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Acoustic scattering from locally perturbed periodic surfaces

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Abstract

We prove well-posedness for the time-harmonic acoustic scattering of plane waves from locally perturbed periodic surfaces in two dimensions under homogeneous Dirichlet boundary conditions. This covers sound-soft acoustic as well as perfectly conducting, TE polarized electromagnetic boundary value problems. Our arguments are based on a variational method in a truncated bounded domain coupled with a boundary integral representation. If the quasi-periodic Green's function to the unperturbed periodic scattering problem is calculated efficiently, then the variational approach can be used for a numerical scheme based on coupling finite elements with a boundary element algorithm.

Even for a general 2D rough-surface problem, it turns out that the Green's function defined with the radiation condition ASR satisfies the Sommerfeld radiation condition over the half plane. Based on this result, for a local perturbation of a periodic surface, the scattered wave of an incoming plane wave is the sum of the scattered wave for the unperturbed periodic surface plus an additional scattered wave satisfying Sommerfeld's condition on the half plane. Whereas the scattered wave for the unperturbed periodic surface has a far field consisting of a finite number of propagating plane waves, the additional field contributes to the far field by a far-field pattern defined in the half-plane directions similarly to the pattern known for bounded obstacles.

1 Introduction

The scattering theory in periodic structures has many applications in near-field optics, micro-electronics, non-destructive testing, and the design of photonic crystals. We refer to [27] for an introduction and historical remarks on the electromagnetic theory of gratings. Over the last twenty years, significant progress has been made concerning the mathematical analysis and the numerical approximation of grating diffraction problems for the case of incident acoustic or electromagnetic waves, using integral equation methods (e.g. [25, 26, 28]) and variational methods (e.g. [5, 14, 15, 19]). This paper is concerned with the analysis of plane waves scattered at a one-dimensional perfectly conducting grating with local perturbation. Physically, the local perturbation of a perfectly periodic surface can be used to model optical devices with localized defects, for instance, unmade or distorted grooves on the surface of diffraction gratings.

The diffracted field for a plane wave incident onto a perfect grating is well-known to be quasi-periodic, due to the periodicity of the scattering surface and the quasi-periodicity of the incoming wave. The presence of defects will break down the quasi-periodicity property, leading to essential difficulties in the reduction of the analysis and simulation to problems over bounded domains. A limited number of approaches have been proposed so far for treating grating problems with local perturbations. To solve transmission problems for periodic interfaces perturbed by compact aperiodic inclusions below the interface, Ammari and Bao [1] propose an integral equation approach. This integral equation is defined over $\mathbb{R}^2$ or $\mathbb{R}^3$ and includes a Fourier transform as well as a kernel function, which is defined by the solution of a family of variational equations. The approach relies on strong a priori assumptions (for

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instance, absence of surface waves and the unique solvability of a periodic equation; see [1, (3.1)]), and the mathematical analysis of the unique solvability and the decay behaviour of the unperturbed fields seem still to be unclear in general. Bonnet-Bendhia and Ramdani [4] treat such compact inclusions if the space beneath a planar interface is filled with media periodic in the interface direction. They employ Floquet-Bloch transforms, variational formulations, and boundary integral techniques. Joly et al. established an exact boundary condition with a map of Dirichlet-to-Neumann type for numerically solving an inhomogeneous source problem in a closed periodic waveguide with a local junction [18] and then extended the approach to an open waveguide, where the unperturbed medium was periodic in two directions [16]. Here and in other publications, the Floquet-Bloch transform was employed to handle scattering problems in a locally perturbed periodic medium; see [12] for a line defect, Haddar and Nguyen [17] in periodic layered medium as well as Lechleiter and Zhang [21] for locally perturbed sound-soft surfaces. The resulting numerical schemes of [17,21] require the calculation of inverse and forward Floquet-Bloch transforms or variational equations for Floquet-Bloch transformed solutions. Sun and Zheng [29] apply perfectly matched layers to reduce the boundary value problem over a periodic metallic interface with a local aperiodic perturbation, and the resulting wave-guide problem can be solved by the methods of [18].

Motivated by recent studies on wave scattering from flat surfaces with local perturbations [2, 3, 30], in this paper we prove that the total field can be uniquely decomposed into three parts (see Thm. 3.1): the incoming wave v^{in}, the reflected field v^{sc} corresponding to the unperturbed periodic scattering interface, and the perturbed wave u_0 caused by the presence of local perturbations. This, in particular, implies that a local perturbation cannot give rise to any surface wave. Moreover, we verify that u_0 satisfies the half-space Sommerfeld radiation condition (see Definition 2.1). In the case of flat surfaces with local perturbations, it is easy to prove that u_0 fulfills the strong Sommerfeld radiation condition uniformly for all outgoing directions in the upper half space. This follows straightforwardly from the properties of the corresponding Green's function in the half space. The characterization of the asymptotic behavior of u_0 in a periodic background medium seems to be missing in the literature and turns out to be non-trivial. We shall prove that even the non-quasi-periodic Green's function G to perfectly conducting gratings with aperiodic Lipschitz interfaces fulfills this half-space Sommerfeld radiation condition. For the locally perturbed periodic grating, this enables us to establish an equivalent variational formulation over a bounded domain containing the defect. The formulation is based on a boundary integral representation of u_0 in terms of the quasi-periodic Green's function G. Thanks to solvability results for general rough surfaces [8], we show that $u = v^{in} + v^{sc} + u_0$ is the unique solution in certain weighted Sobolev spaces over a strip above the scattering surface. By the classical grating theory, the reflected field v^{sc} fulfills the Upward Rayleigh Expansion Radiation. Hence, the scattered field $v^{sc} + u_0$ still satisfies the Upward Angular Spectral Representation ([8, 9]) or, equivalently, the Upward Propagating Radiation Condition of [11]. The estimates leading to Sommerfeld's radiation condition can be used as well to derive the far-field pattern of u_0. Note that the notion of far-field patterns can be used to model the inverse problems of finding the defect surface from measured far-field data (compare the different notion of far-field measurement in e.g. [22]).

The decomposition of the scattered fields into reflected fields and Sommerfeld type outgoing fields also applies to other cases of local perturbations, e.g., a bounded obstacle embedded in periodic background media, including inhomogeneous periodic layered media. Hence, the proposed approach can be used to handle general grating diffraction problems with defects. This requires the determination of the solution for the unperturbed periodic surface and an efficient forward solver for computing the Green's function G to the unperturbed grating diffraction problems, i.e., the computation of the total fields incited by incoming point-source waves. Since such kind of incident waves are not quasi-periodic, one can apply the Floquet-Bloch transform to the calculation of G; see e.g. [20].

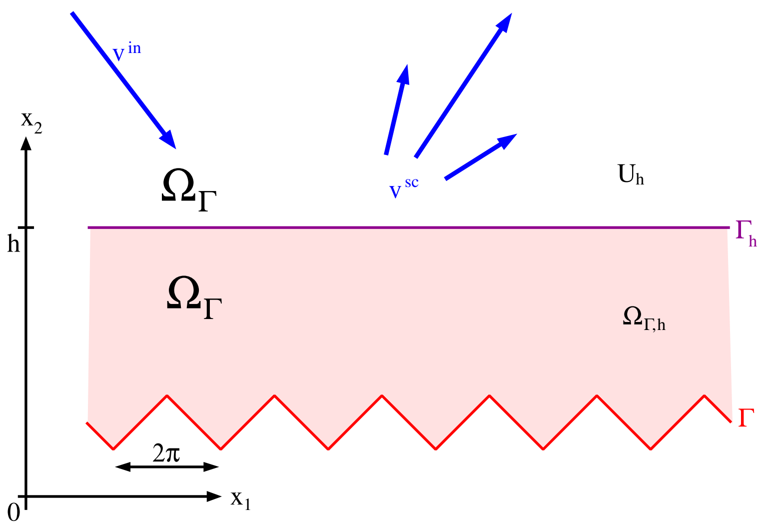

Figure 1: Geometry of unperturbed grating.

The remaining part of this paper is organized as follows. In the subsequent Sect. 2 we recall solvability results for the scattering of plane and point-source waves from perfectly conducting gratings. The half-plane Sommerfeld radiation condition will be given in Definition 2.1. Sect. 3 is devoted to the analysis of a variational formulation over a bounded truncated domain, which is equivalent to the scattering problem. Uniqueness and existence of weak solutions will be reported in Lem. 3.1 and Thm. 3.1. The proof for the Sommerfeld radiation condition of the Green's function to the unperturbed periodic and to aperiodic scattering problems, respectively, will be postponed to the Appendix in Sect. 4.

2 Scattering from periodic surfaces

2.1 Plane-wave incidence

Assume a time-harmonic acoustic wave is incident onto a sound-soft periodic surface $\Gamma \subset \mathbb{R}^2$ of an isotropic homogeneous background medium. Suppose that Γ is Lipschitz continuous, and that the incident wave is a time-harmonic plane wave of the form $v^{in}(x)\exp(-i\omega t)$, incited at the angular frequency $\omega > 0$. The spatially dependent function v^{in} takes the form

$$v^{in}(x) = \exp\Big(ik\,(\sin\theta, -\cos\theta)\cdot x\Big), \tag{2.1}$$

where $\theta \in [0, \pi/2)$ denotes the angle of incidence, where $k := \omega/c_0$ is the wave number, and $c_0 > 0$ the speed of sound. We suppose that Γ is bounded in x_2 and that, without loss of generality, the scattering surface Γ is 2π-periodic in x_1. The unbounded region above Γ, which we denote by Ω_Γ (cf. Fig. 1), is supposed to fulfill the following geometrical condition:

$$(x_1, x_2) \in \Omega_\Gamma \quad \Rightarrow \quad (x_1, x_2 + s) \in \Omega_\Gamma \quad \text{for all } s > 0. \tag{2.2}$$

The wave propagation is then governed by the boundary value problem for the Helmholtz equation

$$\Delta v + k^2 v = 0 \quad \text{in } \Omega_\Gamma, \qquad v = 0 \quad \text{on } \Gamma, \tag{2.3}$$

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where the total field $v = v^{in} + v^{sc}$ is the sum of the incident field v^{in} and a scattered field v^{sc}, which satisfies a radiation condition.

Let $\alpha := k\sin\theta$. Obviously, the incident field is α-quasi-periodic in the sense that $v^{in}(x)\exp(-i\alpha x_1)$ is 2π-periodic with respect to x_1 in Ω_Γ. The periodicity of the structure together with the form of the incident wave implies that the total field v must also be α-quasi-periodic. This implies that

$$v(x_1 + 2\pi\, n, x_2) = \exp(i2\pi\alpha n)\, v(x_1, x_2), \quad \text{for all } n \in \mathbb{Z}.$$

Since the domain Ω_Γ is unbounded in the x_2-direction, a radiation condition must be imposed at infinity to ensure well-posedness of the scattering problem. In other words, setting $U_h := \{x \in \mathbb{R}^2 : x_2 > h\}$, we require the scattered acoustic field v^{sc} to admit the upward Rayleigh expansion condition: There exist coefficients $v_n \in \mathbb{C}$ depending on v^{sc} such that

$$v^{sc}(x) = \sum_{n\in\mathbb{Z}^2} v_n\, \exp(i\alpha_n x_1 + i\beta_n x_2), \quad x \in U_h, \tag{2.4}$$

for any $h > \max\{x_2 : x \in \Gamma\}$ and with the parameters $\alpha_n := n + \alpha \in \mathbb{R}$ and $\beta_n \in \mathbb{C}$ defined by

$$\beta_n = \beta_n(k) := \begin{cases} (k^2 - |\alpha_n|^2)^{\frac{1}{2}} & \text{if } \; |\alpha_n| \le k, \\ i(|\alpha_n|^2 - k^2)^{\frac{1}{2}} & \text{if } \; |\alpha_n| > k. \end{cases}$$

Uniqueness and existence of our scattering problem (2.1)-(2.4) are stated as follows.

Lemma 2.1. *Assume Condition (2.2) is fulfilled. Then, for any fixed $k > 0$ and $\theta \in [0, \pi/2)$ and for $\alpha = k\sin\theta$, there exists a unique α-quasi-periodic variational solution $v = v^{in} + v^{sc} \in H^1_{loc}(\Omega_\Gamma)$ to the scattering problem (2.1)-(2.4).*

The above well-posedness result was first proved by Kirsch [19] for periodic surfaces given by a C^2-smooth function and then by Elschner and Yamamoto [15] for Lipschitz graphs. Chandler-Wilde and Monk [9] proved uniqueness and existence for rough surface scattering problems if the incident wave is generated by a compact source term and if condition (2.2) holds. The case of plane-wave incidence was treated in [8] for sound-soft rough surfaces in two dimensions, from which Lem. 2.1 follows. The uniqueness proofs in the above mentioned papers depend heavily on the use of Rellich's identity under the condition (2.2). Uniqueness to scattering problems in periodic structures cannot hold in the general case. We refer to [5] for non-uniqueness examples in inhomogeneous periodic media.

2.2 Point-source incidence

We now fix a $y \in \Omega_\Gamma$ and consider the case where the incident wave G^{in} is a non-quasiperiodic cylindrical wave of the form:

$$G^{in}(x) = G^{in}(x; y) := \Phi(x; y) := \frac{i}{4} H_0^{(1)}(k|x-y|), \quad x \ne y, \quad x \in \Omega_\Gamma. \tag{2.5}$$

Here $H_0^{(1)}(\cdot)$ stands for the Hankel function of the first kind and of order zero. The function $\Phi(x; y)$ is the free-space fundamental solution of the Helmholtz equation $(\Delta + k^2)u = 0$. Since the incoming wave G^{in} is no longer quasi-periodic, the Rayleigh expansion condition (2.4) is not applicable to point source incidence of the form (2.5). Instead we suppose that the scattered field G^{sc} satisfies the upward Angular Spectrum Representation (ASR) proposed in [9]:

$$G^{sc}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\Big(i[(x_2 - h)\sqrt{k^2 - \xi^2} + x_1\xi]\Big)\; \hat{G}^{sc}_h(\xi)\, \mathrm{d}\xi, \quad x \in U_h, \tag{2.6}$$

for all $h > \max\{x_2 : x \in \Gamma\}$. Here, $\sqrt{k^2 - \xi^2} = i\sqrt{\xi^2 - k^2}$ when $\xi^2 > k^2$, and $\hat{G}_h^{sc}(\xi)$ denotes the Fourier transform of $G^{sc}(x_1, h)$ with respect to x_1, i.e.,

$$\hat{G}_h^{sc}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-ix_1\xi) G^{sc}(x_1, h)\, \mathrm{d}x_1, \quad \xi \in \mathbb{R}.$$

The radiation condition (2.6) is equivalent to the UPRC of [9] and, if Γ is periodic and the solution is α-quasi-periodic (see [7]), to the Rayleigh expansion condition (2.4).

Denote the infinite strip between Γ and the straight line $\Gamma_h := \{x \in \mathbb{R}^2 : x_2 = h\}$ by $\Omega_{\Gamma,h} := \{x \in \Omega_\Gamma : x_2 < h\}$ and, again, let $U_h := \{x \in \mathbb{R}^2 : x_2 > h\}$ (cf. Fig. 1). For point-source incidence (2.5), we look for the scattered field in the weighted Sobolev space $V_{h,\varrho}$ defined as the closure of all the $u|_{\Omega_{\Gamma,h}}$ with $u \in C_0^\infty(\Omega_\Gamma)$ w.r.t. the norm

$$\|u\|_{V_{h,\varrho}} := \left[\int_{\Omega_{\Gamma,h}} \left\{ \left|(1+|x_1|^2)^{\varrho/2} u(x)\right|^2 + \left|\nabla \left[(1+|x_1|^2)^{\varrho/2} u(x)\right]\right|^2 \right\} \mathrm{d}x \right]^{1/2}.$$

One can also employ the following norm equivalent to $||\cdot||_{V_{h,\varrho}}$:

$$||u|| := \left[\int_{\Omega_{\Gamma,h}} (1+|x_1|^2)^\varrho \left\{ \left|u(x)\right|^2 + \left|\nabla u(x)\right|^2 \right\} \mathrm{d}x \right]^{1/2}, \quad u \in V_{h,\varrho}.$$

Moreover, we introduce $H_\varrho^s(\mathbb{R}) := (1 + x_1^2)^{-\varrho/2} H^s(\mathbb{R})$ for $\varrho, s \in \mathbb{R}$ endowed with the norm

$$||u||_{H_\varrho^s(\mathbb{R})} := \left\|(1 + x_1^2)^{\varrho/2} u(x_1)\right\|_{H^s(\mathbb{R})}.$$

We have the identity $V_{h,\varrho} = H_\varrho^1(\Omega_{\Gamma,h}) \cap \{u : u|_\Gamma = 0\}$ and, if $\varrho = 0$, the equality $H_\varrho^s(\mathbb{R}) = H^s(\mathbb{R})$, where the $H^s(\mathbb{R})$ are the usual non-weighted Sobolev spaces.

Theorem 2.1. *Under the condition (2.2), the scattering problem*

$$\Delta G(\cdot\,; y) + k^2 G(\cdot\,; y) = \delta_y \text{ on } \Omega_\Gamma, \quad G(\cdot\,; y) = 0 \text{ on } \Gamma, \quad G(\cdot\,; y) - G^{in}(\cdot\,; y) \text{ satisfies ASR },$$

due to the incident point-source wave $G^{in}(x; y)$ with $y \in \Omega_\Gamma$, has exactly one variational solution $G = G^{in} + G^{sc}$ with the scattered field $G^{sc}(x; y)$ such that

$$G^{sc}(\cdot\,; y) \in H_\varrho^1(\Omega_{\Gamma,h}) \qquad \textit{for all } h > \max\{x_2 : x \in \Gamma\}, \quad -1 < \varrho < 0.$$

Clearly, the function G in Thm. 2.1 is the Green's function of the boundary value problem (2.3) with radiation condition ASR. The proof of Thm. 2.1 relies essentially on the decay property of G^{in} on Γ. Its proof can be carried out following the arguments of [8, Thm. 4.1] by transforming the original boundary value problem to an inhomogeneous Helmholtz equation with homogeneous Dirichlet boundary condition on Γ and with an inhomogeneous source term in weighted Sobolev spaces (also cf. [22, Sect. 2.3]). For three dimensions, it was proved in [8] that $G^{sc}(\cdot\,; y) \in H_\varrho^1(\Omega_{\Gamma,h})$ with $\varrho \in (-1, -1/2)$. The two-dimensional case can be treated analogously; see also the arguments presented in Sect. 4.

For any $r > 0$, write $S_r^\Gamma := S_r := \{x \in \Omega_\Gamma : |x| = r\}$. Below we shall prove that, for point-source incidence, the upward ASR (2.6) is equivalent to the Sommerfeld outgoing radiation condition in a half plane, which is defined as follows.

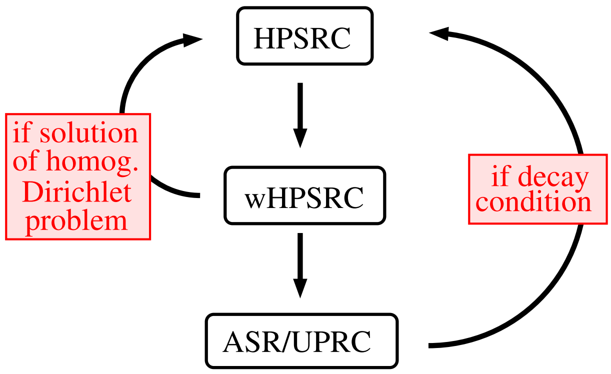

Figure 2: Radiation conditions.

Definition 2.1. *Let $v \in C^\infty(\Omega_\Gamma \cap \{x \in \mathbb{R}^2 : |x| > R\})$ for a sufficiently large $R > 0$. Then we say that v satisfies the half-plane Sommerfeld radiation condition (HPSRC) if, for any $h > \max\{x_2 : x \in \Gamma\}$, the function v is in $H^1_\varrho(\Omega_{\Gamma,h} \cap \{x \in \mathbb{R}^2 : |x_1| > R\})$ with some $\varrho > -1$ and if*

$$\sup_{x \in S_r \cap U_h} r^{1/2} \left|\partial_\nu v(x) - ikv(x)\right| \to 0, \ r \to \infty, \qquad \sup_{x \in \Omega_\Gamma \cap U_h : |x| \geq R} |x|^{1/2} \left|v(x)\right| < \infty. \tag{2.7}$$

If v satisfies the HPSRC with (2.7) replaced by

$$\int_{S_r \cap U_h} |\partial_\nu v - ikv|^2 \,\mathrm{d}s \to 0, \ r \to \infty, \qquad \sup_{0<r} \int_{S_r \cap U_h} |v|^2 \,\mathrm{d}s < \infty, \tag{2.8}$$

then we shall say that v fulfills the weak half-plane Sommerfeld radiation condition (wHPSRC).

The integrals in (2.8) are defined over $S_r \cap U_h$ rather than S_r, because the normal derivative $\partial_\nu v$ on $S_r \cap \Omega_{\Gamma,h}$ might not exist in the L^2-sense. The dependencies between the different radiation conditions are shown in Fig. 2. Obviously, HPSRC implies wHPSRC. On the other hand, any function $v = u_0$ over the domain Ω_Γ (or the perturbed domain Ω_Λ in Sect. 3) satisfying the wHPSRC and $v|_\Gamma = 0$ (or $v|_\Lambda = 0$) can be represented as (3.5), and the subsequent Lem. 2.4 implies the HPSRC. Furthermore, note that the wHPSRC is stronger than the ASR (cf. (2.6)). Indeed, using (3.4) over a half plane, a representation like (4.1) can be shown for v, and v satisfies the UPRC and, equivalently (cf. [8]), the ASR. Vice versa, the ASR together with the decay condition $v|_{\Gamma_h \cap \{x \in \mathbb{R}^2 : |x| > R\}} \in L^2_\varrho$, with a ϱ s.t. $1/2 < \varrho < 1$, implies the HPSRC (cf. the proof of Lem. 4.2). Hence in many cases, HPSRC, wHPSRC, and ASR are equivalent.

The function $x \to \Phi(x; y)$ with $y \in \mathbb{R}^2 \setminus \overline{\Omega}_\Gamma$ satisfies (2.7). For functions satisfying the HPSRC, we define the far-field pattern over the direction $\hat{x} = (\cos\theta, \sin\theta) \in \mathbb{S}_+$ with $\mathbb{S}_+ := \{x \in \mathbb{R}^2 : x_2 > 0, |x| = 1\}$ and $\theta \in (0, \pi)$.

Definition 2.2. *Let $v \in C^\infty(\Omega_\Gamma \cap \{x \in \mathbb{R}^2 : |x| > R\})$ for a sufficiently large $R > 0$. We shall call the continuous function $v_\infty \in C(\overline{\mathbb{S}_+})$ the far-field pattern of v if there is an $h > 0$ s.t.*

$$\sup_{x = r\hat{x} \in S_r \cap U_h} \left| v(x) - \frac{\exp(ikr)}{r^{1/2}} v_\infty(\hat{x}) \right| r^{1/2} \longrightarrow 0, \quad r \to \infty. \tag{2.9}$$

The lemma below shows that the scattered field caused by G^{in} also fulfills the stronger condition of Def. 2.1 and admits an asymptotics like in (2.9).

Lemma 2.2. *For any fixed $y\in\Omega_\Gamma$, the Green's function $G(\cdot\,;y)$ of Thm. 2.1 satisfies the HPSRC and has a far-field pattern in $C(\overline{\mathbb{S}_+})$. Moreover, $G(\cdot\,;y)\in H^1_\varrho(\Omega_{\Gamma,h}\cap\{x\in\mathbb{R}^2:\,|x_1|>R\})$ for any $|\varrho|<1$ and $R>|y_1|$.*

The assertion of Lem. 2.2, for Γ the graph of a $C^{1,1}$-smooth function and for an incident wave with compactly supported source in Ω_Γ at a positive distance from Γ, is contained already in [10, Thm. 5.1] but without proofs. In the Appendix we shall present a proof valid for Lipschitz (non-periodic) rough surfaces under the condition (2.2). In the special case $\Gamma=\Gamma_0:=\{x\in\mathbb{R}^2:\,x_2=0\}$, Lem. 2.2 follows straightforwardly from the explicit formula

$$G(x;y)=\Phi(x;y)-\Phi(x;y^*),\quad y^*:=(y_1,-y_2).$$

Remark 2.1. *Unfortunately, the assertion of Lem. 2.2 does not hold for the scattered field generated by plane-wave incidence, due to the appearance of propagating wave modes, which do not decay at infinity.*

In the Appendix we shall prove the following lemmata.

Lemma 2.3. *Suppose $l_j,\ j=1,2$ are non-negative integers. The assertion of Lem. 2.2 holds for $G(\cdot\,;y)$ replaced by the derivative $\partial^{l_1}_{y_1}\partial^{l_2}_{y_2}G(\cdot\,;y)$.*

Lemma 2.4. *Suppose that S_R is a circular arc around the midpoint $(0,0)$ and of radius R in Ω_Γ with end points located at Γ. Further suppose $f\in H^s(S_R),\ s\in\mathbb{R}$, and $l_j,\ j=1,2$ are non-negative integers. Then the function $v(x):=\int_{S_R}\partial^{l_1}_{y_1}\partial^{l_2}_{y_2}G(x;y)f(y)\,\mathrm{d}s(y)$ for $x\in\Omega_\Gamma$ with $|x|>R$ fulfills the HPSRC and has a far-field pattern in the space $C(\overline{\mathbb{S}_+})$.*

We shall use the Green's function to define single and double layer potential operators over the circular arc S_R. The behaviour of the Green's function close to the end points, however, is hard to predict. To guarantee a degree of smoothness suitable for a numerical discretization, an additional condition on Γ like the following (2.10) would be helpful. For fixed R and any of the two end points of the arc $\tilde{y}\in S_R\cap\Gamma$, we assume that in a neighbourhood of $\tilde{y}$ the curve Γ is a linear segment $[\tilde{y}_a,\tilde{y}_b]$, i.e., there are points $\tilde{y}_a,\tilde{y}_b\in\mathbb{R}^2$ and an $\varepsilon_L>0$

$$\Gamma\cap\{y\in\mathbb{R}^2:\ |y-\tilde{y}|\le\varepsilon_L\}=\{\mu\tilde{y}_a+(1-\mu)\tilde{y}_b:\ 0\le\mu\le 1\},\quad \tilde{y}\cdot(\tilde{y}_b-\tilde{y}_a)\neq 0. \tag{2.10}$$

Here the assumption on the non-vanishing scalar product is equivalent to the condition that the angle between Γ and S_R at $\tilde{y}$ is strictly in $(0,\pi)$. With (2.10) the Green's function G is locally the sum of the Green's function Ψ for the rotated half plane (cf. (4.2)) plus a function G_r such that the mapping $S_R\ni y\mapsto G_r(.;y)\in H^1_{loc}(\Omega_\Gamma)$ is continuous (cf. the proofs in the Appendix in Sect. 4).

3 Scattering from locally perturbed periodic surfaces

Now consider a one-dimensional Lipschitz curve $\Lambda\subset\mathbb{R}^2$ different from Γ and suppose (2.2) also for Λ. The curve Λ is said to be a local perturbation of the periodic interface Γ if Λ coincides with Γ in $\{x\in\mathbb{R}^2:\,|x_1|>R\}$ for some fixed $R>0$. In other words, Λ differs from Γ in a compact set which may stand for a defect of Γ. The presence of the defect causes a perturbation u of the total wave field $v=v^{in}+v^{sc}$ that corresponds to the perfectly periodic interface Γ. In this section we study the relation between the perturbed and unperturbed scattering problems. We derive a variational equation for the solution u based on the solution v.

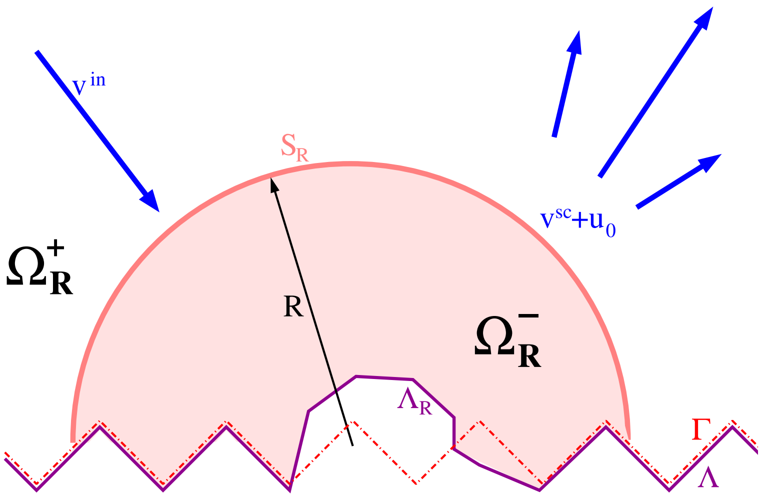

Figure 3: Geometry of locally perturbed grating.

We keep the notation used in Sect. 2. Let Ω_Λ be the region above Λ and set (cf. Fig. 3)

$$\Lambda_R := \{x \in \Lambda\colon |x| < R\}, \quad \Omega_R^- := \{x \in \Omega_\Lambda\colon |x| < R\}, \quad \Omega_R^+ := \{x \in \Omega_\Lambda\colon |x| > R\}.$$

Assume a plane wave $v^{in}(x;\theta)$ of the form (2.1) is incident onto Λ from Ω_Λ. We seek the total field $u \in H^1_{loc}(\Omega_\Lambda)$ in the form

$$u(x) = v^{in}(x;\theta) + v^{sc}(x;\theta) + u_0(x;\theta) = v(x) + u_0(x), \quad x \in \Omega_\Lambda \cap \Omega_\Gamma,$$

where $v = v^{in} + v^{sc}$ is the total field generated by the unperturbed surface Γ, and the unknown function u_0 incited by the defect is supposed to satisfy the HPSRC. Since both u and v vanish on $\Lambda\backslash\Lambda_R$, the function u_0 should also vanish on $\Lambda\backslash\Lambda_R$. Define the energy space X_R over the truncated domain Ω_R^- as $X_R := \{u \in H^1(\Omega_R^-)\colon u = 0 \text{ on } \Lambda_R\}$, which is equipped with the usual H^1-norm

$$\|u\|_{X_R} := \Big(\int_{\Omega_R^-} \{|\nabla u|^2 + |u|^2\}\, \mathrm{d}x \Big)^{1/2}.$$

Let S_R^Λ be defined in the same way as S_R^Γ with Γ replaced by Λ. Obviously, for large R we have $S_R^\Lambda = S_R^\Gamma$. Hence, for notational convenience, we drop the indices Γ and Λ and write $S_R = S_R^\Lambda = S_R^\Gamma$. Introduce the Sobolev spaces on the open arc (see, e.g., [24]):

$$\begin{aligned} H^{1/2}(S_R) &:= \{u|_{S_R}\colon u \in H^{1/2}(\partial\Omega_R^-)\}, \\ \widetilde{H}^{1/2}(S_R) &:= \{u \in H^{1/2}(\partial\Omega_R^-)\colon \mathrm{supp}\,(u) \subset S_R\}. \end{aligned}$$

Then we denote the dual space of $\widetilde{H}^{1/2}(S_R)$ by $H^{-1/2}(S_R)$, and the dual space of $H^{1/2}(S_R)$ by $\widetilde{H}^{-1/2}(S_R)$. It is easy to derive the following variational formulation for u:

$$\int_{\Omega_R^-} \{\nabla u \cdot \nabla \overline{\phi} - k^2 u \overline{\phi}\}\ \mathrm{d}x - \int_{S_R} \partial_\nu u \overline{\phi}\, \mathrm{d}s = 0 \quad \text{for all } \phi \in X_R, \tag{3.1}$$

where ν is the unit normal on S_R pointing into Ω_R^+. Choosing $R_1 > R$ and applying Green's formula to u_0, we see

$$u_0(x) = \left(-\int_{S_{R_1}} + \int_{S_R} \right) \big[u_0(y)\partial_{\nu(y)} G(x;y) - \partial_{\nu(y)} u_0(y) G(x;y) \big]\, \mathrm{d}s(y), \tag{3.2}$$

for $x\in\Omega^-_{R_1}\cap\Omega^+_R$. Note that both, u_0 and $G=G^\Gamma$, vanish on $\Lambda\backslash\Lambda_R$.

Taking $h>\max\{x_2: x\in\Gamma\}$ and making use of the wHPSRC of u_0 and G yield

$$\int_{S_{R_1}\cap U_h}\big[u_0(y)\partial_{\nu(y)}G(x;y)-\partial_{\nu(y)}u_0(y)G(x;y)\big]\,\mathrm{d}s(y)= \tag{3.3}$$
$$\int_{S_{R_1}\cap U_h}\big\{u_0(y)[\partial_{\nu(y)}G(y;x)-ikG(y;x)]-[\partial_{\nu(y)}u_0(y)-iku_0(y)]G(y;x)\big\}\,\mathrm{d}s(y)\to 0$$

as $R_1\to\infty$. Here we have used the symmetry $G(x;y)=G(y;x)$ (cf. e.g. [22, Thm. 7]), which can be proved following the lines in the proof of [23, Thm. 3.14] and using Lem. 2.2. Further, the integral over the remaining part $S_{R_1,h}:=S_{R_1}\cap\Omega_{\Gamma,h}$ of S_{R_1} can be estimated by

$$\begin{aligned}
&\int_{S_{R_1,h}}\big[u_0(y)\partial_{\nu(y)}G(x;y)-\partial_{\nu(y)}u_0(y)G(x;y)\big]\,\mathrm{d}s(y)\\
\le\ & ||u_0(1+|y_1|^\varrho)||_{H^{1/2}(S_{R_1,h})}\ ||\partial_{\nu(y)}G(x;\cdot)(1+|y_1|^{-\varrho})||_{\widetilde{H}^{-1/2}(S_{R_1,h})}\\
&+||\partial_\nu u_0(1+|y_1|^\varrho)||_{\widetilde{H}^{-1/2}(S_{R_1,h})}\ ||G(x;\cdot)(1+|y_1|^{-\varrho})||_{H^{1/2}(S_{R_1,h})}\\
\le\ & ||u_0||_{H^{1/2}_\varrho(S_{R_1,h})}||\partial_{\nu(y)}G(x;\cdot)||_{\widetilde{H}^{-1/2}_{-\varrho}(S_{R_1,h})}+||\partial_\nu u_0||_{\widetilde{H}^{-1/2}_\varrho(S_{R_1,h})}||G(x;\cdot)||_{H^{1/2}_{-\varrho}(S_{R_1,h})}\\
\le\ & C\,||u_0||_{H^1_\varrho(\Sigma_{R_1,h})}\ ||G(x;\cdot)||_{H^1_{-\varrho}(\Sigma_{R_1,h})}.
\end{aligned}$$

Here, we choose $\varrho\in(-1,0)$ from the wHPSRC for u_0 and take $\Sigma_{R_1,h}\subset\Omega_{\Gamma,h}$ as a small region with fixed area that contains $S_{R_1,h}$ inside. In view of the wHPSRC relation $u_0\in H^1_\varrho(\{x\in\Omega_{\Gamma,h}: |x_1|>R\})$ and of the fact that $G(x;\cdot)\in H^1_{-\varrho}(\{x\in\Omega_{\Gamma,h}: |x_1|>R\})$, the right-hand side of the previous inequality tends to zero as $R_1\to\infty$. This together with (3.3) implies that

$$\int_{S_{R_1}}\big[u_0(y)\partial_{\nu(y)}G(x;y)-\partial_{\nu(y)}u_0(y)G(x;y)\big]\,\mathrm{d}s(y)\to 0\quad\text{as } R_1\to\infty. \tag{3.4}$$

Hence, letting $R_1\to\infty$ in (3.2), we can represent the function u_0 as

$$u_0(x)=\int_{S_R}\big[u_0(y)\partial_{\nu(y)}G(x;y)-\partial_{\nu(y)}u_0(y)G(x;y)\big]\,\mathrm{d}s(y),\quad x\in\Omega^+_R. \tag{3.5}$$

Taking the limit $x\to S_R$ in (3.5) and setting $p:=\partial_\nu u_0|_{S_R}\in H^{-1/2}(S_R)$ and $q:=u_0|_{S_R}\in\widetilde{H}^{1/2}(S_R)$ we arrive at the integral equation

$$(\frac{1}{2}I-\mathcal{D})q+\mathcal{S}p=0\quad\text{on}\quad S_R. \tag{3.6}$$

Here $\mathcal{D}$ and $\mathcal{S}$ are the double and single layer potentials over S_R, respectively, defined by

$$\begin{aligned}
(\mathcal{S}p)(x)\ &:=\ \int_{S_R}G(x;y)p(y)\,\mathrm{d}s(y),\qquad x\in S_R,\\
(\mathcal{D}q)(x)\ &:=\ \int_{S_R}\partial_{\nu(y)}G(x;y)q(y)\,\mathrm{d}s(y),\qquad x\in S_R.
\end{aligned}$$

Note that the classical jump relations apply for the special Green's function. Indeed, on Ω_Γ the function G is locally the sum of the classical full-space Green's function Φ plus an analytic function since the solution of the Helmholtz equation is analytic away from the boundary. The equations (3.1) and (3.6) give the variational formulation for the unknown solution $u \in X_R$ and $p \in H^{-1/2}(S_R)$:

$$A\big((u,p),(\varphi,\chi)\big) := \begin{pmatrix} a_1\big((u,p),(\varphi,\chi)\big) \\ a_2\big((u,p),(\varphi,\chi)\big) \end{pmatrix} = \begin{pmatrix} \int_{S_R} \partial_\nu v \, \overline{\varphi} \, \mathrm{d}s \\ \int_{S_R} (\frac{1}{2} I - \mathcal{D})(v|_{S_R}) \overline{\chi} \, \mathrm{d}s \end{pmatrix} \tag{3.7}$$

for all $(\varphi,\chi) \in X_R \times H^{-1/2}(S_R)$, where

$$a_1\big((u,p),(\varphi,\chi)\big) := \int_{\Omega_R^-} \big\{ \nabla u \cdot \nabla \overline{\varphi} - k^2 u \overline{\varphi} \big\} \, \mathrm{d}x - \int_{S_R} p \, \overline{\varphi} \, \mathrm{d}s,$$

$$a_2\big((u,p),(\varphi,\chi)\big) := \int_{S_R} \left[(\frac{1}{2} I - \mathcal{D})(u|_{S_R}) + \mathcal{S} p \right] \overline{\chi} \, \mathrm{d}s.$$

Altogether, we have shown that (u,p) is a solution of (3.7). Recall that $u \in X_R$ is the restriction to Ω_R^- of the solution u to the Helmholtz problem

$$\Delta[u - v^{in}] + k^2 [u - v^{in}] = 0 \text{ in } \Omega_\Lambda, \quad u = 0 \text{ on } \Lambda, \quad u - v^{in} \text{ satisfies ASR} \tag{3.8}$$

in the variational sense of [8, Thm. 4.1]. The difference $u_0 := u - v^{in} - v^{sc}$ satisfies the HPSRC by assumption and the solution function p is the trace of the normal derivative of u_0 on S_R. On the other hand, a solution $u \in X_R$ obtained from (3.7), can be extended from Ω_R^- to Ω_Λ via $u = v^{in} + v^{sc} + u_0$, where u_0 is expressed over $\{x \in \mathbb{R}^2 : |x| > R\}$ by (3.5) with the traces $u_0|_{S_R}$ and $\partial_\nu u_0|_{S_R}$ replaced by $(u - v)|_{S_R}$ and the solution p of (3.7), respectively. Moreover, the extension is a solution of the Helmholtz equation and thus analytic at the points of S_R (observe that the second variational equation in (3.7) yields the continuity of the extension over S_R and the first equation that of the normal derivatives), and the difference of the extension and the solution v satisfies the HPSRC due to Lem. 2.4.

Denoting the domain enclosed between $\Gamma_R := \{x \in \Gamma : |x| < R\}$ and S_R by $\widetilde{\Omega}_R^-$, we state the uniqueness and existence of solutions to (3.7) as follows.

Lemma 3.1. *Suppose the squared wavenumber k^2 is not an eigenvalue for the negative Laplacian over the domain $\widetilde{\Omega}_R^-$. Then there exists a unique solution $(u,p) \in X_R \times H^{-1/2}(S_R)$ of the variational equation (3.7).*

Proof. By arguing the same way as in [3], one can prove that the sesqui-linear form (3.7) is strongly elliptic over the space $X_R \times H^{-1/2}(S_R)$. Let us prove that the null space is trivial. The condition $a_1((u,p),(\varphi,\chi)) = 0$ yields that u satisfies the Helmholtz equation in Ω_R^- and that $\partial_\nu u = p$ over S_R. Introducing the function $\tilde{u} := \int_{S_R} \{\partial_\nu G(\cdot\,;y) u(y) - G(\cdot\,;y) p(y)\}$ over $\Omega_\Gamma \setminus S_R$, the condition $a_2((u,p),(\varphi,\chi)) = 0$ yields that the trace on S_R of u from Ω_R^- coincides with the trace of $\tilde{u}$ from Ω_R^+. Consequently, the jump relation for the integrals in the definition of $\tilde{u}$ implies that the trace on S_R of $\tilde{u}$ from $\widetilde{\Omega}_R^-$ is zero. In other words, the restriction $\tilde{u}|_{\widetilde{\Omega}_R^-}$ is a solution of the homogeneous Dirichlet problem for the Helmholtz equation over $\widetilde{\Omega}_R^-$. If there is no non-trivial solution of the Dirichlet problem, then $\tilde{u}|_{\widetilde{\Omega}_R^-} = 0$ and the trace on S_R of $\partial_\nu \tilde{u}$ from $\widetilde{\Omega}_R^-$ vanishes. The jump relation for the integrals in the definition of $\tilde{u}$ implies that the trace of $\partial_\nu \tilde{u}$ from Ω_R^+ is equal to p. If we define the function w by $w(x) := u(x)$ for $x \in \Omega_R^-$ and $w(x) := \tilde{u}(x)$ for $x \in \Omega_R^+$, then w and $\partial_\nu w$ are continuous over S_R. In other words, w is a solution of the homogeneous Dirichlet problem for the Helmholtz equation over $\Omega_\Lambda = \Omega_R^+ \cup S_R \cup \Omega_R^-$, which satisfies the radiation condition. The uniqueness of the solution to this boundary value problem (cf. [8, Thm. 4.1]) implies $w = 0$ s.t. the solutions u and $p = \partial_\nu u$ vanish. Hence, the null space of the operator defined by the left-hand side of (3.7) is trivial. Applying Fredholm's alternative, we obtain existence and uniqueness of weak solutions to (3.7). ∎

For $h>\max\{x_2: x\in\Lambda\}$, denote the strip between Λ and the straight line $\Gamma_h := \{x\in\mathbb{R}^2: x_2=h\}$ by $\Omega_{\Lambda,h}$. The space defined as $V_{h,\varrho}$ with $\Omega_{\Gamma,h}$ replaced by $\Omega_{\Lambda,h}$ is denoted by $V'_{h,\varrho}$. Well-posedness of the perturbed scattering problem is stated below, where the incoming wave is allowed to be either a plane wave or a point-source wave.

Theorem 3.1. *The locally perturbed wave scattering problem (3.8) admits a unique solution u such that $u-v^{in}\in H^1_{loc}(\Omega_\Lambda)$ and that the difference $u-v^{in}-v^{sc}$ fulfills the HPSRC and has a far-field pattern in $C(\overline{\mathbb{S}_+})$. Moreover, the restriction $(u-v^{in})|_{\Omega_{\Lambda,h}}$ is the unique variational solution in the weighted Sobolev space $V'_{h,\varrho}$. Here $-1<\varrho<-1/2$ for incident plane waves and $-1<\varrho<0$ for incident point-source waves.*

Proof. First we assume that the squared wavenumber is not an eigenvalue for the negative Laplacian over $\widetilde{\Omega}_R^-$. It is easy to check that a plane wave belongs to $H^1_\varrho(\Omega_{\Lambda,h})$ for any $h>\max\{x_2: x\in\Lambda\}$ and $\varrho\in(-1,-1/2)$, and that a point-source wave away from the source lies in the weighted Sobolev space with the index $\varrho\in(-1,0)$. Under the condition (2.2), the locally perturbed scattering problem admits a unique solution u such that $u-v^{in}$ satisfies the ASR (2.6) and belongs to the same space as the incoming wave (cf. [8, Thm. 4.1]). On the other hand, for the unique solution u to the variational problem (3.7) the difference $u-v^{in}=v^{sc}+u_0$ can be extended to a solution over Ω_Λ. In particular, the extension of u_0 for $|x|>R$ is given by (3.5). In view of Lem. 2.4, u_0 fulfills the HPSRC and has a far-field pattern. Moreover, $v^{sc}+u_0$ is in $H^1_\varrho(\Omega_{\Lambda,h})$ and satisfies the ASR (2.6), since both v^{sc} and u_0 are in $H^1_\varrho(\Omega_{\Lambda,h})$ and fulfill the ASR. Thm. 3.1 then follows from the uniqueness result of [8, Thm. 4.1].

If k^2 is a Dirichlet eigenvalue of $-\Delta$ over $\widetilde{\Omega}_R^-$, then by changing the artificial boundary S_R we arrive at a modified domain, for which k^2 is not an eigenvalue, and the above proof goes through with this modification. Indeed, we choose a smooth open curve S'_R intersecting Γ in the end points of S_R and located above Γ with a tiny vertical diameter. We denote the domain enclosed between S'_R and Γ by $\widetilde{\Omega}'_R := \widetilde{\Omega}_R^{-\prime}$. Then it is easy to show that, for a small ε, we get $\varepsilon\int_{\widetilde{\Omega}'_R}|\nabla u|^2>\int_{\widetilde{\Omega}'_R}|u|^2$ for all H^1 functions vanishing on the boundary of $\widetilde{\Omega}'_R$. The constant ε is less than a constant times the square root of the tiny diameter. Consequently, the sesqui-linear form $(u,v)\mapsto\int_{\widetilde{\Omega}'_R}\nabla u\cdot\overline{\nabla u}-k^2u\cdot\bar{v}$ is coercive for all $k^2<\varepsilon^{-1}$, and k^2 cannot be an eigenvalue of the negative Laplacian.

Unfortunately, S'_R might intersect the curve Λ. For parameter values $\mu\in[0,1]$, we consider the curves

$$S_R(\mu) := \left\{\left(x_1,\mu x'_2+(1-\mu)x_2\right): (x_1,x_2)\in S_R,\ (x_1,x'_2)\in S'_R\right\}$$

and the domains $\Omega_R(\mu)$ enclosed between $S_R(\mu)$ and $\Gamma_R:=\{x\in\Gamma: |x|<R\}$. If μ is less than a suitable $\varepsilon>0$, then $S_R(\mu)$ is close to S_R and does not intersect Λ. Transforming the Helmholtz operator over $\Omega_R(\mu)$ to a reference domain and considering appropriate Sobolev spaces, we obtain a family of operators, which depend analytically on μ and which are Fredholm with index zero. Since the operator is invertible for $\mu=1$, we conclude that there is only a countable set of μ such that the operator is not invertible. Consequently, there exist a $\mu<\varepsilon$ such that the operator is invertible and k^2 is not a Dirichlet eigenvalue of $-\Delta$. The curve $S_R(\mu)$ does not intersect Λ. ■

4 Appendix: Proofs of Lemmata 2.2 – 2.4

In this section, we suppose that Γ is a 2D rough surface, which usually means a non-local perturbation of an infinite planar boundary surface such that the surface lies within a finite distance of the boundary line of the half plane. A periodic surface is a special rough surface. Altogether, we suppose in two dimensions that Γ is Lipschitz, fulfills the condition (2.2), lies above the straight line $\{x\in\mathbb{R}^2:\, x_2=0\}$, and is bounded in the x_2-direction. All other definitions from the previous sections are retained. We prepare our proofs with two technical lemmata.

Lemma 4.1. *Let $x\in\mathbb{R}^2_+$ and $\Gamma_0=\{z\in\mathbb{R}^2:\, z_2=0\}$. Suppose that $g\in L^2_\varrho(\Gamma_0)$ with $1/2<\varrho<1$. If $n+\varrho>1/2$, then there is constant $C>0$ s.t.*

$$\int_{\{z\in\Gamma_0:\,|z_1|>1\}} \frac{|g(z)|}{|x-z|^n}\,\mathrm{d}s(z) \le C\|g\|_{L^2_\varrho(\Gamma_0)}\frac{1}{|x|^n}.$$

Proof. It follows from $g\in L^2_\varrho(\Gamma_0)$ that

$$\Big|\int_{\Gamma_0}\frac{|g(z)|}{|x-z|^n}\,\mathrm{d}s(z)\Big|^2 \le ||g||^2_{L^2_\varrho(\Gamma_0)}\int_{\{z\in\Gamma_0:\,|z_1|>1\}}\frac{1}{|x-z|^{2n}\,|z_1|^{2\varrho}}\,\mathrm{d}s(z)\,.$$

Hence, we only need to estimate the integral on the right-hand side. Without loss of generality, we suppose that $x=(0,x_2)$ lying on the positive x_2-axis, so that $|x|=x_2$. Denote the angle formed by $x-z$ and the positive x_2-axis by $\varphi\in(0,\pi/2)$. Then it is easy to see that $x_2=|x-z|\cos\varphi$ and $|z_1|=x_2\tan\varphi$. Changing variables, we find

$$\begin{aligned}
\int_{\{z\in\Gamma_0:\,|z_1|>1\}}\frac{1}{|x-z|^{2n}\,|z_1|^{2\varrho}}\,\mathrm{d}s(z) \;&\le\; \frac{C}{|x_2|^{2(n+\varrho-1/2)}}\int_{\arctan(1/x_2)}^{\pi/2}\frac{(\cos\varphi)^{2(n-1)}}{(\tan\varphi)^{2\varrho}}\,\mathrm{d}\varphi\\
&\le\; \frac{C}{|x_2|^{2(n+\varrho-1/2)}}\left\{1+\int_{\arctan(1/x_2)}^{\pi/2}\varphi^{-2\varrho}\,\mathrm{d}\varphi\right\}\\
&\le\; \frac{C}{|x_2|^{2(n+\varrho-1/2)}}\left\{1+\arctan(1/x_2)^{-2\varrho+1}\right\} \;\le\; \frac{C}{|x_2|^{2n}}.
\end{aligned}$$

This finishes the proof of Lem. 4.1. ∎

Lemma 4.2. *Consider fixed numbers h, h', and ϱ s.t. $h>h'>0$ and $1/2<\varrho<1$. Choose a function $f\in L^1(S_R)$ and suppose that $g_y\in L^2_\varrho(\Omega_{\Gamma,h'})$, $y\in S_R$, is a family of functions, which depend continuously on y. Extend g_y to Ω_Γ by $g_y(x):=0$ for $x_2>h'$. By w denote the y dependent solution of the homogeneous Dirichlet problem (cf. [8, Thm. 4.1]) for $\Delta w(\cdot\,;y)+k^2w(\cdot\,;y)=g_y$ over the domain Ω_Γ s.t. $w(\cdot\,;y)$ satisfies the condition ASR. Then the functions $w(\cdot\,;y)$, $y\in S_R$ and $w_I(\cdot):=\int_{S_R}w(\cdot\,;y)f(y)\mathrm{d}y$ defined over Ω_Γ satisfy the HPSRC and have a far-field pattern in $C(\overline{\mathbb{S}_+})$.*

Proof. We only prove the more involved case of w_I. From [8, Thm. 4.1] we infer that the family of solutions $S_R\ni y\mapsto w(\cdot\,;y)\in V_{h,\varrho}$ is continuous for the fixed ϱ. Hence, $w_I\in V_{h,\varrho}$, and, for the HPSRC, it remains to prove (2.7) with $v=w_I$. We set $\Gamma_h:=\{x\in\mathbb{R}^2:\, x_2=h\}$ and observe that the Dirichlet data $g_{h,y}=w(\cdot\,;y)|_{\Gamma_h}$ is analytic as a Helmholtz solution and belongs to the space $H^{1/2}_\varrho(\Gamma_h)$ for the ϱ with $1/2<\varrho<1$ and depends continuously on $y\in S_R$. Hence, $g_h:=w_I|_{\Gamma_h}\in L^2_\varrho(\Gamma_h)\subset H^{1/2}_\varrho(\Gamma_h)$, and g_h is analytic. Moreover, since $\operatorname{supp} g_y\subseteq\overline{\Omega}_{\Gamma,h'}$, the function w_I over the set $U_h:=\{x\in\mathbb{R}^2:\, x_2>h\}$ can be written as

$$\begin{aligned}
w_I(x) \;&=\; \int_{\Gamma_h}\frac{\partial\Phi_h(x;z)}{\partial z_2}g_h(z)\,\mathrm{d}s(z),\quad x\in U_h, && (4.1)\\
\Phi_h(x;z) \;&:=\; \Phi(x;z)-\Phi(x;z_h^*) \;=\; \frac{i}{4}H_0^{(1)}(k|x-z|)-\frac{i}{4}H_0^{(1)}(k|x-z_h^*|), && (4.2)\\
\frac{\partial\Phi_h(x;z)}{\partial z_2} \;&=\; 2\frac{\partial\Phi(x;z)}{\partial z_2}\ \text{for } z\in\Gamma_h,
\end{aligned}$$

which is known as the Upward Propagating Radiation Condition (UPRC) (see [9]). Here, z_h^* denotes the image of z with respect to reflection by the line Γ_h, and the function $\Phi_h(x;z)$ is the Green's function to the Helmholtz equation with the Dirichlet boundary condition on Γ_h. The improper integral in the above expression of w_I can be understood as the duality between $H_\varrho^{1/2}(\Gamma_h)$ and its dual space $H_{-\varrho}^{-1/2}(\Gamma_h)$ for our ϱ; we refer to [8] for the equivalence of the UPRC and ASR in weighted Sobolev spaces.

Using a twice differentiable cut-off function, we can represent g_h as the sum of two functions, the first with compact support and the second with support in $\{z\in\Gamma_0\colon |z_1|>1\}$. Correspondingly, w_I is the sum of the two integrals of the type (4.1) with g_h replaced by the two functions adding up to g_h. For both integrals, we have to prove the HPSRC. The case of w_I with compact support concerns a classical double layer potential with layer function from the trace space $H^{1/2}$. The resulting w_I fulfills the classical full-space Sommerfeld condition and has the well-known far-field pattern for all directions θ with $|\theta|=1$. The boundedness of the norms in $H_\varrho^1(\Omega_{\Gamma,h}\cap\{x\in\mathbb{R}^2\colon\ |x_1|>R\})$ with $-1<\varrho<1$ follows from the estimate $|\partial_{z_2}\Phi(x;z)|\le C(1+|x_2|)|x|^{-3/2}$, valid for z in a bounded set and for $|x|>R$ with sufficiently large R (cf. the subsequent formulas (4.3) and (4.5)). Consequently, without loss of generality we may suppose that the support of g_h on Γ_h is contained in the set $\{z\in\Gamma_0\colon |z_1|>1\}$, which allows us to apply Lem. 4.1.

Straightforward calculations show that for $x\in U_h$ and $z=(z_1,z_2)\in\Gamma_h$,

$$\frac{\partial\Phi_h(x;z)}{\partial z_2}\Big|_{z_2=h} \;=\; \frac{ik(x_2-z_2)H_1^{(1)}(k|x-z|)}{2|x-z|}\Big|_{z_2=h}. \tag{4.3}$$

Write $x=r(\cos\theta,\sin\theta)$, $s(r,z):=k|x(r)-z|$ and $\hat{x}:=x/r=(\cos\theta,\sin\theta)$. Here and thereafter, $H_n^{(1)}$ denotes the Hankel function of the first kind of order $n\in\mathbb{Z}$. Then we may rewrite the previous identity as

$$\frac{\partial\Phi_h(x;z)}{\partial z_2}\Big|_{z_2=h}=\frac{ik^2(r\sin\theta-h)}{2}\,\frac{H_1^{(1)}(s(r,z))}{s(r,z)}\Big|_{z_2=h}. \tag{4.4}$$

Below we shall write $s=s(r,z)$ for notational simplicity and make use of the asymptotic behavior of the Hankel functions for large argument as follows (cf. e.g. (3.59) in [13]):

$$\begin{aligned} H_n^{(1)}(s) &= \sqrt{\frac{2}{\pi s}}\,e^{i(s-(2n+1)/4\pi)}+\mathcal{O}(|s|^{-3/2}),\\ (H_n^{(1)})'(s) &= i\sqrt{\frac{2}{\pi s}}\,e^{i(s-(2n+1)/4\pi)}+\mathcal{O}(|s|^{-3/2}). \end{aligned} \tag{4.5}$$

We choose a $h''>h$ and consider $x\in\mathbb{R}^2$ with $x_2>h''$. Thus $s>k(h''-h)>0$, and the identity (4.4) implies that there exists a constant $C>0$ such that

$$\left|\frac{\partial\Phi_h(x;z)}{\partial z_2}\right|\le\frac{C\,r}{s^{3/2}}\qquad\text{for all}\quad x_2>h'', z_2=h.$$

Hence, by Lem. 4.1 we obtain

$$|w_I(x)|\le\int_{\Gamma_h}\frac{C\,r}{s(r,z)^{3/2}}|g_h(z)|\,\mathrm{d}s(z)\le C\,\|g_h\|_{L^2_\varrho(\Gamma_h)}r^{-1/2},$$

leading to the boundedness $\sup_{r>1}\sup_{x\in S_r\cap U_h} r^{1/2}|w_I(x)|<\infty$.

Further, through direct calculations we obtain

$$\frac{\partial}{\partial r}\frac{\partial\Phi_h(x;z)}{\partial z_2}=\frac{ik^2\sin\theta}{2}\frac{H_1^{(1)}(s)}{s}+\frac{ik^2(r\sin\theta-h)}{2}\frac{d}{ds}\left(\frac{H_1^{(1)}(s)}{s}\right)\frac{ds(r,z)}{dr}. \tag{4.6}$$

As $s\to\infty$, it holds that (cf. (4.5))

$$\frac{d}{ds}\left(\frac{H_1^{(1)}(s)}{s}\right)=\frac{H_1^{(1)'}(s)\,s-H_1^{(1)}(s)}{s^2}=i\frac{H_1^{(1)}(s)}{s}+\mathcal{O}(s^{-5/2}). \tag{4.7}$$

It is easy to check that, for $z_2=h$,

$$\begin{aligned}
\frac{ds(r)}{dr} &= \frac{k(r-\hat{x}\cdot z)}{|x-z|} && = k+k\frac{|x|-\hat{x}\cdot z-|x-z|}{|x-z|}\,,\\
\frac{ds(r)}{dr}-k &= k\frac{\dfrac{|x|^2-|x-z|^2}{|x|+|x-z|}-\hat{x}\cdot z}{|x-z|} && = k\,\frac{\dfrac{2|x|\hat{x}\cdot z-|z|^2}{|x|+|x-z|}-\hat{x}\cdot z}{|x-z|}\\
&= k\frac{\dfrac{-|z|^2}{|x|+|x-z|}+\dfrac{2|x|-|x|-|x-z|}{|x|+|x-z|}\hat{x}\cdot z}{|x-z|}\\
&= k\frac{\dfrac{-|z|^2}{|x|+|x-z|}+\dfrac{2|x|\hat{x}\cdot z-|z|^2}{(|x|+|x-z|)^2}\hat{x}\cdot z}{|x-z|}\,,\\
\left|\frac{ds(r)}{dr}-k\right| &\le C\frac{1+|z_1|}{s}.
\end{aligned} \tag{4.8}$$

Here the constant $C>0$ is independent of z with $z_2=h$ and of $x\in U_h$. Combining the relations (4.7) and (4.8) yields that, for $s\to\infty$,

$$\begin{aligned}
\frac{d}{ds}\left(\frac{H_1^{(1)}(s)}{s}\right)\frac{ds(r)}{dr}-ik\frac{H_1^{(1)}(s)}{s} &= i\frac{H_1^{(1)}(s)}{s}\left[\frac{ds(r,z)}{dr}-k\right]+\mathcal{O}(s^{-5/2})\\
&= i\frac{H_1^{(1)}(s)}{s}\mathcal{O}\left(\frac{1+|z_1|}{s}\right)+\mathcal{O}(s^{-5/2})\\
&= (1+|z_1|)\,\mathcal{O}(s^{-5/2}).
\end{aligned} \tag{4.9}$$

Now we deduce from (4.4),(4.6), and (4.9) that, for $z\in\Gamma_h$ and a suitable constant $C>0$,

$$\begin{aligned}
\left|\left(\frac{\partial}{\partial r}-ik\right)\frac{\partial\Phi_h(x;z)}{\partial z_2}\right| &\le C\left(\frac{1}{s^{3/2}}+\frac{r\,(1+|z_1|)}{s^{5/2}}\right) && (4.10)\\
&\le C\left(\frac{1}{s^{3/2}}+\frac{r\,(1+r)}{s^{5/2}}+\frac{r}{s^{3/2}}\right) && (4.11)
\end{aligned}$$

as $s\to\infty$. Again using Lem. 4.1, we get as $r\to\infty$ that

$$\begin{aligned}
|\partial_r w_I(x)-ikw_I(x)| &\le \int_{\{z\in\mathbb{R}^2:\,z_2=h\}}\left|(\partial_r-ik)\frac{\partial\Phi_h(x;z)}{\partial z_2}g_h(z)\right|\,\mathrm{d}s(z)\\
&\le C\|g_h\|_{L^2_\varrho(\Gamma_h)}|x|^{-1/2}
\end{aligned} \tag{4.12}$$

We choose $\varepsilon>0$ and prove that there is a constant C independent of ε s.t. the supremum over $x\in S_r\cap U_h$ of the expression $r^{1/2}|\partial_r w_I(x) - ikw_I(x)|$ is less than $C\varepsilon$ whenever r is larger than a suitable threshold. We choose an approximation $\tilde g_h$ of g_h over Γ_h with compact support s.t. $\|\tilde g_h - g_h\|_{L^2_\varrho(\Gamma_h)}<\varepsilon$ and define $\tilde w_I$ by the integral on the right-hand side of (4.1) with g_h replaced by $\tilde g_h$. Then the proof of (4.12) implies

$$\sup_{x\in S_r\cap U_h} r^{1/2}\left|\partial_\nu[w_I(x)-\tilde w_I(x)] - ik[w_I(x)-\tilde w_I(x)]\right| \le C\varepsilon. \tag{4.13}$$

On the other hand, the derivation of (4.12) implies

$$\sup_{x\in S_r\cap U_h} r^{1/2}\left|\partial_\nu \tilde w_I - ik\tilde w_I\right|^2 \le C_{g_h}\, r^{-1} \le \varepsilon \tag{4.14}$$

if r is larger than a suitable threshold. Indeed, for the function $\tilde g_h$ with bounded support, we can restrict the integration to $\{z\in\Gamma_h : |z_1|<C\}$ in the definition of $\tilde w_I$ (compare the right-hand side of (4.1)). Instead of the bound (4.11), we can use (4.10) with $|z_1|$ replaced by C, which leads us to (4.14). Combining (4.13) and (4.14), we get that the supremum over $x\in S_r\cap U_h$ of the expression $r^{1/2}|\partial_r w_I(x) - ikw_I(x)|$ is less than $(C+1)\varepsilon$ if r is sufficiently large. The proof of (2.7) for $v=w_I$ is completed.

Next we have to prove the existence of the far-field pattern. We prove it for the representation of w_I by the right-hand side of (4.1). The relations (4.3) and (4.5) lead to

$$\begin{aligned} w_I(x) &= \int_{\Gamma_h}\left\{ c\frac{e^{ik|x-z|}x_2}{|x-z|^{3/2}} + \mathcal{O}\big(|x-z|^{-3/2}\big)\right\} g_h(z)\mathrm{d}s(z) \\ &= c\int_{\Gamma_h}\frac{e^{ik|x-z|}x_2}{|x-z|^{3/2}} g_{L,h}(z)\mathrm{d}s(z) + \mathcal{O}\big(\|g_h-g_{L,h}\|\,|x|^{-1/2}\big) + \mathcal{O}\big(|x|^{-3/2}\big)\,, \\ g_{L,h}(z) &:= \begin{cases} g_h(z) & \text{if } -L<z_1<L \\ 0 & \text{else}\end{cases} \end{aligned}$$

(cf. Lem. 4.1), where c is an appropriate constant. Using that, for fixed L and $|x|>>L$,

$$\begin{aligned} \frac{1}{|x-z|^{3/2}} &= \frac{1}{|x|^{3/2}} + \mathcal{O}_L(|x|^{-5/2}), \\ |x-z| &= |x|\sqrt{1-2[x/|x|]\cdot z/|x| + |z|^2/|x|^2} \\ &= |x|\{1-[x/|x|]\cdot z/|x| + \mathcal{O}_L(|x|^{-2})\}, \\ \exp(ik|x-z|) &= \exp(ik|x|)\exp(-ik[x/|x|]\cdot z)\left[1+\mathcal{O}_L(|x|^{-1})\right], \end{aligned}$$

and setting $x=r\hat x$ with $r:=|x|$ and $\hat x\in S_R$, we arrive at

$$\begin{aligned} w_I(x) &= c\frac{e^{ikr}}{r^{1/2}}\,\hat x_2\, e^{-ikh\hat x_2}\int_{-L}^{L} e^{-ikz_1\hat x_1} g_h(z_1,h)\mathrm{d}z_1 \\ &\quad + \mathcal{O}\Big(\|g_h-g_{L,h}\|_{L^2_\varrho(\Gamma_h)}\,|x|^{-1/2}\Big) + \mathcal{O}_L\big(|x|^{-3/2}\big)\,. \end{aligned}$$

Here the $\mathcal{O}_L$ terms denote usual $\mathcal{O}$ expressions defined with constants depending on L. Now we get that $g_h\in L^1(\Gamma_h)\subset L^2_\varrho(\Gamma_h)$ is valid for $\varrho\in(1/2,1)$. So we obtain

$$\begin{aligned} w_I(x) &= \frac{e^{ikr}}{r^{1/2}}\,c\,\hat x_2 e^{-ikh\hat x_2}\int_{\mathbb{R}} e^{-ikz_1\hat x_1} g_h(z_1,h)\mathrm{d}z_1 \\ &\quad + \mathcal{O}\Big(\|g_h-g_{L,h}\|_{L^2_\varrho(\Gamma_h)}\,r^{-1/2}\Big) + \mathcal{O}_L\big(r^{-3/2}\big)\,, \end{aligned}$$

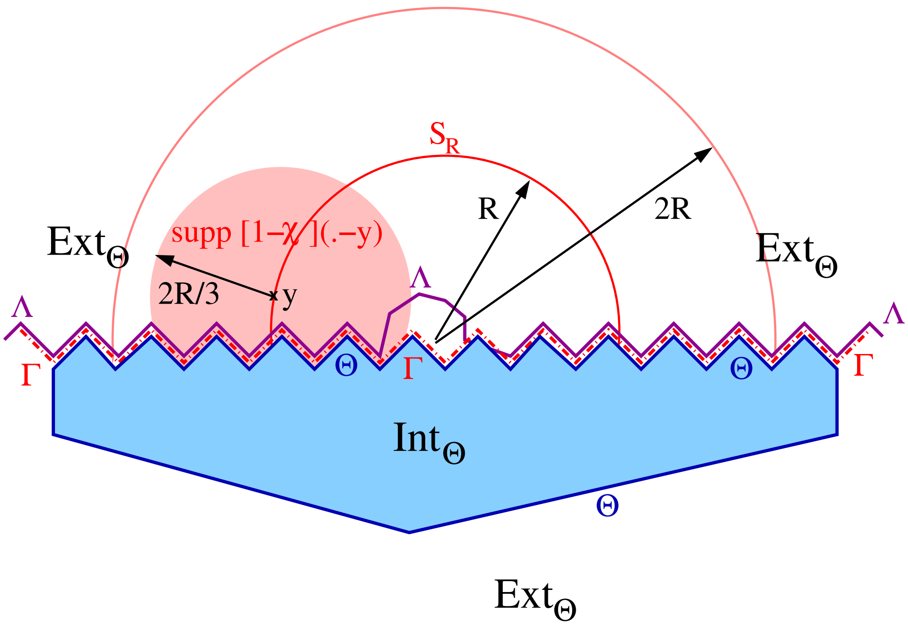

Figure 4: Closed curve Θ having common part with periodic profile curve Γ.

where the second term on the right-hand side is smaller than $\varepsilon/2$ for sufficiently large L. Fixing such an L, the third term is less than $\varepsilon/2$ if r is sufficiently large. All these estimates are uniform w.r.t. $\hat{x}$ s.t. the multiplicator of $\exp(ikr)r^{-1/2}$ in the first term on the right-hand side is the far-field pattern of the function w_I. ■

Below we present a proof to Lem. 2.2 by adapting the arguments in [8, Rem. 5.6]. Our approach has the merit that the constructed Green's function depends continuously on the source position y and does not rely on the distance between y and Γ.

Proof of Lemma 2.2. Without loss of generality we may fix $R>0$ and $y\in\Omega_\Gamma$ such that $|y|\leq R$. For a radius $r>0$, we denote the circle $\{x\in\mathbb{R}^2 : |x|<r\}$ by B_r. We consider a simple, bounded, and closed Lipschitz curve $\Theta\subset\mathbb{R}^2\setminus\Omega_\Gamma$ s.t. $\Gamma\cap B_{2R}\subseteq\Theta\cap B_{2R}$. By $G^\Theta(x;y)$ we denote the Green's function for the Dirichlet boundary problem with classical Sommerfeld radiation condition for the Helmholtz equation over the domain Ext_Θ exterior to Θ (cf. Fig. 4). Furthermore, we fix a cut-off function as

$$\chi(x)=\left\{\begin{array}{ll} 0 & \text{if} \quad |x|<R/3 \\ 1 & \text{if} \quad |x|>2R/3 \end{array}\right. ; \quad |\nabla^{|\alpha|}\chi|<C, \quad |\alpha|=0,1,2.$$

Recall that Γ is located between Γ_h and Γ_0. Then we shall prove

$$\begin{aligned} G(x;y) &= G^\Theta(x;y)-G^\Theta\big(x+(0,H);y\big)+u(x;y)+w(x;y), \\ u(x;y) &:= -\chi(x-y)G^\Theta(x;y)+G^\Theta\big(x+(0,H);y\big), \\ g_y(x) &:= -\left(\Delta_x+k^2\right)u(x;y), \end{aligned} \tag{4.15}$$

where H is a fixed positive constant s.t. $\max\{y,h\}\leq H/2$ and $w(\cdot\,;y)$ is the solution of the homogeneous Dirichlet problem for $\Delta w(\cdot\,;y)+k^2w(\cdot\,;y)=g_y$ under the condition ASR. Concerning the term $G^\Theta(x+(0,H);y)$, we observe that, for $x_0\in\Theta$, we get $G^\Theta([x_0-(0,H)]+(0,H);y)=G^\Theta(x_0;y)$, i.e., the boundary behaviour of $G^\Theta(x_0;y),\ x_0\in\Theta$ is shifted by H into the negative x_2 direction. Moreover, the weak singularity of the Green's function at the source point appears for $x+(0,H)=y$, i.e., at $x=y-(0,H)$. In other words, the function $(x,y)\mapsto G^\Theta(x+(0,H);y)$ is the Green's function $G^{\Theta-(0,H)}(x,y-(0,H))$ of the domain $Ext_\Theta-(0,H)$ at the source point $y-(0,H)$. In particular,

$G^{\Theta}(x+(0,H);y)$ is an analytic function on Ω_Γ. Clearly, the support of the right-hand side g_y over Ω_Γ is contained in the the compact set $\operatorname{supp}[1-\chi](\cdot-y) \subseteq \{x \in \mathbb{R}^2 : |x-y| \le 2R/3\}$ (cf. Fig. 4) and

$$g_y = -\sum_{j=1}^{2} \Big\{ 2\partial_{x_j}\chi(\cdot-y)\ \partial_{x_j}G^{\Theta}(\cdot\,;y) + \partial_{x_j}^2\chi(\cdot-y)\,G^{\Theta}(\cdot\,;y) \Big\}$$

is in $L^2_\varrho(\Omega_\Gamma)$. Therefore, the solution function $w(\cdot\,;y)$ is the solution of the variational equation of [8, Thm. 4.1].

Let us define G by the right-hand side of (4.15). Then the equation $\Delta G(\cdot\,;y) + k^2 G(\cdot\,;y) = \delta_y$ follows from the Green's function property of $G^{\Theta}(\cdot\,;y)$ and $G^{\Theta}(\cdot+(0,H);y)$ and from the definition of w using g_y. The boundary condition is fulfilled since $w(\cdot\,;y)|_\Gamma = 0$ holds for the solution of a homogeneous Dirichlet problem, since $G^{\Theta}(x;y) - G^{\Theta}(x+(0,H);y) + u(x;y)$ vanishes for x with $\chi(x-y) = 1$, and since, for $|y| \le R$ and for any $x \in \Gamma$ with $\chi(x-y) \ne 1$, we get $\chi(x-y)G^{\Theta}(x;y) = G^{\Theta}(x;y) = 0$ by the Dirichlet condition for the Green's function G^{Θ}. The condition ASR is satisfied as we shall prove the stronger HPSRC below. In other words, the right-hand side (4.15) is really the Green's function $G(x;y)$ for the domain Ω_Γ.

Let us prove the radiation condition and the existence of the far-field pattern for the terms on the right-hand side of (4.15). Lem. 4.2 implies the HPSRC and the existence of the far-field pattern for $w(\cdot\,;y)$. The Green's functions $G^{\Theta}(\cdot\,;y)$ and $G^{\Theta}(\cdot+(0,H);y)$ satisfy the classical full-space Sommerfeld condition implying (2.7) and have a far-field pattern even uniformly in all directions θ with $|\theta| = 1$. The boundedness of the $H^1_\varrho(\Omega_{\Gamma,h} \cap \{x \in \mathbb{R}^2 : |x_1| > R\})$-norms of $G^{\Theta}(\cdot\,;y) - G^{\Theta}(\cdot+(0,H);y)$ for $-1 < \varrho < 1$ follows from $\Phi(x+(0,H);y) = \Phi(x;y-(0,H))$ (cf. (2.5) for the definition of Φ) and from the estimate

$$\big|\partial_{y_1}^{l_1}\partial_{y_2}^{l_2}\Phi(x;y) - \partial_{y_1}^{l_1}\partial_{y_2}^{l_2}\Phi\big(x;y-(0,H)\big)\big| \le C\frac{1+|x_2|}{|x|^{3/2}} \tag{4.16}$$

valid for fixed integers $l_1, l_2 \ge 0$, for any y from a bounded set, and for any $x > R$ with sufficiently large R (see below and also [6, 7]). Indeed, we can represent $G^{\Theta}(\cdot\,;y)$ by the representation formula as the sum of a single and double layer operator over a bounded smooth curve Θ' enclosing Θ. The weight functions in these potentials are smooth. Consequently, $G^{\Theta}(\cdot\,;y) - G^{\Theta}(\cdot+(0,H);y)$ is equal to the difference of the representation formula minus the same formula with the same weights but on the curve Θ shifted by H in the direction of the negative x_2 axis. Applying (4.16), we get the estimate $|G^{\Theta}(x;y) - G^{\Theta}(x+(0,H);y)| \le C|x|^{-3/2}$ for large values of $|x|$ with $x_2 < h$. Similarly, we can prove the estimate for the difference of the gradients $|\nabla_x G^{\Theta}(x;y) - \nabla_x G^{\Theta}(x+(0,H);y)| \le C|x|^{-3/2}$ for large values of $|x|$ with $x_2 < h$. It is easy to see that these estimates imply the boundedness of the $H^1_\varrho(\Omega_{\Gamma,h} \cap \{x \in \mathbb{R}^2 : |x_1| > R\})$-norms of the functions $G^{\Theta}(\cdot\,;y) - G^{\Theta}(\cdot+(0,H);y)$.

For the proof of (4.16), we observe that $\partial_{y_1}^{l_1}\partial_{y_2}^{l_2}\Phi(x;y)$ is a derivative of the function $H_0^{(1)}(k|x-y|)$ multiplied by a rational function depending on the arguments $|x-y|^{1/2}$, x_1, x_2, y_1, and y_2. The derivatives of order higher than one can be reduced to the zero and first order derivative using Bessel's differential equation. In view of (4.5), we can replace the derivatives of the Hankel functions by the expression $\exp(i(k|x-y| - (2n+1)/4\pi))$. Simple estimates of the difference for the expressions with y and that with y replaced by $y-(0,H)$ gives the estimate on the right-hand side of (4.16). Indeed, estimates like

$$\begin{aligned}
&\Big|\,|x-y|^{1/2} - \big|x - \big(y-(0,H)\big)\big|^{1/2}\Big| \\
&= \frac{\big|-2x_2H + |y|^2 - |y-(0,H)|^2\big|}{\Big(|x-y|^{1/2} + \big|x-\big(y-(0,H)\big)\big|^{1/2}\Big)\big(|x-y| + \big|x-\big(y-(0,H)\big)\big|\big)} \le C|x|^{-1/2}
\end{aligned}$$

and

$$
\begin{aligned}
&\Big|\exp\big(ik|x-y|\big) - \exp\Big(ik\big|x-\big(y-(0,H)\big)\big|\Big)\Big| \\
&= \;\Big|\exp\Big(ik\big(|x-y| + \big|x-\big(y-(0,H)\big)\big|\,\big)/2\Big) \\
&\qquad \times 2i\sin\Big(k\big(|x-y| - \big|x-\big(y-(0,H)\big)\big|\,\big)/2\Big)\Big| \\
&= \;\Big|\exp\Big(ik\big(|x-y| + \big|x-\big(y-(0,H)\big)\big|\,\big)/2\Big) \\
&\qquad \times 2\sin\left(\frac{k}{2}\,\frac{|y|^2 - |y-(0,H)|^2 + 2Hx_2}{|x-y| + \big|x-\big(y-(0,H)\big)\big|}\right)\Bigg| \;\le\; C\frac{1+|x_2|}{|x|},
\end{aligned}
$$

lead us to the additional factor $(1+|x_2|)|x|^{-1}$ in $C(1+|x_2|)|x|^{-3/2}$ in comparison to an estimate by $C|x|^{-1/2}$ following directly from (4.5) applied to a single derivative of Φ. ■

We note that, using the approach of approximating the boundary curve of [8], Lem. 2.2 even holds for a larger class of non-smooth surfaces, namely, for graphs of arbitrary bounded continuous functions. It follows from (4.16) that the function $v = G^{\Theta}(\cdot\,;y) - G^{\Theta}(\cdot\,;y^*)$ decays faster than $G^{\Theta}(\cdot\,;y)$ in U_h. As a consequence of the proof of Lem. 2.2, we obtain the following well-posedness result on rough surface scattering problems.

Corollary 4.1. *Suppose that Γ is Lipschitz continuous, the domain Ω_Γ fulfills the condition (2.2) and that $f_\Gamma \in H^{1/2}_{\varrho}(\Gamma)$ with $\varrho > 1/2$. Moreover, suppose there exists an extension $w \in H^1_{\varrho}(\Omega_\Gamma)$ of f_Γ (i.e., $w|_\Gamma = f_\Gamma$) s.t., additionally, $\Delta w \in L^2_{\varrho}(\Omega_\Gamma)$. Then the boundary value problem $v = f_\Gamma$ on Γ for $\Delta v + k^2 v = 0$ in Ω_Γ under the condition ASR admits a unique solution $v \in H^1_{loc}(\Omega_\Gamma)$, which satisfies the HPSRC and has a far-field pattern in $C(\overline{\mathbb{S}_+})$.*

We do not know whether the condition on the index of decay $\varrho > 1/2$ is sharp. For instance, the function $\Phi(\cdot\,;y)|_\Gamma$ for $y \in \mathbb{R}^2 \setminus \overline{\Omega_\Gamma}$ belongs to $H^{1/2}_{\varrho}(\Gamma)$ with $\varrho < 0$, and $\Phi(\cdot\,;y)$ still fulfills the HPSRC.

Proof of Lemma 2.3. Replacing $G(x;y)$ by $\partial^{l_1}_{y_1}\partial^{l_1}_{y_1} G(x;y)$ in the proof of Lem. 2.2, we conclude that the modified right-hand side of (4.15) satisfies the properties of a differentiated Green's function together with the HPSRC. Applying the inverse operator $[\partial^{l_1}_{y_1}\partial^{l_1}_{y_1}]^{-1}$, i.e. integrations w.r.t. the variables y_1 and y_2, we define a new Green's function satisfying the HPSRC. From the uniqueness of the Green's function, we obtain that the modified right-hand side of (4.15) is indeed the differentiated Green's function $G(x;y)$. Hence, $\partial^{l_1}_{y_1}\partial^{l_1}_{y_1} G(x;y)$ is equal to the modified right-hand side of (4.15), and the HPSRC is satisfied. The far-field pattern exists as well. ■

Proof of Lemma 2.4. Applying integration by parts along the boundary S_R, we get a new integral representation with higher order derivatives on G but with smoother weight function f. Without loss of generality we may suppose $f \in L^1(\Gamma_0)$. Now the proof of Lem. 2.2 implies Lem. 2.4 if we apply Lem. 4.2 with w_I equal to the v of Lem. 2.4. ■

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