ON THE DETERMINATION OF POINT SOURCES BY BOUNDARY OBSERVATIONS: UNIQUENESS, STABILITY AND RECONSTRUCTION

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ABSTRACT. We consider the problem

$$u''(x,t) = u_{xx}(x,t) + \lambda(t)\sum_{k=1}^{N}\alpha_k\delta(x-\xi_k), \qquad 0<x<1,\ 0<t<T$$
$$u(x,0) = u'(x,0) = 0, \qquad 0<x<1$$
$$u(0,t) = u(1,t) = 0, \qquad 0<t<T,$$

where $u'(x,t) = \frac{\partial u}{\partial t}(x,t)$, $u''(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t)$, and $\lambda \in C^1[0,T]$, $\alpha_k \neq 0, \in \mathbb{R}$, $\xi_k \in (0,1)$, and $\delta(\cdot - \xi_k)$ is Dirac's delta function at ξ_k, $1 \le k \le n$. Our task consists in the determination of N, α_k, ξ_k, $1 \le k \le N$ from the boundary observation $\frac{\partial u}{\partial x}(0,t)$, $0<t<T$, where λ and $T>0$ are given. We prove the uniqueness, give a stability estimate and provide a scheme for reconstructing α_1, α_2, ξ_1, ξ_2 from $\frac{\partial u}{\partial x}(0,t)$, $0<t<T$ in the case $N=2$.

§1. Introduction.

We discuss a wave equation with point dislocation sources:

$$u''(x,t) = \Delta u(x,t) + \lambda(t)\sum_{k=1}^{N}\alpha_k\delta(x-\xi_k), \qquad x\in\Omega,\ 0<t<T$$
$$u(x,0) = u'(x,0) = 0, \qquad x\in\Omega$$
$$u(x,t) = 0, \qquad x\in\partial\Omega,\ 0<t<T.$$

Here $u'(x,t) = \frac{\partial u}{\partial t}(x,t)$, $u''(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t)$, $\Omega \subset \mathbb{R}^r$, $r \ge 1$, is a bounded domain with C^2-boundary, Δ is the Laplacian, $\lambda \in C^1[0,T]$, $\alpha_k \in \mathbb{R}$, $\xi_k \in \Omega$, $1 \le k \le N$ and $\delta(\cdot - \xi_k)$ is Dirac's delta function at ξ_k, that is, $<\delta(\cdot-\xi_k), \phi> = \phi(\xi_k)$ for $\phi \in C_0^\infty(\Omega)$. Here and henceforth $<\cdot,\cdot>$ denotes the duality pairing between $(C_0^\infty(\Omega))'$ and $C_0^\infty(\Omega)$ and $\cdot'$ denotes the dual space. Moreover we assume that $\alpha_k \neq 0$ and ξ_k are distinct for $1 \le k \le N$.

In this system, the N point sources are assumed to start the vibration. For instance, this kind of point sources can be related with models in seismology (e.g., Aki and Richards [1]).

Here we are engaged with the determination of point dislocation sources from boundary measurements: Let $\lambda = \lambda(t)$ and $T>0$, $\Gamma \subset \partial\Omega$ be given. Then determine $N \in \mathbb{N}$, $\alpha_k \neq 0, \in \mathbb{R}$, $\xi_k \in \Omega$, $1 \le k \le N$, from the normal derivative $\frac{\partial u}{\partial \nu}(x,t)$, $x \in \Gamma$, $0<t<T$. Our subjects are

(I) (Uniqueness) Does $\frac{\partial u}{\partial \nu}(x,t)$, $x\in\Gamma$, $0<t<T$ determine $N\in\mathbb{N}$, $\alpha_k\in\mathbb{R}$, $\xi_k\in\Omega$, $1\le k\le N$ uniquely?

(II) (Stability) Do small variations in $\frac{\partial u}{\partial \nu}(x,t)$, $x\in\Gamma$, $0<t<T$, imply small deviations in N, α_k, ξ_k?

(III) (Reconstruction of locations of sources) Can we reconstruct N, α_k and ξ_k, $1\le k\le N$ from $\frac{\partial u}{\partial \nu}(x,t)$, $x\in\Gamma$, $0<t<T$ in a stable manner?

In this paper, we treat the one-dimensional case, that is, $r=1$. With some modification in terms of the observability inequality in the Hilbert Uniqueness Method (e.g., Komornik [6], Lions [7]), we can extend our methodology to the multidimensional case. The results here are announced without proofs in Bruckner and Yamamoto [2] where Theorem 3 is, however, shown in a simpler form.

More precisely, we mainly consider

$$(1.1)\qquad \begin{cases} u''(x,t) = u_{xx}(x,t) + \lambda(t)\sum_{k=1}^{N}\alpha_k\delta(x-\xi_k), & 0<x<1,\ 0<t<T \\ u(x,0) = u'(x,0) = 0, & 0<x<1 \\ u(0,t) = u(1,t) = 0, & 0<t<T. \end{cases}$$

In the one-dimensional case, we observe the derivative $\frac{\partial u}{\partial x}(0,t)$, $0<t<T$ and we study

(I) (Uniqueness) Does $\frac{\partial u}{\partial x}(0,t)$, $0<t<T$, determine $N\in\mathbb{N}$, $\alpha_k\in\mathbb{R}$, $\xi_k\in(0,1)$, $1\le k\le N$ uniquely?

(II) (Stability) Do small variations in $\frac{\partial u}{\partial x}(0,t)$, $0<t<T$, imply small deviations in N, α_k, ξ_k?

(III) (Reconstruction of locations of sources) Let us fix $N\in\mathbb{N}$. Can we give a scheme for reconstructing $\alpha_k\in\mathbb{R}$, $\xi_k\in(0,1)$, $1\le k\le N$ from $\frac{\partial u}{\partial x}(0,t)$, $0<t<T$?

In this paper, we adopt the observation at one end point $x=0$. If we can make observations at $x=0,1$, then we are able to sharpen some of our results by noting that we can determine ξ_k from the observation over the time interval, for which waves from ξ_k reach the nearest of both end points. In Bruckner and Yamamoto [3], we considered an interior pointwise observation $u(x_0,t)$, $0<t<T$, at a fixed point $x_0\in(0,1)$ for determining ξ_k, and in this case uniqueness and stability are very sensitive to the choice of x_0. On the other hand, for the determination of L^2-source functions, not delta functions, in hyperbolic equations, we refer to Grasselli and Yamamoto [5], Yamamoto [10], [11], [12].

If one takes the observation function from $L^2(0,T)$, our reconstruction problem is well-posed. In the case of pointwise observations treated in [3], the analogous problem was proved to be mildly ill-posed, and a regularization was considered. Moreover, in this paper, the question of existence to the inverse problem is answered partially by presenting a reconstruction scheme in a special case.

The remainder of this paper is composed of eight Sections. In §2 we discuss the existence and regularity of solutions to the direct problem (1.1). In §§3-5, we give answers to the questions of uniqueness, stability and reconstruction, respectively, for the inverse problem. In §6, we show key lemmata needed for the proofs of our main results announced in §§3-5. In §§7-9 we prove the main results.

§2. Existence and regularity of solutions to the direct problem.

Henceforth let us define vectors P and Q by

$$(2.1)\qquad \begin{cases} P = \{N, \alpha_1, ..., \alpha_N, \xi_1, ..., \xi_N\} \in \mathbb{N}\times\mathbb{R}^N\times(0,1)^N, \\ Q = \{M, \beta_1, ..., \beta_M, \eta_1, ..., \eta_M\} \in \mathbb{N}\times\mathbb{R}^M\times(0,1)^M \end{cases}$$

with

$$\begin{cases} \alpha_k \ne 0,\ 1 \le k \le N,\ 0 < \xi_1 < .... < \xi_N < 1 \\ \beta_k \ne 0,\ 1 \le k \le M,\ 0 < \eta_1 < .... < \eta_M < 1 \end{cases} \tag{2.2}$$

and assume

$$\lambda \in C^1[0,T]. \tag{2.3}$$

Then we show the

Proposition 1. *For a given P, there exists a unique weak solution to (1.1):*

$$u = u(P) \in C^1([0,T]; L^2(0,1)) \cap C^0([0,T]; H_0^1(0,1)).$$

Proof. It is sufficient to assume that $N = 1$ and $\alpha_1 = 1$. We set $u = u(P)$. By the eigenfunction expansion, the integration by parts and (2.3), we see that u is represented by

$$\begin{aligned} u(x,t) = &\sum_{k=1}^{\infty} \frac{2}{k^2\pi^2} \sin k\pi\xi_1 \sin k\pi x \\ &\times \left(\lambda(t) - \lambda(0)\cos k\pi t - \int_0^t \lambda'(t-s)\cos k\pi s ds\right) \equiv \sum_{k=1}^{\infty} s_k(x,t) \end{aligned} \tag{2.4}$$

in the sense of distributions in (x,t). By Parseval's equality, we have

$$\left\| \sum_{k=1}^{\infty} \frac{\partial s_k(x,t)}{\partial x} \right\|^2_{L^2(0,1)} < \infty.$$

This implies that the series $\sum_{k=1}^{\infty} \frac{\partial s_k}{\partial x}$ is convergent in $L^\infty(0,T; H_0^1(0,1))$, namely, we see that $u \in C^0([0,T]; H_0^1(0,1))$. Similarly we can see that $u \in C^1([0,T]; L^2(0,1))$. For our observation $\frac{\partial u}{\partial x}(0,t)$, $0 < t < T$, we can prove the

Proposition 2.

$$\frac{\partial u}{\partial x}(0,\cdot) \in L^2(0,T).$$

The proof is given in §6.

§3. Uniqueness.

For the uniqueness and the stability, the length T of observation time has to be large, because of the finiteness of the wave propagation speed. Henceforth we always assume

$$T \ge 1. \tag{3.1}$$

Moreover we assume

$$\lambda \in C^1[0,T], \quad \lambda(0) \ne 0. \tag{3.2}$$

In other words, T must be larger than one, equal to the travelling time of a wave from $x = 0$ to $x = 1$. We are ready to state the

Theorem 1 (Uniqueness). *Under (2.1), (2.2), (3.1), (3.2), if*

$$\frac{\partial u(P)}{\partial x}(0,t) = \frac{\partial u(Q)}{\partial x}(0,t), \quad 0 < t < T,$$

then $P = Q$, namely, $M = N$, $\beta_k = \alpha_k$, $\xi_k = \eta_k$, $1 \le k \le N$.

The condition (3.1) for T is the best possible. In fact, we can prove the

Proposition 3. *For $T < 1$, let $u_T = u_T(x,t)$ be the solution to*

$$\begin{aligned} u''(x,t) &= u_{xx}(x,t) + \delta(x - T), \qquad 0 < x < 1,\ t > 0 \\ u(x,0) &= u'(x,0) = 0, \qquad 0 < x < 1 \\ u(0,t) &= u(1,t) = 0, \qquad t > 0. \end{aligned}$$

Then $\frac{\partial u_T}{\partial x}(0,t) = 0$, $0 < t < T$.

Consequently for $T < 1$, the derivative $\frac{\partial u}{\partial x}(0,t)$, $0 < t < T$, cannot distinguish the source term $\delta(\cdot - T)$ from $\delta(\cdot - T_1)$ with $T < T_1 < 1$.

§4. Stability.

Here for simplicity, we are concerned only with locations of point sources. That is, in this Section we assume

$$M = N, \quad \beta_k = \alpha_k, \quad 1 \le k \le N. \tag{4.1}$$

Our purpose is to measure the "distance" between a pair of N point sources $\{\xi_1, ..., \xi_N\}$ and $\{\eta_1, ..., \eta_N\}$. We recall

$$\begin{aligned} &0 \equiv \xi_0 < \xi_1 < ..... < \xi_N < \xi_{N+1} \equiv 1 \\ &0 < \eta_1 < .... < \eta_N < 1. \end{aligned} \tag{4.2}$$

Henceforth we regard $\{\xi_i\}_{1 \le i \le N}$ and $\{\eta_i\}_{1 \le i \le N}$ as known and unknown point sources, respectively. We would like to estimate $\{\eta_i\}_{1 \le i \le N}$ from $\{\xi_i\}_{1 \le i \le N}$.

Let us choose an arbitrary $\epsilon > 0$ such that

$$0 < \epsilon < \frac{1}{2} \min_{1 \le i \le N+1} |\xi_i - \xi_{i-1}|. \tag{4.3}$$

In order to obtain an actual estimate, we assume an a-priori information for $\{\eta_i\}_{1 \le i \le N}$:

$$|\xi_i - \eta_i| \le \epsilon, \quad 1 \le i \le N. \tag{4.4}$$

The condition (4.4) implies that $\{\xi_i\}_{1 \le i \le N}$ and $\{\eta_i\}_{1 \le i \le N}$ separate and are not very far mutually. By (4.3), we see

$$\gamma(\epsilon, \xi_j) \equiv \min_{1 \le i \le N+1} |\xi_i - \xi_{i-1} - 2\epsilon| > 0. \tag{4.5}$$

Then we can state conditional stability in the sense that the constant in the estimate depends on $\epsilon > 0$ and $\{\xi_i\}_{1 \le i \le N}$.

Theorem 2 (Conditional stability). *Under (3.1), (3.2) and (4.1)-(4.4), there exists a constant $C = C(T, \lambda) > 0$ such that*

$$\sum_{i=1}^{N} |\xi_i - \eta_i| \le \frac{C}{\sqrt{\gamma(\epsilon, \xi_j)}} \left\| \frac{\partial u(P)}{\partial x}(0, \cdot) - \frac{\partial u(Q)}{\partial x}(0, \cdot) \right\|_{L^2(0,T)}.$$

Here we note that $\gamma(\epsilon, \xi_j)$ is increasing as $\epsilon \to 0$, so that the constant in the estimate is decreasing.

§5. Reconstruction.

In this Section, for the sake of simplicity, we treat only two point sources, although our way is applicable to the reconstruction of many point sources:

$$N = 2. \tag{5.1}$$

In a succeeding paper, we will consider the reconstruction in the general case. Let us set

$$P = \{\alpha_1, \alpha_2, \xi_1, \xi_2\}, \quad \alpha_1, \alpha_2 \in \mathbb{R}, \ne 0,\ 0 < \xi_1 < \xi_2 < 1, \tag{5.2}$$

and

$$\frac{\partial u(P)}{\partial x}(0, t) = h(t), \quad 0 < t < 1. \tag{5.3}$$

In (5.3) we notice: For the uniqueness, $T \ge 1$ is required (Theorem 1), so that it is natural to consider the reconstruction in the case $T = 1$. Our task is to present a scheme for calculating $P = \{\alpha_1, \alpha_2, \xi_1, \xi_2\}$ from h. To this end, we introduce an operator from $L^2(0, 1)$ to $L^2(0, 1)$ by

$$(Lf)(t) = \int_0^t \lambda'(t - s) f(s) ds, \quad 0 < t < 1. \tag{5.4}$$

Then by $\lambda(0) \ne 0$, we see that $-(\lambda(0) + L)^{-1}$ corresponds to the solution of a Volterra equation of the second kind, and therefore, $-(\lambda(0) + L)^{-1}$ is a bounded operator from $L^2(0, 1)$ to $L^2(0, 1)$. In this Section, for simplicity, we further assume

$$\int_0^1 ((\lambda(0) + L)^{-1} \lambda)(t) dt \ne 0. \tag{5.5}$$

Example. The condition (5.5) with (3.2) are satisfied if

$$\lambda(t) = \cos \omega t, \quad t > 0 \quad \text{with } \omega \in \mathbb{R} \tag{5.6}$$

or

$$\lambda(t) = \frac{-t + c}{c}, \quad t > 0 \quad \text{with } c \ne 0. \tag{5.7}$$

In fact, in the both cases, we have $((\lambda(0)+L)^{-1}\lambda)(t)=1$.

Henceforth we denote the $L^2(0,1)$-scalar product by $(\cdot,\cdot)$: $(\phi,\psi)=\int_0^1\phi(t)\psi(t)dt$. Moreover, let for $k\ge 0$

$$e_k(t)=\cos k\pi t \tag{5.8}$$

and

$$\psi_k(t)=((\lambda(0)+L^*)^{-1}e_k)(t), \tag{5.9}$$

where $L^*: L^2(0,1)\longrightarrow L^2(0,1)$ is the adjoint operator of L:

$$(L^*\psi)(t)=\int_t^1\lambda'(s-t)\psi(s)ds, \quad 0<t<1,$$

and $(\lambda(0)+L^*)^{-1}e_k$ is equivalent to the solution ψ_k of a Volterra equation of the second kind:

$$\lambda(0)\psi_k(t)+\int_t^1\lambda'(s-t)\psi_k(s)ds=\cos k\pi t, \quad 0<t<1.$$

We note that (5.5) is rewritten as

$$(\lambda,\psi_0)\ne 0. \tag{5.5'}$$

Now we can state the

Theorem 3 (Reconstruction). *Assuming (5.1), (5.2) and (5.5') with (3.2), we have the following equations with respect to ξ_1, ξ_2, α_1 and α_2:*

$$\alpha_1\sin k\pi\xi_1+\alpha_2\sin k\pi\xi_2=k\pi\left\{\frac{(h,\psi_0)}{(\lambda,\psi_0)}(\lambda,\psi_k)-(h,\psi_k)\right\} \tag{5.10}$$

for $k\ge 1$.

In particular, let us assume that the strength is one and that both point sources are in the left half interval:

$$\alpha_1=\alpha_2=1, \quad 0<\xi_1<\xi_2\le\frac{1}{2}. \tag{5.11}$$

Then $(\xi_1,\xi_2)\in(0,\frac{1}{2}]^2$ with $\xi_1<\xi_2$ can be reconstructed from the data $h(t)$, $0<t<1$, given by (5.3) as follows.

First Step. *Calculate*

$$\begin{cases} a:=\pi\dfrac{(h,\psi_0)}{(\lambda,\psi_0)}(\lambda,\psi_1)-\pi(h,\psi_1) \\ b:=3\pi\dfrac{(h,\psi_0)}{(\lambda,\psi_0)}(\lambda,\psi_3)-3\pi(h,\psi_3). \end{cases} \tag{5.12}$$

Second Step. *Solve*

$$\theta^2 - a\theta + \frac{b + 4a^3 - 3a}{12a} = 0. \tag{5.13}$$

Let us denote the roots by θ_1, θ_2 *and suppose* $\theta_1 < \theta_2$.

Third Step.

$$\xi_1 = \frac{1}{\pi}\mathit{Arcsin}\ \theta_1, \quad \xi_2 = \frac{1}{\pi}\mathit{Arcsin}\ \theta_2. \tag{5.14}$$

This reconstruction is stable for $L^2(0,1)$-errors in the data $\frac{\partial u(P)}{\partial x}(0,\cdot) = h$, which is different from the case of an interior pointwise observation (Bruckner and Yamamoto [3]).

§6. Key lemmata.

In this Section, we establish three key lemmata for the proofs of Theorems 1 and 2. Let us first introduce function spaces. Identifying $L^2(0,1)$ with its dual, we define $H^{-1}(0,1)$ by the dual of $H_0^1(0,1)$: $H^{-1}(0,1) = (H_0^1(0,1))'$. Henceforth let $\|\cdot\|_X$ be the norm in a given Banach space X. Moreover let

$$^0H^1(0,T) = \{u \in H^1(0,T); u(T) = 0\} \tag{6.1}$$

be a Hilbert space with the norm

$$\|\psi\|_{^0H^1(0,T)} = \left(\int_0^T \psi'(t)^2 dt\right)^{\frac{1}{2}}. \tag{6.2}$$

Here we note that the norm $\|\psi\|_{^0H^1(0,T)}$ is equivalent to the norm $\|\psi\|_{H^1(0,T)} = \left(\int_0^T \psi(t)^2 + \psi'(t)^2 dt\right)^{\frac{1}{2}}$ for $\psi \in {}^0H^1(0,T)$. Let $H_{-1}(0,T)$ be the dual of $^0H^1(0,T)$ provided that the dual of $L^2(0,T)$ is identified with itself:

$$^0H^1(0,T) \subset L^2(0,T) \subset (^0H^1(0,T))' = H_{-1}(0,T). \tag{6.3}$$

Here the embeddings $^0H^1(0,T) \subset L^2(0,T)$ and $L^2(0,T) \subset H_{-1}(0,T)$ are bounded and dense.

We denote the duality pairing between $H_{-1}(0,T)$ and $^0H^1(0,T)$ by ${}_{H_{-1}}\!<\cdot,\cdot>_{^0H^1}$. Then we have

$${}_{H_{-1}}\!< \psi, u >_{^0H^1} = (\psi, u)_{L^2(0,T)}, \quad \psi \in L^2(0,T),\ u \in {}^0H^1(0,T). \tag{6.4}$$

By the definition of $H_{-1}(0,T)$, we have

$$\|\psi\|_{H_{-1}} = \sup\left\{ \left|{}_{H_{-1}}\!< \psi, \phi >_{^0H^1}\right|; \phi \in {}^0H^1(0,T),\ \|\phi\|_{^0H^1(0,T)} = 1 \right\}. \tag{6.5}$$

Next, let us consider an auxiliary system:

$$
\begin{cases} v''(x,t) = v_{xx}(x,t), & 0 < x < 1,\ 0 < t < T \\ v(x,0) = 0, \quad v'(x,0) = a(x), & 0 < x < 1 \\ v(0,t) = v(1,t) = 0, & 0 < t < T, \end{cases} \tag{6.6}
$$

where $a \in H^{-1}(0,1)$. There exists a unique weak solution

$$
v = v(a) \in C^0([0,T]; L^2(0,1)) \cap C^1([0,T]; H^{-1}(0,1))
$$

(e.g., Theorem 1.1 in Komornik [6], Theorem 3.8.2 in Lions and Magenes [8]). Furthermore, similarly to Théorème I.6.3 in Lions [7] (see also Grasselli and Yamamoto [5]), we can prove the

Lemma 1. *On the assumption (3.1), there exists a constant $C_1 > 0$ independent of a, such that*

$$
C_1^{-1}\|a\|_{H^{-1}(0,1)} \le \left\| \frac{\partial v(a)}{\partial x}(0,\cdot) \right\|_{H_{-1}(0,T)} \le C_1 \|a\|_{H^{-1}(0,1)} \tag{6.7}
$$

for all $a \in H^{-1}(0,1)$.

Let us define

$$
(K\psi)(t) = \int_0^t \lambda(t-s)\psi(s)ds, \quad 0 < t < T \tag{6.8}
$$

for $\psi \in L^2(0,T)$. Then K is a bounded operator from $L^2(0,T)$ to itself. Moreover, we have the

Lemma 2. *Under the assumption (3.2), we can uniquely extend the operator $K : L^2(0,T) \longrightarrow L^2(0,T)$ to an operator defined on $H_{-1}(0,T)$. Denoting the extended operator again by K, we choose a constant $C_2 = C_2(T,\lambda) > 0$ such that*

$$
C_2^{-1}\|K\psi\|_{L^2(0,T)} \le \|\psi\|_{H_{-1}(0,T)} \le C_2\|K\psi\|_{L^2(0,T)}, \quad \psi \in H_{-1}(0,T). \tag{6.9}
$$

Next, for $\lambda \in C^1[0,T]$, we consider

$$
\begin{cases} u''(x,t) = u_{xx}(x,t) + \lambda(t)a(x), & 0 < x < 1,\ 0 < t < T \\ u(x,0) = u'(x,0) = 0, & 0 < x < 1 \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \end{cases} \tag{6.10}
$$

where $a \in H^{-1}(0,1)$. Then there exists a unique weak solution

$$
u = u(a) \in C^0([0,T]; L^2(0,1)) \cap C^1([0,T]; H^{-1}(0,1))
$$

(e.g., Theorem 3.8.2 in Lions and Magenes [8]). Moreover, we can prove Duhamel's principle:

Lemma 3.

$$u(a)(x,t) = \int_0^t \lambda(t-s)v(a)(x,s)ds, \quad 0<x<1,\, 0<t<T$$

for $a \in H^{-1}(0,1)$.

Proof of Lemma 2. First we show the

Lemma 4. *Let us define*

$$(F\zeta)(t) = \lambda(0)\zeta(t) + \int_t^T \lambda'(s-t)\zeta(s)ds, \quad 0<t<T$$

for $\zeta \in {}^0H^1(0,T)$. Then F is surjective and an isomorphism from ${}^0H^1(0,T)$ to ${}^0H^1(0,T)$.

Proof of Lemma 4. Since F is a Volterra operator of the second kind, we easily see that

$$C_3^{-1}\|F\zeta\|_{{}^0H^1(0,T)} \le \|\zeta\|_{{}^0H^1(0,T)} \le C_3\|F\zeta\|_{{}^0H^1(0,T)}, \quad \zeta \in {}^0H^1(0,T)$$

where the constant $C_3 = C_3(T,\lambda) > 0$ is independent of ζ. Finally $(F\zeta)(T) = \lambda(0)\zeta(T) = 0$ if $\zeta \in {}^0H^1(0,T)$, so that F is seen to be surjective. The proof of Lemma 4 is complete.

Now we will complete the proof of Lemma 2. Let $\psi \in L^2(0,T)$ and $\zeta \in {}^0H^1(0,T)$. Then we have

$$((K\psi)', \zeta)_{L^2(0,T)} = \int_0^T (\lambda(0)\psi(t) + \int_0^t \lambda'(t-s)\psi(s)ds)\zeta(t)dt$$

$$= \int_0^T \lambda(0)\psi(t)\zeta(t)dt + \int_0^T \psi(s)\left(\int_s^T \lambda'(t-s)\zeta(t)dt\right)ds$$

by change of orders of integrations: $\int_0^T \left(\int_0^t ds\right)dt = \int_0^T \left(\int_s^T dt\right)ds$. Therefore, recalling the definition of F, we obtain

$$((K\psi)', \zeta)_{L^2(0,T)} = (\psi, F\zeta)_{L^2(0,T)}.$$

On the other hand, for $\psi \in L^2(0,T)$ and $\zeta \in {}^0H^1(0,T)$, since $(K\psi)(0) = \zeta(T) = 0$, the application of integration by parts yields

$$((K\psi)', \zeta)_{L^2(0,T)} = -(K\psi, \zeta')_{L^2(0,T)}.$$

Consequently we obtain

$$(K\psi, \zeta')_{L^2(0,T)} = -(\psi, F\zeta)_{L^2(0,T)}, \quad \psi \in L^2(0,T),\, \zeta \in {}^0H^1(0,T). \tag{6.11}$$

By (6.4) and the definition (6.5) of $\|\psi\|_{H_{-1}(0,T)}$, we have

$$\|\psi\|_{H_{-1}(0,T)} = \sup_{\|\phi\|_{{}^0H^1(0,T)}=1} |(\psi,\phi)_{L^2(0,T)}|$$

for $\psi \in L^2(0,T)$. By Lemma 4, there exists a constant $C_4 = C_4(T,\lambda) > 0$ such that

$$C_4^{-1}\|\psi\|_{H_{-1}(0,T)} \le \sup_{\|\zeta\|_{{}^0H^1(0,T)}=1} |(\psi, F\zeta)_{L^2(0,T)}| \le C_4\|\psi\|_{H_{-1}(0,T)}. \tag{6.12}$$

On the other hand, by (6.2) and (6.11), we have

$$\|K\psi\|_{L^2(0,T)} = \sup_{\|\theta\|_{L^2(0,T)}=1} |(K\psi,\theta)_{L^2(0,T)}|$$
$$= \sup_{\|\zeta\|_{{}^0H^1(0,T)}=1} |(K\psi,\zeta')_{L^2(0,T)}| = \sup_{\|\zeta\|_{{}^0H^1(0,T)}=1} |(\psi,F\zeta)_{L^2(0,T)}|.$$

Therefore, from (6.12) we obtain

$$C_4^{-1}\|\psi\|_{H^{-1}(0,T)} \le \|K\psi\|_{L^2(0,T)} \le C_4\|\psi\|_{H^{-1}(0,T)}, \quad \psi \in L^2(0,T). \tag{6.13}$$

Since $L^2(0,T)$ is dense in $H_{-1}(0,T)$, the inequalities (6.13) imply that we can uniquely extend $K : L^2(0,T) \longrightarrow L^2(0,T)$ to a bounded operator defined on $H_{-1}(0,T)$ to $L^2(0,T)$ and that (6.13) is true for all $\psi \in H_{-1}(0,T)$. Thus the proof of Lemma 2 is complete.

Proof of Lemma 3. For smooth a, this is nothing but Duhamel's principle. In fact, if $a \in C_0^\infty(0,1)$, then we can directly verify that the right hand side satisfies (6.10). For $a \in H^{-1}(0,1)$, noting that $C_0^\infty(0,1)$ is dense in $H^{-1}(0,1)$, by the regularity of $u(a)$ and $v(a)$, we can complete the proof of Lemma 3.

§7. Proof of Theorem 1.

By Sobolev's embedding, we see that $a \equiv \sum_{k=1}^N \alpha_k\delta(\cdot - \xi_k)$, $b \equiv \sum_{k=1}^M \beta_k\delta(\cdot - \eta_k) \in H^{-1}(0,1)$. By Lemmata 1 and 3, we have

$$\left(\frac{\partial u(a)}{\partial x} - \frac{\partial u(b)}{\partial x}\right)(0,\cdot) = K\left(\frac{\partial v(a)}{\partial x}(0,\cdot) - \frac{\partial v(b)}{\partial x}(0,\cdot)\right) \tag{7.1}$$

as functions in $L^2(0,T)$. Since $\frac{\partial u(P)}{\partial x}(0,t) = \frac{\partial u(Q)}{\partial x}(0,t)$, $0 < t < T$, means that $\frac{\partial u(a)}{\partial x}(0,t) = \frac{\partial u(b)}{\partial x}(0,t)$, $0 < t < T$, we apply Lemma 2 to obtain $\frac{\partial v(a-b)}{\partial x}(0,t) = 0$, $0 < t < T$, in the sense of $H_{-1}(0,T)$. Therefore Lemma 1 implies $a = b$ in $H^{-1}(0,1)$. Noting (2.1) and (2.2), the equality $a = b$ in $H^{-1}(0,1)$ is equivalent to $P = Q$, the conclusion of Theorem 1.

§8. Proof of Theorem 2.

We set

$$a = \sum_{k=1}^{N} \delta(\cdot - \xi_k), \quad b = \sum_{k=1}^{N} \delta(\cdot - \eta_k). \tag{8.1}$$

Then by (3.1) and (3.2), we apply Lemma 2 in (7.1) to obtain

$$\left\| \frac{\partial v(a-b)}{\partial x}(0,\cdot) \right\|_{H_{-1}(0,T)} \le C_2 \left\| \frac{\partial u(a-b)}{\partial x}(0,\cdot) \right\|_{L^2(0,T)}.$$

Henceforth constants $C = C(T, \lambda) > 0$ are independent of a and b. Therefore Lemma 1 implies

$$\|a-b\|_{H^{-1}(0,1)} \le C \left\| \frac{\partial u(P)}{\partial x}(0,\cdot) - \frac{\partial u(Q)}{\partial x}(0,\cdot) \right\|_{L^2(0,T)}. \tag{8.2}$$

As test functions, we set

$$\phi_i(x) = \begin{cases} \frac{2}{\xi_i - \xi_{i-1} + 2\epsilon}\left(x - \frac{\xi_i+\xi_{i-1}}{2}\right), & \frac{\xi_i+\xi_{i-1}}{2} \le x \le \xi_i + \epsilon \\ \frac{-2}{\xi_{i+1} - \xi_i - 2\epsilon}\left(x - \frac{\xi_i+\xi_{i+1}}{2}\right), & \xi_i + \epsilon < x < \frac{\xi_i+\xi_{i+1}}{2} \\ 0, & \text{otherwise} \end{cases} \tag{8.3}$$

for $1 \le i \le N$. Here we recall that $\xi_0 = 0$ and $\xi_{N+1} = 1$. By (4.3) and (4.4), for $1 \le i \le N$, we can readily verify that $\phi_i \in H_0^1(0,1)$,

$$\phi_i(\xi_j) = \phi_i(\eta_j) = 0, \quad j \ne i, \tag{8.4}$$

$$\|\phi_i\|_{H_0^1(0,1)} = \sqrt{2}\left(\frac{1}{\xi_i - \xi_{i-1} + 2\epsilon} + \frac{1}{\xi_{i+1} - \xi_i - 2\epsilon} \right)^{\frac{1}{2}} \tag{8.5}$$

and

$$\phi_i'(x) = \frac{2}{\xi_i - \xi_{i-1} + 2\epsilon}, \quad \xi_i - \epsilon < x < \xi_i + \epsilon. \tag{8.6}$$

By the definition of $\|\cdot\|_{H^{-1}(0,1)}$, we have

$$\left| {}_{H^{-1}} < a - b, \phi >_{H_0^1} \right| \le \|a-b\|_{H^{-1}(0,1)} \|\phi\|_{H_0^1(0,1)}$$

for any $\phi \in H_0^1(0,1)$, so that for $\phi = \phi_i$ by using (8.4), we obtain

$$|\phi_i(\xi_i) - \phi_i(\eta_i)| \le \|\phi_i\|_{H_0^1(0,1)} \|a-b\|_{H^{-1}(0,1)},$$

namely,

$$\begin{aligned} &|\xi_i - \eta_i| \\ \le& \frac{\xi_i - \xi_{i-1} + 2\epsilon}{2}\sqrt{2}\left(\frac{1}{\xi_i - \xi_{i-1} + 2\epsilon} + \frac{1}{\xi_{i+1} - \xi_i - 2\epsilon} \right)^{\frac{1}{2}} \|a-b\|_{H^{-1}(0,1)} \\ \le& C\left(1 + \frac{1}{\gamma(\epsilon, \xi_j)}\right)^{\frac{1}{2}} \|a-b\|_{H^{-1}(0,1)} \end{aligned}$$

by (8.5), (8,6) and (4.5). Thus (8.2) completes the proof of Theorem 2.

§9. Proof of Theorem 3.

By (2.4), recalling (5.4) and (5.8), we have

$$\begin{aligned} u(P)(x,t) =& \sum_{j=1}^{\infty} \frac{2}{j^2\pi^2}(\alpha_1 \sin j\pi\xi_1 + \alpha_2 \sin j\pi\xi_2) \sin j\pi x \\ & \times(\lambda(t) - (\lambda(0)+L)e_j(t)), \quad 0 < x < \xi_1,\, 0 < t < 1, \end{aligned} \tag{9.1}$$

with $P = \{\alpha_1, \alpha_2, \xi_1, \xi_2\}$. On the other hand,

$$\sum_{j=1}^{\infty} \frac{2 \sin j\pi\xi \sin j\pi x}{j^2\pi^2} = x(1-\xi), \quad 0 < x < \xi$$

(e.g., Prudnikov, Brychkov and Marichev [p.743, 9]), so that

$$\begin{aligned} & -u(P)(x,t) + \lambda(t)x(\alpha_1(1-\xi_1) + \alpha_2(1-\xi_2)) \\ =& \sum_{j=1}^{\infty} \frac{2}{j^2\pi^2}(\alpha_1 \sin j\pi\xi_1 + \alpha_2 \sin j\pi\xi_2)(\lambda(0)+L)e_j(t) \sin j\pi x, \\ & 0 < x < \xi_1,\, 0 < t < 1. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} & -\frac{\partial u(P)}{\partial x}(x,t) + \lambda(t)(\alpha_1(1-\xi_1) + \alpha_2(1-\xi_2)) \\ =& \frac{\partial}{\partial x}\left(\sum_{j=1}^{\infty} \frac{2}{j^2\pi^2}(\alpha_1 \sin j\pi\xi_1 + \alpha_2 \sin j\pi\xi_2)(\lambda(0)+L)e_j(t) \sin j\pi x\right), \\ & 0 < x < \xi_1,\, 0 < t < 1. \end{aligned} \tag{9.2}$$

Now we will prove that we can exchange $\frac{\partial}{\partial x}$ and $\sum_{j=1}^{\infty}$ at the right hand side. Since $\{1, \sqrt{2}e_j\}_{j\ge 1}$ forms an orthonormal basis in $L^2(0,1)$ and $(\lambda(0)+L) : L^2(0,1) \longrightarrow L^2(0,1)$ is surjetive and an isomorphism, we see that

$$C^{-1}\left(\sum_{j=1}^{\infty} a_j^2\right)^{\frac{1}{2}} \le \left\| \sum_{j=1}^{\infty} a_j(\lambda(0)+L)e_j \right\|_{L^2_t(0,1)} \le C\left(\sum_{j=1}^{\infty} a_j^2\right)^{\frac{1}{2}}$$

for $a_j \in \mathbb{R}$, $j \ge 1$ (e.g., Gohberg and Kreĭn [4]). Here $\|\cdot\|_{L^2_t(0,1)}$ means that we take the $L^2(0,1)$-norm of functions in t. For simplicity, we set

$$a_j(x) = \frac{2}{j\pi}(\alpha_1 \sin j\pi\xi_1 + \alpha_2 \sin j\pi\xi_2) \cos j\pi x, \quad j \ge 1.$$

For any $p \in \mathbb{N}$, we have

$$\frac{\partial}{\partial x}\left(\sum_{j=1}^{p} \frac{2}{j^2\pi^2}(\alpha_1 \sin j\pi\xi_1 + \alpha_2 \sin j\pi\xi_2) \sin j\pi x(\lambda(0) + L)e_j(t)\right),$$
$$= \sum_{j=1}^{p} a_j(x)(\lambda(0) + L)e_j(t)$$

and for any $x \in (0, 1)$,

$$\left\|\sum_{j=1}^{p} a_j(x)(\lambda(0) + L)e_j\right\|^2_{L^2_t(0,1)} \le C\sum_{j=1}^{p} |a_j(x)|^2 \le \frac{4(|\alpha_1| + |\alpha_2|)^2 C}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{j^2}.$$

Therefore we obtain

$$\sup_{p\ge 1} \max_{0\le x\le 1} \left\|\sum_{j=1}^{p} a_j(x)(\lambda(0) + L)e_j\right\|_{L^2_t(0,1)} < \infty,$$

which implies that

$$\sum_{j=1}^{\infty} \frac{\partial}{\partial x}\left(\frac{2}{j^2\pi^2}(\alpha_1 \sin j\pi\xi_1 + \alpha_2 \sin j\pi\xi_2)(\lambda(0) + L)e_j(t) \sin j\pi x\right)$$

is convergent in $C^0([0, \xi_1], L^2_t(0, 1))$. Consequently we can exchange $\frac{\partial}{\partial x}$ and $\sum_{j=1}^{\infty}$ in (9.2) to obtain

$$-\frac{\partial u(P)}{\partial x}(x, t) + \lambda(t)(\alpha_1(1 - \xi_1) + \alpha_2(1 - \xi_2))$$
$$= \sum_{j=1}^{\infty} \frac{2}{j\pi}(\alpha_1 \sin j\pi\xi_1 + \alpha_2 \sin j\pi\xi_2) \cos j\pi x(\lambda(0) + L)e_j(t)$$

in $C^0([0, \xi_1], L^2_t(0, 1))$, so that we can substitute $x = 0$:

$$\begin{aligned} &- h(t) + \lambda(t)(\alpha_1(1 - \xi_1) + \alpha_2(1 - \xi_2)) \\ &= \sum_{j=1}^{\infty} \frac{2}{j\pi}(\alpha_1 \sin j\pi\xi_1 + \alpha_2 \sin j\pi\xi_2)(\lambda(0) + L)e_j(t) \end{aligned} \tag{9.3}$$

where the series is convergent in $L^2_t(0, 1)$. On the other hand, we note

$$((\lambda(0) + L)e_j, (\lambda(0) + L^*)^{-1}e_k)_{L^2_t(0,1)} = \begin{cases} 0, & k \ne j,\ k, j \ge 1 \\ \frac{1}{2}, & k = j \end{cases}$$

and

$$((\lambda(0)+L)e_j,(\lambda(0)+L^*)^{-1}1)_{L_t^2(0,1)}=0, \quad j\ge 1$$

by $(e_j,e_k)_{L_t^2(0,1)}=\begin{cases}\frac{1}{2}, & k=j\\ 0, & k\ne j\end{cases}$ and $(e_j,1)_{L_t^2(0,1)}=0$.

Therefore in (9.3) taking $L_t^2(0,1)$-scalar products with $\psi_j=(\lambda(0)+L^*)^{-1}e_j$, we obtain

$$-(h,\psi_0)_{L_t^2(0,1)}+(\alpha_1(1-\xi_1)+\alpha_2(1-\xi_2))(\lambda,\psi_0)_{L_t^2(0,1)}=0 \tag{9.4}$$

and

$$\begin{aligned}&-(h,\psi_k)_{L_t^2(0,1)}+(\alpha_1(1-\xi_1)+\alpha_2(1-\xi_2))(\lambda,\psi_k)_{L_t^2(0,1)}\\ &=\frac{1}{k\pi}(\alpha_1\sin k\pi\xi_1+\alpha_2\sin k\pi\xi_2), \quad k\ge 1.\end{aligned} \tag{9.5}$$

By (5.5') and (9.4), we have

$$\alpha_1(1-\xi_1)+\alpha_2(1-\xi_2)=\frac{(h,\psi_0)_{L_t^2(0,1)}}{(\lambda,\psi_0)_{L_t^2(0,1)}},$$

with which we combine (9.5) and we obtain (5.10) for $k\ge 1$.

Finally let us assume (5.11). Then from (5.10) with $k=1,3$ we can derive

$$\sin\pi\xi_1+\sin\pi\xi_2=\pi\left\{\frac{(h,\psi_0)_{L_t^2(0,1)}}{(\lambda,\psi_0)_{L_t^2(0,1)}}(\lambda,\psi_1)_{L_t^2(0,1)}-(h,\psi_1)_{L_t^2(0,1)}\right\} \tag{9.6}$$

and

$$\sin 3\pi\xi_1+\sin 3\pi\xi_2=3\pi\left\{\frac{(h,\psi_0)_{L_t^2(0,1)}}{(\lambda,\psi_0)_{L_t^2(0,1)}}(\lambda,\psi_3)_{L_t^2(0,1)}-(h,\psi_3)_{L_t^2(0,1)}\right\}. \tag{9.7}$$

Recalling (5.5') and (5.12), we see from (9.6) and (9.7) that $a=\sin\pi\xi_1+\sin\pi\xi_2$ and $b=\sin 3\pi\xi_1+\sin 3\pi\xi_2$. By $\sin 3\rho=3\sin\rho-4\sin^3\rho$, we have

$$a=\sin\pi\xi_1+\sin\pi\xi_2, \quad \sin\pi\xi_1\sin\pi\xi_2=\frac{b+4a^3-3a}{12a},$$

and so the roots θ_1, θ_2 of (5.13) are equal to $\sin\pi\xi_1$ and $\sin\pi\xi_2$, respectively. Thus by $0<\xi_1<\xi_2\le\frac{1}{2}$, the equalities (5.14) follow and the proof of Theorem 3 is complete.

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