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Approximation and optimal control of dissipative solutions to the Ericksen–Leslie system

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Abstract

We analyze the Ericksen–Leslie system equipped with the Oseen–Frank energy in three space dimensions. Recently, the author introduced the concept of dissipative solutions. These solutions show several advantages in comparison to the earlier introduced measure-valued solutions. In this article, we argue that dissipative solutions can be numerically approximated by a relative simple scheme, which fulfills the norm-restriction on the director in every step. We introduce a semi-discrete scheme and derive an approximated version of the relative-energy inequality for solutions of this scheme. Passing to the limit in the semi-discretization, we attain dissipative solutions. Additionally, we introduce an optimal control scheme, show the existence of an optimal control and a possible approximation strategy. We prove that the cost functional is lower semi-continuous with respect to the convergence of this approximation and argue that an optimal control is attained in the case that there exists a solution admitting additional regularity.

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1 Introduction

Solutions to nonlinear partial differential equations are often not explicitly computable. Therefore, numerical schemes are needed to determine a good approximation of the actual solution. Such a numerical approximation should rely on the analysis of the continuum system, in order to resemble its properties and especially converge in some sense to a solution for vanishing discretization parameters. In a recent series of papers, different generalized solution concepts for the *Ericksen–Leslie system* equipped with the *Oseen–Frank energy* were investigated. The phletropa of different solution concepts to the Ericksen–Leslie system ranges from strong [14, 15, 17] over weak [4, 10, 25], and measure-valued [22] to *dissipative solutions* [21].

In the paper at hand, we introduce a semi-discrete scheme approximating dissipative solutions [21] of the Ericksen–Leslie system. Additionally, an *optimal control problem for dissipative solutions* to the system is investigated.

The Ericksen–Leslie equations describe the evolution of a nematic liquid crystal under flow. Nematic liquid crystals are anisotropic fluids. The rod- or disk-like molecules build, or are dispersed in, a fluid and are directionally ordered. This ordering and its direction has a substantial influence on the properties of the material such as light scattering or rheology. This gives rise to many applications, where *liquid crystal displays* are only the most prominent ones. Other important application possibly arise in semiconductor devices, Nanotechnology [19], or light-driven motors [31]. Due to its simplicity and its good agreement with experiments (see [3, Sec. 11.1 page 463]), the Ericksen–Leslie model is one of the most common models to describe nematic liquid crystals [9]. In nematic liquid crystals, the molecules tend to be aligned in a common direction, at least in equilibrium situations. This predominant direction is described by a unit vector, the so-called director, henceforth denoted by $\boldsymbol{d}$. The director can be seen as the local average over the directions of a set of molecules contained in a local volume. The evolution of the flow-field is modeled by a Navier–Stokes-like equation and the alignment of the molecules is modeled by an evolution equation with nonlinear principle part resulting from the *variational derivative of a nonconvex free energy*.

This model has been in the focus of mathematics and physics research for decades [9]. In the physics comunity, it has been observed in analytical [18] as well as experimental studies [13] that the model predicts defects [1] and possibly *exits the nematic phase* by developing a biaxial character, *i.e.*, two predominant directions of the molecules [33]. This gives a hint, why a global existence theory for the full Ericksen–Leslie model was missing in the mathematical literature until very recently. On the one hand, there have been many works proving global existence of weak solutions to simplified systems [25, 26] or the full system [4], but always equipped with the one-constant approximation of the Oseen–Frank energy and approximating the norm restriction via a double-well potential. On the other hand, there are also some local well-posedness results for the full Oseen–Frank energy with a pointwise norm restriction [11], [16].

Recently, the author introduced the concept of measure-valued solutions to the full Ericksen–Leslie system equipped with the Oseen–Frank energy and point-wise norm restriction [22]. This is the *first global solution concept* to this model and moreover a generalization of classical solutions since these measure-valued solutions enjoy the weak-strong (or rather measure-valued-strong) uniqueness [23], *i.e.*, they coincide with a local strong solution emanating from the same initial data, as long as such a strong solution exists. The *expectation of the measure-valued solution* fulfills the so-called dissipative formulation (see [21] for details). To get this formulation, the solution concept is not relaxed in terms of parametrized measures, but the equality is relaxed to an inequality. The dissipative solution concept agrees with the physical modeling since the director was initially modeled as the local average over a set of molecules and is now the average of the measure-valued solution. Indeed, while a measure-valued solution captures every possible effect influencing the liquid crystal (possibly exiting the nematic phase during the evolution), a dissipative solution only captures the *quantity of interest*, the locally averaged direction of the molecules. Therefore, we investigate in the paper at hand the numerical approximation of a dissipative solution via a semi-discrete scheme.

This generalized solution concept of dissipative solutions relies on a *relative energy inequality*, which compares the dissipative solution in a certain way (via the relative energy) with more regular functions solving the equation possibly only approximately (see Definition 3).

The solutions to the introduced approximate scheme solve an associated *approximate relative energy inequality* (see Theorem 11). This results in a new concept of convergence of a numerical scheme: Instead of showing the convergence of solutions to a numerical scheme directly, we prove that the distance between the solution to the numerical scheme and a regular test function, measured in terms of the relative energy is bounded by certain norms of the test functions, the difference of the initial values, and how well the test functions solve the Ericksen–Leslie system. In case that the dissipative solution fulfills additional regularity requirements, which can be expected at least locally in time, the solutions to the numerical scheme converge to the more regular solution of the original system in a stronger sense .

The proposed semi-discrete scheme would also be an appropriate choice for the discretization of a simplified version of the Ericksen–Leslie system (compare to the second approximation scheme in [2]). Due to its anisotropic properties, one can not expect to attain a weak solutions as the limit of the sequence of solutions to the approximate scheme since it dramatically lacks coercivity. But it is still possible to prove the convergence to a dissipative solution. This seems to be an indicator that dissipative solution are a valid concept for anisotropic systems, which often lack coercivity properties on the whole space.

In the second part of the paper, we propose an optimal control scheme. The aim is to use an electromagnetic field to stir the evolution of the liquid crystal to a desired end state. The optimal control problem consists of a convex cost functional and as a constraint the solutions should fulfill the properties of a dissipative solution to the Ericksen–Leslie equations. In a first result, existence of an optimal control to this problem is shown to exist via standard variational arguments. This problem is approximated by problems consisting of the same cost functional with an restricted set of admissible controls and the constraints are approximated as introduced in the first part of this article. The approximation strategy is somehow contrary to the often advocated strategy „first regularize, then optimize“. Following this approach would mean to introduce the regularization and penalization terms as in the proof of the existence of measure-valued solutions (see [22]). But this technique bears several disadvantages. Due to the high-order regularization, a high-order finite element scheme has to be adopted. Additionally, the penalization due to the double-well potential requires a fine handling of the discretization, regularization, and penalization parameters (see also [2]). The question of convergence for vanishing regularization, penalization, and discretization limit and their interchange remains widely open. In contrast to this the proposed scheme does not suffer from this shortcomings, but the sense of convergence of this scheme for vanishing discretization parameter is a very weak one.

We can prove that the sequence of minimizers to the approximate problems is lower semi-continuous and in the case that the optimal solution enjoys additional regularity, the solutions to our scheme converge to an optimal control. The additional regularity requirement on the minimizer is enough to deduce uniqueness of the dissipative solution, such that the asserted convergence makes sense.

In the context of Γ or Moscow convergence, *i.e.*, in the context of lower-semi-continuity and attainability, one can assert that the cost functional is lower-semi continuous with respect to the convergence of the approximate optimal control scheme. But the optimal solution is only known to be attainable under additional assumptions on an optimal solution. Like in the context of solvability of partial differential equations, one can talk about a weak-strong optimal control scheme in comparison to weak-strong uniqueness.

It seems remarkable that even though the control enters the system nonlinearly, weak convergence of the control is enough to go to the limit in the formulation.

Plan of the paper: In Section 1.1, we introduce some notation including elements of tensor calculus required for a concise presentation of the results. In Section 2.1, the full Ericksen–Leslie model is introduced and in Section 2.2 the general Oseen–Frank energy as well as the energy contribution due to electromagnetic effects. Section 3 collects the definition of a dissipative solution and the main result. Section 4 provides the proof of the main result by introducing the semi-discrete scheme and approximate relative energy inequality (see Section 4.1), derive *a priori* estimates, and extract converging subsequences (see Section 4.2), prove the approximate relative energy estimate, and show its convergence to the continuous one (see Section 4.3). In the last section (see Section 5), we introduce the continuous optimal control problem, prove the existence of an optimal control (see Section 5.1), introduce the approximate optimal control problem and show that the cost functional is lower semi-continuous with respect to the convergence of the solutions to the approximate scheme. Under additional regularity assumptions, the solutions to the approximate problems even converge to an optimal solution of the continuous one (see Section 5.2).

1.1 Notation

Vectors of $\mathbb{R}^3$ are denoted by bold small Latin letters. Matrices of $\mathbb{R}^{3\times 3}$ are denoted by bold capital Latin letters. We also use tensors of higher order, which are denoted by bold capital Greek letters. Moreover, numbers are denoted be small Latin or Greek letters, and capital Latin letters are reserved for potentials. The euclidean scalar product in $\mathbb{R}^3$ is denoted by a dot $\boldsymbol{a}\cdot\boldsymbol{b} := \boldsymbol{a}^T\boldsymbol{b} = \sum_{i=1}^3 \boldsymbol{a}_i\boldsymbol{b}_i$, for $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$ and the Frobenius product in $\mathbb{R}^{3\times 3}$ by a double point $\boldsymbol{A}:\boldsymbol{B} := \operatorname{tr}(\boldsymbol{A}^T\boldsymbol{B}) = \sum_{i,j=1}^3 \boldsymbol{A}_{ij}\boldsymbol{B}_{ij}$, for $\boldsymbol{A},\boldsymbol{B}\in\mathbb{R}^{3\times 3}$. Additionally, the scalar product in the space of tensors of order three is denoted by three dots

$$\boldsymbol{\Upsilon} \,\vdots\, \boldsymbol{\Gamma} := \left[\sum_{j,k,l=1}^{3} \boldsymbol{\Upsilon}_{jkl}\boldsymbol{\Gamma}_{jkl}\right], \quad \boldsymbol{\Upsilon}\in\mathbb{R}^{3\times3\times3}, \boldsymbol{\Gamma}\in\mathbb{R}^{3\times3\times3}.$$

The associated norms are all denoted by $|\cdot|$, where also the norms of tensors of higher order are denoted in the same way

$$|\boldsymbol{\Lambda}|^2 := \sum_{i,j,k,l=1}^{3} \boldsymbol{\Lambda}_{ijkl}^2, \quad \text{for } \boldsymbol{\Lambda}\in\mathbb{R}^{3^4} \quad \text{and} \quad |\boldsymbol{\Theta}|^2 := \sum_{i,j,k,l,m,n=1}^{3} \boldsymbol{\Theta}_{ijklmn}^2, \quad \text{for } \boldsymbol{\Theta}\in\mathbb{R}^{3^6},$$

respectively. Similar, we define the products of tensors of different order. The product of a tensor of fourth and third order with a matrix is defined by

$$\boldsymbol{\Lambda}:\boldsymbol{A} := \left[\sum_{k,l=1}^{3} \boldsymbol{\Lambda}_{ijkl}\boldsymbol{A}_{kl}\right]_{i,j=1}^{3}, \boldsymbol{\Gamma}:\boldsymbol{A} := \left[\sum_{j,k=1}^{3} \boldsymbol{\Gamma}_{ijk}\boldsymbol{A}_{jk}\right]_{i=1}^{3}, \boldsymbol{\Lambda}\in\mathbb{R}^{3^4}, \boldsymbol{\Gamma}\in\mathbb{R}^{3^3}, \boldsymbol{A}\in\mathbb{R}^{3\times3}.$$

The product of a tensor of sixth order and a matrix or a tensor of third order is defined via

$$\boldsymbol{A}:\boldsymbol{\Theta} := \left[\sum_{i,j=1}^{3} \boldsymbol{A}_{ij}\boldsymbol{\Theta}_{ijklmn}\right]_{k,l,m,n=1}^{3}, \boldsymbol{\Theta}\,\vdots\,\boldsymbol{\Gamma} := \left[\sum_{l,m,n=1}^{3} \boldsymbol{\Theta}_{ijklmn}\boldsymbol{\Gamma}_{lmn}\right]_{i,j,k=1}^{3}, \boldsymbol{\Theta}\in\mathbb{R}^{3^6}, \boldsymbol{A}\in\mathbb{R}^{3\times3}, \boldsymbol{\Gamma}\in\mathbb{R}^{3^3}.$$

The product of a vector and a tensor of fourth order is defined differently. The definition is adjusted to the cases of this work:

$$\boldsymbol{a}\cdot\boldsymbol{\Theta} := \left[\sum_{k=1}^{3} \boldsymbol{a}_k\boldsymbol{\Theta}_{ijklmn}\right]_{i,j,l,m,n=1}^{3}, \boldsymbol{\Theta}\in\mathbb{R}^{3^6}, \boldsymbol{a}\in\mathbb{R}^3.$$

The standard matrix and matrix-vector multiplication is written without an extra sign for brevity,

$$\boldsymbol{A}\boldsymbol{B} = \left[\sum_{j=1}^{3} \boldsymbol{A}_{ij}\boldsymbol{B}_{jk}\right]_{i,k=1}^{3}, \quad \boldsymbol{A}\boldsymbol{a} = \left[\sum_{j=1}^{3} \boldsymbol{A}_{ij}\boldsymbol{a}_j\right]_{i=1}^{3}, \quad \boldsymbol{A}\in\mathbb{R}^{3\times3}, \boldsymbol{B}\in\mathbb{R}^{3\times3}, \boldsymbol{a}\in\mathbb{R}^3.$$

The outer vector product is given by $\boldsymbol{a}\otimes\boldsymbol{b} := \boldsymbol{a}\boldsymbol{b}^T = [\boldsymbol{a}_i\boldsymbol{b}_j]_{i,j=1}^3$ for two vectors $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$ and by $\boldsymbol{A}\otimes\boldsymbol{a} := \boldsymbol{A}\boldsymbol{a}^T = [\boldsymbol{A}_{ij}\boldsymbol{a}_k]_{i,j,k=1}^3$ for a matrix $\boldsymbol{A}\in\mathbb{R}^{3\times3}$ and a vector $\boldsymbol{a}\in\mathbb{R}^3$. The symmetric and skew-symmetric parts of a matrix are given by $\boldsymbol{A}_{\text{sym}} := \frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T)$ and $\boldsymbol{A}_{\text{skw}} := \frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}^T)$, respectively ($\boldsymbol{A}\in\mathbb{R}^{3\times3}$). For the product of two matrices $\boldsymbol{A},\boldsymbol{B}\in\mathbb{R}^{3\times3}$, we observe

$$\boldsymbol{A}:\boldsymbol{B} = \boldsymbol{A}:\boldsymbol{B}_{\text{sym}}, \quad \text{if } \boldsymbol{A}^T = \boldsymbol{A} \quad \text{and} \quad \boldsymbol{A}:\boldsymbol{B} = \boldsymbol{A}:\boldsymbol{B}_{\text{skw}}, \quad \text{if } \boldsymbol{A}^T = -\boldsymbol{A}.$$

Furthermore, it holds $\boldsymbol{A}^T\boldsymbol{B} : \boldsymbol{C} = \boldsymbol{B} : \boldsymbol{A}\boldsymbol{C}$ for $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C} \in \mathbb{R}^{3\times 3}$ and $\boldsymbol{a}\otimes\boldsymbol{b} : \boldsymbol{A} = \boldsymbol{a}\cdot\boldsymbol{A}\boldsymbol{b}$ for $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^3$, $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ and hence $\boldsymbol{a}\otimes\boldsymbol{a} : \boldsymbol{A} = \boldsymbol{a}\cdot\boldsymbol{A}\boldsymbol{a} = \boldsymbol{a}\cdot\boldsymbol{A}_{\text{sym}}\boldsymbol{a}$.

We use the Nabla symbol ∇ for real-valued functions $f : \mathbb{R}^3 \to \mathbb{R}$ and vector-valued functions $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ denoting

$$\nabla f := \left[\frac{\partial f}{\partial \boldsymbol{x}_i}\right]_{i=1}^3 , \quad \nabla \boldsymbol{f} := \left[\frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{x}_j}\right]_{i,j=1}^3 .$$

The divergence of a vector-valued $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ and a matrix-valued function $\boldsymbol{A} : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ is defined by

$$\nabla\cdot \boldsymbol{f} := \sum_{i=1}^3 \frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{x}_i} = \operatorname{tr}(\nabla \boldsymbol{f}), \quad \nabla\cdot \boldsymbol{A} := \left[\sum_{j=1}^3 \frac{\partial \boldsymbol{A}_{ij}}{\partial \boldsymbol{x}_j}\right]_{i=1}^3 .$$

Additionally, we abbreviate $\nabla\nabla$ by ∇^2. For a given tensor of fourth order, we abbreviate the associated second order operator by $\Delta_{\boldsymbol{\Lambda}}\boldsymbol{d} := \nabla\cdot\boldsymbol{\Lambda} : \nabla\boldsymbol{d}$ acting on functions $\boldsymbol{d} \in \mathscr{C}^2(\Omega;\mathbb{R}^3)$.

Throughout this paper, let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficiently regular boundary $\partial\Omega$. We rely on the usual notation for spaces of continuous functions, Lebesgue and Sobolev spaces. Spaces of vector-valued functions are emphasized by bold letters, for example $\boldsymbol{L}^p(\Omega) := L^p(\Omega;\mathbb{R}^3)$, $\boldsymbol{W}^{k,p}(\Omega) := W^{k,p}(\Omega;\mathbb{R}^3)$. The standard inner product in $L^2(\Omega;\mathbb{R}^3)$ is just denoted by $(\cdot,\cdot)$, in $L^2(\Omega;\mathbb{R}^{3\times 3})$ by $(\cdot;\cdot)$, and in $L^2(\Omega;\mathbb{R}^{3\times 3\times 3})$ by $(\cdot\,\vdots\,\cdot)$. The space of smooth solenoidal functions with compact support in Ω is denoted by $\mathscr{C}^\infty_{c,\sigma}(\Omega;\mathbb{R}^3)$. By $\boldsymbol{L}^p_\sigma(\Omega)$, $\boldsymbol{H}^1_{0,\sigma}(\Omega)$, and $\boldsymbol{W}^{1,p}_{0,\sigma}(\Omega)$, we denote the closure of $\mathscr{C}^\infty_{c,\sigma}(\Omega;\mathbb{R}^3)$ with respect to the norm of $\boldsymbol{L}^p(\Omega)$, $\boldsymbol{H}^1(\Omega)$, and $\boldsymbol{W}^{1,p}(\Omega)$, respectively. The dual space of a Banach space V is always denoted by V^* and equipped with the standard norm; the duality pairing is denoted by $\langle\cdot,\cdot\rangle$. The duality pairing between $\boldsymbol{L}^p(\Omega)$ and $\boldsymbol{L}^q(\Omega)$ (with $1/p+1/q=1$), however, is denoted by $(\cdot,\cdot)$, $(\cdot;\cdot)$, or $(\cdot\,\vdots\,\cdot)$.

The cross product of two vectors is denoted by $\times$. We introduce the notation $[\cdot]_{\boldsymbol{X}}$, which is defined via

$$[\cdot]_{\boldsymbol{X}} : \mathbb{R}^d \to \mathbb{R}^{d\times d}, \quad [\boldsymbol{h}]_{\boldsymbol{X}} := \begin{pmatrix} 0 & -\boldsymbol{h}_3 & \boldsymbol{h}_2 \\ \boldsymbol{h}_3 & 0 & -\boldsymbol{h}_1 \\ -\boldsymbol{h}_2 & \boldsymbol{h}_1 & 0 \end{pmatrix} .$$

The i-th component of the vector $\boldsymbol{h} \in \mathbb{R}^3$ is denoted by $\boldsymbol{h}_i$. The mapping $[\cdot]_{\boldsymbol{X}}$ has some nice properties, for instance

$$[\boldsymbol{a}]_{\boldsymbol{X}}\boldsymbol{b} = \boldsymbol{a}\times\boldsymbol{b}, \quad [\boldsymbol{a}]_{\boldsymbol{X}}^T[\boldsymbol{b}]_{\boldsymbol{X}} = (\boldsymbol{a}\cdot\boldsymbol{b})I - \boldsymbol{b}\otimes\boldsymbol{a},$$

for all $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^3$, where I denotes the identity matrix in $\mathbb{R}^{3\times 3}$ or

$$[\boldsymbol{a}]_{\boldsymbol{X}} : \nabla\boldsymbol{b} = [\boldsymbol{a}]_{\boldsymbol{X}} : (\nabla\boldsymbol{b})_{\text{skw}} = \boldsymbol{a}\cdot\nabla\times\boldsymbol{b}, \quad \nabla\cdot[\boldsymbol{a}]_{\boldsymbol{X}} = -\nabla\times\boldsymbol{a}, \quad \frac{1}{2}[\nabla\times\boldsymbol{a}]_{\boldsymbol{X}} = (\nabla\boldsymbol{a})_{\text{skw}},$$

for all $\boldsymbol{a},\boldsymbol{b} \in \mathscr{C}^1(\overline{\Omega})$. Displaying the cross product by this matrix makes the operation associative.

For a given Banach space V, Bochner–Lebesgue spaces are denoted by $L^p(0,T;V)$. Moreover, $W^{1,p}(0,T;V)$ denotes the Banach space of abstract functions in $L^p(0,T;V)$ whose weak time derivative exists and is again in $L^p(0,T;V)$ (see also Diestel and Uhl [8, Section II.2] or Roubíček [34, Section 1.5] for more details). By $\mathscr{AC}([0,T];V)$, $\mathscr{C}([0,T];V)$, and $\mathscr{C}_w([0,T];V)$, we denote the spaces of abstract functions mapping $[0,T]$ into V that are absolutely continuous, continuous, and continuous with respect to the weak topology in V, respectively. We often omit the time interval $(0,T)$ and the domain Ω and just write, *e.g.*, $L^p(\boldsymbol{W}^{k,p})$ for brevity. Finally, by $c > 0$, we denote a generic positive constant.

2 Model

This section introduces the considered Ericksen–Leslie system and the associated energy.

2.1 Governing equations

Let Ω be a bounded domain with sufficiently regular boundary $\partial\Omega$. We consider the Ericksen–Leslie model as introduced in [22]. The governing equations read as

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p + \nabla\cdot\left(\nabla \boldsymbol{d}^T \frac{\partial F}{\partial \nabla \boldsymbol{d}}(\boldsymbol{d},\nabla \boldsymbol{d})\right) - \nabla\cdot \boldsymbol{T}^L = \boldsymbol{g}, \tag{2.1a}$$

$$\boldsymbol{d}\times\left(\partial_t \boldsymbol{d} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{d} - (\nabla \boldsymbol{v})_{\text{skw}}\boldsymbol{d} + \lambda(\nabla \boldsymbol{v})_{\text{sym}}\boldsymbol{d} + \boldsymbol{q}\right) = 0, \tag{2.1b}$$

$$\nabla\cdot \boldsymbol{v} = 0, \tag{2.1c}$$

$$|\boldsymbol{d}| = 1. \tag{2.1d}$$

The variable $\boldsymbol{v} : \overline{\Omega}\times[0,T]\to\mathbb{R}^3$ denotes the velocity of the fluid, $\boldsymbol{d} : \overline{\Omega}\times[0,T]\to\mathbb{R}^3$ represents the director, *i.e.*, the orientation of the rod-like molecules, and $p : \overline{\Omega}\times[0,T]\to\mathbb{R}$ denotes the pressure. The Helmholtz free energy potential F, which is described rigorously in the next section, is assumed to depend only on the director and its gradient, $F = F(\boldsymbol{d},\nabla\boldsymbol{d})$. The free energy functional $\mathscr{F}$ is defined by

$$\mathscr{F} : \boldsymbol{H}^1 \to \mathbb{R}, \quad \mathscr{F}(\boldsymbol{d}) := \int_\Omega F(\boldsymbol{d},\nabla \boldsymbol{d})\,\mathrm{d}\boldsymbol{x},$$

and $\boldsymbol{q}$ is its variational derivative (see Furihata and Matsuo [12, Section 2.1]),

$$\boldsymbol{q} := \frac{\delta\mathscr{F}}{\delta \boldsymbol{d}}(\boldsymbol{d}) = \frac{\partial F}{\partial \boldsymbol{d}}(\boldsymbol{d},\nabla\boldsymbol{d}) - \nabla\cdot\frac{\partial F}{\partial\nabla\boldsymbol{d}}(\boldsymbol{d},\nabla\boldsymbol{d}). \tag{2.2a}$$

The elastic part of the stress tensor, *i.e.*, $\nabla\boldsymbol{d}^T\partial F/\partial\nabla\boldsymbol{d}$ is named after Ericksen and the dissipative Leslie tensor is given by

$$\begin{aligned}\boldsymbol{T}_1^L = {}& \mu_1(\boldsymbol{d}\cdot(\nabla\boldsymbol{v})_{\text{sym}}\boldsymbol{d})\boldsymbol{d}\otimes\boldsymbol{d} + \mu_4(\nabla\boldsymbol{v})_{\text{sym}} + (\mu_5+\mu_6)\left(\boldsymbol{d}\otimes(\nabla\boldsymbol{v})_{\text{sym}}\boldsymbol{d}\right)_{\text{sym}} \\ & + (\mu_2+\mu_3)\left(\boldsymbol{d}\otimes\boldsymbol{e}\right)_{\text{sym}} + \lambda\left(\boldsymbol{d}\otimes(\nabla\boldsymbol{v})_{\text{sym}}\boldsymbol{d}\right)_{\text{skw}} + (\boldsymbol{d}\otimes\boldsymbol{e})_{\text{skw}},\end{aligned} \tag{2.2b}$$

where

$$\boldsymbol{e} := \partial_t\boldsymbol{d} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{d} - (\nabla\boldsymbol{v})_{\text{skw}}\boldsymbol{d}. \tag{2.2c}$$

We emphasis that Parodi's law is always assumed

$$\lambda = \mu_2 + \mu_3. \tag{2.2d}$$

It follows from Onsager's reciprocal relation and is essential to prove the energy inequality (13). Formally, equation (2.1b) can be taken in the cross product with $\boldsymbol{d}$ itself, this leads to

$$-(I-\boldsymbol{d}\otimes\boldsymbol{d})(\partial_t\boldsymbol{d} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{d} - ((\nabla\boldsymbol{v})_{\text{sym}}\boldsymbol{d} + \lambda(\nabla\boldsymbol{v})_{\text{sym}}\boldsymbol{d} + \boldsymbol{q}) = 0.$$

The norm restriction $|\boldsymbol{d}| = 1$ on the director implies $\partial_t|\boldsymbol{d}|^2 = (\boldsymbol{v}\cdot\nabla)|\boldsymbol{d}|^2 = \boldsymbol{d}\cdot(\nabla\boldsymbol{v})_{\text{skw}}\boldsymbol{d} = 0$ such that $\boldsymbol{e}\cdot\boldsymbol{d} = 0$. Hence, we may infer $\boldsymbol{e} = -(I-\boldsymbol{d}\otimes\boldsymbol{d})(\lambda(\nabla\boldsymbol{v})_{\text{sym}}\boldsymbol{d} + \boldsymbol{q})$. Inserting this into (2.2b), yields

$$\begin{aligned}\boldsymbol{T}^L = {}& (\mu_1+\lambda^2)(\boldsymbol{d}\cdot(\nabla\boldsymbol{v})_{\text{sym}}\boldsymbol{d})\boldsymbol{d}\otimes\boldsymbol{d} + \mu_4(\nabla\boldsymbol{v})_{\text{sym}} + (\mu_5+\mu_6-\lambda^2)\left(\boldsymbol{d}\otimes(\nabla\boldsymbol{v})_{\text{sym}}\boldsymbol{d}\right)_{\text{sym}} \\ & - \lambda(\boldsymbol{d}\otimes(I-\boldsymbol{d}\otimes\boldsymbol{d})\boldsymbol{q})_{\text{sym}} - (\boldsymbol{d}\otimes\boldsymbol{q})_{\text{skw}}.\end{aligned} \tag{2.2e}$$

We choose to work with the formulation (2.2e), which is equivalent to (2.2b), but more suitable for our purposes. Indeed, replacing the time derivative in $\boldsymbol{e}$ allows to write the semi-discrete scheme (see (4.2) below) as an explicit ordinary differential equation. This is essential to deduce existence of solutions to such a problem.

To ensure the dissipative character of the system, we assume that

$$\mu_4 > 0, \quad (\mu_5+\mu_6) - \lambda^2 > 0, \quad \mu_1 + \lambda^2 > 0. \tag{2.2f}$$

Finally, we impose boundary and initial conditions as follows:

$$\boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x}\in\Omega, \qquad \boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{0} \qquad \text{for } (t,\boldsymbol{x})\in[0,T]\times\partial\Omega, \tag{2.3a}$$

$$\boldsymbol{d}(\boldsymbol{x},0) = \boldsymbol{d}_0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x}\in\Omega, \qquad \boldsymbol{d}(\boldsymbol{x},t) = \boldsymbol{d}_1(\boldsymbol{x}) \qquad \text{for } (t,\boldsymbol{x})\in[0,T]\times\partial\Omega. \tag{2.3b}$$

We always assume that $\boldsymbol{d}_1 = \boldsymbol{d}_0$ on $\partial\Omega$, which is a compatibility condition providing regularity.

2.2 The general Oseen–Frank energy and electromagnetic field effects

The *Oseen–Frank* energy is given by (see Leslie [24])

$$F^{OF}(\boldsymbol{d},\nabla\boldsymbol{d}) := \frac{K_1}{2}(\nabla\cdot\boldsymbol{d})^2 + \frac{K_2}{2}(\boldsymbol{d}\cdot\nabla\times\boldsymbol{d})^2 + \frac{K_3}{2}|\boldsymbol{d}\times\nabla\times\boldsymbol{d}|^2\,, \tag{2.4}$$

where $K_1, K_2, K_3 > 0$. This energy can be reformulated using the norm one restriction, to

$$F^{OF}(\boldsymbol{d},\nabla\boldsymbol{d}) := \frac{k_1}{2}(\nabla\cdot\boldsymbol{d})^2 + \frac{k_2}{2}|\nabla\times\boldsymbol{d}|^2 + \frac{k_3}{2}|\boldsymbol{d}|^2(\nabla\cdot\boldsymbol{d})^2 + \frac{k_4}{2}(\boldsymbol{d}\cdot\nabla\times\boldsymbol{d})^2 + \frac{k_5}{2}|\boldsymbol{d}\times\nabla\times\boldsymbol{d}|^2\,, \tag{2.5}$$

where $k_1 = k_3 = K_1/2$, $k_2 = \min\{K_2, K_3\}/2$, $k_4 = K_2 - k_2$, and $k_5 = K_3 - k_2$ are again non-negative constants, with $k_1, k_2 > 0$. We remark that $|\boldsymbol{d}|^2|\nabla\times\boldsymbol{d}|^2 = (\boldsymbol{d}\cdot\nabla\times\boldsymbol{d})^2 + |\boldsymbol{d}\times\nabla\times\boldsymbol{d}|^2$.

To abbreviate, we define the tensor of order 4, $\boldsymbol{\Lambda} \in \mathbb{R}^{3^4}$ and a tensor of order 6, $\boldsymbol{\Theta} \in \mathbb{R}^{3^6}$ via

$$\boldsymbol{\Lambda}_{ijkl} := k_1\boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{kl} + k_2(\boldsymbol{\delta}_{ik}\boldsymbol{\delta}_{jl} - \boldsymbol{\delta}_{il}\boldsymbol{\delta}_{jk})\,, \tag{2.6}$$

and

$$\begin{aligned}\boldsymbol{\Theta}_{ijklmn} := {}& k_3\boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{lm}\boldsymbol{\delta}_{kn} + k_5\left(\boldsymbol{\delta}_{il}\boldsymbol{\delta}_{mn}\boldsymbol{\delta}_{jk} - \boldsymbol{\delta}_{mi}\boldsymbol{\delta}_{ln}\boldsymbol{\delta}_{jk} - \boldsymbol{\delta}_{lj}\boldsymbol{\delta}_{mn}\boldsymbol{\delta}_{ik} + \boldsymbol{\delta}_{jm}\boldsymbol{\delta}_{ln}\boldsymbol{\delta}_{ik}\right)\\ &+ k_4\left(\boldsymbol{\delta}_{kn}\boldsymbol{\delta}_{jm}\boldsymbol{\delta}_{il} + \boldsymbol{\delta}_{km}\boldsymbol{\delta}_{jl}\boldsymbol{\delta}_{in} + \boldsymbol{\delta}_{kl}\boldsymbol{\delta}_{jn}\boldsymbol{\delta}_{im} - \boldsymbol{\delta}_{kn}\boldsymbol{\delta}_{jl}\boldsymbol{\delta}_{im} - \boldsymbol{\delta}_{km}\boldsymbol{\delta}_{jn}\boldsymbol{\delta}_{il} - \boldsymbol{\delta}_{kl}\boldsymbol{\delta}_{jm}\boldsymbol{\delta}_{in}\right)\,,\end{aligned} \tag{2.7}$$

respectively. The free energy can be written as

$$2F^{OF}(\boldsymbol{d},\nabla\boldsymbol{d}) = \nabla\boldsymbol{d} : \boldsymbol{\Lambda} : \nabla\boldsymbol{d} + \nabla\boldsymbol{d}\otimes\boldsymbol{d} \,\vdots\, \boldsymbol{\Theta} \,\vdots\, \nabla\boldsymbol{d}\otimes\boldsymbol{d}\,.$$

The Tensor $\boldsymbol{\Lambda}$ is strongly elliptic, *i.e.* there is an $\eta > 0$ such that $\boldsymbol{a}\otimes\boldsymbol{b} : \boldsymbol{\Lambda} : \boldsymbol{a}\otimes\boldsymbol{b} \geq \eta|\boldsymbol{a}|^2|\boldsymbol{b}|^2$ for all $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^3$. Indeed, it holds

$$\boldsymbol{a}\otimes\boldsymbol{b} : \boldsymbol{\Lambda} : \boldsymbol{a}\otimes\boldsymbol{b} = k_1(\boldsymbol{a}\cdot\boldsymbol{b})^2 + k_2(|\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2) \geq \min\{k_1,k_2\}|\boldsymbol{a}|^2|\boldsymbol{b}|^2\,.$$

The second order differential operator $\Delta_{\boldsymbol{\Lambda}}$ introduced by a strongly elliptic tensor is coercive on $\boldsymbol{H}_0^1$, *i.e.*, there exists a constant $k > 0$ such that

$$k\|\nabla\boldsymbol{d}\|_{\boldsymbol{L}^2}^2 \leq \frac{1}{2}(\nabla\boldsymbol{d};\boldsymbol{\Lambda}:\nabla\boldsymbol{d}) \tag{2.8}$$

holds for all $\boldsymbol{d} \in \boldsymbol{H}_0^1$ (see [5, Proposition 13.1]).

Remark 1. *In comparisson to the proof of existence of measure-valued solutions (see [22]), it is sufficient to assume that k_3, k_4, $k_5 \geq 0$. The strict inequality is not necessary for the proof of dissipative solutions. This seems to be an artificial generalization considering the reformulation of* (2.4) *to* (2.5)*, but it relies on the fact that the director can be bounded in the L^∞-norm for the proposed scheme (see* (4.2) *and Proposition 12).*

In the sequel of this article, we want to investigate how a dissipative solution can be controlled by means of an electromagnetic field. This seems to be a good way to control the evolution of a nematic liquid crystal. At least in the case of the famous application in liquid crystal displays, the material is controlled in this way.

The model will be extended by an electromagnetic field influencing the dynamics of the liquid crystal. Therefore, the model is adapted by adding an electromagnetic potential to the free energy. The adapted free energy potential for a magnetic field $\boldsymbol{H}$ is given by [7, Section 3.2]

$$F_{\boldsymbol{H}}(\boldsymbol{d},\nabla\boldsymbol{d},\boldsymbol{H}) = F^{OF}(\boldsymbol{d},\nabla\boldsymbol{d}) - \frac{\chi_\parallel}{2}(\boldsymbol{d}\cdot\boldsymbol{H})^2 - \frac{\chi_\perp}{2}|\boldsymbol{d}\times\boldsymbol{H}|^2\,, \tag{2.9}$$

where the free energy potential F^{OF} is given in (2.5). The associated variational derivative is given via the definition (2.2a) by (see [20])

$$\begin{aligned} \boldsymbol{q} = & -k_1\nabla\nabla\cdot\boldsymbol{d}+k_2\nabla\times\nabla\times\boldsymbol{d}-k_3\nabla(\nabla\cdot\boldsymbol{d}|\boldsymbol{d}|^2)-k_4\nabla\cdot([\boldsymbol{d}]_{\boldsymbol{X}}(\boldsymbol{d}\cdot\nabla\times\boldsymbol{d}))-4k_5\nabla\cdot((\nabla\boldsymbol{d})_{\text{skw}}\boldsymbol{d}\otimes\boldsymbol{d})_{\text{skw}} \\ & +k_3(\nabla\cdot\boldsymbol{d})^2\boldsymbol{d}+k_4(\boldsymbol{d}\cdot\nabla\times\boldsymbol{d})\nabla\times\boldsymbol{d}+4k_5(\nabla\boldsymbol{d})_{\text{skw}}^T(\nabla\boldsymbol{d})_{\text{skw}}\boldsymbol{d}-\chi_{\|}(\boldsymbol{d}\cdot\boldsymbol{H})\boldsymbol{H}+\chi_{\perp}\boldsymbol{H}\times(\boldsymbol{H}\times\boldsymbol{d}) \\ = & -\Delta_{\boldsymbol{\Lambda}}\boldsymbol{d}-\nabla\cdot(\boldsymbol{d}\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\boldsymbol{d}\otimes\boldsymbol{d})+\nabla\boldsymbol{d}:\boldsymbol{\Theta}\,\vdots\,\nabla\boldsymbol{d}\otimes\boldsymbol{d}-\chi_{\|}(\boldsymbol{d}\cdot\boldsymbol{H})\boldsymbol{H}+\chi_{\perp}\boldsymbol{H}\times(\boldsymbol{H}\times\boldsymbol{d})\,. \end{aligned} \tag{2.10}$$

Here $\chi_{\|}$ and $\chi_{\perp}$ denote the constants measuring the magnetic susceptibility parallel and orthogonal to the director, both constants are negative $\chi_{\|},\chi_{\perp}<0$ due to the diamagnetic properties of liquid crystal (compare [7, Section 3.2]). In usual nematic liquid crystals, it holds that $\chi_{\|}>\chi_{\perp}$, such that $|\chi_{\|}|<|\chi_{\perp}|$ which agrees with the naive perception since molecules that are not aligned should experience a bigger force. Using the calculation rules in Section 1.1, we observe for $\boldsymbol{d}$ with $|\boldsymbol{d}|=1$ that

$$-\frac{\chi_{\|}}{2}(\boldsymbol{d}\cdot\boldsymbol{H})^2-\frac{\chi_{\perp}}{2}|\boldsymbol{d}\times\boldsymbol{H}|^2=-\frac{\chi_{\|}}{2}(\boldsymbol{d}\cdot\boldsymbol{H})^2-\frac{\chi_{\perp}}{2}(|\boldsymbol{d}|^2|\boldsymbol{H}|^2-(\boldsymbol{d}\cdot\boldsymbol{H})^2)=-\frac{\chi_{\|}-\chi_{\perp}}{2}(\boldsymbol{d}\cdot\boldsymbol{H})^2-\frac{\chi_{\perp}}{2}|\boldsymbol{H}|^2\,.$$

Since $\boldsymbol{H}$ is given and $\chi_{\|}-\chi_{\perp}>0$, this energy part is minimized if $\boldsymbol{d}$ is parallel to $\boldsymbol{H}$. Clearly, the molecules are then aligned along the direction of the magnetic field $\boldsymbol{H}$.

Remark 2. *The influence of an electric field can be modeled similar (see [7, Section 3.3.1]). For a static electric field $\boldsymbol{E}$, the free energy potential is adapted via*

$$F_{\boldsymbol{H},\boldsymbol{E}}(\boldsymbol{d},\nabla\boldsymbol{d},\boldsymbol{H},\boldsymbol{E})=F(\boldsymbol{d},\nabla\boldsymbol{d})-\frac{\chi_{\|}}{2}(\boldsymbol{d}\cdot\boldsymbol{H})^2-\frac{\chi_{\perp}}{2}|\boldsymbol{d}\times\boldsymbol{H}|^2-\frac{\varepsilon_{\|}}{8\pi}(\boldsymbol{d}\cdot\boldsymbol{E})^2-\frac{\varepsilon_{\perp}}{8\pi}|\boldsymbol{d}\times\boldsymbol{E}|^2\,,$$

where $\varepsilon_{\|}$ and $\varepsilon_{\perp}$ are the static dielectric constants measured along and perpendicular to the director, respectively. Mathematically, such an influence of an electric field can be handled similar to the magnetic field and the calculations below (see also [21, Remark 5.3 and Remark 5.5]). Therefore, we only consider the influence of a magnetic field in this article.

The magnetic field is assumed to be static. If it is time-dependent, its evolution should be determined by Maxwell's equations. This would heavily impede the mathematical theory, so this additional difficulty is left for future work. Nevertheless, the magnetic field should fulfill the standard source-free assumption $\nabla\cdot\boldsymbol{H}=0$.

3 Dissipative solvability concept and main result

This section is devoted to the introduction of dissipative solutions and the assertion of the main result.

3.1 Relative energy and dissipative solutions

The concept of dissipative solutions heavily relies on the formulation of an appropriate relative energy for the Oseen–Frank energy. This relative energy serves as a natural comparing tool for two different solutions $(\boldsymbol{v},\boldsymbol{d})$ and $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})$. The relative energy is defined by

$$\begin{aligned} \mathscr{E}(\boldsymbol{v},\boldsymbol{d},\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}}):=&\frac{1}{2}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{\boldsymbol{L}^2}^2+\frac{1}{2}\left(\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}};\boldsymbol{\Lambda}:(\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}})\right) \\ &+\frac{1}{2}\Big(\nabla\boldsymbol{d}\otimes\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\,\vdots\,\boldsymbol{\Theta}\,\vdots\,(\nabla\boldsymbol{d}\otimes\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}})\Big) \\ &-\frac{\chi_{\|}}{2}\|\boldsymbol{d}\cdot\boldsymbol{H}-\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2-\frac{\chi_{\perp}}{2}\|\boldsymbol{d}\times\boldsymbol{H}-\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2 \end{aligned} \tag{3.1}$$

and the relative dissipation by

$$\begin{aligned} \mathscr{W}(\boldsymbol{v},\boldsymbol{d}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}):=&(\mu_1+\lambda^2)\|\boldsymbol{d}\cdot(\nabla\boldsymbol{v})_{\text{sym}}\boldsymbol{d}-\tilde{\boldsymbol{d}}\cdot(\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^2}^2+\mu_4\|(\nabla\boldsymbol{v})_{\text{sym}}-(\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\|_{\boldsymbol{L}^2}^2 \\ &+(\mu_5+\mu_6-\lambda^2))\|(\nabla\boldsymbol{v})_{\text{sym}}\boldsymbol{d}-(\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^2}^2+\|\boldsymbol{d}\times\boldsymbol{q}-\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}\|_{\boldsymbol{L}^2}^2\,. \end{aligned} \tag{3.2}$$

Note that the relative dissipation intrinsically depends on $\boldsymbol{H}$ and $\tilde{\boldsymbol{H}}$ too. The definition of the variational derivatives $\boldsymbol{q}$ and $\tilde{\boldsymbol{q}}$ (see (2.10)) inherit the dependence on $\boldsymbol{H}$ and $\tilde{\boldsymbol{H}}$, respectively. The variational derivative $\tilde{\boldsymbol{q}}$ is defined by (2.10), where $\boldsymbol{d}$ and $\boldsymbol{H}$ are replaced by $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{H}}$, respectively. Additionally, we remark that $\chi_\parallel, \chi_\perp < 0$ such that all terms in (3.1) are positive. The same holds true for the terms in (3.2).

Inserting the definitions of the tensors $\boldsymbol{\Lambda}$ and $\boldsymbol{\Theta}$, the relative energy can be expressed as

$$\begin{aligned}\mathscr{E}(\boldsymbol{v},\boldsymbol{d},\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}}) = &+\frac{k_1}{2}\|\nabla\cdot\boldsymbol{d}-\nabla\cdot\tilde{\boldsymbol{d}}\|_{L^2}^2 + k_2\|(\nabla\boldsymbol{d})_{\text{skw}}-(\nabla\tilde{\boldsymbol{d}})_{\text{skw}}\|_{\boldsymbol{L}^2}^2 + \frac{k_3}{2}\|(\nabla\cdot\boldsymbol{d})\boldsymbol{d}-(\nabla\cdot\tilde{\boldsymbol{d}})\tilde{\boldsymbol{d}}\|_{L^2}^2\\ &+\frac{k_4}{2}\|\boldsymbol{d}\cdot\nabla\times\boldsymbol{d}-\tilde{\boldsymbol{d}}\cdot\nabla\times\tilde{\boldsymbol{d}}\|_{L^2}^2 + 2k_5\|(\nabla\boldsymbol{d})_{\text{skw}}\boldsymbol{d}-(\nabla\tilde{\boldsymbol{d}})_{\text{skw}}\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^2}^2 + \frac{1}{2}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{\boldsymbol{L}^2}^2\\ &-\frac{\chi_\parallel}{2}\|\boldsymbol{d}\cdot\boldsymbol{H}-\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}}\|_{L^2}^2 - \frac{\chi_\perp}{2}\|\boldsymbol{d}\times\boldsymbol{H}-\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2 .\end{aligned}$$

We always assume that $(\boldsymbol{v},\boldsymbol{d})$ and $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})$ fulfill the regularity requirements (3.3) below such that (3.1) and (3.2) are well defined. In the following $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})$ are even assumed to fulfill (3.5) below.

Definition 3 (Dissipative solution). *Let $\boldsymbol{H}\in\boldsymbol{L}^3$. The triple $(\boldsymbol{v},\boldsymbol{d},\boldsymbol{q})$ consisting of the velocity field $\boldsymbol{v}$, the director field $\boldsymbol{d}$ and the variational derivative $\boldsymbol{q}$ is said to be a dissipative solution to* (2.1) *with magnetic field $\boldsymbol{H}$ if*

$$\boldsymbol{v}\in\mathscr{C}_w(0,T;\boldsymbol{L}^2_\sigma)\cap L^2(0,T;\boldsymbol{H}^1_{0,\sigma})\cap W^{1,2}(0,T;(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*), \tag{3.3a}$$

$$\boldsymbol{d}\in\mathscr{C}_w(0,T;\boldsymbol{H}^1)\cap W^{1,2}(0,T;\boldsymbol{L}^{3/2}) \text{ with } |\boldsymbol{d}(\boldsymbol{x},t)|=1 \text{ a. e. in } \Omega\times(0,T), \tag{3.3b}$$

$$\boldsymbol{d}\times\boldsymbol{q}\in L^2(0,T;\boldsymbol{L}^2) \tag{3.3c}$$

and if

$$\begin{aligned}&\frac{1}{2}\mathscr{E}(\boldsymbol{v}(t),\boldsymbol{d}(t),\boldsymbol{H}|\tilde{\boldsymbol{v}}(t),\tilde{\boldsymbol{d}}(t),\tilde{\boldsymbol{H}}) + \|\boldsymbol{H}-\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^3}^2 + \frac{1}{2}\int_0^t \mathscr{W}(\boldsymbol{v}(s),\boldsymbol{d}(s)|\tilde{\boldsymbol{v}}(s),\tilde{\boldsymbol{d}}(s))\exp\left(\int_s^t\mathscr{K}(\tau)\,d\tau\right)ds\\ &\le \mathscr{D}_0(\boldsymbol{v}(0),\boldsymbol{d}(0),\boldsymbol{H}|\tilde{\boldsymbol{v}}(0),\tilde{\boldsymbol{d}}(0),\tilde{\boldsymbol{H}})\exp\left(\int_0^t\mathscr{K}(s)\,ds\right)\\ &\quad+\int_0^t\left(\mathscr{A}(\tilde{\boldsymbol{v}}(s),\tilde{\boldsymbol{d}}(s)),\begin{pmatrix}\tilde{\boldsymbol{v}}(s)-\boldsymbol{v}(s)\\ \boldsymbol{d}(s)\times(\tilde{\boldsymbol{q}}(s)-\boldsymbol{q}(s)+\boldsymbol{a}(\boldsymbol{d}(s),\boldsymbol{H}|\tilde{\boldsymbol{d}}(s),\tilde{\boldsymbol{H}}))\end{pmatrix}\right)\exp\left(\int_s^t\mathscr{K}(\tau)\,d\tau\right)ds\end{aligned} \tag{3.4}$$

for all test functions $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})$ with

$$\begin{aligned}&\tilde{\boldsymbol{v}}\in L^\infty(0,T;\boldsymbol{L}^2_\sigma)\cap L^2(0,T;\boldsymbol{L}^\infty)\cap L^2(0,T;\boldsymbol{W}^{1,3}_{0,\sigma})\cap W^{1,2}(0,T;(\boldsymbol{H}^1_{0,\sigma})^*),\\ &\tilde{\boldsymbol{d}}\in L^\infty(0,T;\boldsymbol{W}^{1,3}\cap\boldsymbol{L}^\infty)\cap L^2(0,T;\boldsymbol{W}^{2,3})\cap L^4(0,T;\boldsymbol{W}^{1,6})\cap W^{1,1}(0,T;\boldsymbol{W}^{1,3}\cap\boldsymbol{L}^\infty)\cap W^{1,2}(0,T;\boldsymbol{L}^3)\end{aligned} \tag{3.5}$$

and $|\tilde{\boldsymbol{d}}|=1$ a.e. in $\Omega\times(0,T)$, $\operatorname{tr}(\tilde{\boldsymbol{d}})=\boldsymbol{d}_1$, $\tilde{\boldsymbol{H}}\in\boldsymbol{L}^\infty$, $\tilde{\boldsymbol{q}}$ given by (2.10) *with $\boldsymbol{d}$ and $\boldsymbol{H}$ replaced by $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{H}}$, respectively, as well as*

$$\int_0^T\left(\boldsymbol{d}(t)\times\left(\partial_t\boldsymbol{d}(t)+(\boldsymbol{v}(t)\cdot\nabla)\boldsymbol{d}(t)-(\nabla\boldsymbol{v}(t))_{\text{skw}}\boldsymbol{d}(t)+\lambda(\nabla\boldsymbol{v}(t))_{\text{sym}}\boldsymbol{d}(t)+\boldsymbol{q}(t)\right),\boldsymbol{\zeta}(t)\right)dt=0 \tag{3.6}$$

for $\boldsymbol{\zeta}\in L^2(0,T;\boldsymbol{L}^3)$ with $|\boldsymbol{d}(\boldsymbol{x},t)|=1$ a.e. in $\Omega\times(0,T)$, and an estimate for the time derivative of $\boldsymbol{v}$, i.e.,

$$\|\partial_t\boldsymbol{v}\|_{L^2((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*)}\le c\left(\|\boldsymbol{g}\|_{L^2((\boldsymbol{H}^1_{0,\sigma})^*)},\|\boldsymbol{v}_0\|_{\boldsymbol{L}^2_\sigma},\|\boldsymbol{d}_0\|_{\boldsymbol{H}^1},\|\boldsymbol{d}_0\|_{\boldsymbol{L}^\infty},\|\boldsymbol{d}_1\|_{\boldsymbol{H}^{3/2}(\partial\Omega)}\right), \tag{3.7}$$

where c is a constant depending on the right-hand side $\boldsymbol{g}$ and the initial and boundary values.

The potential $\mathscr{K}$ is given by

$$\mathscr{K}(s)=C\left(\|\tilde{\boldsymbol{v}}(s)\|_{\boldsymbol{L}^\infty}^2+\|\nabla\tilde{\boldsymbol{v}}(s)\|_{\boldsymbol{L}^3}^2+\|\tilde{\boldsymbol{q}}\|_{\boldsymbol{L}^3}^2+\|\partial_t\tilde{\boldsymbol{d}}(s)\|_{\boldsymbol{L}^\infty}+\|\partial_t\tilde{\boldsymbol{d}}(s)\|_{\boldsymbol{W}^{1,3}}+\|\partial_t\tilde{\boldsymbol{d}}(s)\|_{\boldsymbol{L}^3}^2\right), \tag{3.8}$$

where C is a possible large constant depending on the norms $\|\boldsymbol{d}\|_{L^\infty(\boldsymbol{L}^\infty)}$, $\|\tilde{\boldsymbol{d}}\|_{L^\infty(\boldsymbol{L}^\infty)}$, which are just 1 due to (3.3) *and* (3.5)*. Additionally, the constant C depends on $\|\nabla\tilde{\boldsymbol{d}}\|_{L^\infty(\boldsymbol{L}^3)}$, $\|\boldsymbol{H}\|_{\boldsymbol{L}^3}$, and $\|\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^\infty}$. It is obvious that $\mathscr{K}$ is bounded in $L^1(0,T)$ due to the regularity assumptions* (3.5)*. The potential $\mathscr{K}$ can be seen as a measure for the regularity of the test functions $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})$. The potential $\mathscr{D}_0$ measures the distance in the initial point in an appropriate way, it is given by*

$$\mathscr{D}_0(\boldsymbol{v},\boldsymbol{d},\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}}) = \mathscr{E}(\boldsymbol{v},\boldsymbol{d},\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}}) + \frac{1}{2k}\left\|(\boldsymbol{d}-\tilde{\boldsymbol{d}})\cdot\boldsymbol{\Theta} \vdots \nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\right\|_{\boldsymbol{L}^2}^2 + \left((\nabla\boldsymbol{d}-\nabla\tilde{\boldsymbol{d}})\otimes(\boldsymbol{d}-\tilde{\boldsymbol{d}}) \vdots \boldsymbol{\Theta} \vdots \nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\right) \tag{3.9}$$

The operator $\mathscr{A}$ incorporates the classical formulation (2.1) *evaluated at the test functions $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})$ and, thus measures how well the test function approximates a strong solution to* (2.1)*. It is given by*

$$\mathscr{A}(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}) = \begin{pmatrix} \partial_t\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}} + \nabla\cdot(\nabla\boldsymbol{d}^T\frac{\partial F}{\partial\nabla\boldsymbol{d}}(\tilde{\boldsymbol{d}},\nabla\tilde{\boldsymbol{d}})) - \nabla\cdot\tilde{\boldsymbol{T}}^L - \boldsymbol{g} \\ \tilde{\boldsymbol{d}}\times\left(\partial_t\tilde{\boldsymbol{d}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}} - (\nabla\tilde{\boldsymbol{d}})_{\text{skw}}\tilde{\boldsymbol{d}} + \lambda(\nabla\tilde{\boldsymbol{d}})_{\text{sym}}\tilde{\boldsymbol{d}} + \tilde{\boldsymbol{q}}\right) \end{pmatrix} \tag{3.10}$$

and $\boldsymbol{a}(\boldsymbol{d},\boldsymbol{H}|\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})$ is given by

$$\boldsymbol{a}(\boldsymbol{d},\boldsymbol{H}|\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}}) := \frac{1}{k}\left((\tilde{\boldsymbol{d}}-\boldsymbol{d})\cdot\boldsymbol{\Theta} \vdots \nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\right) : \boldsymbol{\Theta} \vdots \nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}} + \chi_\parallel(\tilde{\boldsymbol{H}}-\boldsymbol{H})(\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}}) - \chi_\perp(\tilde{\boldsymbol{H}}-\boldsymbol{H})\times(\tilde{\boldsymbol{H}}\times\tilde{\boldsymbol{d}}). \tag{3.11}$$

Remark 4. *Note that the definition on $\mathscr{K}$ differs from the one in [21]. More precisely, the norms $\|\nabla\tilde{\boldsymbol{v}}\|_{L^1(\boldsymbol{L}^\infty)}$ and $\|\nabla\tilde{\boldsymbol{d}}\|_{L^1(\boldsymbol{L}^\infty)}$ are missing. Consequently, we need to assume less regularity for the test functions (compare* (3.5)*) and the solution concept becomes stronger in the sense that less regularity is needed to get uniqueness.*

3.2 Main result

Theorem 5. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with sufficiently regular boundary $\partial\Omega$ and let the assumptions* (2.2d), (2.2f), (2.5)*, and* (2.9) *be fulfilled. For every $\boldsymbol{v}_0\in\boldsymbol{L}^2_\sigma$, $\boldsymbol{d}_0\in\boldsymbol{H}^1$ with $|\boldsymbol{d}_o|=1$ a.e. in $\Omega\times(0,T)$ and $\text{tr}(\boldsymbol{d}_0)=\boldsymbol{d}_1$ with $\boldsymbol{d}_1\in\boldsymbol{H}^{s-1/2}(\partial\Omega)$ with $s\in[5/2,3]$ as well as $\boldsymbol{g}\in L^2(0,T;(\boldsymbol{H}^1_{0,\sigma})^*)$ and $\boldsymbol{H}\in\boldsymbol{L}^3$, there exists a dissipative solution in the sense of Definition 3.*

Remark 6. *In contrast to the proof in [21], the existence proof in this article is not relying on the existence of measure valued solutions and therewith a regularization and penalization technique. The result is proven via the convergence of a semi-discrete scheme and thus appropriate for a decent numerical approximation. The boundary condition $\boldsymbol{d}_1$ is chosen regular enough such that $\mathbb{E}\boldsymbol{d}_1\in\boldsymbol{H}^s(\Omega)$ with $s\in(5/2,3]$, which grants that $\boldsymbol{H}^s(\Omega)\hookrightarrow\boldsymbol{W}^{2,3}(\Omega)$. This implies that $\tilde{\boldsymbol{d}}=\mathbb{E}\boldsymbol{d}_1$ is a possible test function fulfilling* (3.5) *and the regularity assumptions on the test functions actually make sense.*

Remark 7 (Young measure interpretation)**.** *The variational derivative $\boldsymbol{q}$ can be identified in a measure-valued sense (see [22] and [21]). There exists a generalized Gradient Young measure*

$$\begin{aligned} \{\nu^o_{(\boldsymbol{x},t)}\} &\subset \mathscr{P}(\mathbb{R}^{3\times3}),\ \text{a. e. in } \Omega\times(0,T),\\ \{m_t\} &\subset \mathscr{M}^+(\overline{\Omega}),\ \text{a. e. in } (0,T),\\ \{\nu^\infty_{(\boldsymbol{x},t)}\} &\subset \mathscr{P}(\overline{B}_3\times\mathscr{S}^{3^2-1}),\ m_t\text{-a. e. in } \overline{\Omega} \text{ and a. e. in } (0,T), \end{aligned}$$

and a classical Young measure $\{\mu_{(\boldsymbol{x},t)}\} \subset \mathscr{P}(\mathbb{R}^3)$ *, a. e. in* $\Omega \times (0,T)$ *, such that*

$$\int_0^T \langle\!\langle \nu_t, \left(\boldsymbol{\Upsilon} : \left(\boldsymbol{S}(\boldsymbol{\Lambda} : \boldsymbol{S} + \boldsymbol{h} \cdot \boldsymbol{\Theta} \vdots \boldsymbol{S} \otimes \boldsymbol{h})^T\right)\right) \cdot \boldsymbol{\psi}(t) \rangle\!\rangle \, dt + \int_0^T \langle\!\langle \nu_t, (\boldsymbol{h} \times (\boldsymbol{h} \cdot \boldsymbol{\Theta} \vdots \boldsymbol{S} \otimes \boldsymbol{h})) \cdot \boldsymbol{\psi}(t) \rangle\!\rangle \, dt$$
$$+ \int_0^T ([\boldsymbol{d}(t)]_{\boldsymbol{X}} (\boldsymbol{\Lambda} : \nabla \boldsymbol{d}(t) + \boldsymbol{d}(t) \cdot \boldsymbol{\Theta} \vdots \nabla \boldsymbol{d}(t) \otimes \boldsymbol{d}(t)) ; \nabla \boldsymbol{\psi}(t)) \, dt$$
$$+ \int_0^t \left(\langle \mu_t, \boldsymbol{d}(t) \times \left(\chi_{\parallel} \boldsymbol{H}(\boldsymbol{d}(t) \cdot \boldsymbol{H}) - \chi_{\perp} \boldsymbol{H} \times (\boldsymbol{H} \times \boldsymbol{d}(t)) \right) \rangle \right) ds = \int_0^T (\boldsymbol{d}(t) \times \boldsymbol{q}(t), \boldsymbol{\psi}(t)) \, dt$$

for all $\boldsymbol{\psi} \in \mathscr{C}_c^\infty(\Omega \times (0,T))$*. Note that the measure* m_t *is mutually singular in* $\overline{\Omega}$*, i.e., it is supported on a set of Lebesgue measure zero. The tensor* $\boldsymbol{\Upsilon} \in \mathbb{R}^{3\times3\times3}$ *is the Levi–Civita tensor defined in [21, Section 1.1]. The dual pairings are defined as*

$$\langle \mu_{(\boldsymbol{x},t)}, f(\boldsymbol{x},t) \rangle := \int_{\mathbb{R}^3} f(\boldsymbol{x},t,\boldsymbol{H}) \mu_{(\boldsymbol{x},t)}(d\boldsymbol{H})$$

and

$$\langle\!\langle \nu_t, f(\boldsymbol{h},\boldsymbol{S}) \rangle\!\rangle := \int_\Omega \int_{\mathbb{R}^{3\times3}} f(\boldsymbol{x},t,\boldsymbol{d}(\boldsymbol{x},t),\boldsymbol{S}) \nu^o_{(\boldsymbol{x},t)}(d\boldsymbol{S}) \, d\boldsymbol{x} + \int_{\overline{\Omega}} \int_{\mathscr{S}^{3^2-1} \times \overline{B}_3} \tilde{f}(\boldsymbol{x},t,\tilde{\boldsymbol{h}},\tilde{\boldsymbol{S}}) \nu^\infty_{(\boldsymbol{x},t)}(d\tilde{\boldsymbol{S}}, d\tilde{\boldsymbol{h}}) m_t(d\boldsymbol{x}) \, .$$

The transformed function $\tilde{f} : \overline{\Omega} \times [0,T] \times B_3 \times B_{3\times3} \to \mathbb{R}$*, the so-called recession function is given by*

$$\tilde{f}(\boldsymbol{x},t,\tilde{\boldsymbol{h}},\tilde{\boldsymbol{S}}) := f(\boldsymbol{x},t,\frac{\tilde{\boldsymbol{h}}}{\sqrt{1-|\tilde{\boldsymbol{h}}|^2}},\frac{\tilde{\boldsymbol{S}}}{\sqrt{1-|\tilde{\boldsymbol{S}}|^2}})(1-|\tilde{\boldsymbol{h}}|^2)(1-|\tilde{\boldsymbol{S}}|^2) \, .$$

See [22] for further details on generalized Gradient Young measures. For the existence theory, the Young measure μ *is just a point measure at the considered magnetic field, i.e.,* $\mu_{(\boldsymbol{x},t)} = \delta_{\boldsymbol{H}(\boldsymbol{x},t)}$*. But in the case of the optimal control problem in Section 5, we also need to relax the control in the definition of a solution. Instead of introducing the measure-valued formulation in the definition 3, the function* $\boldsymbol{q} \in L^2(0,T;\boldsymbol{L}^2)$ *itself is inserted in the definition.*

Remark 8 (Subdifferential interpretation)**.** *The variational derivative* $\boldsymbol{q}$ *can also be interpreted as an element of a suitable subdifferential of the free energy* (2.9)*. The sense of this subdifferential has to be rather weak, to include the vector* $\boldsymbol{q}$*. It should take into account the weak convergence result for* $\{\boldsymbol{q}_n\}$ *as well as the geometric properties, i.e., as a subdifferential of an energy on the manifold* $\mathbb{S}^2$*,* $\boldsymbol{q}$ *should be an element of its cotangent. Following the proof of the convergence of* $\{\boldsymbol{q}_n\}$*, this subdifferential should be defined similar to the Bouligand subdifferential (see [35]), but more general.*

4 Convergence of a semi-discrete scheme

In the following we introduce a semi-discrete scheme and show its convergence to a dissipative solution.

4.1 Semi-discrete scheme and approximate relative energy inequality

We consider two general Galerkin-schemes, one for the discretization of the Navier–Stokes-like equation and one for the director equation.
Let $\{W_n\}_{n\in\mathbb{N}}$ be such that $W_n \subset W_{n+1}$ and $W_n \subset \boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma}$ for all $n \in \mathbb{N}$ and $\overline{\lim_{n\to\infty} W_n} = \boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma}$. Let P_n be the $\boldsymbol{L}^2_\sigma$-projection onto W_n. We additionally assume that the projection P_n is stable in the $\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma}$-sense, *i.e.*, there exists a $c > 0$ such that (see [29, Appendix Theorem 4.11, Lemma 4.26])

$$\|P_n \boldsymbol{w}\|_{\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma}} \leq c \|\boldsymbol{w}\|_{\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma}} \text{ for all } \boldsymbol{w} \in \boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma} \text{ and } n \in \mathbb{N} \, . \tag{4.1a}$$

Let $\{Z_n\}_{n\in\mathbb{N}}$ be such that $Z_n \subset \boldsymbol{H}_0^1 \cap \boldsymbol{L}^\infty$ and $Z_n \subset Z_{n+1}$ for all $n \in \mathbb{N}$ and $\overline{\lim_{n\to\infty} Z_n} = \boldsymbol{H}_0^1 \cap \boldsymbol{L}^\infty$. Let R_n be the $\boldsymbol{L}^2$-projection onto Z_n. We additionally assume that the projection R_n is $\boldsymbol{H}_0^1$ and $\boldsymbol{L}^\infty$-stable, *i.e.*, there exists a $c > 0$ such that

$$\|R_n \boldsymbol{z}\|_{\boldsymbol{H}_0^1} \leq c\|\boldsymbol{z}\|_{\boldsymbol{H}_0^1} \text{ as well as } \|R_n \boldsymbol{z}\|_{\boldsymbol{L}^\infty} \leq c\|\boldsymbol{z}\|_{\boldsymbol{L}^\infty} \text{ for all } \boldsymbol{z} \in \boldsymbol{H}_0^1 \cap \boldsymbol{L}^\infty \text{ and } n \in \mathbb{N}\,. \tag{4.1b}$$

Remark 9. *As a sequence of linear spaces fulfilling the assumption on $\{W_n\}$, the spaces spanned by eigenfunctions of the Stokes operator can be chosen (see [22]). It is also possible to replace the $\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma}$-regularity assumption on the spaces W_n by $\boldsymbol{H}^1_{0,\sigma} \cap \boldsymbol{L}^\infty$. This is rather fulfilled by linear finite elements, but the sense of the divergence-free condition has to be redefined or added as a constraint to the system in such a case.*

For a domain Ω with sufficiently regular boundary $\partial\Omega$, a standard Finite Element scheme, i.e., linear finite elements on a quasi-uniform triangulation, fulfills the above assumptions on the sequence of spaces $\{Z_n\}$ (see Thomée [36, Lemma 5.1], Ciarlet [6, Section 3.3], or Nitsche [32]).

Proposition 10 (Extension operator)**.** *There exists a linear continuous operator $\mathbb{E} : \boldsymbol{H}^{s-1/2}(\partial\Omega) \to \boldsymbol{H}^s(\Omega)$ with $s \in [5/2, 3]$, where Ω is of class $\mathscr{C}^{2,1}$. This operator is the right-inverse of the trace operator, i.e. for all $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\partial\Omega)$, it holds $\mathbb{E}\boldsymbol{g} = \boldsymbol{g}$ on $\partial\Omega$ in the sense of the trace operator. There exists a constant $c > 0$ such that*

$$\|\mathbb{E}\boldsymbol{g}\|_{\boldsymbol{H}^s(\Omega)} \leq c\|\boldsymbol{g}\|_{\boldsymbol{H}^{s-1/2}(\partial\Omega)} \quad \text{for } \boldsymbol{g} \in \boldsymbol{H}^{s-1/2}(\partial\Omega) \text{ for } s \in [5/2, 3]\,.$$

The Sobolev exponent is chosen in such a way that the regularity of the associated test functions (3.5) *is achieved (see Remark 6).*

Proof. Let Ω be of class $\mathscr{C}^{2,1}$. The extension operator is defined via the solution operator of the problem

$$-\Delta_{\boldsymbol{\Lambda}} \boldsymbol{d} = 0 \quad \text{in } \Omega\,, \qquad \boldsymbol{d} = \boldsymbol{g} \quad \text{on } \partial\Omega\,.$$

This problem is uniquely solvable for a tensor enjoying the strong ellipticity (see McLean [30, Theorem 4.10] and (2.8)). The associated solution operator is linear and continuous and the regularity of this problem asserts (compare McLean [30, Theorem 4.21])

$$\mathbb{E} : \boldsymbol{H}^{s-1/2}(\partial\Omega) \to \boldsymbol{H}^s(\Omega) \text{ for } s \in [5/2, 3]\,.$$

□

The approximate system is similar to the one in [10]. Let $n \in \mathbb{N}$ be fixed. As usual, we consider the ansatz

$$\boldsymbol{v}_n(t) = \sum_{i=1}^n v_n^i(t)\boldsymbol{w}_i, \quad \boldsymbol{d}_n(t) = \mathbb{E}\boldsymbol{d}_1 + \sum_{i=1}^n d_n^i(t)\boldsymbol{z}_i$$

with $(v_n^i, d_n^i) \in \mathscr{AC}([0,T])$ for all $i = 1, \dots, n$.

Our approximation reads as: Find $(\boldsymbol{v}_n, \boldsymbol{d}_n) \in \mathscr{AC}([0,T]; W_n \times Z_n)$ such that

$$\begin{aligned} (\partial_t \boldsymbol{v}_n, \boldsymbol{w}) + ((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{v}_n, \boldsymbol{w}) - (\nabla \boldsymbol{d}_n^T(|\boldsymbol{d}_n|^2 I - \boldsymbol{d}_n \otimes \boldsymbol{d}_n)\boldsymbol{q}_n, \boldsymbol{w}) + \left(\boldsymbol{T}_n^L : \nabla \boldsymbol{w}\right) &= \langle \boldsymbol{g}, \boldsymbol{w}\rangle\,, \\ \boldsymbol{v}_n(0) &= P_n \boldsymbol{v}_0\,, \end{aligned} \tag{4.2a}$$

$$\begin{aligned} \left(\partial_t \boldsymbol{d}_n + \left((|\boldsymbol{d}_n|^2 I - \boldsymbol{d}_n \otimes \boldsymbol{d}_n\right)\left((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{d}_n - (\nabla \boldsymbol{v}_n)_{\text{skw}}\boldsymbol{d}_n + \lambda(\nabla \boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n + \boldsymbol{q}_n\right), \boldsymbol{z}\right) &= 0\,, \\ \boldsymbol{d}_n(0) - \mathbb{E}\boldsymbol{d}_1 &= R_n(\boldsymbol{d}_0 - \mathbb{E}\boldsymbol{d}_1) \end{aligned} \tag{4.2b}$$

holds for all $\boldsymbol{w} \in W_n$ and $\boldsymbol{z} \in Z_n$, where $\boldsymbol{q}_n$ is given by the projection of the variational derivative of the free energy

$$\begin{aligned} \boldsymbol{q}_n :=\, & R_n\left(-\Delta_{\boldsymbol{\Lambda}} \boldsymbol{d}_n - \nabla \cdot (\boldsymbol{d}_n \cdot \boldsymbol{\Theta} \,\vdots\, \nabla \boldsymbol{d}_n \otimes \boldsymbol{d}_n) + \nabla \boldsymbol{d}_n : \boldsymbol{\Theta} \,\vdots\, \nabla \boldsymbol{d}_n \otimes \boldsymbol{d}_n\right) \\ & - R_n\left(\chi_{\|}(\boldsymbol{d}_n \cdot \boldsymbol{H})\boldsymbol{H} - \chi_{\perp}\boldsymbol{H} \times (\boldsymbol{H} \times \boldsymbol{d}_n)\right) + \gamma \boldsymbol{d}_n\,, \end{aligned} \tag{4.2c}$$

with $\gamma \in \mathbb{R}$, which is chosen to vanish for simplicity, *i.e.*, $\gamma = 0$, and

$$\begin{aligned} \boldsymbol{T}_n^L := {}& (\mu_1+\lambda^2)(\boldsymbol{d}_n\cdot(\nabla\boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n)(\boldsymbol{d}_n\otimes\boldsymbol{d}_n)+\mu_4(\nabla\boldsymbol{v}_n)_{\text{sym}}+(\mu_5+\mu_6-\lambda^2)\left(\boldsymbol{d}_n\otimes(\nabla\boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n\right)_{\text{sym}} \\ & -\lambda\left(\boldsymbol{d}_n\otimes(|\boldsymbol{d}_n|^2 I-\boldsymbol{d}_n\otimes\boldsymbol{d}_n)\boldsymbol{q}_n\right)_{\text{sym}}-|\boldsymbol{d}_n|^2\left(\boldsymbol{d}_n\otimes\boldsymbol{q}_n\right)_{\text{skw}} \end{aligned} \tag{4.2d}$$

is the approximate Leslie stress.

Note that in comparison to formulation (2.2e), we replaced 1 by $|\boldsymbol{d}_n|^2$ in the second line of (4.2d). In the limit this should be the same, which motivates this choice for the approximate system. By substituting $\boldsymbol{e}_n$, we replace the time derivative $\partial_t\boldsymbol{d}_n$ in (4.2a) and this allows to write the system (4.2) as an ordinary differential equation in finite dimensions. The solvability of this approximate system is rather standard and we refer to [10] for more details. We also replaced the Ericksen-stress $\nabla\cdot(\nabla\boldsymbol{d}^T\partial F/\partial\nabla\boldsymbol{d})$ by $-\nabla\boldsymbol{d}_n(|\boldsymbol{d}_n|^2 I-\boldsymbol{d}_n\otimes\boldsymbol{d}_n)\boldsymbol{q}_n$. This is motivated by the integration-by-parts formula [10, Equation 21] and since for the continuous system it should hold $|\boldsymbol{d}_n| = 1$ as well as $\nabla|\boldsymbol{d}_n|^2 = 2\nabla\boldsymbol{d}_n^T\boldsymbol{d}_n = 0$. Choosing the Ericksen–stress as in (4.2a) assures that the energy equality is valid in the approximate setting. The term $(|\boldsymbol{d}_n|^2 I-\boldsymbol{d}_n\otimes\boldsymbol{d}_n)$ can also be written as $-\boldsymbol{d}_n\times\boldsymbol{d}_n\times$ (compare to the second approximation scheme in [2]).

Note that there is a free parameter in the system, γ can be chosen arbitrarily, since it does not change the other parts of the system. It can be used as a normalizing constant, for example to achieve $(\boldsymbol{q}_n,\boldsymbol{d}_n) = 0$. That $\boldsymbol{q}_n$ is only defined up to an additive shift by $\boldsymbol{d}_n$ corresponds to the fact that $\boldsymbol{q}_n$ should approximate the derivative of F taking values in the sphere and this derivative is an element of the cotangent space of the sphere. Since it holds $\boldsymbol{h}\cdot\boldsymbol{d}$ for every element $\boldsymbol{h}$ in the tangent space at $\boldsymbol{d}$, the cotangent space can be chosen arbitrarily in the direction $\boldsymbol{d}$.

Theorem 11. *Let* Ω *be a bounded domain with sufficiently regular boundary and let the assumption* (2.2d)*,* (2.2f) *and* (2.5) *be fulfilled. For the solutions* $(\boldsymbol{v}_n,\boldsymbol{d}_n)$ *to the semi-discrete approximate problem* (4.2)*, it holds under the Assumption* (4.1) *on the discrete spaces that*

$$\begin{aligned} & \frac{1}{2}\mathscr{E}(\boldsymbol{v}_n(t),\boldsymbol{d}_n(t)|\tilde{\boldsymbol{v}}(t),\tilde{\boldsymbol{d}}(t))+\|\boldsymbol{H}-\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2+\frac{1}{2}\int_0^t\mathscr{W}(\boldsymbol{v}_n(t),\boldsymbol{d}_n(t)|\tilde{\boldsymbol{v}}(t),\tilde{\boldsymbol{d}}(t))\exp\left(\int_s^t\mathscr{K}(\tau)\,d\tau\right)ds \\ & \leq \mathscr{D}_0(\boldsymbol{v}_n(0),\boldsymbol{d}_n(0),\boldsymbol{H}|\tilde{\boldsymbol{v}}(0),\tilde{\boldsymbol{d}}(0),\tilde{\boldsymbol{H}})\exp\left(\int_0^t\mathscr{K}(s)\,ds\right) \\ & \quad+\int_0^t\left[\left(\mathscr{A}_n(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}),\begin{pmatrix}\tilde{\boldsymbol{v}}-\boldsymbol{v}_n\\ \boldsymbol{d}_n\times(\tilde{\boldsymbol{q}}_n-\boldsymbol{q}_n+\boldsymbol{a}_n)\end{pmatrix}\right)+\langle(I-R_n)\partial_t\tilde{\boldsymbol{d}},\boldsymbol{q}(\tilde{\boldsymbol{d}})-\boldsymbol{q}(\boldsymbol{d}_n)\rangle\right]\exp\left(\int_s^t\mathscr{K}(\tau)\,d\tau\right)ds \\ & \quad+\int_0^t\left[\left(\mathscr{A}_n(\boldsymbol{v}_n,\boldsymbol{d}_n),\begin{pmatrix}P_n\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}\\ \boldsymbol{d}_n\times(R_n\boldsymbol{a}_n-\boldsymbol{a}_n)\end{pmatrix}\right)+\left(\partial_t\boldsymbol{d}_n(|\boldsymbol{d}_n|^2-1),R_n\boldsymbol{a}_n-\boldsymbol{a}_n\right)\right]\exp\left(\int_s^t\mathscr{K}(\tau)\,d\tau\right)ds \\ & \quad+\int_0^t\left((1-|\tilde{\boldsymbol{d}}|^2)\partial_t\tilde{\boldsymbol{d}}+\frac{1}{2}\partial_t|\tilde{\boldsymbol{d}}|^2\tilde{\boldsymbol{d}},\tilde{\boldsymbol{q}}_n-\boldsymbol{q}_n+\boldsymbol{a}_n\right)\exp\left(\int_s^t\mathscr{K}(\tau)\,d\tau\right)ds \end{aligned} \tag{4.3}$$

for all $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})$ *fulfilling* (3.5) *as well as* $\tilde{\boldsymbol{v}}\in L^2(0,T;\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})$ *and* $\operatorname{tr}(\tilde{\boldsymbol{d}})=\boldsymbol{d}_1$. *Here* $\mathscr{A}_n$ *is given similar to* (3.10) *by*

$$\mathscr{A}_n(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})=\begin{pmatrix}\partial_t\tilde{\boldsymbol{v}}+(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}-\nabla\tilde{\boldsymbol{d}}^T(|\tilde{\boldsymbol{d}}|^2 I-\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}})\tilde{\boldsymbol{q}}_n-\nabla\cdot\tilde{\boldsymbol{T}}_n^L-\boldsymbol{g}\\ \tilde{\boldsymbol{d}}\times\left(\partial_t\tilde{\boldsymbol{d}}(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}}-(\nabla\tilde{\boldsymbol{v}})_{\text{skw}}\tilde{\boldsymbol{d}}+\lambda(\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\tilde{\boldsymbol{d}}+\tilde{\boldsymbol{q}}_n\right)\end{pmatrix} \tag{4.4}$$

and $\tilde{\boldsymbol{q}}_n$ *by*

$$\tilde{\boldsymbol{q}}_n := R_n\left(-\Delta_{\boldsymbol{\Lambda}}\tilde{\boldsymbol{d}}-\nabla\cdot(\tilde{\boldsymbol{d}}\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}})+\nabla\tilde{\boldsymbol{d}}:\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}-\chi_{\|}(\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}})\tilde{\boldsymbol{H}}+\chi_{\perp}\tilde{\boldsymbol{H}}\times(\tilde{\boldsymbol{H}}\times\tilde{\boldsymbol{d}})\right) \tag{4.5}$$

as well as $\boldsymbol{a}_n$ *by*

$$\boldsymbol{a}_n := \frac{1}{k}\left((\tilde{\boldsymbol{d}}-\boldsymbol{d}_n))\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\right):\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}+\chi_{\|}(\tilde{\boldsymbol{H}}-\boldsymbol{H})(\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}})-\chi_{\perp}(\tilde{\boldsymbol{H}}-\boldsymbol{H})\times(\tilde{\boldsymbol{H}}\times\tilde{\boldsymbol{d}})\,. \tag{4.6}$$

The term $\tilde{\boldsymbol{T}}_n^L$ *is given by* $\boldsymbol{T}_n^L$ *with* $\boldsymbol{d}_n$, $\boldsymbol{v}_n$, *and* $\boldsymbol{q}_n$ *replaced by* $\tilde{\boldsymbol{d}}$, $\tilde{\boldsymbol{v}}$, *and* $\tilde{\boldsymbol{q}}_n$, *respectively. Additionally, the abbreviation* $\langle \cdot, \boldsymbol{q}(\cdot)\rangle$ *is defined for all* $\boldsymbol{l} \in \boldsymbol{H}^1 \cap \boldsymbol{L}^\infty$ *and* $\boldsymbol{h} \in \boldsymbol{H}^1 \cap \boldsymbol{L}^\infty$ *via*

$$\langle \boldsymbol{l}, \boldsymbol{q}(\boldsymbol{h})\rangle := (\nabla \boldsymbol{l}; \boldsymbol{\Lambda} : \nabla \boldsymbol{h} + \boldsymbol{h} \cdot \boldsymbol{\Theta} \,\vdots\, \nabla \boldsymbol{h} \otimes \boldsymbol{h}) + \big(\boldsymbol{l}, \nabla \boldsymbol{h} : \boldsymbol{\Theta} \,\vdots\, \nabla \boldsymbol{h} - \chi_\parallel \boldsymbol{H}(\boldsymbol{h} \cdot \boldsymbol{H}) + \chi_\perp \boldsymbol{H} \times (\boldsymbol{H} \times \boldsymbol{h})\big)\,. \tag{4.7}$$

The proof of Theorem 11 is executed in Section 4.3. Beforehand, we derive *a priori* estimates In Section 4.2 and extract a subsequence converging in appropriate spaces to be able to go to the limit in inequality (4.3).

4.2 *A priori* estimates and converging subsequence

In a first step, we show that the approximate solution obeys the norm restriction almost everywhere.

Proposition 12. *Let* $(\boldsymbol{v}_n, \boldsymbol{d}_n)$ *be a solution to the approximate system* (4.2)*. Then it holds* $|\boldsymbol{d}_n(\boldsymbol{x},t)| \le c$ *a.e. in* $\Omega \times (0,T)$ *and* $|\boldsymbol{d}(\boldsymbol{x},t)| \to 1$ *as* $n \to \infty$ *a.e. in* $\Omega \times (0,T)$.

Proof. Multiplying (4.2b) with $\boldsymbol{d}_n$ and integrating in time yields

$$|\boldsymbol{d}_n(\boldsymbol{x},t)|^2 = |R_n(\boldsymbol{d}_0 - \mathbb{E}\boldsymbol{d}_0)(\boldsymbol{x}) + \mathbb{E}\boldsymbol{d}_0(\boldsymbol{x})|^2 \quad \text{for a.e. } (\boldsymbol{x},t) \in \Omega \times (0,T)\,.$$

Note that $(|\boldsymbol{d}_n|^2 I - \boldsymbol{d}_n \otimes \boldsymbol{d}_n)\boldsymbol{d}_n = 0$. Since R_n is a stable projection in $\boldsymbol{L}^\infty$, we observe

$$\|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^\infty)} \le \|R_n(\boldsymbol{d}_0 - \mathbb{E}\boldsymbol{d}_0) + \mathbb{E}\boldsymbol{d}_0\|_{\boldsymbol{L}^\infty} \le c\|\boldsymbol{d}_0 - \mathbb{E}\boldsymbol{d}_0\|_{\boldsymbol{L}^\infty} + \|\,\mathbb{E}\boldsymbol{d}_0\|_{\boldsymbol{L}^\infty} \le c\,.$$

The initial datum is assumed to fulfill the unit-vector restriction. Since $R_n(\boldsymbol{d}_0 - \mathbb{E}\boldsymbol{d}_0) + \mathbb{E}\boldsymbol{d}_0 \to \boldsymbol{d}_0$ as $n \to \infty$ and $|\boldsymbol{d}_0| = 1$, it holds $|\boldsymbol{d}_n(\boldsymbol{x},t)| \to 1$ as $n \to \infty$ a.e. in $\Omega \times (0,T)$.

□

Proposition 13. *Let the assumptions of Theorem 11 be fulfilled and let* $(\boldsymbol{v}_n, \boldsymbol{d}_n)$ *be a solution to the semi-discrete problem* (4.2)*. Then the energy equality*

$$\begin{aligned}
\|\boldsymbol{v}_n(t)\|_{\boldsymbol{L}^2}^2 + \mathscr{F}(\boldsymbol{d}_n(t)) &+ \int_0^t \big[(\mu_1 + \lambda^2)\|\boldsymbol{d}_n \cdot (\nabla \boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n\|_{L^2}^2 + \mu_4\|(\nabla \boldsymbol{v}_n)_{\text{sym}}\|_{\boldsymbol{L}^2}^2\big]\, ds \\
&+ \int_0^t \big[(\mu_5 + \mu_6 - \lambda^2)\|(\nabla \boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n\|_{\boldsymbol{L}^2}^2 + \|\boldsymbol{d}_n \times \boldsymbol{q}_n\|_{\boldsymbol{L}^2}^2\big]\, ds \\
&= \|\boldsymbol{v}_n(0)\|_{\boldsymbol{L}^2}^2 + \mathscr{F}(\boldsymbol{d}_n(0)) + \int_0^t \langle \boldsymbol{g}, \boldsymbol{v}_n\rangle\, ds
\end{aligned}$$

is valid for every $t \in [0,T]$*. We omit the dependence on* s *under the time integral for brevity.*

Proof. The proof is very similar to the proof of Proposition [10, Proposition 2]. We test equation (4.2a) with $\boldsymbol{v}_n$ and equation (4.2b) with $\boldsymbol{q}_n$ and add them up

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 + (\partial_t \boldsymbol{d}_n, \boldsymbol{q}_n) - \langle \nabla \boldsymbol{d}_n^T(|\boldsymbol{d}_n|^2 I - \boldsymbol{d}_n \otimes \boldsymbol{d}_n)\boldsymbol{q}_n, \boldsymbol{v}_n\rangle \\
&\quad + ((|\boldsymbol{d}_n|^2 I - \boldsymbol{d}_n \otimes \boldsymbol{d}_n)(\boldsymbol{v}_n \cdot \nabla)\boldsymbol{d}_n, \boldsymbol{q}_n) + (\boldsymbol{T}_n^L; \nabla \boldsymbol{v}_n) \\
&\quad - ((|\boldsymbol{d}_n|^2 I - \boldsymbol{d}_n \otimes \boldsymbol{d}_n)(\nabla \boldsymbol{v}_n)_{\text{skw}}\boldsymbol{d}_n - \lambda(\nabla \boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n, \boldsymbol{q}_n) + \big(\boldsymbol{q}_n, (|\boldsymbol{d}_n|^2 I - \boldsymbol{d}_n \otimes \boldsymbol{d}_n)\boldsymbol{q}_n\big) = \langle \boldsymbol{g}, \boldsymbol{v}_n\rangle\,.
\end{aligned}$$

Note that $\boldsymbol{q}_n$ is an element of Z_n since the projection R_n is applied. Inserting the Definition of $\boldsymbol{T}_n^L$ yields

$$\begin{aligned}
(\boldsymbol{T}_n^L; \nabla \boldsymbol{v}_n) = {}& (\mu_1 + \lambda^2)\|\boldsymbol{d}_n \cdot (\nabla \boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n\|_{L^2}^2 + \mu_4\|(\nabla \boldsymbol{v}_n)_{\text{sym}}\|_{\boldsymbol{L}^2}^2 + (\mu_5 + \mu_6 - \lambda^2)\|(\nabla \boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n\|_{\boldsymbol{L}^2}^2 \\
&+ ((|\boldsymbol{d}_n|^2 I - \boldsymbol{d}_n \otimes \boldsymbol{d}_n)(\nabla \boldsymbol{v}_n)_{\text{skw}}\boldsymbol{d}_n, \boldsymbol{q}_n) - \lambda((|\boldsymbol{d}_n|^2 I - \boldsymbol{d}_n \otimes \boldsymbol{d}_n)(\nabla \boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n, \boldsymbol{q}_n)\,,
\end{aligned}$$

where we employed $\boldsymbol{d}_n \cdot (\nabla \boldsymbol{v}_n)_{\text{skw}}\boldsymbol{d}_n = 0$. We find $(\partial_t \boldsymbol{d}_n, \boldsymbol{q}_n) = \mathrm{d}\mathscr{F}(\boldsymbol{d}_n)/\mathrm{d}t$ by the chain rule [10, Equation 33]. Note that the prescribed boundary values $\boldsymbol{d}_1$ are constant in time. Employing the equation $|\boldsymbol{d}_n|^2 I - \boldsymbol{d}_n \otimes \boldsymbol{d}_n = [\boldsymbol{d}_n]_{\boldsymbol{X}}^T[\boldsymbol{d}_n]_{\boldsymbol{X}}$ and integrating in time yields the assertion.

□

Corollary 14. *Let the assumptions of Theorem 11 be fulfilled and let* $(\boldsymbol{v}_n, \boldsymbol{d}_n)$ *be a solution to the semi-discrete problem* (4.2). *There exists a possible small* $\eta > 0$ *and two constants* $c, C > 0$ *such that*

$$\eta \|\boldsymbol{d}_n\|_{\boldsymbol{H}^1}^2 - c\|\boldsymbol{d}_1\|_{\boldsymbol{H}^{1/2}(\partial\Omega)}^2 \leq \frac{k_1}{2}\|\nabla\cdot\boldsymbol{d}_n\|_{L^2}^2 + \frac{k_2}{2}\|\nabla\times\boldsymbol{d}_n\|_{\boldsymbol{L}^2}^2 \leq \mathscr{F}(\boldsymbol{d}_n) \leq C, \tag{4.8a}$$

$$\frac{k_3}{2}\|(\nabla\cdot\boldsymbol{d}_n)\boldsymbol{d}_n\|_{\boldsymbol{L}^2}^2 + \frac{k_4}{2}\|\boldsymbol{d}_n\cdot\nabla\times\boldsymbol{d}_n\|_{L^2}^2 + \frac{k_5}{2}\|\boldsymbol{d}_n\times\nabla\times\boldsymbol{d}_n\|_{\boldsymbol{L}^2}^2 \leq \mathscr{F}(\boldsymbol{d}_n) \leq C. \tag{4.8b}$$

Proof. The first inequality holds since $\boldsymbol{\Lambda}$ (see (2.6)) is a strongly elliptic tensor. For all $\boldsymbol{\varphi} \in \boldsymbol{H}_0^1$ this implies that $\|\nabla\boldsymbol{\varphi}(t)\|_{\boldsymbol{L}^2}^2 \leq c\,(\nabla\boldsymbol{\varphi}(t);\boldsymbol{\Lambda}:\boldsymbol{\varphi}(t))$ holds for a.e. $t \in (0,T)$ for some $c > 0$. Since $\boldsymbol{d}_n - \mathbb{E}\boldsymbol{d}_1 \in \boldsymbol{H}_0^1$, this implies that there exists a $\eta > 0$ such that

$$\eta\|\nabla\boldsymbol{d}_n\|_{\boldsymbol{L}^2}^2 \leq \eta\|\nabla(\boldsymbol{d}_n - \mathbb{E}\boldsymbol{d}_1)\|_{\boldsymbol{L}^2}^2 + \eta\|\nabla\mathbb{E}\boldsymbol{d}_1\|_{\boldsymbol{L}^2}^2 \leq \frac{1}{4}\,(\nabla\boldsymbol{d}_n - \nabla\mathbb{E}\boldsymbol{d}_1;\boldsymbol{\Lambda}:(\nabla\boldsymbol{d}_n - \nabla\mathbb{E}\boldsymbol{d}_1)) + \eta\|\nabla\mathbb{E}\boldsymbol{d}_1\|_{\boldsymbol{L}^2}$$
$$\leq \frac{1}{2}\,(\nabla\boldsymbol{d}_n;\boldsymbol{\Lambda}:\nabla\boldsymbol{d}_n) + c\|\boldsymbol{d}_1\|_{\boldsymbol{H}^{1/2}(\partial\Omega)}.$$

In the last estimate we used Young's inequality and the property of the extension operator $\mathbb{E}$ (see Proposition 10). With Poincaré's estimate and again the extension operator $\mathbb{E}$ (see Proposition 10) we find

$$\|\boldsymbol{d}_n\|_{\boldsymbol{L}^2} \leq \|\boldsymbol{d}_n - \mathbb{E}\boldsymbol{d}_1\|_{\boldsymbol{L}^2} + \|\mathbb{E}\boldsymbol{d}_1\|_{\boldsymbol{L}^2} \leq c\|\nabla(\boldsymbol{d}_n - \mathbb{E}\boldsymbol{d}_1)\|_{\boldsymbol{L}^2}^2 + \|\mathbb{E}\boldsymbol{d}_1\|_{\boldsymbol{L}^2} \leq c(\|\nabla\boldsymbol{d}_n\|_{\boldsymbol{L}^2} + \|\boldsymbol{d}_1\|_{\boldsymbol{H}^{1/2}(\partial\Omega)})$$

and thus (4.8a). The estimate (4.8b) follows from the definition of $\mathscr{F}$ (see (2.5)) and Proposition 13. □

Proposition 15. *Let the assumptions of Theorem 11 be fulfilled and let* $(\boldsymbol{v}_n, \boldsymbol{d}_n)$ *be a solution to the semi-discrete problem* (4.2). *Then there exists a constant* $c > 0$ *such that*

$$\|\partial_t\boldsymbol{v}_n\|_{L^2((\boldsymbol{H}^2\cap\boldsymbol{H}_{0,\sigma}^1)^*)} + \|\partial_t\boldsymbol{d}_n\|_{L^2(\boldsymbol{L}^{3/2})} \leq c$$

for all $n \in \mathbb{N}$.

Proof. The bound on the sequence $\{\partial_t\boldsymbol{v}_n\}$ follows from similar arguments as in [10, Proposition 3]. For $\boldsymbol{\varphi} \in L^2(0,T;\boldsymbol{H}_0^1)$, we test equation (4.2b) with $R_n\boldsymbol{\varphi}$. Note that the projection R_n is necessary since equation (4.2b) is only well-defined for test functions with values in Z_n. It holds

$$\|\partial_t\boldsymbol{d}_n\|_{L^2(\boldsymbol{L}^{3/2})} \leq$$
$$\sup_{\|\boldsymbol{\varphi}\|_{L^2(\boldsymbol{L}^3)}=1}\Big(\|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^\infty)}^2\Big(\|\boldsymbol{v}_n\|_{L^2(\boldsymbol{L}^6)}\|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{H}^1)} + \|(\nabla\boldsymbol{v}_n)_{\text{skw}}\|_{L^2(\boldsymbol{L}^2)}\|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^6)}\Big)\|R_n\boldsymbol{\varphi}\|_{L^2(\boldsymbol{L}^3)}$$
$$+\Big(|\lambda|\|(\nabla\boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n\|_{L^2(\boldsymbol{L}^2)}\|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^\infty)}^2 + \|\boldsymbol{d}_n\times\boldsymbol{q}_n\|_{L^2(\boldsymbol{L}^2)}\|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^\infty)}\Big)\|R_n\boldsymbol{\varphi}\|_{L^2(\boldsymbol{L}^2)}\Big). \tag{4.9}$$

It is essential that the $\boldsymbol{L}^2$-projection R_n is $\boldsymbol{L}^\infty$-stable. This together with an interpolation argument between $\boldsymbol{L}^2$ and $\boldsymbol{L}^\infty$ grants $\|R_n\boldsymbol{\varphi}\|_{L^2(\boldsymbol{L}^3)} \leq \|\boldsymbol{\varphi}\|_{L^2(\boldsymbol{L}^3)}$. All terms on the right-hand side of the previous estimate are bounded in regard of Proposition 13, Corollary 14, and Korn's inequality [30, Theorem 10.1]. □

Proposition 16. *Let the assumptions of Theorem 11 be fulfilled and let* $\{(\boldsymbol{v}_n, \boldsymbol{d}_n)\}$ *be the sequence of solutions* *to the semi-discrete problems* (4.2)*. Then there exists a subsequence, which is not relabeled such that*

$$\boldsymbol{v}_n \overset{*}{\rightharpoonup} \boldsymbol{v} \qquad \text{in } L^\infty(0,T;\boldsymbol{L}^2_\sigma)\,, \tag{4.10a}$$

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \qquad \text{in } L^2(0,T;\boldsymbol{H}^1_{0,\sigma})\,, \tag{4.10b}$$

$$\boldsymbol{d}_n \times \boldsymbol{q}_n \rightharpoonup \boldsymbol{d} \times \boldsymbol{q} \qquad \text{in } L^2(0,T;\boldsymbol{L}^2)\,, \tag{4.10c}$$

$$(\nabla \boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n \rightharpoonup (\nabla \boldsymbol{v})_{\text{sym}}\boldsymbol{d} \qquad \text{in } L^2(0,T;\boldsymbol{L}^2)\,, \tag{4.10d}$$

$$\boldsymbol{d}_n \cdot (\nabla \boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n \rightharpoonup \boldsymbol{d} \cdot (\nabla \boldsymbol{v})_{\text{sym}}\boldsymbol{d} \qquad \text{in } L^2(0,T;L^2)\,, \tag{4.10e}$$

$$\partial_t \boldsymbol{v}_n \rightharpoonup \partial_t \boldsymbol{v} \qquad \text{in } L^2(0,T;(\boldsymbol{H}^2 \cap \boldsymbol{H}^1_{0,\sigma})^*)\,, \tag{4.10f}$$

$$\partial_t \boldsymbol{d}_n \rightharpoonup \partial_t \boldsymbol{d} \qquad \text{in } L^2(0,T;\boldsymbol{H}^{-1})\,, \tag{4.10g}$$

$$\boldsymbol{d}_n \overset{*}{\rightharpoonup} \boldsymbol{d} \qquad \text{in } L^\infty(0,T;\boldsymbol{H}^1)\,. \tag{4.10h}$$

$$\boldsymbol{d}_n \to \boldsymbol{d} \qquad \text{in } L^q(0,T;\boldsymbol{L}^r) \text{ for any } q \in [1,\infty)\,, r \in [1,6)\,, \tag{4.10i}$$

$$k_3(\nabla \cdot \boldsymbol{d}_n)\boldsymbol{d}_n \overset{*}{\rightharpoonup} k_3(\nabla \cdot \boldsymbol{d})\boldsymbol{d} \qquad \text{in } L^\infty(0,T;\boldsymbol{L}^2)\,. \tag{4.10j}$$

$$k_4\boldsymbol{d}_n \cdot \nabla \times \boldsymbol{d}_n \overset{*}{\rightharpoonup} k_4\boldsymbol{d} \cdot \nabla \times \boldsymbol{d} \qquad \text{in } L^\infty(0,T;L^2)\,. \tag{4.10k}$$

$$k_5\boldsymbol{d}_n \times \nabla \times \boldsymbol{d}_n \overset{*}{\rightharpoonup} k_5\boldsymbol{d} \times \nabla \times \boldsymbol{d} \qquad \text{in } L^\infty(0,T;\boldsymbol{L}^2)\,. \tag{4.10l}$$

Proof. The existence of a weakly and weakly* converging subsequence follows by standard arguments from the energy equality of Proposition 13 and Proposition 15. Note that we can bound the right-hand side of the energy equality in Proposition 13. Indeed due to Young's inequality and Korn's inequality [30, Theorem 10.1], we find

$$\langle \boldsymbol{g}, \boldsymbol{v}_n \rangle \leq \|\boldsymbol{v}_n\|_{\boldsymbol{H}^1_0} \|\boldsymbol{g}\|_{(\boldsymbol{H}^1_{0,\sigma})^*} \leq \frac{\mu_4}{2c^2_{\text{Korn}}} \|\boldsymbol{v}_n\|^2_{\boldsymbol{H}^1_0} + \frac{c^2_{\text{Korn}}}{2\mu_4} \|\boldsymbol{g}\|^2_{(\boldsymbol{H}^1_{0,\sigma})^*} \leq \frac{\mu_4}{2} \|(\nabla \boldsymbol{v}_n)_{\text{sym}}\|^2_{\boldsymbol{L}^2} + \frac{c^2_{\text{Korn}}}{2\mu_4} \|\boldsymbol{g}\|^2_{(\boldsymbol{H}^1_{0,\sigma})^*}\,,$$

where c_{Korn} is the constant due to Korn's inequality. The first term on the right-hand side of the above inequality chain can be absorbed in the left-hand side of the energy equality in Proposition 13. Thus, every term on the left-hand side of (13) is bounded for every $t \in [0,T]$ and for every $n \in \mathbb{N}$. Taking the supremum over t in every term individually grants the boundedness of the terms in the above indicated norms (see (4.10)).

The strong convergence follows from the Lions–Aubin compactness lemma (see Lions [27, Théorème 1.5.2]). For $\boldsymbol{d}_n$, we observe that $\boldsymbol{H}^1$ is compactly embedded in $\boldsymbol{L}^2$, which implies strong convergence in $L^2(0,T;\boldsymbol{L}^2)$ and together with the boundedness in $L^\infty(0,T;\boldsymbol{H}^1)$ also in $L^q(0,T;\boldsymbol{L}^r)$ for any $q \in [1,\infty)$ and any $r \in [2,6)$. This strong convergence allows to identify the limits in (4.10d) and (4.10e). Corollary 14 grants the weak convergences (4.10j)-(4.10l) and the strong convergence (4.10i) allows to identify the limits. For the limit in (4.10c), we initially only get that $\boldsymbol{d}_n \times R_n\boldsymbol{q}_n \rightharpoonup \boldsymbol{a}$, for some $\boldsymbol{a} \in L^2(0,T;\boldsymbol{L}^2)$. Due to the strong convergence of $\boldsymbol{d}_n$ to $\boldsymbol{d}$, it holds

$$0 = \boldsymbol{d}_n \cdot [\boldsymbol{d}_n]_{\boldsymbol{X}} R_n\boldsymbol{q}_n = \boldsymbol{d}_n \cdot (\boldsymbol{d}_n \times R_n\boldsymbol{q}_n) \rightharpoonup \boldsymbol{d} \cdot \boldsymbol{a}\,.$$

The vector $\boldsymbol{a}$ is thus point-wise orthogonal to $\boldsymbol{d}$ in the usual Euclidean sense. Hence, there exists a vector $\boldsymbol{q}$ such that $\boldsymbol{a} = \boldsymbol{d} \times \boldsymbol{q}$, which is the assertion of (4.10c). The constants k_3, k_4, and k_5 are inserted in the convergence results (4.10j)–(4.10l) since these constants can also vanish. In this case, no convergence can be deduced. □

Corollary 17. *Let the assumptions of Theorem 11 be fulfilled and let* $\{(\boldsymbol{v}_n, \boldsymbol{d}_n)\}$ *be the sequence of solutions to the semi-discrete problems* (4.2)*. Then there exists a subsequence, which is not relabel such that*

$$\boldsymbol{v}_n \to \boldsymbol{v} \qquad \text{in } \mathscr{C}_w([0,T];\boldsymbol{L}^2_\sigma)\,, \tag{4.11a}$$

$$\boldsymbol{d}_n \to \boldsymbol{d} \qquad \text{in } \mathscr{C}_w([0,T];\boldsymbol{H}^1)\,. \tag{4.11b}$$

$$\boldsymbol{\Theta} \,\vdots\, \nabla \boldsymbol{d}_n \otimes \boldsymbol{d}_n \to \boldsymbol{\Theta} \,\vdots\, \nabla \boldsymbol{d} \otimes \boldsymbol{d} \qquad \text{in } \mathscr{C}_w([0,T];\boldsymbol{L}^2)\,. \tag{4.11c}$$

Proof. Due to the estimate of the time derivative in Proposition 15, we observe for the solution of the approximate Navier–Stokes-like equation that

$$\boldsymbol{v}_n \in W^{1,2}(0,T;(\boldsymbol{H}^2\cap \boldsymbol{H}^1_{0,\sigma})^*) \subset \mathscr{AC}([0,T];(\boldsymbol{H}^2\cap \boldsymbol{H}^1_{0,\sigma})^*) \subset \mathscr{C}_w([0,T];(\boldsymbol{H}^2\cap \boldsymbol{H}^1_{0,\sigma})^*)\,.$$

The boundedness in $L^\infty(0,T;\boldsymbol{L}^2_\sigma)$ implies with a standard lemma (see [28, Page 297]) the convergence (4.11a). With similar arguments, we observe the second asserted convergence of Corollary 17. The compact embedding $\boldsymbol{H}^1 \hookrightarrow^c \boldsymbol{L}^5$ grants that $\boldsymbol{d}_n \to \boldsymbol{d}$ in $\mathscr{C}([0,T];\boldsymbol{L}^5)$. Together with (4.11b), this implies the convergence $\boldsymbol{\Theta} \,\vdots\, \nabla \boldsymbol{d}_n \otimes \boldsymbol{d}_n \to \boldsymbol{\Theta} \,\vdots\, \nabla \boldsymbol{d} \otimes \boldsymbol{d}$ in $\mathscr{C}_w([0,T];\boldsymbol{L}^{10/7})$ and, we may conclude again with a standard lemma (see [28, Page 297]) the third asserted convergence of Corollary 17. □

Remark 18. *The assertions of Corollary 17 are essential to go to the limit in the relative energy. The weak-lower semi-continuity of the $\boldsymbol{L}^2$-norm grants that $\liminf_{n\in\mathbb{N}} \mathscr{E}_n \geq \mathscr{E}$. Note that this is the only step, where we needed the boundedness of the time derivatives of the sequence $\{\boldsymbol{v}_n\}$.*

4.3 Proof of the approximate relative energy inequality and its convergence

We are going to show that the appropriate dissipative formulation (4.3) is fulfilled by the solution of the semi-discrete problem (4.2).Then we prove that in the limit a dissipative solutions (see Definition 3) is attained. Therefore, the aim is to go to the limit with the discretization parameter.

First, we collect associated integration-by-parts formulas. These are very similar to the ones in [23, Proposition 5.1] and [21, Proposition 5.4], but somehow simpler. There are no measures involved, since we consider a problem discretized in space and we omit the additional difficulty of introducing the cross product.

Corollary 19. *For $(\boldsymbol{v}_n,\boldsymbol{d}_n) \in \mathscr{AC}(0,T;W_n\times Z_n)$, $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})$ fulfilling* (3.5)*, and $\boldsymbol{H}\in L^2, \tilde{\boldsymbol{H}} \in \boldsymbol{L}^\infty$, it holds*

$$(\boldsymbol{v}_n(t),\tilde{\boldsymbol{v}}(t)) - (\boldsymbol{v}_n(0),\tilde{\boldsymbol{v}}(0)) = \int_0^t \left[(\boldsymbol{v}_n(s),\partial_t\tilde{\boldsymbol{v}}(s)) + (\partial_t\boldsymbol{v}_n(s),\tilde{\boldsymbol{v}}(s))\right] ds\,, \tag{4.12a}$$

$$\big(\nabla\boldsymbol{d}_n(t);\boldsymbol{\Lambda}:\nabla\tilde{\boldsymbol{d}}(t)\big) - \big(\nabla\boldsymbol{d}_n(0);\boldsymbol{\Lambda}:\nabla\tilde{\boldsymbol{d}}(0)\big) = \int_0^t \left[\big(-\Delta_{\boldsymbol{\Lambda}}\boldsymbol{d}_n,\partial_t\tilde{\boldsymbol{d}}\big) + \big(\partial_t\boldsymbol{d}_n,-\Delta_{\boldsymbol{\Lambda}}\tilde{\boldsymbol{d}}\big)\right] ds\,, \tag{4.12b}$$

$$\begin{aligned}
&\big(\nabla\boldsymbol{d}_n(t)\otimes\boldsymbol{d}_n(t)\,\vdots\,\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(t)\otimes\tilde{\boldsymbol{d}}(t)\big) - \big(\nabla\boldsymbol{d}_n(0)\otimes\boldsymbol{d}_n(0)\,\vdots\,\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(0)\otimes\tilde{\boldsymbol{d}}(0)\big)\\
&\quad - \big((\nabla\boldsymbol{d}_n(t)-\nabla\tilde{\boldsymbol{d}}(t))\otimes(\boldsymbol{d}_n(t)-\tilde{\boldsymbol{d}}(t))\big)\,\vdots\,\boldsymbol{\Theta}\,\vdots\,(\nabla\tilde{\boldsymbol{d}}(t)\otimes\tilde{\boldsymbol{d}}(t))\\
&\quad + \big((\nabla\boldsymbol{d}_n(0)-\nabla\tilde{\boldsymbol{d}}(0))\otimes(\boldsymbol{d}_n(0)-\tilde{\boldsymbol{d}}(0))\big)\,\vdots\,\boldsymbol{\Theta}\,\vdots\,(\nabla\tilde{\boldsymbol{d}}(0)\otimes\tilde{\boldsymbol{d}}(0))\\
&\geq \int_0^t \big(\partial_t\boldsymbol{d}_n,-\nabla\cdot\big(\tilde{\boldsymbol{d}}\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\big)+\nabla\tilde{\boldsymbol{d}}:\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\big)\,ds\\
&\quad + \int_0^t \big(-\nabla\cdot(\boldsymbol{d}_n\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\boldsymbol{d}_n\otimes\boldsymbol{d}_n)+\nabla\boldsymbol{d}_n:\boldsymbol{\Theta}\,\vdots\,\nabla\boldsymbol{d}_n\otimes\boldsymbol{d}_n,\partial_t\tilde{\boldsymbol{d}}\big)\,ds\\
&\quad - c\int_0^t ((1+\|\tilde{\boldsymbol{d}}\|_{L^\infty(\boldsymbol{W}^{1,3})})\|\partial_t\tilde{\boldsymbol{d}}(s)\|_{L^\infty}+\|\nabla\partial_t\tilde{\boldsymbol{d}}(s)\|_{\boldsymbol{L}^3})\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,0|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},0)\,ds\,,
\end{aligned} \tag{4.12c}$$

$$\begin{aligned}
&\big\|(\boldsymbol{d}_n(t)-\tilde{\boldsymbol{d}}(t))\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(t)\otimes\tilde{\boldsymbol{d}}(t)\big\|^2_{\boldsymbol{L}^2} - \big\|(\boldsymbol{d}_n(0)-\tilde{\boldsymbol{d}}(0))\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(0)\otimes\tilde{\boldsymbol{d}}(0)\big\|^2_{\boldsymbol{L}^2}\\
&\leq 2\int_0^t \big(\partial_s\boldsymbol{d}_n-\partial_s\tilde{\boldsymbol{d}},\big((\boldsymbol{d}_n-\tilde{\boldsymbol{d}})\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\big):\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\big)\,ds + \int_0^t \mathscr{K}\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,0|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},0)\,ds\,,
\end{aligned} \tag{4.12d}$$

and

$$\begin{aligned}
&\chi_\parallel\big(\boldsymbol{d}_n(t)\cdot\boldsymbol{H},\tilde{\boldsymbol{d}}(t)\cdot\tilde{\boldsymbol{H}}\big)+\chi_\perp\big(\boldsymbol{d}_n(t)\times\boldsymbol{H},\tilde{\boldsymbol{d}}(t)\times\tilde{\boldsymbol{H}}\big)-\chi_\parallel\big(\boldsymbol{d}_n(0)\cdot\boldsymbol{H},\tilde{\boldsymbol{d}}(0)\cdot\tilde{\boldsymbol{H}}\big)-\chi_\perp\big(\boldsymbol{d}_n(0)\times\boldsymbol{H},\tilde{\boldsymbol{d}}(0)\times\tilde{\boldsymbol{H}}\big)\\
&\geq \int_0^t \left[\big(\partial_t\boldsymbol{d}_n,\chi_\parallel\tilde{\boldsymbol{H}}(\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}})-\chi_\perp\tilde{\boldsymbol{H}}\times(\tilde{\boldsymbol{H}}\times\tilde{\boldsymbol{d}})\big)+\big(\partial_t\tilde{\boldsymbol{d}},\chi_\parallel\boldsymbol{H}(\boldsymbol{d}_n\cdot\boldsymbol{H})-\chi_\perp\boldsymbol{H}\times(\boldsymbol{H}\times\boldsymbol{d}_n)\big)\right] ds\\
&\quad - \int_0^t \|\partial_t\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^\infty}\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})\,ds\\
&\quad + \int_0^t \left[\big(\partial_t\boldsymbol{d}_n-\partial_t\tilde{\boldsymbol{d}},\chi_\parallel(\boldsymbol{H}-\tilde{\boldsymbol{H}})(\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}})-\chi_\perp(\boldsymbol{H}-\tilde{\boldsymbol{H}})\times(\tilde{\boldsymbol{H}}\times\tilde{\boldsymbol{d}})\big)-\|\partial_t\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^\infty}\|\tilde{\boldsymbol{H}}-\boldsymbol{H}\|^2_{\boldsymbol{L}^2}\right] ds\,,
\end{aligned} \tag{4.12e}$$

for all $t \in [0,T]$.

Proof. The Integration by parts formulas (4.12a), (4.12b) and (4.12c) are rather standard and simpler than the ones in [23, Proposition 5.1] and [21, Proposition 5.3]. Note that the regularity in space is no problem anymore due to the discretization. For a proof in the continuous case, involving measures, we refer to [20].

The integration-by-parts formula (4.12d) and (4.12e) can be proven in a similar fashion. We start with formula (4.12d). The fundamental theorem of calculus grants that

$$\begin{aligned}&\left\|(\boldsymbol{d}_n(t)-\tilde{\boldsymbol{d}}(t))\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(t)\otimes\tilde{\boldsymbol{d}}(t)\right\|_{\boldsymbol{L}^2}^2-\left\|(\boldsymbol{d}_n(0)-\tilde{\boldsymbol{d}}(0))\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(0)\otimes\tilde{\boldsymbol{d}}(0)\right\|_{\boldsymbol{L}^2}^2\\&=\int_0^t\partial_s\left|(\boldsymbol{d}_n(s)-\tilde{\boldsymbol{d}}(s))\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(s)\otimes\tilde{\boldsymbol{d}}(s)\right|^2\mathrm{d}s\,.\end{aligned}$$

The chain rule implies that

$$\begin{aligned}&\partial_s\left|(\boldsymbol{d}_n(s)-\tilde{\boldsymbol{d}}(s))\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(s)\otimes\tilde{\boldsymbol{d}}(s)\right|^2\\&\quad=2\left((\partial_s\boldsymbol{d}_n(s)-\partial_s\tilde{\boldsymbol{d}}(s))\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(s)\otimes\tilde{\boldsymbol{d}}(s)\right):\left((\boldsymbol{d}_n(s)-\tilde{\boldsymbol{d}}(s))\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(s)\otimes\tilde{\boldsymbol{d}}(s)\right)\\&\qquad+2\left((\boldsymbol{d}_n(s)-\tilde{\boldsymbol{d}}(s))\cdot\boldsymbol{\Theta}\,\vdots\,\partial_s(\nabla\tilde{\boldsymbol{d}}(s)\otimes\tilde{\boldsymbol{d}}(s))\right):\left((\boldsymbol{d}_n(s)-\tilde{\boldsymbol{d}}(s))\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(s)\otimes\tilde{\boldsymbol{d}}(s)\right)\,,\end{aligned}$$

where the second line can be estimated by

$$\begin{aligned}&\left((\boldsymbol{d}_n(s)-\tilde{\boldsymbol{d}}(s))\cdot\boldsymbol{\Theta}\,\vdots\,\partial_s(\nabla\tilde{\boldsymbol{d}}(s)\otimes\tilde{\boldsymbol{d}}(s))\right):\left((\boldsymbol{d}_n(s)-\tilde{\boldsymbol{d}}(s))\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(s)\otimes\tilde{\boldsymbol{d}}(s)\right)\\&\le\|\boldsymbol{d}_n(s)-\tilde{\boldsymbol{d}}(s)\|_{\boldsymbol{L}^6}^2|\boldsymbol{\Theta}|^2\left(\|\partial_s\nabla\tilde{\boldsymbol{d}}(s)\|_{\boldsymbol{L}^3}\|\tilde{\boldsymbol{d}}(s)\|_{\boldsymbol{L}^\infty}+\|\nabla\tilde{\boldsymbol{d}}(s)\|_{\boldsymbol{L}^3}\|\partial_s\tilde{\boldsymbol{d}}(s)\|_{\boldsymbol{L}^\infty}\right)\|\nabla\tilde{\boldsymbol{d}}(s)\|_{\boldsymbol{L}^3}\|\tilde{\boldsymbol{d}}(s)\|_{\boldsymbol{L}^\infty}\,,\end{aligned}$$

which proves the assertion.

With respect to the integration-by-parts formula (4.12e), the fundamental theorem of calculus grants that

$$\begin{aligned}&\chi_\parallel\left(\boldsymbol{d}_n(t)\cdot\boldsymbol{H},\tilde{\boldsymbol{d}}(t)\cdot\tilde{\boldsymbol{H}}\right)+\chi_\perp\left(\boldsymbol{d}_n(t)\times\boldsymbol{H},\tilde{\boldsymbol{d}}(t)\times\tilde{\boldsymbol{H}}\right)-\chi_\parallel\left(\boldsymbol{d}_n(0)\cdot\boldsymbol{H},\tilde{\boldsymbol{d}}(0)\cdot\tilde{\boldsymbol{H}}\right)-\chi_\perp\left(\boldsymbol{d}_n(0)\times\boldsymbol{H},\tilde{\boldsymbol{d}}(0)\times\tilde{\boldsymbol{H}}\right)\\&=\int_0^t\left[\left(\partial_t\boldsymbol{d}_n,\chi_\parallel\boldsymbol{H}(\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}})-\chi_\perp\boldsymbol{H}\times(\tilde{\boldsymbol{H}}\times\tilde{\boldsymbol{d}})\right)+\left(\partial_t\tilde{\boldsymbol{d}},\chi_\parallel\tilde{\boldsymbol{H}}(\boldsymbol{d}_n\cdot\boldsymbol{H})-\chi_\perp\tilde{\boldsymbol{H}}\times(\boldsymbol{H}\times\boldsymbol{d}_n)\right)\right]\mathrm{d}s\,.\end{aligned}$$

From a rearrangement, we may infer

$$\begin{aligned}&\chi_\parallel\left(\boldsymbol{d}_n(t)\cdot\boldsymbol{H},\tilde{\boldsymbol{d}}(t)\cdot\tilde{\boldsymbol{H}}\right)+\chi_\perp\left(\boldsymbol{d}_n(t)\times\boldsymbol{H},\tilde{\boldsymbol{d}}(t)\times\tilde{\boldsymbol{H}}\right)-\chi_\parallel\left(\boldsymbol{d}_n(0)\cdot\boldsymbol{H},\tilde{\boldsymbol{d}}(0)\cdot\tilde{\boldsymbol{H}}\right)-\chi_\perp\left(\boldsymbol{d}_n(0)\times\boldsymbol{H},\tilde{\boldsymbol{d}}(0)\times\tilde{\boldsymbol{H}}\right)\\&=\int_0^t\left[\left(\partial_t\boldsymbol{d}_n,\chi_\parallel\tilde{\boldsymbol{H}}(\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}})-\chi_\perp\tilde{\boldsymbol{H}}\times(\tilde{\boldsymbol{H}}\times\tilde{\boldsymbol{d}})\right)+\left(\partial_t\tilde{\boldsymbol{d}},\chi_\parallel\boldsymbol{H}(\boldsymbol{d}_n\cdot\boldsymbol{H})-\chi_\perp\boldsymbol{H}\times(\boldsymbol{H}\times\boldsymbol{d}_n)\right)\right]\mathrm{d}s\\&\quad+\int_0^t\left(\partial_t\boldsymbol{d}_n-\partial_t\tilde{\boldsymbol{d}},\chi_\parallel(\boldsymbol{H}-\tilde{\boldsymbol{H}})(\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}})-\chi_\perp(\boldsymbol{H}-\tilde{\boldsymbol{H}})\times(\tilde{\boldsymbol{H}}\times\tilde{\boldsymbol{d}})\right)\mathrm{d}s\\&\quad+\int_0^t\left(\partial_t\tilde{\boldsymbol{d}},\chi_\parallel(\tilde{\boldsymbol{H}}-\boldsymbol{H})(\boldsymbol{d}_n\cdot\boldsymbol{H}-\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}})-\chi_\perp(\tilde{\boldsymbol{H}}-\boldsymbol{H})\times(\boldsymbol{H}\times\boldsymbol{d}_n-\tilde{\boldsymbol{H}}\times\tilde{\boldsymbol{d}})\right)\mathrm{d}s\,.\end{aligned}$$

Estimating the last line

$$\begin{aligned}&\int_0^t\left(\partial_t\tilde{\boldsymbol{d}},-\chi_\parallel(\tilde{\boldsymbol{H}}-\boldsymbol{H})(\boldsymbol{d}_n\cdot\boldsymbol{H}-\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}})-\chi_\perp(\tilde{\boldsymbol{H}}-\boldsymbol{H})\times(\boldsymbol{H}\times\boldsymbol{d}_n-\tilde{\boldsymbol{H}}\times\tilde{\boldsymbol{d}})\right)\mathrm{d}s\\&\le\int_0^t\|\partial_t\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^\infty}\left(-\chi_\parallel\|\boldsymbol{d}_n\cdot\boldsymbol{H}-\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2-\chi_\perp\|\boldsymbol{H}\times\boldsymbol{d}_n-\tilde{\boldsymbol{H}}\times\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^2}^2+c\|\tilde{\boldsymbol{H}}-\boldsymbol{H}\|_{\boldsymbol{L}^2}^2\right)\mathrm{d}s\end{aligned}$$

proves the assertion. □

We are now ready to prove Theorem 11.

Proof of Theorem 11. We split the proof in several steps. In the **first step**, we find similar to [23, Corollary 5.1] that

$$
\begin{aligned}
&\left((\nabla \boldsymbol{d}_n(t)-\nabla \tilde{\boldsymbol{d}}(t))\otimes(\boldsymbol{d}_n(t)-\tilde{\boldsymbol{d}}(t))\,\vdots\,\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(t)\otimes\tilde{\boldsymbol{d}}(t)\right)\\
&\le \frac{k}{2}\left\|\nabla \boldsymbol{d}_n(t)-\nabla\tilde{\boldsymbol{d}}(t)\right\|_{\boldsymbol{L}^2}^2+\frac{1}{2k}\|(\boldsymbol{d}_n(t)-\tilde{\boldsymbol{d}}(t))\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(t)\otimes\tilde{\boldsymbol{d}}(t)\|_{\boldsymbol{L}^2}^2\\
&\le \frac{1}{4}\left(\nabla \boldsymbol{d}_n(t)-\nabla\tilde{\boldsymbol{d}}(t);\boldsymbol{\Lambda}:(\nabla \boldsymbol{d}_n(t)-\nabla\tilde{\boldsymbol{d}}(t))\right)+\frac{1}{2k}\|(\boldsymbol{d}_n(t)-\tilde{\boldsymbol{d}}(t))\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(t)\otimes\tilde{\boldsymbol{d}}(t)\|_{\boldsymbol{L}^2}^2 .
\end{aligned}
$$

Here k is the coercivity constant for the strongly elliptic tensor $\boldsymbol{\Lambda}$ (see (2.8)). Additionally, we may infer from the integration-by-parts formulae (4.12) that

$$
\begin{aligned}
&-\left[\left(\nabla \boldsymbol{d}_n;\boldsymbol{\Lambda}:\nabla\tilde{\boldsymbol{d}}\right)+\left(\nabla \boldsymbol{d}_n\otimes \boldsymbol{d}_n\,\vdots\,\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\right)-\chi_{\|}\left(\boldsymbol{d}\cdot\boldsymbol{H},\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}}\right)-\chi_{\perp}\left(\boldsymbol{d}\times\boldsymbol{H},\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{H}}\right)\right]\Big|_0^t\\
&\le -\int_0^t\left[(\partial_t \boldsymbol{d}_n,\tilde{\boldsymbol{q}}_n)+(\boldsymbol{q}_n,\partial_t\tilde{\boldsymbol{d}})\right]\mathrm{d}s+\left((\nabla \boldsymbol{d}(0)-\nabla\tilde{\boldsymbol{d}}(0))\otimes(\boldsymbol{d}(0)-\tilde{\boldsymbol{d}}(0))\,\vdots\,\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}(0)\otimes\tilde{\boldsymbol{d}}(0)\right)\\
&\quad+\frac{1}{k}\left\|\left((\boldsymbol{d}_n(0)-\tilde{\boldsymbol{d}}(0))\cdot\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\right):\boldsymbol{\Theta}\,\vdots\,\nabla\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\right\|_{\boldsymbol{L}^2}^2+\int_0^t\mathscr{K}\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})\,\mathrm{d}s\\
&\quad+\int_0^t\left[\left(\partial_t\boldsymbol{d}-\partial_t\tilde{\boldsymbol{d}},\boldsymbol{a}_n\right)+c\|\partial_t\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^\infty}\|\tilde{\boldsymbol{H}}-\boldsymbol{H}\|_{\boldsymbol{L}^2}^2+\langle(R_n-I)\partial_t\tilde{\boldsymbol{d}},\boldsymbol{q}(\boldsymbol{d}_n)\rangle\right]\mathrm{d}s+\frac{1}{2}\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,0|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},0)(t)\,.
\end{aligned}
\tag{4.13}
$$

holds for all $t\in[0,T]$. We used the abbreviation (4.7). Note that in the definition of $\boldsymbol{q}_n$, the projection R_n appears. Therefore, the term $(\boldsymbol{q}_n,\partial_t\tilde{\boldsymbol{d}})$ is simultaneously added and subtracted. This leads to the second term appearing on the right-hand side of (4.13) and the second to the last term on the right-hand side of (4.13) incorporating the difference of the projection and the identity, *i.e.*, (R_n-I). This projection may be inserted in the case of $\tilde{\boldsymbol{q}}_n$ since $\partial_t\boldsymbol{d}_n$ is an element of the appropriate subspace.

In the **second step**, we recall the shifted energy inequality for the test functions $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})$ (see [21]). We add and simultaneously subtract the equations (4.2a) and equation (4.2b) evaluated at $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})$ tested with $\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{q}}_n$, respectively. This leads to

$$
\begin{aligned}
&\frac{1}{2}\|\tilde{\boldsymbol{v}}(t)\|_{\boldsymbol{L}^2}^2+\mathscr{F}(\tilde{\boldsymbol{d}}(t))+\int_0^t\left[(\mu_1+\lambda^2)\|\tilde{\boldsymbol{d}}\cdot(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}\tilde{\boldsymbol{d}}\|_{L^2}^2+\mu_4\|(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}\|_{\boldsymbol{L}^2}^2\right]\mathrm{d}s\\
&+\int_0^t\left[(\mu_5+\mu_6-\lambda^2)\|(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^2}^2+\|\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n\|_{\boldsymbol{L}^2}^2-\langle(I-R_n)\partial_t\tilde{\boldsymbol{d}},\boldsymbol{q}(\tilde{\boldsymbol{d}})\rangle\right]\mathrm{d}s\\
&\qquad=\left(\frac{1}{2}\|\tilde{\boldsymbol{v}}(0)\|_{\boldsymbol{L}^2}^2+\mathscr{F}(\tilde{\boldsymbol{d}}(0))\right)+\int_0^t\left[\langle\boldsymbol{g}(s),\tilde{\boldsymbol{v}}(s)\rangle+\left(\tilde{\mathscr{A}}(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}),\begin{pmatrix}\tilde{\boldsymbol{v}}(s)\\ \tilde{\boldsymbol{q}}_n(s)\end{pmatrix}\right)\right]\mathrm{d}s\,.
\end{aligned}
$$

Note that we have to add and subtract the variational derivative of $\tilde{\boldsymbol{q}}$ to be able to use the chain rule to get $\mathscr{F}(\boldsymbol{d}\tilde{(t)})$ on the left-hand side. This leads to the last term on the left-hand side. Here $\tilde{\mathscr{A}_n}$ is given by

$$
\tilde{\mathscr{A}_n}(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})=\begin{pmatrix}\partial_t\tilde{\boldsymbol{v}}+(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}-\nabla\tilde{\boldsymbol{d}}^T(|\tilde{\boldsymbol{d}}|^2-\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}})\tilde{\boldsymbol{q}}_n-\nabla\cdot\tilde{\boldsymbol{T}}_n^L-\boldsymbol{g}\\ \partial_t\tilde{\boldsymbol{d}}+\left(|\tilde{\boldsymbol{d}}|^2I-\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\right)\left((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}}-(\nabla\tilde{\boldsymbol{v}})_{\mathrm{skw}}\tilde{\boldsymbol{d}}+\lambda(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}\tilde{\boldsymbol{d}}+\tilde{\boldsymbol{q}}_n\right)\end{pmatrix}.
\tag{4.14}
$$

With the identity

$$
\partial_t\tilde{\boldsymbol{d}}=(1-|\tilde{\boldsymbol{d}}|^2)\partial_t\tilde{\boldsymbol{d}}+\frac{1}{2}\partial_t|\tilde{\boldsymbol{d}}|^2\tilde{\boldsymbol{d}}-\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{d}}\times\partial_t\tilde{\boldsymbol{d}}\,,
\tag{4.15}
$$

we find

$$\begin{aligned}
&\frac{1}{2}\|\tilde{\boldsymbol{v}}(t)\|_{\boldsymbol{L}^2}^2+\mathscr{F}(\tilde{\boldsymbol{d}}(t))+\int_0^t\left[(\mu_1+\lambda^2)\|\tilde{\boldsymbol{d}}\cdot(\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\tilde{\boldsymbol{d}}\|_{L^2}^2+\mu_4\|(\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\|_{\boldsymbol{L}^2}^2\right]\mathrm{d}s\\
&+\int_0^t\left[(\mu_5+\mu_6-\lambda^2)\|(\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^2}^2+\|\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n\|_{\boldsymbol{L}^2}^2\right]\mathrm{d}s\\
&\quad=\left(\frac{1}{2}\|\tilde{\boldsymbol{v}}(0)\|_{\boldsymbol{L}^2}^2+\mathscr{F}(\tilde{\boldsymbol{d}}(0))\right)+\int_0^t\left[\langle\boldsymbol{g},\tilde{\boldsymbol{v}}\rangle+\left(\mathscr{A}_n(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}),\begin{pmatrix}\tilde{\boldsymbol{v}}\\ \tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n\end{pmatrix}\right)\right]\mathrm{d}s\\
&\quad+\int_0^t\left[\left((1-|\tilde{\boldsymbol{d}}|^2)\partial_t\tilde{\boldsymbol{d}}+\frac{1}{2}\partial_t|\tilde{\boldsymbol{d}}|^2\tilde{\boldsymbol{d}},\tilde{\boldsymbol{q}}_n\right)+\langle(I-R_n)\partial_t\tilde{\boldsymbol{d}},\boldsymbol{q}(\tilde{\boldsymbol{d}})\rangle\right]\mathrm{d}s\,.
\end{aligned}\tag{4.16}$$

Here $\mathscr{A}_n$ is given by (4.4).

In the same way by adding and subtracting equation (4.2a) evaluated at $\tilde{\boldsymbol{v}}$ and tested with $\boldsymbol{v}_n$ and equation (4.2b) evaluated at $\tilde{\boldsymbol{d}}$ and tested with $\boldsymbol{q}_n$, we obtain that

$$\begin{aligned}
&-\int_0^t\left[(\partial_t\tilde{\boldsymbol{v}},\boldsymbol{v}_n)+\left(\partial_t\tilde{\boldsymbol{d}},\boldsymbol{q}_n\right)+\mu_4\left((\nabla\tilde{\boldsymbol{v}})_{\text{sym}},(\nabla\boldsymbol{v}_n)_{\text{sym}}\right)+\left(\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n,\tilde{\boldsymbol{d}}\times\boldsymbol{q}_n\right)\right]\mathrm{d}s\\
&=\int_0^t\left[((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}},\boldsymbol{v}_n)+(\mu_1+\lambda^2)\left(\tilde{\boldsymbol{d}}\cdot(\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\tilde{\boldsymbol{d}},\tilde{\boldsymbol{d}}\cdot(\nabla\boldsymbol{v}_n)_{\text{sym}}\tilde{\boldsymbol{d}}\right)-\langle\boldsymbol{g},\boldsymbol{v}_n\rangle\right]\mathrm{d}s\\
&+\int_0^t\left[(\mu_5+\mu_6-\lambda^2)\left((\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\tilde{\boldsymbol{d}},(\nabla\boldsymbol{v}_n)_{\text{sym}}\tilde{\boldsymbol{d}}\right)-\lambda\left(\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n,\tilde{\boldsymbol{d}}\times(\nabla\boldsymbol{v}_n)_{\text{sym}}\tilde{\boldsymbol{d}}\right)\right]\mathrm{d}s\\
&-\int_0^t\left[\left(\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n,\tilde{\boldsymbol{d}}\times(\nabla\boldsymbol{v}_n)_{\text{skw}}\tilde{\boldsymbol{d}}\right)+\left(\tilde{\boldsymbol{d}}\times\left((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}}-(\nabla\tilde{\boldsymbol{v}})_{\text{skw}}\tilde{\boldsymbol{d}}\right),\tilde{\boldsymbol{d}}\times\boldsymbol{q}_n\right)\right]\mathrm{d}s\\
&-\int_0^t\left[\left(\lambda\tilde{\boldsymbol{d}}\times(\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\tilde{\boldsymbol{d}},\tilde{\boldsymbol{d}}\times\boldsymbol{q}_n\right)+\left(\nabla\tilde{\boldsymbol{d}}^T\left(|\tilde{\boldsymbol{d}}|^2I-\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\right)\tilde{\boldsymbol{q}}_n,\boldsymbol{v}_n\right)\right]\mathrm{d}s\\
&+\int_0^t\left[\left((1-|\tilde{\boldsymbol{d}}|^2)\partial_t\tilde{\boldsymbol{d}}+\frac{1}{2}\partial_t|\tilde{\boldsymbol{d}}|^2\tilde{\boldsymbol{d}},\boldsymbol{q}_n\right)-\left(\mathscr{A}_n(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}),\begin{pmatrix}\boldsymbol{v}_n\\ \tilde{\boldsymbol{d}}\times\boldsymbol{q}_n\end{pmatrix}\right)\right]\mathrm{d}s\,.
\end{aligned}\tag{4.17}$$

Note that equation (4.17) is valid for all test functions $(\boldsymbol{v}_n,\boldsymbol{q}_n)$ since basically it is just zero. Again, we employ (4.15) to replace $\tilde{\mathscr{A}_n}$ by the last line of (4.17).

In the **third step**, we want to derive a similar equation for the solutions $(\boldsymbol{v}_n,\boldsymbol{d}_n)$ of the approximate system (4.2). One is tempted to test equation (4.2a) with $\tilde{\boldsymbol{v}}$ and equation (4.2b) with $\tilde{\boldsymbol{q}}_n$. But the equation (4.2a) does not hold for this test function. Therefore, we test the equations with the associated projected value, *i.e.*, equation (4.2a) tested with $P_n\tilde{\boldsymbol{v}}$ and add and subtract the equation (4.2a) tested with $\tilde{\boldsymbol{v}}$ simultaneously. This leads to

$$\begin{aligned}
&-\int_0^t\left[(\partial_t\boldsymbol{v}_n,\tilde{\boldsymbol{v}})+(\partial_t\boldsymbol{d}_n,\tilde{\boldsymbol{q}}_n)+\mu_4\left((\nabla\boldsymbol{v}_n)_{\text{sym}},(\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\right)+(\boldsymbol{d}_n\times\boldsymbol{q}_n,\boldsymbol{d}_n\times\tilde{\boldsymbol{q}}_n)\right]\mathrm{d}s\\
&=\int_0^t\left[((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\tilde{\boldsymbol{v}})+(\mu_1+\lambda^2)\left(\boldsymbol{d}_n\cdot(\nabla\boldsymbol{v}_n)_{\text{sym}}\tilde{\boldsymbol{d}},\boldsymbol{d}_n\cdot(\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\boldsymbol{d}_n\right)-\langle\boldsymbol{g},\boldsymbol{v}_n\rangle\right]\mathrm{d}s\\
&+\int_0^t\left[(\mu_5+\mu_6-\lambda^2)\left((\nabla\boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n,(\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\boldsymbol{d}_n\right)-\lambda\left(\boldsymbol{d}_n\times\boldsymbol{q}_n,\boldsymbol{d}_n\times(\nabla\tilde{\boldsymbol{v}})_{\text{sym}}\boldsymbol{d}_n\right)\right]\mathrm{d}s\\
&-\int_0^t\left[(\boldsymbol{d}_n\times\boldsymbol{q}_n,\boldsymbol{d}_n\times(\nabla\tilde{\boldsymbol{v}})_{\text{skw}}\boldsymbol{d}_n)+(\boldsymbol{d}_n\times((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{d}_n-(\nabla\boldsymbol{v}_n)_{\text{skw}}\boldsymbol{d}_n),\boldsymbol{d}_n\times\tilde{\boldsymbol{q}}_n)\right]\mathrm{d}s\\
&-\int_0^t\left[\left(\lambda\boldsymbol{d}_n\times(\nabla\boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n,\boldsymbol{d}_n\times\tilde{\boldsymbol{q}}_n\right)+\left(\nabla\boldsymbol{d}_n^T\left(|\boldsymbol{d}_n|^2I-\boldsymbol{d}_n\otimes\boldsymbol{d}_n\right)\boldsymbol{q}_n,\tilde{\boldsymbol{v}}\right)\right]\mathrm{d}s\\
&+\int_0^t\left(\mathscr{A}_n(\boldsymbol{v}_n,\boldsymbol{d}_n),\begin{pmatrix}P_n\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}\\ 0\end{pmatrix}\right)\mathrm{d}s\,.
\end{aligned}\tag{4.18}$$

In the **fourth step**, the results of the previous steps are put together to get the relative energy inequality. Calculating the relative energy $\mathscr{E}$ and the relative dissipation $\mathscr{W}$, inserting the energy equality of Proposition 13 and the shifted energy equality (4.16) yields

$$
\begin{aligned}
&\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})(t)+\int_0^t \mathscr{W}(\boldsymbol{v}_n,\boldsymbol{d}_n|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})\,\mathrm{d}s \le \mathscr{D}_0(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})(0)\\
&-2\int_0^t \left[(\mu_1+\lambda^2)(\boldsymbol{d}_n\cdot(\nabla\boldsymbol{v}_n)_{\mathrm{sym}}\boldsymbol{d}_n,\tilde{\boldsymbol{d}}\cdot(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}\tilde{\boldsymbol{d}})+(\mu_5+\mu_6-\lambda^2)((\nabla\boldsymbol{v}_n)_{\mathrm{sym}}\boldsymbol{d}_n,(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}\tilde{\boldsymbol{d}})\right]\mathrm{d}s\\
&-2\int_0^t \left[\mu_4((\nabla\boldsymbol{v}_n)_{\mathrm{sym}};(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}})+(\boldsymbol{d}_n\times\boldsymbol{q}_n,\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n)\right]\mathrm{d}s+\int_0^t\langle \boldsymbol{g},\boldsymbol{v}+\tilde{\boldsymbol{v}}\rangle\,\mathrm{d}s\\
&-\int_0^t[(\boldsymbol{v}_n,\partial_t\tilde{\boldsymbol{v}})+(\partial_t\boldsymbol{v}_n,\tilde{\boldsymbol{v}})]\,\mathrm{d}s-\int_0^t\left[(\partial_t\boldsymbol{d}_n,\tilde{\boldsymbol{q}}_n)+(\boldsymbol{q}_n,\partial_t\tilde{\boldsymbol{d}})\right]\mathrm{d}s+\frac{1}{2}\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,0|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},0)(t)\\
&+\int_0^t\left[\mathscr{K}\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})+\left(\mathscr{A}_n(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}),\begin{pmatrix}\tilde{\boldsymbol{v}}\\ \tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n\end{pmatrix}\right)+\langle(I-R_n)\partial_t\tilde{\boldsymbol{d}},\boldsymbol{q}(\tilde{\boldsymbol{d}})-\boldsymbol{q}(\boldsymbol{d}_n)\rangle\right]\mathrm{d}s\\
&+\int_0^t\left[\left(\partial_t\boldsymbol{d}-\partial_t\tilde{\boldsymbol{d}},\boldsymbol{a}_n\right)-c\|\partial_t\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^\infty}\|\tilde{\boldsymbol{H}}-\boldsymbol{H}\|^2_{\boldsymbol{L}^2}+\left((1-|\tilde{\boldsymbol{d}}|^2)\partial_t\tilde{\boldsymbol{d}}+\frac{1}{2}\partial_t|\tilde{\boldsymbol{d}}|^2\tilde{\boldsymbol{d}},\tilde{\boldsymbol{q}}_n\right)\right]\mathrm{d}s,
\end{aligned}
\tag{4.19}
$$

where $\mathscr{D}_0$ is given in (3.9). Note also the definition (4.7). Inserting now equation (4.17) and (4.18), respectively, yields

$$
\begin{aligned}
&\frac{1}{2}\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})(t)+\int_0^t \mathscr{W}(\boldsymbol{v}_n,\boldsymbol{d}_n|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})\,\mathrm{d}s\\
&\le \mathscr{D}_0(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})(0)+\int_0^t\mathscr{K}\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})\,\mathrm{d}s+\int_0^t[((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}},\boldsymbol{v}_n)+((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\tilde{\boldsymbol{v}})]\,\mathrm{d}s\\
&\quad+(\mu_1+\lambda^2)\int_0^t\left(\tilde{\boldsymbol{d}}\cdot(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}\tilde{\boldsymbol{d}},(\nabla\boldsymbol{v}_n)_{\mathrm{sym}}:\left(\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}-\boldsymbol{d}_n\otimes\boldsymbol{d}_n\right)\right)\mathrm{d}s\\
&\quad+(\mu_1+\lambda^2)\int_0^t\left(\boldsymbol{d}_n\cdot(\nabla\boldsymbol{v}_n)_{\mathrm{sym}}\boldsymbol{d}_n,(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}:\left(\boldsymbol{d}_n\otimes\boldsymbol{d}_n-\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}}\right)\right)\mathrm{d}s\\
&\quad+(\mu_5+\mu_6-\lambda^2)\int_0^t\left[\left((\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}\tilde{\boldsymbol{d}},(\nabla\boldsymbol{v}_n)_{\mathrm{sym}}(\tilde{\boldsymbol{d}}-\boldsymbol{d}_n)\right)+\left((\nabla\boldsymbol{v}_n)_{\mathrm{sym}}\boldsymbol{d}_n,(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}(\boldsymbol{d}_n-\tilde{\boldsymbol{d}})\right)\right]\mathrm{d}s\\
&\quad-\int_0^t\left[2\left(\boldsymbol{d}_n\times\boldsymbol{q}_n,\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n\right)-\left(\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n,\tilde{\boldsymbol{d}}\times\boldsymbol{q}_n\right)-(\boldsymbol{d}_n\times\boldsymbol{q}_n,\boldsymbol{d}_n\times\tilde{\boldsymbol{q}}_n)\right]\mathrm{d}s\\
&\quad-\int_0^t\left[\left(\tilde{\boldsymbol{d}}\times(\nabla\tilde{\boldsymbol{v}})_{\mathrm{skw}}\tilde{\boldsymbol{d}},\tilde{\boldsymbol{d}}\times\boldsymbol{q}_n\right)-(\boldsymbol{d}_n\times(\nabla\tilde{\boldsymbol{v}})_{\mathrm{skw}}\boldsymbol{d}_n,\boldsymbol{d}_n\times\boldsymbol{q}_n)\right]\mathrm{d}s\\
&\quad-\int_0^t\left[(\boldsymbol{d}_n\times(\nabla\boldsymbol{v}_n)_{\mathrm{skw}}\boldsymbol{d}_n,\boldsymbol{d}_n\times\tilde{\boldsymbol{q}}_n)-\left(\tilde{\boldsymbol{d}}\times(\nabla\boldsymbol{v}_n)_{\mathrm{skw}}\tilde{\boldsymbol{d}},\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n\right)\right]\mathrm{d}s\\
&\quad-\lambda\int_0^t\left[\left(\tilde{\boldsymbol{d}}\times(\nabla\boldsymbol{v}_n)_{\mathrm{sym}}\tilde{\boldsymbol{d}},\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n\right)+\left(\boldsymbol{d}_n\times(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}\boldsymbol{d}_n,\boldsymbol{d}_n\times\boldsymbol{q}_n\right)\right]\mathrm{d}s\\
&\quad+\lambda\int_0^t\left[\left(\tilde{\boldsymbol{d}}\times(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}\tilde{\boldsymbol{d}},\tilde{\boldsymbol{d}}\times\boldsymbol{q}_n\right)+\left(\boldsymbol{d}_n\times(\nabla\boldsymbol{v}_n)_{\mathrm{sym}}\boldsymbol{d}_n,\boldsymbol{d}_n\times\tilde{\boldsymbol{q}}_n\right)\right]\mathrm{d}s\\
&\quad+\int_0^t\left[(\tilde{\boldsymbol{d}}\times(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}},\tilde{\boldsymbol{d}}\times\boldsymbol{q}_n)+(\boldsymbol{d}_n\times(\boldsymbol{v}_n\cdot\nabla)\boldsymbol{d}_n,\boldsymbol{d}_n\times\tilde{\boldsymbol{q}}_n)\right]\mathrm{d}s\\
&\quad-\int_0^t\left[(\nabla\tilde{\boldsymbol{d}}^T(|\tilde{\boldsymbol{d}}|^2I-\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}})\tilde{\boldsymbol{q}}_n,\boldsymbol{v}_n)+(\nabla\boldsymbol{d}_n^T(|\boldsymbol{d}_n|^2I-\boldsymbol{d}_n\otimes\boldsymbol{d}_n)\boldsymbol{q}_n,\tilde{\boldsymbol{v}})\right]\mathrm{d}s\\
&\quad+\int_0^t\left[\left(\partial_t\boldsymbol{d}-\partial_t\tilde{\boldsymbol{d}},\boldsymbol{a}_n\right)+\left((1-|\tilde{\boldsymbol{d}}|^2)\partial_t\tilde{\boldsymbol{d}}+\frac{1}{2}\partial_t|\tilde{\boldsymbol{d}}|^2\tilde{\boldsymbol{d}},\tilde{\boldsymbol{q}}_n-\boldsymbol{q}_n\right)+\langle(I-R_n)\partial_t\tilde{\boldsymbol{d}},\boldsymbol{q}(\tilde{\boldsymbol{d}})-\boldsymbol{q}(\boldsymbol{d}_n)\rangle\right]\mathrm{d}s\\
&\quad+\int_0^t\left[\left(\mathscr{A}_n(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}),\begin{pmatrix}\tilde{\boldsymbol{v}}-\boldsymbol{v}_n\\ \tilde{\boldsymbol{d}}\times(\tilde{\boldsymbol{q}}-\boldsymbol{q}_n)\end{pmatrix}\right)-c\|\partial_t\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^\infty}\|\tilde{\boldsymbol{H}}-\boldsymbol{H}\|^2_{\boldsymbol{L}^2}+\left(\mathscr{A}_n(\boldsymbol{v}_n,\boldsymbol{d}_n),\begin{pmatrix}P_n\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}\\ 0\end{pmatrix}\right)\right]\mathrm{d}s\\
&=\sum_{i=1}^{3}I_i+(\mu_1+\lambda^2)(I_4+I_5)+(\mu_5+\mu_6-\lambda^2)I_6-\sum_{1=7}^{9}I_i-\lambda I_{10}+\lambda I_{11}+I_{12}-I_{13}+\sum_{i=14}^{15}I_i\,.
\end{aligned}
\tag{4.20}
$$

The remaining part of the proof consists of appropriately estimating the right hand side of the foregoing inequality similar to the proof of [23, Corollary 6.1].

In **step five**, the different dissipative terms, *i.e.*,the terms I_3–I_{13}, are estimated. The terms I_3–I_6 are already estimated in [23], this implies that

$$I_3+(\mu_1+\lambda^2)(I_4+I_5)+(\mu_5+\mu_6-\lambda^2)I_6 \le \delta\int_0^t \mathscr{W}(\boldsymbol{v}_n,\boldsymbol{d}_n|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})\,\mathrm{d}s+\int_0^t \mathscr{K}\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,0|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},0)\,\mathrm{d}s\,.$$

The terms I_7–I_{13} are estimated similar. We exemplify the calculations for the term $I_{12}-I_{13}$. Using the properties of the cross product (see Section 1.1), we observe for the term $I_{12}-I_{13}$ that

$$\begin{aligned}
I_{12}-I_{13} = {}& \int_0^t \left[(\tilde{\boldsymbol{d}}\times(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}},\tilde{\boldsymbol{d}}\times(\boldsymbol{q}_n-\tilde{\boldsymbol{q}}_n))+(\boldsymbol{d}_n\times((\boldsymbol{v}_n-\tilde{\boldsymbol{v}})\cdot\nabla)\boldsymbol{d}_n,\boldsymbol{d}_n\times\tilde{\boldsymbol{q}}_n)\right]\mathrm{d}s\\
&+\int_0^t \left[(\tilde{\boldsymbol{d}}\times((\tilde{\boldsymbol{v}}-\boldsymbol{v}_n)\cdot\nabla)\tilde{\boldsymbol{d}},\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n)-(\boldsymbol{d}_n\times(\tilde{\boldsymbol{v}}\cdot\nabla)\boldsymbol{d}_n,\boldsymbol{d}_n\times(\boldsymbol{q}_n-\tilde{\boldsymbol{q}}_n))\right]\mathrm{d}s\\
={}&\int_0^t(\tilde{\boldsymbol{d}}\times(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}},(\tilde{\boldsymbol{d}}-\boldsymbol{d}_n)\times(\boldsymbol{q}_n-\tilde{\boldsymbol{q}}_n))\,\mathrm{d}s+\int_0^t(\tilde{\boldsymbol{d}}\times((\boldsymbol{v}_n-\tilde{\boldsymbol{v}})\cdot\nabla)\tilde{\boldsymbol{d}},(\boldsymbol{d}_n-\tilde{\boldsymbol{d}})\times\tilde{\boldsymbol{q}}_n)\,\mathrm{d}s\\
&+\int_0^t\left[((\boldsymbol{d}_n-\tilde{\boldsymbol{d}})\times((\boldsymbol{v}_n-\tilde{\boldsymbol{v}})\cdot\nabla)\tilde{\boldsymbol{d}}+\boldsymbol{d}_n\times((\boldsymbol{v}_n-\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{d}_n-\tilde{\boldsymbol{d}}),\boldsymbol{d}_n\times\tilde{\boldsymbol{q}}_n)\right]\mathrm{d}s\\
&+\int_0^t\left[((\tilde{\boldsymbol{d}}-\boldsymbol{d}_n)\times(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}}+\boldsymbol{d}_n\times(\tilde{\boldsymbol{v}}\cdot\nabla)(\tilde{\boldsymbol{d}}-\boldsymbol{d}_n),\boldsymbol{d}_n\times\boldsymbol{q}_n-\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n+(\tilde{\boldsymbol{d}}-\boldsymbol{d}_n)\times\tilde{\boldsymbol{q}}_n))\right]\mathrm{d}s\,.
\end{aligned}$$

We keep the first term on the right-hand side of the second equality and estimate the remaining terms

$$\begin{aligned}
I_{12}-I_{13}\le{}&\int_0^t(\tilde{\boldsymbol{d}}\times(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}},(\tilde{\boldsymbol{d}}-\boldsymbol{d}_n)\times(\boldsymbol{q}_n-\tilde{\boldsymbol{q}}_n))\,\mathrm{d}s\\
&+\delta\int_0^t\|\boldsymbol{v}_n-\tilde{\boldsymbol{v}}\|_{\boldsymbol{L}^6}^2\,\mathrm{d}s+C_\delta\|\tilde{\boldsymbol{d}}\|_{L^\infty(\boldsymbol{L}^\infty)}^2\int_0^t\|\tilde{\boldsymbol{q}}_n\|_{\boldsymbol{L}^3}^2\|\nabla\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^3}^2\|\boldsymbol{d}_n-\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^6}^2\,\mathrm{d}s\\
&+C_\delta\|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^\infty)}^2\int_0^t\|\tilde{\boldsymbol{q}}_n\|_{\boldsymbol{L}^3}^2\left(\|\nabla\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^3}^2\|\boldsymbol{d}_n-\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^6}^2+\|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^\infty)}^2\|\nabla\boldsymbol{d}_n-\nabla\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^2}^2\right)\mathrm{d}s\\
&+\delta\int_0^t\|\boldsymbol{d}_n\times\boldsymbol{q}_n-\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{q}}_n\|_{\boldsymbol{L}^2}^2\,\mathrm{d}s\\
&+C_\delta\int_0^t\|\tilde{\boldsymbol{v}}\|_{\boldsymbol{L}^\infty}^2\left(\|\nabla\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^3}^2\|\boldsymbol{d}_n-\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^6}^2+\|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^\infty)}^2\|\nabla\boldsymbol{d}_n-\nabla\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^2}^2\right)\mathrm{d}s\\
&+\int_0^t\|\tilde{\boldsymbol{v}}\|_{\boldsymbol{L}^\infty}\|\tilde{\boldsymbol{q}}_n\|_{\boldsymbol{L}^3}\left(\|\nabla\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^3}\|\boldsymbol{d}_n-\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^6}^2+\|\boldsymbol{d}_n\|_{\boldsymbol{L}^\infty}\|\nabla\boldsymbol{d}_n-\nabla\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^2}\|\boldsymbol{d}_n-\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^6}\right)\mathrm{d}s\,.
\end{aligned}$$

Similar estimates for the terms I_7–I_{11} let us conclude that

$$\begin{aligned}
-\sum_{1=7}^{9}I_i-\lambda I_{10}+\lambda I_{11}+I_{12}-I_{13}\le{}&\delta\int_0^t\mathscr{W}(\boldsymbol{v}_n,\boldsymbol{d}_n|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})\,\mathrm{d}s+\int_0^t\mathscr{K}\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})\,\mathrm{d}s\\
&\int_0^t\left(\tilde{\boldsymbol{d}}\times\left((\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{d}}-(\nabla\tilde{\boldsymbol{v}})_{\mathrm{skw}}\tilde{\boldsymbol{d}}+\lambda(\nabla\tilde{\boldsymbol{v}})_{\mathrm{sym}}\tilde{\boldsymbol{d}}+\tilde{\boldsymbol{q}}_n\right),(\tilde{\boldsymbol{d}}-\boldsymbol{d}_n)\times(\boldsymbol{q}_n-\tilde{\boldsymbol{q}}_n)\right)\mathrm{d}s\,.
\end{aligned}$$

Adding and subtracting the term $\int_0^t\tilde{\boldsymbol{d}}\times\partial_t\tilde{\boldsymbol{d}},(\tilde{\boldsymbol{d}}-\boldsymbol{d}_n)\times(\boldsymbol{q}_n-\tilde{\boldsymbol{q}}_n)\,\mathrm{d}s$ leads to

$$\begin{aligned}
-\sum_{1=7}^{9}I_i-\lambda I_{10}+\lambda I_{11}+I_{12}-I_{13}\le{}&\delta\int_0^t\mathscr{W}(\boldsymbol{v}_n,\boldsymbol{d}_n|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})\,\mathrm{d}s+\int_0^t\mathscr{K}\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})\,\mathrm{d}s\\
&-\int_0^t(\tilde{\boldsymbol{d}}\times\partial_t\tilde{\boldsymbol{d}},(\tilde{\boldsymbol{d}}-\boldsymbol{d}_n)\times(\boldsymbol{q}_n-\tilde{\boldsymbol{q}}_n))\,\mathrm{d}s+\int_0^t\left(\mathscr{A}_n(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}),\begin{pmatrix}0\\(\tilde{\boldsymbol{d}}-\boldsymbol{d}_n)\times(\boldsymbol{q}_n-\tilde{\boldsymbol{q}}_n)\end{pmatrix}\right)\mathrm{d}s\,.
\end{aligned}$$

It remains to estimate the first term in the second line of the previous inequality. It results in the **sixth step**, where the difference of the variational derivatives are estimated. First we observe that the definition (4.2c) of $\boldsymbol{q}_n$

and $\tilde{\boldsymbol{q}}_n$ incorporates the projections R_n. Since R_n is an $\boldsymbol{L}^2$-projection, we find with the definition (2.10) that

$$\begin{aligned}
&\int_0^t (\tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}}, (\tilde{\boldsymbol{d}} - \boldsymbol{d}_n) \times (\boldsymbol{q}_n - \tilde{\boldsymbol{q}}_n))\, \mathrm{d}s \\
&= -\int_0^t ((\tilde{\boldsymbol{d}} - \boldsymbol{d}_n) \times \tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}}, \boldsymbol{q}_n - \tilde{\boldsymbol{q}}_n)\, \mathrm{d}s \\
&= \int_0^t \left(R_n\left((\tilde{\boldsymbol{d}} - \boldsymbol{d}_n) \times \tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}}\right), \Delta_{\boldsymbol{\Lambda}}(\boldsymbol{d}_n - \tilde{\boldsymbol{d}})\right) \mathrm{d}s \\
&\quad + \int_0^t \left(R_n\left((\tilde{\boldsymbol{d}} - \boldsymbol{d}_n) \times \tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}}\right), \nabla\cdot \left(\boldsymbol{d}_n \cdot \boldsymbol{\Theta} \vdots \nabla \boldsymbol{d}_n \otimes \boldsymbol{d}_n - \tilde{\boldsymbol{d}} \cdot \vdots \boldsymbol{\Theta} \vdots \nabla \tilde{\boldsymbol{d}} \otimes \tilde{\boldsymbol{d}}\right)\right) \mathrm{d}s \\
&\quad - \int_0^t \left(R_n\left((\tilde{\boldsymbol{d}} - \boldsymbol{d}_n) \times \tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}}\right), \nabla \boldsymbol{d}_n : \boldsymbol{\Theta} \vdots \nabla \boldsymbol{d}_n \otimes \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}} : \vdots \boldsymbol{\Theta} \vdots \nabla \tilde{\boldsymbol{d}} \otimes \tilde{\boldsymbol{d}}\ \right) \mathrm{d}s \\
&\quad + \int_0^t \left(R_n\left((\tilde{\boldsymbol{d}} - \boldsymbol{d}_n) \times \tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}}\right), \chi_\parallel \left((\boldsymbol{d}_n \cdot \boldsymbol{H})\boldsymbol{H} - (\tilde{\boldsymbol{d}} \cdot \tilde{\boldsymbol{H}})\tilde{\boldsymbol{H}}\right) - \chi_\perp \left(\boldsymbol{H} \times (\boldsymbol{H} \times \boldsymbol{d}_n) - \tilde{\boldsymbol{H}} \times (\tilde{\boldsymbol{H}} \times \tilde{\boldsymbol{d}})\right)\right) \mathrm{d}s\,.
\end{aligned} \tag{4.21}$$

Rearranging the right-hand side leads to

$$\boldsymbol{d}_n \cdot \boldsymbol{\Theta} \vdots \nabla \boldsymbol{d}_n \otimes \boldsymbol{d}_n - \tilde{\boldsymbol{d}} \cdot \boldsymbol{\Theta} \vdots \nabla \tilde{\boldsymbol{d}} \otimes \tilde{\boldsymbol{d}} = \boldsymbol{d}_n \cdot \boldsymbol{\Theta} \vdots \left(\nabla \boldsymbol{d}_n \otimes \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}} \otimes \tilde{\boldsymbol{d}}\right) + (\boldsymbol{d}_n - \tilde{\boldsymbol{d}}) \cdot \boldsymbol{\Theta} \vdots \nabla \tilde{\boldsymbol{d}} \otimes \tilde{\boldsymbol{d}}$$

and

$$\begin{aligned}
\nabla \boldsymbol{d}_n : \boldsymbol{\Theta} \vdots \nabla \boldsymbol{d}_n \otimes \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}} : \boldsymbol{\Theta} \vdots \nabla \tilde{\boldsymbol{d}} \otimes \tilde{\boldsymbol{d}} &= (\nabla \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}}) : \boldsymbol{\Theta} \vdots \left(\nabla \boldsymbol{d}_n \otimes \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}} \otimes \tilde{\boldsymbol{d}}\right) \\
&\quad + \nabla \tilde{\boldsymbol{d}} : \boldsymbol{\Theta} \vdots \left(\nabla \boldsymbol{d}_n \otimes \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}} \otimes \tilde{\boldsymbol{d}}\right) + (\nabla \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}}) : \boldsymbol{\Theta} \vdots \nabla \tilde{\boldsymbol{d}} \otimes \tilde{\boldsymbol{d}}
\end{aligned}$$

as well as

$$\begin{aligned}
&\chi_\parallel \left((\boldsymbol{d}_n \cdot \boldsymbol{H})\boldsymbol{H} - (\tilde{\boldsymbol{d}} \cdot \tilde{\boldsymbol{H}})\tilde{\boldsymbol{H}}\right) - \chi_\perp \left(\boldsymbol{H} \times (\boldsymbol{H} \times \boldsymbol{d}_n) - \tilde{\boldsymbol{H}} \times (\tilde{\boldsymbol{H}} \times \tilde{\boldsymbol{d}})\right) \\
&\quad = \chi_\parallel \left((\boldsymbol{d}_n \cdot \boldsymbol{H} - \tilde{\boldsymbol{d}} \cdot \tilde{\boldsymbol{H}})\boldsymbol{H} + (\tilde{\boldsymbol{d}} \cdot \tilde{\boldsymbol{H}})(\boldsymbol{H} - \tilde{\boldsymbol{H}})\right) - \chi_\perp \left(\boldsymbol{H} \times (\boldsymbol{H} \times \boldsymbol{d}_n - \tilde{\boldsymbol{H}} \times \tilde{\boldsymbol{d}}) + (\boldsymbol{H} - \tilde{\boldsymbol{H}}) \times (\tilde{\boldsymbol{H}} \times \tilde{\boldsymbol{d}})\right).
\end{aligned}$$

Note that the $\boldsymbol{L}^2$-projection R_n is stable in $\boldsymbol{H}^1$ and $\boldsymbol{L}^\infty$. By a simple interpolation argument between $\boldsymbol{L}^\infty$ and $\boldsymbol{L}^2$, the stability of the $\boldsymbol{L}^2$-projection in $\boldsymbol{L}^6$ follows immediately. Performing an integration-by-parts on the terms in the first two lines on the right-hand side of (4.21) and estimating yields

$$\begin{aligned}
&\int_0^t (\tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}}, (\tilde{\boldsymbol{d}} - \boldsymbol{d}_n) \times (\boldsymbol{q}_n - \tilde{\boldsymbol{q}}_n))\, \mathrm{d}s \\
&\leq \int_0^t \|(\tilde{\boldsymbol{d}} - \boldsymbol{d}_n) \times (\tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}})\|_{\boldsymbol{H}_0^1} |\boldsymbol{\Lambda}| \|\nabla \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^2}\, \mathrm{d}s \\
&\quad + \int_0^t \|(\tilde{\boldsymbol{d}} - \boldsymbol{d}_n) \times (\tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}})\|_{\boldsymbol{H}_0^1} \|\boldsymbol{d}_n\|_{\boldsymbol{L}^\infty} \left\|\boldsymbol{\Theta} \vdots \left(\nabla \boldsymbol{d}_n \otimes \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}} \otimes \tilde{\boldsymbol{d}}\right)\right\|_{\boldsymbol{L}^2} \mathrm{d}s \\
&\quad + \int_0^t \|(\tilde{\boldsymbol{d}} - \boldsymbol{d}_n) \times (\tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}})\|_{\boldsymbol{H}_0^1} \|\boldsymbol{d}_n - \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^6} |\boldsymbol{\Theta}| \|\nabla \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^3} \|\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^\infty}\, \mathrm{d}s \\
&\quad + \int_0^t \|(\tilde{\boldsymbol{d}} - \boldsymbol{d}_n) \times (\tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}})\|_{\boldsymbol{L}^\infty} \|\nabla \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^2} \left\|\boldsymbol{\Theta} \vdots \left(\nabla \boldsymbol{d}_n \otimes \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}} \otimes \tilde{\boldsymbol{d}}\right)\right\|_{\boldsymbol{L}^2} \mathrm{d}s \\
&\quad + \int_0^t \|\tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^\infty} \|\boldsymbol{d}_n - \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^6} \|\nabla \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^3} \left\|\boldsymbol{\Theta} \vdots \left(\nabla \boldsymbol{d}_n \otimes \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}} \otimes \tilde{\boldsymbol{d}}\right)\right\|_{\boldsymbol{L}^2} \mathrm{d}s \\
&\quad + \int_0^t \|\tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^\infty} \|\boldsymbol{d}_n - \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^6} \left\|\nabla \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}}\right\|_{\boldsymbol{L}^2} |\boldsymbol{\Theta}| \|\nabla \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^3} \|\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^\infty}\, \mathrm{d}s \\
&\quad + \int_0^t \|\tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^\infty} \|\boldsymbol{d}_n - \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^6} \|\boldsymbol{H}\|_{\boldsymbol{L}^3} \left(\chi_\parallel \|\boldsymbol{d}_n \cdot \boldsymbol{H} - \tilde{\boldsymbol{d}} \cdot \tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2} + \chi_\perp \|\boldsymbol{H} \times \boldsymbol{d}_n - \tilde{\boldsymbol{H}} \times \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^2}\right) \mathrm{d}s \\
&\quad + \int_0^t \|\tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^\infty} \|\boldsymbol{d}_n - \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^6} \|\boldsymbol{H} - \tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2} \left(\chi_\parallel \|\tilde{\boldsymbol{d}} \cdot \tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^3} + \chi_\perp \|\tilde{\boldsymbol{H}} \times \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^3}\right) \mathrm{d}s\,,
\end{aligned}$$

from where we may conclude with the algebraic relation in [23, Proposition A.1] that

$$\int_0^t (\tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}}, (\tilde{\boldsymbol{d}} - \boldsymbol{d}_n) \times (\boldsymbol{q}_n - \tilde{\boldsymbol{q}}_n))\,\mathrm{d}s \le \int_0^t \mathscr{K}\left(\mathscr{E}(\boldsymbol{v}_n, \boldsymbol{d}_n, \boldsymbol{H}|\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}, \tilde{\boldsymbol{H}}) + \|\boldsymbol{H} - \tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2\right)\mathrm{d}s\,.$$

The **seventh step** consists of estimating the contribution due to the non-convex character of the considered potential. To rearrange the term I_{14}, we use equation (4.2b) tested with $R_n \boldsymbol{a}_n$ and add as well as subtract equation (4.2b) tested with $\boldsymbol{a}_n$. Additionally, we add and subtract the term $\int_0^t (\mathscr{A}_n(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}), (0, \boldsymbol{d}_n \times \boldsymbol{a}_n)^T)\,\mathrm{d}s$, with $\boldsymbol{a}_n$ given in (4.6). This yields

$$\begin{aligned}
&\int_0^t \left(\partial_t \tilde{\boldsymbol{d}} - \partial_t \boldsymbol{d}_n, \boldsymbol{a}_n\right)\mathrm{d}s \\
&= \int_0^t \left(\boldsymbol{d}_n \times \boldsymbol{a}_n, \tilde{\boldsymbol{d}} \times (\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{d}} - \boldsymbol{d}_n \times (\boldsymbol{v}_n \cdot \nabla)\boldsymbol{d}_n\right)\mathrm{d}s - \int_0^t \left(\boldsymbol{d}_n \times \boldsymbol{a}_n, \tilde{\boldsymbol{d}} \times (\nabla \tilde{\boldsymbol{v}})_{\mathrm{skw}}\tilde{\boldsymbol{d}} - \boldsymbol{d}_n \times (\nabla \boldsymbol{v}_n)_{\mathrm{skw}}\boldsymbol{d}_n\right)\mathrm{d}s \\
&\quad + \int_0^t \left(\boldsymbol{d}_n \times \boldsymbol{a}_n, \lambda\left(\tilde{\boldsymbol{d}} \times (\nabla \tilde{\boldsymbol{v}})_{\mathrm{sym}}\tilde{\boldsymbol{d}} - \boldsymbol{d}_n \times (\nabla \boldsymbol{v}_n)_{\mathrm{sym}}\boldsymbol{d}_n\right) + \tilde{\boldsymbol{d}} \times \tilde{\boldsymbol{q}}_n - \boldsymbol{d}_n \times \boldsymbol{q}_n\right)\mathrm{d}s \\
&\quad + \int_0^t \left[\left(\mathscr{A}_n(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}), \begin{pmatrix} 0 \\ \boldsymbol{d}_n \times \boldsymbol{a}_n \end{pmatrix}\right) + \left(\tilde{\mathscr{A}_n}(\boldsymbol{v}_n, \boldsymbol{d}_n), \begin{pmatrix} 0 \\ R_n \boldsymbol{a}_n - \boldsymbol{a}_n \end{pmatrix}\right)\right]\mathrm{d}s + \int_0^t \left(\partial_t \tilde{\boldsymbol{d}} + \boldsymbol{d}_n \times \tilde{\boldsymbol{d}} \times \partial_t \tilde{\boldsymbol{d}}, \boldsymbol{a}_n\right)\mathrm{d}s
\end{aligned} \tag{4.22}$$

where $\tilde{\mathscr{A}_n}$ is defined in (4.14). The estimates for the terms in the first two lines on the right-hand side of (4.22) are similar. Therefore, we exemplify the estimates for the first term on the right-hand side. We observe after some rearrangements

$$\begin{aligned}
&\int_0^t \left(\boldsymbol{d}_n \times \boldsymbol{a}_n, \tilde{\boldsymbol{d}} \times (\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{d}} - \boldsymbol{d}_n \times (\boldsymbol{v}_n \cdot \nabla)\boldsymbol{d}_n\right)\mathrm{d}s \\
&= \int_0^t \left(\boldsymbol{d}_n \times \boldsymbol{a}_n, (\tilde{\boldsymbol{d}} - \boldsymbol{d}_n) \times (\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{d}} + \boldsymbol{d}_n \times (\tilde{\boldsymbol{v}} \cdot \nabla)(\tilde{\boldsymbol{d}} - \boldsymbol{d}_n)\right)\mathrm{d}s \\
&\quad + \int_0^t \left[\left(\boldsymbol{d}_n \times \boldsymbol{a}_n, \boldsymbol{d}_n \times ((\tilde{\boldsymbol{v}} - \boldsymbol{v}_n) \cdot \nabla)(\boldsymbol{d}_n - \tilde{\boldsymbol{d}})\right) + \left(\boldsymbol{d}_n \times \boldsymbol{a}_n, \boldsymbol{d}_n \times ((\tilde{\boldsymbol{v}} - \boldsymbol{v}_n) \cdot \nabla)\tilde{\boldsymbol{d}}\right)\right]\mathrm{d}s
\end{aligned}$$

that this term can be estimated by

$$\begin{aligned}
&\int_0^t \left(\boldsymbol{d}_n \times \boldsymbol{a}_n, \tilde{\boldsymbol{d}} \times (\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{d}} - \boldsymbol{d}_n \times (\boldsymbol{v}_n \cdot \nabla)\boldsymbol{d}_n\right)\mathrm{d}s \\
&\le \|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^\infty)} \|\nabla \tilde{\boldsymbol{d}}\|_{L^\infty(\boldsymbol{L}^3)} \int_0^t \|\tilde{\boldsymbol{v}}\|_{\boldsymbol{L}^\infty} \|\boldsymbol{d}_n - \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^6} \|\boldsymbol{a}_n\|_{\boldsymbol{L}^2}\,\mathrm{d}s + \|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^\infty)}^2 \int_0^t \|\tilde{\boldsymbol{v}}\|_{\boldsymbol{L}^\infty} \|\boldsymbol{a}_n\|_{\boldsymbol{L}^2} \|\nabla \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^2}\,\mathrm{d}s \\
&\quad + \delta \int_0^t \|\boldsymbol{v}_n - \tilde{\boldsymbol{v}}\|_{\boldsymbol{L}^6}^2\,\mathrm{d}s + C_\delta \|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^\infty)}^4 \int_0^t \|\boldsymbol{a}_n\|_{\boldsymbol{L}^3}^2 \|\nabla \boldsymbol{d}_n - \nabla \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^2}^2\,\mathrm{d}s + C_\delta \|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^\infty)}^4 \int_0^t \|\nabla \tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^3} \|\boldsymbol{a}_n\|_{\boldsymbol{L}^2}^2\,\mathrm{d}s\,.
\end{aligned}$$

For the abbreviation $\boldsymbol{a}_n$ (see (4.6)), we observe

$$\|\boldsymbol{a}_n(t)\|_{\boldsymbol{L}^2}^2 \le \frac{1}{k} \|\tilde{\boldsymbol{d}}\|_{L^\infty(\boldsymbol{L}^\infty)}^2 \|\nabla \tilde{\boldsymbol{d}}(t)\|_{\boldsymbol{L}^6}^4 \|\boldsymbol{d}_n(t) - \tilde{\boldsymbol{d}}(t)\|_{\boldsymbol{L}^6}^2 + \left(\chi_\parallel + \chi_\perp\right) \|\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^\infty}^2 \|\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^\infty}^2 \|\boldsymbol{H} - \tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2\,, \tag{4.23}$$

and

$$\begin{aligned}
\|\boldsymbol{a}_n(t)\|_{\boldsymbol{L}^3}^2 \le{}& \frac{1}{k} \|\tilde{\boldsymbol{d}}\|_{L^\infty(\boldsymbol{L}^\infty)}^2 \|\nabla \tilde{\boldsymbol{d}}(t)\|_{\boldsymbol{L}^6}^4 \left(\|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^\infty)} + \|\tilde{\boldsymbol{d}}\|_{L^\infty(\boldsymbol{L}^\infty)}\right)^2 \\
&+ \left(\chi_\parallel + \chi_\perp\right) \|\tilde{\boldsymbol{d}}\|_{L^\infty(\boldsymbol{L}^\infty)}^2 \|\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^\infty}^2 \left(\|\boldsymbol{H}\|_{\boldsymbol{L}^3} + \|\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^3}\right)^2\,.
\end{aligned} \tag{4.24}$$

From where we may infer that

$$\begin{aligned}
&\int_0^t \left(\boldsymbol{d}_n \times \boldsymbol{a}_n, \tilde{\boldsymbol{d}} \times (\tilde{\boldsymbol{v}} \cdot \nabla)\tilde{\boldsymbol{d}} - \boldsymbol{d}_n \times (\boldsymbol{v}_n \cdot \nabla)\boldsymbol{d}_n\right)\mathrm{d}s \le \\
&\qquad \int_0^t \mathscr{K}\left(\mathscr{E}(\boldsymbol{v}_n, \boldsymbol{d}_n, \boldsymbol{H}|\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}, \tilde{\boldsymbol{H}}) + \|\boldsymbol{H} - \tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2\right)\mathrm{d}s + \delta \int_0^t \mathscr{W}(\boldsymbol{v}_n, \boldsymbol{d}_n|\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}})\,\mathrm{d}s\,.
\end{aligned}$$

Again, we exemplified the calculations for the first terms, the other follow similar such that we observe from (4.22)

$$\int_0^t \left(\partial_t\tilde{\boldsymbol{d}} - \partial_t\boldsymbol{d}_n, \boldsymbol{a}_n\right) \mathrm{d}s \leq \int_0^t \mathscr{K}\left(\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}}) + \|\boldsymbol{H}-\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^3}^2\right)\mathrm{d}s + \delta\int_0^t \mathscr{W}(\boldsymbol{v}_n,\boldsymbol{d}_n|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})\,\mathrm{d}s$$
$$+\int_0^t \left[\left(\mathscr{A}(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}), \begin{pmatrix} 0 \\ \boldsymbol{d}_n\times\boldsymbol{a}_n\end{pmatrix}\right) + \left(\mathscr{A}_n(\boldsymbol{v}_n,\boldsymbol{d}_n), \begin{pmatrix} 0 \\ \boldsymbol{d}_n\times(R_n\boldsymbol{a}_n-\boldsymbol{a}_n)\end{pmatrix}\right)\right]\mathrm{d}s$$
$$+\int_0^t \left[(\partial_t\boldsymbol{d}_n(|\boldsymbol{d}_n|^2-1), R_n\boldsymbol{a}_n-\boldsymbol{a}_n) + \left(\partial_t\tilde{\boldsymbol{d}} + \boldsymbol{d}_n\times\tilde{\boldsymbol{d}}\times\partial_t\tilde{\boldsymbol{d}}, \boldsymbol{a}_n\right)\right]\mathrm{d}s\,.$$

Note that $\tilde{\mathscr{A}_n}(\boldsymbol{v}_n,\boldsymbol{d}_n)\cdot(0,R_n\boldsymbol{a}_n-\boldsymbol{a}_n)^T = \mathscr{A}_n(\boldsymbol{v}_n,\boldsymbol{d}_n)\cdot(0,\boldsymbol{d}_n\times(R_n\boldsymbol{a}_n-\boldsymbol{a}_n))^T + (\partial_t\boldsymbol{d}_n(|\boldsymbol{d}_n|^2-1),R_n\boldsymbol{a}_n-\boldsymbol{a}_n)$ due to the properties of the cross product stated in Section 1.1 and the property $\partial_t\boldsymbol{d}_n\cdot\boldsymbol{d}_n = 0$ of the solution to the approximate scheme (compare to (4.15)).

The last term on the right hand side can be transformed via (4.15) to

$$\int_0^t \left(\partial_t\tilde{\boldsymbol{d}} + \boldsymbol{d}_n\times\tilde{\boldsymbol{d}}\times\partial_t\tilde{\boldsymbol{d}}, \boldsymbol{a}_n\right)\mathrm{d}s = \int_0^t \left((1-|\tilde{\boldsymbol{d}}|^2)\partial_t\tilde{\boldsymbol{d}} + \frac{1}{2}\partial_t|\tilde{\boldsymbol{d}}|^2\tilde{\boldsymbol{d}}, \boldsymbol{a}_n\right)\mathrm{d}s$$
$$+\int_0^t \left(\boldsymbol{d}_n\times\tilde{\boldsymbol{d}}\times\partial_t\tilde{\boldsymbol{d}} - \tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{d}}\times\partial_t\tilde{\boldsymbol{d}}, \boldsymbol{a}_n\right)\mathrm{d}s$$

and the second line can be estimated by

$$\int_0^t \left(\boldsymbol{d}_n\times\tilde{\boldsymbol{d}}\times\partial_t\tilde{\boldsymbol{d}} - \tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{d}}\times\partial_t\tilde{\boldsymbol{d}}, \boldsymbol{a}_n\right)\mathrm{d}s = \int_0^t \left((\boldsymbol{d}_n-\tilde{\boldsymbol{d}})\times\tilde{\boldsymbol{d}}\times\partial_t\tilde{\boldsymbol{d}}, \boldsymbol{a}_n\right)\mathrm{d}s$$
$$\leq \|\tilde{\boldsymbol{d}}\|_{L^\infty(\boldsymbol{L}^\infty)}\int_0^t \|\partial_t\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^3}\|\boldsymbol{a}_n\|_{\boldsymbol{L}^2}\|\boldsymbol{d}_n-\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^6}\,\mathrm{d}s\,.$$

Inserting everything back into (4.22) yields for the term I_{14} with (4.23) that

$$I_{14} \leq \int_0^t \left[\left(\mathscr{A}(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}), \begin{pmatrix} 0 \\ \boldsymbol{d}_n\times\boldsymbol{a}_n\end{pmatrix}\right) + \left(\mathscr{A}_n(\boldsymbol{v}_n,\boldsymbol{d}_n), \begin{pmatrix} 0 \\ \boldsymbol{d}_n\times(R_n\boldsymbol{a}_n-\boldsymbol{a}_n)\end{pmatrix}\right)\right]\mathrm{d}s$$
$$+\int_0^t \left[\left((1-|\tilde{\boldsymbol{d}}|^2)\partial_t\tilde{\boldsymbol{d}} + \frac{1}{2}\partial_t|\tilde{\boldsymbol{d}}|^2\tilde{\boldsymbol{d}}, \boldsymbol{a}_n\right) + (\partial_t\boldsymbol{d}_n(|\boldsymbol{d}_n|^2-1), R_n\boldsymbol{a}_n-\boldsymbol{a}_n)\right]\mathrm{d}s$$
$$+\int_0^t \mathscr{K}\left(\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}}) + \|\boldsymbol{H}-\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2\right)\mathrm{d}s + \delta\int_0^t \mathscr{W}(\boldsymbol{v}_n,\boldsymbol{d}_n|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})\,\mathrm{d}s\,.$$

In the **eighth and last step**, Gronwall's lemma is applied yielding the approximate relative energy inequality. Inserting all the calculations and estimates back into (4.20), yields

$$\frac{1}{2}\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})(t) + \|\boldsymbol{H}-\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2 + (1-\delta)\int_0^t \mathscr{W}(\boldsymbol{v}_n,\boldsymbol{d}_n|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})\,\mathrm{d}s \leq \mathscr{D}_0(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})(0)$$
$$+\int_0^t \left[\mathscr{K}\left(\mathscr{E}(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}|\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}}) + \|\boldsymbol{H}-\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2\right) + \langle (I-R_n)\partial_t\tilde{\boldsymbol{d}}, \boldsymbol{q}(\tilde{\boldsymbol{d}})-\boldsymbol{q}(\boldsymbol{d}_n)\rangle\right]\mathrm{d}s$$
$$+\int_0^t \left[\left(\mathscr{A}(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}}), \begin{pmatrix} \tilde{\boldsymbol{v}}-\boldsymbol{v}_n \\ \boldsymbol{d}_n\times(\tilde{\boldsymbol{q}}_n-\boldsymbol{q}_n+\boldsymbol{a}_n)\end{pmatrix}\right) + \left(\mathscr{A}_n(\boldsymbol{v}_n,\boldsymbol{d}_n), \begin{pmatrix} P_n\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}} \\ \boldsymbol{d}_n\times(R_n\boldsymbol{a}_n-\boldsymbol{a}_n)\end{pmatrix}\right)\right]\mathrm{d}s$$
$$+\int_0^t \left[\left((1-|\tilde{\boldsymbol{d}}|^2)\partial_t\tilde{\boldsymbol{d}} + \frac{1}{2}\partial_t|\tilde{\boldsymbol{d}}|^2\tilde{\boldsymbol{d}}, \tilde{\boldsymbol{q}}_n-\boldsymbol{q}_n+\boldsymbol{a}_n\right) + (\partial_t\boldsymbol{d}_n(|\boldsymbol{d}_n|^2-1), R_n\boldsymbol{a}_n-\boldsymbol{a}_n)\right]\mathrm{d}s$$

Note that we added $\|\boldsymbol{H}-\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2$ to both sides of the inequality and used that this term is not time dependent. Choosing $\delta = 1/2$ allows to absorb $\mathscr{W}$ on the left hand side and Gronwall's Lemma provides the assertion.

□

Proof of Theorem 5. We argue that (4.3) converges to (3.4) as $n\to\infty$. First, we observe that for $\tilde{\boldsymbol{d}}$ fulfilling $|\tilde{\boldsymbol{d}}| = 1$ the terms $(1-|\tilde{\boldsymbol{d}}|^2)$ and $\partial_t|\tilde{\boldsymbol{d}}|^2$ vanish. From (3.5), (2.2a), and (4.24), we may infer $\|R_n\boldsymbol{a}_n-\boldsymbol{a}_n\|_{\boldsymbol{L}^3}\to 0$

as $n \to \infty$. Additionally, we find $\|P_n\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}\|_{\boldsymbol{H}^2\cap \boldsymbol{H}^1_{0,\sigma}} \to 0$ as $n\to\infty$. All terms in (4.2a) and (4.2b) are bounded in $(\boldsymbol{H}^2\cap \boldsymbol{H}^1_{0,\sigma})^*$ and $\boldsymbol{L}^{3/2}$, respectively. Therefore, the terms

$$
\begin{aligned}
\int_0^t \left(\mathscr{A}_n(\boldsymbol{v}_n,\boldsymbol{d}_n), \begin{pmatrix} P_n\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}} \\ \boldsymbol{d}_n \times (R_n\boldsymbol{a}_n - \boldsymbol{a}_n)\end{pmatrix}\right) \le {}& \\
& \left(\|\partial_t \boldsymbol{v}_n\|_{L^2((\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})^*)} + c\|\boldsymbol{v}_n\|_{L^\infty(\boldsymbol{L}^2)}\|\boldsymbol{v}_n\|_{L^2(\boldsymbol{H}^1)} + \|\boldsymbol{g}\|_{L^2((\boldsymbol{H}^1_{0,\sigma})^*)}\right)\|P_n\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}\|_{L^2(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})} \\
& + \left(2c\|\nabla \boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^2)}\|\boldsymbol{d}_n\|^2_{L^\infty(\boldsymbol{L}^\infty)}\|\boldsymbol{q}_n\|_{L^2(\boldsymbol{L}^2)} + c\|\boldsymbol{T}^L_n\|_{L^2(\boldsymbol{L}^2)}\right)\|P_n\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}\|_{L^2(\boldsymbol{H}^2\cap\boldsymbol{H}^1_{0,\sigma})} \\
& + \|\boldsymbol{d}_n\|^2_{L^\infty(\boldsymbol{L}^\infty)}\left(\|\partial_t \boldsymbol{d}_n\|_{L^2(L^{3/2})} + \|\boldsymbol{v}_n\|_{L^2(\boldsymbol{L}^6)}\|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{H}^1)}\right)\|R_n\boldsymbol{a}_n - \boldsymbol{a}_n\|_{L^3} \\
& + \|\boldsymbol{d}_n\|^2_{L^\infty(\boldsymbol{L}^\infty)}\left((1+|\lambda|)\|\boldsymbol{v}_n\|_{L^2(\boldsymbol{H}^1)}\|\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^\infty)} + \|\boldsymbol{q}_n\|_{L^2(\boldsymbol{L}^2)}\right)\|R_n\boldsymbol{a}_n - \boldsymbol{a}_n\|_{L^2}
\end{aligned} \tag{4.25}
$$

vanish as $n\to\infty$. Similarly, we find

$$
\begin{aligned}
&\left(\nabla(R_n - I)\partial_t\tilde{\boldsymbol{d}};\boldsymbol{\Lambda} : \nabla\boldsymbol{d}_n + \boldsymbol{d}_n\cdot\boldsymbol{\Theta} \,\vdots\, \nabla\boldsymbol{d}_n\otimes\boldsymbol{d}_n\right) \\
&+ \left((R_n - I)\partial_t\tilde{\boldsymbol{d}}, \nabla\boldsymbol{d}_n : \boldsymbol{\Theta} \,\vdots\, \nabla\boldsymbol{d}_n - \chi_\parallel\boldsymbol{H}(\boldsymbol{d}_n\cdot\boldsymbol{H}) + \chi_\perp\boldsymbol{H}\times(\boldsymbol{H}\times\boldsymbol{d}_n)\right) + \left(\partial_t\boldsymbol{d}_n(|\boldsymbol{d}_n|^2-1), R_n\boldsymbol{a}_n - \boldsymbol{a}_n\right) \\
&\le \|(R_n - I)\partial_t\tilde{\boldsymbol{d}}\|_{\boldsymbol{H}^1_0}\left(|\boldsymbol{\Lambda}|\|\boldsymbol{d}_n\|_{\boldsymbol{H}^1} + |\boldsymbol{\Theta}|\|\boldsymbol{d}_n\|^2_{\boldsymbol{L}^\infty}\|\boldsymbol{d}_n\|_{\boldsymbol{H}^1}\right) \\
&\quad + \|(R_n - I)\partial_t\tilde{\boldsymbol{d}}\|_{\boldsymbol{L}^\infty}\left(|\boldsymbol{\Theta}|\|\boldsymbol{d}_n\|^2_{\boldsymbol{H}^1} + (\chi_\parallel + \chi_\perp)\|\boldsymbol{H}\|^2_{\boldsymbol{L}^2}\right)\|\boldsymbol{d}_n\|_{\boldsymbol{L}^\infty} + \|\partial_t\boldsymbol{d}_n\|_{\boldsymbol{L}^{3/2}}(\|\boldsymbol{d}_n\|^2_{\boldsymbol{L}^\infty}+1)\|(R_n - I)\boldsymbol{a}_n\|_{\boldsymbol{L}^3}\,.
\end{aligned}
$$

The right-hand side vanishes as $n\to\infty$ since the $\boldsymbol{L}^2$-projection R_n is assumed to be stable in $\boldsymbol{H}^1_0$, $\boldsymbol{L}^\infty$, and thus also in $\boldsymbol{L}^3$. Note that the boundary values are constant in time such that $\partial_t\tilde{\boldsymbol{d}} \in \boldsymbol{H}^1_0$. The stability of the projection R_n in $\boldsymbol{L}^3$ yields that $\tilde{\boldsymbol{q}}_n \to \tilde{\boldsymbol{q}}$ in $\boldsymbol{L}^3$ as $n\to\infty$. Due to $|\tilde{\boldsymbol{d}}| = 1$, we find for the Ericksen stress that $\nabla\tilde{\boldsymbol{d}}^T(|\tilde{\boldsymbol{d}}|^2 I - \tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{d}})\tilde{\boldsymbol{q}} = \nabla\tilde{\boldsymbol{d}}^T\tilde{\boldsymbol{q}}$. Due to an integration-by-parts formula (see [10, Equation 21]) this is equivalent to the formulation in (2.1). Concerning the Leslie stress, we observe by the definition that in general $\tilde{\boldsymbol{T}}^L_n \neq \tilde{\boldsymbol{T}}^L$. Especially, it holds for $\tilde{\boldsymbol{T}}^L$ given in (2.2e) and $\tilde{\boldsymbol{T}}^L_n$ (see (4.2d) and Theorem 11)

$$
\tilde{\boldsymbol{T}}^L_n - \tilde{\boldsymbol{T}}^L = (1 - |\tilde{\boldsymbol{d}}|^2)\left((\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{q}})_{\mathrm{skw}} + \lambda(\tilde{\boldsymbol{d}}\otimes\tilde{\boldsymbol{q}})_{\mathrm{sym}}\right) + |\tilde{\boldsymbol{d}}|^2\left((\tilde{\boldsymbol{d}}\otimes(\tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}_n))_{\mathrm{skw}} + \lambda(\tilde{\boldsymbol{d}}\otimes(\tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}_n))_{\mathrm{sym}}\right)\,.
$$

Due to $|\tilde{\boldsymbol{d}}| = 1$ and $\tilde{\boldsymbol{q}}_n \to \tilde{\boldsymbol{q}}$ in $L^2(0,T;\boldsymbol{L}^3)$, we find $\tilde{\boldsymbol{T}}^L_n \to \tilde{\boldsymbol{T}}^L$ in $L^2(0,T;\boldsymbol{L}^3)$. Taken together, the approximate relative energy inequality (4.3) converges to the continuous relative energy inequality (3.1).

Taking in equation (4.2b) the test function $R_n([\boldsymbol{d}_n]^T_{\boldsymbol{X}}\boldsymbol{\zeta})$ for $\boldsymbol{\zeta}\in\mathscr{C}^\infty(\Omega\times(0,T))$, and adding and subtracting $[\boldsymbol{d}_n]^T_{\boldsymbol{X}}\boldsymbol{\zeta}$ yields

$$
\begin{aligned}
&\left(\boldsymbol{d}_n\times\left(\partial_t\boldsymbol{d}_n + |\boldsymbol{d}_n|^2\left((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{d}_n - (\nabla\boldsymbol{v}_n)_{\mathrm{skw}}\boldsymbol{d}_n + \lambda(\nabla\boldsymbol{v}_n)_{\mathrm{sym}}\boldsymbol{d}_n + \boldsymbol{q}_n\right)\right),\boldsymbol{\zeta}\right) \\
&= \left(\partial_t\boldsymbol{d}_n + \left(|\boldsymbol{d}_n|^2 I - \boldsymbol{d}_n\otimes\boldsymbol{d}_n\right)((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{d}_n - (\nabla\boldsymbol{v}_n)_{\mathrm{skw}}\boldsymbol{d}_n + \lambda(\nabla\boldsymbol{v}_n)_{\mathrm{sym}}\boldsymbol{d}_n + \boldsymbol{q}_n), (I - R_n)(\boldsymbol{d}_n\times\boldsymbol{\zeta})\right)
\end{aligned}
$$

The right-hand side can be estimated using (4.9)

$$
\begin{aligned}
&\left(\partial_t\boldsymbol{d}_n + \left(|\boldsymbol{d}_n|^2 I - \boldsymbol{d}_n\otimes\boldsymbol{d}_n\right)((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{d}_n - (\nabla\boldsymbol{v}_n)_{\mathrm{skw}}\boldsymbol{d}_n + \lambda(\nabla\boldsymbol{v}_n)_{\mathrm{sym}}\boldsymbol{d}_n + \boldsymbol{q}_n), (R_n - I)(\boldsymbol{d}_n\times\boldsymbol{\zeta})\right) \\
&\le \|(R_n - I)(\boldsymbol{d}_n\times\boldsymbol{\zeta})\|_{L^2(\boldsymbol{L}^3)}\left(\|\partial_t\boldsymbol{d}_n\|_{L^2(\boldsymbol{L}^{3/2})} + \|\boldsymbol{d}_n\|^2_{L^\infty(\boldsymbol{L}^\infty)}\|\boldsymbol{v}_n\|_{L^2(\boldsymbol{L}^6)}\|\nabla\boldsymbol{d}_n\|_{L^\infty(\boldsymbol{L}^2)}\right) \\
&\quad + \|(R_n - I)(\boldsymbol{d}_n\times\boldsymbol{\zeta})\|_{L^2(\boldsymbol{L}^2)}\|\boldsymbol{d}_n\|^2_{L^\infty(\boldsymbol{L}^\infty)}\left((1+|\lambda|)\|\boldsymbol{v}_n\|_{L^2(\boldsymbol{H}^1)}\|\boldsymbol{d}_n\|^2_{L^\infty(\boldsymbol{L}^\infty)} + c\|\boldsymbol{d}_n\times\boldsymbol{q}_n\|_{L^2(\boldsymbol{L}^2)}\right)
\end{aligned}
$$

such that the right-hand side converges to zero as $n\to\infty$. Furthermore, we observe

$$
\begin{aligned}
&\left(\boldsymbol{d}_n\times\left(\partial_t\boldsymbol{d}_n + |\boldsymbol{d}_n|^2\left((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{d}_n - (\nabla\boldsymbol{v}_n)_{\mathrm{skw}}\boldsymbol{d}_n + \lambda(\nabla\boldsymbol{v}_n)_{\mathrm{sym}}\boldsymbol{d}_n + \boldsymbol{q}_n\right)\right),\boldsymbol{\zeta}\right) \\
&= \left(\boldsymbol{d}_n\times\left(\partial_t\boldsymbol{d}_n + (\boldsymbol{v}_n\cdot\nabla)\boldsymbol{d}_n - (\nabla\boldsymbol{v}_n)_{\mathrm{skw}}\boldsymbol{d}_n + \lambda(\nabla\boldsymbol{v}_n)_{\mathrm{sym}}\boldsymbol{d}_n + \boldsymbol{q}_n\right),\boldsymbol{\zeta}\right) \\
&\quad + \left(\boldsymbol{d}_n\times\left((|\boldsymbol{d}_n|^2 - 1)\left((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{d}_n - (\nabla\boldsymbol{v}_n)_{\mathrm{skw}}\boldsymbol{d}_n + \lambda(\nabla\boldsymbol{v}_n)_{\mathrm{sym}}\boldsymbol{d}_n + \boldsymbol{q}_n\right)\right),\boldsymbol{\zeta}\right)\,,
\end{aligned}
$$

where the last line can again be estimated by

$$\big(\boldsymbol{d}_n \times \big((|\boldsymbol{d}_n|^2-1)\big((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{d}_n-(\nabla\boldsymbol{v}_n)_{\text{skw}}\boldsymbol{d}_n+\lambda(\nabla\boldsymbol{v}_n)_{\text{sym}}\boldsymbol{d}_n+\boldsymbol{q}_n\big)\big),\boldsymbol{\zeta}\big)$$
$$\leq \|\boldsymbol{d}_n\|_{\boldsymbol{L}^\infty}\big\||\boldsymbol{d}_n|^2-1\big\|_{\boldsymbol{L}^3}\left(\|\boldsymbol{v}_n\|_{\boldsymbol{L}^6}\|\boldsymbol{d}_n\|_{\boldsymbol{H}^1}+(1+|\lambda|)\|\boldsymbol{v}_n\|_{\boldsymbol{H}^1}\|\boldsymbol{d}_n\|_{\boldsymbol{L}^6}+\|\boldsymbol{q}_n\|_{\boldsymbol{L}^{3/2}}\right).$$

Due to Proposition 12 and Lebesgue's theorem of dominated convergence, we observe that $\||\boldsymbol{d}_n|^2-1\|_{\boldsymbol{L}^3}\to 0$ as $n\to\infty$ such that the right-hand side converges to zero. The bound on the time derivative of $\boldsymbol{v}$ (see (3.7)) follows from Proposition 15. Taken together, the semi-discrete approximation scheme convergences to a dissipative solution in the sense of Definition 3.

□

5 Optimal control

In this section, we are going to introduce an optimal control problem, where the Ericksen–Leslie equations equipped with the Oseen–Frank energy acts as a constraint. This constraint is inserted via the dissipative solvability concept in form of an inequality constraint and an additional equality constraint due to the director equation (3.6). The general goal of this section is to approximate the optimal control problem in the way that the cost functional remains the same throughout the approximation and the dissipative solutions are approximated in the way introduced in the first part of this article.

5.1 Optimal control problem

We introduce an end time problem, *i.e.*, the goal is to "be near a desired state at a given time T using a small control $\boldsymbol{H}$". Consider the following optimal control problem for $\boldsymbol{v}_T\in\boldsymbol{L}^2_\sigma$ and $\boldsymbol{d}_T\in\boldsymbol{H}^1$:

Continuous optimal control problem

$$\min_{\|\boldsymbol{H}\|_{\boldsymbol{L}^3}\leq c_{\boldsymbol{H}}} J(\boldsymbol{v},\boldsymbol{d},\boldsymbol{H}) := \min_{\|\boldsymbol{H}\|_{\boldsymbol{L}^3}\leq c_{\boldsymbol{H}}} \|\boldsymbol{v}(T)-\boldsymbol{v}_T\|^2_{\boldsymbol{L}^2}+\|\boldsymbol{d}(T)-\boldsymbol{d}_T\|^2_{\boldsymbol{H}^1}+\gamma\|\boldsymbol{H}\|^2_{\boldsymbol{L}^2}. \tag{5.1}$$

for $\boldsymbol{v}_T\in\boldsymbol{L}^2_\sigma$ and $\boldsymbol{d}_T\in\boldsymbol{H}^1$ given such that there exists an $\boldsymbol{d}\times\boldsymbol{q}$, which fulfills together with $(\boldsymbol{v},\boldsymbol{d})$ the Definition 3 for the magnetic field $\boldsymbol{H}$, where the dependence of $\mathscr{K}$ in Definition 3 on the $\|\boldsymbol{H}\|_{\boldsymbol{L}^3}$-norm is replaced by $c_{\boldsymbol{H}}$.

First we want to argue that there exists an optimal control. The proof relies on the standard procedure of variational calculus.

Proposition 20. *The continuous optimal control problem* (5.1) *possesses an optimal control $\boldsymbol{H}^*$ and an associated optimal state $(\boldsymbol{v}^*,\boldsymbol{d}^*)$.*

Remark 21. *The optimal control $\boldsymbol{H}^*$ is not necessarily unique. Even the associated state $(\boldsymbol{v}^*,\boldsymbol{d}^*)$ to an optimal control $\boldsymbol{H}^*$ may not be unique since the uniqueness for dissipative solutions is not known. The uniqueness of the optimal control is not known, even though the cost functional J in convex in $\boldsymbol{H}$, because the solution set fulfilling Definition 3 is not known to be convex with respect to $\boldsymbol{H}$.*

Proof. First, we observe that the set of possible solutions is not empty. Indeed for every $\boldsymbol{H}\in\boldsymbol{L}^2$ with $\|\boldsymbol{H}\|_{\boldsymbol{L}^3}\leq c_{\boldsymbol{H}}$, there exists a dissipative solution (see Theorem (5)).

In the following, we consider a minimizing sequence $(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}_n)$ of the optimization problem 5.1. Choosing $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}},\tilde{\boldsymbol{H}})=(0,\mathbb{E}\boldsymbol{d}_1,0)$ in the dissipative formulation (3) grants *a priori* estimates. Here $\boldsymbol{d}_1\in\boldsymbol{H}^{s-1/2}(\partial\Omega)$ with

$s \in [5/2,3]$ is the boundary condition (see (2.3)) and $\mathbb{E}$ is the extension operator (see Proposition 10). Note that $\mathbb{E}\boldsymbol{d}_1 \in \boldsymbol{H}^s$ with $s \geq 5/2$ such that $\mathbb{E}\boldsymbol{d}_1 \in \boldsymbol{W}^{2,3}$. Extracting weakly converging subsequences provides all convergences in (4.10) and

$$\boldsymbol{H}_n \rightharpoonup \boldsymbol{H} \qquad \text{in } L^2(0,T;\boldsymbol{L}^2)\,. \tag{5.2}$$

The arguments to validate this convergences are essentially the same as in Proposition 16. Note the additional convergence (5.2) due to the boundedness of the cost functional J.

In the same way as in Corollary 17, we observe the convergences in (4.11). This allows to go to the limit in Definition 3. Note that the test functions $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})$ remain fixed. Indeed, the relative energy $\mathscr{E}$ (see (3.1) is lower-semi-continuous with respect to the convergences (4.11). Especially the convergence (4.10i) and (5.2) imply that

$$\begin{aligned} -\chi_{\|}\|\boldsymbol{d}\cdot\boldsymbol{H}-\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}}\|_{L^2}^2-\chi_{\perp}\|\boldsymbol{d}\times\boldsymbol{H}-\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2+\|\boldsymbol{H}-\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2 \leq \\ \liminf_{n\to\infty}\left(-\chi_{\|}\|\boldsymbol{d}_n\cdot\boldsymbol{H}_n-\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{H}}\|_{L^2}^2-\chi_{\perp}\|\boldsymbol{d}_n\times\boldsymbol{H}_n-\tilde{\boldsymbol{d}}\times\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2+\|\boldsymbol{H}_n-\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2}^2\right)\,. \end{aligned}$$

The same holds true for the relative dissipation $\mathscr{W}$ (see (3.2)) and the convergences (4.10b)-(4.10e). In the term $\mathscr{A}$, the weak convergences (4.10) and (5.2) suffice to go to the limit in $(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{d}_n\times\boldsymbol{q}_n)$. The term $\mathscr{K}$ (see (3.8)) almost exclusively depends on the test functions $(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{d}})$. To go to the limit in $\mathscr{K}$ it is crucial that we replaced the $\|\boldsymbol{H}\|_{\boldsymbol{L}^3}$-norm by $c_{\boldsymbol{H}}$ in (5.1). Together, we may infer that the limit $(\boldsymbol{v},\boldsymbol{d},\boldsymbol{H})$ fulfills Definition 3. From (4.11a), (4.11b), and (5.2), we conclude

$$J(\boldsymbol{v},\boldsymbol{d},\boldsymbol{H}) \leq \liminf_{n\to\infty} J(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}_n)\,. \tag{5.3}$$

Therefore, the infimum is actually attained and $\boldsymbol{H}$ is an optimal control, whereas $(\boldsymbol{v},\boldsymbol{d})$ is the associated optimal state. □

5.2 Approximation of the optimal control problem

The continuous problem (5.1) is approximated by approximating the state equation and state inequality (due to the dissipative solvability concept) in the way executed in Section 4. The approximate problems read as

Approximate optimal control problem

$$\min_{\boldsymbol{H}_n\in Z_n,\,\|\boldsymbol{H}_n\|_{\boldsymbol{L}^3}\leq c_{\boldsymbol{H}}} J(\boldsymbol{v}_n,\boldsymbol{d}_n,\boldsymbol{H}_n) := \min_{\boldsymbol{H}_n\in Z_n,\,\|\boldsymbol{H}_n\|_{\boldsymbol{L}^3}\leq c_{\boldsymbol{H}}} \|\boldsymbol{v}_n(T)-\boldsymbol{v}_T\|_{\boldsymbol{L}^2}^2+\|\boldsymbol{d}_n(T)-\boldsymbol{d}_T\|_{\boldsymbol{H}^1}^2+\gamma\|\boldsymbol{H}_n\|_{\boldsymbol{L}^2}^2\,. \tag{5.4}$$

for $\boldsymbol{v}_T \in \boldsymbol{L}^2_\sigma$ and $\boldsymbol{d}_T \in \boldsymbol{H}^1$ given such that $(\boldsymbol{v}_n,\boldsymbol{d}_n)$ solves the approximate scheme (4.2).

The optimal control problem (5.4) is an optimal control problem for ordinary differential equations. Note that the inequality is replaced by an equality and the solution operator becomes differentiable.

Therefore, we do not comment on the solvability of this problem. We rather assume that problem (5.4) has a solution for every $n \in \mathbb{N}$ and show that a subsequence converges to an optimal control in a suitable sense. This suitable sense is rather weak, since we can only show that such sequence converges under additional assumption on the optimal control. Nevertheless, it is possible to show the lower-semi-continuity of the cost functional for every approximating sequence. It is remarkable that even though the control enters the system nonlinearly, only weak convergence of the control is needed to go to the limit in the formulation. Thus, boundedness of the control in some $\boldsymbol{L}^p$ spaces suffice to use some weak compactness arguments and go to the limit for a subsequence.

Proposition 22. *Let for every $n \in \mathbb{N}$ the triple $(\boldsymbol{v}_n, \boldsymbol{d}_n, \boldsymbol{H}_n)$ be a solution to the optimal control problem* (5.4)*. There exists $(\boldsymbol{v}, \boldsymbol{d}, \boldsymbol{H})$ fulfilling Definition 3 and a subsequence $\{(\boldsymbol{v}_n, \boldsymbol{d}_n, \boldsymbol{H}_n)\}_{n\in\mathbb{N}}$ converging in the sense of* (4.10) *and* (5.2) *to $(\boldsymbol{v}, \boldsymbol{d}, \boldsymbol{H})$. Furthermore, the inequality* (5.3) *holds.*

Assume additionally that there exists an optimal control and state $(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}, \tilde{\boldsymbol{H}})$ of problem 5.1 fulfilling additional regularity assumptions, i.e., (3.5) *with $|\tilde{\boldsymbol{d}}| = 1$ a.e. in $\Omega \times (0,T)$ and $\tilde{\boldsymbol{H}} \in \boldsymbol{L}^\infty$. Then the limit $(\boldsymbol{v}, \boldsymbol{d}, \boldsymbol{H})$ of the sequence $\{(\boldsymbol{v}_n, \boldsymbol{d}_n, \boldsymbol{H}_n)\}$ is an optimal control and state, i.e., $J(\boldsymbol{v}_n, \boldsymbol{d}_n, \boldsymbol{H}_n) \to J(\boldsymbol{v}, \boldsymbol{d}, \boldsymbol{H}) = J(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}, \tilde{\boldsymbol{H}})$ as $n \to \infty$.*

Proof. Since $(\boldsymbol{v}_n, \boldsymbol{d}_n)$ solves the approximate scheme (4.2) for every given $\boldsymbol{H}_n$ and for all $n \in \mathbb{N}$, it can be shown in the same fashion as in the proof of Theorem 11 that the approximate relative energy inequality (4.3) is fulfilled for $(\boldsymbol{v}_n, \boldsymbol{d}_n)$ with $\boldsymbol{H} = \boldsymbol{H}_n$. The boundedness of the cost functional J implies the existence of an $\boldsymbol{H} \in \boldsymbol{L}^3$ with $\|\boldsymbol{H}\|_{\boldsymbol{L}^3} \le c_{\boldsymbol{H}}$ and the weak convergence $\boldsymbol{H}_n \rightharpoonup \boldsymbol{H}$ in $\boldsymbol{L}^2$. With the same line of reasoning as in the proof of Theorem 5, we can show that there exists $(\boldsymbol{v}, \boldsymbol{d}, \boldsymbol{d} \times \boldsymbol{q})$, which obey (3.3) and a subsequence converging to this triple in the sense of Theorem 5 such that $(\boldsymbol{v}, \boldsymbol{d}, \boldsymbol{d} \times \boldsymbol{q})$ fulfills Definition 3 for $\boldsymbol{H}$. The inequality (5.3) follows again by lower-semi-continuity with respect to the convergences (4.10) and (5.2).

Assume now that there exists an optimal control and state $(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}, \tilde{\boldsymbol{H}})$ of problem (5.1) fulfilling additional regularity assumptions, *i.e.*, (3.5) with $|\tilde{\boldsymbol{d}}| = 1$ a.e. in $\Omega \times (0,T)$ and $\tilde{\boldsymbol{H}} \in \boldsymbol{L}^\infty$. We define $\tilde{\boldsymbol{H}}_n := R_n \tilde{\boldsymbol{H}}$. Then there exists a solution $(\tilde{\boldsymbol{v}}_n, \tilde{\boldsymbol{d}}_n)$ to the approximate problem (4.2) with magnetic field $\tilde{\boldsymbol{H}}_n$ for every $n \in \mathbb{N}$. The same line of reasoning as in the first part of the proof implies that there exists a subsequence $\{(\tilde{\boldsymbol{v}}_n, \tilde{\boldsymbol{d}}_n)\}_{n\in\mathbb{N}}$ that converges to a dissipative solution for the magnetic field $\tilde{\boldsymbol{H}}$. Due to the additional regularity assumptions on $(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}, \tilde{\boldsymbol{H}})$, this dissipative solution is known to be unique such that from the approximate relative energy inequality (4.3) and $\tilde{\boldsymbol{H}}_n \to \tilde{\boldsymbol{H}}$ in $\boldsymbol{L}^2$, we may infer that $(\tilde{\boldsymbol{v}}_n, \tilde{\boldsymbol{d}}_n, \tilde{\boldsymbol{H}}_n) \to (\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}, \tilde{\boldsymbol{H}})$ in $L^2(0,T; \boldsymbol{H}^1_{0,\sigma}) \times L^2(0,T; \boldsymbol{H}^1) \times \boldsymbol{L}^2$ as $n \to \infty$. In this case even a stronger convergence result follows, inserting $(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}, \tilde{\boldsymbol{H}})$ as test functions in the approximate relative energy inequality (4.3) even grants norm convergence such that (see the proof of Theorem 5)

$$\tilde{\boldsymbol{v}}_n \to \tilde{\boldsymbol{v}} \quad \text{in } \mathscr{C}([0,T]; \boldsymbol{L}^2_\sigma) \quad \text{and} \quad \tilde{\boldsymbol{d}}_n \to \tilde{\boldsymbol{d}} \quad \text{in } \mathscr{C}([0,T]; \boldsymbol{H}^1) \,. \tag{5.5}$$

Observe that $(\tilde{\boldsymbol{v}}_n, \tilde{\boldsymbol{d}}_n, \tilde{\boldsymbol{H}}_n)$ is a solution to the approximate optimal control problem (5.4) such that it holds

$$J(\boldsymbol{v}_n, \boldsymbol{d}_n, \boldsymbol{H}_n) \le J(\tilde{\boldsymbol{v}}_n, \tilde{\boldsymbol{d}}_n, \tilde{\boldsymbol{H}}_n)$$

for all $n \in \mathbb{N}$, where $(\boldsymbol{v}_n, \boldsymbol{d}_n, \boldsymbol{H}_n)$ is assumed to solve (5.4). From the stronger convergence (5.5) and $R_n \tilde{\boldsymbol{H}} \to \tilde{\boldsymbol{H}}$ in $\boldsymbol{L}^2$ we may conclude for the cost functional that $J(\tilde{\boldsymbol{v}}_n, \tilde{\boldsymbol{d}}_n, \tilde{\boldsymbol{H}}_n) \to J(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}, \tilde{\boldsymbol{H}})$. The inequality (5.3) and the observation that $J(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{d}}, \tilde{\boldsymbol{H}})$ is actually a minimum of (5.1) imply the assertion. □

Remark 23. *Without this additional regularity assumption in Proposition 22, the solution of the problem is not known to be unique. In this case, it is not clear what convergence to the solution means. It may be possible to consider the convergence of the sets of solutions to each other, for the convergence of the control parameter. But a difficulty arises since the solutions sets are not expected to be convex.*

Thus, the weak-strong optimal control scheme concept seems to be the best possibility up to now.

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