\int_{A_\delta}\frac{|u_h(x)-u_h(y)|^p+|u(x)-u(y)|^p}{|x-y|^{d+sp}}$$
$$+2\int_{\mathbb{R}^{2d}\backslash A_\delta}\frac{|u(x)-u_h(x)-u(y)+u_h(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y\,.$$

The first integral can be estimated by

$$2^{p+2}\left\|\nabla u\right\|_\infty^p\int_{A_\delta}\frac{1}{|x-y|^{d+sp-p}}=2^{p+2}\left\|\nabla u\right\|_\infty^p|2B|\left|S^{d-1}\right|\delta^{p-sp}\,.$$

The second integral converges to 0 as $h\to 0$ as it is bounded by

$$\delta^{-d-sp}4\left\|u-u_h\right\|\to 0\,.$$

Hence, we have shown that $\lim_{h\to 0}[u(\cdot)-u(\cdot-h)]_{s,p}\leq C\delta^{p-sp}$ for every $\delta>0$, implying (43). For arbitrary $u\in W^{s,p}(\mathbb{R}^d)$ the lemma follows from a standard approximation argument. □

Remark 38. Via the triangle inequality, the last Lemma implies that $h\mapsto\left\|u(\cdot)-u(\cdot-h)\right\|_{s,p}$ is continuous:

$$\left|\left\|u(\cdot)-u(\cdot-h_1)\right\|_{s,p}-\left\|u(\cdot)-u(\cdot-h_2)\right\|_{s,p}\right|\leq\left\|u(\cdot-h_1)-u(\cdot-h_2)\right\|_{s,p}\,.$$

Proof of Lemma 14. First note that it is well known that

$$\left\|u*\eta_k\right\|_{L^p(\mathbb{R}^d)}\leq\left\|u\right\|_{L^p(\mathbb{R}^d)}\qquad\text{and}\qquad\lim_{k\to\infty}\left\|u*\eta_k-u\right\|_{L^p(\mathbb{R}^d)}=0$$

and it only remains to show

$$[u*\eta_k]_{s,p}\leq[u]_{s,p}\qquad\text{and}\qquad\lim_{k\to\infty}[u*\eta_k-u]_{s,p}=0\,.$$

The inequality can be easily verified from the fact that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\left|\int_{\mathbb{R}^d}\left(\eta_k(z)u(x-z)-\eta_k(z)u(y-z)\right)\mathrm{d}z\right|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\leq\left\|\eta_k\right\|_{L^1(\mathbb{R}^d)}^{p/p^*}\int_{\mathbb{R}^d}\eta_k(z)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|(u(x-z)-u(y-z))|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&\quad=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|(u(x)-u(y))|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

The limit behavior follows from Lemma 37, Remark 38 and the following calculation:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\left|\int_{\mathbb{R}^d}\left(\eta_k(z)\left(u(x-z)-u(x)\right)-\eta_k(z)\left(u(y-z)-u(y)\right)\right)\mathrm{d}z\right|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&\leq\left\|\eta_k\right\|_{L^1(\mathbb{R}^d)}^{p/p^*}\int_{\mathbb{R}^d}\eta_k(z)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|(u(x-z)-u(x)-u(y-z)+u(y))|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&\leq\left\|\eta_k\right\|_{L^1(\mathbb{R}^d)}^{p/p^*}\int_{\mathbb{R}^d}\eta_k(z)\,[u(\cdot)-u(\cdot-z)]_{s,p}\,\mathrm{d}z\\
&\to 0\quad\text{as }k\to\infty\,.
\end{aligned}$$

□

A.2 Proof of Remark 16

The Remark is a consequence of the following Lemma.

Lemma 39. *Let $\varphi \in C_c^1(\mathbb{R}^d)$. Then for every $\varepsilon > 0$, $p \in (1,\infty)$, $s \in (0,1)$ and $u \in W^{s,p}(\mathbb{Z}_\varepsilon^d)$ it holds $\varphi u \in W^{s,p}(\mathbb{Z}_\varepsilon^d)$ and there exists some $C > 0$ which does not depend on ε such that*

$$\|\varphi u\|_{s,p,\varepsilon} \leq C \|u\|_{s,p,\varepsilon} \|\varphi\|_{C_0^1(\mathbb{R}^d)} \,. \tag{44}$$

Proof. Writing $\delta_f(x,y) := |f(x) - f(y)|$, we first observe that

$$\sum_{x\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\sum_{y\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{\delta_{u\varphi}(x,y)^p}{|x-y|^{d+ps}} \leq \sum_{x\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\sum_{y\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{|u(x)|\,\delta_\varphi(x,y) + \delta_u(x,y)\,|\varphi(y)|}{|x-y|^{d+ps}}\delta_{u\varphi}(x,y)^{p-1} \,.$$

Let $B(x) := \left\{y \in \mathbb{R}^d \,:\, |x-y| < 1\right\}$ with complement $B^{\complement}(x)$. Then, for every $x \in \mathbb{Z}_\varepsilon^d$ we find

$$\begin{aligned}\sum_{y\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{\delta_\varphi(x,y)^p}{|x-y|^{d+ps}} &\leq \sum_{y\in B(x)\cap\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{\|\nabla\varphi\|_\infty^p}{|x-y|^{d+ps-p}} + \sum_{y\in B^{\complement}(x)\cap\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{\|\varphi\|_\infty^p}{|x-y|^{d+ps}} \\ &\leq C\left(\|\nabla\varphi\|_\infty^p + \|\varphi\|_\infty^p\right) .\end{aligned}$$

Furthermore, note that

$$\begin{aligned}&\sum_{x\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\sum_{y\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{|u(x)|\,\delta_\varphi(x,y)}{|x-y|^{d+ps}}\,|\delta_{u\varphi}(x,y)|^{p-1} \\ &\qquad\leq \sum_{x\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}|u(x)|\left(\sum_{y\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{\delta_\varphi(x,y)^p}{|x-y|^{d+ps}}\right)^{\frac{1}{p}}\left(\sum_{y\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{|\delta_{u\varphi}(x,y)|^p}{|x-y|^{d+ps}}\right)^{\frac{p-1}{p}} \\ &\qquad\leq C\left(\|\nabla\varphi\|_\infty^p + \|\varphi\|_\infty^p\right)\|u\|_{L^p(\mathbb{Z}_\varepsilon^d)}\left(\sum_{x\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\sum_{y\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{|\delta_{u\varphi}(x,y)|^p}{|x-y|^{d+ps}}\right)^{\frac{p-1}{p}}\end{aligned}$$

as well as

$$\sum_{x\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\sum_{y\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{\delta_u(x,y)\,|\varphi(y)|}{|x-y|^{d+ps}}\,|\delta_{u\varphi}(x,y)|^{p-1} \leq \|\varphi\|_\infty^p\,[u]_{s,p,\varepsilon}\left(\sum_{x\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\sum_{y\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{|\delta_{u\varphi}(x,y)|^p}{|x-y|^{d+ps}}\right)^{\frac{p-1}{p}} \,.$$

Hence we obtain (44). □

A.3 Proof of Theorem 17

We prove Theorem 17 after three auxiliary lemmas. The first Lemma is an equivalent to Lemma 6.1 in [8].

Lemma 40. *Let $1 \leq p < \infty$, $s \in (0,1)$. There exists C depending only on s, p and d such that*

$$\sum_{y\in E^{\complement}\cap\mathbb{Z}_\varepsilon^d}\frac{\varepsilon^d}{|x-y|^{d+sp}}\geq C\,|E|^{-sp/d}$$

for every $\varepsilon>0$, every $x\in\mathbb{Z}_\varepsilon^d$ and every measurable set $E\subset\mathbb{R}^d$ with finite measure.

Proof. Let $\rho := 2d^{\frac{1}{d}}\left(|E|_\varepsilon\right)^{\frac{1}{d}}$, where $|E|_\varepsilon = \varepsilon^d\sharp\left\{E\cap\mathbb{Z}_\varepsilon^d\right\}$, see the beginning of Section 3. Then, for every $\tilde\rho\geq\rho$ we find $|B_{\tilde\rho}(x)|_\varepsilon\geq|E|_\varepsilon$ and hence

$$\begin{aligned}\left|E^{\complement}\cap B_{\tilde\rho}(x)\right|_\varepsilon &= |B_{\tilde\rho}(x)|_\varepsilon-|E\cap B_{\tilde\rho}(x)|_\varepsilon\geq|E|_\varepsilon-|E\cap B_{\tilde\rho}(x)|_\varepsilon\\ &\geq\left|E\cap B_{\tilde\rho}^{\complement}(x)\right|_\varepsilon .\end{aligned}$$

Hence, we infer that

$$\begin{aligned}\sum_{y\in E^{\complement}\cap\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{1}{|x-y|^{d+sp}} &= \sum_{y\in E^{\complement}\cap B_{\tilde\rho}(x)}^{\varepsilon}\frac{1}{|x-y|^{d+sp}}+\sum_{y\in E^{\complement}\cap B_{\tilde\rho}^{\complement}(x)}^{\varepsilon}\frac{1}{|x-y|^{d+sp}}\\ &\geq\frac{\left|E^{\complement}\cap B_{\tilde\rho}(x)\right|_\varepsilon}{|\tilde\rho|^{d+sp}}+\sum_{y\in E^{\complement}\cap B_{\tilde\rho}^{\complement}(x)}^{\varepsilon}\frac{1}{|x-y|^{d+sp}}\\ &\geq\frac{\left|E\cap B_{\tilde\rho}^{\complement}(x)\right|_\varepsilon}{|\tilde\rho|^{d+sp}}+\sum_{y\in E^{\complement}\cap B_{\tilde\rho}^{\complement}(x)}^{\varepsilon}\frac{1}{|x-y|^{d+sp}}\\ &\geq\sum_{y\in E\cap B_{\tilde\rho}^{\complement}(x)}^{\varepsilon}\frac{1}{|x-y|^{d+sp}}+\sum_{y\in E^{\complement}\cap B_{\tilde\rho}^{\complement}(x)}^{\varepsilon}\frac{1}{|x-y|^{d+sp}}\\ &=\sum_{y\in B_{\tilde\rho}^{\complement}(x)}^{\varepsilon}\frac{1}{|x-y|^{d+sp}} .\end{aligned}$$

Next, we consider cells $C_\varepsilon(z):=z+\varepsilon(-\frac{1}{2},\frac{1}{2})$, $z\in\mathbb{Z}_\varepsilon^d\backslash\{0\}$. On each of these cells, we want to estimate the ratio between the maximal and the minimal value of the function $f(y)=|y|^{-d-ps}$. Due to the polynomial decay of this function, the closer one of the cells $C_\varepsilon(z)$ lies next to 0, the higher will be the ratio in f. The biggest value that f can attain on $\mathbb{R}^d\backslash C_\varepsilon(0)$ is ε^{-d-ps}. Furthermore, all neighboring cells to $C_\varepsilon(0)$ lie within the cube $\left(-\frac{3}{2}\varepsilon,\frac{3}{2}\varepsilon\right)$ and the minimal value of f is on this domain is the value of f is $\left(\frac{3}{2}d^{\frac{1}{d}}\varepsilon\right)^{-d-sp}$. Hence we obtain

$$\inf_{z\in\mathbb{Z}_\varepsilon^d\backslash\{0\}}\inf_{y\in C_\varepsilon(z)}|y|^{-d-sp}\left(\sup_{y\in C_\varepsilon(z)}|y|^{-d-sp}\right)^{-1}\geq\left(\frac{\varepsilon}{2}\right)^{d+sp}\left(\frac{3}{2}d^{\frac{1}{d}}\varepsilon\right)^{-d-sp}=\left(3d^{\frac{1}{d}}\right)^{-d-sp},$$

and we conclude that

$$\sum_{y\in E^{\complement}\cap\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{1}{|x-y|^{d+sp}}\geq\sum_{y\in B_{\tilde\rho}^{\complement}(x)}^{\varepsilon}\frac{1}{|x-y|^{d+sp}}\geq\left(3d^{\frac{1}{d}}\right)^{-d-sp}\int_{y\in B_{\tilde\rho}^{\complement}(x)}\frac{1}{|x-y|^{d+sp}}\mathrm{d}y .$$

Now the theorem follows from integration using polar coordinates. □

Lemma 41 ([8, Lemma 6.2]). *Let $s \in (0,1)$ and $p \in [1,\infty)$ be such that $sp < d$. Fix $T > 1$ and let $N \in \mathbb{Z}$, and*

$$a_k \text{ be a non-increasing sequence such that } a_k = 0 \text{ for every } k \geq N\,.$$

Then,

$$\sum_{k\in\mathbb{Z}} a_k^{(d-sp)/d} T^k \leq C \sum_{k\in\mathbb{Z},\, a_k\neq 0} a_{k+1} a_k^{-sp/d} T^k\,,$$

for a suitable constant $C = C(d,s,p,T)$, independent of N.

We are now in the position to prove the following variant of [8], Lemma 6.3.

Lemma 42. *Let $s \in (0,1)$ and $p \in [1,\infty)$ be such that $sp < d$. Let $f \in L^\infty(\mathbb{Z}^d_\varepsilon)$ be compactly supported. For any $k \in \mathbb{Z}$ let*

$$a_k := \left|\left\{|f| > 2^k\right\}\right|\,.$$

Then,

$$\sum_{x\in\mathbb{Z}^d_\varepsilon}^{\varepsilon} \sum_{y\in\mathbb{Z}^d_\varepsilon}^{\varepsilon} \frac{|f(x)-f(y)|^p}{|x-y|^{d+ps}} \geq C \sum_{a_k\neq 0} 2^{pk} a_{k+1} a_k^{-sp/d}$$

for some suitable constant $C = C(d,s,p) > 0$, which depends not on ε.

Proof. We first emphasize that $\left||f(x)|-|f(y)|\right| \leq |f(x)-f(y)|$ and hence we only consider $f \geq 0$, possibly replacing f by $|f|$.

We define

$$A_k := \left\{f > 2^k\right\} \quad \text{with} \quad A_{k+1} \subset A_k$$
$$a_k := \left|\left\{f > 2^k\right\}\right| \quad \text{with} \quad a_{k+1} \leq a_k\,.$$

We define

$$D_k := A_k \backslash A_{k+1} = \left\{2^{k+1} \geq f > 2^k\right\} \quad \text{and} \quad d_k := |D_k| \quad \text{with}$$
$$d_k \text{ and } a_k \text{ are bounded and they become zero when } k \text{ is large enough,}$$

since f is bounded. We define $D_{-\infty} = \{f = 0\}$ and further observe that the sets D_k are mutually disjoint and

$$D_{-\infty} \cup \bigcup_{l\in\mathbb{Z},l\leq k} D_l = A^{\complement}_{k+1}\,, \qquad \bigcup_{l\in\mathbb{Z},l\geq k} D_l = A_k\,. \tag{45}$$

As a consequence, we have

$$a_k = \sum_{l=k}^{\infty} d_l\,, \qquad d_k = a_k - \sum_{l=k+1}^{\infty} d_l\,. \tag{46}$$

The first equality implies that the series $\sum_{l\geq k} d_l$ are convergent. For convenience of notation, in the following we write for arbitrary expressions $g(y)$

$$\sum_{j=-\infty}^{i-2} \sum_{y\in D_j}^{\varepsilon} g(y) := \sum_{y\in D_{-\infty}}^{\varepsilon} g(y) + \sum_{\substack{l\in\mathbb{Z}\\ l\leq i-2}} \sum_{y\in D_j}^{\varepsilon} g(y)$$

Now, we fix $i \in \mathbb{Z}$ and $x \in D_i$. For every $j \in \mathbb{Z}$, $j \le i-2$ and every $y \in D_j$ we have
$$|f(x) - f(y)| \ge 2^i - 2^{j+1} \ge 2^i - 2^{i-1} = 2^{i-1}$$
and hence by the first equality in (45) it holds
$$\sum_{j=-\infty}^{i-2} \sum_{y\in D_j}^{\varepsilon} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \ge 2^{p(i-1)} \sum_{j=-\infty}^{i-2} \sum_{y\in D_j}^{\varepsilon} \frac{1}{|x-y|^{d+sp}} \ge 2^{p(i-1)} \sum_{y\in A_{i-1}^{\complement}}^{\varepsilon} \frac{1}{|x-y|^{d+sp}}\,.$$
Therefore, by Lemma 40, there exists a constant c_0 such that for every $i \in \mathbb{Z}$ and every $x \in D_i$ it holds
$$\sum_{j=-\infty}^{i-2} \sum_{y\in D_j}^{\varepsilon} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \ge c_0 2^{pi} a_{i-1}^{-sp/d}\,,$$
$$\text{and} \qquad \sum_{j=-\infty}^{i-2} \sum_{y\in D_j}^{\varepsilon} \sum_{x\in D_i}^{\varepsilon} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \ge c_0 2^{pi} a_{i-1}^{-sp/d} d_i\,.$$
Summing up the last inequality over $i \in \mathbb{Z}$ we have on one side
$$\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} \sum_{j=-\infty}^{i-2} \sum_{y\in D_j}^{\varepsilon} \sum_{x\in D_i}^{\varepsilon} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \ge c_0 \sum_{l\in\mathbb{Z},\, a_{l-1}\neq 0} 2^{pl} a_{l-1}^{-sp/d} d_l =: c_0 S\,, \tag{47}$$
implying S to be bounded, and on the other hand, using (46), we have
$$\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} \sum_{j=-\infty}^{i-2} \sum_{y\in D_j}^{\varepsilon} \sum_{x\in D_i}^{\varepsilon} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \ge c_0 \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} \left(2^{pi} a_{i-1}^{-sp/d} a_i - \sum_{l=i+1}^{\infty} 2^{pi} a_{i-1}^{-sp/d} d_l \right)\,, \tag{48}$$
where we estimate the second sum by S through
$$\begin{aligned} \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} \sum_{\substack{l\in\mathbb{Z}\\ l\ge i+1}} 2^{pi} a_{i-1}^{-sp/d} d_l &= \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} \sum_{\substack{l\in\mathbb{Z}\\ l\ge i+1\\ a_{i-1}d_l\neq 0}} 2^{pi} a_{i-1}^{-sp/d} d_l \\ &\le \sum_{i\in\mathbb{Z}} \sum_{\substack{l\in\mathbb{Z}\\ l\ge i+1\\ a_{l-1}\neq 0}} 2^{pi} a_{i-1}^{-sp/d} d_l \\ &= \sum_{\substack{l\in\mathbb{Z}\\ a_{l-1}\neq 0}} \sum_{\substack{i\in\mathbb{Z}\\ i\le l-1}} 2^{pi} a_{i-1}^{-sp/d} d_l \\ &\le \sum_{\substack{l\in\mathbb{Z}\\ a_{l-1}\neq 0}} \sum_{\substack{i\in\mathbb{Z}\\ i\le l-1}} 2^{pi} a_{l-1}^{-sp/d} d_l \le S\,. \end{aligned}$$

Using the last estimate in (48), we obtain

$$\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}}\sum_{j=-\infty}^{i-2}\sum_{y\in D_j}^{\varepsilon}\sum_{x\in D_i}^{\varepsilon}\frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}\geq c_0\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}}2^{pi}a_{i-1}^{-sp/d}a_i-c_0S$$

and using estimate (47) we find upon relabeling c_0 that

$$\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}}\sum_{j=-\infty}^{i-2}\sum_{y\in D_j}^{\varepsilon}\sum_{x\in D_i}^{\varepsilon}\frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}\geq c_0\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}}2^{pi}a_{i-1}^{-sp/d}a_i\,.$$

On the other hand, it clearly holds that

$$\sum_{y\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\sum_{x\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}\geq\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}}\sum_{j=-\infty}^{i-2}\sum_{y\in D_j}^{\varepsilon}\sum_{x\in D_i}^{\varepsilon}\frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}$$

and hence the lemma follows. □

We are now in the position to prove the first Sobolev theorem.

Proof of Theorem 17. It suffices to prove the claim for

$$[f]_{s,p,\varepsilon}^p=\sum_{x\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\sum_{y\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{|f(x)-f(y)|^p}{|x-y|^{d+ps}}<\infty$$

and for $f\in L^\infty(\mathbb{Z}_\varepsilon^d)$. Indeed, for arbitrary $f\in W^{s,p}(\mathbb{Z}_\varepsilon^d)$, with $f_N:=\max\{-N,\min\{N,f\}\}$ we obtain that

$$\lim_{N\to\infty}\sum_{x\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\sum_{y\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{|f_N(x)-f_N(y)|^p}{|x-y|^{d+ps}}=\sum_{x\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\sum_{y\in\mathbb{Z}_\varepsilon^d}^{\varepsilon}\frac{|f(x)-f(y)|^p}{|x-y|^{d+ps}}$$

due to the dominated convergence theorem and pointwise convergence $f_N\to f$.

We recall the definitions

$$A_k:=\left\{|f|>2^k\right\}\quad\text{with}\quad A_{k+1}\subset A_k$$
$$a_k:=\left|\left\{|f|>2^k\right\}\right|\quad\text{with}\quad a_{k+1}\leq a_k$$

from the proof of Lemma 42 and obtain

$$\|f\|_{L^{p^*}(\mathbb{Z}_\varepsilon^d)}^{p^\star}=\sum_{k\in\mathbb{Z}}\sum_{x\in A_k\setminus A_{k+1}}^{\varepsilon}|f(x)|^{p^\star}\leq\sum_{k\in\mathbb{Z}}\sum_{x\in A_k\setminus A_{k+1}}^{\varepsilon}\left|2^{k+1}\right|^{p^\star}\leq\sum_{k\in\mathbb{Z}}2^{(k+1)p^\star}a_k\,.$$

Using $p/p^\star=(d-sp)/d=1-sp/d<1$ we can conclude with Lemma 41 that

$$\begin{aligned}\|f\|_{L^{p^*}(\mathbb{Z}_\varepsilon^d)}^{p}&\leq 2^p\left(\sum_{k\in\mathbb{Z}}2^{kp^\star}a_k\right)^{\frac{p}{p^\star}}\leq 2^p\sum_{k\in\mathbb{Z}}2^{kp}a_k^{(d-sp)/d}\\&\leq C\sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}}2^{kp}a_{k+1}a_k^{\frac{-sp}{d}}\,.\end{aligned}$$

It only remains to apply Lemma 42 and relabeling the constant C to find (18) in case $q = p^\star$. In case $q = \theta p + (1-\theta)p^\star$, $\theta \in (0,1)$, we obtain from Hölder's inequality and the case $q = p^\star$ that

$$\sum_{x\in\mathbb{Z}^d_\varepsilon}^{\varepsilon} |f(x)|^q = \sum_{x\in\mathbb{Z}^d_\varepsilon}^{\varepsilon} |f(x)|^{\theta p} |f(x)|^{(1-\theta)p^\star} \leq \left(\sum_{x\in\mathbb{Z}^d_\varepsilon}^{\varepsilon} |f(x)|^p\right)^{\theta} \left(\sum_{x\in\mathbb{Z}^d_\varepsilon}^{\varepsilon} |f(x)|^{p^\star}\right)^{1-\theta}$$
$$= \|f\|^{p\theta}_{L^p(\mathbb{Z}^d_\varepsilon)} \|f\|^{(1-\theta)p^\star}_{L^{p^\star}(\mathbb{Z}^d_\varepsilon)} \leq \|f\|^{p\theta}_{L^p(\mathbb{Z}^d_\varepsilon)} [f]^{(1-\theta)p^\star}_{s,p,\varepsilon} \leq \|f\|^{p\theta}_{s,p,\varepsilon} \|f\|^{(1-\theta)p^\star}_{s,p,\varepsilon} = \|f\|^{q}_{s,p,\varepsilon} .$$

□

A.4 Proof of Theorem 19

Proof. Since $\boldsymbol{Q}$ is a uniform extension domain, the family $\mathcal{R}^*_\varepsilon u^\varepsilon$ is precompact if and only if $\mathcal{R}^*_\varepsilon \mathcal{E}_\varepsilon u^\varepsilon$ is compact, where we recall the operator $\mathcal{E}_\varepsilon$ from Definition 15. We will apply the Frechet-Kolmogorov(-Riesz) theorem to prove compactness of $\mathcal{R}^*_\varepsilon \mathcal{E}_\varepsilon u^\varepsilon$. More precisely, it suffices to verify the following three properties:

$$\sup_{\varepsilon>0} \|\mathcal{R}^*_\varepsilon \mathcal{E}_\varepsilon u^\varepsilon\|_{L^q(\mathbb{R}^d)} < \infty , \qquad \lim_{R\to\infty} \sup_{\varepsilon>0} \|\mathcal{R}^*_\varepsilon \mathcal{E}_\varepsilon u^\varepsilon\|_{L^q(\mathbb{R}^d\setminus B_R(0))} = 0 , \tag{49}$$

$$\lim_{|h|\to 0} \sup_{\varepsilon>0} \|\mathcal{R}^*_\varepsilon \mathcal{E}_\varepsilon u^\varepsilon(\cdot) - \mathcal{R}^*_\varepsilon \mathcal{E}_\varepsilon u^\varepsilon(\cdot + h)\|_{L^q(\mathbb{R}^d)} \to 0 . \tag{50}$$

Note that the conditions in (49) are satisfied due to Theorem 17 and Remark 16. Thus, it only remains to show (50).

For $h \in \mathbb{R}^d$ we write $\tau_h u(x) := u(x+h)$, whenever this is well defined. Moreover, for every $\varepsilon > 0$ we define

$$\|u\|_{p,\varepsilon} := \left(\sum_{x\in\mathbb{Z}^d_\varepsilon}^{\varepsilon} |u(x)|^p\right)^{1/p} .$$

We first prove the Theorem in case $q = p$. Let $h \in \mathbb{Z}^d_\varepsilon$ and $\eta := 10h$. We define $B_{\eta,\varepsilon} := \left\{y \in \mathbb{Z}^d_\varepsilon \,:\, |y| \leq |\eta|\right\}$ and $B_\eta := \left\{y \in \mathbb{R}^d \,:\, |y| < |\eta|\right\}$. Since $h \in \mathbb{Z}^d_\varepsilon$ we always have $\eta \geq 10\varepsilon$ and hence we have

$$C_B := \sup_{\varepsilon,\eta} \left(\frac{|B_{\eta,\varepsilon}|_\varepsilon}{|B_\eta|} + \frac{|B_\eta|}{|B_{\eta,\varepsilon}|_\varepsilon}\right) < +\infty$$

and

$$\tilde{C}_B := \sup_{\varepsilon,\eta} \left(\sum_{y\in B_{\eta,\varepsilon}}^{\varepsilon} |y|^{(d+ps)/(p-1)}\right) \Big/ \left(\int_{B_\eta} |y|^{(d+ps)/(p-1)}\,\mathrm{d}y\right) < +\infty$$

We find

$$\|u - \tau_h u\|_{p,\varepsilon} \leq |B_{\eta,\varepsilon}|^{-1}_\varepsilon \sum_{y\in B_{\eta,\varepsilon}}^{\varepsilon} \left(\|u - \tau_y u\|_{p,\varepsilon} + \|\tau_h u - \tau_y u\|_{p,\varepsilon}\right) .$$

In order to estimate the right-hand side, we apply Hölder's inequality and obtain

$$\begin{aligned}\sum_{y\in B_{\eta,\varepsilon}}^{\varepsilon} \|u-\tau_y u\|_{p,\varepsilon} &\leq \left(\sum_{y\in B_{\eta,\varepsilon}}^{\varepsilon} \frac{\|u-\tau_y u\|_{p,\varepsilon}^p}{|y|^{d+ps}}\right)^{\frac{1}{p}} \left(\sum_{y\in B_{\eta,\varepsilon}}^{\varepsilon} |y|^{(d+ps)/(p-1)}\right)^{\frac{p-1}{p}} \\ &\leq \left(\sum_{y\in B_{\eta,\varepsilon}}^{\varepsilon}\sum_{x\in\cap\mathbb{Z}_\varepsilon^d}^{\varepsilon} \frac{(u(x)-u(x+y))^p}{|y|^{d+ps}}\right)^{\frac{1}{p}} \tilde{C}_B^{\frac{p-1}{p}} \left(\int_{B_\eta} |y|^{(d+ps)/(p-1)}\,\mathrm{d}y\right)^{\frac{p-1}{p}} \\ &= C\,\|u\|_{W^{s,p}(\mathbb{Z}_\varepsilon^d)}\,|B_\eta|\,|\eta|^s\,.\end{aligned}$$

Also with $B_{2\eta,\varepsilon}(h) := \left\{y\in\mathbb{Z}_\varepsilon^d\,:\,|y-h|\leq 2|\eta|\right\}$ we get from Hölder's inequality

$$\begin{aligned}\sum_{y\in B_{\eta,\varepsilon}}^{\varepsilon} \|\tau_h u-\tau_y u\|_{p,\varepsilon} &\leq \left(\sum_{y\in B_{\eta,\varepsilon}}^{\varepsilon} \frac{\|\tau_h u-\tau_y u\|_{p,\varepsilon}^p}{|y-h|^{d+ps}}\right)^{\frac{1}{p}} \left(\sum_{y\in B_{\eta,\varepsilon}}^{\varepsilon} |y-h|^{(d+ps)/(p-1)}\right)^{\frac{p-1}{p}} \\ &\leq C\,\|u\|_{W^{s,p}(\mathbb{Z}_\varepsilon^d)} \left(\sum_{y\in B_{2\eta,\varepsilon(h)}}^{\varepsilon} |y-h|^{(d+ps)/(p-1)}\right)^{\frac{p-1}{p}} \\ &= 2^{d+ps}C\,\|u\|_{W^{s,p}(\mathbb{Z}_\varepsilon^d)}\,|B_\eta|\,|\eta|^s\,.\end{aligned}$$

This implies

$$\|u-\tau_h u\|_{p,\varepsilon} \leq C\,\|u\|_{W^{s,p}(\mathbb{Z}_\varepsilon^d)}\,|h|^s\,. \tag{51}$$

Now, let $C_\varepsilon := [-\varepsilon,\varepsilon]^d$ be the cube of size ε and let $h\in\mathbb{R}^d\backslash C_\varepsilon$. Further, let $\mathbb{Z}_{\varepsilon,h}^d := \left\{z\in\mathbb{Z}_\varepsilon^d\,:\,(z+C_\varepsilon)\cap(h+C_\varepsilon)\neq\emptyset\right\}$ and for every $z\in\mathbb{Z}_{\varepsilon,h}^d$ let $V(z,h)=|(z+C_\varepsilon)\cap(h+C_\varepsilon)|$. Then we find

$$\begin{aligned}\|\mathcal{R}_\varepsilon^* u-\tau_h\mathcal{R}_\varepsilon^* u\|_{L^p(\mathbb{R}^d)} &\leq \sum_{z\in\mathbb{Z}_{\varepsilon,h}^d} V(z,h)\,\|\mathcal{R}_\varepsilon^* u-\tau_z\mathcal{R}_\varepsilon^* u\|_{L^p(\mathbb{R}^d)} \\ &= \sum_{z\in\mathbb{Z}_{\varepsilon,h}^d} V(z,h)\,\|u-\tau_z u\|_{L^p(\mathbb{Z}_\varepsilon^d)} \\ &\overset{(51)}{\leq} C\sum_{z\in\mathbb{Z}_{\varepsilon,h}^d} V(z,h)\,\|u\|_{W^{s,p}(\mathbb{Z}_\varepsilon^d)}\,|z|^s \\ &\leq C\sum_{z\in\mathbb{Z}_{\varepsilon,h}^d} V(z,h)\,\|u\|_{W^{s,p}(\mathbb{Z}_\varepsilon^d)}\,|2h|^s \\ &\leq C\,\|u\|_{W^{s,p}(\mathbb{Z}_\varepsilon^d)}\,|h|^s\,.\end{aligned}$$

Now, let $h\in C_\varepsilon$. Like above, we obtain

$$\|\mathcal{R}_\varepsilon^* u-\tau_h\mathcal{R}_\varepsilon^* u\|_{L^p(\mathbb{R}^d)} \leq C\sum_{z\in\mathbb{Z}_{\varepsilon,h}^d} V(z,h)\,\|u\|_{W^{s,p}(\mathbb{Z}_\varepsilon^d)}\,|z|^s\,.$$

However, this time we find $V(z,h) \to 0$ uniformly and linearly in $|h| \to 0$. Hence, we have

$$\left\|\mathcal{R}_\varepsilon^* u - \tau_h \mathcal{R}_\varepsilon^* u\right\|_{L^p(\mathbb{R}^d)} \leq C \begin{cases} |h|^s & \text{if } h \in \mathbb{R}^d \backslash C_\varepsilon \\ |h| & \text{if } h \in C_\varepsilon \end{cases} .$$

Since C does not depend on ε, we infer

$$\left\|\mathcal{R}_\varepsilon^* u - \tau_h \mathcal{R}_\varepsilon^* u\right\|_{L^p(\mathbb{R}^d)} \leq C \begin{cases} |h|^s & \text{if } |h| > 1 \\ |h| & \text{if } |h| \leq 1 \end{cases} . \tag{52}$$

This implies (50) in case $p = q$.

In case $q < p$, we use Remark 16 and let $\tilde{\boldsymbol{Q}}$ denote the common support of $\mathcal{E}_\varepsilon u^\varepsilon$. We then obtain by Hölder's inequality

$$\left\|\mathcal{R}_\varepsilon^* \mathcal{E}_\varepsilon u^\varepsilon - \tau_h \mathcal{R}_\varepsilon^* \mathcal{E}_\varepsilon u^\varepsilon\right\|_{L^q(\mathbb{R}^d)} \leq \left|\tilde{\boldsymbol{Q}}\right|^{\frac{p-q}{p}} \left\|\mathcal{R}_\varepsilon^* \mathcal{E}_\varepsilon u^\varepsilon - \tau_h \mathcal{R}_\varepsilon^* \mathcal{E}_\varepsilon u^\varepsilon\right\|_{L^p(\mathbb{R}^d)}^{\frac{q}{p}} ,$$

and hence compactness by (52).

In case $q \in (p, p^\star)$ we use the same trick as in the proof of Theorem 17: we have for $f = u - \tau_h u$ and for $q = \theta p + (1-\theta) p^\star$ that

$$\sum_{x \in Q^\varepsilon}^{\varepsilon} \left|\left(u - \tau_h u\right)(x)\right|^q \leq \left\|u - \tau_h u\right\|_{L^p(Q^\varepsilon)}^{p\theta} \left\|u - \tau_h u\right\|_{L^{p^\star}(Q^\varepsilon)}^{(1-\theta)p^\star} ,$$

and hence (50) follows from Theorem 17 and (52). □

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