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Pierluigi Colli[1], Gianni Gilardi[1], Jürgen Sprekels[2]

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[1] Dipartimento di Matematica "F. Casorati"
Università di Pavia
and Research Associate at the IMATI – C.N.R. Pavia
Via Ferrata, 5
27100 Pavia
Italy
E-Mail: pierluigi.colli@unipv.it
gianni.gilardi@unipv.it

[2] Department of Mathematics
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin
Germany
and
Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: juergen.sprekels@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

In the recent paper "Well-posedness and regularity for a generalized fractional Cahn–Hilliard system" by the same authors, general well-posedness results have been established for a class of evolutionary systems of two equations having the structure of a viscous Cahn–Hilliard system, in which nonlinearities of double-well type occur. The operators appearing in the system equations are fractional versions in the spectral sense of general linear operators A, B having compact resolvents, which are densely defined, unbounded, selfadjoint, and monotone in a Hilbert space of functions defined in a smooth domain. In this work we complement the results given in quoted paper by studying a distributed control problem for this evolutionary system. The main difficulty in the analysis is to establish a rigorous Fréchet differentiability result for the associated control-to-state mapping. This seems only to be possible if the state stays bounded, which, in turn, makes it necessary to postulate an additional global boundedness assumption. One typical situation, in which this assumption is satisfied, arises when B is the negative Laplacian with zero Dirichlet boundary conditions and the nonlinearity is smooth with polynomial growth of at most order four. Also a case with logarithmic nonlinearity can be handled. Under the global boundedness assumption, we establish existence and first-order necessary optimality conditions for the optimal control problem in terms of a variational inequality and the associated adjoint state system.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ denote an open, bounded, and connected set with smooth boundary Γ and outward normal derivative ∂_ν, let $T > 0$ be a final time, and let $H := L^2(\Omega)$ denote the Hilbert space of square-integrable real-valued functions defined on Ω, endowed with the standard inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, respectively. We set $Q_t := \Omega \times (0,t)$ for $0 < t < T$ and $Q := \Omega \times (0,T)$. We investigate in this paper the following abstract distributed optimal control problem:

(CP) Minimize the tracking-type cost functional

$$\mathcal{J}((\mu,y),u) := \frac{\alpha_1}{2}\|y(T)-y_\Omega\|^2 + \frac{\alpha_2}{2}\int_0^T \|y(t)-y_Q(t)\|^2\,dt + \frac{\alpha_3}{2}\int_0^T \|\mu(t)-\mu_Q(t)\|^2\,dt + \frac{\alpha_4}{2}\int_0^T \|u(t)\|^2\,dt \tag{1.1}$$

over the admissible set

$$\mathcal{U}_{\rm ad} := \left\{ u \in H^1(0,T;L^2(\Omega)) : |u| \le \rho_1 \text{ a.e. in } Q, \quad \|u\|_{H^1(0,T;L^2(\Omega))} \le \rho_2 \right\}, \tag{1.2}$$

subject to the evolutionary state system

$$\partial_t y + A^{2r}\mu = 0, \tag{1.3}$$

$$\tau\partial_t y + B^{2\sigma} y + f'(y) = \mu + u, \tag{1.4}$$

$$y(0) = y_0. \tag{1.5}$$

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Here, ρ_1 and ρ_2 are fixed positive constants; α_i, $i = 1, 2, 3, 4$, are nonnegative coefficients but not all zero, and the given target functions satisfy $y_\Omega \in H$ and $y_Q, \mu_Q \in L^2(0, T; H)$. The linear operators A^{2r} and $B^{2\sigma}$, with $r > 0$ and $\sigma > 0$, denote fractional powers (in the spectral sense) of operators A and B. We will give a proper definition of such operators in the next section. Throughout this paper, we generally assume:

(A1) $A : D(A) \subset H \to H$ and $B : D(B) \subset H \to H$ are unbounded, monotone, and selfadjoint linear operators with compact resolvents.

This assumption implies that there are sequences $\{\lambda_j\}$ and $\{\lambda'_j\}$ of eigenvalues and orthonormal sequences $\{e_j\}$ and $\{e'_j\}$ of corresponding eigenvectors, that is,

$$Ae_j = \lambda_j e_j, \quad Be'_j = \lambda'_j e'_j, \quad \text{and} \quad (e_i, e_j) = (e'_i, e'_j) = \delta_{ij}, \quad \text{for } i, j = 1, 2, \dots, \tag{1.6}$$

such that

$$0 \le \lambda_1 \le \lambda_2 \le \dots, \quad \text{and} \quad 0 \le \lambda'_1 \le \lambda'_2 \le \dots, \quad \text{with} \quad \lim_{j\to\infty} \lambda_j = \lim_{j\to\infty} \lambda'_j = +\infty, \tag{1.7}$$

$$\{e_j\} \text{ and } \{e'_j\} \text{ are complete systems in } H. \tag{1.8}$$

Note that the state system (1.3)–(1.5) can be seen as a generalization of the famous Cahn–Hilliard system which models a phase separation process taking place in the container Ω. In this case, one typically has $A^{2r} = B^{2\sigma} = -\Delta$ with zero Neumann or Dirichlet boundary conditions, and the unknown functions y and μ stand for the *order parameter* (usually a scaled density of one of the involved phases) and the *chemical potential* associated with the phase transition, respectively. Moreover, f denotes a double-well potential. Typical and physically significant examples for f are the so-called *classical regular potential*, the *logarithmic double-well potential*, and the *double obstacle potential*, which are given, in this order, by

$$f_{reg}(r) := \frac{1}{4}(r^2 - 1)^2, \quad r \in \mathbb{R}, \tag{1.9}$$

$$f_{log}(r) := \big((1 + r)\ln(1 + r) + (1 - r)\ln(1 - r)\big) - c_1 r^2, \quad r \in (-1, 1), \tag{1.10}$$

$$f_{2obs}(r) := -c_2 r^2 \quad \text{if } |r| \le 1 \quad \text{and} \quad f_{2obs}(r) := +\infty \quad \text{if } |r| > 1. \tag{1.11}$$

Here, the constants $c_1 > 1$ and $c_2 > 0$ in (1.10) and (1.11) are such that f_{log} and f_{2obs} are nonconvex. Notice that in the case of the nondifferentiable potential (1.11) the state equation (1.4) has to be understood as a variational inequality. We also note that τ is a nonnegative parameter, where for the classical Cahn–Hilliard system one has $\tau = 0$ (the *nonviscous* case), while $\tau > 0$ corresponds to the *viscous* case.

In the recent paper [20], general well-posedness and regularity results for the state system (1.3)–(1.5) have been established for both the viscous and nonviscous cases and for nonlinearities that include all of the three cases (1.9)–(1.11). It turned out that the first eigenvalue λ_1 of A plays an important role in the analysis. Indeed, the main assumption for the operators A, B besides **(A1)** was the following:

(A2) Either

(i) $\lambda_1 > 0$

or

(ii) $0 = \lambda_1 < \lambda_2$, and e_1 is a constant and belongs to the domain of B^σ.

For our analysis of the optimal control problem **(CP)**, the general assumptions **(A1)** and **(A2)** are not sufficient. Indeed, in order to be able to prove that the control-to-state operator $\mathcal{S} : u \mapsto (\mu, y)$ is

Fréchet differentiable between suitable Banach spaces, it seems to be indispensable to assume that f is smooth in its domain (which means that the potential (1.11) is not admitted) and to have at disposal an $L^\infty(Q)$ bound for both the state component y and the functions $f^{(i)}(y)$, for $i = 1, 2, 3$. In the case of the logarithmic potential (1.10), this means that we need to separate y away from the critical arguments ± 1. We will discuss in Section 3 three situations in which appropriate boundedness conditions for y and the derivatives $f'(y)$ can be guaranteed, where one of these cases applies to the logarithmic potential.

Under these boundedness assumptions, we will be able to show the Fréchet differentiability of the control-to-state operator $\mathcal{S}$ (cf. Section 4) and to derive first-order necessary optimality conditions (cf. Section 5).

Let us add a few remarks on the existing literature. There exist numerous contributions on viscous/-nonviscous, local/nonlocal, convective/nonconvective Cahn–Hilliard systems for the classical (non-fractional) case $A = B = -\Delta$, $2r = 2\sigma = 1$, or some nonlocal counterparts, where various types of boundary conditions (e.g., Dirichlet, Neumann, dynamic) and different assumptions on the nonlinearity f were considered. We refer the interested reader to the recent paper [17] for a selection of associated references. Some papers also address the coupled Cahn–Hilliard/Navier–Stokes system (see, e.g, [23, 24] and the references given therein).

The literature on optimal control problems for non-fractional Cahn–Hilliard system is still less numerous. The case of Dirichlet and/or Neumann boundary conditions for various types of such systems were the subject of, e.g., the works [12, 14, 16, 22, 42, 45, 46], while the case of dynamic boundary conditions was studied in [9–11, 13, 15, 18, 19, 21, 27]. The optimal control of convective Cahn–Hilliard systems was addressed in [39, 43, 44], while the papers [25, 26, 29–33, 36] were concerned with coupled Cahn–Hilliard/Navier–Stokes systems.

There are only a few contributions to the theory of Cahn–Hilliard systems involving fractional operators. In the connection of well-posedness and regularity results, we refer to [1, 2] for the case of the fractional negative Laplacian with zero Dirichlet boundary conditions; general operators other than the negative Laplacian have apparently only studied in [20]. As of now, aspects of optimal control have been scarcely dealt with even for simpler linear evolutionary systems involving fractional operators; for such systems, some identification problems were addressed in the recent contributions [28, 38, 41], while for optimal control problems for such cases we refer to [5] (for the stationary – elliptic – case, see also [3, 4]). However, to the authors' best knowledge, the present paper appears to be the first contribution that addresses optimal control problems for Cahn–Hilliard systems with general fractional order operators.

The paper is organized as follows: the subsequent Section 2 brings some auxiliary functional analytic material, while in Section 3 some preparatory results concerning the state system (1.3)–(1.5) are discussed. In Section 4, the Fréchet differentiability of the control-to-state operator is shown, and in the final Section 5, we then prove an existence result for the optimal control problem and establish the first-order necessary conditions of optimality.

Throughout the paper, for a general Banach space X we denote by $\|\cdot\|_X$ and X^* its norm and dual space, respectively. However, particular symbols are adopted for the spaces we introduce in the next section.

2 Fractional powers and auxiliary results

In this section, we collect some auxiliary material concerning functional analytic notions. To this end, we generally assume that the conditions **(A1)** and **(A2)** are satisfied. At this point, some remarks on the assumption **(A2)** are in order.

Remark 2.1. First, the meaning of **(A2)**,(i) is clear, and this condition is satisfied for the more usual elliptic operators with zero Dirichlet boundary conditions (however, also zero mixed boundary conditions could be considered, with proper definitions of the domains of the operators). For instance, A can be the Laplace operator $-\Delta$ with domain $D(-\Delta) = H^2(\Omega) \cap H^1_0(\Omega)$ or the bi-harmonic operator Δ^2 with the domain $D(\Delta^2) = H^4(\Omega) \cap H^2_0(\Omega)$. The second case **(A2)**,(ii), in which the strict inequality means that the first eigenvalue $\lambda_1 = 0$ is simple, arises in both of the following important situations: A is the Laplace operator $-\Delta$ with zero Neumann boundary conditions, which corresponds to the choice $D(-\Delta) = \{v \in H^2(\Omega) : \ \partial_\nu v = 0 \text{ on } \Gamma\}$, or A is the bi-harmonic operator Δ^2 with the boundary conditions encoded in the definition of the domain $D(\Delta^2) = \{v \in H^4(\Omega) : \ \partial_\nu v = \partial_\nu \Delta v = 0 \text{ on } \Gamma\}$. Indeed, Ω is assumed to be bounded, smooth and connected.

Using the facts summarized in (1.6)–(1.8), we can define the powers of A and B for an arbitrary positive real exponent. For the first operator, we have

$$V_A^r := D(A^r) = \Big\{ v \in H : \ \sum_{j=1}^\infty |\lambda_j^r (v, e_j)|^2 < +\infty \Big\}, \tag{2.1}$$

$$A^r v = \sum_{j=1}^\infty \lambda_j^r (v, e_j) e_j \quad \text{for } v \in V_A^r, \tag{2.2}$$

the series being convergent in the strong topology of H, due to the properties (2.1) of the coefficients. In principle, we can endow V_A^r with the (graph) norm and inner product

$$\|v\|^2_{gr,A,r} := (v,v)_{gr,A,r} \quad \text{and} \quad (v,w)_{gr,A,r} := (v,w) + (A^r v, A^r w) \quad \text{for } v,w \in V_A^r. \tag{2.3}$$

This makes V_A^r a Hilbert space. However, we can choose any equivalent Hilbert norm. Indeed, in view of assumption **(A2)**, it is more convenient to work with the Hilbert norm

$$\|v\|^2_{A,r} := \begin{cases} \|A^r v\|^2 = \displaystyle\sum_{j=1}^\infty |\lambda_j^r (v, e_j)|^2 & \text{if } \lambda_1 > 0, \\[2ex] |(v,e_1)|^2 + \|A^r v\|^2 = |(v,e_1)|^2 + \displaystyle\sum_{j=2}^\infty |\lambda_j^r (v,e_j)|^2 & \text{if } \lambda_1 = 0. \end{cases} \tag{2.4}$$

In [20, Prop. 3.1] it has been shown that this norm is equivalent to the graph norm defined in (2.3), and we always will work with the norm (2.4) instead of (2.3). We also use the corresponding inner product in V_A^r given by

$$(v,w)_{A,r} = (A^r v, A^r w) \quad \text{or} \quad (v,w)_{A,r} = (v,e_1)(w,e_1) + (A^r v, A^r w),$$
$$\text{depending on whether } \lambda_1 > 0 \text{ or } \lambda_1 = 0, \quad \text{for } v,w \in V_A^r. \tag{2.5}$$

Remark 2.2. Observe that in the case $\lambda_1 = 0$ the constant value of e_1 equals one of the numbers $\pm|\Omega|^{-1/2}$, where $|\Omega|$ is the volume of Ω. It follows for every $v \in H$ that the first term $(v, e_1)e_1$ of the Fourier series of v is the constant function whose value is the mean value of v, which is defined by

$$\mathrm{mean}(v) := \frac{1}{|\Omega|} \int_\Omega v \,. \tag{2.6}$$

Moreover, the first terms of the sums appearing in (2.4) and (2.5) are given by

$$|(v, e_1)|^2 = |\Omega| \, (\mathrm{mean}\, v)^2 \quad \text{for every } v \in H,$$
$$(v, e_1)(w, e_1) = |\Omega| \, (\mathrm{mean}\, v)(\mathrm{mean}\, w) \quad \text{for every } v, w \in H.$$

In the same way as for A, starting from (1.6)–(1.8) for B, we can define the power B^σ of B for every $\sigma > 0$, where for V_B^σ we choose the graph norm. We therefore set

$$V_B^\sigma := D(B^\sigma), \quad \text{with the norm } \|\cdot\|_{B,\sigma} \text{ associated to the inner product}$$
$$(v, w)_{B,\sigma} := (v, w) + (B^\sigma v, B^\sigma w) \quad \text{for } v, w \in V_B^\sigma. \tag{2.7}$$

Remark 2.3. Let us briefly comment on the condition **(A2)**,(ii). We notice that the condition that e_1 be a constant belonging to V_B^σ holds true for many operators having a domain that involve Neumann boundary conditions. This is the case, for instance, if B is the Laplace operator with domain $D(-\Delta) = \{v \in H^2(\Omega) : \partial_\nu v = 0 \text{ on } \Gamma\}$. On the contrary, if $B = -\Delta$ with domain $D(-\Delta) := H^2(\Omega) \cap H_0^1(\Omega)$, then $D(B)$ does not contain any nonzero constant functions. However, V_B^σ does contain every constant function provided that $\sigma \in (0, 1/4)$, since V_B^σ is in this case a subspace of the usual Sobolev-Slobodeckij space $H^{2\sigma}(\Omega)$.

To resume our preparations, we observe that if r_i and σ_i are arbitrary positive exponents, then it is easily seen that we have the "Green type" formulas

$$(A^{r_1+r_2} v, w) = (A^{r_1} v, A^{r_2} w) \quad \text{for every } v \in V_A^{r_1+r_2} \text{ and } w \in V_A^{r_2}, \tag{2.8}$$
$$(B^{\sigma_1+\sigma_2} v, w) = (B^{\sigma_1} v, B^{\sigma_2} w) \quad \text{for every } v \in V_B^{\sigma_1+\sigma_2} \text{ and } w \in V_B^{\sigma_2}. \tag{2.9}$$

The next step is the introduction of some spaces with negative exponents. We set

$$V_A^{-r} := (V_A^r)^* \quad \text{for } r > 0, \tag{2.10}$$

and endow V_A^{-r} with the dual norm $\|\cdot\|_{A,-r}$ of $\|\cdot\|_{A,r}$. We use the symbol $\langle \cdot\,, \cdot \rangle_{A,r}$ for the duality pairing between V_A^{-r} and V_A^r and identify H with a subspace of V_A^{-r} in the usual sense, i.e., in order that $\langle z, v \rangle_{A,r} = (z, v)$ for every $z \in H$ and $v \in V_B^\sigma$. Similarly, we set

$$V_B^{-\sigma} := (V_B^\sigma)^* \quad \text{for } \sigma > 0. \tag{2.11}$$

As V_B^σ is dense in H, we have the analogous embedding

$$H \subset V_B^{-\sigma}. \tag{2.12}$$

Observe that the following embedding results are valid:

$$\text{the embeddings } V_A^{r_2} \subset V_A^{r_1} \subset H \text{ are dense and compact for } 0 < r_1 < r_2; \tag{2.13}$$
$$\text{the embeddings } H \subset V_A^{-r_1} \subset V_A^{-r_2} \text{ are dense and compact for } 0 < r_1 < r_2; \tag{2.14}$$
$$\text{the embeddings } V_B^{\sigma_2} \subset V_B^{\sigma_1} \subset H \text{ are dense and compact for } 0 < \sigma_1 < \sigma_2. \tag{2.15}$$

We also note the validity of the Poincaré type inequality (see [20, formula (3.5)])

$$\|v\| \le \widehat{c}\,\|A^r v\| \quad \text{for every } v \in V_A^r \text{ with } \mathrm{mean}(v) = 0. \tag{2.16}$$

At this point, we introduce the Riesz isomorphism $\mathcal{R}_r : V_A^r \to V_A^{-r}$ associated with the inner product (2.5), which is given by

$$\langle \mathcal{R}_r v, w \rangle_{A,r} = (v, w)_{A,r} \quad \text{for every } v, w \in V_A^r. \tag{2.17}$$

Moreover, we set

$$V_0^r := V_A^r \quad \text{and} \quad V_0^{-r} := V_A^{-r} \quad \text{if } \lambda_1 > 0, \tag{2.18}$$

$$V_0^r := \{ v \in V_A^r : \ \mathrm{mean}(v) = 0 \} \ \text{ and}$$
$$V_0^{-r} := \{ v \in V_A^{-r} : \ \langle v, 1 \rangle_{A,r} = 0 \} \quad \text{if } \lambda_1 = 0\,. \tag{2.19}$$

According to [20, Prop. 3.2], $\mathcal{R}_r$ maps V_0^r onto V_0^{-r} and extends to V_0^r the restriction of A^{2r} to V_0^{2r}. In view of this result, it is reasonable to use a proper notation for the restrictions of $\mathcal{R}_r$ and $\mathcal{R}_r^{-1}$ to the subspaces V_0^r and V_0^{-r}, respectively. We set

$$A_0^{2r} := (\mathcal{R}_r)_{|V_0^r} \quad \text{and} \quad A_0^{-2r} := (\mathcal{R}_r^{-1})_{|V_0^{-r}}\,, \tag{2.20}$$

where the index 0 has no meaning if $\lambda_1 > 0$ (since then $V_0^{\pm r} = V_A^{\pm r}$), while it reflects the zero mean value condition in the case $\lambda_1 = 0$. We thus have

$$A_0^{2r} \in \mathcal{L}(V_0^r, V_0^{-r}), \quad A_0^{-2r} \in \mathcal{L}(V_0^{-r}, V_0^r) \quad \text{and} \quad A_0^{-2r} = (A_0^{2r})^{-1}\,, \tag{2.21}$$

$$\langle A_0^{2r} v, w \rangle_{A,r} = (v, w)_{A,r} = (A^r v, A^r w) \quad \text{for every } v \in V_0^r \text{ and } w \in V_A^r\,, \tag{2.22}$$

$$\langle f, A_0^{-2r} f \rangle_{A,r} = \|A_0^{-2r} f\|_{A,r}^2 = \|f\|_{A,-r}^2 \quad \text{for every } f \in V_0^{-r}. \tag{2.23}$$

Notice that (2.23) implies that

$$\langle f', A_0^{-2r} f \rangle_{A,r} = \frac{1}{2}\,\frac{d}{dt}\,\|f\|_{A,-r}^2 \quad \text{a.e. in } (0,T), \ \text{for every } f \in H^1(0,T; V_0^{-r}). \tag{2.24}$$

Moreover, by virtue of [20, Prop. 3.3], we have

$$\bigl(A^r A_0^{-2r} f, A^r v \bigr) = \langle f, v \rangle_{A,r} \quad \text{for every } f \in V_0^{-r} \text{ and } v \in V_A^r. \tag{2.25}$$

In addition (see [20, Prop. 3.4]), the operator $A^{2r} \in \mathcal{L}(V_A^{2r}, H)$ can be extended in a unique way to a continuous linear operator, still termed A^{2r}, from V_A^r into V_0^{-r}, and we have

$$\|A^{2r} v\|_{A,-r} \le \|A^r v\| \quad \text{for every } v \in V_A^r. \tag{2.26}$$

As a final preparation, we now introduce some notations concerning interpolating functions.

Interpolants. Let N be a positive integer and Z be one of the spaces H, V_A^r, V_B^σ. We set $h_N := T/N$ and $I_N^n := ((n-1)h_N, nh_N)$ for $n = 1, \dots, N$. Given $z = (z_0, z_1, \dots, z_N) \in Z^{N+1}$, we define the piecewise constant and piecewise linear interpolants

$$\overline{z}_{h_N} \in L^\infty(0,T;Z), \quad \underline{z}_{h_N} \in L^\infty(0,T;Z) \quad \text{and} \quad \widehat{z}_{h_N} \in W^{1,\infty}(0,T;Z)$$

by setting

$$\overline{z}_{h_N}(t) = z^n \quad \text{and} \quad \underline{z}_{h_N}(t) = z^{n-1} \quad \text{for a.a. } t \in I_N^n,\ n = 1, \dots, N, \tag{2.27}$$

$$\widehat{z}_{h_N}(0) = z_0 \quad \text{and} \quad \partial_t \widehat{z}_{h_N}(t) = \frac{z^{n+1} - z^n}{h_N} \quad \text{for a.a. } t \in I_N^n,\ n = 1, \dots, N. \tag{2.28}$$

For the reader's convenience, we summarize some well-known relations between the finite set of values and the interpolants. We have that

$$\|\overline{z}_{h_N}\|_{L^\infty(0,T;Z)} = \max_{n=1,\dots,N} \|z^n\|_Z \,, \quad \|\underline{z}_{h_N}\|_{L^\infty(0,T;Z)} = \max_{n=0,\dots,N-1} \|z^n\|_Z \,, \tag{2.29}$$

$$\|\partial_t \widehat{z}_{h_N}\|_{L^\infty(0,T;Z)} = \max_{0 \le n \le N-1} \|(z^{n+1} - z^n)/h_N\|_Z \,, \tag{2.30}$$

$$\|\overline{z}_{h_N}\|^2_{L^2(0,T;Z)} = h_N \sum_{n=1}^{N} \|z^n\|_Z^2 \,, \quad \|\underline{z}_{h_N}\|^2_{L^2(0,T;Z)} = h_N \sum_{n=0}^{N-1} \|z^n\|_Z^2 \,, \tag{2.31}$$

$$\|\partial_t \widehat{z}_{h_N}\|^2_{L^2(0,T;Z)} = h_N \sum_{n=0}^{N-1} \|(z^{n+1} - z^n)/h_N\|_Z^2 \,, \tag{2.32}$$

$$\|\widehat{z}_{h_N}\|_{L^\infty(0,T;Z)} = \max_{n=1,\dots,N} \max\{\|z^{n-1}\|_Z, \|z^n\|_Z\} = \max\{\|z_0\|_Z, \|\overline{z}_{h_N}\|_{L^\infty(0,T;Z)}\} \,, \tag{2.33}$$

$$\|\widehat{z}_{h_N}\|^2_{L^2(0,T;Z)} \le h_N \sum_{n=1}^{N} \big(\|z^{n-1}\|_Z^2 + \|z^n\|_Z^2 \big) \le h_N \|z_0\|_Z^2 + 2 \|\overline{z}_{h_N}\|^2_{L^2(0,T;Z)} \,. \tag{2.34}$$

Moreover, it holds that

$$\|\overline{z}_{h_N} - \widehat{z}_{h_N}\|_{L^\infty(0,T;Z)} = \max_{n=0,\dots,N-1} \|z^{n+1} - z^n\|_Z = h_N \, \|\partial_t \widehat{z}_{h_N}\|_{L^\infty(0,T;Z)} \,, \tag{2.35}$$

$$\|\overline{z}_{h_N} - \widehat{z}_{h_N}\|^2_{L^2(0,T;Z)} = \frac{h_N}{3} \sum_{n=0}^{N-1} \|z^{n+1} - z^n\|_Z^2 = \frac{h_N^2}{3} \, \|\partial_t \widehat{z}_{h_N}\|^2_{L^2(0,T;Z)} \,, \tag{2.36}$$

and similar identities for the difference $\underline{z}_{h_N} - \widehat{z}_{h_N}$. As a consequence, we also have the inequalities

$$\|\overline{z}_{h_N} - \underline{z}_{h_N}\|_{L^\infty(0,T;Z)} \le 2h_N \, \|\partial_t \widehat{z}_{h_N}\|_{L^\infty(0,T;Z)} \,, \tag{2.37}$$

$$\|\overline{z}_{h_N} - \underline{z}_{h_N}\|^2_{L^2(0,T;Z)} \le \frac{4h_N^2}{3} \, \|\partial_t \widehat{z}_{h_N}\|^2_{L^2(0,T;Z)} \,. \tag{2.38}$$

Finally, we observe that

$$h_N \sum_{n=0}^{N-1} \|(z^{n+1} - z^n)/h_N\|_Z^2 \le \|\partial_t z\|^2_{L^2(0,T;Z)}$$
$$\text{if } z \in H^1(0,T;Z) \quad \text{and} \quad z^n = z(nh_N) \quad \text{for} \quad n = 0, \dots, N. \tag{2.39}$$

Throughout the paper, we make use of the elementary identity and inequalities

$$a(a-b) = \frac{1}{2}\, a^2 + \frac{1}{2}\, (a-b)^2 - \frac{1}{2}\, b^2 \ge \frac{1}{2}\, a^2 - \frac{1}{2}\, b^2 \quad \text{for every } a, b \in \mathbb{R}, \tag{2.40}$$

$$ab \le \delta a^2 + \frac{1}{4\delta}\, b^2 \quad \text{for every } a, b \in \mathbb{R} \text{ and } \delta > 0, \tag{2.41}$$

and quote (2.41) as the Young inequality. We also take advantage of the summation by parts formula

$$\sum_{n=0}^{k-1} a_{n+1}(b_{n+1}-b_n) = a_k b_k - a_1 b_0 - \sum_{n=1}^{k-1}(a_{n+1}-a_n)b_n\,, \tag{2.42}$$

which is valid for arbitrary real numbers $a_1,\dots,a_k$ and $b_0,\dots,b_k$. We also account for the discrete Gronwall lemma in the following form (see, e.g., [34, Prop. 2.2.1]): for nonnegative real numbers M and $a_n, b_n, n = 0,\dots,N$,

$$a_k \le M + \sum_{n=0}^{k-1} b_n a_n \quad \text{for } k = 0,\dots,N \qquad \text{implies}$$

$$a_k \le M \exp\Bigl(\sum_{n=0}^{k-1} b_n\Bigr) \quad \text{for } k = 0,\dots,N. \tag{2.43}$$

In (2.42)–(2.43) it is understood that a sum vanishes if the corresponding set of indices is empty.

3 General assumptions and the state system

In this section, we state our general assumptions and discuss the properties of the state system (1.3)–(1.5). Besides **(A1)** and **(A2)**, we generally assume for the data of the state system:

(A3) r, σ, and τ are fixed positive real numbers.

(A4) $f = f_1 + f_2$, where f_1, f_2 and f satisfy:
$f_1 \in C^3(D(f_1))$, $D(f_1)$ being an open interval, and $f_1'' \ge 0$ in $D(f_1)$;
$f_2 \in C^3(\mathbb{R})$, and f_2' is Lipschitz continuous on $\mathbb{R}$;
$\liminf_{|s|\nearrow+\infty} \frac{f(s)}{s^2} > 0$.

(A5) $y_0 \in V_B^{2\sigma}$ and $f'(y_0) \in H$.

Notice that **(A4)** holds true for the classical regular potential (1.9), for which we have $D(f_1) = \mathbb{R}$. In general, if $D(f_1) \ne \mathbb{R}$, then it is understood that f_1 also stands for its l.s.c. extension in the sum $f = f_1 + f_2$. This is the case for the logarithmic potential (1.10), for which we have $D(f_1) = (-1,1)$, and its l.s.c. extension is given by setting $f_1(\pm 1) := 2\ln(2)$ and $f_1(r) := +\infty$ if $|r| > 1$. In cases like this, the growth condition at infinity for f is trivially satisfied. Finally, we remark that assumption **(A4)** excludes the double obstacle potential (1.11), whose effective domain is the closed interval $[-1,1]$.

For the quantities entering the cost functional and the admissible set $\mathcal{U}_{\rm ad}$ (see (1.1) and (1.2)), we generally assume:

(A6) $y_\Omega \in L^2(\Omega)$, $y_Q, \mu_Q \in L^2(Q)$, the constants $\alpha_i \ge 0$, $i = 1,2,3,4$, are not all equal to zero, $\rho_1 > 0$, and $\rho_2 > 0$.

Finally, we denote the control space by

$$\mathcal{X} \,:=\, H^1(0,T;L^2(\Omega)) \cap L^\infty(Q), \tag{3.1}$$

and make an assumption which is rather a denotation, since $\mathcal{U}_{\rm ad}$ is a bounded subset of $\mathcal{X}$:

(A7) The constant $R > 0$ is such that $\mathcal{U}_{\mathrm{ad}} \subset \mathcal{U}_R := \{u \in \mathcal{X} : \|u\|_{\mathcal{X}} < R\}$.

With the above assumptions, we are now ready to cite a well-posedness result for the state system (1.3)–(1.5) which is a special case of the general results [20, Thm. 2.6 and Thm. 2.8]. To this end, we recall the weak notion of solution to the system (1.3)–(1.5) introduced in [20]. Namely, we look for a pair of functions (μ, y) satisfying the variational (in)equalities

$$(\partial_t y(t), v) + (A^r \mu(t), A^r v) = 0 \quad \text{for a.e. } t \in (0,T) \text{ and every } v \in V_A^r, \tag{3.2}$$

$$(\tau \partial_t y(t), y(t) - v) + (B^\sigma y(t), B^\sigma(y(t) - v)) + \int_\Omega f_1(y(t)) \, + (f_2'(y(t)), y(t) - v)$$

$$\leq \, (\mu(t) + u(t), y(t) - v) + \int_\Omega f_1(v) \quad \text{for a.e. } t \in (0,T) \text{ and every } v \in V_B^\sigma, \tag{3.3}$$

$$y(0) = y_0 \, . \tag{3.4}$$

We have the following result.

Theorem 3.1. *Suppose that the general assumptions* **(A1)**–**(A5)** *and* **(A7)** *are fulfilled. Then the weak state system* (3.2)–(3.4) *has for every* $u \in \mathcal{U}_R$ *a unique solution* (μ, y) *such that*

$$\mu \in L^\infty(0,T; V_A^{2r}), \tag{3.5}$$
$$y \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V_B^\sigma), \tag{3.6}$$
$$f_1(y) \in L^1(Q). \tag{3.7}$$

Moreover, there are constants $K_1 > 0$ *and* $K_2 > 0$*, which depend only on the data of the state system and* R*, such that the following holds true:*

(i) *Whenever* $u \in \mathcal{U}_R$ *is given, then the associated solution* (μ, y) *satisfies*

$$\|\mu\|_{L^\infty(0,T;V_A^{2r})} \, + \, \|y\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V_B^\sigma)} \, \leq \, K_1. \tag{3.8}$$

(ii) *Whenever* $u_i \in \mathcal{U}_R$, $i = 1,2$*, are given and* (μ_i, y_i), $i = 1,2$*, are the associated solutions, then*

$$\|y_1 - y_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V_B^\sigma)} \, \leq \, K_2 \, \|u_1 - u_2\|_{L^2(0,T;H)} \, . \tag{3.9}$$

Remark 3.2. Note that the regularity (3.7) can be improved up to

$$f_1(y) \in L^\infty(0,T; L^1(\Omega)).$$

Indeed, first, f_1 is bounded from below by an affine function, so that

$$\int_\Omega f_1(y(t)) \geq -c \int_\Omega \big(1 + |y(t)|\big)$$

for some constant $c > 0$ and for a.a. $t \in (0,T)$, and the last term is bounded since $y \in L^\infty(0,T;H)$. On the other hand, thanks to (3.3), $\int_\Omega f_1(y)$ is bounded from above by an $L^\infty(0,T)$-function (cf. (3.5) –(3.6)).

Theorem 3.1 ensures that the control-to-state operator

$$\mathcal{S} : u \mapsto \mathcal{S}(u) := (\mu, y) \tag{3.10}$$

is well defined as a mapping from $\mathcal{U}_R \subset \mathcal{X}$ into the Banach space specified by the regularity conditions (3.5), (3.6).

The following global boundedness condition is crucial for the analysis of the control problem.

(GB) There is a constant $K_3 > 0$, which depends only on the data of the state system and R, such that the following holds true: whenever $(\mu, y) = \mathcal{S}(u)$ for some $u \in \mathcal{U}_R$, then

$$\|y\|_{L^\infty(Q)} + \max_{i=0,1,2,3} \|f_1^{(i)}(y)\|_{L^\infty(Q)} \le K_3. \tag{3.11}$$

Remark 3.3. We observe that under the condition **(GB)** we have that $f_1'(y) \in L^\infty(Q)$, and the variational inequality (3.3) is easily seen to be equivalent to the variational equation

$$(\tau\partial_t y(t), v) + (B^\sigma y(t), B^\sigma(v)) + (f_1'(y(t)), v) + (f_2'(y(t)), v) = (\mu(t) + u(t), v)$$
$$\text{for a.e. } t \in (0,T) \text{ and every } v \in V_B^\sigma. \tag{3.12}$$

A fortiori, by virtue of the bounds (3.8) and a comparison in equation (3.12), we have $B^{2\sigma} y = \mu + u - \tau\,\partial_t y - f'(y) \in L^\infty(0,T;H)$, whence we can infer the additional regularity

$$y \in L^\infty(0,T;V_B^{2\sigma}). \tag{3.13}$$

In particular, under the condition (3.13) the solution (μ, y) is strong and, in particular, (1.4) is valid almost everywhere in Q.

Examples. The condition **(GB)** seems to be very restrictive and requires a case-to-case analysis. We now give some sufficient conditions under which it holds true. In all of the following three examples, we have $B = -\Delta$ with either zero Dirichlet or zero Neumann boundary condition. Then, it turns out that $V_B^1 \subset H^2(\Omega)$, and thus, by regularity, $V_B^\sigma \subset H^{2\sigma}(\Omega)$ for all $\sigma \in \mathbb{N}$. Interpolation shows that then also $V_B^\sigma \subset H^{2\sigma}(\Omega)$ for all noninteger $\sigma > 0$. We also notice that $V_{-\Delta}^{1/2}$ is equal to $H_0^1(\Omega)$ for Dirichlet boundary conditions or $H^1(\Omega)$ in the case of Neumann boundary conditions.

1. We begin with the logarithmic potential (1.10). Recall that in this case we have $f_1(r) = (1+r)\ln(1+r) + (1-r)\ln(1-r)$ for $r \in (-1,1)$, $f_1(\pm 1) = 2\ln(2)$, and $f_1(r) = +\infty$ if $r \notin [-1,1]$. Hence it follows from the variational inequality (3.3) that the corresponding solution component y must satisfy $y \in [-1,1]$ almost everywhere. In particular, $\|f_2'(y)\|_{L^\infty(Q)}$ is bounded. Now assume that $B = -\Delta$ with zero Neumann boundary condition, $2\sigma = 1$, and

$$-1 < \inf_{x\in\Omega} y_0(x), \quad \sup_{x\in\Omega} y_0(x) < +1. \tag{3.14}$$

Moreover, assume that the embedding

$$V_A^{2r} \subset L^\infty(\Omega) \tag{3.15}$$

holds true. This is the case, for instance, if $A = -\Delta$ with zero Dirichlet or Neumann condition and $r > 3/8$. Indeed, we then have (see above) $V_A^{2r} \subset H^{4r}(\Omega)$ and $4r > 3/2$, which implies that $H^{4r}(\Omega) \subset L^\infty(\Omega)$. Now let $(\mu, y) = \mathcal{S}(u)$ for some $u \in \mathcal{U}_R$. If (3.15) is satisfied, then we can infer from (3.8) that there is some global constant $M > 0$ such that $\|\mu + u - f_2'(y)\|_{L^\infty(Q)} \le M$. By the form of the derivative f_1' of the logarithmic potential, there are constants $r_*, r^* \in (-1,1)$ with $r_* \le y_0 \le r^*$ a.e. in Ω such that

$$f_1'(r) + M \le 0 \quad \forall\, r \in (-1, r_*) \quad \text{and} \quad f_1'(r) - M \ge 0 \quad \forall\, r \in (r^*, 1).$$

Now, recall that $V_{-\Delta}^{1/2} = H^1(\Omega)$. We thus may insert $v = y(t) - (y(t) - r^*)^+ \in H^1(\Omega)$ in the variational inequality (3.3), where $(y(t) - r^*)^+$ is the positive part of $y(t) - r^*$. We then find for almost every $t \in (0,T)$ the inequality

$$\begin{aligned} &\frac{\tau}{2}\frac{d}{dt}\|(y(t)-r^*)^+\|^2 + \int_\Omega |\nabla(y(t)-r^*)^+|^2 \\ &\le \int_\Omega \big[f_1(y(t)-(y(t)-r^*)^+) - f_1(y(t)) + (\mu(t)+u(t)-f_2'(y(t)))(y(t)-r^*)^+\big]. \end{aligned} \tag{3.16}$$

We claim that the integrand of the integral on the right-hand side is nonpositive. To this end, we put

$$\Omega_+(t) := \{x \in \Omega : \, y(x,t) > r^*\}, \quad \Omega_-(t) = \{x \in \Omega : \, y(x,t) \le r^*\}.$$

Obviously, $(y(t)-r^*)^+ = 0$ on $\Omega_-(t)$, and thus the integrand is zero on $\Omega_-(t)$. On the other hand, in $\Omega_+(t)$ we have $(y(t)-r^*)^+ = y(t) - r^*$, and thus the integrand equals

$$f_1(r^*) - f_1(y(t)) + (\mu(t)+u(t)-f_2'(y(t)))(y(t)-r^*).$$

Now $r^* \in (-1,1)$, and thus f_1 is differentiable at r^*. Hence, invoking the convexity of f_1, we have in $\Omega_+(t)$ that $f_1(r^*) - f_1(y(t)) \le -f_1'(r^*)(y(t)-r^*)$. Now, by construction, it holds that $\mu(t) + u(t) - f_2'(y(t)) - f_1'(r^*) \le 0$, which implies that the integrand is nonpositive also in this case, as claimed. In conclusion, the expression on the right-hand side of (3.16) is nonpositive. At this point, we integrate (3.16) over $(0,t)$, where $t \in (0,T]$ is arbitrary. Since $(y_0 - r^*)^+ = 0$ by assumption, we obtain that $(y - r^*)^+ = 0$ a.e. in Q, which implies $y \le r^*$ a.e. in Q. Similarly, we obtain that $y \ge r_*$ a.e. in Q. With this, the condition (3.11), i.e., the validity of **(GB)**, is shown.

We conclude this examples with the remark that the above argumentation remains valid for every potential $f_1 \in C^1(-1,1) \cap C^0([-1,1])$ which is convex on $[-1,1]$ and satisfies

$$\lim_{r \searrow -1} f_1'(r) = -\infty, \quad \lim_{r \nearrow +1} f_1'(r) = +\infty,$$

where it is understood that f_1 is extended to the whole of $\mathbb{R}$ by putting $f_1(r) = +\infty$ for $r \notin [-1,1]$.

2. Next, we assume that $f_1 \in C^3(\mathbb{R})$, which is satisfied for the classical potential (1.9). In this case, $V_B^\sigma \subset H^{2\sigma}(\Omega)$, and it holds $H^{2\sigma}(\Omega) \subset L^\infty(\Omega)$ (and thus $y \in L^\infty(Q)$ with (3.11) whenever $(\mu, y) = \mathcal{S}(u)$ for some $u \in \mathcal{U}_R$) if $\sigma > 3/4$.

3. The following result shows that the condition $\sigma > 3/4$ is not optimal if the nonlinearity satisfies a suitable growth condition, which is met by, e.g., the classical regular potential (1.9).

Proposition 3.4. *Let $B = -\Delta$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$, let $f \in C^3(\mathbb{R})$, and suppose that the general assumptions* **(A1)–(A5)** *and* **(A7)** *are fulfilled. In addition, assume that there is some $\widehat{C}_1 > 0$ such that*

$$|f'(s)| \le \widehat{C}_1 \left(1 + |s|^3\right) \quad \forall\, s \in \mathbb{R}. \tag{3.17}$$

Then the condition **(GB)** *holds true whenever $\frac{9}{20} < \sigma \le \frac{3}{4}$.*

Proof. We show the result only for $\frac{9}{20} < \sigma < \frac{3}{4}$ (the case $\sigma = \frac{3}{4}$ can be treated in a similar way). We then have

$$V_B^\sigma \subset H^{2\sigma}(\Omega) \subset L^p(\Omega) \quad \text{with} \quad -\frac{3}{p} = 2\sigma - \frac{3}{2}, \quad \text{i.e.,} \quad p = \frac{6}{3-4\sigma}\,. \tag{3.18}$$

We notice that (3.18) holds true also in the case of Neumann boundary conditions. However, we have to assume Dirichlet boundary conditions later on. From (1.4), we infer that $B^{2\sigma}y = g - f'(y)$ with $g := \mu + u - \tau\,\partial_t y$, where, owing to (3.8) and (3.17),

$$\|g\|_{L^\infty(0,T;H)} + \|f'(y)\|_{L^\infty(0,T;L^{p/3}(\Omega))} \le C_1\,, \tag{3.19}$$

with a global constant $C_1 > 0$. We now distinguish between the two cases $p/3 \ge 2$ and $p/3 < 2$, which, by virtue of (3.18), occur if $\sigma \ge 1/2$ and $\sigma < 1/2$, respectively.

Assume first that $\sigma \ge 1/2$. Then, by (3.19), $B^{2\sigma}y \in L^\infty(0,T;H)$, whence

$$y \in L^\infty(0,T;V_B^{2\sigma}) \subset L^\infty(0,T;H^{4\sigma}(\Omega)) \subset L^\infty(Q),$$

since $4\sigma \ge 2$. Therefore, (3.11) is valid.

Assume now that $\sigma < 1/2$. Then, we only have $B^{2\sigma}y \in L^\infty(0,T;L^{p/3}(\Omega))$. We now claim that the following implication is valid:

If $v \in H$ and $B^s v \in L^q(\Omega)$ with $s \in (0,1)$ and $q > \frac{3}{2s}$, then $v \in L^\infty(\Omega)$ and $\|v\|_{L^\infty(\Omega)} \le C_2$, where C_2 depends only on s, q and Ω. (3.20)

To prove this claim, we note that $\lambda_1' > 0$ (see (1.6)) in our situation, and thus we have $B^s w = 0$ for $w \in V_B^\sigma$ if and only if $w = 0$. Therefore, we must have $v = \tilde{v}_+ - \tilde{v}_-$, where $\tilde{v}_\pm \in V_B^\sigma$ is the (unique) weak solution to the fractional Dirichlet problem $B^s\tilde{v}_\pm = (B^s v)_\pm$. At this point, as we are dealing with Dirichlet boundary conditions, we can apply the results of [7, Thm. 4.1 and Sect. 2.1], which imply that the estimate

$$0 \le \tilde{v}_\pm \le \kappa\,\|(B^s v)_\pm\|_q\,\mathcal{B}_q(\varphi) \quad \text{in } \Omega, \tag{3.21}$$

holds true. Here, the constant $\kappa > 0$ depends on s, q, and Ω, $\varphi \in C^0(\bar{\Omega})$ is the first (positive) eigenfunction of B^s (or, equivalently, of B, i.e., we have $\varphi = e_1'$), and $\mathcal{B}_q$ is a suitable continuous function on $[0,+\infty)$ depending on q. The claim thus holds true.

We now choose $s = 2\sigma$, so that $s \in (0,1)$, as well as $q = p/3$. Then we can apply (3.20) provided that $q > \frac{3}{2s}$, i.e., $2s > \frac{3}{q}$, which, in view of (3.18), just means that $\sigma > \frac{9}{20}$. □

Remark 3.5. Observe that if $B = -\Delta$ with zero Dirichlet boundary condition and $\sigma > \frac{9}{20}$, then the assumption **(A2)** can only be fulfilled if $\lambda_1 > 0$. Indeed, if $\lambda_1 = 0$, then **(A2)**,(ii) necessitates that the constant functions belong to $V_B^\sigma \subset H^{2\sigma}(\Omega)$, which in turn requires that $0 < \sigma < 1/4$.

In the following, we will always assume that the condition **(GB)** is satisfied and account for Remark 3.3. We now improve the stability estimate (3.9) established in Theorem 3.1.

Theorem 3.6. *Suppose that* **(A1)**–**(A5)**, **(A7)** *and* **(GB)** *are satisfied. Then there is a constant $K_4 > 0$, which depends only on the data of the state system and R, such that the following holds true: whenever $u_i \in \mathcal{U}_R$, $i = 1,2$, are given and $(\mu_i, y_i) = \mathcal{S}(u_i)$, $i = 1,2$, are the associated solutions to the state system* (1.3)–(1.5)*, then it holds, for every $t \in (0,T]$,*

$$\begin{aligned}&\|\mu_1 - \mu_2\|_{L^2(0,t;V_A^{2r})} + \|y_1 - y_2\|_{H^1(0,t;H)\cap L^\infty(0,t;V_B^\sigma)}\\&\le K_4\,\|u_1 - u_2\|_{L^2(0,t;H)}\,.\end{aligned} \tag{3.22}$$

Proof. The functions $u := u_1 - u_2$, $y := y_1 - y_2$, $\mu := \mu_1 - \mu_2$, obviously satisfy the system

$$\partial_t y + A^{2r}\mu = 0 \quad \text{a.e. in } Q, \tag{3.23}$$

$$\tau\,\partial_t y + B^{2\sigma} y + f'(y_1) - f'(y_2) = \mu + u \quad \text{a.e. in } Q, \tag{3.24}$$

$$y(0) = 0 \quad \text{a.e. in } \Omega. \tag{3.25}$$

In the following, C_i, $i \in \mathbb{N}$, denote constants that depend only on the data of the state system and R. We multiply (3.23) by μ and (3.24) by $\partial_t y$, add the resulting identities, and integrate over Q_t, where $t \in (0, T]$ is arbitrary. Rearranging terms and applying Young's inequality, we then obtain the inequality

$$\tau \int_0^t \|\partial_t y(s)\|^2\,ds + \int_0^t \|A^r\mu(s)\|^2\,ds + \frac{1}{2}\|B^\sigma y(t)\|^2 = \int_0^t\int_\Omega \partial_t y\big(u - (f'(y_1) - f'(y_2)\big)$$
$$\le \frac{\tau}{2}\int_0^t \|\partial_t y(s)\|^2\,ds + C_1\left(\|u\|^2_{L^2(Q_t)} + \|f'(y_1) - f'(y_2)\|^2_{L^2(Q_t)}\right). \tag{3.26}$$

Now observe that $|f'(y_1) - f'(y_2)| \le K_3|y|$ a.e. in Q, by (3.11). Hence, if we add the term $\int_{Q_t} y\,\partial_t y$ to both sides of (3.26) and apply Young's inequality appropriately, then we readily infer from Gronwall's lemma the estimate

$$\|A^r\mu\|_{L^2(0,t;H)} + \|y\|_{H^1(0,t;H)\cap L^\infty(0,t;V_B^\sigma)} \le C_2\,\|u\|_{L^2(0,t;H)}\,, \tag{3.27}$$

whence, by virtue of (3.23), also

$$\|A^{2r}\mu\|_{L^2(0,t;H)} \le C_2\,\|u\|_{L^2(0,t;H)}\,. \tag{3.28}$$

It remains to show the estimate

$$\|\mu\|_{L^2(0,t;V_A^{2r})} \le C_3\,\|u\|_{L^2(0,t;H)}\,. \tag{3.29}$$

According to (2.4), this follows directly from (3.28) if $\lambda_1 > 0$, while in the case $\lambda_1 = 0$ we have to estimate the mean value $\mathrm{mean}\,(\mu)$. Now, by **(A2)**, the constant function $\mathbf{1}(x) \equiv 1$ belongs to V_B^σ. Moreover, we have in this case that $A^r\mathbf{1} = 0$, and it follows from (3.23) that $\mathrm{mean}\,(\partial_t y) = 0$, almost everywhere on $(0, T)$. We thus can integrate (3.24) over Ω to see that we have almost everywhere in $(0, T)$ the estimate

$$\Big|\int_\Omega \mu(t)\Big| \le \Big|\tau\int_\Omega \partial_t y(t) + (B^\sigma y(t), B^\sigma\mathbf{1}) - \int_\Omega u(t) + \int_\Omega (f'(y_1(t)) - f'(y_2(t)))\Big|$$
$$\le C_4\left(\|B^\sigma y(t)\| + \|u(t)\| + \|y(t)\|\right),$$

and (3.27) implies that

$$\|\mathrm{mean}\,(\mu)\|_{L^2(0,t)} \le C_5\,\|u\|_{L^2(0,t;H)},$$

whence (3.29) follows.

□

4 Differentiability of the control-to-state mapping

In this section, we prove that the control-to-state mapping $\mathcal{S} : u \mapsto (\mu, y)$ is Fréchet differentiable from the space $\mathcal{X}$ defined in (3.1) into a suitable Banach space $\mathcal{Y}$. To this end, we assume that the general assumptions **(A1)**–**(A5)**, **(A7)**, and **(GB)** are satisfied, and we suppose that a fixed $\bar{u} \in \mathcal{U}_R$ is given and that $(\bar{y}, \bar{\mu}) = \mathcal{S}(\bar{u})$. We then consider for an arbitrary $k \in \mathcal{X}$ the linearized system

$$\partial_t \xi + A^{2r}\eta = 0 \quad \text{in } Q, \tag{4.1}$$

$$\tau\, \partial_t \xi + B^{2\sigma}\xi + f''(\bar{y})\xi = \eta + k \quad \text{in } Q, \tag{4.2}$$

$$\xi(0) = 0 \quad \text{in } \Omega. \tag{4.3}$$

More precisely, we consider the following weak version of the system (4.1)–(4.3):

$$(\partial_t \xi(t), v) + (A^r \eta(t), A^r v) = 0 \quad \text{for a.e. } t \in (0,T) \text{ and all } v \in V_A^r, \tag{4.4}$$

$$(\tau\, \partial_t \xi(t), v) + (B^\sigma \xi(t), B^\sigma v) + (f''(\bar{y}(t))\xi(t), v) = (\eta(t) + k(t), v)$$
$$\text{for a.e. } t \in (0,T) \text{ and all } v \in V_B^\sigma, \tag{4.5}$$

$$\xi(0) = 0. \tag{4.6}$$

If this system admits a unique solution (η, ξ), and if the Fréchet derivative $D\mathcal{S}(\bar{u})$ of $\mathcal{S}$ at $\bar{u}$ exists, then we should have that $D\mathcal{S}(\bar{u})(k) = (\eta, \xi)$. Observe that $\bar{y}$ enjoys the regularity (3.6), and the global bounds (3.8) and (3.11) are satisfied for $y = \bar{y}$. We have the following result.

Theorem 4.1. *Under the given assumptions, the linearized system* (4.4)–(4.6) *admits for every $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$ and every $k \in \mathcal{X}$ a unique solution (η, ξ) such that*

$$\eta \in L^2(0,T;V_A^r), \quad \xi \in H^1(0,T;H) \cap L^\infty(0,T;V_B^\sigma). \tag{4.7}$$

Moreover, there is a constant $K_5 > 0$, which depends only on the data of the state system and $R > 0$, such that

$$\|\eta\|_{L^2(0,T;V_A^r)} + \|\xi\|_{H^1(0,T;H)\cap L^\infty(0,T;V_B^\sigma)} \le K_5\, \|k\|_{L^\infty(Q)}\,. \tag{4.8}$$

Proof. We prove the assertion in a number of separate steps.

Step 1. Discretization. We fix an integer $N > 1$, set $h_N := T/N$ and $t_N^n := n\, h_N$, $n = 0, \ldots, N$, and notice that by virtue of the global bound (3.11) the linear operators

$$P_N^n : H \to H; \quad v \mapsto P_N^n\, v := f''(\bar{y}(\cdot, t_N^n))v, \tag{4.9}$$

are continuous, where with $\widehat{C} := K_3$ it holds

$$\|P_N^n\|_{\mathcal{L}(H,H)} \le \widehat{C} \quad \forall\, N \in \mathbb{N}, \quad 0 \le n \le N.$$

The discrete problem then consists in finding two $(N+1)$-tuples $(\xi_N^0, \ldots, \xi_N^N)$ and $(\eta_N^0, \ldots, \eta_N^N)$ satisfying

$$\xi_N^0 = \eta_N^0 = 0\,, \quad (\xi_N^1, \ldots, \xi_N^N) \in (V_B^{2\sigma})^N, \quad (\eta_N^1, \ldots, \eta_N^N) \in (V_A^{2r})^N, \tag{4.10}$$

and

$$\frac{\xi_N^{n+1} - \xi_N^n}{h_N} + \eta_N^{n+1} + A^{2r}\eta_N^{n+1} = \eta_N^n, \tag{4.11}$$

$$\tau\, \frac{\xi_N^{n+1} - \xi_N^n}{h_N} + \left(\widehat{C} I + B^{2\sigma} + P_N^{n+1}\right)(\xi_N^{n+1}) = \widehat{C}\, \xi_N^n + \eta_N^{n+1} + k_N^{n+1}, \tag{4.12}$$

for $n = 0, 1, \ldots, N-1$, where $I : H \to H$ is the identity and

$$k_N^n := k(nh_N) \quad \text{for } n = 0, 1, \ldots, N. \tag{4.13}$$

In view of (3.1), note that k is continuous from $[0, T]$ to H, so that the above definition is meaningful. The problem (4.10)–(4.12) can be solved inductively for $n = 0, \ldots, N-1$ in the following way: let (η_N^n, ξ_N^n) be given in $V_A^{2r} \times V_B^{2\sigma}$. We first rewrite the above equations in the form

$$h_N \left(I + A^{2r}\right) \eta_N^{n+1} + \xi_N^{n+1} \;=\; \xi_N^n + h_N \eta_N^n, \tag{4.14}$$

$$\Big((\widehat{C} + (\tau/h_N))I + B^{2\sigma} + P_N^{n+1}\Big)\,(\xi_N^{n+1}) \;=\; (\widehat{C} + (\tau/h_N))\xi_N^n + \eta_N^{n+1} + k_N^{n+1}. \tag{4.15}$$

Next, we observe that the operator $\mathcal{A} := \widehat{C}\,I + P_N^{n+1} : H \to H$ is monotone and continuous. On the other hand, the unbounded operator $B^{2\sigma}$ is monotone in H, and $I + B^{2\sigma} : V_B^{2\sigma} \to H$ is surjective, whence it follows that $B^{2\sigma}$ is maximal monotone. Therefore, the sum $\mathcal{A} + B^{2\sigma}$ is also maximal monotone (see, e.g., [6, Cor. 2.1 p. 35]). It follows that $(\tau/h_N)I + \mathcal{A} + B^{2\sigma}$, i.e., the operator that acts on ξ_N^{n+1} in (4.12), is linear, surjective and one-to-one from $V_B^{2\sigma}$ onto H. Therefore, (4.12) can be rewritten in the equivalent form

$$\xi_N^{n+1} = \left(L_N I + B^{2\sigma} + P_N^{n+1}\right)^{-1} \left(L_N \xi_N^n + \eta_N^{n+1} + k_N^{n+1}\right), \tag{4.16}$$

where, for brevity, we have set $L_N := \widehat{C} + (\tau/h_N)$. By accounting for (4.14), we conclude that problem (4.11)–(4.12) is equivalent to the system obtained by coupling (4.16) with the equation

$$h_N \left(I + A^{2r}\right) \eta_N^{n+1} + \left(L_N I + B^{2\sigma} + P_N^{n+1}\right)^{-1} \left(L_N \xi_N^n + \eta_N^{n+1} + k_N^{n+1}\right) \;=\; \xi_N^n + h_N \eta_N^n$$

or

$$\begin{aligned} & h_N \left(I + A^{2r}\right) \eta_N^{n+1} + \left(L_N I + B^{2\sigma} + P_N^{n+1}\right)^{-1} \eta_N^{n+1} \\ & = \xi_N^n + h_N \eta_N^n - \left(L_N I + B^{2\sigma} + P_N^{n+1}\right)^{-1} \left(L_N \xi_N^n + k_N^{n+1}\right). \end{aligned} \tag{4.17}$$

By arguing as before, we see that the operator acting on η_N^{n+1} on the left-hand side of (4.17) is surjective and one-to-one from V_A^{2r} onto H, so that the equation can be uniquely solved for η_N^{n+1} in V_A^{2r}. Inserting the solution in (4.16), we directly find that $\xi_N^{n+1} \in V_B^{2\sigma}$.

Now that the discrete problem is solved, we can start estimating. In the following, the (possibly different) values of the constants termed C_i, $i \in \mathbb{N}$, are independent of the parameters $h_N = T/N$ and $n \in \mathbb{N}$. Also, in order to avoid an overloaded notation, we omit the index N in the expressions ξ_N^n and η_N^n, writing it only at the end of each estimate.

Moreover, we also express the bounds we find in terms of the interpolants. According to the notation introduced in Section 2, and recalling that $\xi_N^0 = \eta_N^0 = 0$, we remark at once that the discrete problem also reads

$$\widehat{\xi}_{h_N} \in W^{1,\infty}(0,T; V_B^\sigma), \quad \underline{\xi}_{h_N}, \overline{\xi}_{h_N} \in L^\infty(0,T; V_B^{2\sigma}), \tag{4.18}$$

$$\underline{\eta}_{h_N}, \overline{\eta}_{h_N} \in L^\infty(0,T; V_A^{2r}), \tag{4.19}$$

$$\partial_t \widehat{\xi}_{h_N} + \overline{\eta}_{h_N} + A^{2r} \overline{\eta}_{h_N} \;=\; \underline{\eta}_{h_N} \quad \text{a.e. in } (0,T), \tag{4.20}$$

$$\tau\,\partial_t \widehat{\xi}_{h_N} + \Big(\widehat{C} I + B^{2\sigma} + P_N\Big)\,(\overline{\xi}_{h_N}) \;=\; \widehat{C}\underline{\xi}_{h_N} + \overline{\eta}_{h_N} + \overline{k}_{h_N} \quad \text{a.e. in } (0,T), \tag{4.21}$$

$$\widehat{\xi}_{h_N}(0) = 0\,, \tag{4.22}$$

where it is understood that

$$\left(P_N\overline{\xi}_{h_N}\right)(\cdot,t) = P_N^{n+1}\xi_N^{n+1} \quad \text{for a.e. } t\in(t_N^n,t_N^{n+1}), \quad 0\le n\le N-1. \tag{4.23}$$

Step 2. First a priori estimate. We test (4.11) and (4.12) (by taking the scalar product in H) by $h_N\eta^{n+1}$ and $\xi^{n+1}-\xi^n$, respectively, and add the resulting identities. Noting an obvious cancellation, we obtain the equation

$$\begin{aligned} &h_N\,(\eta^{n+1}-\eta^n,\eta^{n+1}) + h_N\,(A^{2r}\eta^{n+1},\eta^{n+1}) + \tau\,h_N\,\left\|\tfrac{\xi^{n+1}-\xi^n}{h_N}\right\|^2 \\ &+ (B^{2\sigma}\xi^{n+1},\xi^{n+1}-\xi^n) + \big((\widehat{C}I+P_N^{n+1})(\xi^{n+1}),\xi^{n+1}-\xi^n\big) \\ &= \widehat{C}(\xi^n,\xi^{n+1}-\xi^n) + (k^{n+1},\xi^{n+1}-\xi^n). \end{aligned} \tag{4.24}$$

Now, we observe that

$$\left(\widehat{C}\,\xi^{n+1},\xi^{n+1}-\xi^n\right) = \frac{\widehat{C}}{2}\,\|\xi^{n+1}\|^2 - \frac{\widehat{C}}{2}\,\|\xi^n\|^2 + \frac{\widehat{C}}{2}\,\|\xi^{n+1}-\xi^n\|^2\,. \tag{4.25}$$

Moreover, by Young's inequality it holds that

$$\begin{aligned} &\left(\widehat{C}\,\xi^n + k^{n+1} - P_N^{n+1}\xi^{n+1},\xi^{n+1}-\xi^n\right) \\ &\le \frac{\tau}{2}\,h_N\left\|\tfrac{\xi^{n+1}-\xi^n}{h_N}\right\|^2 + \frac{1}{2\tau}\,h_N\left\|\widehat{C}\,\xi^n + k^{n+1} - P_N^{n+1}\xi^{n+1}\right\|^2 \\ &\le \frac{\tau}{2}\,h_N\left\|\tfrac{\xi^{n+1}-\xi^n}{h_N}\right\|^2 + C_1\,h_N\left(\|\xi^n\|^2 + \|k\|^2_{L^\infty(Q)} + \|\xi^{n+1}\|^2\right), \end{aligned} \tag{4.26}$$

where C_1 depends only on τ and $\widehat{C}$. Combining (4.24)–(4.26), we deduce that

$$\begin{aligned} &\frac{h_N}{2}\,\|\eta^{n+1}\| - \frac{h_N}{2}\,\|\eta^n\|^2 + \frac{h_N}{2}\,\|\eta^{n+1}-\eta^n\|^2 + h_N\,\|A^r\eta^{n+1}\|^2 \\ &+ \frac{\tau}{2}\,h_N\left\|\tfrac{\xi^{n+1}-\xi^n}{h_N}\right\|^2 + \frac12\,\|B^\sigma\xi^{n+1}\|^2 + \frac12\,\|B^\sigma(\xi^{n+1}-\xi^n)\|^2 - \frac12\,\|B^\sigma\xi^n\|^2 \\ &+ \frac{\widehat{C}}{2}\,\|\xi^{n+1}\|^2 - \frac{\widehat{C}}{2}\,\|\xi^n\|^2 + \frac{\widehat{C}}{2}\,\|\xi^{n+1}-\xi^n\|^2 \\ &\le C_1\,h_N\left(\|\xi^n\|^2 + \|k\|^2_{L^\infty(Q)} + \|\xi^{n+1}\|^2\right). \end{aligned} \tag{4.27}$$

Then, we sum up for $n=0,\dots,\ell-1$ with $\ell\le N$, obtaining the inequality

$$\begin{aligned} &\frac{h_N}{2}\,\|\eta^\ell\|^2 + \frac12\sum_{n=0}^{\ell-1} h_N\,\|\eta^{n+1}-\eta^n\|^2 + \sum_{n=0}^{\ell-1} h_N\,\|A^r\eta^{n+1}\|^2 \\ &\quad + \frac{\tau}{2}\sum_{n=0}^{\ell-1} h_N\left\|\tfrac{\xi^{n+1}-\xi^n}{h_N}\right\|^2 + \frac12\,\|B^\sigma\xi^\ell\|^2 + \frac12\sum_{n=0}^{\ell-1}\|B^\sigma(\xi^{n+1}-\xi^n)\|^2 \\ &\quad + \Big(\frac{\widehat{C}}{2} - C_1\,h_N\Big)\,\|\xi^\ell\|^2 + \frac{\widehat{C}}{2}\sum_{n=0}^{\ell-1}\|\xi^{n+1}-\xi^n\|^2 \\ &\le C_1\,\ell\,h_N\,\|k\|^2_{L^\infty(Q)} + 2\,C_1\sum_{n=0}^{\ell-1} h_N\,\|\xi^n\|^2\,. \end{aligned} \tag{4.28}$$

At this point, we fix any $N_0 \in \mathbb{N}$ such that $N_0 \geq 4\, C_1\, T/\widehat{C}$. With this choice, we have for any integer $N \geq N_0$ that $\frac{\widehat{C}}{2} - C_1\, h_N \geq \frac{\widehat{C}}{4}$. Since also $\ell\, h_N \leq T$, we conclude from the discrete Gronwall lemma that for any such $N \in \mathbb{N}$ it holds the bound

$$\begin{aligned} & h_N\, \|\eta^\ell\|^2 + \sum_{n=0}^{\ell-1} h_N\, \|\eta^{n+1} - \eta^n\|^2 + \sum_{n=0}^{\ell-1} h_N\, \|A^r \eta^{n+1}\|^2 + \sum_{n=0}^{\ell-1} h_N\, \left\| \tfrac{\xi^{n+1}-\xi^n}{h_N} \right\|^2 \\ & \quad + \|B^\sigma \xi^\ell\|^2 + \sum_{n=0}^{\ell-1} \|B^\sigma (\xi^{n+1} - \xi^n)\|^2 + \|\xi^\ell\|^2 + \sum_{n=0}^{\ell-1} \|\xi^{n+1} - \xi^n\|^2 \\ & \leq C_2\, \|k\|^2_{L^\infty(Q)} \leq C_3\,. \end{aligned} \tag{4.29}$$

Since this holds for $\ell = 0, \ldots, N$, we obtain in terms of the interpolants, by neglecting the first contribution and recalling that $\mu^0 = 0$ and the definition (2.7) of the norm in V_B^σ, that

$$\begin{aligned} & \|\overline{\eta}_{h_N} - \underline{\eta}_{h_N}\|_{L^2(0,T;H)} + \|A^r \overline{\eta}_{h_N}\|_{L^2(0,T;H)} + \|A^r \underline{\eta}_{h_N}\|_{L^2(0,T;H)} + \|\partial_t \widehat{\xi}_{h_N}\|_{L^2(0,T;H)} \\ & \quad + \|\underline{\xi}_{h_N}\|_{L^\infty(0,T;V_B^\sigma)} + \|\overline{\xi}_{h_N}\|_{L^\infty(0,T;V_B^\sigma)} + h_N^{-1/2} \|\overline{\xi}_{h_N} - \underline{\xi}_{h_N}\|_{L^\infty(0,T;V_B^\sigma)} \\ & \leq C_4\, \|k\|_{L^\infty(Q)} \leq C_5\,. \end{aligned} \tag{4.30}$$

Step 3. Second a priori estimate. Let $N \geq N_0$. We want to improve the estimate for $A^r \overline{\eta}_{h_N}$ given by (4.30) and show that

$$\|\overline{\eta}_{h_N}\|_{L^2(0,T;V_A^r)} + \|\underline{\eta}_{h_N}\|_{L^2(0,T;V_A^r)} \leq C_6 \|k\|_{L^\infty(Q)} \leq C_7\,. \tag{4.31}$$

By recalling (2.4), we see that there is nothing to prove if $\lambda_1 > 0$. Assume now that $0 = \lambda_1 < \lambda_2$. We then have to estimate the mean value of $\overline{\eta}_{h_N}$. To this end, we recall that e_1 is a constant and belongs to V_B^σ. Thus, the function $\mathbf{1}(x) \equiv 1$ also belongs to V_B^σ. Integrating the equation (4.21) over Ω, we therefore obtain almost everywhere on $(0,T)$ the identity

$$\int_\Omega \overline{\eta}_{h_N} = \int_\Omega \Big(-\overline{k}_{h_N} + \widehat{C} \left(\overline{\xi}_{h_N} - \underline{\xi}_{h_N} \right) + P_N(\overline{\xi}_{h_N}) + \tau\, \partial_t \widehat{\xi}_{h_N} \Big) + \big(B^\sigma \overline{\xi}_{h_N}, B^\sigma \mathbf{1} \big)\,. \tag{4.32}$$

Applying the Cauchy–Schwarz inequality to the expressions on the right-hand side, we readily conclude from (4.30) the bound

$$\begin{aligned} \big\| \mathrm{mean}(\overline{\eta}_{h_N}) \big\|^2_{L^2(0,T)} & \leq C_8 \Big(\|k\|^2_{L^\infty(Q)} + \|\overline{\xi}_{h_N}\|^2_{L^2(Q)} + \|\underline{\xi}_{h_N}\|^2_{L^2(Q)} + \big\| B^\sigma \overline{\xi}_{h_N} \big\|^2_{L^2(Q)} \\ & \qquad + \big\| \partial_t \widehat{\xi}_{h_N} \big\|^2_{L^2(Q)} \Big) \\ & \leq C_9\, \|k\|^2_{L^\infty(Q)} \leq C_{10}\,, \end{aligned} \tag{4.33}$$

and the claim (4.31) is proved as far as $\overline{\eta}_{h_N}$ is concerned. But as $\|\overline{\eta}_{h_N} - \underline{\eta}_{h_N}\|_{L^2(Q)}$ is by (4.30) bounded, and since $A^r \eta_N^0 = A^r 0 = 0$, it also holds true for $\underline{\eta}_{h_N}$.

Step 4. Existence. Combining the estimates (4.30) and (4.31), recalling (2.36), and using standard weak and weak-star compactness results, we see that there are functions ξ and η such that, at least for suitable subsequences which are again indexed by N,

$$\overline{\xi}_{h_N} \to \xi\,, \quad \underline{\xi}_{h_N} \to \xi\,, \quad \widehat{\xi}_{h_N} \to \xi, \quad \text{all weakly star in } L^\infty(0,T;V_B^\sigma), \tag{4.34}$$

$$\partial_t \widehat{\xi}_{h_N} \to \partial_t \xi \quad \text{weakly in } L^2(0,T;H), \tag{4.35}$$

$$\overline{\eta}_{h_N} \to \eta \quad \text{weakly in } L^2(0,T;V_A^r), \tag{4.36}$$

as $N \to \infty$. Moreover, owing to the compact embedding $V_B^\sigma \subset H$ (see (2.15)) and to well-known strong compactness results (see, e.g., [40, Sect. 8, Cor. 4]), we obtain from (4.34)–(4.35) that

$$\widehat{\xi}_{h_N} \to \xi \quad \text{strongly in } C([0,T];H), \tag{4.37}$$

whence it follows that $\xi(0) = 0$ and, using (2.36),

$$\overline{\xi}_{h_N} \to \xi, \quad \underline{\xi}_{h_N} \to \xi, \quad \text{both strongly in } L^2(0,T;H). \tag{4.38}$$

Next, we prove that

$$\underline{\eta}_{h_N} \to \eta \quad \text{weakly in } L^2(0,T;V_A^r). \tag{4.39}$$

By (4.30) and (4.36), it suffices to check that

$${}_{L^2(0,T;V_A^{-r})}\langle v, \overline{\eta}_{h_N} - \underline{\eta}_{h_N} \rangle_{L^2(0,T;V_A^r)} \to 0 \quad \text{as } N \to \infty, \tag{4.40}$$

for every v belonging to a dense subspace $\mathcal{V}$ of $L^2(0,T;V_A^{-r})$, where we can take $\mathcal{V} = C_c^1(0,T;H)$ since H is dense in V_A^{-r} (see (2.14)). So, we fix $v \in C_c^1(0,T;H)$ and choose $\delta > 0$ such that $v(t) = 0$ for $t \in [0,T] \setminus (\delta, T-\delta)$. If $h_N \in (0, \delta/2)$, then we have

$$\begin{aligned}
&\big| {}_{L^2(0,T;V_A^{-r})}\langle v, \overline{\eta}_{h_N} - \underline{\eta}_{h_N} \rangle_{L^2(0,T;V_A^r)} \big| = \Big| \int_{h_N}^T (\overline{\eta}_{h_N} - \underline{\eta}_{h_N})(t)\, v(t)\, dt \Big| \\
&= \Big| \int_{h_N}^T \big(\overline{\eta}_{h_N}(t) - \overline{\eta}_{h_N}(t-h_N) \big)\, v(t)\, dt \Big| \\
&= \Big| \int_{h_N}^T \overline{\eta}_{h_N}(t)\, v(t)\, dt - \int_0^{T-h_N} \overline{\eta}_{h_N}(t)\, v(t+h_N)\, dt \Big| \\
&= \Big| \int_{h_N}^{T-h_N} \overline{\eta}_{h_N}(t)\, (v(t) - v(t+h_N))\, dt \Big| \le T^{1/2}\, \|\overline{\eta}_{h_N}\|_{L^2(0,T;H)}\, \|v'\|_{L^\infty(0,T;H)}\, h_N\,,
\end{aligned}$$

and (4.40) follows.

We now show that

$$P_N(\overline{\xi}_{h_N}) \to f''(\bar{y})\xi \quad \text{strongly in } L^1(Q) \text{ as } N \to \infty. \tag{4.41}$$

Indeed, employing the global bounds (3.11), we have, for almost every $(x,t) \in \Omega \times (t_N^{n-1}, t_N^n)$, where $1 \le n \le N$,

$$\begin{aligned}
&\big| P_N(\overline{\xi}_{h_N})(x,t) - f''(\bar{y}(x,t))\xi(x,t) \big| = |f''(\bar{y}(x,t_N^n))\xi_N^n(x) - f''(\bar{y}(x,t))\xi(x,t)| \\
&\le |f''(\bar{y}(x,t_N^n)) - f''(\bar{y}(x,t))|\, |\xi(x,t)| + |f''(\bar{y}(x,t_N^n))|\, |\xi_N^n(x) - \xi(x,t)| \\
&\le \widehat{C}\, \big|\overline{\xi}_{h_N}(x,t) - \xi(x,t)\big| + \widehat{C}\, |\xi(x,t)|\, |\bar{y}(x,t_N^n) - \bar{y}(x,t)| \\
&\le \widehat{C}\, \big|\overline{\xi}_{h_N}(x,t) - \xi(x,t)\big| + \widehat{C}\, |\xi(x,t)| \int_{t_N^{n-1}}^{t_N^n} |\partial_t \bar{y}(x,s)|\, ds \\
&\le \widehat{C}\, \big|\overline{\xi}_{h_N}(x,t) - \xi(x,t)\big| + \widehat{C}\, h_N^{1/2}\, |\xi(x,t)| \Big(\int_{t_N^{n-1}}^{t_N^n} |\partial_t \bar{y}(x,s)|^2\, ds \Big)^{1/2}.
\end{aligned} \tag{4.42}$$

The claim (4.41) then follows from (4.38) and a simple calculation on the last term by recalling that ξ and $\partial_t \bar{y}$ belong to $L^2(Q)$.

Therefore, we can pass to the limit as $N \to \infty$ in the weak time-integrated versions of (4.20) and (4.21) (written with bounded time-dependent test functions) to conclude that the pair (η, ξ) solves the variational equations (4.4) and (4.5). Since also $\xi(0) = 0$, the existence part of the assertion is shown. Moreover, the continuity estimate (4.8) is a direct consequence of (4.30), (4.31) and the semicontinuity of norms.

Step 5. Uniqueness. To show uniqueness, suppose that the system (4.4)–(4.6) has two solutions (η_i, ξ_i), $i = 1, 2$, with the regularity (4.7). Then the pair (η, ξ) with $\eta = \eta_1 - \eta_2$, $\xi = \xi_1 - \xi_2$, solves the system (4.4)–(4.6), where in this case $k \equiv 0$. We then test (4.4) by η and (4.5) by $\partial_t \xi$ and add the resulting equations to arrive at the identity

$$\int_0^t \|A^r \eta(s)\|^2 \, ds \, + \, \frac{\tau}{2} \, \|B^\sigma \xi(t)\|^2 \, + \, \tau \int_0^t \!\! \int_\Omega |\partial_t \xi|^2 \; = \; - \int_0^t \!\! \int_\Omega f''(\bar{y}) \, \xi \, \partial_t \xi \, , \tag{4.43}$$

which is valid for every $t \in [0, T]$. Now we add the term $\int_0^t \!\int_\Omega \xi \, \partial_t \xi$ to both sides of (4.43) and apply Young's inequality appropriately to the resulting right-hand side. It then follows from Gronwall's lemma that $A^r \eta = \xi = 0$. But then, by virtue of (4.5), also $\eta = 0$. This concludes the proof of Theorem 4.1. □

After these preparations, the road is paved for proving the Fréchet differentiability of the control-to-state operator $\mathcal{S}$. We need, however, yet another assumption.

(A8) V_B^σ is continuously embedded in $L^4(\Omega)$.

Observe that this condition is fulfilled if, e.g., $B = -\Delta$ with zero Dirichlet or Neumann boundary conditions and $\sigma \geq 3/8$. Indeed, by virtue of (3.18), we have in this case $V_B^\sigma \subset H^{2\sigma}(\Omega) \subset L^4(\Omega)$ if $-\frac{3}{4} \leq 2\sigma - \frac{3}{2}$, i.e., if $\sigma \geq 3/8$.

Recalling the statement of Theorem 4.1, we show the following result.

Theorem 4.2. *Suppose that the assumptions* **(A1)–(A5)**, **(A7)**, **(A8)**, *and* **(GB)** *are fulfilled. Then the control-to-state operator* $\mathcal{S} : u \mapsto \mathcal{S}(u) = (\mu, y)$ *is Fréchet differentiable in* $\mathcal{U}_R$ *when viewed as a mapping between the spaces* $\mathcal{X} = H^1(0, T; H) \cap L^\infty(Q)$ *and* $\mathcal{Y} := L^2(0, T; V_A^r) \times (H^1(0, T; H) \cap L^\infty(0, T; V_B^\sigma))$. *Moreover, whenever* $\bar{u} \in \mathcal{U}_R$ *with* $(\bar{\mu}, \bar{y}) = \mathcal{S}(\bar{u})$ *is given, then the Fréchet derivative* $D\mathcal{S}(\bar{u}) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *of* $\mathcal{S}$ *at* $\bar{u}$ *is specifed by the identity* $D\mathcal{S}(\bar{u})(k) = (\eta, \xi)$, *where* (η, ξ) *is the unique solution to the weak formulation* (4.4)–(4.6) *of the linearized system.*

Proof. Since $\mathcal{U}_R$ is open, there is some $\Lambda > 0$ such that $\bar{u} + k \in \mathcal{U}_R$ whenever $k \in \mathcal{X}$ and $\|k\|_{\mathcal{X}} \leq \Lambda$. In the following, we consider only such perturbations k, for which we define the quantities

$$(\mu^k, y^k) := \mathcal{S}(\bar{u} + k), \quad \rho^k := \mu^k - \bar{\mu} - \eta^k, \quad z^k := y^k - \bar{y} - \xi^k,$$

where $(\eta^k, \xi^k) = (\eta, \xi)$ denotes the unique solution to the system (4.4)–(4.6). Obviously, we have $\rho^k \in L^2(0, T; V_A^r)$ and $z^k \in H^1(0, T; H) \cap L^\infty(0, T; V_B^\sigma)$. Moreover, it turns out that

$$(\partial_t z^k(t), v) + \big(A^r \rho^k(t), A^r v\big) = 0 \quad \text{for a.e. } t \in (0, T) \text{ and all } v \in V_A^r, \tag{4.44}$$

$$\tau \, (\partial_t z^k(t), v) + \big(B^\sigma z^k(t), B^\sigma v\big) \, + \, (f'(y^k(t)) - f'(\bar{y}(t)) - f''(\bar{y}(t)) \xi^k(t), v)$$
$$= \; (\rho^k(t), v) \quad \text{for a.e. } t \in (0, T) \text{ and all } v \in V_B^\sigma, \tag{4.45}$$

$$z^k(0) = 0. \tag{4.46}$$

In addition, by Taylor's theorem and (3.11), we have almost everywhere in Q that

$$\left|f'(y^k) - f'(\bar{y}) - f''(\bar{y})\xi^k\right| \,\le\, C_1 \left(|z^k| \,+\, |y^k - \bar{y}|^2\right), \tag{4.47}$$

where, here and in the remainder of the proof, the constants $C_i > 0,\ i \in \mathbb{N}$, depend only on the data of the problem and R, but not on the special choice of $k \in \mathcal{X}$ with $\|k\|_{\mathcal{X}} \le \Lambda$. Using (3.22) in Theorem 3.6 and the continuity of the embedding $V_B^\sigma \subset L^4(\Omega)$, we infer that, for any $t \in (0,T]$,

$$\|y^k - \bar{y}\|_{L^\infty(0,t;L^4(\Omega))} \,\le\, C_2\, \|k\|_{L^2(0,t;H)}\,. \tag{4.48}$$

Now recall that by (4.8) the mapping $k \mapsto (\eta^k, \xi^k)$ is continuous from $\mathcal{X}$ into $\mathcal{Y}$. According to the notion of Fréchet differentiability, it therefore suffices to construct an increasing function $Z : (0,\Lambda) \to (0,+\infty)$ such that $\lim_{\lambda\searrow 0} \frac{Z(\lambda)}{\lambda^2} = 0$ and

$$\|\rho^k\|^2_{L^2(0,T;V_A^r)} \,+\, \|z^k\|^2_{H^1(0,T;H)\cap L^\infty(0,T;V_B^\sigma)} \,\le\, Z\left(\|k\|_{L^2(0,T;H)}\right)\,. \tag{4.49}$$

At this point, we test (4.44) by $\rho^k(t)$, (4.45) by $\partial_t z^k(t)$, add the resulting equations, and integrate over Q_t, where $t \in (0,T]$. In addition, we add the term $\int_0^t\!\int_\Omega z^k\, \partial_t z^k$ to both sides of the result. Invoking (4.47), we then obtain the inequality

$$\begin{aligned} &\frac{1}{2}\left(\|z^k(t)\|^2 \,+\, \|B^\sigma z^k(t)\|^2\right) \,+ \tau \int_0^t\!\!\int_\Omega |\partial_t z^k|^2 \,+ \int_0^t \|A^r \rho^k(s)\|^2\, ds \\ &\le\, C_3 \int_0^t\!\!\int_\Omega |z^k|\,|\partial_t z^k| \,+\, C_4 \int_0^t\!\!\int_\Omega |\partial_t z^k|\,\left|y^k - \bar{y}\right|^2 \,=:\, I_1 + I_2, \end{aligned} \tag{4.50}$$

with obvious notation. Now, by Young's inequality,

$$I_1 \,\le\, \frac{\tau}{4} \int_0^t\!\!\int_\Omega |\partial_t z^k|^2 \,+\, C_5 \int_0^t\!\!\int_\Omega |z^k|^2\,,$$

while, by also using Hölder's inequality and (4.48),

$$I_2 \,\le\, C_4 \int_0^t \|\partial_t z^k(s)\|\, \|y^k(s) - \bar{y}(s)\|^2_{L^4(\Omega)}\, ds \,\le\, \frac{\tau}{4} \int_0^t\!\!\int_\Omega |\partial_t z^k|^2 \,+\, C_6\, \|k\|^4_{L^2(0,T;H)}\,.$$

Employing Gronwall's lemma, we thus conclude from (4.50) the estimate

$$\|z^k\|^2_{H^1(0,T;H)\cap L^\infty(0,T;V_B^\sigma)} \,+\, \|A^r \rho^k\|^2_{L^2(0,T;H)} \,\le\, C_7\, \|k\|^4_{L^2(0,T;H)}\,. \tag{4.51}$$

At this point, we have to distinguish between two cases. Assume first that $\lambda_1 > 0$. In this case, we have $\|\rho^k\|_{L^2(0,T;V_A^r)} \,\le\, C_8\, \|A^r \rho^k\|_{L^2(0,T;H)}$, and thus (4.49) follows from (4.51) with $Z(\lambda) = (1 + C_8) C_7\, \lambda^4$.

Assume now that $\lambda_1 = 0$. In this case, we need to estimate the mean value of ρ^k. To this end, we observe that **(A1)** implies that for $\lambda_1 = 0$ we have $\mathbf{1} \in V_A^r \cap V_B^\sigma$ and $A^r \mathbf{1} = 0$. From this it immediately follows that $\operatorname{mean}(\partial_t z^k(t)) = 0$ for almost every $t \in (0,T)$. Thus, inserting $v = \mathbf{1} \in V_B^\sigma$ in (4.45) and applying the Cauchy–Schwarz inequality and the inequality (4.47), we find that for a.a. $t \in (0,T)$ it holds that

$$\begin{aligned} \Big|\int_\Omega \rho^k(t)\Big| \,&\le\, \int_\Omega |B^\sigma z^k(t)|\,|B^\sigma \mathbf{1}| \,+\, C_1 \int_\Omega \left(|z^k(t)| \,+\, |y^k(t) - \bar{y}(t)|^2\right) \\ &\le\, C_9 \left(\|B^\sigma z^k(t)\| \,+\, \|z^k(t)\| \,+\, \|y^k(t) - \bar{y}(t)\|^2_{L^4(\Omega)}\right), \end{aligned}$$

and it follows the estimate

$$\| \operatorname{mean}(\rho^k)\|_{L^2(0,T)} \le C_{10} \left(\|z^k\|_{L^2(0,T;H)} + \|B^\sigma z^k\|_{L^2(0,T;H)} + \|y^k - \bar{y}\|^2_{L^\infty(0,T;L^4(\Omega))} \right) .$$

In view of (4.51) and (4.48), and by recalling (2.4) and Remark 2.2, this yields that

$$\|\rho^k\|^2_{L^2(0,T;V_A^r)} \le C_{11} \left(\|A^r \rho^k\|^2_{L^2(0,T;H)} + \|\operatorname{mean}(\rho^k)\|^2_{L^2(0,T)} \right) \le C_{12} \|k\|^4_{L^2(0,T;H)} .$$

In conclusion, the condition (4.49) holds true with the choice $Z(\lambda) = (C_7 + C_{12}) \lambda^4$. With this, the assertion is completely proved. □

Using the above differentiability result and the fact that $\mathcal{U}_{\mathrm{ad}}$ is a closed and convex subset of $\mathcal{X}$, we can infer from the chain rule via a standard argument (which can be omitted here) the following first-order necessary optimality condition:

Corollary 4.3. *Let the assumptions of Theorem 4.2 be satisfied, and assume that $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$ with $(\bar{\mu}, \bar{y}) = \mathcal{S}(\bar{u})$ is a solution to the optimal control problem* **(CP)**. *Then it holds the variational inequality*

$$\alpha_1 \int_\Omega (\bar{y}(T) - y_\Omega)\, \xi(T) \,+\, \alpha_2 \int_0^T\!\!\int_\Omega (\bar{y} - y_Q)\, \xi \,+\, \alpha_3 \int_0^T\!\!\int_\Omega (\bar{\mu} - \mu_Q)\, \eta \\ +\, \alpha_4 \int_0^T\!\!\int_\Omega \bar{u}\,(v - \bar{u}) \;\ge\; 0 \quad \forall\, v \in \mathcal{U}_{\mathrm{ad}}, \tag{4.52}$$

where (η, ξ) is the unique solution to the system (4.4)–(4.6) *associated with $k = v - \bar{u}$.*

5 Existence and first-order optimality conditions

In this section, we state and prove the main results of this paper. We begin with an existence result.

Theorem 5.1. *Suppose that the conditions* **(A1)**–**(A8)** *and* **(GB)** *are fulfilled. Then the optimal control problem* **(CP)** *has a solution.*

Proof. We use the direct method. To this end, let $\{u_n\} \subset \mathcal{U}_{\mathrm{ad}}$ be a minimizing sequence, and let $(\mu_n, y_n) = \mathcal{S}(u_n)$, for $n \in \mathbb{N}$. Then the global bounds (3.8) and (3.11) apply, and there are some $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$, a pair $(\bar{\mu}, \bar{y})$, and some $z \in L^\infty(Q)$, such that, at least for a subsequence which is again indexed by $n \in \mathbb{N}$,

$$u_n \to \bar{u} \quad \text{weakly star in } \mathcal{X}, \tag{5.1}$$

$$\mu_n \to \bar{\mu} \quad \text{weakly star in } L^\infty(0,T;V_A^{2r}), \tag{5.2}$$

$$y_n \to \bar{y} \quad \text{weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V_B^\sigma)\,, \tag{5.3}$$

$$f_1'(y_n) \to z \quad \text{weakly star in } L^\infty(Q). \tag{5.4}$$

We also observe that standard compactness results (see, e.g. [40, Sect. 8, Cor. 4]) imply that we may without loss of generality assume that

$$y_n \to y \quad \text{strongly in } C^0([0,T];H) \text{ and pointwise a.e. in } Q, \tag{5.5}$$

which yields that $\bar{y}(0) = y_0$, in particular. In addition, by **(A4)**, f_2' is Lipschitz continuous on $\mathbb{R}$, which implies that $f_2'(y_n) \to f_2'(\bar{y})$ strongly in $C^0([0,T];H)$; moreover, by the convexity of f_1 it turns out that f_1' induces a maximal monotone graph. It then follows from standard results on maximal monotone operators (see, e.g., [6, Prop. 2.2, p. 38]) that $z = f_1'(\bar{y})$. In summary, we have that $f'(y_n) \to f'(\bar{y})$ weakly star in $L^\infty(0,T;H)$.

Now, we consider the (equivalent) integrated version of (3.2)–(3.4), written for $u = u_n$, $y = y_n$, $\mu = \mu_n$, $n \in \mathbb{N}$, and with time-dependent test functions, and we pass to the limit as $n \to \infty$. We then obtain the analogous formulation for $u = \bar{u}$, $\mu = \bar{\mu}$, $y = \bar{y}$, that is, we have $(\bar{\mu}, \bar{y}) = \mathcal{S}(\bar{u})$. But this means that the pair $((\bar{\mu}, \bar{y}), \bar{u})$ is admissible for the minimization problem **(CP)**. By the semicontinuity properties of the cost functional, it is a minimizer. □

Next, we aim to establish meaningful first-order necessary optimality conditions by eliminating the quantities η and ξ from (4.52) by means of the adjoint state variables. To this end, we consider the adjoint state system which formally reads

$$A^{2r} p - q = \alpha_3(\bar{\mu} - \mu_Q) \quad \text{in } Q, \tag{5.6}$$

$$-(\partial_t p + \tau\, \partial_t q) + B^{2\sigma} q + f''(\bar{y})\, q = \alpha_2(\bar{y} - y_Q) \quad \text{in } Q, \tag{5.7}$$

$$p(T) + \tau\, q(T) = \alpha_1(\bar{y}(T) - y_\Omega) \quad \text{in } \Omega. \tag{5.8}$$

However, we can manage such a system only if the right-hand side of (5.6) satisfies restrictive assumptions which are not fulfilled, in general, because of the presence of the component $\bar{\mu}$. Therefore, we assume $\alpha_3 = 0$ in the sequel. Moreover, we consider a variational formulation of the above formal problem. We recall the definition (2.11) of $V_B^{-\sigma}$ and the embedding $H \subset V_B^{-\sigma}$ (see (2.12)); let us use the simpler notation $\langle\,\cdot\,,\,\cdot\,\rangle$ without indices for the duality pairing between $V_B^{-\sigma}$ and V_B^σ. For the adjoint state (p,q), we require the following regularity conditions:

$$p \in L^2(0,T;V_A^{2r}), \tag{5.9}$$

$$q \in L^2(0,T;V_B^\sigma), \tag{5.10}$$

$$p + \tau q \in H^1(0,T;V_B^{-\sigma}). \tag{5.11}$$

The adjoint problem we consider then reads as follows:

$$(A^r p(t), A^r v) - (q(t), v) = 0 \quad \text{for a.a. } t \in (0,T) \text{ and every } v \in V_A^r, \tag{5.12}$$

$$-\langle \partial_t(p+\tau q)(t), v\rangle + (B^\sigma q(t), B^\sigma v) + (\psi(t) q(t), v) = (g_2(t), v) \quad \text{for a.a. } t \in (0,T) \text{ and every } v \in V_B^\sigma, \tag{5.13}$$

$$(p + \tau q)(T) = g_1, \tag{5.14}$$

where, for brevity, we have set

$$\psi := f''(\bar{y}), \quad g_1 := \alpha_1(\bar{y}(T) - y_\Omega) \quad \text{and} \quad g_2 := \alpha_2(\bar{y} - y_Q). \tag{5.15}$$

We have written for convenience the weak form (5.12), which still makes sense under the weaker regularity requirement $p \in L^2(0,T;V_A^r)$. However, it is immediately seen that such a regularity and (5.12) imply (5.9) and

$$q = A^{2r} p, \tag{5.16}$$

i.e., the equation (5.6) with $\alpha_3 = 0$.

Solving the problem (5.12)–(5.14) requires some preliminary work. It is understood that the assumptions **(A1)–(A8)** and **(GB)** are in force. In particular, we have that $\psi \in L^\infty(Q)$, $g_1 \in L^2(\Omega)$, and $g_2 \in L^2(Q)$. First of all, we give an equivalent formulation.

Proposition 5.2. *The regularity conditions* (5.9)–(5.11) *and problem* (5.12)–(5.14) *are equivalent to* (5.9)–(5.10)*,* (5.12)*, and*

$$\int_0^T \big((p+\tau q)(t), \partial_t v(t)\big)\,dt = -\int_0^T \big(B^\sigma q(t), B^\sigma v(t)\big)\,dt + \int_0^T \big(g_2(t)-\psi(t)q(t), v(t)\big)\,dt + \big(g_1, v(T)\big)$$
$$\textit{for every } v \in H^1(0,T;H)\cap L^2(0,T;V_B^\sigma) \textit{ satisfying } v(0)=0. \tag{5.17}$$

Proof. Before starting, we observe that, for $p \in L^2(0,T;H)$ and $q \in L^2(0,T;V_B^\sigma)$ with $p+\tau q \in H^1(0,T;V_B^{-\sigma})$, the variational equation (5.13) is equivalent to the following integrated version:

$$-\int_0^T \langle \partial_t(p+\tau q)(t), v(t)\rangle\,dt = -\int_0^T \big(B^\sigma q(t), B^\sigma v(t)\big)\,dt + \int_0^T \big(g_2(t)-\psi(t)q(t), v(t)\big)\,dt$$
$$\text{for every } v \in L^2(0,T;V_B^\sigma). \tag{5.18}$$

We also recall an integration-by-parts formula (see, e.g., [18, Lemma 4.5]): if $(\mathcal{V},\mathcal{H},\mathcal{V}^*)$ is a Hilbert triplet and

$$w \in H^1(0,T;\mathcal{H})\cap L^2(0,T;\mathcal{V}) \quad \text{and} \quad z \in H^1(0,T;\mathcal{V}^*)\cap L^2(0,T;\mathcal{H}),$$

then the function $t \mapsto (w(t),z(t))_{\mathcal{H}}$ is absolutely continuous, and for every $t,t' \in [0,T]$ we have that

$$\int_{t'}^t \big\{ \big(\partial_t w(s), z(s)\big)_{\mathcal{H}} + {}_{\mathcal{V}^*}\langle \partial_t z(s), w(s)\rangle_{\mathcal{V}} \big\}\,ds = \big(w(t),z(t)\big)_{\mathcal{H}} - \big(w(t'),z(t')\big)_{\mathcal{H}}. \tag{5.19}$$

Now, we prove the statement. We first assume that (5.9)–(5.11) and (5.12)–(5.14) are valid. Then, we just have to prove that (5.17) holds true. We start from (5.18), with any $v \in H^1(0,T;H)\cap L^2(0,T;V_B^\sigma)$. By applying (5.19), we immediately obtain (5.17) on account of (5.14).

Conversely, assume that (p,q) satisfies (5.9)–(5.10), (5.12), and (5.17). We prove the (apparently) stronger regularity requirement (5.11) and the validity of the formulas (5.13) and (5.14). To this end, we observe that, because of the meaning of the Hilbert triplet $(V_B^\sigma, H, V_B^{-\sigma})$, the conditions (5.11) and (5.13) (or (5.18)) are equivalent to the following properties: $i)$ formula (5.17) holds for every $v \in C_c^\infty(0,T;V_B^\sigma)$ (C^∞ functions with compact support in $(0,T)$); $ii)$ the maps that associates to every $v \in C_c^\infty(0,T;V_B^\sigma)$ the right-hand side of (5.17) (i.e., the same as in (5.18) since $v(T)=0$) is continuous with respect to the topology of $L^2(0,T;V_B^\sigma)$. The former follows from our assumption and it is straightforward to see that the latter is satisfied since $q \in L^2(0,T;V_B^\sigma)$. Hence, both (5.11) and (5.18) are established (the latter first for every $v \in C_c^\infty(0,T;V_B^\sigma)$ by definition, then for every $v \in L^2(0,T;V_B^\sigma)$ by continuity). At this point, we can take (5.17), with $v \in H^1(0,T;H)\cap L^2(0,T;V_B^\sigma)$ satisfying $v(0)=0$, and integrate by parts using (5.19). By comparing with (5.18), we deduce that $\big((p+\tau q)(T)-g_1, v(T)\big)=0$ for every $v \in H^1(0,T;H)\cap L^2(0,T;V_B^\sigma)$ satisfying $v(0)=0$. By choosing $v(t)=tv_0$ with any $v_0 \in V_B^\sigma$ we conclude that (5.14) holds true as well since V_B^σ is dense in H. □

Thus, we are going to solve the new problem given by the previous proposition. The case $\lambda_1 > 0$ is easier, since the operator $A^{2r} \in \mathcal{L}(V_A^{2r}, H)$ has the inverse $A^{-2r} := (A^{2r})^{-1} \in \mathcal{L}(H, V_A^{2r})$, so that we can use (5.16) in order to eliminate p. Hence, we immediately obtain the following lemma:

Lemma 5.3. *Assume $\lambda_1 > 0$. Then, a pair (p, q) satisfying* (5.9)–(5.10) *solves* (5.12) *and* (5.17) *if and only if $p = A^{-2r}q$ with q satisfying*

$$q \in L^2(0,T;V_B^\sigma), \tag{5.20}$$

$$\int_0^T \bigl(A^{-2r}q(t) + \tau q(t), \partial_t v(t)\bigr)\,dt$$
$$= -\int_0^T \bigl(B^\sigma q(t), B^\sigma v(t)\bigr)\,dt + \int_0^T \bigl(g_2(t) - \psi(t)q(t), v(t)\bigr)\,dt + \bigl(g_1, v(T)\bigr)$$
$$\text{for every } v \in H^1(0,T;H) \cap L^2(0,T;V_B^\sigma) \text{ satisfying } v(0) = 0. \tag{5.21}$$

On the contrary, the situation is much more complicated in the case when $\lambda_1 = 0$. To handle this case, we adapt the ideas of [15, Sect. 5]. To this end, we have to introduce some new spaces. We set

$$H_0 := \{v \in H : \ \mathrm{mean}(v) = 0\} \quad \text{and} \quad V_{B,0}^\sigma := V_B^\sigma \cap H_0, \tag{5.22}$$

and notice that $H^1(0,T;H) \cap L^2(0,T;V_{B,0}^\sigma) = H^1(0,T;H_0) \cap L^2(0,T;V_{B,0}^\sigma)$. Moreover, we observe that the operators A_0^{2r} and A_0^{-2r} (see (2.20)) also satisfy that

$$A_0^{2r} : V_0^{2r} \to H_0 \quad \text{and} \quad A_0^{-2r} = (A_0^{2r})^{-1} : H_0 \to V_0^{2r} \quad \text{are isomorphisms.} \tag{5.23}$$

Finally, for simplicity, in the next statement and in its proof, we often use the same notation φ for some real function $\varphi \in H^1(0,T)$ and the function $\varphi\mathbf{1} \in H^1(0,T;H)$.

Lemma 5.4. *Assume that $\lambda_1 = 0$. Then a pair (p, q) satisfying* (5.9)–(5.10) *solves* (5.12) *and* (5.17) *if and only if*

$$p = p_\Omega + A_0^{-2r}q, \tag{5.24}$$

with q and p_Ω given as follows:

$$q \in L^2(0,T;V_{B,0}^\sigma), \tag{5.25}$$

$$\int_0^T \bigl(A_0^{-2r}q(t) + \tau q(t), \partial_t v(t)\bigr)\,dt$$
$$= -\int_0^T \bigl(B^\sigma q(t), B^\sigma v(t)\bigr)\,dt + \int_0^T \bigl(g_2(t) - \psi(t)q(t), v(t)\bigr)\,dt + \bigl(g_1 - \mathrm{mean}(g_1)\mathbf{1}, v(T)\bigr)$$
$$\text{for every } v \in H^1(0,T;H) \cap L^2(0,T;V_{B,0}^\sigma) \text{ satisfying } v(0) = 0, \tag{5.26}$$

$$p_\Omega(t) = \mathrm{mean}(g_1) + \frac{1}{|\Omega|}\int_t^T \Bigl\{\bigl(g_2(s), \mathbf{1}\bigr) - \bigl(B^\sigma q(s), B^\sigma \mathbf{1}\bigr) - \bigl(\psi(s)q(s), \mathbf{1}\bigr)\Bigr\}\,ds$$
$$\text{for every } t \in [0,T]. \tag{5.27}$$

Proof. Assume that (p, q) satisfies (5.9)–(5.10) and solves (5.12) and (5.17). By Proposition 5.2 we can also use the previous formulation (5.12)–(5.14) of the adjoint problem. Testing (5.12) by $v = \mathbf{1} \in V_B^\sigma$ yields

$$\bigl(q(t), \mathbf{1}\bigr) = \bigl(A^r q(t), A^r \mathbf{1}\bigr) = 0 \quad \text{for a.a. } t \in (0,T),$$

since $\lambda_1 = 0$. Thus, q has zero mean value, and (5.25) is a consequence of (5.10). Moreover, in view of (5.11) it turns out that the function

$$t \mapsto \mathrm{mean}((p + \tau q)(t)), \quad t \in [0,T],$$

belongs to $H^1(0,T)$, and in particular it has a continuous representative (termed exactly as it is). We set

$$p_\Omega(t) := \mathrm{mean}((p+\tau q)(t)), \quad \text{for every } t \in [0,T], \tag{5.28}$$

and it turns out that

$$p_\Omega(t) = \mathrm{mean}(p(t)) \quad \text{for a.a. } t \in (0,T).$$

Therefore, by choosing $v = \mathbf{1}$ in (5.13) and using (5.14), we also deduce that

$$- |\Omega|\, p'_\Omega(t) + \big(B^\sigma q(t), B^\sigma \mathbf{1}\big) + \big(\psi(t) q(t), \mathbf{1}\big) = \big(g_2(t), \mathbf{1}\big) \quad \text{for a.a. } t \in (0,T),$$
$$p_\Omega(T) = \mathrm{mean}(g_1).$$

Hence, (5.27) immediately follows. Furthermore, since $A^{2r}\mathbf{1} = 0$, we can write (5.16) in the form

$$q = A^{2r}(p - p_\Omega) = A_0^{2r}(p - p_\Omega),$$

and, owing to the zero mean value property of q, once more we conclude that

$$p - p_\Omega = A_0^{-2r} q,$$

that is, (5.24) holds true. Using this, we compute both sides of (5.17) with zero-mean-value test functions, i.e., $v \in H^1(0,T;H) \cap L^2(0,T;V^\sigma_{B,0})$, such that $v(0) = 0$. Since $p_\Omega(t)$ is space independent, $\mathrm{mean}(g_1)$ is a constant, and $\partial_t v(t)$ and $v(T)$ have zero mean value, we have

$$\int_0^T \big((p+\tau q)(t), \partial_t v(t)\big)\, dt = \int_0^T \big((A_0^{-2r} q + \tau q)(t), \partial_t v(t)\big)\, dt + \int_0^T \big(p_\Omega(t), \partial_t v(t)\big)\, dt$$
$$= \int_0^T \big((A_0^{-2r} q + \tau q)(t), \partial_t v(t)\big)\, dt\,,$$

as well as

$$- \int_0^T \big(B^\sigma q(t), B^\sigma v(t)\big)\, dt + \int_0^T \big(g_2(t) - \psi(t) q(t), v(t)\big)\, dt + \big(g_1, v(T)\big)$$
$$= - \int_0^T \big(B^\sigma q(t), B^\sigma v(t)\big)\, dt + \int_0^T \big(g_2(t) - \psi(t) q(t), v(t)\big)\, dt + \big(g_1 - \mathrm{mean}(g_1)\mathbf{1}, v(T)\big).$$

Hence, (5.17) with such test functions becomes (5.26).

Conversely, assume that p fulfils (5.24) with q satisfying (5.25)–(5.26) and with p_Ω given by (5.27). First of all, observe that (5.28) (which is not required a priori) still holds as a consequence of (5.24), since $A_0^{-2r} q$ has zero mean value. Moreover, (5.10) is trivially implied by (5.25). We now prove the validity of (5.17). To this end, take any $v \in H^1(0,T;H) \cap L^2(0,T;V^\sigma_B)$ with $v(0) = 0$ and split v as follows:

$$v = (v - \varphi\mathbf{1}) + \varphi\mathbf{1} \quad \text{where} \quad \varphi := \mathrm{mean}(v)\,.$$

Then, $v - \varphi\mathbf{1} \in H^1(0,T;H) \cap L^2(0,T;V^\sigma_{B,0})$, and $(v - \varphi\mathbf{1})(0) = 0$. Hence, (5.26) yields

$$\int_0^T \big(A_0^{-2r} q(t) + \tau q(t), \partial_t v(t) - \varphi'(t)\,\mathbf{1}\big)\, dt$$
$$= - \int_0^T \big(B^\sigma q(t), B^\sigma v(t)\big)\, dt + \int_0^T \big(g_2(t) - \psi(t) q(t), v(t)\big)\, dt$$
$$+ \big(g_1 - \mathrm{mean}(g_1)\mathbf{1}, v(T)\big)$$
$$+ \int_0^T \big(B^\sigma q(t), B^\sigma \mathbf{1}\big)\varphi(t)\, dt - \int_0^T \big(g_2(t) - \psi(t) q(t), \mathbf{1}\big)\varphi(t)\, dt$$
$$- \big(g_1 - \mathrm{mean}(g_1)\mathbf{1}, \mathbf{1}\big)\varphi(T),$$

and we note that the last term vanishes. Now, we observe that $\varphi \in H^1(0,T)$ and that $\varphi(0) = 0$ (since $v(0) = 0$). Thus, we multiply (5.27) by $|\Omega|\varphi'(t)$, integrate over $(0,T)$ with respect to t, and perform an integration by parts on the right-hand side. We obtain that

$$\int_0^T \big(p_\Omega(t), \mathbf{1}\big)\, \varphi'(t)\, dt$$
$$= \big(\mathrm{mean}(g_1)\mathbf{1}, \mathbf{1}\big)\varphi(T) - \int_0^T \big(B^\sigma q(t), B^\sigma \mathbf{1}\big)\varphi(t) + \int_0^T \big(g_2(t) - \psi(t)q(t), \mathbf{1}\big)\varphi(t)\, dt\,.$$

By summing up, we deduce that

$$\int_0^T \big(A_0^{-2r} q(t) + \tau q(t), \partial_t v(t) - \varphi'(t)\, \mathbf{1}\big)\, dt + \int_0^T \big(p_\Omega(t), \mathbf{1}\big)\, \varphi'(t)\, dt$$
$$= - \int_0^T \big(B^\sigma q(t), B^\sigma v(t)\big)\, dt + \int_0^T \big(g_2(t) - \psi(t)q(t), v(t)\big)\, dt + \big(g_1, v(T)\big).$$

Notice that the right-hand sides of this identity and of (5.17) coincide. Thus, it suffices to show that the same happens for the left-hand sides. By also accounting for (5.24), and noting that the mean values of both $\partial_t v - \varphi'\mathbf{1}$ and $p - p_\Omega + \tau q$ vanish, we have

$$\int_0^T \big(A_0^{-2r} q(t) + \tau q(t), \partial_t v(t) - \varphi'(t)\, \mathbf{1}\big)\, dt + \int_0^T \big(p_\Omega(t), \mathbf{1}\big)\, \varphi'(t)\, dt$$
$$= \int_0^T \big(p(t) + \tau q(t) - p_\Omega(t), \partial_t v(t) - \varphi'(t)\, \mathbf{1}\big)\, dt + \int_0^T \big(p_\Omega(t), \mathbf{1}\big)\, \varphi'(t)\, dt$$
$$= \int_0^T \big(p(t) + \tau q(t), \partial_t v(t)\big)\, dt - \int_0^T \big(p(t) + \tau q(t), \mathbf{1}\big)\varphi'(t)\, dt + \int_0^T \big(p_\Omega(t), \mathbf{1}\big)\, \varphi'(t)\, dt$$
$$= \int_0^T \big(p(t) + \tau q(t), \partial_t v(t)\big)\, dt - \int_0^T \big(p(t) - p_\Omega(t) + \tau q(t), \mathbf{1}\big)\varphi'(t)\, dt$$
$$= \int_0^T \big(p(t) + \tau q(t), \partial_t v(t)\big)\, dt\,.$$

This completes the proof. □

Lemma 5.5. *The space $V_{B,0}^\sigma$ is dense in H_0. In particular, the Hilbert triplet*

$$(V_{B,0}^\sigma, H_0, V_{B,0}^{-\sigma}), \quad \textit{where} \quad V_{B,0}^{-\sigma} := (V_{B,0}^\sigma)^*,$$

is meaningful.

Proof. We assume that $z \in H_0$ satisfies $(z, v) = 0$ for every $v \in V_{B,0}^\sigma$ and deduce that $z = 0$. Take any $v \in V_B^\sigma$. Then $v - \mathrm{mean}(v)\mathbf{1} \in V_{B,0}^\sigma$, whence $(z, v - \mathrm{mean}(v)\mathbf{1}) = 0$. On the other hand, $(z, \mathrm{mean}(v)\mathbf{1}) = 0$ since $\mathrm{mean}(z) = 0$. Therefore, $(z, v) = 0$. Since this holds for every $v \in V_B^\sigma$ and V_B^σ is dense in H, we conclude that $z = 0$. □

Lemma 5.6. *Let $(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$ be a Hilbert triplet and let $(\,\cdot\,,\,\cdot\,)$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ be the inner product of $\mathcal{H}$ and the duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$, respectively. Moreover, let $\mathcal{A}$ and $\mathcal{B}$ satisfy, with suitable* *positive constants M, λ, and α, the following conditions:*

$$\mathcal{A} \in \mathcal{L}(\mathcal{H};\mathcal{H}) \quad \textit{is symmetric}\,; \tag{5.29}$$

$$\mathcal{B}(t) \in \mathcal{L}(\mathcal{V};\mathcal{V}^*) \quad \textit{for a.a. } t \in (0,T); \tag{5.30}$$

$$\textit{for every } v, w \in \mathcal{V}, \textit{ the function } t \mapsto \langle \mathcal{A}(t)v, w\rangle \textit{ is measurable on } (0,T); \tag{5.31}$$

$$\langle \mathcal{A}v, v\rangle \ge \alpha \, \|v\|_{\mathcal{H}}^2 \quad \textit{for every } v \in \mathcal{H}\,; \tag{5.32}$$

$$\|\mathcal{B}(t)v\|_{\mathcal{V}^*} \le M\, \|v\|_{\mathcal{V}} \quad \textit{for a.a. } t \in (0,T) \textit{ and every } v \in \mathcal{V}\,; \tag{5.33}$$

$$\langle \mathcal{B}(t)v, v\rangle + \lambda \, \|v\|_{\mathcal{H}}^2 \ge \alpha \, \|v\|_{\mathcal{V}}^2 \quad \textit{for a.a. } t \in (0,T) \textit{ and every } v \in \mathcal{V}. \tag{5.34}$$

Then, for every $F \in L^2(0,T;\mathcal{V}^*)$ and every $\gamma \in \mathcal{H}$, there exists a unique $q \in L^2(0,T;\mathcal{V})$ satisfying

$$\int_0^T \big(\mathcal{A}q(t), v'(t)\big)\, dt + \int_0^T \langle \mathcal{B}(t)q(t), v(t)\rangle \, dt$$
$$= \int_0^T \langle F(t), v(t)\rangle + \big(\gamma, v(T)\big)$$
$$\textit{for every } v \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}) \textit{ such that } v(0) = 0. \tag{5.35}$$

Proof. The similar forward problem (presented in a slightly different way, see also [35, Lem. 1.1, p. 44] for a similar equivalence) is solved in [35, Thm. 7.1, p. 70] under even more general assumptions on the structure (in particular, there $\mathcal{A}$ is also allowed to depend on time) and equivalent assumptions on the data. □

Remark 5.7. By arguing as we did for Proposition 5.2, one can easily see that a function $q \in L^2(0,T;\mathcal{V})$ solves (5.35) if and only if it satisfies

$$q \in L^2(0,T;\mathcal{V}), \quad \mathcal{A}q \in H^1(0,T;\mathcal{V}^*), \quad -(\mathcal{A}q)' + \mathcal{B}u = F, \quad \text{and} \quad (\mathcal{A}q)(T) = \gamma,$$

where the abstract equation holds a.e. in $(0,T)$ in the sense of $\mathcal{V}^*$ and the final condition is meaningful since $\mathcal{A}q \in C^0([0,T];\mathcal{V}^*)$.

At this point, we are ready to state a well-posedness result for the adjoint problem in the case $\alpha_3 = 0$, i.e., for the system (5.12)–(5.14). Namely, we have the following theorem.

Theorem 5.8. *Suppose that the conditions* **(A1)**–**(A8)** *and* **(GB)** *are fulfilled. Moreover, assume that $\bar u \in \mathcal{U}_{\rm ad}$, and let $(\bar\mu, \bar y) = \mathcal{S}(\bar u)$ be the corresponding state. Then the adjoint problem* (5.12)–(5.14) *has a unique solution (p,q) satisfying* (5.9)–(5.11).

Proof. Thanks to Proposition 5.2 and Lemmas 5.3 and 5.4, it is sufficient to establish well-posedness for the sub-problems that involve just q, i.e., (5.20)–(5.21) and (5.25)–(5.26) in the cases $\lambda_1 > 0$ and $\lambda_1 = 0$, respectively. However, we can unify these problems by seeing both of them as particular cases of a new one. To this end, we set

$$\mathcal{H} := H, \quad \mathcal{V} := V_B^\sigma, \quad \mathcal{A} := A^{-2r} + \tau I, \quad \text{and} \quad \gamma := g_1, \qquad \text{if } \lambda_1 > 0,$$
$$\mathcal{H} := H_0, \quad \mathcal{V} := V_{B,0}^\sigma, \quad \mathcal{A} := A_0^{-2r} + \tau I, \quad \text{and} \quad \gamma := g_1 - \mathrm{mean}(g_1)\mathbf{1}, \qquad \text{if } \lambda_1 = 0,$$

where I is the identity map of $\mathcal{H}$, and we define $\mathcal{B}(t) \in \mathcal{L}(\mathcal{V};\mathcal{V}^*)$ by

$$\langle \mathcal{B}(t)v, w\rangle := \big(B^\sigma v, B^\sigma w\big) + \big(\psi(t)v, w\big) \quad \text{for a.a. } t \in (0,T) \text{ and every } v, w \in \mathcal{V},$$

in both cases (with different meanings of the notations, e.g., $\mathcal{V}$). Then, each of the problems we have to solve appears in the form (5.35). It is immediately seen that the assumptions of Lemma 5.6 are fulfilled. In particular, (5.33) and (5.34) hold since ψ is bounded. Hence, the lemma provides a unique solution. □

We conclude with the first-order necessary condition for optimality expressed in terms of the adjoint state variables.

Theorem 5.9. *Let the assumptions of Theorem 4.2 be satisfied, and assume that* $\bar{u} \in \mathcal{U}_{\rm ad}$ *is a solution to the optimal control problem* **(CP)** *with* $\alpha_3 = 0$*. Moreover, let* $(\bar{\mu}, \bar{y}) = \mathcal{S}(\bar{u})$ *be the corresponding state, and let* (p, q) *be the unique solution to the related adjoint problem. Then the following variational inequality holds true:*

$$\int_0^T \int_\Omega (q + \alpha_4 \bar{u})(v - \bar{u}) \, \geq \, 0 \quad \forall \, v \in \mathcal{U}_{\rm ad}. \tag{5.36}$$

In particular, if $\alpha_4 \neq 0$*, the optimal control* $\bar{u}$ *is the* $L^2(0,T;H)$*-projection of* $-q/\alpha_4$ *on* $\mathcal{U}_{\rm ad}$.

Proof. Fix any $v \in \mathcal{U}_{\rm ad}$, set $k := v - \bar{u}$, and consider the solutions (η, ξ) and (p, q) to the corresponding linearized system (4.4)–(4.6) and the adjoint system (5.12)–(5.14), respectively. We test (4.4) and (4.5) by $p(t)$ and $q(t)$, respectively. Then, we add the resulting equalities to each other and integrate over $(0, T)$. By recalling the notations (5.15), we obtain that

$$\begin{aligned} &\int_0^T \bigl\{ \bigl(\partial_t \xi(t), p(t)\bigr) + \bigl(A^r \eta(t), A^r p(t)\bigr) \bigr\} \, dt \\ &\quad + \int_0^T \bigl\{ \bigl(\tau \, \partial_t \xi(t), q(t)\bigr) + \bigl(B^\sigma \xi(t), B^\sigma q(t)\bigr) + \bigl(\psi(t)\xi(t), q(t)\bigr) \bigr\} \, dt \\ &= \int_0^T \bigl(\eta(t) + k(t), q(t)\bigr) \, dt \,. \end{aligned}$$

At the same time, by testing (5.12) and (5.13) by $-\eta(t)$ and $-\xi(t)$, summing up and integrating with respect to t, we have that

$$\begin{aligned} &\int_0^T \bigl\{ -\bigl(A^r p(t), A^r \eta(t)\bigr) + \bigl(q(t), \eta(t)\bigr) \bigr\} \, dt \\ &\quad + \int_0^T \bigl\{ \langle \partial_t (p + \tau q)(t), \xi(t) \rangle - \bigl(B^\sigma q(t), B^\sigma \xi(t)\bigr) - \bigl(\psi(t) q(t), \xi(t)\bigr) \bigr\} \, dt \\ &= - \int_0^T \bigl(g_2(t), \xi(t)\bigr) \, dt. \end{aligned}$$

At this point, we add these equations and notice that several cancellations occur. We are left with the following identity:

$$\begin{aligned} &\int_0^T \bigl\{ \bigl(\partial_t \xi(t), (p + \tau q)(t)\bigr) + \langle \partial_t (p + \tau q)(t), \xi(t) \rangle \bigr\} \, dt \\ &= \int_0^T \bigl(k(t), q(t)\bigr) \, dt - \int_0^T \bigl(g_2(t), \xi(t)\bigr) \, dt \,. \end{aligned} \tag{5.37}$$

By applying the integration-by-parts formula (5.19) to the left-hand side, invoking the Cauchy conditions (4.6) and (5.14), and rearranging terms, we deduce that

$$\bigl(g_1, \xi(T)\bigr) + \int_0^T \bigl(g_2(t), \xi(t)\bigr) \, dt = \int_0^T \bigl(q(t), k(t)\bigr) \, dt \,. \tag{5.38}$$

On the other hand, since $\alpha_3 = 0$, the inequality (4.52) given by Corollary 4.3 reads

$$\bigl(g_1, \xi(T)\bigr) + \int_0^T \bigl(g_2(t), \xi(t)\bigr) \, dt + \alpha_4 \int_0^T \bigl(\bar{u}(t), k(t)\bigr) \, dt \geq 0 \,.$$

By replacing the sum of the first two integrals by the right-hand side of (5.38), we obtain (5.36) and the proof is complete. Indeed, the last sentence is just a consequence of the Hilbert projection theorem, since $\mathcal{U}_{\rm ad}$ is a convex and closed subset of $L^2(0,T;H)$. □

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