Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Generalized Sasa–Satsuma equation: Densities approach to new infinite hierarchy of integrable evolution equations

Adrian Ankiewicz[1], Uwe Bandelow[2], Nail Akhmediev[1]

submitted: May 24, 2018

[1] Optical Sciences Group
Research School of Physics and Engineering
The Australian National University
Canberra ACT 2600
Australia
E-Mail: adrian.ankiewicz@anu.edu.au
nna124@rsphysse.anu.edu.au

[2] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: uwe.bandelow@wias-berlin.de

No. 2510

Berlin 2018

2010 *Mathematics Subject Classification.* 37K10, 37K40, 35Q55.

2010 *Physics and Astronomy Classification Scheme.* 02.30.Ik 05.45.Yv 42.65.Tg 47.35.Fg.

Key words and phrases. Sasa-Satsuma equation, integrable equations, solitons .

The authors gratefully acknowledge the support of the Australian Research Council (Discovery Projects DP140100265 and DP150102057) and support from the Volkswagen Stiftung. N. A. is a recipient of the Alexander von Humboldt Award. U.B. acknowledges support by the German Research Foundation in the framework of the Collaborative Research Center 787 'Semiconductor Nanophotonics' under project B5.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Generalized Sasa–Satsuma equation: Densities approach to new infinite hierarchy of integrable evolution equations

Adrian Ankiewicz, Uwe Bandelow, Nail Akhmediev

Abstract

We derive the new infinite Sasa-Satsuma hierarchy of evolution equations using an invariant densities approach. Being significantly simpler than the Lax-pair technique, this approach does not involve ponderous 3×3 matrices. Moreover, it allows us to explicitly obtain operators of many orders involved in the time evolution of the Sasa-Satsuma hierarchy functionals. All these operators are parts of a generalized Sasa-Satsuma equation of infinitely high order. They enter this equation with independent arbitrary real coefficients that govern the evolution pattern of this multi-parameter dynamical system.

1 Introduction

An integrable hierarchy is an infinite sequence of partial differential equations that starts with a particular case. The starting equation could be the Korteweg de-Vries equation (KdV) [1, 2], nonlinear Schrödinger equation (NLSE) [3], Toda lattice [4], certain classes of Painlevé equations [5], etc. Each successive equation in the hierarchy normally takes a more complex form than the previous one. An integrable hierarchy can be considered as a system of commuting Hamiltonian flows [6]. An infinite number of commuting flows can be obtained recursively. Most of the works on integrable hierarchies are written by mathematicians and may not be easily accessible. Consequently, purely mathematical results may not be easily used in practical applications. Even if recursive rules for deriving higher-order equations of a particular hierarchy are provided, obtaining explicit forms of these equations is a different matter. A countable number of these equations has been presented so far. Moreover, some of these hierarchies remain unknown due to obscurities in the definition of 'integrability'. The Painlevé criterion can be useful in finding these special cases [7]. However, using it does not exclude the possibility that another integrable case, not covered by the technique, remains hidden. In mathematical terms, the Painlevé criterion is a necessary but not sufficient condition for finding integrable equations.

One important observation is that members of the hierarchy can be written in the form of one general equation which combines all individual equations into a single one [8, 9]. This general equation can have an infinite number of operators controlling the time evolution of a system [8, 9]. It includes all equations of the hierarchy as particular cases with arbitrary real coefficients which govern the contribution of each operator to the whole. The convenience of such representation lies in the arbitrariness of these coefficients. When all of them are zero except one, we obtain an individual equation of the hierarchy. Having two or more coefficients being nonzero provides more complicated equations that can be of interest due to the special case in physics that such an equation can describe. One example is the Heisenberg spin chain dynamics [7].

Such general equations could be of great importance for physics because higher-order terms in this equation may describe finer effects such as higher-order dispersion or higher-order nonlinearities in wave propagation phenomena. They become important only beyond the basic approximation that is usually described by the lowest-order equation. The brightest example of such approach is soliton science, which started with the KdV [10] and NLS [11] equations. Clearly, the basic properties of solitons have to be known before we can move to such elaborations as pulse compression [12], self-frequency shift [13], wave breaking [14], etc. Particular cases of such step-by-step improvements in wave description are well-known in optics. Starting from simple theory [15], the soliton approach has been advanced with the contributions of higher-order terms in later works [16, 17, 18]. Similar 'upgrades' of the soliton approach [19] have been provided in water wave theory [20, 21, 22].

DOI 10.20347/WIAS.PREPRINT.2510

Unfortunately, not all higher-order terms in these upgrades result in integrable equations. A specific set of coefficients is required for these very special cases. It is indeed fortunate when such an 'upgrade' belongs to a general equation of an integrable hierarchy. The chances are low if there is only one hierarchy that starts with the given base equation. Finding new hierarchies is thus an important task which may significantly improve the accuracy of modelling of physical phenomena. Luckily, there are at least two 'general equations' that have the NLSE as a base. One of these is a Hirota hierarchy [8, 9], while the other one, found recently [23], is a Sasa-Satsuma hierarchy. Both start with the NLSE as the base evolution equation. Thus, both of them could be called NLSE hierarchies. In order to avoid confusion and distinguish them explicitly, we label them here as the generalized Hirota and generalized Sasa-Satsuma equations (SSE).

The first few equations of the Hirota hierarchy are the NLSE [11], the third-order Hirota equation [25], fourth-order Lakshmanan-Porsezian-Daniel (LPD) equation [26] and the quintic extension of this sequence [27, 28]. Higher-order extensions (to sixth-order, etc.) have been presented in explicit forms in [8, 9]. On the other hand, the two starting equations of the Sasa-Satsuma hierarchy are the NLSE and the third-order Sasa-Satsuma equation [29, 30, 31]. Higher-order extensions (fourth-order etc.) have been discovered in [23].

In the present work, we make further progress in dealing with the general Sasa-Satsuma equation. Namely, we derive many equations in the hierarchy using invariant densities. This approach is significantly simpler than the original Lax pair technique developed in [23]. It does not require operations with cumbersome 3×3-matrices. Moreover, recurrent relations in this technique are straightforward and require only the knowledge of the infinite set of invariant densities H_j. These can be found by representing any given evolution equation in the form of continuity equations:

$$\frac{\partial H_j}{\partial t} + \frac{\partial J_j}{\partial x} = 0 \tag{1}$$

of various orders j where J_j are the corresponding invariant flows of the order j. These continuity equations can be constructed in the same way as for the NLSE [24]. This process requires nothing more than purely algebraic transformations. For the SSE in the form

$$iu_t + \frac{u_{xx}}{2} + |u|^2 u = i\epsilon\left[u_{xxx} + 3(|u|^2)_x u + 6|u|^2 u_x\right],$$

for the first order, we have:

$$H_1 = |u|^2$$

and

$$J_1 = \frac{i}{2}(uu_x^* - u^* u_x) - \epsilon\left[(|u|^2)_{xx} - 4|u_x|^2 + 6|u|^4\right].$$

When the free real parameter $\epsilon = 0$, these densities are the same as for the NLSE [24]. The whole infinite set of higher order invariant densities and flows can be found easily. An alternative way of obtaining the invariant densities is using the generating function technique [32]. Below, we consider the invariant densities to be known. As we can see from Section 2, with this approach, the whole set of calculations occupies only one page. Integrability of the higher-order equations is guaranteed, as it is based on the invariants of the SSE.

Our present technique confirms that all terms in the SSE hierarchy found in [23] are correct and it provides an independent and simple way of working with these highly nontrivial extensions of the NLSE and SSE. Again, we stress that the general equation related to this hierarchy contains an infinite number of real parameters. The practical benefit of such approach is that this general equation contains higher-order terms with adjustable coefficients, and these could describe wave propagation phenomena with higher accuracy.

2 Derivation of generalised Sasa-Satsuma equation from invariant densities

As noted above, the results of our first paper [23] can be derived using an even simpler technique that uses invariant densities of the SSE. This can be done in a way that is related to that used in [9] for the Hirota (previously called NLSE) hierarchy. The difference lies in using the set of invariant densities of the SSE instead of invariant densities of the NLSE. Let us write the hierarchy, containing an infinite set of equations, in the form:

$$i\,u_t + \sum_{n=1}^{\infty} (\alpha_{2n} S_{2n} - i\,\alpha_{2n+1} S_{2n+1}) = 0, \tag{2}$$

where S_j are the functionals of the order j for the envelope function $u(x,t)$, and the α_j are arbitrary real coefficients. We stress that the coefficients α_j are not small parameters. They are finite real numbers, thus making our approach far from being just another perturbation analysis. In Eq.(2), we explicitly separated even and odd terms for the reason which will be clear in the following.

The SSE functional can be written in the form:

$$S_3 = \frac{6}{b^2} u_x |u|^2 + \frac{3}{b^2} u(|u|^2)_x + u_{xxx}. \tag{3}$$

Other forms [29, 30, 31] follow from Eq.(3) after simple transformations. The inclusion of the arbitrary real constant b in (3) follows from free scaling on the variable x. In the normalization of [23], $b = 1$, but here we find it more appropriate to use $b = \sqrt{2}$.

The first density of the integral invariant of the SSE is

$$H_1 = |u|^2. \tag{4}$$

The transverse integral (i.e. over x) of this expression represents the conserved mass during evolution.

We define the variational (Frechet) derivative as

$$\mathcal{F} \equiv \frac{\partial}{\partial u^*} - \frac{\partial}{\partial x}\frac{\partial}{\partial u^*_x} + \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial u^*_{xx}} - \cdots \tag{5}$$

The second invariant density (integrand) of the SSE, divided by b, is

$$H_2 = -\frac{1}{2}\left[u_x u^* + u u^*_x\right]. \tag{6}$$

The transverse integral of H_2 represents the conservation of momentum. The Frechet derivative of H_2 is zero.

The third invariant density of the SSE divided by b^2 is

$$H_3 = \frac{u^* u_{xx}}{2} + \frac{u {u^*}_{xx}}{2} + \frac{2}{b^2}|u|^4. \tag{7}$$

Its integral with respect to x is the Hamiltonian, indicating conservation of energy.

Taking the variational derivative produces the NLS term:

$$\mathcal{F}(H_3) = S_2 = \frac{4}{b^2} u|u|^2 + u_{xx}. \tag{8}$$

We now take $b = \sqrt{2}$ to get the usual form of the NLSE, $iu_t + \alpha_2 S_2 = 0$, i.e.

$$iu_t + \alpha_2(2u|u|^2 + u_{xx}) = 0. \tag{9}$$

Note that this normalization matches the form used in [9], $K_2 = 2u|u|^2 + u_{xx}$.

The fourth invariant of the SSE, divided by b^3, is

$$H_4 \;=\; -\frac{1}{2}\left[\frac{10}{b^2} u_x u (u^*)^2 + u_{xxx} u^* + \frac{10}{b^2} u^2 u^* (u^*)_x + u (u^*)_{xxx}\right]. \tag{10}$$

The Frechet derivative of H_4 is also zero.

The invariant density H_5 of the SSE, divided by b^4 is:

$$\begin{aligned} H_5 &= \frac{8|u|^6}{b^4} + \frac{4u^2(u_x^*)^2}{b^2} + \frac{4(u^*)^2u_x^2}{b^2} + \frac{14|u|^2|u_x|^2}{b^2} + \frac{7u^*|u|^2u_{xx}}{b^2} \\ &+ \frac{7u|u|^2u_{xx}^*}{b^2} + \frac{u^*u_{xxxx}}{2} + \frac{uu^*{}_{xxxx}}{2} \end{aligned} \tag{11}$$

Setting $b^2 = 2$, we get:

$$\begin{aligned} H_5 &= 2u^2(u_x^*)^2 + 2\,(u^*)^2\,u_x^2 + 7|u|^2|u_x|^2 + \frac{7}{2}|u|^2u^*u_{xx} + \frac{u^*u_{xxxx}}{2} \\ &+ \frac{uu^*{}_{xxxx}}{2} + 2|u|^6 + \frac{7}{2}|u|^2uu_{xx}^*. \end{aligned} \tag{12}$$

We obtain the higher order terms by using the operator $\mathcal{F}$ from Eq.(5). Thus

$$\mathcal{F}(H_5) = S_4.$$

The 4-th order functional is thus:

$$S_4 = \frac{24}{b^4}|u|^4u + \frac{6}{b^2}u^2u_{xx}^* + \frac{12}{b^2}u|u_x|^2 + \frac{14}{b^2}|u|^2u_{xx} + \frac{8}{b^2}u^*u_x^2 + u_{xxxx}. \tag{13}$$

Then, the functional (13) resembles the one in the LPD equation [9] with the form of the terms, but has different coefficients in front of each of them.

The sixth invariant is

$$\begin{aligned} H_6 &= -\frac{44\,(u^*)^2\,u^3u_x^*}{b^4} - \frac{14\,(u^*)^2\,u_xu_{\mathsf{xx}}}{b^2} - \frac{11u^*u_x^2u_x^*}{b^2} - \frac{u^*u_{5x}}{2} \\ &- u^2\left[\frac{44\,(u^*)^3\,u_x}{b^4} + \frac{14u_x^*u_{xx}^*}{b^2} + \frac{9u^*u^*{}_{xxx}}{b^2}\right] \\ &- u\left[\frac{24u^*u_xu_{xx}^*}{b^2} + \frac{24u^*\,(u_x)^*u_{xx}}{b^2} + \frac{9\,(u^*)^2\,u_{xxx}}{b^2} + \frac{11u_x\,(u_x^*)^2}{b^2} + \frac{u_{5x}^*}{2}\right]. \end{aligned} \tag{14}$$

The Frechet derivative of H_6 is also zero.

The 7-th invariant density in the set is

$$\begin{aligned} H_7 &= \frac{1}{2}\Bigg\{u\left[\frac{u^*}{b^2}\left(73u_xu_{xxx}^* + 98u_{xx}u^*{}_{xx} + 73u_x^*u_{xxx}\right) + \frac{97}{b^2}|u_x|^2u_{xx}^* + \frac{194}{b^4}(u^*)^3u_x^2\right. \\ &+ \left.\frac{22}{b^2}(u^*)^2u_{xxxx} + u_{6x}^* + \frac{45}{b^2}(u_x^*)^2u_{xx}\right] + u^*\left\{\frac{u_x}{b^2}\left[43u^*u_{xxx} + 97u_x^*u_{xx}\right]\right. \\ &+ \left.\frac{45}{b^2}u_x^2u_{xx}^* + \frac{29}{b^2}(u^*)^2u_{xx}^2 + u_{6x}\right\} + \frac{80}{b^6}|u|^8 + 2u^3\frac{u^*}{b^4}\left[76u^*u_{xx}^* + 97(u_x^*)^2\right] \\ &+ u^2\left[\frac{43}{b^2}u_x^*u_{xxx}^* + \frac{572}{b^4}(u^*)^2|u_x|^2 + \frac{152}{b^4}(u^*)^3u_{xx} + \frac{29}{b^2}u^2(u_{xx}^*)^2 + \frac{22}{b^2}u^*u_{xxxx}^*\right]\Bigg\}. \end{aligned}$$

We again obtain the 6-th order functional by taking the variational derivative of H_7:

$$\begin{aligned} S_6 &= \frac{55}{b^4}u^3(u_x^*)^2 + \frac{45}{b^2}u_x^2u_{xx}^* + \frac{32}{b^2}uu_xu_{xxx}^* + \frac{43}{b^2}u^*u_xu_{xxx} + \frac{175}{b^4}(u^*)^2uu_x^2 + \frac{53}{b^2}uu_{xx}u_{xx}^* \\ &+ \frac{31}{b^2}u^*u_{xx}^2 + \frac{20}{b^2}|u|^2u_{xxxx} + \frac{160}{b^6}|u|^6u + \frac{110}{b^4}u^*u^3u_{xx}^* + \frac{330}{b^4}u^*u^2|u_x|^2 + \frac{170}{b^4}|u|^4u_{xx} \\ &+ \frac{8}{b^2}u^2u_{xxxx}^* + \frac{95}{b^2}|u_x|^2u_{xx} + \frac{37}{b^2}u(u_x)^*u_{xxx} + u_{6x}. \end{aligned} \tag{15}$$

Clearly, this process can be continued indefinitely generating the infinite hierarchy of operators of the general Sasa-Satsuma equation (2).

2.1 Operators S_j with normalization $b^2 = 2$

In conclusion, we can give explicitly the general equation (2). We use $b^2 = 2$ throughout. The lowest order functional $S_2[u(x,t)]$ in Eq.(2) is given by

$$S_2 = u_{xx} + 2u|u|^2, \tag{16}$$

while Eq.(3) gives:

$$S_3 = 3u_x|u|^2 + \frac{3}{2}u(|u|^2)_x + u_{xxx}. \tag{17}$$

Using Eq.(13), we obtain:

$$S_4 \ = \ 6|u|^4u + 3u^2u^*_{xx} + 6u|u_x|^2 + 7|u|^2u_{xx} + 4u^*u_x^2 + u_{xxxx}. \tag{18}$$

Re-normalizing the form given in [23]), we obtain:

$$\begin{aligned} S_5 \ &= \ u_{5x} + 20|u|^4u_x + \frac{5}{2}u^2u^*_{xxx} + \frac{25}{2}u(u_xu^*_{xx} + u^*_xu_{xx}) + 10|u|^2u^2u^*_x \\ &+ \ 10|u_x|^2u_x + \frac{15}{2}|u|^2u_{xxx} + 15u^*u_xu_{xx}. \end{aligned} \tag{19}$$

Using Eq.(15), we obtain:

$$\begin{aligned} S_6 \ &= \ \frac{55}{4}u^3(u^*_x)^2 + \frac{45}{2}u_x^2u^*_{xx} + 16uu_xu^*_{xxx} + \frac{43}{2}u^*u_xu_{xxx} + \frac{175}{4}(u^*)^2uu_x^2 \\ &+ \ +\frac{53}{2}uu_{xx}u^*_{xx} + \frac{31}{2}u^*u_{xx}^2 + 10|u|^2u_{xxxx} + 20|u|^6u + \frac{55}{2}u^*u^3u^*_{xx} \\ &+ \ \frac{165}{2}u^*u^2|u_x|^2 + \frac{85}{2}|u|^4u_{xx} + 4u^2u^*_{xxxx} + \frac{95}{2}|u_x|^2u_{xx} + \frac{37}{2}u(u_x)^*u_{xxx} + u_{6x}. \end{aligned} \tag{20}$$

3 Reduction to real-valued forms for odd terms only

For the moment, let us restrict ourselves to the odd-order functionals only. If we additionally specify that the functions $u(x,t)$ should be real-valued, then Eq.(2) reduces to higher order forms of the mKdV equation. In this case, Eq.(17) reduces to the basic mKdV equation,

$$u_t - \ \alpha_3S_3 = 0,$$

with:

$$M_3 = u_{xxx} + 6u_xu^2. \tag{21}$$

For the 5-th order, we obtain

$$u_t - \ \alpha_5S_5 = 0,$$

from Eq.(18):

$$M_5 \ = \ u_{5x} + 10(u_{xxx}u^2 + 4uu_xu_{xx} + u_x^3 + 3u_xu^4). \tag{22}$$

The basic soliton solution of the SSE hierarchy with odd terms only, $viz.$

$$u_t - \ \alpha_3S_3 - \ \alpha_5S_5 - \cdots = 0, \tag{23}$$

is

$$u = p\,\text{sech}\,[p(x+st)], \tag{24}$$

where $s = \sum_{n=1}^{\infty} \alpha_{2n+1} p^{2n}$ represents a velocity. Eq.(24) is real, and in this case the SSE and its basic solution are the same as for the lowest-order mKdV,viz.

$$u_t + \ a_3(u_{xxx} + 6u^2 u_x) = 0.$$

The general equation (23), with odd-numbered functionals only, is essentially the mKdV hierarchy [33] of equations. Thus, it is a (real) subset of the Sasa-Satsuma hierarchy. We expect the form of the solution (24) to be valid for all these equations. Moreover, if $u(x,t)$ is an mKdV solution, then scaling shows that $u' = q\, u(q\, x, q^3\, t)$ is also its solution for any real q, so we can just set $p = 1$ in the above, without loss of generality.

4 Relations between SSE, Hirota (NLS) and mKdV hierarchies

To find the relation between SSE, Hirota (NLS) and mKdV hierarchies, we need a consistent normalization and use $b = \sqrt{2}$ in the results found above; we note that this is different from that used in the earlier paper [23]. We denote our previously-found [9] functionals for the Hirota (NLS) hierarchy by K_j, so that the equations of the infinite hierarchy are

$$i\, u_t + \sum_{n=1}^{\infty} (\alpha_{2n} K_{2n} - i\, \alpha_{2n+1} K_{2n+1}) = 0. \tag{25}$$

The NLSE is obtained from Eq.(25) when all real coefficients except α_2 are zero. Thus:

$$i\, u_t + \alpha_2 K_2 = 0, \tag{26}$$

where the functional $K_2 = u_{xx} + 2u\, |u|^2$.

If $u(x,t)$ in (26) is real, this gives the real form D_2 of the functional K_2,

$$D_2 \equiv \ u_{xx} + 2u^3. \tag{27}$$

We can take the derivative of D_2 with respect to x

$$M_3 = \frac{\partial D_2}{\partial x} = u_{xxx} + 6 u_x u^2. \tag{28}$$

to get the mKdV equation,

$$i\, u_t - i \alpha_3 M_3 = 0.$$

From [9], we have

$$K_3 = u_{xxx} + 6 u_x |u|^2. \tag{29}$$

Plainly, S_3 and K_3 are related through

$$S_3 - K_3 = \left(\frac{3}{2}u\right) \begin{vmatrix} u & u^* \\ u_x & u_x^* \end{vmatrix} = \frac{3}{2} u(u u_x^* - u^* u_x) = \frac{3}{2} u\, W_1,$$

where the determinant W_1 is defined as:

$$W_1 = \begin{vmatrix} u & u^* \\ u_x & u_x^* \end{vmatrix}.$$

If u is real, this determinant is zero.

For u real, the SSE

$$u_t - \ \alpha_3 \left[u_{xxx} + \frac{3}{2} \left(|u|^2\right)_x u + 3|u|^2 u_x \right] = 0$$

reduces to the basic mKdV equation

$$u_t - \alpha_3(u_{3x} + 6u^2\, u_x) = 0,$$

i.e. $u_t - \alpha_3 M_3 = 0$, where $M_3 = u_{3x} + 6u^2\, u_x$. The Hirota functional, $K_3 = u_{xxx} + 6|u|^2 u_x$, also reduces to M_3 for u real.

Converting Eq.(13) to its $b^2 = 2$ form, we get

$$S_4 \quad = \quad 3u^*_{xx}u^2 + 6u|u|^4 + 6|u_x|^2 u + 7|u|^2 u_{xx} + 4u^* u_x^2 + u_{xxxx}.$$

It is clear that Eq.(30) has the form of the 4th member of the Hirota (NLS) hierarchy which is known as the LPD equation, but has different coefficients. Now, from [9]:

$$K_4 \quad = \quad 2u^*_{xx}u^2 + 6u|u|^4 + 4|u_x|^2 u + 8|u|^2 u_{xx} + 6u^* u_x^2 + u_{xxxx}. \tag{30}$$

The difference between S_4 and K_4 is

$$S_4 - K_4 = 2|u_x|^2 u - 2u^* u_x^2 + u^*_{xx} u^2 - |u|^2 u_{xx},$$

so the two fourth order equations differ only by 2 terms:

$$u(S_4 - K_4) \quad = \quad \begin{vmatrix} u & u^* \\ \frac{\partial}{\partial x}\left(u_x u^2\right) & \frac{\partial}{\partial x}\left(u^*_x u^2\right) \end{vmatrix} = u\frac{\partial}{\partial x}\left(u^*_x u^2\right) - u^*\frac{\partial}{\partial x}\left(u_x u^2\right), \tag{31}$$

so

$$S_4 - K_4 = uW_2 + 2u_x W_1,$$

where

$$W_2 = \begin{vmatrix} u & u^* \\ u_{2x} & u^*_{2x} \end{vmatrix}.$$

If u is real, then $S_4 - K_4$ is also zero.

If we take u to be real, then we clearly have $S_4 - K_4 = 0$, and we get

$$S_4 = K_4 = 10u_{xx}u^2 + 6u^5 + 10uu_x^2 + u_{xxxx} \equiv D_4. \tag{32}$$

Now D_4 is the real form which can be used to obtain the 5th order mKdV equation [34],

$$u_t - \alpha_5 M_5 = 0.$$

Thus

$$\frac{\partial D_4}{\partial x} = 30u_x u^4 + 10u_{xxx}u^2 + 40u_x u_{xx} u + 10u_x^3 + u_{5x} = M_5. \tag{33}$$

This is the same as the functional M_5 found by other means. It is also the same as Eq.(22) of the real-valued SSE hierarchy, S_5, found here, and it is the same as the real-valued form of K_5 found in [9].

Indeed, from [9], we have

$$K_5 \quad = \quad u_{5x} + 10|u|^2 u_{xxx} + 10(u\,|u_x|^2)_x + 20u^* u_x u_{xx} + 30|u|^4 u_x. \tag{34}$$

We furthermore find that S_5 and K_5 are related as follows:

$$S_5 - K_5 = \frac{5}{2}u\,W_3 + \frac{5}{2}u_x W_2 + \left(\frac{5}{2}u_{2x} + 10u|u|^2\right) W_1,$$

where

$$W_3 = \begin{vmatrix} u & u^* \\ u_{3x} & u^*_{3x} \end{vmatrix},$$

and W_1 and W_2 defined above. We can generalize these definitions as follows:

$$W_j = \begin{vmatrix} u & u^* \\ u_{jx} & u^*_{jx} \end{vmatrix},$$

so that the W_j determinant involves the j^{th} derivatives of u and its complex conjugate, u^*. When u is real, then each W_j is zero. If u is real, then $S_5 - K_5$ is also zero, so $S_5 = K_5 = M_5$, as noted above.

The real scaled form of S_6 from this present work is:

$$\begin{aligned} S_6 &= 20u^7 + 70u^4 u_{xx} + 140u^3 u_x^2 + 14u^2 u_{xxxx} \\ &+ 14u\left(4u_x u_{xxx} + 3u_{xx}^2\right) + 70u_x^2 u_{xx} + u_{6x}. \end{aligned} \tag{35}$$

The real form of K_6, given in [9] is exactly the same as (35). These two functionals are clearly identical and we label them D_6. Now, we can obtain the 7th order mKdV equation, $u_t - \alpha_7 M_7 = 0$, where

$$\begin{aligned} \frac{\partial D_6}{\partial x} &= u_{7x} + 14u^2\left(u_{5x} + 30u_x^3\right) + 140u^6 u_x + 70u^4 u_{xxx} + 560u^3 u_x u_{xx} \\ &+ 28u(3u_x u_{xxxx} + 5u_{xx} u_{xxx}) + 182u_x u_{xx}^2 + 126u_x^2 u_{xxx} = M_7. \end{aligned}$$

This is the 7th order mKdV functional. It is the same as the real form of K_7. The forms of the mKdV functionals, M_{2j+1}, agree with those found from the recursion operator.

Also, from [9], we have

$$\begin{aligned} K_6 &= u_{6x} + 2\left[30u^*|u_x|^2 + 25(u^*)^2 u_{xx} + u^*_{xxxx}\right] u^2 \\ &+ u\left[12u^* u_{xxxx} + 8u_x u^*_{xxx} + 22|u_{xx}|^2\right] + u\left[18u_{xxx} u^*_x + 70(u^*)^2 u_x^2\right] + 20\,(u_x)^2\, u^*_{xx} \\ &+ 10u_x\left[5u_{xx} u^*_x + 3u^* u_{xxx}\right] + 20u^* u_{xx}^2 + 10u^3\left[(u^*_x)^2 + 2u^* u^*_{xx}\right] + 20u|u|^6. \end{aligned} \tag{36}$$

We furthermore find that S_6 and K_6 are related as follows:

$$\begin{aligned} S_6 - K_6 &= 2u\,W_4 + 8u_x W_3 + \frac{1}{2}\left(9u_{2x} + 15u|u|^2 + \frac{5}{u}u_x^2\right)W_2 \\ &+ \frac{1}{4}\left[15u^2 u^*_x + 105|u|^2 u_x - \frac{10}{u}u_x u_{xx} + 2u_{xxx}\right] W_1, \end{aligned} \tag{37}$$

If u is real, then each W_j is zero and thus $S_6 = K_6$.

In the expressions $S_j - K_j$, the first term is proportional to $u\,W_{j-2}$, while the second term is proportional to $u_x\,W_{j-3}$. Of course, when the K_j and differences are known, the set S_j can be generated. So, the pattern of these hierarchies is now more evident. The results of this section are illustrated schematically, in Table 1.

$$
\begin{array}{|ccccccc|}
\hline
\text{density} & & S_i(\text{complex}) & & \text{real form} & & \text{NLS}(\text{complex}) \\
\hline
H_3 & \longrightarrow & S_2 & \longrightarrow & D_2 & \longleftarrow & K_2[\text{NLS}] = S_2 \\
 & & & & \downarrow \frac{\partial}{\partial x} & & \\
 & & S_3[\text{SSE}] & \longrightarrow & M_3 & \longleftarrow & K_3[\text{Hirota}] \\
H_5 & \longrightarrow & S_4 & \longrightarrow & D_4 & \longleftarrow & K_4[\text{LPD}] \\
 & & & & \downarrow \frac{\partial}{\partial x} & & \\
 & & S_5 & \longrightarrow & M_5 & \longleftarrow & K_5[\text{quintic}] \\
H_7 & \longrightarrow & S_6 & \longrightarrow & D_6 & \longleftarrow & K_6[\text{sextic}] \\
 & & & & \downarrow \frac{\partial}{\partial x} & & \\
 & & S_7 & \longrightarrow & M_7 & \longleftarrow & K_7[\text{heptic}] \\
 & & \cdots\cdots\cdots & & & \cdots\cdots\cdots & \\
H_{2j+1} & \longrightarrow & S_{2j} & \longrightarrow & D_{2j} & \longleftarrow & K_{2j} \\
 & & & & \downarrow \frac{\partial}{\partial x} & & \\
 & & S_{2j+1} & \longrightarrow & M_{2j+1} & \longleftarrow & K_{2j+1} \\
\hline
\end{array}
$$

Table 1: Relations between hierarchies, summarizing the results of section 4. Here H_{2j+1} (Eqs.(7), (13) and (15)) indicates invariant density of basic SSE, K_j indicates j^{th} order functional of NLS hierarchy, S_j indicates j^{th} order functional of the SSE hierarchy (newly presented here), M_j (Eq.(33)) indicates j^{th} order functional of mKdV hierarchy. If the functions u are specified as being real, then both K_{2j} and S_{2j} reduce to the same functional, viz. D_{2j} (Eqs.(27), (32) and (35)) for each j. Each vertical arrow indicates $\frac{\partial}{\partial x}$, so that $\frac{\partial D_{2j}}{\partial x} = M_{2j+1}$; for example see Eqs.(28), (33) and (36).

5 Conclusions

Using densities, we have derived a new general multi-parameter equation that contains, as particular cases, the Sasa-Satsuma and mKdV hierarchies of equations. Real arbitrary parameters in this equation allow one to select any particular equation of these hierarchies and any combination of them. While we have presented the operators involved in this equation in explicit forms up to order 6, the technique given here in principle allows one to conveniently obtain the equation of any even order.

This generalized Sasa-Satsuma equation (2), in addition to the generalized Hirota equation [8, 9], will be useful in improving the accuracy of modelling solitons, breathers and rogue waves and even turbulent phenomena in integrable systems [35]. It will allow inclusion of higher-order effects into mathematical modelling of systems using the form with any number of real parameters to control the dynamics being investigated.

Finding solutions of the original Sasa-Satsuma equation is not easy [36, 37, 38, 39]. It would be even more difficult to find them for the whole generalized equation (2). Nevertheless, integrability means there is a way to find solutions in analytic form. An example of a soliton solution for the whole infinite equation has been given in [23]. We believe this work can be continued and, in a few years, we will be able to see more detailed analysis along this path. This immense work needs collective efforts.

References

[1] S. A. R. Horsley, The KdV hierarchy in optics, J. Opt. bf 18, 085104 (2016).

[2] S. Ghosh, B. Talukdar, J. Shamanna, On the bi-Hamiltonian structure of the KdV hierarchy, Czechoslovak J. Phys., **53**, 425 (2003).

[3] A. Kundu, Integrable Hierarchy of Higher Nonlinear Schrödinger Type Equations, Symmetry, Integrability and Geometry: Methods and Applications, **2**, 078 (2006).

[4] D. Gieseker, The Toda Hierarchy and the KdV Hierarchy, Commun. Math. Phys., **181**, 587 – 603 (1996).

[5] N. Joshi, The Second Painlevé Hierarchy and the Stationary KdV Hierarchy, Publ. RIMS, Kyoto Univ., **40**, 1039 – 1061 (2004).

[6] L. D. Faddeev and L. A. Takhtajan, Hamiltonian Methods in the Theory of Solitons (Springer-Verlag, Berlin, Heidelberg, 1987).

[7] K. Porsezian, M. Daniel and M. Lakshmanan, On the integrability aspects of the one-dimensional classical continuum isotropic biquadratic Heisenberg spin chain, J. Math. Phys., **33**, 1807–1816 (1992).

[8] D. J. Kedziora, A. Ankiewicz, A. Chowdury and N. Akhmediev, Integrable equations of the infinite nonlinear Schrödinger equation hierarchy with time variable coefficients, Chaos, **25**, 103114 (2015).

[9] A. Ankiewicz, D. J. Kedziora, A. Chowdury, U. Bandelow and N. Akhmediev, Infinite hierarchy of nonlinear Schrödinger equations and their solutions, Phys. Rev. E **93**, 012206 (2016).

[10] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, Method for solving the Korteweg-de Vries Equation, Phys. Rev. Lett., **19**, 1095 (1967).

[11] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focussing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP, **34**, 62 (1972).

[12] A. Blanco-Redondo, C. Husko, D. Eades, Y. Zhang, J. Li, T. F. Krauss, B. J. Eggleton Observation of soliton compression in silicon photonic crystals, Nature Commun., Nat. Commun., **5**, 3160 (2014).

[13] F. M. Mitschke and L. F. Mollenauer, Discovery of the soliton self-frequency shift, Opt. Lett., **11**, 659 – 661 (1986).

[14] D. Anderson, M. Desaix, M. Lisak, and M. L. Quiroga–Teixeiro, Wave breaking in nonlinear-optical fibers, J. Opt. Soc. Am. B **9**, 1358 – 1361 (1992).

[15] A. Hasegawa and F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion, Appl. Phys. Lett. **23**, 142 – 144 (1973).

[16] S. B. Cavalcanti, J. C. Cressoni, H. R. da Cruz, and A. S. Gouveia-Neto. Modulation instability in the region of minimum group-velocity dispersion of single-mode optical fibers via an extended nonlinear Schrödinger equation, *Phys. Rev. A*, **43**, 6162 – 6165 (1991).

[17] M. Trippenbach and Y. B. Band. Effects of self-steepening and self-frequency shifting on short-pulse splitting in dispersive nonlinear media, *Phys. Rev. A*, **57**, 4791 – 4803 (1991).

[18] M. J. Potasek, Exact solutions for an extended nonlinear Schrödinger equation, *Phys. Lett. A* **60**, 449 – 452 (1991).

[19] Yuen H. C. & Lake, B. M. Nonlinear dynamics of deep-water gravity waves. *Adv. Appl. Mech.* **22**, 67–228 (1982).

[20] Dysthe, K. B. Note on a modification to the nonlinear Schrodinger equation for application to deep water waves. *Proc. R. Soc. Lond. A* **369**, 105 – 114 (1979).

[21] Yu. V. Sedletskii, The Fourth-Order nonlinear Schrödinger equation for the envelope of Stokes waves on the surface of a finite-depth fluid, Sov. Phys. JETP, **97**, 180 - 193 (2003) [Translated from Zh. Eksper. Teor. Fiziki. **124**, 200 – 213 (2003)].

[22] A. V. Slunyaev, A High-Order Nonlinear Envelope Equation for Gravity Waves in Finite-Depth Water, Sov. Phys. JETP, **101**, 926–941 (2005). [Translated from Zh. Eksper. Teor. Fiziki, **128**, 1061–1077 (2005)].

[23] U. Bandelow, A. Ankiewicz, Sh. Amiranashvili, N. Akhmediev, Sasa-Satsuma hierarchy of integrable evolution equations, to appear in Chaos: Interdisciplinary J. Nonlinear Science, Manuscript No 18365.

[24] N. Akhmediev and A. Ankiewicz, Solitons, nonlinear pulses and beams, Chapman & Hall, London 1997.

[25] R. Hirota, Exact envelope-soliton solutions of a nonlinear wave equation, J. Math. Phys, **14**, 805, (1973).

[26] M. Lakshmanan, K. Porsezian and M. Daniel, Phys. Lett. A, **133**, 483–488 (1988).

[27] S. M. Hoseini and T. R. Marchant, Solitary wave interaction and evolution for a higher-order Hirota equation, Wave Motion, **44**, 92 (2006).

[28] A. Chowdury, D. J. Kedziora, A. Ankiewicz and N. Akhmediev, Breather-to-soliton conversions described by the quintic equation of the nonlinear Schrödinger hierarchy, Phys. Rev. E **91**, 032928 (2015).

[29] N. Sasa and J. Satsuma, New-type of soliton solutions for a higher-order nonlinear Schrödinger equation. *J. of Physical Society of Japan*, **60**, 409 – 417 (1991).

[30] C. Gilson, J. Hientarinta, J. Nimmo, and Y. Ohta, Sasa - Satsuma higher-order nonlinear Schrödinger equation and its bilinearization and multisoliton solutions, *Phys. Rev. E*, **68**, 016614 (2003).

[31] D. Mihalache, L. Torner, F. Moldoveanu, N. C. Panoiu, and N. Truta, Soliton solutions for a perturbed nonlinear Schrödinger equation. *J. of Physics A: Mathematical and General*, **26**, L757 – L765, (1993).

[32] S. Ghosh, A. Kundu and S. Nandy, Soliton solutions, Liouville integrability and gauge equivalence of Sasa–Satsuma equation, J. Math. Phys., **40**, 1993 (1999).

[33] Peter A. Clarkson, Nalini Joshi and Marta Mazzocco, The Lax pair for the mKdV hierarchy, Séminaires & Congrès, **14**, 53 – 64, (2006).

[34] Y. Matsuno, Bilinearization of Nonlinear Evolution Equations. II. Higher-Order Modified Korteweg-de Vries Equations, J. Phys. Soc. Japan. **49**, 787 – 794 (1980).

[35] N. Akhmediev, J. M. Soto-Crespo and N. Devine, Breather turbulence versus soliton turbulence: Rogue waves, probability density functions, and spectral features, Phys. Rev. E **94**, 022212 (2016).

[36] D. Mihalache, N. C. Panoiu, F. Moldoveanu, and D.-M. Baboiu, The Riemann problem method for solving a perturbed nonlinear Schrodinger equation describing pulse propagation in optical fibres, *Journal of Physics A: Mathematical and General*, **27**, 6177 – 6189 (1994).

[37] D. Mihalache, L. Torner, F. Moldoveanu, N. C. Panoiu, and N. Truta, Inverse-scattering approach to femtosecond solitons in monomode optical fibers, *Physical Review E*, **48**, 4699 – 4709 (1993).

[38] U. Bandelow and N. Akhmediev, Persistence of rogue waves in extended nonlinear Schrödinger equations: Integrable Sasa – Satsuma case. *Phys. Lett. A* **376**, 1558 – 1561 (2012).

[39] O. C. Wright III, Sasa - Satsuma equation, unstable plane waves and heteroclinic connections. *Chaos, Solitons & Fractals*, **33**, 374 – 387 (2007).