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# The point charge oscillator: Qualitative and analytical investigations

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# The point charge oscillator: Qualitative and analytical investigations

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**Abstract**

We determine the global phase portrait of a mathematical model describing the point charge oscillator. It shows that the family of closed orbits describing the point charge oscillations has two envelopes: an equilibrium point and a homoclinic orbit to an equilibrium point at infinity. We derive an expression for the growth rate of the primitive period $T_a$ of the oscillation with the amplitude $a$ as $a$ tends to infinity. Finally, we determine an exact relation between period and amplitude by means of the Jacobi elliptic function cn.

## 1 Introduction

Consider a uniformly charged ring with a conducting wire placed along the axis of the ring. Assume that a point charge $q$ whose sign is opposite to the sign of the charge $Q$ of the ring, is confined with the wire. Under the assumption that the loss of energy per oscillation due to radiation is negligibly small, the oscillations of this point charge can be modeled by the following dimensionless scalar nonlinear autonomous differential equation (see e.g. [2, 5])

$$\frac{d^2x}{dt^2} + \frac{x}{\sqrt{1+x^2}^3} = 0. \tag{1.1}$$

Since (1.1) is a conservative system, the initial value problem

$$x(0) = a,\ \frac{dx}{dt}(0) = 0 \tag{1.2}$$

has to any $a,\ 0 < a < +\infty$, a unique solution $x(t,a)$ defined $\forall t$ which represents a periodic solution with (positive) primitive period $T_a$. The parameter $a$ can be interpreted as the amplitude of the periodic solution $x(t,a)$. In [3] the authors apply the method of harmonic balance to derive analytic approximations for the periodic solution $x(t,a)$ and for the corresponding frequency $\omega_a = 2\pi/T_a$ in the cases of first and second order approximations.
For the first-order harmonic balance approximation $\tilde{x}(t,a) = a\cos\tilde{\omega}_a t$, the authors derive in [1] the relation

$$\tilde{\omega}_a = \frac{2}{a\sqrt{\pi}}\sqrt{K(-a^2) - \frac{E(-a^2)}{1+a^2}},$$

which implies for the corresponding approximate period $\tilde{T}_a$ the relation

$$\tilde{T}_a = \frac{a\pi^{3/2}}{\sqrt{K(-a^2) - \frac{E(-a^2)}{1+a^2}}},$$

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where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively, defined by

$$K(k) := \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k\sin^2\varphi}}, \quad E(k) := \int_0^{\pi/2} \sqrt{1 - k\sin^2\varphi}\, d\varphi. \tag{1.3}$$

Further investigations to derive approximations of the relationship between the amplitude $a$ and the corresponding frequency $\omega_a$ can be found in [6, 7, 8].
The goal of our contribution is to describe the global phase portrait of equation (1.1) and to determine the dependence of the primitive period on the amplitude $a$, especially for large $a$.

## 2 Global phase portrait of the trajectories of equation (1.1)

For the sequel we represent equation (1.1) as the system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -\frac{x}{\sqrt{1+x^2}^3} \end{aligned} \tag{2.1}$$

in the $(x, y)$-phase plane. The following properties of system (2.1) can be easily verified.

**Lemma 2.1.** *The origin is the unique equilibrium point $\mathcal{E}$ of system (2.1) in the finite part of the phase plane.*

**Lemma 2.2.** *System (2.1) has the first integral*

$$H(x, y) := \frac{y^2}{2} - \frac{1}{\sqrt{1+x^2}} = c, \quad c \geq -1. \tag{2.2}$$

From Lemma 2.2 we get

**Corollary 2.3.** *The phase portrait of system (2.1) is symmetric with respect to the $x$-axis as well as with respect to the $y$-axis.*

**Corollary 2.4.** *System (2.1) can be rewritten in the form*

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial y}, \\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x} \end{aligned} \tag{2.3}$$

that is, (2.1) is a Hamiltonian system.

**Lemma 2.5.** *The family of orbits $\{\mathcal{O}_c\}$ of system (2.3) defined by*

$$\{\mathcal{O}_c\} := \{(x, y) \in \mathbb{R}^2 : \quad H(x, y) = c\}$$

*consists for $-1 < c < 0$ of closed orbits located in the finite part of the phase plane. This family of closed orbits has two envelopes, the equilibrium point $\mathcal{E} = \mathcal{O}_{-1}$ and the curve $\mathcal{O}_0$ defined by*

$$\mathcal{O}_0 := \left\{(x, y) \in \mathbb{R}^2 : \quad \frac{y^2}{2} - \frac{1}{\sqrt{1+x^2}} = 0\right\}.$$

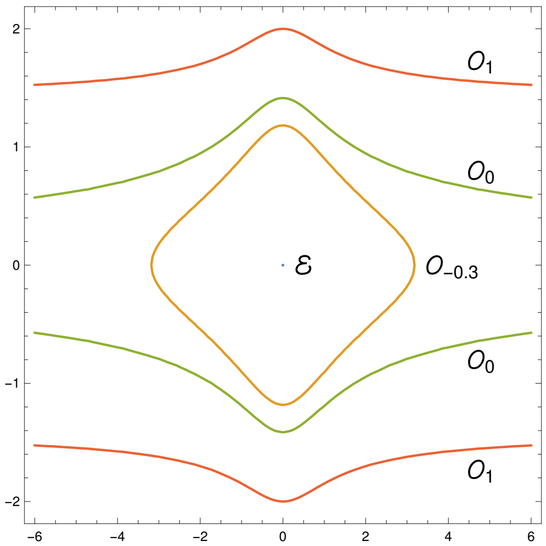


Figure 1: Selected orbits of system (2.1) in a finite part of the phase plane

*For $c > 0$, the family $\{\mathcal{O}_c\}$ consists of the curves*

$$\mathcal{O}_c := \left\{ (x, y) \in \mathbb{R}^2 : \quad y = \pm\sqrt{2c + \frac{2}{\sqrt{1+x^2}}} \right\},$$

*where $|y(x)|$ takes its maximum $\sqrt{2(1+c)}$ at $x = 0$ and satisfies*

$$\lim_{x\to\pm\infty} |y(x)| = \sqrt{2c}$$

*(see Fig. 1).*

Lemma 2.5 implies

**Corollary 2.6.** *The equilibrium point $\mathcal{E}$ of system (2.1) is a center.*

If we consider the velocity $v_c(x, y)$ of a point moving along the closed orbit $\{\mathcal{O}_c\}$ at the point $(x_c, 0)$, then we get from (2.1)

$$v_c^2(x_c, 0) = \frac{x_c^2}{(1 + x_c^2)^3}. \tag{2.4}$$

From (2.2) we obtain

$$x_c^2 = \frac{1 - c^2}{c^2}. \tag{2.5}$$

By (2.4) and (2.5) we have

$$v_c^2(x_c, 0) = c^4(1 - c^2).$$

Therefore, it holds

$$v_c^2(x_c, 0) \to 0 \quad \text{as} \quad c \to 0$$

and we expect that $\mathcal{O}_0$ is a heteroclinic cycle. To give an answer to this conjecture we have to study the equilibria of system (2.1) at infinity. For this purpose we apply transformations mapping the phase plane onto planes tangent to the Poincaré sphere at the equator. By means of the transformation

$$x = \frac{v}{z}, \quad y = \frac{1}{z}$$

we can study the existence of equilibria on the equator located at the “ends” of the $y$-axis. We obtain the system

$$\begin{aligned} \frac{dz}{dt} &= \frac{z^4 v}{\sqrt{z^2+v^2}^3}, \\ \frac{dv}{dt} &= 1 + \frac{v^2 z^3}{\sqrt{z^2+v^2}^3}. \end{aligned} \tag{2.6}$$

From (2.6) we get immediately

**Lemma 2.7.** *There is no equilibrium of system (2.1) at infinity located on the equator of the Poincaré sphere at the "ends" of the $y$-axis.*

Using the transformation

$$x = \frac{1}{z}, \quad y = \frac{u}{z}$$

we are looking for equilibria located on the equator of the Poincaré sphere not located at the “ends” of the $y$-axis. We obtain the system

$$\begin{aligned} \frac{du}{dt} &= -\frac{z^3}{\sqrt{z^2+1}^3} - u^2, \\ \frac{dz}{dt} &= -uz \end{aligned} \tag{2.7}$$

having the unique equilibrium point $u = z = 0$ which is represented in the Poincaré disc as the equilibrium point $\mathcal{E}_{-1}$ (see Fig. 2). We note that this equilibrium point coincides with the equilibrium point $\mathcal{E}_{+1}$. The existence of a unique equilibrium point at infinity implies that the orbit $\mathcal{O}_0$ is a homoclinic orbit and not a heteroclinic orbit as expected.
Taking into account our qualitative results described in Lemma 2.5 we get the following result:

**Theorem 2.8.** *The orbits $\{\mathcal{O}_c\}$ of system (2.3) defined for $c \geq -1$ represent in the Poincaré disc*

- *for $c = -1$ the equilibrium point $\mathcal{E}$ at the origin,*
- *for $-1 < c < 0$ closed orbits with finite period,*
- *for $c \geq 0$ homoclinic orbits to the unique equilibrium point at infinity.*

The corresponding global phase portrait of system (2.1) is represented in Fig. 2.

In the following section we study the closed orbits of the family $\mathcal{O}_c$ which are periodic solutions. Especially, we are interested in the dependence of the primitive period on the amplitude.

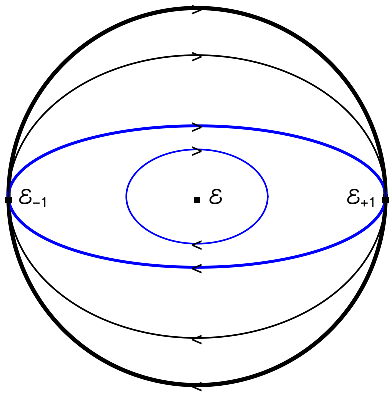


Figure 2: Global phase portrait of system (2.1) in the Poincaré disc

## 3 Periodic orbits and their period

We denote by $\Gamma_a$ the orbit of system (2.1) passing the point $(x = a, y = 0)$. Without loss of generality we may assume $a \geq 0$. $\Gamma_a$ has the representation

$$\Gamma_a := \left\{(x,y) \in \mathbb{R}^2 : \quad \frac{y^2}{2} - \frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+a^2}} = 0\right\}. \tag{3.1}$$

From (3.1) it follows that $\Gamma_a$ is a closed orbit for any $a, 0 < a < +\infty$. Since the parameter $a$ can be interpreted as the amplitude of the closed orbit $\Gamma_a$, we can conclude that the point charge oscillator (2.1) has to any amplitude a unique periodic solution. But from the property that the function $x/\sqrt{1+x^2}^3$ arising in equation (1.1) does not tend to $+\infty$ as $|x|$ tend to $+\infty$, we cannot conclude that the family of closed orbits $\{\Gamma_a\}_{a\geq 0}$ covers the full phase plane.
For the sequel we denote by $T_a$ the (positive) primitive period of $\Gamma_a$. By the symmetry properties of the closed orbit $\Gamma_a$, for the determination of $T_a$ it is sufficient to calculate the time for running along the part of $\Gamma_a$ located in the positive orthant. Using the relation $dt = dx/y$ in (2.1) and the representation of $\Gamma_a$

$$\Gamma_a := \left\{(x,y) \in \mathbb{R}^2 : \quad y = \sqrt{2}\sqrt{\frac{1}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+a^2}}}\right\} \tag{3.2}$$

which follows from (3.1), we get

**Lemma 3.1.**

$$T_a = \frac{4}{\sqrt{2}}\int_0^a \frac{dx}{\sqrt{\frac{1}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+a^2}}}} = 2\sqrt{2}a\int_0^1 \frac{ds}{\sqrt{\frac{1}{\sqrt{1+a^2s^2}} - \frac{1}{\sqrt{1+a^2}}}}. \tag{3.3}$$

Taking into account the relations

$$\int_0^1 \frac{ds}{\sqrt{\frac{1}{\sqrt{1+a^2s^2}} - \frac{1}{\sqrt{1+a^2}}}} = \sqrt{a}\int_0^1 \frac{ds}{\sqrt{\frac{1}{\sqrt{1/a^2+s^2}} - \frac{1}{\sqrt{1/a^2+1}}}}$$

and

$$\lim_{a\to\infty}\int_0^1 \frac{ds}{\sqrt{\frac{1}{\sqrt{1/a^2+s^2}} - \frac{1}{\sqrt{1/a^2+1}}}} = \int_0^1 \sqrt{\frac{s}{1-s}}\,ds = 2$$

we obtain the result:

**Lemma 3.2.** *The primitive period $T_a$ satisfies the relation*

$$T_a = O(a\sqrt{a}) \quad \textit{as} \quad a \to \infty.$$

From Lemma 3.2 we get

**Corollary 3.3.** *The origin is not an isochronous center of system (2.1).*

Another representation of the period can be obtained as follows. We denote by $\Gamma^b$ the trajectory of system (2.1) passing the point $(0, b)$. Without loss of generality we may assume $b \geq 0$. From (2.2) we obtain for $\Gamma^b$ the representation

$$\Gamma^b := \left\{ (x, y) \in \mathbb{R}^2 : \quad \frac{y^2}{2} - \frac{1}{\sqrt{1+x^2}} + 1 - \frac{b^2}{2} = 0 \right\}. \tag{3.4}$$

The expression (3.4) implies that $\Gamma^b$ is a closed orbit as long as $b$ satisfies $0 < b^2 < 2$. Since the parameter $b$ can be interpreted as the maximum velocity on the closed orbit $\Gamma^b$, we can conclude that $\sqrt{2}$ is the maximum velocity on all periodic solutions of (2.1). Taking into account that the closed orbit $\Gamma_a$ intersects the positive $y$-axis at the point $\left(0, \sqrt{2(1 - \frac{1}{\sqrt{1+a^2}})}\right)$, we can conclude that the closed orbits $\Gamma_a$ and $\Gamma^b$ coincide if it holds

$$b = \tilde{b}(a) := \sqrt{2\left(1 - \frac{1}{\sqrt{1+a^2}}\right)} \quad \text{for } a > 0, \tag{3.5}$$

which is equivalent to

$$a = \tilde{a}(b) := \frac{\sqrt{4 - (2 - b^2)^2}}{2 - b^2} \quad \text{for } 1 < b^2 < 2.$$

For the determination of the (positive) primitive period $T^b$ of $\Gamma^b$ we can restrict ourselves to the part of $\Gamma^b$ located in the positive orthant. For its representation we use the relation

$$x = \frac{\sqrt{4 - (y^2 - b^2 + 2)^2}}{y^2 - b^2 + 2} =: f(y, b) \tag{3.6}$$

which follows from (3.4). According to (2.1) and (3.6) we have

$$-\frac{\sqrt{1 + f(y, b)^2}^3}{f(y, b)} dy = dt. \tag{3.7}$$

Taking into account the relation

$$\frac{\sqrt{1 + f(y, b)^2}^3}{f(y, b)} = \frac{8}{(y^2 - b^2 + 2)^2 \sqrt{4 - (y^2 - b^2 + 2)^2}}$$

we get from (3.7)

$$T^b = 32b \int_0^1 \frac{ds}{(y^2 - b^2 + 2)^2 \sqrt{4 - (b^2 s^2 - b^2 + 2)^2}}. \tag{3.8}$$

The advantage of this representation of the period of the closed curves of system (2.1) consists in the fact that the right hand side of (3.8) is an elliptic integral. In the following section we derive an explicit expression for $T^b$ by means of the Jacobi elliptic functions $cn$.

# 4 Analytic relation between amplitude and period of the point charge oscillations by using Jacobi's elliptic function cn

The expression (3.8) for $T^b$ can be rewritten in the form

$$\begin{aligned} T^b &= 32b\int_0^1 \frac{ds}{(b^2s^2-b^2+2)^2\sqrt{4-(b^2s^2-b^2+2)^2}} \\ &= \frac{32}{b^5}\int_0^1 \frac{ds}{(s^2+\frac{2-b^2}{b^2})^2\sqrt{(\frac{4-b^2}{b^2}+s^2)(1-s^2)}}. \end{aligned} \tag{4.1}$$

In [4] we find on page 49 in the relation (213.13) the formula

$$\int_\gamma^\beta \frac{R(s^2)ds}{\sqrt{\alpha^2+s^2)(\beta^2-s^2)}} = g\int_0^{u_1} R(\beta^2 cn^2 u)du, \tag{4.2}$$

where $R$ is any rational function, $cn$ is one of the three Jacobi's elliptic functions $sn, cn$ and $dn$, $\beta$ and $\gamma$ are constants satisfying $\beta > \gamma \geq 0$, $g$ and $u_1$ are defined by the relations

$$g = \frac{1}{\sqrt{\alpha^2+\beta^2}}, \quad u_1 = F(\varphi, k) = \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2 sin^2\theta}}, \tag{4.3}$$

where

$$\varphi = arccos(\gamma/\beta), \quad k^2 = \frac{\beta^2}{\alpha^2+\beta^2}. \tag{4.4}$$

It is clear that the integral in (4.1) is a special case of the integral (4.2). From (4.1) - (4.3) we get

$$\begin{aligned} &\gamma = 0, \quad \beta = 1, \quad \alpha^2 = \frac{4-b^2}{b^2}, \quad R(s^2) = \frac{1}{(s^2+\frac{2-b^2}{b^2})^2}, \\ &g = \frac{b}{\sqrt{4-b^2}}, \quad k = \frac{b}{2}, \quad \varphi = arccos\, 0 = \frac{\pi}{2} \quad u_1 = F(\frac{\pi}{2}, k), \end{aligned} \tag{4.5}$$

where $F(\frac{\pi}{2}, k)$ coincides with the complete elliptic integral of the first kind $K(k)$ introduced in (1.3). From (4.1), (4.2) and (4.5) we obtain the representation

$$T^b = \frac{32}{b^4\sqrt{4-b^2}}\int_0^{K(b/2)} \frac{du}{(cn^2 u + \frac{2-b^2}{b^2})^2}.$$

Using the relation (3.5), we arrive at the result:

**Theorem 4.1.** *To given $a, 0 < a < +\infty$, the closed orbit $\Gamma_a$ with the amplitude $a$ has the primitive period*

$$T_a = \frac{32}{(\tilde{b}(a))^4\sqrt{4-(\tilde{b}(a))^2}}\int_0^{K(\tilde{b}(a)/2)} \frac{du}{\Big(cn^2 u + \frac{2-(\tilde{b}(a))^2}{(\tilde{b}(a))^2}\Big)^2},$$

where the functions $\tilde{b}$ and $K$ are defined in (3.5) and (1.3), respectively.

# 5 Conclusions

The global phase portrait of system (2.1) implies that the closed orbits $\mathcal{O}_c$ describing the point charge oscillations tend to a homoclinic orbit as $c$ tends to $0$. Thus, the period $T_a$ of the corresponding oscillation with the amplitude $a$ grows unboundedly as $a$ tends to $\infty$. By means of analytical investigations we are able to determine the growth rate of $T_a$ as $a$ tends to $\infty$ and to derive an exact relation between the period $T_a$ and the corresponding amplitude $a$ using the Jacobi elliptic function $cn$.

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