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ABSTRACT. In this paper we develop are two-scale model for a nonlinear parabolic system. Assuming a rapidly oscillating inhomogeneity with period ε for one equation we carry out a formal periodic expansion to obtain a homogenized equation coupled to a local in time cell problem. We justify the expansion by deriving an error estimate between the original and the two-scale model and show numerical simulations, which confirm the analytically derived error estimate.

1. INTRODUCTION

Many technical or biological systems exhibit multiple spatial and/or temporal scales, which often can be utilized for a more efficient numerical or semianalytical approximation based on asymptotic analysis or homogenization. While especially spatial homogenization has been well-investigated both analytically and numerically [1, 2, 3, 5], the influence of multiple temporal scales seems to be less investigated analytically. Examples where averaging and homogenization techniques have been used numerically in technical applications are the inductive heating of metal components, where a metal component is heated by a rapidly alternating induction current [8, 10], and models of high-cycle fatigue, where a technical system, e.g., a wind turbine, is subject to a rapidly changing load, while damage occurs on a much larger timescale [6, 11].

The generic temporal multiscale problem studied in this paper is motivated by induction heating and reads

$$\theta_{\varepsilon,t} - \Delta\theta_\varepsilon = \gamma(\theta_\varepsilon)|\nabla h_\varepsilon|^2 \tag{1.1a}$$

$$\varepsilon h_{\varepsilon,t} - \text{div}\,(\gamma(\theta_\varepsilon)\text{grad}\, h_\varepsilon) = u_1(t)\phi_1(\frac{t}{\varepsilon}) + \varepsilon u_2(t)\phi_2(\frac{t}{\varepsilon}) \quad \text{in } Q \tag{1.1b}$$

$$-\frac{\partial\theta_\varepsilon}{\partial\nu} = \theta_\varepsilon \quad \text{in } \Sigma \tag{1.1c}$$

$$h_\varepsilon = 0 \quad \text{in } \Sigma \tag{1.1d}$$

$$\theta_\varepsilon(0) = \theta_I \quad \text{in } \Omega \tag{1.1e}$$

$$h_\varepsilon(0) = h_I \quad \text{in } \Omega. \tag{1.1f}$$

Here, $\Omega \subset \mathbb{R}^3$ is a domain with sufficiently smooth boundary, $Q = \Omega \times (0,T)$ the space-time cylinder with lateral boundary $\Sigma = \partial\Omega \times (0,T)$. The main contributions of the present paper are the derivation of an asymptotic temporal two-scale model for (1.1) consisting of a homogenized coarse scale problem and a periodic local in time cell problem, together with an error analysis confirmed by numerical results. Our approach is inspired by a corresponding spatial homogenization result for phase field models of liquid-solid phase transitions [4]. Further results for formal asymptotic averaging and homogenization approaches with multiple temporal scales can be found in [2, 6, 11] and the references therein. For analytical two-scale convergence results for parabolic problems we refer to [9, 7].

The paper is organized as follows: in Section 2 we consider induction heating as a motivating example for our model problem (1.1). In Section 3 we formulate our main result and use formal asymptotic expansions to derive the temporal two-scale model. The following section is devoted to proving the error estimate. Section 5 contains numerical results which confirm the error analysis.

2. AN APPLICATION PROBLEM – JOULE HEATING

As shown in [8], the Joule heating of a steel rod with cross section $\tilde{\Omega}$ subject to a rapidly oscillating magnetic field with angular frequency $\tilde{\omega}$ can be described by a nonlinear coupled system of parabolic

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PDEs for temperature $\tilde{\theta}$ and magnetic field $\tilde{h}$,

$$
\begin{aligned}
\rho c\tilde{\theta}_{\tilde{t}} - K\tilde{\Delta}\tilde{\theta} &= \tilde{\gamma}(\tilde{\theta})|\tilde{\nabla}\tilde{h}|^2 && \text{in } \tilde{Q}\\
\mu\tilde{h}_{\tilde{t}} - \tilde{\nabla}\cdot\left(\tilde{\gamma}(\tilde{\theta})\tilde{\nabla}\tilde{h}\right) &= 0 && \text{in } \tilde{Q}\\
-K\frac{\partial\tilde{\theta}}{\partial\tilde{\nu}} &= \tilde{\alpha}(\tilde{\theta}-\theta_\Gamma) && \text{in } \tilde{\Sigma}\\
\tilde{h} &= \tilde{u}(\tilde{t})\sin(\tilde{\omega}\tilde{t}) && \text{in } \tilde{\Sigma}\\
\tilde{\theta}(0) &= \theta_0 && \text{in } \tilde{\Omega}\\
\tilde{h}(0) &= h_0 && \text{in } \tilde{\Omega}.
\end{aligned}
$$

with physical constants $\rho,\ c,\ K,\ \theta_\Gamma,\ \tilde{\alpha}$ and a temperature dependent resistivity $\tilde{\gamma}(\tilde{\theta})$. $\tilde{Q} = \tilde{\Omega}\times(0,\tilde{T})$ is the space-time cylinder with lateral boundary $\tilde{\Sigma}$. For a nondimensionalization of the system, we choose a temperature scale θ_s, length scale L, time scale $t_s := \frac{\rho c L^2}{K}$, and introduce non-dimensional quantities

$$
\theta = \frac{\tilde{\theta}-\theta_\Gamma}{\theta_s}\,,\ x = \frac{\tilde{x}}{L}\,,\ t = \frac{\tilde{t}}{t_s}.
$$

Then we obtain the nondimensionalized energy balance

$$
\begin{aligned}
\theta_t - \Delta\theta &= \frac{L^2}{\theta_s K}\tilde{\gamma}(\theta_s\theta+\theta_\Gamma)|\tilde{\nabla}\tilde{h}|^2 && \text{in } Q\\
-\frac{\partial\theta}{\partial\nu} &= \frac{L}{K}\tilde{\alpha}\theta && \text{in } \Sigma\\
\theta(0) &= \frac{\tilde{\theta}_0-\theta_\Gamma}{\theta_s} =: \theta_0 && \text{in } \Omega.
\end{aligned}
$$

For the Maxwell system we introduce a magnetic field scale h_s, define

$$
h(x,t) = \frac{\tilde{h}(\tilde{x},\tilde{t})}{h_s} = \frac{\tilde{h}(xL,t_s t)}{h_s}
$$

and $\eta = \dfrac{\mu}{\gamma_m}\dfrac{L^2}{t_s}$ to obtain

$$
\begin{aligned}
\eta h_t - \text{div}\left(\frac{\tilde{\gamma}(\tilde{\theta})}{\gamma_m}\text{grad } h\right) &= 0 && \text{in } Q\\
h(t) &= \frac{\tilde{u}(t_s t)}{h_s}\sin(\tilde{\omega}t_s t) && \text{in } \Sigma\\
h(0) &= \frac{\tilde{h}_0}{h_s} =: h_0 && \text{in } \Omega.
\end{aligned}
$$

We choose data for a plain carbon steel according to [8],

$$
\rho = 7.85\frac{g}{cm^3},\ c = 0.5096\frac{J}{gK},\ K = 0.5\frac{J}{gK},\ \gamma(\theta) = c_1 + c_2\theta + c_3\theta^2 + c_4\theta^3\frac{Vcm}{A}
$$

with $c_1 = 4.9656\cdot10^{-5}$, $c_2 = 8.4121\cdot10^{-8}$, $c_3 = -3.7246\cdot10^{-11}$, $c_4 = 6.1960\cdot10^{-15}$, and choose $L = 1$cm, $h_s = 1\frac{A}{\text{cm}}$. Then, we have

$$
t_s = \frac{7.85\cdot0.5096}{0.5} \approx 8.0s \quad \gamma_m = \max_{\theta\in[20,1200]}\frac{1}{\sigma(\theta)} \approx 1.1\cdot10^{-4}\frac{Vcm}{A}
$$

$$
\mu = 4\pi10^{-9}\frac{Vs}{Acm} \quad \eta = \frac{4\pi\cdot10^{-9}}{1.1\cdot10^{-4}}\frac{1}{8} \approx 1.4\cdot10^{-5}.
$$

Now, we define

$$\gamma(\theta) = \frac{\tilde\gamma(\theta\cdot\theta_s + \theta_\Gamma)}{\gamma_m}, \quad u(t) = \frac{\hat h(t\cdot t_s)}{h_s}, \quad \omega = \tilde\omega t_s \eta, \quad \beta = \frac{h_s^2\gamma_m}{\theta_s K}\,,\ \alpha = \frac{L}{K}\tilde\alpha$$

and obtain finally

$$\theta_t - \Delta\theta = \beta\gamma(\theta)|\nabla h|^2 \quad \text{in } Q \tag{2.1a}$$

$$\eta h_t - \text{div}\,(\gamma(\theta)\text{grad}\,h) = 0 \quad \text{in } Q \tag{2.1b}$$

$$-\frac{\partial\theta}{\partial\nu} = \alpha\theta \quad \text{in } \Sigma \tag{2.1c}$$

$$h(t) = u(t)\sin\left(\omega\frac{t}{\eta}\right) \quad \text{in } \Sigma \tag{2.1d}$$

$$\theta(0) = \theta_I \quad \text{in } \Omega \tag{2.1e}$$

$$h(0) = h_I \quad \text{in } \Omega. \tag{2.1f}$$

Note that typical annular frequencies are in the range $10^4 - 10^6$, hence we may assume $\omega = \tilde\omega t_s\eta \sim O(1)$. Hence we can separate two time scales. In the sequel we assume $\omega = 2\pi$. Then, for any solution (θ, h) to (2.1) $(\theta, h - u(t)\sin(\omega\frac{t}{\eta}))$ is a solution to the model problem (1.1) with homogeneous boundary conditions by choosing in addition $\alpha = \beta = 1$.

3. Asymptotic expansions, assumptions, and main result

3.1. **Derivation of the two-scale model.** To derive a two-scale model for (1.1), we introduce a new, microscopic time variable $\tau = \frac{t}{\varepsilon}$ with $\tau \in Y = (0,1)$, and assume the existence of asymptotic expansions

$$\theta_\varepsilon(x,t) = \sum_{m=0,1,\dots} \varepsilon^m\theta_m(x,\tau,t) \tag{3.1}$$

$$h_\varepsilon(x,t) = \sum_{m=0,1,\dots} \varepsilon^m h_m(x,\tau,t). \tag{3.2}$$

Moreover, we consider a Taylor expansion for $\gamma(\theta)$ with respect to ε

$$\gamma^\varepsilon := \gamma(\theta_\varepsilon) = \gamma(\theta_0) + \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + \dots \tag{3.3}$$

Note that the validity of this expansion with remainder of order ε^k requires $\gamma(\cdot) \in C^k$.

Substituting (3.1)-(3.3) into the original system (1.1), we compare the coefficients of different powers of ε, starting from the lowest order. We summarize the following equations

$$\begin{aligned}
O\left(\frac{1}{\varepsilon}\right):&\quad \theta_{0,\tau} = 0\\
O(1):&\quad \theta_{1,\tau} + \theta_{0,t} - \Delta\theta_0 = \gamma(\theta_0))\,|\nabla h_0|^2\\
&\quad h_{0,\tau} - \nabla\cdot(\gamma(\theta_0))\nabla h_0) = u_1(t)\phi_1(\frac{t}{\varepsilon})\\
O(\varepsilon):&\quad \theta_{2,\tau} + \theta_{1,t} - \Delta\theta_1 = \gamma_1\,|\nabla h_0|^2 + 2\gamma(\theta_0))\nabla h_0\cdot\nabla h_1\\
&\quad (h_{1,\tau} + h_{0,t}) - \nabla\cdot(\gamma_1\nabla h_0) - \nabla\cdot(\gamma(\theta_0))\nabla h_1) = u_2(t)\phi_2(\frac{t}{\varepsilon}).
\end{aligned}$$

The *problem of first order* consists of the terms of order ε^{-1} in the heat equations, i.e.,

$$\theta_{0,\tau} = 0 \tag{3.4}$$

in $\Omega \times (0,T) \times Y$ with unit interval $Y = (0,1)$. One can immediately infer that the solutions θ_0 are constant with respect to τ and thus $\theta_0 = \theta_0(x,t)$.

The *problem of second order* is given by the terms of order ε^0, i.e.,

$$\theta_{1,\tau} + \theta_{0,t} - \Delta\theta_0 = \gamma(\theta_0))\,|\nabla h_0|^2 \tag{3.5}$$

$$h_{0,\tau} - \nabla\cdot(\gamma(\theta_0))\nabla h_0) = u_1(t)\phi_1(\frac{t}{\varepsilon}) \tag{3.6}$$

in $\Omega \times (0,T) \times Y$, with periodic boundary condition on Y for θ_1, h_0, h_1. Taking the τ-average in (3.5) and noticing θ_0 is τ-independent, we obtain

$$\theta_{0,t} - \Delta\theta_0 = \gamma(\theta_0))\int_0^1 |\nabla h_0(x,\tau,t)|^2\,d\tau \quad \text{in} \quad \Omega \times (0,T)$$

$$\theta_{1,\tau} = \gamma(\theta_0))\,|\nabla h_0(x,\tau,t)|^2 - \gamma(\theta_0))\int_0^1 |\nabla h_0(x,\tau,t)|^2\,d\tau \quad \text{in} \quad \Omega \times Y.$$

The second equation holds true at any time point $t \in (0,T)$.

The *problem of third order* is obtained by the terms of ε^1 in in both equations

$$\theta_{2,\tau} + \theta_{1,t} - \Delta\theta_1 = \gamma_1\,|\nabla h_0|^2 + 2\gamma(\theta_0))\nabla h_0\cdot\nabla h_1$$
$$h_{1,\tau} + h_{0,t} = \nabla\cdot(\gamma_1\nabla h_0) + \nabla\cdot(\gamma(\theta_0))\nabla h_1) + u_2(t)\phi_2(\frac{t}{\varepsilon}).$$

For the two-scale model, we take the leading terms (θ_0, h_0) of the expansion. It consists of the macroscopic heat equation

$$\theta_{0,t} - \Delta\theta_0 = \gamma(\theta_0))\overline{|\nabla h_0|^2} \qquad \text{in} \quad \Omega\times(0,T) \tag{3.7a}$$

$$-\frac{\partial\theta_0}{\partial n} = \theta_0 \qquad \text{on} \quad \partial\Omega\times(0,T) \tag{3.7b}$$

$$\theta_0(x,0) = \theta_I \qquad \text{in} \quad \Omega. \tag{3.7c}$$

with averaged Joule heat term

$$\overline{|\nabla h_0|^2}(x,t) = \int_0^1 |\nabla h_0(x,\tau,t)|^2 d\tau. \tag{3.7d}$$

Second part is a local in time cell problem to compute the heat source,

$$h_{0,\tau} - \nabla\cdot(\gamma(\theta_0))\nabla h_0) = u_1(t)\phi_1(\tau) \qquad \text{in} \quad \Omega\times Y \tag{3.7e}$$

$$h_0(x,\tau,t) = 0 \qquad \text{on} \quad \partial\Omega\times Y \tag{3.7f}$$

$$h_0(x,\tau=0,t) = h_I \qquad \text{in} \quad \Omega. \tag{3.7g}$$

The latter has to be solved for every point $t \in [0,T]$ of the macroscopic time interval.

3.2. **Assumptions and main result.** From now on, we use a superscript $(\cdot)^\varepsilon$ denoting the macroscopic approximation of the microscopic system when $\tau = \frac{t}{\varepsilon}$. For instance, we denote

$$h_0^\varepsilon := h_0(x,\tau,t)|_{\tau=\frac{t}{\varepsilon}}$$

be the solution of the two-scale problem in its macroscopic approximation when $\tau = \frac{t}{\varepsilon}$. Then, the following notations are straightforward

$$
\begin{aligned}
(\partial_\tau h_0)^\varepsilon &= (\partial_\tau h_0(x,\tau,t))\Big|_{\tau=\frac{t}{\varepsilon}} \\
\partial_t h_0^\varepsilon &= \partial_t h_0(x,\tau,t)\Big|_{\tau=\frac{t}{\varepsilon}} + \frac{1}{\varepsilon}\partial_\tau h_0(x,\tau,t)\Big|_{\tau=\frac{t}{\varepsilon}}.
\end{aligned}
\tag{3.8}
$$

We assume the following regularity for the data.

Assumption 3.1.

$$
\begin{aligned}
&(D1) \quad \gamma \in C^1(\mathbb{R}), \quad \gamma_1 \geq \gamma \geq \gamma_0 > 0 \\
&(D2) \quad u_{1,2} \in C^1(0,T) \qquad \textit{and} \qquad \phi_{1,2} \in C^1_\#(Y).
\end{aligned}
$$

Here, γ_0 is some constant and $C^k_\#(Y)$ the space of periodic C^k- functions on $Y := [0,1]$.

The main focus of the present paper is an error analysis between the original system (1.1) and the two-scale system (3.7). To this end we simply assume the existence of solutions to both systems satisfying the following regularity. For the multiscale system a regularity analysis can be found in [12, Theorem 3].

Assumption 3.2.

$$
\begin{aligned}
&(R1) \quad h_\varepsilon \in L^2(0,T;W^{1,\infty}(\Omega)) \cap H^1(0,T;L^\infty(\Omega)) \\
&(R2) \quad h_0 \in W^1_4(0,T;C^0_\#(Y),W^1_4(\Omega)) \cap L^\infty(0,T;C^1_\#(Y),L^2(\Omega)) \\
&(R3) \quad \theta_\varepsilon \in L^2(0,T;W^{1,2}(\Omega)) \cap H^1(0,T;L^2(\Omega)) \\
&(R4) \quad \theta_0 \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)).
\end{aligned}
$$

Theorem 1. *Let Assumptions 3.1 and 3.2 hold true, then the error estimate between the original multiscale system* (1.1) *and the two-scale system* (3.7) *is*

$$
\begin{aligned}
&\|\theta_\varepsilon - \theta_0\|_{L^\infty((0,T);L^2(\Omega))} + \varepsilon^{\frac{1}{2}}\|h_\varepsilon - h_0^\varepsilon\|_{L^\infty((0,T);L^2(\Omega))} \\
&\quad + \|\nabla(\theta_\varepsilon - \theta_0)\|_{L^2((0,T);L^2(\Omega))} + \|\nabla(h_\varepsilon - h_0^\varepsilon)\|_{L^2((0,T);L^2(\Omega))} \leq C\varepsilon^{\frac{1}{2}}
\end{aligned}
$$

where C is a generic constant.

4. Proof of Theorem 1

4.1. **Error estimate of the heat equation.** Taking the difference between the original coupled system and the macroscopic approximation, we obtain, for the heat equation

$$
\partial_t(\theta_\varepsilon - \theta_0) - \Delta(\theta_\varepsilon - \theta_0) = \gamma(\theta_\varepsilon)|\nabla h_\varepsilon|^2 - \gamma(\theta_0)\overline{|\nabla h_0|^2}. \tag{4.1}
$$

Now, we test (4.1) with $\theta_\varepsilon - \theta_0$ and integrate on $\Omega \times (0,t)$ for fixed $t > 0$ and obtain

$$
\begin{aligned}
&\int_0^t\int_\Omega \partial_t(\theta_\varepsilon - \theta_0)(\theta_\varepsilon - \theta_0)dxds - \int_0^t\int_\Omega \Delta(\theta_\varepsilon - \theta_0)(\theta_\varepsilon - \theta_0)dxds \\
&\qquad = \int_0^t\int_\Omega (\gamma(\theta_\varepsilon)|\nabla h_\varepsilon|^2 - \gamma(\theta_0)\overline{|\nabla h_0|^2})(\theta_\varepsilon - \theta_0)dxds.
\end{aligned}
$$

Denoting

$$I := \int_0^t\int_\Omega \partial_s(\theta_\varepsilon - \theta_0)(\theta_\varepsilon - \theta_0)dxds$$

$$II := -\int_0^t\int_\Omega \Delta(\theta_\varepsilon - \theta_0)(\theta_\varepsilon - \theta_0)dxds$$

$$III := \int_0^t\int_\Omega (\gamma(\theta_\varepsilon)|\nabla h_\varepsilon|^2 - \gamma(\theta_0)\overline{|\nabla h_0|^2})(\theta_\varepsilon - \theta_0)dxds$$

we estimate each term separately.

Since θ_ε and θ_0 have the same initial condition θ_I, we have

$$\begin{aligned} I =& \frac{1}{2}\int_0^t \frac{d}{ds}\left(\int_\Omega |\theta_\varepsilon - \theta_0|^2 dx\right) ds \\ =& \frac{1}{2}\|(\theta_\varepsilon - \theta_0)(t)\|^2_{L^2(\Omega)}. \end{aligned}$$

With the aid of Green's formula and the Robin boundary condition for θ_ε and θ_0, we obtain

$$\begin{aligned} II =& \int_0^t\int_\Omega \nabla(\theta_\varepsilon - \theta_0)\cdot\nabla(\theta_\varepsilon - \theta_0)dxds - \int_0^t\int_{\partial\Omega} \frac{\partial(\theta_\varepsilon - \theta_0)}{\partial n}(\theta_\varepsilon - \theta_0)dxds \\ =& \|\nabla(\theta_\varepsilon - \theta_0)\|^2_{L^2(Q_t)} + \|\theta_\varepsilon - \theta_0\|^2_{L^2(\Sigma_t)} \end{aligned}$$

where we have used the abbreviations $Q_t = \Omega \times (0,t)$ and $\Sigma_t = \partial\Omega \times (0,t)$. Together, we obtain

$$I + II \geq \frac{1}{2}\|(\theta_\varepsilon - \theta_0)(t)\|^2_{L^2(\Omega)} + \|\nabla(\theta_\varepsilon - \theta_0)\|^2_{L^2(Q_t)}. \tag{4.2}$$

The remaining heat source term

$$\begin{aligned} III =& \int_0^t\int_\Omega (\gamma(\theta_\varepsilon) - \gamma(\theta_0))|\nabla h_\varepsilon|^2(\theta_\varepsilon - \theta_0)dxds \\ &+ \int_0^t\int_\Omega \gamma(\theta_0)(|\nabla h_\varepsilon|^2 - \overline{|\nabla h_0|^2})(\theta_\varepsilon - \theta_0)dxds \end{aligned}$$

shall be calibrated more carefully. By virtue of the property of u, the regularity of h_ε in Assumption 3.1 and the imbedding between $C^1(\mathbb{R})$ and $C^{0,1}(\mathbb{R})$, we derive the estimate of the first term in III

$$\begin{aligned} \int_0^t\int_\Omega (\gamma(\theta_\varepsilon) - \gamma(\theta_0))|\nabla h_\varepsilon|^2(\theta_\varepsilon - \theta_0)dxds &\overset{(D1)}{\leq} C\int_0^t\int_\Omega |\nabla h_\varepsilon|^2|\theta_\varepsilon - \theta_0|^2 dxds \\ &\overset{(R1)}{\leq} C\|\theta_\varepsilon - \theta_0\|^2_{L^2(0,t;L^2(\Omega))} \end{aligned}$$

with a generic constant C. To derive an upper bound for

$$\int_0^t\int_\Omega \gamma(\theta_0)(|\nabla h_\varepsilon|^2 - \overline{|\nabla h_0|^2})(\theta_\varepsilon - \theta_0)dxds$$

the error estimate of $|\nabla h_0^\varepsilon|^2 - \overline{|\nabla h_0|^2}$ is necessary and provided by the following lemma.

Lemma 4.1. *Let the time interval $I_T := [0,T]$ with a finite $T < \infty, Y := [0,1]$. If $f \in H^1((0,T);C^0_\#(Y))$, then $f \in L^\infty((0,T);C^0_\#(Y))$. Moreover, for all $f_0^\varepsilon(t) := f(t,\frac{t}{\varepsilon})$ and $\overline{f_0}(x) := \int_Y f(t,\tau)d\tau$ there holds*

$$\|f_0^\varepsilon - \overline{f_0}\|_{H^1(0,T)^*} \le C\varepsilon\left(\|f\|_{H^1((0,T);C^0(Y))} + \|f\|_{L^\infty((0,T);C^0(Y))}\right). \tag{4.3}$$

Here, $\|\cdot\|_{H^1(0,T)^*}$ stands for the norm of the dual space of $H^1(0,T)$. We note that Lemma 4.1 is a one-dimensional variant of [4, Lemma 4.1], hence we omit the proof and refer to the reference instead. We apply Lemma 4.1 to the second term of III and obtain

$$\begin{aligned}
&\int_0^t\int_\Omega \gamma(\theta_0)(|\nabla h_\varepsilon|^2 - \overline{|\nabla h_0|^2})(\theta_\varepsilon-\theta_0)dxds \\
&= \int_0^t\int_\Omega \gamma(\theta_0)(|\nabla h_\varepsilon|^2 - |\nabla h_0^\varepsilon|^2)(\theta_\varepsilon-\theta_0)dxds \\
&\quad + \int_0^t\int_\Omega \gamma(\theta_0)(|\nabla h_0^\varepsilon|^2 - \overline{|\nabla h_0|^2})(\theta_\varepsilon-\theta_0)dxds \\
&\le C\big(\overbrace{\varepsilon^2 + \|\theta_\varepsilon-\theta_0\|^2_{L^2(0,t;L^2(\Omega))}}^{(D1),(R1),(R2)} + \overbrace{\|\nabla h_\varepsilon - \nabla h_0^\varepsilon\|^2_{L^2(0,t;L^2(\Omega))}}^{(D1),(R4)}\big)
\end{aligned}$$

with aid of the Cauchy-Schwarz inequality as well as the factorization $|\nabla h_\varepsilon|^2 - |\nabla h_0^\varepsilon|^2 = (|\nabla h_\varepsilon| + |\nabla h_0^\varepsilon|)(|\nabla h_\varepsilon| - |\nabla h_0^\varepsilon|)$. We thus derive

$$III \le C\left(\|\theta_\varepsilon-\theta_0\|^2_{L^2(0,t;L^2(\Omega))} + \varepsilon^2 + \|\nabla h_\varepsilon - \nabla h_0^\varepsilon\|^2_{L^2(0,t;L^2(\Omega))}\right). \tag{4.4}$$

Combining the estimates (4.2) and (4.4), we obtain

Proposition 2. *Let Assumption 3.1 holds true, the estimate for the difference between θ_ε and θ_0 is*

$$\begin{aligned}
&\frac{1}{2}\|(\theta_\varepsilon-\theta_0)(t_0)\|^2_{L^2(\Omega)} + \|\nabla(\theta_\varepsilon-\theta_0)\|^2_{L^2(0,t;L^2(\Omega))} \le \\
&\qquad C\left(\|\theta_\varepsilon-\theta_0\|^2_{L^2(0,t;L^2(\Omega))} + \varepsilon^2 + \|\nabla h_\varepsilon - \nabla h_0^\varepsilon\|^2_{L^2(0,t;L^2(\Omega))}\right)
\end{aligned} \tag{4.5}$$

where C is a generic constant.

4.2. **Error estimate of the Maxwell equation.** In the sequel we will utilize a time-variable cut-off function χ_ε^T enjoying the following properties

Assumption 4.1.

$$\begin{aligned}
&(CO1)\quad \chi_\varepsilon^T \in C_0^1(0,T),\quad 0 \le \chi_\varepsilon^T \le 1,\quad \varepsilon|\chi_\varepsilon^{T'}| \le C \\
&(CO2)\quad \chi_\varepsilon^T(t) = 1 \quad \textit{for} \quad t \in (\varepsilon, T-\varepsilon) \\
&(CO3)\quad \|1-\chi_\varepsilon^T\|_{L^2(0,T)} \le C\varepsilon^{\frac{1}{2}} \quad \textit{and} \quad \|\chi_\varepsilon^{T'}\|_{L^2(0,T)} \le C\varepsilon^{-\frac{1}{2}}.
\end{aligned}$$

Taking the difference between the original coupled system (1.1) and the macroscopic approximation (3.7), we obtain for the Maxwell equation

$$\varepsilon\partial_t h_\varepsilon - (\partial_\tau h_0)^\varepsilon + \nabla\cdot(\gamma(\theta_0)\nabla h_0^\varepsilon) - \nabla(\gamma(\theta_\varepsilon)\nabla h_\varepsilon) = \varepsilon u_2(t)\phi_2(\frac{t}{\varepsilon}) \tag{4.6}$$

with homogeneous boundary condition and zero initial condition.

Choosing $h_\varepsilon - h_0$ as a test function and integrating on $\Omega \times (0,T)$ for arbitrary $t > 0$ we obtain

$$
\begin{aligned}
&- \int_0^t\!\!\int_\Omega (\partial_\tau h_0)^\varepsilon (h_\varepsilon - h_0^\varepsilon) dxds + \int_0^t\!\!\int_\Omega \varepsilon \partial_t h_\varepsilon (h_\varepsilon - h_0^\varepsilon) dxds \\
&+ \int_0^t\!\!\int_\Omega \nabla \cdot (\gamma(\theta_0) \nabla h_0^\varepsilon)(h_\varepsilon - h_0^\varepsilon) dxds - \int_0^t\!\!\int_\Omega \nabla \cdot (\gamma(\theta_\varepsilon) \nabla h_\varepsilon)(h_\varepsilon - h_0^\varepsilon) dxds \\
&\quad = \varepsilon \int_0^t\!\!\int_\Omega u_2(s) \phi_2(\frac{s}{\varepsilon})(h_\varepsilon - h_0^\varepsilon) dxds.
\end{aligned}
$$

Denote the following items

$$
\begin{aligned}
\text{(i)} &:= - \int_0^t\!\!\int_\Omega (\partial_\tau h_0)^\varepsilon (h_\varepsilon - h_0^\varepsilon) dxds \\
\text{(ii)} &:= \int_0^t\!\!\int_\Omega \varepsilon \partial_s h_\varepsilon (h_\varepsilon - h_0^\varepsilon) dxds \\
\text{(iii)} &:= \int_0^t\!\!\int_\Omega \nabla \cdot (\gamma(\theta_0) \nabla h_0^\varepsilon)(h_\varepsilon - h_0^\varepsilon) dxds \\
\text{(iv)} &:= - \int_0^t\!\!\int_\Omega \nabla \cdot (\gamma(\theta_\varepsilon) \nabla h_\varepsilon)(h_\varepsilon - h_0^\varepsilon) dxds \\
\text{(v)} &:= -\varepsilon \int_0^t\!\!\int_\Omega u_2(s) \phi_2(\frac{s}{\varepsilon})(h_\varepsilon - h_0^\varepsilon) dxds.
\end{aligned}
$$

We focus on the first term (i). Since the solutions of the macroscopic problems do not satisfy any boundary condition on the boundary of the microscopic domain, we then need the cut-off function introduced in Assumption 4.1. The following decomposition is important

$$
(\partial_\tau h_0)^\varepsilon = (1 - \chi_\varepsilon^T)(\partial_\tau h_0)^\varepsilon + \varepsilon \chi_\varepsilon^T \partial_t h_0^\varepsilon - \varepsilon \chi_\varepsilon^t (\partial_t h_0)^\varepsilon \tag{4.7}
$$

recalling the formula in (3.8)

$$
\partial_t h_0^\varepsilon = \partial_t h_0(x,\tau,t)\Big|_{\tau=\frac{t}{\varepsilon}} + \frac{1}{\varepsilon} \partial_\tau h_0(x,\tau,t)\Big|_{\tau=\frac{t}{\varepsilon}}
$$

and

$$
(\partial_t h_0)^\varepsilon = \partial_t h_0(x,\tau,t)\Big|_{\tau=\frac{t}{\varepsilon}}.
$$

In particular, we also have

$$
(\partial_\tau h_0)^\varepsilon = \varepsilon \left(\partial_t h_0^\varepsilon - (\partial_t h_0)^\varepsilon \right).
$$

For simplicity's sake, we denote v_ε be the test function $h_\varepsilon - h_0^\varepsilon$ and reformulate the integral in item (i) by

$$\begin{aligned}\int_0^t\int_\Omega (\partial_\tau h_0)^\varepsilon v_\varepsilon dxds = \int_0^t\int_\Omega (1-\chi_\varepsilon^t)(\partial_\tau h_0)^\varepsilon v_\varepsilon dxds - \int_0^t\int_\Omega \varepsilon\chi_\varepsilon^t(\partial_t h_0)^\varepsilon v_\varepsilon dxds\\ + \int_0^t\int_\Omega \varepsilon\chi_\varepsilon^t\partial_t h_0^\varepsilon v_\varepsilon dxds.\end{aligned} \tag{4.8}$$

The first integral on the right-hand side of (4.8) can be estimated by

$$\begin{aligned}&\int_0^t\int_\Omega (1-\chi_\varepsilon^t)(\partial_\tau h_0)^\varepsilon v_\varepsilon dxds\\ &= \Big(\|1-\chi_\varepsilon^t\|_{L^2(0,t)}\|\partial_\tau h_0\|_{L^\infty(0,t;C^0_\#(Y),L^2(\Omega))} + \varepsilon\|\partial_t h_0\|_{L^2(0,t;C^0_\#(Y),L^2(\Omega))}\\ &\quad + \varepsilon\|\chi_\varepsilon'\|_{L^2(0,t)}\|h_0\|_{L^\infty(0,t;C^0_\#(Y),L^2(\Omega))}\Big)\|v_\varepsilon\|_{L^2(Q_t)}\\ &\quad + \|1-\chi_\varepsilon^t\|_{L^2(0,t)}\|h_0\|_{L^\infty(0,t;C^0_\#(Y),L^2(\Omega))}\varepsilon\|\partial_t v_\varepsilon\|_{L^2(Q_t)}\\ &\le C\overbrace{(\varepsilon+\sqrt{\varepsilon})\|v_\varepsilon\|_{L^2(Q_t)}}^{(R2),(CO3)} + C\overbrace{\varepsilon^{\frac{3}{2}}\|\partial_t v_\varepsilon\|_{L^2(Q_t)}}^{(CO3)}\end{aligned}$$

by properties of the cut-off function χ_ε^t using Assumptions 3.1 and 4.1. Using integration by parts, we derive the following equalities for the remaining two terms on the right-hand side of (4.8)

$$\begin{aligned}&\int_0^t\int_\Omega \varepsilon\chi_\varepsilon^t\partial_t h_0^\varepsilon v_\varepsilon dxds\\ &= -\int_0^t\int_\Omega \varepsilon\partial_t\chi_\varepsilon^t h_0^\varepsilon v_\varepsilon dxds - \int_0^t\int_\Omega \varepsilon h_0^\varepsilon\partial_t v_\varepsilon\chi_\varepsilon^t dxds\\ &= -\int_0^t\int_\Omega \varepsilon\partial_t\chi_\varepsilon^t h_0^\varepsilon v_\varepsilon dxds + \int_0^t\int_\Omega \varepsilon h_0^\varepsilon\partial_t v_\varepsilon(1-\chi_\varepsilon^t)dxds - \int_0^t\int_\Omega \varepsilon h_0^\varepsilon\partial_t v_\varepsilon dxds\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}&\int_0^t\int_\Omega (\partial_\tau h_0)^\varepsilon v_\varepsilon dxds\\ &= \int_0^t\int_\Omega \Big(\big((1-\chi_\varepsilon^t)(\partial_\tau h_0)^\varepsilon - \varepsilon\chi_\varepsilon^t(\partial_t h_0)^\varepsilon - \varepsilon\partial_t\chi_\varepsilon^t h_0^\varepsilon\big)v_\varepsilon + (1-\chi_\varepsilon^t)\varepsilon h_0^\varepsilon\partial_t v_\varepsilon\Big)dxds\\ &\quad - \int_0^t\int_\Omega \varepsilon h_0^\varepsilon\partial_t v_\varepsilon dxds.\end{aligned} \tag{4.10}$$

Based on (4.9)-(4.10), it is straight forward to derive that

$$
\begin{aligned}
\text{(i)} = -\int_0^t\!\!\int_\Omega \Big(\big((1-\chi_\varepsilon^t)(\partial_\tau h_0)^\varepsilon - \varepsilon\chi_\varepsilon^t(\partial_t h_0)^\varepsilon - \varepsilon\partial_t\chi_\varepsilon^t h_0^\varepsilon\big)(h_\varepsilon - h_0^\varepsilon) \\
+ (1-\chi_\varepsilon^t)\varepsilon h_0^\varepsilon \partial_s(h_\varepsilon - h_0^\varepsilon)\Big) dxds + \int_0^t\!\!\int_\Omega \varepsilon h_0^\varepsilon \partial_s(h_\varepsilon - h_0^\varepsilon) dxds := \text{(vi)} + \text{(vii)}
\end{aligned}
\tag{4.11}
$$

where we denote

$$
\begin{aligned}
\text{(vi)} &:= -\int_0^t\!\!\int_\Omega (((1-\chi_\varepsilon^t)(\partial_\tau h_0)^\varepsilon - \varepsilon\chi_\varepsilon^t(\partial_t h_0)^\varepsilon - \varepsilon\partial_t\chi_\varepsilon^t h_0^\varepsilon)(h_\varepsilon - h_0^\varepsilon) \\
&\qquad + (1-\chi_\varepsilon^t)\varepsilon h_0^\varepsilon \partial_s(h_\varepsilon - h_0^\varepsilon)) dxds \\
\text{(vii)} &:= \int_0^t\!\!\int_\Omega \varepsilon h_0^\varepsilon \partial_s(h_\varepsilon - h_0^\varepsilon) dxds.
\end{aligned}
$$

Integrating by parts yields

$$
\text{(vii)} = -\varepsilon \int_0^t\!\!\int_\Omega \partial_s h_0^\varepsilon (h_\varepsilon - h_0^\varepsilon) dxds + \int_\Omega \varepsilon h_0^\varepsilon (h_\varepsilon - h_0^\varepsilon) dx\Big|_0^t := \text{(viii)} + \text{(ix)}
\tag{4.12}
$$

where right-hand sides of the above equality are further defined by

$$
\begin{aligned}
\text{(viii)} &:= -\varepsilon \int_0^t\!\!\int_\Omega \partial_s h_0^\varepsilon (h_\varepsilon - h_0^\varepsilon) dxds \\
\text{(ix)} &:= \int_\Omega \varepsilon h_0^\varepsilon (h_\varepsilon - h_0^\varepsilon) dx|_0^t.
\end{aligned}
$$

Combining (4.11) and (4.12) we thus obtain

$$
\text{(i)} = \text{(vi)} + \text{(viii)} + \text{(ix)}.
$$

As for the item (vi), we apply (CO3) to obtain the upper bound. Indeed, by virtue of elementary inequality, we immediately derive

$$
\begin{aligned}
|\text{(vi)}| \le C \bigg(&\int_{Q_{t0\Omega}} (1-\chi_\varepsilon^t)^2 dxds + \int_{Q_{t0\Omega}} \varepsilon^2 dxds + \int_{Q_{t0\Omega}} (1-\chi_\varepsilon^t)^2 \varepsilon^2 dxds \\
&+ \int_{Q_{t0\Omega}} (h_\varepsilon - h_0^\varepsilon)^2 dxds \bigg)
\end{aligned}
$$

which yields

$$
|\text{(vi)}| \le C \overbrace{(\varepsilon + \|h_\varepsilon - h_0^\varepsilon\|_{L^2(Q_t)}^2)}^{(R1, R2, C03)} .
$$

Consequently we obtain via the Cauchy-Schwarz inequality

$$
\begin{aligned}
\text{(ii)} + \text{(viii)} &= \int_0^t\int_\Omega \varepsilon \frac{\partial(h_\varepsilon - h_0^\varepsilon)}{\partial s}(h_\varepsilon - h_0^\varepsilon)dxds \\
&= \frac{1}{2}\varepsilon\|(h_\varepsilon - h_0^\varepsilon)\|^2_{L^2(\Omega)} \\
|\text{(ix)}| &= \left|\int_\Omega \varepsilon h_0^\varepsilon (h_\varepsilon - h_0^\varepsilon)(t_0)dx\right| \\
&\le C\varepsilon + \frac{1}{4}\varepsilon\|(h_\varepsilon - h_0^\varepsilon)(t_0)\|^2_{L^2(\Omega)} \\
|\text{(v)}| &\le C\int_0^t\int_\Omega \varepsilon(h_\varepsilon - h_0^\varepsilon)dxds \\
&\le C\left(\varepsilon^2 + \|h_\varepsilon - h_0^\varepsilon\|^2_{L^2(Q_{t_0}\Omega)}\right).
\end{aligned}
$$

Finally with the aid of the decomposition

$$
\gamma(\theta_0)\nabla h_0^\varepsilon - \gamma(\theta_\varepsilon)\nabla h_\varepsilon = \gamma(\theta_\varepsilon)\nabla(h_0^\varepsilon - h_\varepsilon) - (\gamma(\theta_\varepsilon) - \gamma(\theta_0))\nabla h_0^\varepsilon
$$

we can obtain the estimates for (iii) and (iv) by Assumption 3.1 and 3.2.

More precisely, by the Green's formula, we have

$$
\begin{aligned}
\text{(iii)} + \text{(iv)} &= \int_0^t\int_\Omega \nabla\cdot(\gamma(\theta_\varepsilon))\nabla(h_0^\varepsilon - h_\varepsilon)(h_\varepsilon - h_0^\varepsilon)dxds \\
&\quad - \int_0^t\int_\Omega \nabla\cdot((\gamma(\theta_\varepsilon) - \gamma(\theta_0))\nabla h_0^\varepsilon)(h_\varepsilon - h_0^\varepsilon) \\
&= -\int_0^t\int_\Omega \gamma(\theta_\varepsilon))\nabla(h_0^\varepsilon - h_\varepsilon)\cdot\nabla(h_\varepsilon - h_0^\varepsilon)dxds \\
&\quad - \int_0^t\int_\Omega \nabla\cdot((\gamma(\theta_\varepsilon) - \gamma(\theta_0))\nabla h_0^\varepsilon)(h_\varepsilon - h_0^\varepsilon) \\
&= \overbrace{\int_0^t\int_\Omega \gamma(\theta_\varepsilon)|\nabla(h_0^\varepsilon - h_\varepsilon)|^2dxds}^{:=\text{(iii)}'} \\
&\quad \overbrace{- \int_0^t\int_\Omega \nabla\cdot((\gamma(\theta_\varepsilon) - \gamma(\theta_0))\nabla h_0^\varepsilon)(h_\varepsilon - h_0^\varepsilon)}^{:=\text{(iv)}'}.
\end{aligned}
$$

The lower bound for $\text{(iii)}'$ can be achieved by

$$
\text{(iii)}' \ge u_0\|\nabla(h_0^\varepsilon - h_\varepsilon)\|^2_{L^2(Q_t)}
$$

and the upper bound for $(\mathrm{iv})'$ is

$$|(\mathrm{iv})'| \leq |\int_0^t\int_\Omega (\gamma(\theta_\varepsilon)-\gamma(\theta_0))(h_\varepsilon - h_0^\varepsilon)\nabla h_0^\varepsilon dxds| \\ \leq C\left(\|h_\varepsilon - h_0^\varepsilon\|^2_{L^2(Q_t)} + \|\theta_0-\theta_\varepsilon\|^2_{L^2(Q_t)}\right).$$

With the lower bound for $(\mathrm{iii})'$ and estimates for $(\mathrm{i}), (\mathrm{ii}), (\mathrm{iv})', (\mathrm{v})$, we finally obtain the estimate for the Maxwell equation.

Proposition 3. *Let Assumption 3.1 hold true, then the estimate for the difference between h_ε and h_0^ε is*

$$\frac{1}{2}\varepsilon\|(h_\varepsilon - h_0^\varepsilon)(t)\|^2_{L^2(\Omega)} + \|\nabla(h_\varepsilon - h_0^\varepsilon)\|^2_{L^2(Q_t)} \\ \leq C\left(\|h_\varepsilon - h_0^\varepsilon\|^2_{L^2(Q_t)} + \varepsilon + \|\theta_0-\theta_\varepsilon\|^2_{L^2(Q_t)}\right) \tag{4.13}$$

where C is a generic constant.

4.3. **Error estimate of the coupled system.** Combining the estimates for the heat equation and Maxwell's equation in Propositions 2 and 3, we obtain

$$\|(\theta_\varepsilon-\theta_0)(t)\|^2_{L^2(\Omega)} + \varepsilon\|(h_\varepsilon - h_0^\varepsilon)(t)\|^2_{L^2(\Omega)} \\ + \|\nabla(\theta_\varepsilon-\theta_0)\|^2_{L^2(Q_t)} + \|\nabla(h_\varepsilon - h_0^\varepsilon)\|^2_{L^2(Q_t)} \\ \leq C\left(\varepsilon + \|h_\varepsilon - h_0^\varepsilon\|^2_{L^2(Q_t)} + \|\theta_\varepsilon-\theta_0\|^2_{L^2(Q_t)}\right).$$

By virtue of the Gronwall lemma and the Hölder inequity, we conclude the proof of Theorem 1.

5. Numerical experiments

In this section, we present numerical computations and compare results of the multiscale system to the homogenized equations. For simplicity we focus on the one-dimensional case, i.e. we choose $\Omega = (0,1)$. Then the non-dimensionalized multiscale system reads

$$\theta_{\varepsilon,t} - \theta_{\varepsilon,xx} = \beta\gamma(\theta_\varepsilon)h^2_{\varepsilon,x} \quad \text{in } Q \tag{5.1a}$$

$$\varepsilon h_{\varepsilon,t} - (\gamma(\theta_\varepsilon)h_{\varepsilon,x})_x = u(t)\cos\left(2\pi\frac{t}{\varepsilon}\right) \quad \text{in } Q \tag{5.1b}$$

$$-\theta_{\varepsilon,x} = \alpha\theta_\varepsilon \quad \text{in } \Sigma \tag{5.1c}$$

$$h_\varepsilon = 0 \quad \text{in } \Sigma \tag{5.1d}$$

$$\theta_\varepsilon(0) = \theta_I \quad \text{in } \Omega \tag{5.1e}$$

$$h_\varepsilon(0) = h_I \quad \text{in } \Omega. \tag{5.1f}$$

We choose $\alpha = 5$ and $\beta = 1/20$. In this example, the control $u(t)$ is taken as a constant, i. e. $u(t) = g$ with $g = 200$. In Section 2, the electrical resistivity was scaled by $\gamma_m = 1.1 \cdot 10^{-4}$. The electrical resistivity directly influences the skin effect, a low value of γ results in a lower penetration depth. Since our application is related to induction heating, where heating occurs only in a boundary layer of the domain, we choose $\gamma_m = 1 \cdot 10^{-2}$ for the numerical tests to have a pronounced skin effect.

The one-dimensional homogenized system reads as follows

(5.2a) $$\theta_{0,t} - \theta_{0,xx} = \beta\gamma(\theta_0)\overline{h_{0,x}^2} \quad \text{in} \quad \Omega \times (0,T)$$

(5.2b) $$-\theta_{0,x} = \alpha\theta_0 \quad \text{on} \quad \partial\Omega \times (0,T)$$

(5.2c) $$\theta_0(x,0) = \theta_I \quad \text{in} \quad \Omega$$

(5.2d) $$\overline{h_{0,x}^2}(x,t) = \int_0^1 h_{0,x}(x,\tau,t)^2 d\tau$$

(5.2e) $$h_{0,\tau} - (\gamma(\theta_0)h_{0,x})_x = u(t)\cos(2\pi\tau) \quad \text{in} \quad \Omega \times Y$$

(5.2f) $$h_0(x,\tau,t) = 0 \quad \text{on} \quad \partial\Omega \times Y$$

(5.2g) $$h_0(x,\tau = 0,t) = h_I \quad \text{in} \quad \Omega.$$

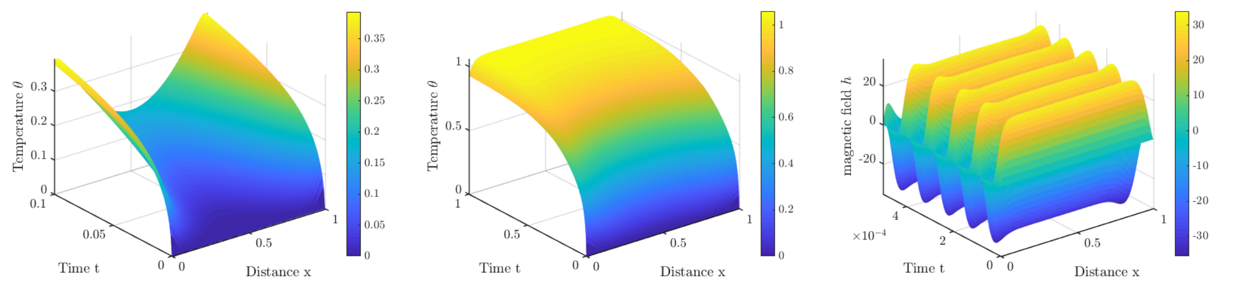
In the homogenized system, equations (5.2e)–(5.2g) depend on the space variable x and the micro time variable $\tau \in Y = (0,1)$ (cf. Sec. 3.1), whereas the macro time variable t is fixed. In current tests, we set $\theta_I \equiv 0$ and $h_I \equiv 0$.

For the discretization of the above systems, we choose an equidistant partition of the space interval, i.e.

$$x_i = i/M, \quad i = 0,\ldots,M.$$

For the time integration, we make use of the Matlab PDE solver `pdepe`, that solves initial-boundary value problems for parabolic-elliptic PDEs in 1D with adaptive time stepping.

We first fix $\varepsilon = 10^{-4}$ and display the numerical results of the multi-scale system in Figure 1. The tem-



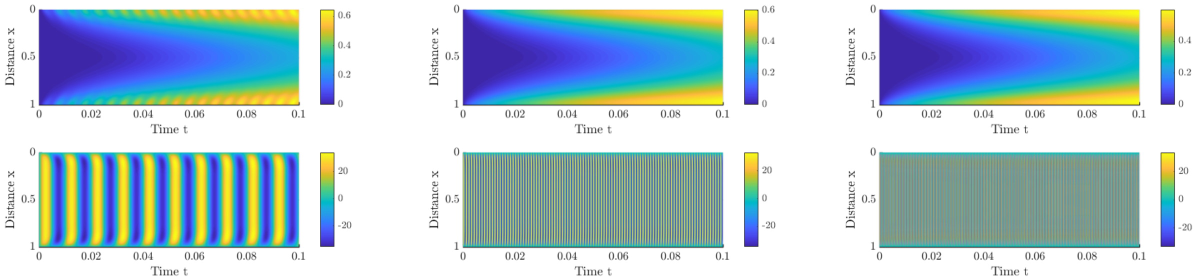
(a) Temperature θ_ε for $t \in (0, 0.1)$ (b) Temperature θ_ε for $t \in (0,1)$ (c) Magnetic field h_ε for $t \in (0, 5\cdot 10^{-4})$

FIGURE 1. Solution of the multi-scale system for $\varepsilon = 10^{-4}$.

perature is depicted in Fig. 1(a) and 1(b) on the time interval $(0, 0.1)$ and on the longer interval $(0,1)$. At the initial stage of the heating, the temperature rises only on the boundary part of the workpiece. At later times, by virtue of thermal conduction, we observe that the temperature rises from the boundary to the interior domain of the steel workpiece. At the end, a homogeneous temperature profile is observed. The magnetic field is shown in Fig. 1(c). In contrast to the temperature, the magnetic field h_ε admits a time scale related to ε.

In the industrial application of induction heating, only a heating of the boundary region of the workpiece is desired, which will require short heating times.

Next, we perform numerical tests for different values of ε. We choose $\varepsilon = 10^{-2}, 10^{-3}$ and 10^{-4}. The final time is fixed as $T = 0.1$. A high value of ε refers to a low frequency, e.g. a value of $\varepsilon = 10^{-2}$ would result in 10 oscillations on the time interval $(0, 0.1)$. In the results of the multiscale system, Figure 2, we depict the magnetic field h_ε and the temperature θ_ε. One can clearly observe the oscillations in the temperature. As ε becomes smaller, the frequency of the oscillations in h_ε rises. Due to the smoothing properties of the heat equation, the oscillations in the temperature are merely visible. For $\varepsilon = 10^{-4}$, there are no oscillations in the temperature visible. But in order to resolve the oscillations in the magnetic field, a very fine discretization of the time interval is necessary, where the number of time steps is proportional to ε^{-1}.



(a) $T = 0.1, \varepsilon = 10^{-2}$ (b) $T = 0.1, \varepsilon = 10^{-3}$ (c) $T = 0.1, \varepsilon = 10^{-4}$

FIGURE 2. Performance of the original system with different ε, top: temperature θ_ε, bottom: magnetic field h_ε.

Next, we compare the results of the multi-scale system and the homogenized system (5.2). In order to solve the homogenized system, we again use the Matlab routine `pdepe`. In each of the time steps taken by the solver, the sub-problem (5.2e)–(5.2g) for $\tau \in (0,1)$ has to be solved and the average of $h_{0,x}^2$ has to be computed according to (5.2d). The average enters the heat equation (5.2a)–(5.2c), that is solved on the macro time interval $(0,T)$. Since for fixed time $t \in (0,T)$ the temperature θ_0 is independent of the micro time τ, equations (5.2e)–(5.2g) are linear. Therefore, a harmonic approach for $h_0(\tau)$ can be used to solve (5.2e)–(5.2g) with respect to time τ, i.e. we assume

$$h_0(x,\tau,t) = h_{\text{cos}}(x,t)\cos(2\pi\tau) + h_{\text{sin}}(x,t)\sin(2\pi\tau).$$

Then (5.2e)–(5.2g) becomes an elliptic system for $h_{\text{cos}}(x,t)$ and $h_{\text{sin}}(x,t)$. The average can be computed according to $\overline{|h_{0,x}|^2} = 1/2(h_{\text{cos},x}^2 + h_{\text{sin},x}^2)$.

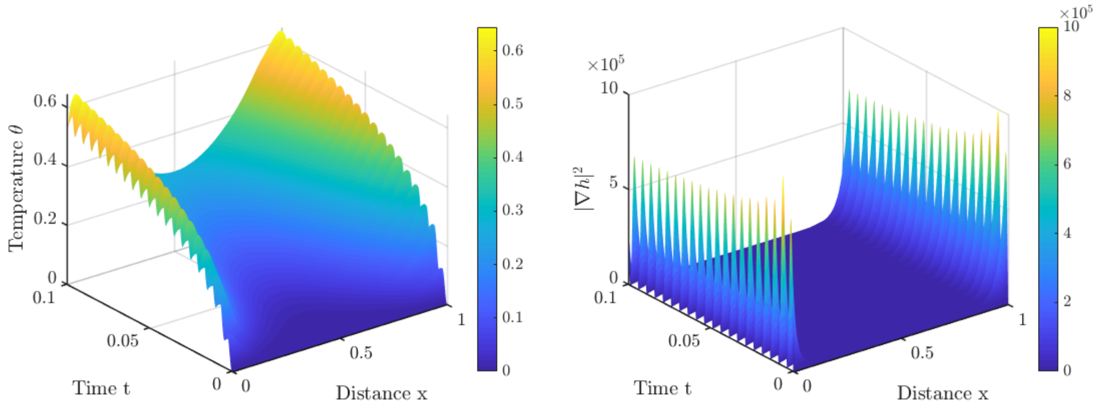
One has to note that the time discretization for the multiscale system must resolve the oscillations of the magnetic field, resulting in a time step size proportional to ε, while the time discretization for the homogenized equations is independent of ε.

In Figure 3 we compare the temperature and the gradient of the magnetic field for the original and the homogenized system for a choice of $\varepsilon = 10^{-2}$.

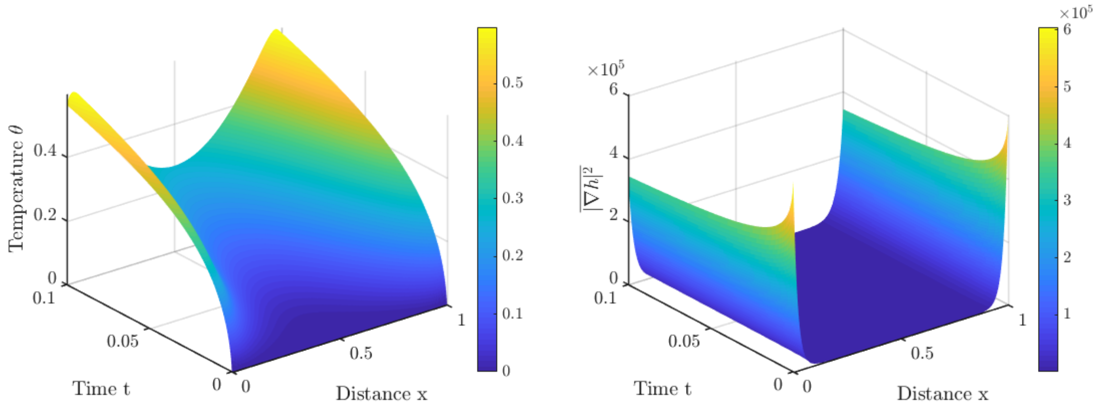
As can be seen, the original system admits an oscillatory behaviour in the temperature due to oscillations in the magnetic field gradient, that represents the heat source. For the homogenized system, the temperature increases smoothly. The averaged magnetic field gradient for the homogenized system is shown on the right of Fig. 3(b). There is a decrease with respect to the macro time t, which results from the temperature dependent parameter $\gamma(\theta_0)$. The same can also be observed for the original problem, Fig. 3(a), resulting in a decrease of the amplitude of $|h_{\varepsilon,x}|^2$.

With a further decrease of ε, the amplitudes of the oscillations in the temperature become much smaller. If we compare the temperature for the homogenized system, Fig. 3(b) with the temperature of the original system with a choice of $\varepsilon = 10^{-4}$, Fig. 1(a), there is no visible difference.

In Figure 4(a), we compare the error between the original and the homogenized system for various values of ε. The L^2-norm of the temperature difference $\|\theta_\varepsilon - \theta_0\|_{L^\infty((0,T);L^2(\Omega))}$ is plotted in blue, the norm of the difference of the temperature gradients, $\|\nabla(\theta_\varepsilon - \theta_0)\|_{L^2((0,T);L^2(\Omega))}$, is shown in red. The triangles indicates the slope $1/2$ and 1. As can be seen, the error between the original and the homogenized system reduces for the temperature and the temperature gradient with respect to ε. While the error



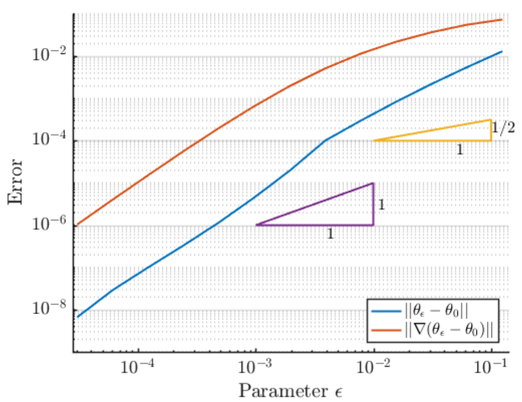
(a) Temperature θ_ε and $|h_{\varepsilon,x}|^2$, original system, T=0.1, $\varepsilon = 10^{-2}$



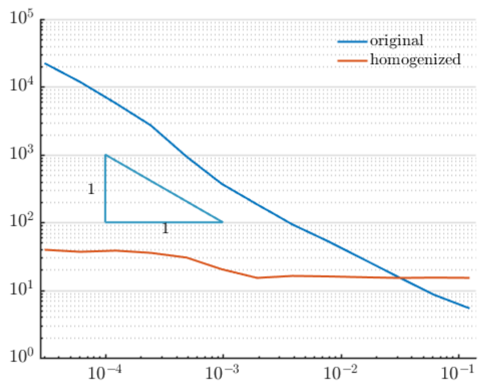
(b) Temperature θ_0 and $\overline{|h_{0,x}|^2}$, homogenized system, T=0.1, $\varepsilon = 10^{-2}$

FIGURE 3. Comparison between original and homogenized system.

estimate in Theorem 1 has a dependence on the error related to $\varepsilon^{\frac{1}{2}}$, the numerical experiments in this section suggest a linear reduction of the error with respect to ε.



(a) Error



(b) Computational time

FIGURE 4. Norms $\|\theta_\varepsilon - \theta_0\|_{L^\infty((0,T);L^2(\Omega))}$ and $\|\nabla(\theta_\varepsilon - \theta_0)\|_{L^2((0,T);L^2(\Omega))}$ and computational time in dependence on ε.

Finally, we compare computational times for the original and the homogenized system in Figure 4(b). As one can see, the computational time for the original problem rises linearly as ε is reduced. This is due to the fact that the time stepping has to resolve the frequency of the magnetic field, that is related to ε^{-1}. In the homogenized system, the computational cost is independent of ε.

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