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Well-posedness and regularity for a generalized fractional Cahn–Hilliard system

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Abstract

In this paper, we investigate a rather general system of two operator equations that has the structure of a viscous or nonviscous Cahn–Hilliard system in which nonlinearities of double-well type occur. Standard cases like regular or logarithmic potentials, as well as non-differentiable potentials involving indicator functions, are admitted. The operators appearing in the system equations are fractional versions of general linear operators A and B, where the latter are densely defined, unbounded, self-adjoint and monotone in a Hilbert space of functions defined in a smooth domain and have compact resolvents. In this connection, we remark the fact that our definition of the fractional power of operators uses the approach via spectral theory. Typical cases are given by standard second-order elliptic differential operators (e.g., the Laplacian) with zero Dirichlet or Neumann boundary conditions, but also other cases like fourth-order systems or systems involving the Stokes operator are covered by the theory. We derive in this paper general well-posedness and regularity results that extend corresponding results which are known for either the non-fractional Laplacian with zero Neumann boundary condition or the fractional Laplacian with zero Dirichlet condition. These results are entirely new if at least one of the operators A and B differs from the Laplacian. It turns out that the first eigenvalue λ_1 of A plays an important und not entirely obvious role: if λ_1 is positive, then the operators A and B may be completely unrelated; if, however, λ_1 equals zero, then it must be simple and the corresponding one-dimensional eigenspace has to consist of the constant functions and to be a subset of the domain of definition of a certain fractional power of B. We are able to show general existence, uniqueness, and regularity results for both these cases, as well as for both the viscous and the nonviscous system.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ denote a bounded, connected and smooth set and H be a Hilbert space of real-valued functions defined on Ω. We investigate in this paper the abstract evolutionary system

$$\partial_t y + A^{2r}\mu = 0, \tag{1.1}$$

$$\tau\partial_t y + B^{2\sigma} y + f'(y) = \mu + u, \tag{1.2}$$

$$y(0) = y_0, \tag{1.3}$$

where A^{2r} and $B^{2\sigma}$, with $r > 0$ and $\sigma > 0$, denote fractional powers of the selfadjoint, monotone and unbounded linear operators A and B, respectively, which are densely defined in H and have compact resolvents. The above system can be seen as a generalization of the famous Cahn–Hilliard system, which models a phase separation process taking place in the container Ω (the list [14, 16, 18, 21, 22, 30, 35, 38] combines basic references with some recent contribution on Cahn–Hilliard systems). In this case, one typically has $A^{2r} = B^{2\sigma} = -\Delta$ with zero Neumann boundary conditions, and

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the unknown functions y and μ stand for the *order parameter* (usually a scaled density of one of the involved phases) and the *chemical potential* associated with the phase transition, respectively. Moreover, f denotes a double-well potential. Typical and physically significant examples for f are the so-called *classical regular potential*, the *logarithmic double-well potential*, and the *double obstacle potential*, which are given, in this order, by

$$f_{reg}(r) := \frac{1}{4}\,(r^2-1)^2\,, \quad r \in \mathbb{R}, \tag{1.4}$$

$$f_{log}(r) := \bigl((1+r)\ln(1+r)+(1-r)\ln(1-r)\bigr) - c_1 r^2\,, \quad r \in (-1,1), \tag{1.5}$$

$$f_{2obs}(r) := -c_2 r^2 \quad \text{if } |r| \le 1 \quad \text{and} \quad f_{2obs}(r) := +\infty \quad \text{if } |r| > 1. \tag{1.6}$$

Here, the constants c_i in (1.5) and (1.6) satisfy $c_1 > 1$ and $c_2 > 0$, so that f_{log} and f_{2obs} are nonconvex. In cases like (1.6), one has to split f into a nondifferentiable convex part $\widehat{\beta}$ (the indicator function of $[-1,1]$, in the present example) and a smooth perturbation $\widehat{\pi}$. Accordingly, one has to replace the derivative of the convex part by the subdifferential and interpret (1.2) as a differential inclusion or, equivalently, as a variational inequality involving $\widehat{\beta}$ rather than its subdifferential. Actually, we will do the latter in this paper. We also note that τ is a nonnegative parameter, where for the classical Cahn–Hilliard system one has $\tau = 0$ (the *nonviscous* case); in this paper, we will handle both the nonviscous case $\tau = 0$ and the *viscous* case $\tau > 0$ simultaneously. Of course, better regularity results are to be expected in the latter case.

Fractional operators are nowadays a very hot topic in the mathematical literature, and it occurs that different variants of fractional operators may be considered and tackled. Let us perform some review of contributions and results. The paper [32] deals with several definitions of the fractional Laplacian (also known as the Riesz fractional derivative operator), which is a core example of a class of nonlocal pseudodifferential operators appearing in various areas of theoretical and applied mathematics. In connection with such fractional operators, fractional Sobolev spaces are revisited and discussed in [19]. The contributions by Servadei and Valdinoci deserve some attention: in [40], a comparison is made between the spectrum of two different fractional Laplacian operators, of which the second one fits in our framework; the paper [41] discusses the regularity of the weak solution to the fractional Laplace equation; the existence of nontrivial solutions for nonlocal semilinear Dirichlet problem is established in [39]; a fractional counterpart to the well-known Brezis–Nirenberg result on the existence of nontrivial solutions to elliptic equations with critical nonlinearities is provided in [42].

The paper [2] presents a construction of harmonic functions on bounded domains for the spectral fractional Laplacian operator having a divergent profile at the boundary. In the contribution [13], a nonlinear pseudodifferential boundary value problem is investigated in a bounded domain with homogeneous Dirichlet boundary conditions, where the square root of the negative Laplace operator is involved. Regularity results and sharp estimates are proved in [15] for fractional elliptic equations. A nonlocal diffusion operator having the fractional Laplacian as a special case is analyzed in [20] on bounded domains, with respect to nonlocal interactions. Fractional Dirichlet and Neumann type boundary problems associated with the fractional Laplacian are investigated in [28], by demonstrating regularity properties with a spectral approach; this analysis is extended to the fractional heat equation in [29]. Obstacle problems for the spectral fractional Laplacian are studied in [34]. By using the Caputo variant of an integral operator with the Riesz kernel, the authors of [36, 37] prove regularity up to the boundary for a Dirichlet-type boundary value problem and study the extremal solutions by extending some well-known results on the extremal solutions when the operator is the Laplacian. Some nonlocal problems involving the fractional $p-$Laplacian and nonlinearities at critical growth are examined in [11].

Fractional porous medium type equations are discussed in [8–10]. The paper [9] deals with existence, uniqueness and asymptotic behavior of the solutions to an integro-differential equation related to porous medium equations in bounded domains; the problem does not have a separate boundary condition, since zero boundary data are implicitly assumed in the definition of the operator. A priori estimates for positive solutions of a porous medium equation are shown in [10], where the spectral fractional Laplacian with zero Dirichlet boundary data is considered; it turns out that the results are influenced by the first eigenvalue and eigenfunction. A quantitative study of nonnegative solutions of the same equation is provided in [8], where the regularity theory is addressed: decay and positivity, Harnack inequalities, interior and boundary regularity, and asymptotic behavior are investigated. Also fractional Schrödinger equations are receiving a good deal of attention, see, e.g., [7] and references therein.

There exist already quite a number of contributions dealing with nonlocal variants of the Cahn–Hilliard system. In [1], the problem of well-posedness for a nonlocal Cahn–Hilliard equation is established by interpreting the problem as a Lipschitz perturbation of a maximal monotone operator in a suitable Hilbertk space. A fractional variant of the Cahn–Hilliard equation settled in a bounded domain and complemented with homogeneous Dirichlet boundary conditions of solid type is introduced in [4]: existence and uniqueness of weak solutions to the related initial-boundary value problem are proved and some significant singular limits are investigated as the order of either of the fractional Laplacians appearing in the system approaches zero. Moreover, in the recent paper [5], for fixed orders of the operators the convergence as time goes to infinity of each solution to a (single) equilibrium is proved. In [3], the authors derive a fractional Cahn–Hilliard equation by considering a gradient flow in the negative order Sobolev space $H^{-\alpha}$, $\alpha \in [0,1]$, where the choice $\alpha = 1$ corresponds to the classical Cahn–Hilliard equation, while the choice $\alpha = 0$ recovers the Allen–Cahn equation; existence and stability estimates are derived in the case where the nonlinearity is a quartic polynomial, as in (1.4). The paper [26] addresses the nonlocal Cahn–Hilliard equation with a singular potential and a constant mobility: among a class of results, in particular the authors can establish the validity of the strict separation property in two dimensions. Another interesting analysis of a nonstandard and nonlocal Cahn–Hilliard system can be found in [17]. Next, in [24] a non-local version of the Cahn–Hilliard equation characterized by the presence of a fractional diffusion operator, and which is subject to fractional dynamic boundary conditions, is studied. The articles [23, 25] treat a doubly nonlocal Cahn–Hilliard equation with special kernels in the operators: well-posedness results, along with regularity, long-time behavior, and global attractors, are investigated in connection with the interaction between the two levels of nonlocality in the operators.

In our approach, which we develop in the subsequent sections, we work with fractional operators defined via spectral theory. This position enables us to deal with powers of a second-order elliptic operator with either Dirichlet or Neumann or Robin boundary conditions, allowing us a wide setting in this respect. Moreover, other operators, such as fourth-order ones or systems involving the Stokes operator, can be covered by the theory.

The aim of the present paper is to prove general well-posedness and regularity theorems that extend the corresponding results known for either the non-fractional Laplacian with zero Neumann boundary condition or the fractional Laplacian with zero Dirichlet condition (cf. [3, 4]). In the development of the theory, one realizes that the first eigenvalue λ_1 of A plays an important und not entirely obvious role. Indeed, it turns out that if λ_1 is positive, then the operators A and B may be completely unrelated. On the other hand, in the case when $\lambda_1 = 0$, then we have to assume that λ_1 is a simple eigenvalue and that the corresponding one-dimensional eigenspace consists of constant functions, on which the proper fractional power of B should operate. This set of assumptions looks like a heavy restriction, but let us notice that the framework is strongly related to the structure of the Cahn–Hilliard system with the natural Neumann homogeneous boundary conditions (that exactly imply conservation of mass). In conclusion, it will turn out that we are able to show well-posedness and regularity results for both the abovementioned situations, as well as for both the viscous and the nonviscous system, under very general assumptions for the convex parts of the potential f (see (1.4)–(1.6)).

Here is a brief outline of the paper. Section 2 contains a precise statement of the problem along with assumptions and main results; some remarks commenting the results and introducing examples of operators are also included. Section 3 is intended to present some auxiliary material about relations among the involved spaces and properties of the operators; all this turns to be a useful toolbox for the following analysis. Section 4 deals with the continuous dependence of the solution on the data, while Section 5 introduces an approximating problem based on the Moreau–Yosida regularizations of the convex functions and on an implicit time discretization of the system, which is fully discussed concerning existence of the discrete solution and uniform a priori estimates for it. Section 6 brings the existence proof, which is carried out by taking the limits with respect to the approximation parameters. Finally, Section 7 is devoted to show the proper estimates ensuring the regularity properties for the solution.

2 Statement of the problem and results

In this section, we state precise assumptions and notations and present our results. First of all, the set $\Omega \subset \mathbb{R}^3$ is assumed to be bounded, connected and smooth, with outward unit normal vector field ν on $\Gamma := \partial\Omega$. Moreover, ∂_ν stands for the corresponding normal derivative. We use the notation

$$H := L^2(\Omega) \tag{2.1}$$

and denote by $\|\cdot\|$ and $(\,\cdot\,,\,\cdot\,)$ the standard norm and inner product of H. Now, we start introducing our assumptions. We first postulate that

$$A : D(A) \subset H \to H \quad \text{and} \quad B : D(B) \subset H \to H \quad \text{are}$$
$$\text{unbounded monotone selfadjoint linear operators with compact resolvents.} \tag{2.2}$$

This assumption implies that there are sequences $\{\lambda_j\}$ and $\{\lambda'_j\}$ of eigenvalues and orthonormal sequences $\{e_j\}$ and $\{e'_j\}$ of corresponding eigenvectors, that is,

$$Ae_j = \lambda_j e_j, \quad Be'_j = \lambda'_j e'_j \quad \text{and} \quad (e_i, e_j) = (e'_i, e'_j) = \delta_{ij} \quad \text{for } i,j = 1,2,\dots \tag{2.3}$$

with δ_{ij} denoting the Kronecker index, such that

$$0 \le \lambda_1 \le \lambda_2 \le \dots \quad \text{and} \quad 0 \le \lambda'_1 \le \lambda'_2 \le \dots \quad \text{with} \quad \lim_{j\to\infty} \lambda_j = \lim_{j\to\infty} \lambda'_j = +\infty, \tag{2.4}$$

$$\{e_j\} \text{ and } \{e'_j\} \text{ are complete systems in } H. \tag{2.5}$$

The above assumptions on A and B allow us to define the powers of A and B for an arbitrary positive real exponent. As far as the first operator is concerned, we have

$$V_A^r := D(A^r) = \Big\{ v \in H : \sum_{j=1}^\infty |\lambda_j^r (v, e_j)|^2 < +\infty \Big\} \quad \text{and} \tag{2.6}$$

$$A^r v = \sum_{j=1}^\infty \lambda_j^r (v, e_j) e_j \quad \text{for } v \in V_A^r, \tag{2.7}$$

the series being convergent in the strong topology of H, due to the properties (2.6) of the coefficients. In principle, we endow V_A^r with the (graph) norm and inner product

$$\|v\|^2_{gr,A,r} := (v,v)_{gr,A,r} \quad \text{and} \quad (v,w)_{gr,A,r} := (v,w) + (A^r v, A^r w) \quad \text{for } v,w \in V_A^r. \tag{2.8}$$

This makes V_A^r a Hilbert space. However, we can choose any equivalent Hilbert norm. Later on, we actually will do that. In the same way, starting from (2.2)–(2.5) for B, we can define the power B^σ of B for every $\sigma > 0$. We therefore set

$$V_B^\sigma := D(B^\sigma), \quad \text{with the norm } \|\cdot\|_{B,\sigma} \text{ associated to the inner product}$$
$$(v,w)_{B,\sigma} := (v,w) + (B^\sigma v, B^\sigma w) \quad \text{for } v,w \in V_B^\sigma. \tag{2.9}$$

If r_i and σ_i are arbitrary positive exponents, it is clear that

$$(A^{r_1+r_2} v, w) = (A^{r_1} v, A^{r_2} w) \quad \text{for every } v \in V_A^{r_1+r_2} \text{ and } w \in V_A^{r_2}, \tag{2.10}$$
$$(B^{\sigma_1+\sigma_2} v, w) = (B^{\sigma_1} v, B^{\sigma_2} w) \quad \text{for every } v \in V_B^{\sigma_1+\sigma_2} \text{ and } w \in V_B^{\sigma_2}. \tag{2.11}$$

From now on, we assume:

$$r \text{ and } \sigma \text{ are fixed positive real numbers.} \tag{2.12}$$

Accordingly, we introduce a space with a negative exponent. We set

$$V_A^{-r} := (V_A^r)^* \quad \text{for } r > 0, \tag{2.13}$$

and use the symbol $\langle \cdot\,, \cdot \rangle_{A,r}$ for the duality pairing between V_A^{-r} and V_A^r. Moreover, we identify H with a subspace of V_A^{-r} in the usual way, i.e., such that

$$\langle v, w \rangle_{A,r} = (v,w) \quad \text{for every } v \in H \text{ and } w \in V_A^r. \tag{2.14}$$

Next, we make the following assumption:

$$\text{Either} \quad \lambda_1 > 0 \quad \text{or} \quad 0 = \lambda_1 < \lambda_2 \text{ and } e_1 \text{ is a constant.} \tag{2.15}$$
$$\text{If} \quad \lambda_1 = 0, \quad \text{the constant functions belong to } V_B^\sigma. \tag{2.16}$$

Remark 2.1. Let us comment on the assumptions (2.15). The meaning of the first case is clear, and such a condition is satisfied by the more usual elliptic operators with Dirichlet boundary conditions (however, also mixed boundary conditions could be considered, with proper definitions of the domains of the operators), for instance: $i)$ A is the Laplace operator $-\Delta$ with domain $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$; $ii)$ A is the bi-harmonic operator Δ^2 with domain: $D(\Delta^2) = H^4(\Omega) \cap H_0^2(\Omega)$. The second case of (2.15), where the strict inequality means that the first eigenvalue $\lambda_1 = 0$ is simple, happens in the following important situations: $i)$ A is the Laplace operator $-\Delta$ with Neumann boundary conditions, which corresponds to the choice $D(-\Delta) = \{v \in H^2(\Omega) : \partial_\nu v = 0\}$; $ii)$ A is the bi-harmonic operator Δ^2 with the boundary conditions corresponding to the following choice of the domain: $D(\Delta^2) = \{v \in H^4(\Omega) : \partial_\nu v = \partial_\nu \Delta v = 0\}$. Indeed, Ω is assumed to be bounded, smooth and connected.

Remark 2.2. We point out that (2.16) is the only condition that involves both operators A and B, i.e., if $\lambda_1 > 0$, these operators are completely unrelated. However, we notice that the assumption on the constant functions is rather mild. Indeed, it holds for many operators whose domain involves Neumann boundary conditions. This is the case, for instance, if B is the Laplace operator with domain $D(-\Delta) = \{v \in H^2(\Omega) : \partial_\nu v = 0\}$. On the contrary, if $B = -\Delta$ with domain $D(-\Delta) := H^2(\Omega) \cap H_0^1(\Omega)$, then $D(B)$ does not contain any nonzero constant functions. However, V_B^σ does contain every constant function provided that $\sigma \in (0, 1/4)$, since it coincides with the usual Sobolev-Slobodeckij space $H^{2\sigma}(\Omega)$. Indeed, the spaces V_A^r and V_B^σ can be seen in the framework of interpolation theory. However, we prefer to avoid this check and deduce all the results we need from our definitions.

Remark 2.3. We have chosen to take $H := L^2(\Omega)$ once and for all, for simplicity. However, it is clear that our assumptions are rather close to an abstract situation and can be adapted to other choices of the space H as well. For instance, one could deal with the Stokes operator with Dirichlet boundary conditions, by taking for H the space of vector-valued functions $v \in (L^2(\Omega))^3$ satisfying $\operatorname{div} v = 0$ in the sense of distributions and defining the operator A as follows: an element $v \in H$ belongs to $D(A)$ if and only if $v \in (H_0^1(\Omega))^3$ and $\Delta v := (\Delta v_i) \in (L^2(\Omega))^3$; for $v \in D(A)$, Av is the L^2-projection on H of $-\Delta v$. In this case, the first assumption of (2.15) is satisfied. Of course, the hypotheses on the structure of the nonlinear terms to be introduced below would have to be adapted to this new situation.

We use assumption (2.15) to define a different Hilbert norm on V_A^r. We set, for $v \in V_A^r$,

$$\|v\|_{A,r}^2 := \begin{cases} \|A^r v\|^2 = \sum_{j=1}^\infty |\lambda_j^r (v, e_j)|^2 & \text{if } \lambda_1 > 0, \\ |(v, e_1)|^2 + \|A^r v\|^2 = |(v, e_1)|^2 + \sum_{j=2}^\infty |\lambda_j^r (v, e_j)|^2 & \text{if } \lambda_1 = 0. \end{cases} \tag{2.17}$$

In the next section we will show that this norm is equivalent to the graph norm defined in (2.8), and we always will use the norm (2.17) rather than (2.8). Of course, we will also use the corresponding inner product in V_A^r and norm in V_A^{-r}. They are given by

$$(v, w)_{A,r} = (A^r v, A^r w) \quad \text{or} \quad (v, w)_{A,r} = (v, e_1)(w, e_1) + (A^r v, A^r w),$$
$$\text{depending on whether } \lambda_1 > 0 \text{ or } \lambda_1 = 0, \quad \text{for } v, w \in V_A^r, \tag{2.18}$$

$$\|\cdot\|_{A,-r} \text{ is the dual norm of } \|\cdot\|_{A,r}. \tag{2.19}$$

Remark 2.4. We notice that in the case $\lambda_1 = 0$ of (2.15) the constant value of e_1 is equal to one of the numbers $\pm|\Omega|^{-1/2}$, where $|\Omega|$ is the volume of Ω. It follows for every $v \in H$ that the first term $(v, e_1)e_1$ of the Fourier series of v is the constant function whose value is the mean value of v, i.e.,

$$\operatorname{mean} v := \frac{1}{|\Omega|} \int_\Omega v\,, \tag{2.20}$$

and that the first terms of the sums appearing in (2.17) and (2.18) are given by

$$|(v, e_1)|^2 = |\Omega|\,(\operatorname{mean} v)^2 \quad \text{for every } v \in H,$$
$$(v, e_1)(w, e_1) = |\Omega|\,(\operatorname{mean} v)(\operatorname{mean} w) \quad \text{for every } v, w \in H.$$

For the other ingredients of our system, we postulate the following properties:

$$\tau \text{ is a nonnegative real number.} \tag{2.21}$$

$$\widehat{\beta} : \mathbb{R} \to [0, +\infty] \quad \text{is convex, proper and l.s.c. with} \quad \widehat{\beta}(0) = 0. \tag{2.22}$$

$$\widehat{\pi}:\mathbb{R}\to\mathbb{R} \quad \text{is of class } C^1 \text{ with a Lipschitz continuous first derivative.} \tag{2.23}$$

$$\text{It holds} \quad \liminf_{|s|\nearrow+\infty} \frac{\widehat{\beta}(s)+\widehat{\pi}(s)}{s^2} > 0. \tag{2.24}$$

We can suppose that $\tau \le 1$ without loss of generality. We remark that the assumptions (2.22)–(2.24) are fulfilled by all of the important potentials (1.4)–(1.6). We set, for convenience,

$$\beta := \partial\widehat{\beta}, \quad \pi := \widehat{\pi}', \quad L_\pi = \text{the Lipschitz constant of } \pi, \quad \text{and} \quad L'_\pi := L_\pi + 1\,. \tag{2.25}$$

Moreover, we term $D(\widehat{\beta})$ and $D(\beta)$ the effective domains of $\widehat{\beta}$ and β, respectively, and, for $r \in D(\beta)$, we use the symbol $\beta^\circ(r)$ for the element of $\beta(r)$ having minimum modulus. Notice that β is a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$.

At this point, we can state the problem under investigation. On account of (2.10)–(2.11), we give a weak formulation of the equations (1.1)–(1.2). Moreover, we present (1.2) as a variational inequality. For the data, we make the following assumptions:

$$u \in H^1(0,T;H) \tag{2.26}$$

$$y_0 \in V_B^\sigma \quad \text{and} \quad \widehat{\beta}(y_0) \in L^1(\Omega) \tag{2.27}$$

$$\text{if } \lambda_1 = 0, \quad m_0 := \operatorname{mean} y_0 \quad \text{belongs to the interior of } D(\beta). \tag{2.28}$$

Notice that no condition on m_0 is required if $\lambda_1 > 0$. Then, we set

$$Q := \Omega \times (0,T) \tag{2.29}$$

and look for a pair (y,μ) satisfying

$$y \in H^1(0,T;V_A^{-r}) \cap L^\infty(0,T;V_B^\sigma) \quad \text{and} \quad \tau\partial_t y \in L^2(0,T;H), \tag{2.30}$$

$$\mu \in L^2(0,T;V_A^r), \tag{2.31}$$

$$\widehat{\beta}(y) \in L^1(Q), \tag{2.32}$$

and solving the system

$$\langle \partial_t y(t), v\rangle_{A,r} + (A^r\mu(t), A^r v) = 0 \quad \text{for every } v \in V_A^r \text{ and a.e. } t \in (0,T), \tag{2.33}$$

$$\begin{aligned} &\tau\big(\partial_t y(t), y(t)-v\big) + \big(B^\sigma y(t), B^\sigma(y(t)-v)\big) \\ &\quad + \int_\Omega \widehat{\beta}(y(t)) + \big(\pi(y(t)) - u(t), y(t)-v\big) \le \big(\mu(t), y(t)-v\big) + \int_\Omega \widehat{\beta}(v) \\ &\quad \text{for every } v \in V_B^\sigma \text{ and a.e. } t \in (0,T), \end{aligned} \tag{2.34}$$

$$y(0) = y_0\,. \tag{2.35}$$

Of course, it is understood that

$$\int_\Omega \widehat{\beta}(v) = +\infty \quad \text{whenever} \quad \widehat{\beta}(v) \notin L^1(\Omega).$$

A similar agreement also holds for integrals of the type $\int_Q \widehat{\beta}(v)$ whenever $v \in L^2(Q)$ but $\widehat{\beta}(v) \notin L^1(Q)$.

Now, let us notice that (2.34) is equivalent to its time-integrated variant, that is,

$$\tau \int_0^T \big(\partial_t y(t), y(t) - v\big)\, dt + \int_0^T \big(B^\sigma y(t), B^\sigma (y(t) - v(t))\big)\, dt$$
$$+ \int_Q \widehat{\beta}(y) + \int_0^T \big(\pi(y(t)) - u(t), y(t) - v(t)\big)\, dt$$
$$\leq \int_0^T \big(\mu(t), y(t) - v(t)\big)\, dt + \int_Q \widehat{\beta}(v) \quad \text{for every } v \in L^2(0,T;V_B^\sigma). \tag{2.36}$$

We also remark that, if $\lambda_1 = 0$, then $A^r(1) = 0$ by (2.15), so that (2.33) implies that

$$\frac{d}{dt}\int_\Omega y(t) = 0 \quad \text{for a.a. } t \in (0,T), \quad \text{i.e.,} \quad \operatorname{mean} y(t) = m_0 \quad \text{for every } t \in [0,T]. \tag{2.37}$$

Finally, let us note that if $\lambda_1 = 0$, then the condition (2.28) on m_0 ensures the existence of some $\delta_0 > 0$ satisfying

$$[m_0 - \delta_0, m_0 + \delta_0] \subset D(\beta). \tag{2.38}$$

Remark 2.5. According to the definition of subdifferential (cf., e.g., [12] or [6]), the precise meaning of the inequality (2.34) is that there exists some element $\chi \in L^2(0,T;(V_B^\sigma)^*)$ such that

$$\chi := \mu - \tau\partial_t y - B^{2\sigma} y - \pi(y) + u \in \partial\Phi(y) \quad \text{a.e. in } (0,T),$$

where $\partial\Phi$ is the subdifferential of the convex function $\Phi : V_B^\sigma \to [0,+\infty]$ defined by

$$\Phi(v) := \int_\Omega \widehat{\beta}(v) \quad \text{if } \widehat{\beta}(v) \in L^1(\Omega), \quad \Phi(v) := +\infty \quad \text{otherwise},$$

and actually the subdifferential $\partial\Phi$ is a maximal monotone operator from V_B^σ to $(V_B^\sigma)^*$. In this sense, (2.34) turns out to be a slight generalization of (1.2).

Here is our well-posedness and continuous dependence result.

Theorem 2.6. *Let the assumptions* (2.2), (2.12), (2.15)–(2.16) *and* (2.21)–(2.24) *on the structure of the system, and* (2.26)–(2.28) *on the data, be fulfilled. Then there exists a unique pair (y,μ) satisfying* (2.30)–(2.32) *and solving problem* (2.33)–(2.35)*. Moreover, this solution satisfies the estimate*

$$\|y\|_{H^1(0,T;V_A^{-r})\cap L^\infty(0,T;V_B^\sigma)} + \|\mu\|_{L^2(0,T;V_A^r)} + \|\widehat{\beta}(y)\|_{L^1(Q)} + \|\tau^{1/2}\partial_t y\|_{L^2(0,T;H)} \leq K_1, \tag{2.39}$$

with a constant K_1 that depends only on the structure of the system, the norms of the data corresponding to (2.26)–(2.27)*, the width δ_0 satisfying* (2.38) *if $\lambda_1 = 0$, and T. Moreover, if u_i, $i = 1,2$, are two choices of u and (y_i,μ_i) are the corresponding solutions, then we have*

$$\|y_1 - y_2\|_{L^\infty(0,T;V_A^{-r})\cap L^2(0,T;V_B^\sigma)} + \|\tau^{1/2}(y_1 - y_2)\|_{L^\infty(0,T;H)} \leq K_2 \|u_1 - u_2\|_{L^2(0,T;H)}, \tag{2.40}$$

with a constant K_2 that depends only on the operators A^r and B^σ, the Lipschitz constant L_π, and T.

Remark 2.7. More generally, we could take two different initial values $y_{0,1}$ and $y_{0,2}$, by assuming that they have the same mean value if $\lambda_1 = 0$. Then, the right-hand side of (2.40) has to be modified by adding two contributions involving $d_0 := y_{0,1} - y_{0,2}$, which are proportional to $\|d_0\|_{A,-r}$ and to $\tau^{1/2}\|d_0\|$.

Under additional assumptions on the data, we have stronger regularity results in both the viscous and nonviscous cases. Namely, we also assume that either $\tau > 0$ and

$$y_0 \in V_B^{2\sigma} \quad \text{and} \quad \beta^\circ(y_0) \in H \tag{2.41}$$

or $\tau = 0$ and

$$y_0 \in V_B^{2\sigma} \quad \text{and} \quad \|\mu_0^\lambda(t)\|_{A,r} \le M_0, \quad \text{where} \tag{2.42}$$
$$\mu_0^\lambda(t) := B^{2\sigma} y_0 + (\beta_\lambda + \pi)(y_0) - u(t), \tag{2.43}$$

for some constant M_0 and every sufficiently small $\lambda > 0$ and $t > 0$, β_λ being the Yosida approximation of β at the level λ (see, e.g., [12, p. 28]). More precisely, it is assumed that the element $\mu_0^\lambda(t)$ (which is well defined by (2.43) due to the first assumption on y_0) belongs to V_A^r and satisfies the above estimate. Of course, this assumption is very restrictive. However, we can give sufficient conditions for it. One possibility is to assume that each of the four contributions to the right-hand side of (2.43) satisfies bounds like (2.42), separately, and that A^r is a local operator in order to deal with the term $\beta_\lambda(y_0)$. For instance, if A^r is the Laplace operator with Dirichlet boundary conditions and β is single-valued and smooth in the interior of its domain, then one can assume that $y_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and that $\min y_0 > \inf D(\beta)$ and $\max y_0 < \sup D(\beta)$. These assumptions keep $\beta_\lambda(y_0)$ bounded in $H^2(\Omega)$, indeed.

Theorem 2.8. *In addition to the assumptions of Theorem 2.6, suppose that either $\tau > 0$ and* (2.41) *or $\tau = 0$ and* (2.42)–(2.43) *are fulfilled. Then the unique solution (y, μ) also satisfies the regularity properties*

$$\partial_t y \in L^\infty(0,T;V_A^{-r}) \cap L^2(0,T;V_B^\sigma) \quad \textit{and} \quad \mu \in L^\infty(0,T;V_A^r) \quad \textit{if } \tau \ge 0, \tag{2.44}$$
$$\partial_t y \in L^\infty(0,T;H) \quad \textit{and} \quad \mu \in L^\infty(0,T;V_A^{2r}) \quad \textit{if } \tau > 0, \tag{2.45}$$

as well as the estimate

$$\|\partial_t y\|_{L^\infty(0,T;V_A^{-r}) \cap L^2(0,T;V_B^\sigma)} + \|\mu\|_{L^\infty(0,T;V_A^r)} + \|\tau^{1/2}\partial_t y\|_{L^\infty(0,T;H)} + \|\tau^{1/2}\mu\|_{L^\infty(0,T;V_A^{2r})} \le K_3, \tag{2.46}$$

with a constant K_3 that depends only on the structure of the system, the norms of the data, the width δ_0 satisfying (2.38) *if $\lambda_1 = 0$, the constant M_0 satisfying* (2.42) *if $\tau = 0$, and T.*

The remainder of the paper is organized as follows. The next section collects some notations and tools that will prove to be useful in the sequel. The uniqueness and continuous dependence result is proved in Section 4, while the existence of a solution and its regularity are proved in the last two Sections 6 and 7 and are prepared by the study of the approximating problem introduced in Section 5.

3 Auxiliary material

Here, we add some comments on the spaces defined in the previous section. Moreover, we introduce some new spaces and operators, as well as some notations and properties concerning interpolating functions. First of all, we stress the following facts:

$$\text{The embeddings } V_A^{r_2} \subset V_A^{r_1} \subset H \text{ are dense and compact for } 0 < r_1 < r_2. \tag{3.1}$$
$$\text{The embeddings } H \subset V_A^{-r_1} \subset V_A^{-r_2} \text{ are dense and compact for } 0 < r_1 < r_2. \tag{3.2}$$
$$\text{The embeddings } V_B^{\sigma_2} \subset V_B^{\sigma_1} \subset H \text{ are dense and compact for } 0 < \sigma_1 < \sigma_2. \tag{3.3}$$

Let us comment on just the first embedding of (3.1), since the second one and (3.3) are similar and (3.2) follows as a consequence of (3.1). The density is clear. For compactness, notice that $\lim_{j\to\infty}\lambda_j^{r_1-r_2}=0$, so that the mapping that to each $\{c_j\}\in\ell^2$ associates $\{\lambda_j^{r_1-r_2}c_j\}$ is compact from ℓ^2 into itself.

From the continuous embedding $H\subset V_A^{-r}$ and the compact embedding $V_B^\sigma\subset H$ given by (3.2)–(3.3), it follows that, for every $\delta>0$, there exists a constant c_δ such that

$$\|v\|^2\leq\delta\,\|B^\sigma v\|^2+c_\delta\|v\|^2_{V_A^{-r}}\quad\text{for every } v\in V_B^\sigma. \tag{3.4}$$

Proposition 3.1. *The norms* (2.8) *and* (2.17) *on* V_A^r *are equivalent.*

Proof. Take any $v\in H$. Then, v can be represented in the form

$$v=\sum_{j=1}^\infty c_je_j\quad\text{in } H,\quad\text{where } c_j:=(v,e_j)\ \text{ for all } j\in\mathbb{N},$$

and where the sequence $\{c_j\}_{j\geq1}$ belongs to ℓ^2. On the other hand, by the definition of V_A^r, A^r and $\|\cdot\|_{gr,A,r}$, we have, for every $v\in H$,

$$v\in V_A^r\quad\text{if and only if}\quad\sum_{j=1}^\infty|\lambda_j^rc_j|^2<+\infty,\quad\text{and}\quad\|v\|^2_{gr,A,r}=\|v\|^2+\sum_{j=1}^\infty|\lambda_j^rc_j|^2.$$

Therefore, by recalling (2.17), we conclude that

$$\|v\|_{A,r}\leq\|v\|_{gr,A,r}.$$

Now, suppose that $\lambda_1>0$. Then we have

$$\lambda_1^{2r}\|v\|^2=\lambda_1^{2r}\sum_{j=1}^\infty|c_j|^2\leq\sum_{j=1}^\infty|\lambda_j^rc_j|^2=\|A^rv\|^2,$$

since $\lambda_j\geq\lambda_1$ for every j, whence immediately

$$\|v\|^2_{gr,A,r}=\|v\|^2+\|A^rv\|^2\leq\Bigl(\frac{1}{\lambda_1^{2r}}+1\Bigr)\|A^rv\|^2=\Bigl(\frac{1}{\lambda_1^{2r}}+1\Bigr)\|v\|^2_{A,r}\,.$$

If, instead, $\lambda_1=0$, then we recall that $A^r(1)=0$ and thus

$$\begin{aligned}&\|v\|^2_{gr,A,r}\leq2\|\mathrm{mean}\,v\|^2_{gr,A,r}+2\|v-\mathrm{mean}\,v\|^2_{gr,A,r}\\&=2\|\mathrm{mean}\,v\|^2+2\bigl(\|v-\mathrm{mean}\,v\|^2+\|A^r(v-\mathrm{mean}\,v)\|^2\bigr)\\&=2|\Omega|\,|\,\mathrm{mean}\,v|^2+2\|v-\mathrm{mean}\,v\|^2+2\|A^rv\|^2\,.\end{aligned}$$

Thus, on accout of Remark 2.4, the desired inequality follows if we prove that, for some constant $\widehat{c}>0$, it holds the Poincaré type inequality

$$\|v\|\leq\widehat{c}\,\|A^rv\|\quad\text{for every } v\in V_A^r \text{ with } \mathrm{mean}\,v=0. \tag{3.5}$$

This is an easy consequence of the compact embedding $V_A^r\subset H$ (see (3.1)). However, we prove it for the reader's convenience. By contradiction, there exists a sequence $\{v_n\}$ in V_A^r satisfying

$$\|v_n\|>n\|A^rv_n\|\quad\text{and}\quad\mathrm{mean}\,v_n=0\quad\text{for every } n\geq1.$$

Clearly, we have that $v_n \neq 0$, so that we can define $w_n := v_n/\|v_n\|$. Then, $\|w_n\| = 1$, $\|A^r w_n\| < 1/n$ and $\operatorname{mean} w_n = 0$ for every n. In particular, $\{w_n\}$ is bounded in V_A^r, whence we have

$$w_{n_k} \to w \quad \text{weakly in } V_A^r$$

for some subsequence and some $w \in V_A^r$. By the compact embedding $V_A^r \subset H$, we infer that w_{n_k} converges to w strongly in H, whence $\|w\| = 1$ and $\operatorname{mean} w = 0$. On the other hand, we also have that $A^r w = 0$ since $\|A^r w_n\| < 1/n$ for every n. Therefore, w is a constant. Hence, the above conclusions $\|w\| = 1$ and $\operatorname{mean} w = 0$ yield a contradiction. □

At this point, we introduce the Riesz isomorphism $\mathcal{R}_r : V_A^r \to V_A^{-r}$ associated with the inner product (2.18), which acts as follows:

$$\langle \mathcal{R}_r v, w \rangle_{A,r} = (v, w)_{A,r} \quad \text{for every } v, w \in V_A^r. \tag{3.6}$$

Moreover, we set

$$V_0^r := V_A^r \quad \text{and} \quad V_0^{-r} := V_A^{-r} \quad \text{if } \lambda_1 > 0,$$
$$V_0^r := \{v \in V_A^r : \ \operatorname{mean} v = 0\} \quad \text{and} \quad V_0^{-r} := \{v \in V_A^{-r} : \ \langle v, 1 \rangle_{A,r} = 0\} \quad \text{if } \lambda_1 = 0\,. \tag{3.7}$$

Proposition 3.2. *The Riesz isomorphism $\mathcal{R}_r$ maps V_0^r onto V_0^{-r}. Moreover, $\mathcal{R}_r$ extends to V_0^r the restriction of A^{2r} to V_0^{2r}.*

Proof. Let us deal with the first assertion. If $\lambda_1 > 0$, there is nothing to prove. Thus, assume that $\lambda_1 = 0$. Then, on account of Remark 2.4, we have that

$$\langle \mathcal{R}_r v, w \rangle_{A,r} = (v, e_1)(w, e_1) + (A^r v, A^r w) = |\Omega|\,(\operatorname{mean} v)(\operatorname{mean} w) + (A^r v, A^r w)$$

for every $v, w \in V_A^r$. In particular, if $v \in V_0^r$, then we have $\operatorname{mean} v = 0$. Moreover, $A^r(1) = 0$ since $\lambda_1 = 0$. Hence,

$$\langle \mathcal{R}_r v, 1 \rangle_{A,r} = (A^r v, A^r(1)) = 0 \quad \text{for every } w \in V_A^r.$$

This shows that $\mathcal{R}_r v \in V_0^{-r}$ for every $v \in V_0^r$. Now, we fix any $f \in V_0^{-r}$ and prove that the element $v := \mathcal{R}_r^{-1} f$ of V_A^r belongs to V_0^r. We have, indeed,

$$0 = \langle f, 1 \rangle_{A,r} = \langle \mathcal{R}_r v, 1 \rangle_{A,r} = |\Omega|\,(\operatorname{mean} v)(\operatorname{mean} 1) + (A^r v, A^r(1)) = |\Omega| \ \operatorname{mean} v.$$

This concludes the proof of the first assertion of the statement. The second one means that, for every $v \in V_0^{2r}$, the elements $\mathcal{R}_r v \in V_A^{-r}$ and $A^{2r} v \in H$ coincide in the sense of the embedding $H \subset V_A^{-r}$. Thus, we fix $v \in V_0^{2r}$ and $w \in V_A^r$. In both cases $\lambda_1 > 0$ and $\lambda_1 = 0$ (in the latter since $\operatorname{mean} v = 0$), we have by the definition (2.18) of the inner product that

$$\langle \mathcal{R}_r v, w \rangle_{A,r} = (A^r v, A^r w) = \sum_{j=1}^{\infty} \big(\lambda_j^r (v, e_j)\big)\big(\lambda_j^r (w, e_j)\big)$$
$$= \sum_{j=1}^{\infty} \big(\lambda_j^{2r} (v, e_j)\big)(w, e_j) = (A^{2r} v, w) = \langle A^{2r} v, w \rangle_{A,r}\,.$$

As $w \in V_A^r$ is arbitrary, we conclude that $\mathcal{R}_r v = A^{2r} v$. □

Due to the above result, it is reasonable to use a proper notation for the restrictions of $\mathcal{R}_r$ and $\mathcal{R}_r^{-1}$ to the subspaces V_0^r and V_0^{-r}, respectively. We set

$$A_0^{2r} := (\mathcal{R}_r)_{|V_0^r} \quad \text{and} \quad A_0^{-2r} := (\mathcal{R}_r^{-1})_{|V_0^{-r}}\,, \tag{3.8}$$

where the index 0 means nothing if $\lambda_1 > 0$ (since then $V_0^{\pm r} = V_A^{\pm r}$), while it reminds the zero mean value condition in the case $\lambda_1 = 0$. We thus have

$$A_0^{2r} \in \mathcal{L}(V_0^r, V_0^{-r}), \quad A_0^{-2r} \in \mathcal{L}(V_0^{-r}, V_0^r) \quad \text{and} \quad A_0^{-2r} = (A_0^{2r})^{-1}\,, \tag{3.9}$$

$$\langle A_0^{2r} v, w\rangle_{A,r} = (v,w)_{A,r} = (A^r v, A^r w) \quad \text{for every } v \in V_0^r \text{ and } w \in V_A^r\,, \tag{3.10}$$

$$\langle f, A_0^{-2r} f\rangle_{A,r} = \|A_0^{-2r} f\|_{A,r}^2 = \|f\|_{A,-r}^2 \quad \text{for every } f \in V_0^{-r}. \tag{3.11}$$

Notice that (3.11) implies that

$$\langle f', A_0^{-2r} f\rangle_{A,r} = \frac{1}{2}\,\frac{d}{dt}\,\|f\|_{A,-r}^2 \quad \text{a.e. in } (0,T), \text{ for every } f \in H^1(0,T;V_0^{-r}). \tag{3.12}$$

Proposition 3.3. *We have*

$$\big(A^r A_0^{-2r} f, A^r v\big) = \langle f, v\rangle_{A,r} \quad \textit{for every } f \in V_0^{-r} \textit{ and } v \in V_A^r. \tag{3.13}$$

Proof. We first notice that $(e_i, e_j)_{A,r} = (\lambda_i^r e_i, \lambda_j^r e_j) = \lambda_j^{2r}\delta_{ij}$ for $i,j \geq 2$, so that the system $\{\lambda_j^{-r} e_j\}_{j\geq 2}$ is orthonormal in V_A^r. It follows that

$$\begin{aligned}&\text{the series} \quad \textstyle\sum_{j=2}^\infty c_j e_j = \sum_{j=2}^\infty (\lambda_j^r c_j)(\lambda_j^{-r} e_j) \quad \text{converges in } V_A^r\\ &\text{if and only if} \quad \textstyle\sum_{j=2}^\infty |\lambda_j^r c_j|^2 < +\infty \quad \text{or} \quad \sum_{j=1}^\infty |\lambda_j^r c_j|^2 < +\infty\,.\end{aligned}$$

On the other hand, if $v \in V_A^r$, we have both

$$\sum_{j=1}^\infty |\lambda_j^r (v,e_j)|^2 < +\infty \quad \text{and} \quad \sum_{j=1}^\infty (v,e_j)e_j = v \quad \text{in } H.$$

We conclude that

$$\sum_{j=1}^\infty (v,e_j)e_j = v \quad \text{in } V_A^r \quad \text{for every } v \in V_A^r. \tag{3.14}$$

In particular, if we set, for convenience,

$$j_0 = 1 \quad \text{if } \lambda_1 > 0 \quad \text{and} \quad j_0 = 2 \quad \text{if } \lambda_1 = 0, \tag{3.15}$$

then we have that

$$\sum_{j=j_0}^\infty (v,e_j)e_j = v \quad \text{in } V_0^r \quad \text{for every } v \in V_0^r. \tag{3.16}$$

Next, notice that $e_j \in V_A^{2r} \cap V_0^r = V_0^{2r}$ if $j \geq j_0$, whence, by Proposition 3.2,

$$A_0^{2r} e_j = \mathcal{R}_r e_j = A^{2r} e_j = \lambda_j^{2r} e_j \quad \text{for every } j \geq j_0.$$

Now, take any $f \in V_0^{-r}$ and set $z := A_0^{-2r} f$. Then $z \in V_0^r$ so that (3.16) holds for z. Therefore, since $A_0^{2r} \in \mathcal{L}(V_0^r, V_0^{-r})$, we deduce that

$$f = A_0^{2r} z = \sum_{j=j_0}^\infty (z,e_j) A_0^{2r} e_j = \sum_{j=j_0}^\infty \lambda_j^{2r} (z,e_j) e_j \quad \text{in } V_0^{-r},$$

whence also

$$\langle f, e_i\rangle_{A,r} = \sum_{j=j_0}^{\infty} \lambda_j^{2r}(z, e_j)\langle e_j, e_i\rangle_{A,r} = \sum_{j=j_0}^{\infty} \lambda_j^{2r}(z, e_j)(e_j, e_i) = \lambda_i^{2r}(z, e_i) \quad \text{for every } i \ge j_0.$$

Hence, the above series expansion becomes

$$f = \sum_{j=j_0}^{\infty} \langle f, e_j\rangle_{A,r}\, e_j \quad \text{in } V_0^{-r} \quad \text{for every } f \in V_0^{-r}. \tag{3.17}$$

At this point, we can easily conclude. Indeed, on the one side, the formulas (3.17) and (3.14), combined with $A_0^{-2r} \in \mathcal{L}(V_0^{-r}, V_0^r)$ and $A^r \in \mathcal{L}(V_A^r, H)$, ensure that

$$\begin{aligned}
&\left(A^r A_0^{-2r} f, A^r v\right) = \Big(A^r A_0^{-2r} \textstyle\sum_{j=j_0}^{\infty} \langle f, e_j\rangle_{A,r} e_j, A^r \sum_{j=1}^{\infty} (v, e_j) e_j\Big) \\
&= \Big(\textstyle\sum_{j=j_0}^{\infty} \langle f, e_j\rangle_{A,r}\, A^r A_0^{-2r} e_j, \sum_{j=1}^{\infty} (v, e_j) A^r e_j\Big) \\
&= \Big(\textstyle\sum_{j=j_0}^{\infty} \langle f, e_j\rangle_{A,r} \lambda_j^{-r} e_j, \sum_{j=1}^{\infty} (v, e_j) \lambda_j^r e_j\Big) = \textstyle\sum_{j=j_0}^{\infty} \langle f, e_j\rangle_{A,r} (v, e_j) \\
&= \langle f, \textstyle\sum_{j=j_0}^{\infty} (v, e_j) e_j\rangle_{A,r} \quad \text{for every } f \in V_0^{-r},\ \ v \in V_A^r.
\end{aligned}$$

On the other hand, the last expression is equal to $\langle f, v\rangle_{A,r}$ in both the cases $\lambda_1 > 0$ and $\lambda_1 = 0$, since the assumption $f \in V_0^{-r}$ implies that $\langle f, 1\rangle_{A,r} = 0$ in the latter. □

Proposition 3.4. *The operator $A^{2r} \in \mathcal{L}(V_A^{2r}, H)$ can be extended in a unique way to a continuous linear operator, still termed A^{2r}, from V_A^r into V_0^{-r}. Moreover,*

$$\|A^{2r} v\|_{A,-r} \le \|A^r v\| \quad \textit{for every } v \in V_A^r. \tag{3.18}$$

Proof. For $v \in V_A^{2r}$ and $w \in V_A^r$, we have that

$$\begin{aligned}
\langle A^{2r} v, w\rangle_{A,r} &= (A^{2r} v, w) = \sum_{j=1}^{\infty} (\lambda_j^{2r}(v, e_j))(w, e_j) \\
&= \sum_{j=1}^{\infty} (\lambda_j^r(v, e_j))(\lambda_j^r(w, e_j)) = (A^r v, A^r w) \le \|v\|_{A,r} \|w\|_{A,r}\,.
\end{aligned}$$

We deduce that

$$\|A^{2r} v\|_{A,-r} \le \|v\|_{A,r} \quad \text{for every } v \in V_A^{2r}.$$

This shows that the mapping $V_A^{2r} \ni v \mapsto A^{2r} v \in V_A^{-r}$ is continuous if V_A^{2r} is endowed with the topology induced by V_A^r. On the other hand, V_A^{2r} is dense in V_A^r (see (3.1)). Thus, the existence of a unique extension $A^{2r} \in \mathcal{L}(V_A^r, V_A^{-r})$ follows, and we have

$$\langle A^{2r} v, w\rangle_{A,r} = (A^r v, A^r w) \quad \text{for every } v, w \in V_A^r. \tag{3.19}$$

We immediately infer that

$$|\langle A^{2r} v, w\rangle_{A,r}| \le \|A^r v\|\, \|A^r w\| \le \|A^r v\|\, \|w\|_{A,r} \quad \text{for every } v, w \in V_A^r,$$

whence (3.18) clearly follows. Thus, it remains to verify that $A^{2r} v \in V_0^{-r}$ for every $v \in V_A^r$ if $\lambda_1 = 0$ (since there is nothing to prove if $\lambda_1 > 0$). For every $v \in V_A^r$, we have

$$\langle A^{2r} v, 1\rangle_{A,r} = (A^r v, A^r(1)) = 0,$$

since $\lambda_1 = 0$ implies that $A^r(1) = 0$ by (2.15). Hence, it turns out that $A^{2r} v \in V_0^{-r}$, as claimed. □

Proposition 3.5. *For every $f \in V_A^{-r}$, we have the representations*

$$\|f\|_{A,-r}^2 = \sum_{j=1}^{\infty} |\lambda_j^{-r} \langle f, e_j \rangle_{A,r}|^2 \quad \textit{if } \lambda_1 > 0, \tag{3.20}$$

$$\|f\|_{A,-r}^2 = |\langle f, e_1 \rangle_{A,r}|^2 + \sum_{j=2}^{\infty} |\lambda_j^{-r} \langle f, e_j \rangle_{A,r}|^2 \quad \textit{if } \lambda_1 = 0. \tag{3.21}$$

Proof. Assume first that $\lambda_1 > 0$ and set $w := \mathcal{R}_r^{-1} f$. Then the definition (2.17) yields that

$$\|w\|_{A,r}^2 = \sum_{j=1}^{\infty} |\lambda_j^r (w, e_j)|^2 \,.$$

On the other hand, by the definition of the Riesz operator $\mathcal{R}_r$, we have that

$$\langle \mathcal{R}_r w, v \rangle_{A,r} = (w, v)_{A,r} = (A^r w, A^r v) = \sum_{j=1}^{\infty} \lambda_j^{2r} (w, e_j)(v, e_j) \quad \text{for every } v \in V_A^r.$$

In particular, it also holds the identity

$$\langle f, e_i \rangle_{A,r} = \lambda_i^{2r} (w, e_i) \quad \text{for every } i \geq 1.$$

Therefore, by recalling (2.17), we deduce that

$$\|f\|_{A,-r}^2 = \|w\|_{A,r}^2 = \sum_{j=1}^{\infty} |\lambda_j^r (w, e_j)|^2 = \sum_{j=1}^{\infty} |\lambda_j^r \, \lambda_j^{-2r} \langle f, e_j \rangle_{A,r}|^2 = \sum_{j=1}^{\infty} |\lambda_j^{-r} \langle f, e_j \rangle_{A,r}|^2 \,,$$

that is, (3.20) is valid. If, instead, $\lambda_1 = 0$, then the same calculation with λ_1 replaced by 1 yields that

$$\|f\|_{A,-r}^2 = |(w, e_1)|^2 + \sum_{j=2}^{\infty} |\lambda_j^{-r} \langle f, e_j \rangle_{A,r}|^2.$$

On the other hand, we have that

$$\langle f, e_1 \rangle_{A,r} = \langle \mathcal{R}_r w, e_1 \rangle_{A,r} = (w, e_1)_{A,r} = (w, e_1)(e_1, e_1) + (A^r w, A^r e_1) = (w, e_1),$$

since $A^r e_1 = 0$. Therefore, (3.21) follows as well. □

Proposition 3.6. *For every $\eta > 0$ and $v \in V_A^\eta$, there holds the interpolation inequality*

$$\|v\| \leq \|v\|_{A,\eta}^{\vartheta} \, \|v\|_{A,-r}^{1-\vartheta}, \quad \textit{where} \quad \vartheta = \frac{r}{r+\eta} \,. \tag{3.22}$$

Proof. Set $c_j := (v, e_j)$ for $j \geq 1$, for brevity, and first assume that $\lambda_1 > 0$. Then we have

$$\|v\|_{A,\eta}^2 = \sum_{j=1}^{\infty} |\lambda_j^\eta c_j|^2 \quad \text{and} \quad \|v\|_{A,-r}^2 = \sum_{j=1}^{\infty} |\lambda_j^{-r} c_j|^2,$$

thanks to (3.20). Therefore, by using the Hölder inequality for infinite sums and noticing that $(1-\vartheta)r/\vartheta = \eta$, we find that

$$\begin{aligned}\|v\|^2 &= \sum_{j=1}^{\infty} c_j^2 = \sum_{j=1}^{\infty} \lambda_j^{2(1-\vartheta)r} c_j^{2\vartheta} \lambda_j^{-2(1-\vartheta)r} c_j^{2(1-\vartheta)} \\ &\le \Big(\sum_{j=1}^{\infty} |\lambda_j^{2(1-\vartheta)r} c_j^{2\vartheta}|^{\frac{1}{\vartheta}}\Big)^{\vartheta} \Big(\sum_{j=1}^{\infty} |\lambda_j^{-2(1-\vartheta)r} c_j^{2(1-\vartheta)}|^{\frac{1}{1-\vartheta}}\Big)^{1-\vartheta} \\ &= \Big(\sum_{j=1}^{\infty} |\lambda_j^{\eta} c_j|^2\Big)^{\vartheta} \Big(\sum_{j=1}^{\infty} |\lambda_j^{-r} c_j|^2\Big)^{1-\vartheta} = \|v\|_{A,\eta}^{2\vartheta}\, \|v\|_{A,-r}^{2(1-\vartheta)} .\end{aligned}$$

Assume now that $\lambda_1 = 0$. Then the same calculation with λ_1 replaced by 1 yields that

$$\|v\|^2 = \sum_{j=1}^{\infty} c_j^2 \le \Big(c_1^2 + \sum_{j=2}^{\infty} |\lambda_j^{\eta} c_j|^2\Big)^{\vartheta} \Big(c_1^2 + \sum_{j=2}^{\infty} |\lambda_j^{-r} c_j|^2\Big)^{1-\vartheta} = \|v\|_{A,\eta}^{2\vartheta}\, \|v\|_{A,-r}^{2(1-\vartheta)}.$$

Hence, the inequality (3.22) holds true in any case. □

Remark 3.7. By simply applying the above result and owing to the Hölder inequality, we deduce that

$$\|v\|_{L^2(0,T;H)} \le \|v\|_{L^2(0,T;V_A^{\eta})}^{\vartheta}\, \|v\|_{L^2(0,T;V_A^{-r})}^{1-\vartheta} \quad \text{for every } v \in L^2(0,T;V_A^{\eta}), \tag{3.23}$$

with the same ϑ as in (3.22).

Now, we introduce some notations concerning interpolating functions.

Notation 3.8. Let N be a positive integer and Z be one of the spaces H, V_A^r, V_B^σ. We set $h := T/N$ and $I_n := ((n-1)h, nh)$ for $n = 1, \dots, N$. Given $z = (z_0, z_1, \dots, z_N) \in Z^{N+1}$, we define the piecewise constant and piecewise linear interpolants

$$\overline{z}_h \in L^\infty(0,T;Z), \quad \underline{z}_h \in L^\infty(0,T;Z) \quad \text{and} \quad \widehat{z}_h \in W^{1,\infty}(0,T;Z)$$

by setting

$$\overline{z}_h(t) = z^n \quad \text{and} \quad \underline{z}_h(t) = z^{n-1} \quad \text{for a.a. } t \in I_n,\ n = 1, \dots, N, \tag{3.24}$$

$$\widehat{z}_h(0) = z_0 \quad \text{and} \quad \partial_t \widehat{z}_h(t) = \frac{z^n - z^{n-1}}{h} \quad \text{for a.a. } t \in I_n,\ n = 1, \dots, N. \tag{3.25}$$

For the reader's convenience, we summarize the relations between the finite set of values and the interpolants in the following proposition, whose proof follows from straightforward computation:

Proposition 3.9. *With Notation 3.8, we have that*

$$\|\overline{z}_h\|_{L^\infty(0,T;Z)} = \max_{n=1,\dots,N} \|z^n\|_Z\,, \quad \|\underline{z}_h\|_{L^\infty(0,T;Z)} = \max_{n=0,\dots,N-1} \|z^n\|_Z\,, \tag{3.26}$$

$$\|\partial_t \widehat{z}_h\|_{L^\infty(0,T;Z)} = \max_{n=0,\dots,N-1} \|(z^{n+1} - z^n)/h\|_Z\,, \tag{3.27}$$

$$\|\overline{z}_h\|_{L^2(0,T;Z)}^2 = h \sum_{n=1}^{N} \|z^n\|_Z^2\,, \quad \|\underline{z}_h\|_{L^2(0,T;Z)}^2 = h \sum_{n=0}^{N-1} \|z^n\|_Z^2\,, \tag{3.28}$$

$$\|\partial_t\widehat{z}_h\|^2_{L^2(0,T;Z)} = h\sum_{n=0}^{N-1}\|(z^{n+1}-z^n)/h\|^2_Z\,, \tag{3.29}$$

$$\|\widehat{z}_h\|_{L^\infty(0,T;Z)} = \max_{n=1,\dots,N}\max\{\|z^{n-1}\|_Z,\|z^n\|_Z\} = \max\{\|z_0\|_Z,\|\overline{z}_h\|_{L^\infty(0,T;Z)}\}\,, \tag{3.30}$$

$$\|\widehat{z}_h\|^2_{L^2(0,T;Z)} \le h\sum_{n=1}^{N}\bigl(\|z^{n-1}\|^2_Z+\|z^n\|^2_Z\bigr) \le h\|z_0\|^2_Z + 2\|\overline{z}_h\|^2_{L^2(0,T;Z)}\,. \tag{3.31}$$

Moreover, it holds that

$$\|\overline{z}_h-\widehat{z}_h\|_{L^\infty(0,T;Z)} = \max_{n=0,\dots,N-1}\|z^{n+1}-z^n\|_Z = h\,\|\partial_t\widehat{z}_h\|_{L^\infty(0,T;Z)}\,, \tag{3.32}$$

$$\|\overline{z}_h-\widehat{z}_h\|^2_{L^2(0,T;Z)} = \frac{h}{3}\sum_{n=0}^{N-1}\|z^{n+1}-z^n\|^2_Z = \frac{h^2}{3}\,\|\partial_t\widehat{z}_h\|^2_{L^2(0,T;Z)}\,, \tag{3.33}$$

and similar identities for the difference $\underline{z}_h-\widehat{z}_h$. As a consequence, we have the inequalities

$$\|\overline{z}_h-\underline{z}_h\|_{L^\infty(0,T;Z)} \le 2h\,\|\partial_t\widehat{z}_h\|_{L^\infty(0,T;Z)}\,, \tag{3.34}$$

$$\|\overline{z}_h-\underline{z}_h\|^2_{L^2(0,T;Z)} \le \frac{4h^2}{3}\,\|\partial_t\widehat{z}_h\|^2_{L^2(0,T;Z)}\,. \tag{3.35}$$

Finally, we have that

$$h\sum_{n=0}^{N-1}\|(z^{n+1}-z^n)/h\|^2_Z \le \|\partial_t z\|^2_{L^2(0,T;Z)}$$
$$\textit{if } z\in H^1(0,T;Z) \quad \textit{and} \quad z^n = z(nh) \textit{ for } n=0,\dots,N. \tag{3.36}$$

Throughout the paper, we make use of the elementary identity and inequalities

$$a(a-b) = \frac{1}{2}\,a^2+\frac{1}{2}\,(a-b)^2-\frac{1}{2}\,b^2 \ge \frac{1}{2}\,a^2-\frac{1}{2}\,b^2 \quad \text{for every } a,b\in\mathbb{R}, \tag{3.37}$$

$$ab \le \delta a^2+\frac{1}{4\delta}\,b^2 \quad \text{for every } a,b\in\mathbb{R} \text{ and } \delta>0, \tag{3.38}$$

and quote (3.38) as the Young inequality. We also take advantage of the summation by parts formula

$$\sum_{n=0}^{k-1}a_{n+1}(b_{n+1}-b_n) = a_kb_k-a_1b_0-\sum_{n=1}^{k-1}(a_{n+1}-a_n)b_n\,, \tag{3.39}$$

which is valid for arbitrary real numbers $a_1,\dots,a_k$ and $b_0,\dots,b_k$. We also account for the discrete Gronwall lemma in the following form (see, e.g., [31, Prop. 2.2.1]): for nonnegative real numbers M and $a_n, b_n, n=0,\dots,N$,

$$a_k \le M+\sum_{n=0}^{k-1}b_na_n \quad \text{for } k=0,\dots,N \qquad \text{implies}$$
$$a_k \le M\exp\Bigl(\sum_{n=0}^{k-1}b_n\Bigr) \quad \text{for } k=0,\dots,N. \tag{3.40}$$

In (3.39)–(3.40) it is understood that a sum vanishes if the corresponding set of indices is empty.

Finally, we state a general rule that we follow throughout the paper as far as the constants are concerned. We always use a small-case italic c without subscripts for different constants that may only depend on the final time T, the operators A^r and B^σ, the shape of the nonlinearities β and π, and the properties of the data involved in the statements at hand. Thus, the values of such constants do not depend on τ, nor on the regularization parameter λ or the time step h we introduce in Section 5, and it is clear that they might change from line to line and even in the same formula or chain of inequalities. In contrast, we use different symbols (e.g., capital letters like M_0 in (2.42)) for precise values of constants we want to refer to.

4 Continuous dependence and uniqueness

This section is devoted to the proof of the uniqueness and the continuous dependence stated in Theorem 2.6. More precisely, we prove just the continuous dependence, since uniqueness follows as a consequence. Moreover, we consider only the case of the same initial datum, for simplicity. However, the case of different initial data sketched in Remark 2.7 could be treated in the same way with only minor changes.

We pick two data u_i, $i = 1, 2$, and the corresponding solutions (y_i, μ_i), and set for convenience $u := u_1 - u_2$, $y := y_1 - y_2$ and $\mu := \mu_1 - \mu_2$. Now, we write equation (2.33) at the time s for these solutions and take the difference. Then, we test it by $v = A_0^{-2r} y(s)$ by observing that $y(s) \in V_0^{-r}$ since $y \in L^2(0, T; H)$ and $\operatorname{mean} y(s) = 0$ if $\lambda_1 = 0$ by the conservation property (2.37), so that v is a well-defined element of V_A^r. Moreover, $A_0^{-2r} y \in L^\infty(0, T; V_A^r)$, since $y \in L^\infty(0, T; V_A^{-r})$ by (2.30). Integrating over $(0, t)$ with respect to s, where $t \in (0, T)$ is arbitrary, we obtain the identity

$$\int_0^t \langle \partial_t y(s), A_0^{-2r} y(s) \rangle_{A,r} \, ds + \int_0^t \big(A^r \mu(s), A^r A_0^{-2r} y(s) \big) \, ds = 0 \, .$$

Now, we apply (3.12) and (3.13), noting that $\mu \in L^2(0, T; V_A^r)$. Thus, the above identity becomes

$$\frac{1}{2} \, \|y(t)\|_{A,-r}^2 + \int_0^t (y(s), \mu(s)) \, ds = 0 \tag{4.1}$$

where the duality product of (3.13) has been replaced by the inner product here, since both y and μ are H-valued. At the same time, we write (2.34) for u_i and (y_i, μ_i), $i = 1, 2$, test them by y_2 and y_1, respectively, add the resulting inequalities to each other, and integrate over $(0, t)$ as before. Then, the terms involving $\widehat{\beta}$ cancel out, and we obtain (after rearranging) that

$$\begin{aligned} &\frac{\tau}{2} \, \|y(t)\|^2 + \int_0^t \|B^\sigma y(s)\|^2 \, ds - \int_0^t \big(\mu(s), y(s) \big) \, ds \\ &\leq \int_0^t \big(u(s), y(s) \big) \, ds \; - \int_0^t \big(\pi(y_1(s)) - \pi(y_2(s)), y(s) \big) \, ds \, . \end{aligned}$$

By adding this to (4.1), and accounting for the Lipschitz continuity of π and the Schwarz and Young inequalities, we deduce that (with L'_π given by (2.25))

$$\frac{1}{2} \, \|y(t)\|_{A,-r}^2 + \frac{\tau}{2} \, \|y(t)\|^2 + \int_0^t \|B^\sigma y(s)\|^2 \, ds \leq \frac{1}{4} \int_0^t \|u(s)\|^2 \, ds + L'_\pi \int_0^t \|y(s)\|^2 \, ds. \tag{4.2}$$

At this point, we recall the compacness inequality (3.4). Thus, we have that

$$L'_\pi \int_0^t \|y(s)\|^2\,ds \le \frac{1}{2}\int_0^t \|B^\sigma y(s)\|^2\,ds + c\int_0^t \|y(s)\|_{A,-r}^2\,ds.$$

By combining this with (4.2) and applying the Gronwall lemma, we conclude that the desired estimate (2.40) holds true with a constant K_2 as in the statement.

5 Approximation

In this section we deal with an approximation of problem (2.33)–(2.35) and solve it by a time discretization procedure. We first introduce the Moreau–Yosida regularizations $\widehat{\beta}_\lambda$ and β_λ of $\widehat{\beta}$ of β at the level $\lambda > 0$ (see, e.g., [12, p. 28 and p. 39]). By accounting for assumptions (2.22)–(2.24), we have

$$\widehat{\beta}_\lambda(s) = \int_0^s \beta_\lambda(s')\,ds' \quad \text{and} \quad 0 \le \widehat{\beta}_\lambda(s) \le \widehat{\beta}(s) \quad \text{for every } s \in \mathbb{R}. \tag{5.1}$$

$$\widehat{\beta}_\lambda(s) + \widehat{\pi}(s) \ge \alpha\, s^2 - C$$
$$\text{for some constants } \alpha, C > 0, \text{ every } s \in \mathbb{R} \text{ and } \lambda > 0 \text{ small enough.} \tag{5.2}$$

Moreover, we recall that β_λ is Lipschitz continuous, so that $\widehat{\beta}_\lambda$ grows at most quadratically, and that the following properties hold true:

$$\widehat{\beta}_{\lambda'}(s) \ge \widehat{\beta}_{\lambda''}(s) \quad \text{if } \lambda' \le \lambda'' \quad \text{and} \quad \lim_{\lambda \searrow 0} \widehat{\beta}_\lambda(s) = \widehat{\beta}(s) \quad \text{for every } s \in \mathbb{R}, \tag{5.3}$$

$$|\beta_\lambda(s)| \le |\beta^\circ(s)| \quad \text{for every } s \in D(\beta). \tag{5.4}$$

By replacing $\widehat{\beta}$ in (2.34) by $\widehat{\beta}_\lambda$, we obtain the following system:

$$\langle \partial_t y^\lambda(t), v\rangle_{A,r} + (A^r\mu^\lambda(t), A^r v) = 0 \quad \text{for every } v \in V_A^r \text{ and for a.a. } t \in (0,T), \tag{5.5}$$

$$\begin{aligned}
&\tau\big(\partial_t y^\lambda(t), y^\lambda(t) - v\big) + \big(B^\sigma y^\lambda(t), B^\sigma(y^\lambda(t) - v)\big) \\
&\quad + \int_\Omega \widehat{\beta}_\lambda(y^\lambda(t)) + \big(\pi(y^\lambda(t)) - u(t), y^\lambda(t) - v\big) \\
&\le \big(\mu^\lambda(t), y^\lambda(t) - v\big) + \int_\Omega \widehat{\beta}_\lambda(v) \quad \text{for every } v \in V_B^\sigma \text{ and for a.a. } t \in (0,T),
\end{aligned} \tag{5.6}$$

$$y^\lambda(0) = y_0. \tag{5.7}$$

We stress that (5.6) is equivalent to both the time-integrated variational inequality

$$\begin{aligned}
&\tau\int_0^T \big(\partial_t y^\lambda(t), y^\lambda(t) - v(t)\big)\,dt + \int_0^T \big(B^\sigma y^\lambda(t), B^\sigma(y^\lambda(t) - v(t))\big)\,dt \\
&\quad + \int_Q \widehat{\beta}_\lambda(y^\lambda) + \int_0^T \big(\pi(y^\lambda(t)) - u(t), y^\lambda(t) - v(t)\big)\,dt \\
&\le \int_0^T \big(\mu^\lambda(t), y^\lambda(t) - v(t)\big)\,dt + \int_Q \widehat{\beta}_\lambda(v) \quad \text{for every } v \in L^2(0,T;V_B^\sigma),
\end{aligned} \tag{5.8}$$

and the pointwise variational equation (since $\widehat{\beta}_\lambda$ is differentiable and β_λ is its derivative)

$$\big(B^\sigma y^\lambda(t), B^\sigma v\big) + \big(\beta_\lambda(y^\lambda(t)) + \pi(y^\lambda(t)) - u(t), v\big) = \big(\mu^\lambda(t), v\big)$$
$$\text{for every } v \in V_B^\sigma \text{ and for a.a. } t \in (0,T). \tag{5.9}$$

Theorem 5.1. *Under the assumptions of Theorem 2.6, problem* (5.5)–(5.7) *has a unique solution satisfying* (2.30)–(2.31).

Uniqueness follows from Theorem 2.6, since β_λ and $\widehat{\beta}_\lambda$ satisfy the properties we have postulated for β and $\widehat{\beta}$. So, we just have to prove the existence of a solution, and the remainder of the section is devoted to this proof. To this end, we solve a proper discrete problem and take the limits of the interpolants as the time step tends to zero.

The discrete problem. We fix an integer $N > 1$ and set $h := T/N$. Then, the discrete problem consists in finding two $(N+1)$-tuples $(y^0, \ldots, y^N)$ and $(\mu^0, \ldots, \mu^N)$ satisfying

$$y^0 = y_0\,, \quad \mu^0 = 0, \quad (y^1, \ldots, y^N) \in (V_B^{2\sigma})^N \quad \text{and} \quad (\mu^1, \ldots, \mu^N) \in (V_A^{2r})^N \tag{5.10}$$

and solving

$$\frac{y^{n+1} - y^n}{h} + \mu^{n+1} + A^{2r}\mu^{n+1} \;=\; \mu^n, \tag{5.11}$$

$$\tau\,\frac{y^{n+1} - y^n}{h} + (L'_\pi I + B^{2\sigma} + \beta_\lambda + \pi)(y^{n+1}) \;=\; L'_\pi y^n + \mu^{n+1} + u^{n+1}, \tag{5.12}$$

for $n = 0, 1, \ldots, N-1$, where $I : H \to H$ is the identity, L'_π is given by (2.25), and

$$u^n := u(nh) \quad \text{for } n = 0, 1, \ldots, N. \tag{5.13}$$

This problem can be solved inductively for $n = 0, \ldots, N-1$ in the following way: let (y^n, μ^n) be given in $V_B^\sigma \times V_A^{2r}$. We first rewrite the above equations in the form

$$h(I + A^{2r})\mu^{n+1} + y^{n+1} \;=\; y^n + h\mu^n, \tag{5.14}$$

$$((L'_\pi + (\tau/h))I + B^{2\sigma} + \beta_\lambda + \pi)(y^{n+1}) \;=\; (L'_\pi + (\tau/h))y^n + \mu^{n+1} + u^{n+1}. \tag{5.15}$$

Next, we observe that the operator $\mathcal{A}_\lambda := L_\pi I + \beta_\lambda + \pi : H \to H$ is monotone and continuous. On the other hand, the unbounded operator $B^{2\sigma}$ is monotone in H, and $I + B^{2\sigma}$ is surjective, whence it follows that $B^{2\sigma}$ is maximal monotone. Therefore, the sum $\mathcal{A}_\lambda + B^{2\sigma}$ is also maximal monotone (see, e.g., [6, Cor. 2.1 p. 35]). It follows that $(1 + (\tau/h))I + \mathcal{A}_\lambda + B^{2\sigma}$, i.e., the operator that acts on y^{n+1} in (5.12), is surjective and one-to-one from $V_B^{2\sigma}$ onto H. Therefore, (5.12) can be rewritten in the equivalent form

$$y^{n+1} = (L_h I + B^{2\sigma} + \beta_\lambda + \pi)^{-1}\big(L_h y^n + \mu^{n+1} + u^{n+1}\big), \tag{5.16}$$

where, for brevity, we have set $L_h := L'_\pi + (\tau/h)$. By accounting for (5.14), we conclude that problem (5.11)–(5.12) is equivalent to the system obtained by coupling (5.16) with the equation

$$h(I + A^{2r})\mu^{n+1} + (L_h I + B^{2\sigma} + \beta_\lambda + \pi)^{-1}\big(L_h y^n + \mu^{n+1} + u^{n+1}\big) \;=\; y^n + h\mu^n. \tag{5.17}$$

Arguing as before, we see that the operator acting on μ^{n+1} on the left-hand side of (5.17) is surjective and one-to-one from V_A^{2r} onto H, so that the equation can be uniquely solved for μ^{n+1} in V_A^{2r}. Inserting the solution in (5.16), we directly find that $y^{n+1} \in V_B^{2\sigma}$.

Once the discrete problem is solved, we can start estimating. According to the general rule stated at the end of Section 3, the (possibly different) values of the constants termed c are independent of the three parameters h, λ and τ. Moreover, we also express the bounds we find in terms of the interpolants. According to Notation 3.8, and recalling that $y^0 = y_0 \in V_B^\sigma$ and that $\mu^0 = 0$ (see (2.27) and (5.10)), we remark at once that the discrete problem also reads

$$\widehat{y}_h \in W^{1,\infty}(0,T;V_B^\sigma), \quad \underline{y}_h \in L^\infty(0,T;V_B^\sigma) \quad \text{and} \quad \overline{y}_h \in L^\infty(0,T;V_B^{2\sigma}), \tag{5.18}$$

$$\underline{\mu}_h\,, \overline{\mu}_h \in L^\infty(0,T;V_A^{2r}), \tag{5.19}$$

$$\partial_t \widehat{y}_h + \overline{\mu}_h + A^{2r}\overline{\mu}_h \;=\; \underline{\mu}_h \quad \text{a.e. in } (0,T), \tag{5.20}$$

$$\tau\,\partial_t \widehat{y}_h + (L_\pi' I + B^{2\sigma} + \beta_\lambda + \pi)(\overline{y}_h) \;=\; L_\pi' \underline{y}_h + \overline{\mu}_h + \overline{u}_h \quad \text{a.e. in } (0,T), \tag{5.21}$$

$$\widehat{y}_h(0) = y_0\,. \tag{5.22}$$

First a priori estimate. We test (5.11) and (5.12) (by taking the scalar product in H) by $h\mu^{n+1}$ and $y^{n+1} - y^n$, respectively, and add the resulting identities. Noting an obvious cancellation, we obtain the equation

$$\begin{aligned} & h(\mu^{n+1} - \mu^n, \mu^{n+1}) + h(A^{2r}\mu^{n+1}, \mu^{n+1}) + \frac{\tau}{h}\,\|y^{n+1} - y^n\|^2 \\ & + (B^{2\sigma}y^{n+1}, y^{n+1} - y^n) + \big((L_\pi' I + \beta_\lambda + \pi)(y^{n+1}), y^{n+1} - y^n\big) \\ & = \; L_\pi'(y^n, y^{n+1} - y^n) + (u^{n+1}, y^{n+1} - y^n). \end{aligned}$$

Now, we observe that the function $r \mapsto \frac{L_\pi'}{2}\, r^2 + \widehat{\beta}_\lambda(r) + \widehat{\pi}(r)$ is convex on $\mathbb{R}$, since $\widehat{\beta}_\lambda$ is convex and $|\pi'| \le L_\pi$. Thus, we have that

$$\begin{aligned} & \big((L_\pi' I + \beta_\lambda + \pi)(y^{n+1}), y^{n+1} - y^n\big) \\ & \ge \frac{L_\pi'}{2}\,\|y^{n+1}\|^2 + \int_\Omega \big(\widehat{\beta}_\lambda(y^{n+1}) + \widehat{\pi}(y^{n+1})\big) - \frac{L_\pi'}{2}\,\|y^n\|^2 - \int_\Omega \big(\widehat{\beta}_\lambda(y^n) + \widehat{\pi}(y^n)\big). \end{aligned}$$

By using this inequality and formulas (2.10)–(2.11), and applying the identity (3.37) in two terms on the left-hand side and in the first one on the right-hand side, we deduce that

$$\begin{aligned} & \frac{h}{2}\,\|\mu^{n+1}\|^2 + \frac{h}{2}\,\|\mu^{n+1} - \mu^n\|^2 - \frac{h}{2}\,\|\mu^n\|^2 + h\|A^r\mu^{n+1}\|^2 \\ & \quad + \frac{\tau}{h}\,\|y^{n+1} - y^n\|^2 + \frac{1}{2}\,\|B^\sigma y^{n+1}\|^2 + \frac{1}{2}\,\|B^\sigma(y^{n+1} - y^n)\|^2 - \frac{1}{2}\,\|B^\sigma y^n\|^2 \\ & \quad + \frac{L_\pi'}{2}\,\|y^{n+1}\|^2 + \int_\Omega \big(\widehat{\beta}_\lambda(y^{n+1}) + \widehat{\pi}(y^{n+1})\big) - \frac{L_\pi'}{2}\,\|y^n\|^2 - \int_\Omega \big(\widehat{\beta}_\lambda(y^n) + \widehat{\pi}(y^n)\big) \\ & \le \frac{L_\pi'}{2}\,\|y^{n+1}\|^2 - \frac{L_\pi'}{2}\,\|y^n\|^2 - \frac{L_\pi'}{2}\,\|y^{n+1} - y^n\|^2 + (u^{n+1}, y^{n+1} - y^n). \end{aligned}$$

Then, we first rearrange and then sum up for $n = 0, \dots, k-1$ with $k \le N$, employing summation by parts (see (3.39)) in the last term. Using (5.3), we then arrive at the inequality

$$\begin{aligned} & \frac{h}{2}\,\|\mu^k\|^2 + \sum_{n=0}^{k-1} \frac{h}{2}\,\|\mu^{n+1} - \mu^n\|^2 + \sum_{n=0}^{k-1} h\|A^r\mu^{n+1}\|^2 \\ & + \tau \sum_{n=0}^{k-1} h \Big\| \frac{y^{n+1} - y^n}{h} \Big\|^2 + \frac{1}{2}\,\|B^\sigma y^k\|^2 + \sum_{n=0}^{k-1} \frac{1}{2}\,\|B^\sigma(y^{n+1} - y^n)\|^2 \\ & + \int_\Omega \big(\widehat{\beta}_\lambda(y^k) + \widehat{\pi}(y^k)\big) - \int_\Omega \big(\widehat{\beta}(y_0) + \widehat{\pi}(y_0)\big) + \frac{L_\pi'}{2} \sum_{n=0}^{k-1} \|y^{n+1} - y^n\|^2 \end{aligned}$$

$$\leq (u^k, y^k) - (u^1, y_0) - \sum_{n=1}^{k-1} (u^{n+1} - u^n, y^n). \tag{5.23}$$

Now, we observe that (5.2) implies that

$$\int_\Omega \bigl(\widehat{\beta}_\lambda(y^k) + \widehat{\pi}(y^k)\bigr) \geq \frac{1}{2} \int_\Omega \bigl(\widehat{\beta}_\lambda(y^k) + \widehat{\pi}(y^k)\bigr) + \frac{\alpha}{2} \, \|y^k\|^2 - c \,,$$

for sufficiently small $\lambda > 0$. In particular, the above integral is bounded from below. We treat the right-hand side of (5.23) by using the Young and Schwarz inequalities for finite sums, as well as (3.36). We obtain

$$\begin{aligned}
&(u^k, y^k) - (u^1, y_0) - \sum_{n=1}^{k-1} (u^{n+1} - u^n, y^n) \\
&\leq \frac{\alpha}{4} \, \|y^k\|^2 + c \, \|u^k\|^2 + \|y_0\|^2 + \|u^1\|^2 + \sum_{n=1}^{k-1} h \Bigl\| \frac{u^{n+1} - u^n}{h} \Bigr\|^2 + \sum_{n=1}^{k-1} h \|y^n\|^2 \\
&\leq \frac{\alpha}{4} \, \|y^k\|^2 + \|y_0\|^2 + c \, \|u\|^2_{L^\infty(0,T;H)} + \|\partial_t u\|^2_{L^2(0,T;H)} + \sum_{n=1}^{k-1} h \|y^n\|^2.
\end{aligned}$$

By combining the last two estimates with (5.23) and (2.27) and recalling that $L'_\pi \geq 1$, we infer that

$$\begin{aligned}
&\frac{h}{2} \, \|\mu^k\|^2 + \sum_{n=0}^{k-1} \frac{h}{2} \, \|\mu^{n+1} - \mu^n\|^2 + \sum_{n=0}^{k-1} h \|A^r \mu^{n+1}\|^2 + \tau \sum_{n=0}^{k-1} h \Bigl\| \frac{y^{n+1} - y^n}{h} \Bigr\|^2 \\
&\quad + \frac{1}{2} \, \|B^\sigma y^k\|^2 + \frac{\alpha}{4} \, \|y^k\|^2 + \frac{1}{2} \int_\Omega \bigl(\widehat{\beta}_\lambda(y^k) + \widehat{\pi}(y^k)\bigr) \\
&\quad + \sum_{n=0}^{k-1} \frac{1}{2} \, \|B^\sigma (y^{n+1} - y^n)\|^2 + \frac{1}{2} \sum_{n=0}^{k-1} \|y^{n+1} - y^n\|^2 \\
&\leq \sum_{n=1}^{k-1} h \|y^n\|^2 + c \,.
\end{aligned}$$

Since this holds for $k = 0, \ldots, N$, and as the last integral on the left-hand side is bounded from below, we can apply the discrete Gronwall lemma (3.40) and conclude that

$$\begin{aligned}
&h \, \|\mu^k\|^2 + \sum_{n=0}^{k-1} \frac{h}{2} \, \|\mu^{n+1} - \mu^n\|^2 + \sum_{n=0}^{k-1} h \|A^r \mu^{n+1}\|^2 + \tau \sum_{n=0}^{k-1} h \Bigl\| \frac{y^{n+1} - y^n}{h} \Bigr\|^2 \\
&\quad + \|y^k\|^2_{B,\sigma} + \int_\Omega \bigl(\widehat{\beta}_\lambda(y^k) + \widehat{\pi}(y^k)\bigr) + \sum_{n=0}^{k-1} \|B^\sigma (y^{n+1} - y^n)\|^2 + \sum_{n=0}^{k-1} \|y^{n+1} - y^n\|^2 \\
&\leq c \quad \text{for } k = 0, \ldots, N.
\end{aligned} \tag{5.24}$$

In terms of the interpolants, by neglecting the first contribution and recalling that $\mu^0 = 0$, we have on account of Proposition 3.9 that

$$\begin{aligned}
&\|\overline{\mu}_h - \underline{\mu}_h\|_{L^2(0,T;H)} + \|A^r \overline{\mu}_h\|_{L^2(0,T;H)} + \|A^r \underline{\mu}_h\|_{L^2(0,T;H)} \\
&\quad + \|\underline{y}_h\|_{L^\infty(0,T;V_B^\sigma)} + \|\overline{y}_h\|_{L^\infty(0,T;V_B^\sigma)} + \|\widehat{y}_h\|_{L^\infty(0,T;V_B^\sigma)} \\
&\quad + \tau^{1/2} \|\partial_t \widehat{y}_h\|_{L^2(0,T;H)} + \|\widehat{\beta}_\lambda(\overline{y}_h) + \widehat{\pi}(\overline{y}_h)\|_{L^\infty(0,T;L^1(\Omega))} \\
&\quad + h^{-1/2} \|B^\sigma (\overline{y}_h - \underline{y}_h)\|_{L^2(0,T;H)} + h^{-1/2} \|\overline{y}_h - \underline{y}_h\|_{L^2(0,T;H)} \leq c \,.
\end{aligned} \tag{5.25}$$

Due to (2.23), we easily infer that $\|\widehat{\pi}(\overline{y}_h)\|_{L^\infty(0,T;L^1(\Omega))} \le c\,(\|\overline{y}_h\|^2_{L^\infty(0,T;H)}+1) \le c$, whence we deduce that

$$\|\widehat{\beta}_\lambda(\overline{y}_h)\|_{L^\infty(0,T;L^1(\Omega))} \le c\,. \tag{5.26}$$

Second a priori estimate. By recalling (5.20) and applying Proposition 3.4, we immediately obtain

$$\begin{aligned}
&\|\partial_t\widehat{y}_h\|_{L^2(0,T;V_A^{-r})} \le \|\underline{\mu}_h-\overline{\mu}_h\|_{L^2(0,T;V_A^{-r})} + \|A^{2r}\overline{\mu}_h\|_{L^2(0,T;V_A^{-r})}\\
&\le c\,\|\underline{\mu}_h-\overline{\mu}_h\|_{L^2(0,T;H)} + c\,\|A^r\overline{\mu}_h\|_{L^2(0,T;H)}\,.
\end{aligned}$$

Hence, (5.25) implies that

$$\|\partial_t\widehat{y}_h\|_{L^2(0,T;V_A^{-r})} \le c\,. \tag{5.27}$$

Consequence. By combining (5.25) and (5.27) with the application of (3.33) and its analogue to $\overline{y}_h$, $\underline{y}_h$ and $\widehat{y}_h$, we deduce that

$$\|\overline{y}_h-\widehat{y}_h\|_{L^2(0,T;V_A^{-r})} + \|\underline{y}_h-\widehat{y}_h\|_{L^2(0,T;V_A^{-r})} \le c\,h. \tag{5.28}$$

Third a priori estimate. We want to improve the estimate for $A^r\overline{\mu}_h$ given by (5.25) and show that

$$\|\overline{\mu}_h\|_{L^2(0,T;V_A^r)} + \|\underline{\mu}_h\|_{L^2(0,T;V_A^r)} \le c\,. \tag{5.29}$$

By recalling (2.15) and (2.17), we see that there is nothing to prove if $\lambda_1>0$. On the contrary, if $\lambda_1=0$, we have to estimate the mean value of $\overline{\mu}_h$. Thus, we assume $\lambda_1=0$ and first derive an estimate of $\beta_\lambda(\overline{y}_h)$. To this end, we recall that $m_0\in V_B^\sigma$ by (2.16) and that we have postulated the interior assumption (2.28). Then, we test (5.12) by $y^{n+1}-m_0$ (see (2.28)) and use the inequality

$$\beta_\lambda(s)(s-m_0) \ge \delta_0|\beta_\lambda(s)| - C_0, \tag{5.30}$$

which holds for some $C_0>0$ and every $s\in\mathbb{R}$ and $\lambda\in(0,1)$, where δ_0 is the same as in (2.38) (cf. [33, Appendix, Prop. A.1]; see also [27, p. 908] for a detailed proof). By partially using (5.24) as well, we have

$$\begin{aligned}
&\int_\Omega \bigl(\delta_0|\beta_\lambda(y^{n+1})| - C_0\bigr) \le \int_\Omega \beta_\lambda(y^{n+1})(y^{n+1}-m_0)\\
&= -\tau\Bigl(\frac{y^{n+1}-y^n}{h}, y^{n+1}-m_0\Bigr) - L'_\pi(y^{n+1}-y^n, y^{n+1}-m_0)\\
&\quad - \bigl(B^{2\sigma}y^{n+1}, y^{n+1}-m_0\bigr) - \bigl(\pi(y^{n+1}), y^{n+1}-m_0\bigr) + (\mu^{n+1}+u^{n+1}, y^{n+1}-m_0)\\
&\le c\,\tau\Bigl\|\frac{y^{n+1}-y^n}{h}\Bigr\|\,(\|y^{n+1}\|+1) + c(\|y^{n+1}\|^2+\|y^n\|^2+1) + c\,\|u^{n+1}\|\,(\|y^{n+1}\|+1)\\
&\quad + \bigl|\bigl(B^{2\sigma}y^{n+1}, y^{n+1}-m_0\bigr)\bigr| + (\mu^{n+1}, y^{n+1}-m_0)\\
&\le c\,\tau\Bigl\|\frac{y^{n+1}-y^n}{h}\Bigr\| + c\,\|u^{n+1}\| + c\\
&\quad + \bigl|\bigl(B^{2\sigma}y^{n+1}, y^{n+1}-m_0\bigr)\bigr| + (\mu^{n+1}, y^{n+1}-m_0)\,.
\end{aligned} \tag{5.31}$$

We now estimate the last two terms. For the first one, we owe to assumption (2.16) just mentioned and property (2.11). By recalling (5.24) once more, we see that

$$\bigl|\bigl(B^{2\sigma}y^{n+1}, y^{n+1}-m_0\bigr)\bigr| = \bigl|\bigl(B^\sigma y^{n+1}, B^\sigma y^{n+1} - B^\sigma m_0\bigr)\bigr| \le c\,.$$

We deal with the other term by first observing that (5.11) and the assumption $\lambda_1 = 0$ in (2.15) imply that

$$\mathrm{mean}(y^{n+1} + h\mu^{n+1}) - \mathrm{mean}(y^n + h\mu^n) = -\frac{h}{|\Omega|}\,(A^r\mu^{n+1}, A^r(1)) = 0 \quad ,$$

for $n = 0, \ldots, N-1$, whence $\mathrm{mean}(y^{n+1} + h\mu^{n+1}) = m_0$ for every n, since $\mu^0 = 0$ (see (5.10)). Hence, by taking advantage of the the Poincaré inequality (3.5) and (5.24), we obtain the estimate

$$\begin{aligned}
&(\mu^{n+1}, y^{n+1} - m_0) = (\mu^{n+1} - \mathrm{mean}\,\mu^{n+1}, y^{n+1} - m_0) + (\mathrm{mean}\,\mu^{n+1}, y^{n+1} - m_0) \\
&\le \widehat{c}\,\|A^r\mu^{n+1}\|\,\|y^{n+1} - m_0\| + (\mathrm{mean}\,\mu^{n+1}, -h\mu^{n+1}) \\
&\le c\,\|A^r\mu^{n+1}\| - |\Omega|\,h\,(\mathrm{mean}\,\mu^{n+1})^2 \le c\,\|A^r\mu^{n+1}\|\,.
\end{aligned}$$

Therefore, (5.31) becomes

$$\|\beta_\lambda(y^{n+1})\|_{L^1(\Omega)} \le c\Big(\tau\,\Big\|\frac{y^{n+1} - y^n}{h}\Big\| + \|u^{n+1}\| + \|A^r\mu^{n+1}\| + 1\Big). \tag{5.32}$$

Now, we square (5.32), multiply by h and sum up over $n = 0, \ldots, k-1$ with $k \le N$. We deduce that

$$\begin{aligned}
&\sum_{n=0}^{k-1} h\|\beta_\lambda(y^{n+1})\|^2_{L^1(\Omega)} \\
&\le c\,\tau\,h\sum_{n=0}^{k-1}\Big\|\frac{y^{n+1} - y^n}{h}\Big\|^2 + c\,h\sum_{n=0}^{k-1}\|u^{n+1}\|^2 + c\,h\sum_{n=0}^{k-1}\|A^r\mu^{n+1}\|^2 + c\,.
\end{aligned}$$

Thanks to (5.24), the right-hand side is bounded, and we conclude that

$$\|\beta_\lambda(\overline{y}_h)\|_{L^2(0,T;L^1(\Omega))} \le c\,. \tag{5.33}$$

At this point, we simply integrate (5.21) over Ω and have a.e. in $(0,T)$

$$\begin{aligned}
|\Omega|\,\mathrm{mean}\,\overline{\mu}_h \; &= \tau\int_\Omega \partial_t\widehat{y}_h + L'_\pi\int_\Omega(\overline{y}_h - \underline{y}_h) + \big(B^\sigma\overline{y}_h, B^\sigma(1)\big) \\
&\quad + \int_\Omega \beta_\lambda(\overline{y}_h) + \int_\Omega \pi(\overline{y}_h) - \int_\Omega \overline{u}_h\,.
\end{aligned}$$

Thus, $\mathrm{mean}\,\overline{\mu}_h$ is bounded in $L^2(0,T)$, thanks to (5.25), (5.33) and (2.26). This completes the proof of the desired estimate (5.29) as far as $\overline{\mu}_h$ is concerned. Since $A^r\mu^0 = A^r0 = 0$ and $\overline{\mu}_h - \underline{\mu}_h$ is bounded in $L^2(0,T;H)$ by virtue of (5.25), the same estimate holds for $\underline{\mu}_h$. Hence, (5.29) holds also in the case $\lambda_1 = 0$.

Limit. Collecting the estimates (5.25)–(5.29), and using standard weak and weak-star compactness results, we see that there are functions y^λ and μ^λ such that

$$\overline{y}_h \to y^\lambda\,, \quad \underline{y}_h \to y^\lambda\,, \quad \text{and} \quad \widehat{y}_h \to y^\lambda \quad \text{weakly star in } L^\infty(0,T;V_B^\sigma), \tag{5.34}$$

$$\partial_t\widehat{y}_h \to \partial_t y^\lambda \quad \text{weakly in } L^2(0,T;V_A^{-r}), \tag{5.35}$$

$$\partial_t\widehat{y}_h \to \partial_t y^\lambda \quad \text{weakly in } L^2(0,T;H) \quad \text{if } \tau > 0, \tag{5.36}$$

$$\overline{\mu}_h \to \mu^\lambda \quad \text{weakly in } L^2(0,T;V_A^r), \tag{5.37}$$

as $h \searrow 0$ (more precisely, as $N \to \infty$), at least for some (not relabeled) subsequence, provided that $\lambda > 0$ is small enough. By letting h tend to zero in (5.22), we see that y^λ satisfies (5.7). Now, we prove that

$$\underline{\mu}_h \to \mu^\lambda \quad \text{weakly in } L^2(0,T;V_A^r). \tag{5.38}$$

By (5.25) and (5.37), it suffices to check that

$$_{L^2(0,T;V_A^{-r})}\langle v, \overline{\mu}_h - \underline{\mu}_h \rangle_{L^2(0,T;V_A^r)} \to 0 \quad \text{as } h \searrow 0, \tag{5.39}$$

for every v belonging to a dense subspace $\mathcal{S}$ of $L^2(0,T;V_A^{-r})$, where we can take $\mathcal{S} = C_c^1(0,T;H)$ since H is dense in V_A^{-r} (see (3.2)). So, we fix $v \in C_c^1(0,T;H)$ and choose $\delta > 0$ such that $v(t) = 0$ for $t \in [0,T] \setminus (\delta, T-\delta)$. If $h \in (0, \delta/2)$, then we have

$$\begin{aligned}
&\big|_{L^2(0,T;V_A^{-r})}\langle v, \overline{\mu}_h - \underline{\mu}_h \rangle_{L^2(0,T;V_A^r)}\big| = \Big| \int_h^T (\overline{\mu}_h - \underline{\mu}_h)(t)\, v(t)\, dt \Big| \\
&= \Big| \int_h^T \big(\overline{\mu}_h(t) - \overline{\mu}_h(t-h) \big)\, v(t)\, dt \Big| = \Big| \int_h^T \overline{\mu}_h(t)\, v(t)\, dt - \int_0^{T-h} \overline{\mu}_h(t)\, v(t+h)\, dt \Big| \\
&= \Big| \int_h^{T-h} \overline{\mu}_h(t)\, (v(t) - v(t+h))\, dt \Big| \le \|\overline{\mu}_h\|_{L^2(0,T;H)}\, \|v'\|_{L^\infty(0,T;H)}\, h^{1/2} ,
\end{aligned}$$

and (5.39) follows. Therefore, (5.38) is proved and the pair (y^λ, μ^λ) solves (5.5). In order to deal with the nonlinear terms of (5.21), we owe to the compact embedding $V_B^\sigma \subset H$ (see (3.3)) and to well-known strong compactness results (see, e.g., [43, Sect. 8, Cor. 4]). From (5.34)–(5.35) we deduce that

$$\widehat{y}_h \to y^\lambda \quad \text{strongly in } L^\infty(0,T;H). \tag{5.40}$$

This and (5.25) (see the last term on the left-hand side) imply that

$$\overline{y}_h \to y^\lambda \quad \text{strongly in } L^2(0,T;H). \tag{5.41}$$

By Lipschitz continuity, we infer that also

$$(\beta_\lambda + \pi)(\overline{y}_h) \to (\beta_\lambda + \pi)(y^\lambda) \quad \text{strongly in } L^2(0,T;H).$$

Moreover, as we can assume that $\overline{y}_h$ converges to y^λ a.e. in Q and $\widehat{\beta}_\lambda$ grows at most quadratically, we can also apply (5.26) and Fatou's lemma to deduce that

$$\int_\Omega \widehat{\beta}_\lambda(y^\lambda(t)) \le \liminf_{h \searrow 0} \int_\Omega \widehat{\beta}_\lambda(\overline{y}_h(t)) \le c \quad \text{for a.a. } t \in (0,T), \tag{5.42}$$

whence

$$\|\widehat{\beta}_\lambda(y^\lambda)\|_{L^\infty(0,T;L^1(\Omega))} \le c\,. \tag{5.43}$$

Therefore, we can pass to the limit in the time-integrated version of (5.21) (written with time-dependent test functions) and deduce that the pair (y^λ, μ^λ) also solves (5.8), which is equivalent to (5.6). This concludes the proof of Theorem 5.1.

6 Existence

This section is devoted to the proof of the existence part of Theorem 2.6. Just by the semicontinuity of the norms, all of the uniform estimates we have established for the interpolants of the discrete solution hold true for the (unique) solution to the approximating problem. Therefore, from (5.25)–(5.27), (5.29) and (5.43), we deduce that

$$\|y^\lambda\|_{H^1(0,T;V_A^{-r})\cap L^\infty(0,T;V_B^\sigma)} + \|\mu^\lambda\|_{L^2(0,T;V_A^r)} \\ +\tau^{1/2}\|\partial_t y^\lambda\|_{L^2(0,T;H)} + \|\widehat{\beta}_\lambda(y^\lambda)\|_{L^\infty(0,T;L^1(\Omega))} \le c \tag{6.1}$$

for $\lambda > 0$ small enough. We infer that there exist a strictly decreasing sequence $\lambda_n \searrow 0$ and a pair (y,μ) satisfying, as $n \nearrow \infty$,

$$y^{\lambda_n} \to y \quad \text{weakly star in } H^1(0,T;V_A^{-r}) \cap L^\infty(0,T;V_B^\sigma)\,, \tag{6.2}$$

$$\mu^{\lambda_n} \to \mu \quad \text{weakly in } L^2(0,T;V_A^r)\,, \tag{6.3}$$

$$\partial_t y^{\lambda_n} \to \partial_t y \quad \text{weakly in } L^2(0,T;H) \quad \text{if } \tau > 0\,. \tag{6.4}$$

Then, it is immediately seen that (y,μ) solves (2.33) and that y satisfies the initial condition (2.35). Now, we prove that the variational inequality (2.34) holds true as well. To this end, we first owe to the compact embedding $V_B^\sigma \subset H$ (see (3.3)) and, e.g., to [43, Sect. 8, Cor. 4]), and deduce that we also have, at least for another subsequence, that

$$y^{\lambda_n} \to y \quad \text{strongly in } L^\infty(0,T;H) \quad \text{and} \quad \text{a.e. in } Q. \tag{6.5}$$

This implies that $\pi(y^{\lambda_n})$ converges to $\pi(y)$ in the same space, by Lipschitz continuity. Now, we use (6.5) once more to show that

$$\int_Q \widehat{\beta}(y) \le \liminf_{n\to\infty} \int_Q \widehat{\beta}_{\lambda_n}(y^{\lambda_n}) < +\infty. \tag{6.6}$$

We notice that the right-hand side of (6.6) actually is finite thanks to the bound for $\widehat{\beta}_\lambda(y^\lambda)$ given by (6.1). In particular, the requirement $\widehat{\beta}(y) \in L^1(Q)$ (see (2.32)) is fulfilled once the first inequality of (6.6) is established. In order to prove it, we take arbitrary indices m and n with $n > m$. Then $\lambda_n < \lambda_m$, and we can apply (5.3). We deduce that

$$\widehat{\beta}_{\lambda_m}(y^{\lambda_n}) \le \widehat{\beta}_{\lambda_n}(y^{\lambda_n}) \quad \text{a.e. in } Q\text{, for every } n > m,$$

whence also (since $\widehat{\beta}_{\lambda_m}$ is continuous)

$$\widehat{\beta}_{\lambda_m}(y) = \lim_{n\to\infty} \widehat{\beta}_{\lambda_m}(y^{\lambda_n}) = \liminf_{n\to\infty} \widehat{\beta}_{\lambda_m}(y^{\lambda_n}) \le \liminf_{n\to\infty} \widehat{\beta}_{\lambda_n}(y^{\lambda_n}) \quad \text{a.e. in } Q.$$

On the other hand, we have, by virtue of the second property stated in (5.3),

$$\widehat{\beta}(y) = \lim_{m\to\infty} \widehat{\beta}_{\lambda_m}(y) \quad \text{a.e. in } Q. \tag{6.7}$$

Thus,

$$\widehat{\beta}(y) \le \liminf_{n\to\infty} \widehat{\beta}_{\lambda_n}(y^{\lambda_n}) \quad \text{a.e. in } Q, \tag{6.8}$$

and (6.6) follows from Fatou's lemma. Next, we have that

$$\begin{aligned}
&\int_0^T \big(B^\sigma y(t), B^\sigma(y(t)-v(t))\big)\,dt \\
&\le \liminf_{n\to\infty} \int_0^T \big(B^\sigma y^{\lambda_n}(t), B^\sigma y^{\lambda_n}(t)\big)\,dt - \lim_{n\to\infty} \int_0^T \big(B^\sigma y^{\lambda_n}(t), B^\sigma v(t)\big)\,dt \\
&= \liminf_{n\to\infty} \int_0^T \big(B^\sigma y^{\lambda_n}(t), B^\sigma(y^{\lambda_n}(t)-v(t))\big)\,dt
\end{aligned}$$

for every $v \in L^2(0,T;V_B^\sigma)$, since $B^\sigma y^\lambda$ converges to $B^\sigma y$ weakly in $L^2(0,T;H)$ by (6.2). At this point, we can let n tend to infinity in (5.8) written with $\lambda = \lambda_n$. By also accounting for (6.3), (6.5) and (5.3), we see that, for every $v \in L^2(0,T;V_B^\sigma)$, we have

$$\begin{aligned}
&\int_Q \widehat{\beta}(y) + \int_0^T \big(B^\sigma y(t), B^\sigma(y(t)-v(t))\big)\,dt \\
&\le \liminf_{n\to\infty} \int_Q \widehat{\beta}_{\lambda_n}(y^{\lambda_n}) + \liminf_{n\to\infty} \int_0^T \big(B^\sigma y^{\lambda_n}(t), B^\sigma(y^{\lambda_n}(t)-v(t))\big)\,dt \\
&\le \liminf_{n\to\infty} \Big(\int_Q \widehat{\beta}_{\lambda_n}(y^{\lambda_n}) + \int_0^T \big(B^\sigma y^{\lambda_n}(t), B^\sigma(y^{\lambda_n}(t)-v(t))\big)\,dt \Big) \\
&\le \lim_{n\to\infty} \Big(\int_0^T \big(-\tau\partial_t y^{\lambda_n}(t) - \pi(y^{\lambda_n}(t)) + u(t) + \mu^{\lambda_n}(t), y^{\lambda_n}(t) - v(t)\big)\,dt + \int_Q \widehat{\beta}_{\lambda_n}(v) \Big) \\
&= \int_0^T \big(-\tau\partial_t y(t) - \pi(y(t)) + u(t) + \mu(t), y(t) - v(t)\big)\,dt + \int_Q \widehat{\beta}(v).
\end{aligned}$$

Thus, (2.36) holds true. Since (2.36) is equivalent to (2.34), the proof of Theorem 2.6 is complete.

7 Regularity

This section is devoted to the proof of Theorem 2.8. Coming back to the proofs of Theorems 2.6 and 5.1, we see that it is sufficient to establish some estimates on the solution to the discrete problem in one of the forms (5.11)–(5.12) and (5.20)–(5.21), uniformly with respect to both h and λ. Of course, we can account for the estimates proved in Section 5.

First regularity estimate. We prove the uniform estimate

$$\|A^r \overline{\mu}_h\|_{L^\infty(0,T;H)} + \|B^\sigma \partial_t \widehat{y}_h\|_{L^2(0,T;H)} + \tau^{1/2} \|\partial_t \widehat{y}_h\|_{L^\infty(0,T;H)} \le c\,, \tag{7.1}$$

with a constant c that does not depend on h, λ and τ (like the constant K_3 in the statement of the theorem). We test (5.11) by $\mu^{n+1} - \mu^n$. On account of (2.10), we obtain

$$\Big(\frac{y^{n+1}-y^n}{h}, \mu^{n+1} - \mu^n\Big) + \|\mu^{n+1} - \mu^n\|^2 + \big(A^r \mu^{n+1}, A^r(\mu^{n+1} - \mu^n)\big) = 0. \tag{7.2}$$

Now, we perform a discrete differentiation on (5.12). Precisely, we write it for both (y^n, μ^n) and

(y^{n-1}, μ^{n-1}), take the difference, divide by h and rearrange. We have for $1 \le n < N$

$$\begin{aligned} &\frac{1}{h}\Big(\tau\,\frac{y^{n+1}-y^n}{h} - \tau\,\frac{y^n-y^{n-1}}{h}\Big) + L_\pi'\Big(\frac{y^{n+1}-y^n}{h} - \frac{y^n-y^{n-1}}{h}\Big) \\ &\quad + B^{2\sigma}\,\frac{y^{n+1}-y^n}{h} + \frac{1}{h}\big(\beta_\lambda(y^{n+1}) - \beta_\lambda(y^n)\big) \\ &= \frac{\mu^{n+1}-\mu^n}{h} + \frac{u^{n+1}-u^n}{h} - \frac{1}{h}\big(\pi(y^{n+1}) - \pi(y^n)\big) \end{aligned}$$

and test this equality by $y^{n+1} - y^n$. On account of (2.11), we obtain

$$\begin{aligned} &\tau\Big(\frac{y^{n+1}-y^n}{h}, \frac{y^{n+1}-y^n}{h} - \frac{y^n-y^{n-1}}{h}\Big) \\ &\quad + L_\pi' h\Big(\frac{y^{n+1}-y^n}{h}, \frac{y^{n+1}-y^n}{h} - \frac{y^n-y^{n-1}}{h}\Big) \\ &\quad + h\,\Big\|B^\sigma\,\frac{y^{n+1}-y^n}{h}\Big\|^2 + \frac{1}{h}\big(\beta_\lambda(y^{n+1}) - \beta_\lambda(y^n), y^{n+1}-y^n\big) \\ &= \Big(\frac{\mu^{n+1}-\mu^n}{h}, y^{n+1}-y^{n-1}\Big) \\ &\quad + \Big(\frac{u^{n+1}-u^n}{h}, y^{n+1}-y^n\Big) - \frac{1}{h}\big(\pi(y^{n+1}) - \pi(y^n), y^{n+1}-y^n\big). \end{aligned} \tag{7.3}$$

Now, we add this to (7.2) and notice that two terms cancel each other and that the term involving β_λ is nonnegative by monotonicity. Thus, thanks to the identity (3.37), applying the Schwarz and Young inequalities to the remaining terms on the right-hand side, and accounting for the Lipschitz continuity of π, we deduce that

$$\begin{aligned} &\|\mu^{n+1}-\mu^n\|^2 + \frac{1}{2}\|A^r\mu^{n+1}\|^2 + \frac{1}{2}\,\|A^r(\mu^{n+1}-\mu^n)\|^2 - \frac{1}{2}\,\|A^r\mu^n\|^2 \\ &\quad + \frac{\tau}{2}\left(\Big\|\frac{y^{n+1}-y^n}{h}\Big\|^2 - \Big\|\frac{y^n-y^{n-1}}{h}\Big\|^2\right) + \frac{\tau}{2}\,\Big\|\frac{y^{n+1}-y^n}{h} - \frac{y^n-y^{n-1}}{h}\Big\|^2 \\ &\quad + L_\pi'\,\frac{h}{2}\left(\Big\|\frac{y^{n+1}-y^n}{h}\Big\|^2 - \Big\|\frac{y^n-y^{n-1}}{h}\Big\|^2\right) \\ &\quad + L_\pi'\,\frac{h}{2}\,\Big\|\frac{y^{n+1}-y^n}{h} - \frac{y^n-y^{n-1}}{h}\Big\|^2 + h\,\Big\|B^\sigma\,\frac{y^{n+1}-y^n}{h}\Big\|^2 \\ &\le \frac{h}{2}\,\Big\|\frac{u^{n+1}-u^n}{h}\Big\|^2 + \frac{h}{2}\,\Big\|\frac{y^{n+1}-y^n}{h}\Big\|^2 + \frac{L_\pi h}{2}\,\Big\|\frac{y^{n+1}-y^n}{h}\Big\|^2. \end{aligned}$$

Summing up for $n = 1, \dots, k-1$ with $k \le N$, and omitting a number of nonnegative terms on the left-hand side, we infer that

$$\begin{aligned} &\frac{1}{2}\|A^r\mu^k\|^2 + \frac{\tau}{2}\,\Big\|\frac{y^k-y^{k-1}}{h}\Big\|^2 + \sum_{n=1}^{k-1} h\,\Big\|B^\sigma\,\frac{y^{n+1}-y^n}{h}\Big\|^2 \\ &\le \frac{1}{2}\,\|A^r\mu^1\|^2 + \frac{\tau}{2}\,\Big\|\frac{y^1-y_0}{h}\Big\|^2 + L_\pi'\,\frac{h}{2}\,\Big\|\frac{y^1-y_0}{h}\Big\|^2 \\ &\quad + \sum_{n=1}^{k-1} h\Big\|\frac{u^{n+1}-u^n}{h}\Big\|^2 + \frac{L_\pi'}{2}\sum_{n=1}^{k-1} h\Big\|\frac{y^{n+1}-y^n}{h}\Big\|^2. \end{aligned}$$

At this point, we use (3.36), the compactness inequality (3.4) and the estimate (5.27), to control the last two terms on the right-hand side:

$$\sum_{n=1}^{k-1} h\Big\| \frac{u^{n+1}-u^n}{h} \Big\|^2 + \frac{L_\pi'}{2} \sum_{n=1}^{k-1} h\Big\| \frac{y^{n+1}-y^n}{h} \Big\|^2$$
$$\le \|\partial_t u\|_{L^2(0,T;H)}^2 + \frac{1}{2} \sum_{n=1}^{k-1} h \Big\| B^\sigma \, \frac{y^{n+1}-y^n}{h} \Big\|^2 + c \sum_{n=1}^{k-1} h \Big\| \frac{y^{n+1}-y^n}{h} \Big\|_{A,-r}^2$$
$$= c + \frac{1}{2} \sum_{n=1}^{k-1} h \Big\| B^\sigma \, \frac{y^{n+1}-y^n}{h} \Big\|^2 + c \, \|\partial_t \widehat{y}_h\|_{L^2(0,T;A,-r)}^2 \le \frac{1}{2} \sum_{n=1}^{k-1} h \Big\| B^\sigma \, \frac{y^{n+1}-y^n}{h} \Big\|^2 + c \, .$$

Therefore, on account of Proposition 3.9, the above inequality becomes

$$\|A^r \overline{\mu}_h\|_{L^\infty(0,T;H)}^2 + \tau \, \Big\| \partial_t \widehat{y}_h \Big\|_{L^\infty(0,T;H)}^2 + \|B^\sigma \partial_t \widehat{y}_h\|_{L^2(0,T;H)}^2$$
$$\le c \Big(\|A^r \mu^1\|^2 + \tau \, \Big\| \frac{y^1 - y_0}{h} \Big\|^2 + h \, \Big\| \frac{y^1 - y_0}{h} \Big\|^2 + 1 \Big) \, , \tag{7.4}$$

and (7.1) will follow whenever we estimate the right-hand side of (7.4). To this end, we write (5.11) and (5.12) with $n = 0$. We also rearrange the latter, recall that $y^0 = y_0$ and $\mu^0 = 0$, and set for convenience $\mathcal{A}_\lambda := L_\pi' I + \beta_\lambda + \pi$. We have

$$\frac{y^1 - y_0}{h} + \mu^1 + A^{2r} \mu^1 = 0 \tag{7.5}$$

$$\tau \, \frac{y^1 - y_0}{h} + \mathcal{A}_\lambda(y^1) - \mathcal{A}_\lambda(y_0) + B^{2\sigma}(y^1 - y_0)$$
$$= \mu^1 + \big(u^1 - B^{2\sigma} y_0 - \beta_\lambda(y_0) - \pi(y_0) \big). \tag{7.6}$$

Now, we test (7.5) by μ^1 and (7.6) by $(y^1 - y_0)/h = -(\mu^1 + A^{2r}\mu^1)$, by choosing the first or second expression according to our convenience. In view of (2.10)–(2.11), and noting that $(\mathcal{A}_\lambda(y^1) - \mathcal{A}_\lambda(y_0), y^1 - y_0) \ge 0$ since β_λ is monotone and L_π is the Lipschitz constant of π, we obtain

$$\Big(\frac{y^1 - y_0}{h}, \mu^1 \Big) + \|\mu^1\|^2 + \|A^r \mu^1\|^2 = 0 \tag{7.7}$$

first, and then

$$\tau \Big\| \frac{y^1 - y_0}{h} \Big\|^2 + h \, \|B^\sigma (y^1 - y_0)\|^2$$
$$\le \Big(\mu^1, \frac{y^1 - y_0}{h} \Big) + \Big(u^1 - B^{2\sigma} y_0 - \beta_\lambda(y_0) - \pi(y_0), \frac{y^1 - y_0}{h} \Big) \tag{7.8}$$

or, alternatively,

$$\tau \Big\| \frac{y^1 - y_0}{h} \Big\|^2 + h \, \|B^\sigma (y^1 - y_0\|^2$$
$$\le \Big(\mu^1, \frac{y^1 - y_0}{h} \Big) - \big(u^1 - B^{2\sigma} y_0 - \beta_\lambda(y_0) - \pi(y_0), \mu^1 + A^{2r} \mu^1 \big) \, . \tag{7.9}$$

Now, we distinguish the two cases of the statement of Theorem 2.8. We first assume $\tau > 0$ and (2.41). Then, we add (7.7) and (7.8), by noticing that two terms cancel each other. Moreover, we omit a nonnegative term on the left-hand side and use the Schwarz and Young inequalities on the right-hand side. We then have that

$$\|\mu^1\|^2 + \|A^r\mu^1\|^2 + \tau\Big\|\frac{y^1-y_0}{h}\Big\|^2 \le \frac{\tau}{2}\,\Big\|\frac{y^1-y_0}{h}\Big\|^2 + \frac{1}{2\tau}\|u^1 - B^{2\sigma}y_0 - \beta_\lambda(y_0) - \pi(y_0)\|^2.$$

By accounting for (2.26), which implies that $\|u^1\| = \|u(h)\| \le c$, (2.41) and (5.4), we see that the last norm is bounded uniformly with respect to λ. Therefore, the right-hand side of (7.4) is bounded, too. Now, we assume $\tau = 0$ and (2.42)–(2.43). Then, we add (7.7) and (7.9) and similarly have that

$$\begin{aligned}
&\|\mu^1\|^2 + \|A^r\mu^1\|^2 \le -\big(u^1 - B^{2\sigma}y_0 - \beta_\lambda(y_0) - \pi(y_0), \mu^1 + A^{2r}\mu^1\big) \\
&\le \frac{1}{2}\,\|\mu^1\|^2 + \frac{1}{2}\,\|u^1 - B^{2\sigma}y_0 - \beta_\lambda(y_0) - \pi(y_0)\|^2 \\
&\quad + \frac{1}{2}\,\|A^r\mu^1\|^2 + \frac{1}{2}\,\|A^r(u^1 - B^{2\sigma}y_0 - \beta_\lambda(y_0) - \pi(y_0))\|^2.
\end{aligned}$$

Hence, the sought bound is ensured by (2.42)–(2.43), since $u^1 = u(h)$. Therefore, (7.1) is established in any case.

Consequence. By applying the compactness inequality (3.4), we obtain

$$\|\partial_t\widehat{y}_h(t)\|^2 \le \|B^\sigma\partial_t\widehat{y}_h(t)\|^2 + c\,\|\partial_t\widehat{y}_h(t)\|^2_{A,-r} \quad \text{for a.a. } t\in(0,T).$$

On the other hand, we have that $\|\partial_t\widehat{y}_h\|_{L^2(0,T;V_A^{-r})} \le c$, by virtue of (5.27). Therefore, we deduce from (7.1) that

$$\|\partial_t\widehat{y}_h\|^2_{L^2(0,T;H)} \le \|B^\sigma\partial_t\widehat{y}_h\|^2_{L^2(0,T;H)} + c\,\|\partial_t\widehat{y}_h\|^2_{L^2(0,T;V_A^{-r})} \le c\,,$$

as well as

$$\|\partial_t\widehat{y}_h\|^2_{L^2(0,T;V_B^\sigma)} = \|\partial_t\widehat{y}_h\|^2_{L^2(0,T;H)} + \|B^\sigma\partial_t\widehat{y}_h\|^2_{L^2(0,T;H)} \le c\,.$$

This implies that

$$\partial_t y \in L^2(0,T;V_B^\sigma) \quad \text{and} \quad \|\partial_t y\|_{L^2(0,T;V_B^\sigma)} \le c\,,$$

which is a part of (2.44) and (2.46).

Second regularity estimate. We now prove the inequalities

$$\|\overline{\mu}_h\|_{L^\infty(0,T;H)} \le c \quad \text{and} \quad \|\underline{\mu}_h\|_{L^\infty(0,T;H)} \le c\,, \tag{7.10}$$

the latter being a consequence of the former since $\mu^0 = 0$. If $\lambda_1 > 0$, then $\|v\| \le \|A^r v\|$ for every $v \in V_A^r$, so that (7.1) also implies that

$$\|\overline{\mu}_h\|_{L^\infty(0,T;H)} \le c\,\|A^r\overline{\mu}_h\|_{L^\infty(0,T;H)} \le c\,,$$

and the first claim of (7.10) is proved for the case $\lambda_1 > 0$. In the case $\lambda_1 = 0$, we only have (see (2.17) and Remark 2.4)

$$\|\overline{\mu}_h - \operatorname{mean}\overline{\mu}_h\|_{L^\infty(0,T;H)} \le c\,\|A^r\overline{\mu}_h\|_{L^\infty(0,T;H)} \le c\,.$$

Thus, in order to achieve (7.10), we have to estimate the mean value. To this end, we recall (5.32), which can be written in the form

$$\|\beta_\lambda(\overline{y}_h(t))\|_{L^1(\Omega)} \le c\big(\tau\,\|\partial_t\widehat{y}_h(t)\| + \|\overline{u}_h(t)\| + \|A^r\overline{\mu}_h(t)\| + 1\big) \le c \quad \text{for a.a. } t\in(0,T).$$

From (7.1) and (2.26), we deduce that

$$\|\beta_\lambda(\overline{y}_h)\|_{L^\infty(0,T;L^1(\Omega))} \le c\,.$$

At this point, we simply integrate (5.21) over Ω to obtain, almost everywhere in $(0,T)$,

$$\begin{aligned} |\Omega|\,\mathrm{mean}\,\overline{\mu}_h \;&= \tau \int_\Omega \partial_t \widehat{y}_h + L'_\pi \int_\Omega (\overline{y}_h - \underline{y}_h) + \bigl(B^\sigma \overline{y}_h, B^\sigma(1)\bigr) \\ &\quad + \int_\Omega \beta_\lambda(\overline{y}_h) + \int_\Omega \pi(\overline{y}_h) - \int_\Omega \overline{u}_h\,. \end{aligned}$$

Thus, $\mathrm{mean}\,\overline{\mu}_h$ is bounded in $L^\infty(0,T)$ thanks to (5.25) and (2.26). This concludes the proof of (7.10).

Conclusion. From (7.1) and (7.10), we infer that

$$\mu \in L^\infty(0,T;V_A^r) \quad \text{and} \quad \|\mu\|_{L^\infty(0,T;V_A^r)} \le c\,,$$

which is another claim of (2.44) and (2.46). Moreover, by recalling (5.20) and Proposition 3.4, we deduce that

$$\begin{aligned} &\|\partial_t \widehat{y}_h\|_{L^\infty(0,T;V_A^{-r})} \le \|A^{2r}\widehat{\mu}_h\|_{L^\infty(0,T;V_A^{-r})} + c\,\|\underline{\mu}_h - \overline{\mu}_h\|_{L^\infty(0,T;H)} \\ &\le \|A^r \widehat{\mu}_h\|_{L^\infty(0,T;H)} + c \;\le\; c\,, \end{aligned}$$

which yields that

$$\partial_t y \in L^\infty(0,T;V_A^{-r}) \quad \text{and} \quad \|\partial_t y\|_{L^\infty(0,T;V_A^{-r})} \le c\,.$$

Now, we assume that $\tau > 0$, in addition. Then (5.21), (2.46) and (7.10) give that

$$\tau^{1/2}\|A^{2r}\overline{\mu}_h\|_{L^\infty(0,T;H)} \le \tau^{1/2}\|\partial_t \widehat{y}_h\|_{L^\infty(0,T;H)} + \tau^{1/2}\|\underline{\mu}_h - \overline{\mu}_h\|_{L^\infty(0,T;H)} \le c\,,$$

whence

$$\partial_t y \in L^\infty(0,T;H), \;\; \mu \in L^\infty(0,T;V_A^{2r}) \quad \text{and} \quad \|\tau^{1/2}\partial_t y\|_{L^\infty(0,T;H)} + \|\tau^{1/2}\mu\|_{L^\infty(0,T;V_A^{2r})} \le c\,.$$

This concludes the proof of Theorem 2.8.

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