ON THE IDENTIFICATION OF SOIL TRANSMISSIVITY FROM MEASUREMENTS OF THE GROUNDWATER LEVEL

GOTTFRIED BRUCKNER, SYBILLE HANDROCK–MEYER, HARTMUT LANGMACH

Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, D–10117 Berlin, Germany.

ABSTRACT. This paper is devoted to the inverse problem of identifying a spatially varying coefficient in a linear elliptic differential equation describing the filtration of groundwater. Practice suggests that the gradient of the piezometric head, i.e., Darcy's velocity, may have discontinuities and the transmissivity coefficient is a piecewise constant function.

For solving this problem we have used a direct method of G. Vainikko. Starting at a weak formulation of the problem a suitable discretization is obtained by the method of minimal error. If necessary this method can be combined with Tikhonov's regularization.

The main difficulty consists in generating distributed state observations from measurements of the ground water level. For this step we propose an optimized data preparation procedure using additional information like knowledge of the sought parameter values at some points and lower and upper bounds for the parameter.

First numerical tests show that locally sufficiently many measurements provide locally satisfactory results.

1. Introduction

The two–dimensional steady flow in an isotropic and confined aquifer is governed by the linear elliptic boundary value problem (cf. e. g. [15])

$$\begin{aligned} -\boldsymbol{\nabla}\cdot(a(x,y)\boldsymbol{\nabla}h(x,y)) &= f(x,y) \quad (x,y)\in\Omega\subset\mathbf{R}^2 && (1.1)\\ h(x,y) &= z(x,y) \quad (x,y)\in\partial\Omega_1 && (1.2)\\ a(x,y)\boldsymbol{\nabla}h(x,y)\cdot\boldsymbol{\nu}(x,y) &= g(x,y) \quad (x,y)\in\partial\Omega_2=\partial\Omega\backslash\partial\Omega_1\,, && (1.3)\end{aligned}$$

where Ω is a bounded domain with piecewise smooth boundary and $\boldsymbol{\nu}=\boldsymbol{\nu}(x,y)$ is the outer unit normal on $\partial\Omega_2$. In the sequel, we confine ourselves to the special case that $\partial\Omega_1$ has a positive Lebesgue measure and $z(x,y)=const.$ Physically, $h(x,y)$ can be interpreted as the groundwater level (piezometric head of ground water) in Ω, and $a(x,y)$ as transmissivity coefficient. The function $f(x,y)$ characterizes sources or sinks in Ω. By z the ground water level on $\partial\Omega_1$ and by $g(x,y)$ the inflow or outflow through $\partial\Omega_2$ are denoted. The direct (forward) problem consists in the following:
Given $f,z,g,a.$ Find $h.$
In our case the direct problem (1.1)–(1.3) is well–posed in the sense of Hadamard, (i.e. there exists a unique solution h which continuously depends on the data f,z,g,a) ([10]).
Now let us formulate the inverse problem:
Given $f,z,g,h.$ Find $a.$
An inverse problem is ill–posed in general. Due to the lack of continuous dependence on the data (i.e. due to the lack of stability) difficulties arise by using noisy data.

There are a lot of investigations concerning the estimation of the diffusivity from potential measurements in the intransient and transient cases. Direct and indirect methods for solving such problems can be found in [1],[2], [3], [4], [5], [6], [7], [8], [9], [12], [13]; cf. also the survey paper of J. Sprekels [14].

To reduce the high computational expense of those methods, in this paper a direct inversion is proposed being numerically cheap but very sensitive with respect to errors. Therefore, it is combined with an optimized data preparation procedure.

For the identification of the positive, piecewise continuous transmissivity a method of G. Vainikko ([16], [17], [18]) is used. Starting at the weak formulation of the problem this method consists in a finite element discretization of an operator equation in suitable

Hilbert spaces where the operator depends on the measured data. The considered projection method, the so–called method of least error, takes advantage of the simple form of the adjoint operator. The procedure is combined with Tikhonov's regularization.

This approach needs one measurement at each node.

But in practice, only very few measurements are at our disposal such that data gained by interpolation are very erroneous and not in accordance with the a priori information on the coefficient.

To meet those difficulties the method of Vainikko is combined with a method of "data smoothing" whose stabilizing effect consists in restricting the possible data set by a "smoothing" process. The goal of this method is an optimal utilization of the given information concerning both the coefficient and the data.

"New" data are sought, optimally fitting the "old" ones and satisfying the discretized state equation with a certain tolerance. The state equation is built using an a priori guess of the transmissivity. One gets a constrained minimization problem that is solved by the method of Lagrange multipliers and Newton's method. (Concerning similar considerations, cf. Parker's book [11].)

Numerical tests with locally sufficiently many measurements have provided better local results than the pure direct method. In that case, for slightly disturbed data (1 %) and small jumps of the transmissivity the computational results are satisfactory.

The paper is organized as follows. In Section 2 a short survey of Vainikko's method is given and the question of identifiability is considered in the case of constant Dirichlet conditions on one part of the boundary and Neumann conditions on the other one. Section 3 deals with the data preparation. Finally, some numerical experiments are presented in Section 4.

2. Vainikko's method

2.1. **Formulation as an operator equation.** Let $u(x,y) \equiv h(x,y) - z$ in (1.1)–(1.3), where $z = const$ and consider the system

$$\begin{aligned} -\boldsymbol{\nabla} \cdot (a(x,y)\boldsymbol{\nabla} u(x,y)) &= f(x,y) \quad (x,y) \in \Omega \subset \mathbf{R}^2 \\ u(x,y) &= 0 \qquad\quad\; (x,y) \in \partial\Omega_1 \\ a(x,y)\boldsymbol{\nabla} u(x,y) \cdot \boldsymbol{\nu}(x,y) &= g(x,y) \quad (x,y) \in \partial\Omega_2 \,. \end{aligned}$$

Here $\partial\Omega_2 \subset \partial\Omega$ can be empty.

The introduction of the subspace

$$H^1(\Omega, \partial\Omega_1) = \{w \in H^1(\Omega) : w(x,y) = 0 \quad \text{for } (x,y) \in \partial\Omega_1\} \subset H^1(\Omega)$$

yields the following weak formulation of the inverse problem: For given u find $a \in L^2(\Omega)$ such that

$$\int\limits_\Omega a \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} w \, dx\, dy = \int\limits_\Omega f w \, dx \, dy + \int\limits_{\partial\Omega_2} g w \, dS \text{ for all } w \in H^1(\Omega, \partial\Omega_1) \,. \tag{2.1}$$

The problem (2.1) makes sense for $u \in W^{1,\infty}(\Omega)$, $a \in L^2(\Omega)$, $g \in H^{-1/2}(\partial\Omega_2)$ and $f \in H^{-1}(\Omega)$. Let G be the space of gradients of the functions $w \in H^1(\Omega, \partial\Omega_1)$:

$$G = G(\Omega, \partial\Omega_1) = \{\boldsymbol{\nabla} w : w \in H^1(\Omega, \partial\Omega_1)\} \subset (L^2(\Omega))^2 \,.$$

Furthermore, using the orthoprojector

$$Q_G : (L^2(\Omega))^2 \to G$$

we define an operator $T \in \mathcal{L}(L^2(\Omega), G)$ by

$$Ta = Q_G(a\boldsymbol{\nabla} u)\,, \quad a \in L^2(\Omega)$$

and consider the operator equation

$$Ta = \boldsymbol{\nabla}\psi\,, \tag{2.2}$$

where $\psi \in H^1(\Omega, \partial\Omega_1)$ is the solution to the following direct problem

$$\begin{aligned} -\,\Delta\psi(x,y) &= f(x,y) \quad (x,y) \in \Omega \\ \psi(x,y) &= 0 \qquad\quad\; (x,y) \in \partial\Omega_1 \\ \boldsymbol{\nabla}\psi(x,y)\cdot\boldsymbol{\nu}(x,y) &= g(x,y) \quad (x,y) \in \partial\Omega_2\,. \end{aligned}$$

The problem (2.1) is equivalent to the operator equation (2.2). The adjoint operator

$$T^*\boldsymbol{\nabla} w = \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} w$$

where $T^* \in \mathcal{L}(G, L^2(\Omega))$, has the following properties:

1°. The range $R(T^*)$ is not closed.

2°. The operator T^* as multiplication operator in G is not compact if $\mid \boldsymbol{\nabla} u \mid \geq c_0 > 0$.

The properties 1° and 2° then are valid for the operator $T \in \mathcal{L}(L^2(\Omega), G)$, too.

Therefore, (2.2) is an ill–posed problem with a non–compact operator, i.e. discretization and regularization schemes for ill–posed problems with a compact operator cannot be used. Since the adjoint operator has a very simple form we are interested in discretization schemes which use T^* and not T itself.

2.2. **Identifiability.** In the theory of inverse problems, a very important question is to know whether the observation data contain sufficient information for identifying the unknown parameters. In our consideration the identifiability is equivalent to the uniqueness of the transmissivity coefficient $a(x,y)$. We reduce our investigations to the problem (1.1)–(1.3), where $z(x,y) = const$ and follow the identifiability concept of Vainikko and Kunisch ([16]).

Definition 2.1. *The transmissivity coefficient $a(x,y)$ is called L^1–identifiable on a subdomain $\tilde{\Omega} \subset \Omega$ from the data $u(x,y) \in W^{1,\infty}(\Omega)$ if any solution $a \in L^1(\Omega)$ to the homogeneous problem*

$$\int_\Omega a\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} w \, dx\, dy = 0 \quad \textit{for all} \quad w \in W^{1,\infty}(\Omega, \partial\Omega_1), \tag{2.3}$$

with

$$W^{1,\infty}(\Omega, \partial\Omega_1) = \{w \in W^{1,\infty}(\Omega) : w(x,y) = 0 \quad \textit{for} \quad (x,y) \in \partial\Omega_1\} \subset W^{1,\infty}(\Omega)$$

yields $a(x,y) = 0$ almost everywhere in $\tilde{\Omega}$.

It is clear that the L^1–identifiability of the coefficient a from the data $u \in W^{1,\infty}(\Omega)$ on $\tilde{\Omega}$ implies the L^2–identifiability of a on the same set $\tilde{\Omega}$.

We consider the case that $u \in C(\overline{\Omega})$ but Darcy's velocity ∇u may have discontinuities on piecewise smooth curves K_r $(r = 1, \cdots, l)$ in Ω. Introducing the notation $K = \cup_{r=1}^{l} K_r$ we assume that

$$u \in W^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega_{\varepsilon,K}) \quad \text{for all} \quad \varepsilon > 0, \tag{2.4}$$

where $\Omega_{\varepsilon,K}$ is the set of all points $(x,y) \in \Omega \backslash K$ for which the distance from (x,y) to the nearest non–smoothness point of $\partial\Omega$ or K_r $(r = 1, \cdots, l)$ as well as the distance from (x,y) to the nearest intersection point of a pair of curves $\partial\Omega$, K_r $(r = 1, \cdots, l)$ are greater than ε.

The identifiability is dependent on the behavior of the streamlines which are solutions of the Cauchy–problem

$$\frac{dy}{dx} = \frac{u_y}{u_x} \qquad y(x_0) = y_0. \tag{2.5}$$

Therefore, the identifiability is a property of the solution u to the direct problem. We write the solution of (2.5) through $\mathbf{P}_0 = (x_0, y_0)$ in the form

$$\mathbf{P} = (x,y) = \boldsymbol{\varphi}(t, \mathbf{P}_0) = (\varphi_1(t, \mathbf{P}_0), \varphi_2(t, \mathbf{P}_0))^T \quad t \in (t^-_{\mathbf{P}_0}, t^+_{\mathbf{P}_0}),$$

where $(t^-_{\mathbf{P}_0}, t^+_{\mathbf{P}_0})$ is a finite or an infinite time interval.

Furthermore, we assume

(A) The validity of the compatibility condition: Let $\tilde{\mathbf{P}} \in K_r$ be a smoothness point of K_r, $\boldsymbol{\nu}_r(\tilde{\mathbf{P}})$ a unit normal to K_r at $\tilde{\mathbf{P}}$, $\hat{a}$ a positive piecewise smooth function with jumps on K, and let (2.4) holds. Then

$$\lim_{\substack{(\mathbf{P}-\tilde{\mathbf{P}})\cdot\boldsymbol{\nu}_r(\tilde{\mathbf{P}})>0 \\ \mathbf{P}\to\tilde{\mathbf{P}}}} \hat{a}(\mathbf{P})\, \nabla u(\mathbf{P}) \cdot \boldsymbol{\nu}_r(\tilde{\mathbf{P}}) = \lim_{\substack{(\mathbf{P}-\tilde{\mathbf{P}})\cdot\boldsymbol{\nu}_r(\tilde{\mathbf{P}})<0 \\ \mathbf{P}\to\tilde{\mathbf{P}}}} \hat{a}(\mathbf{P})\, \nabla u(\mathbf{P}) \cdot \boldsymbol{\nu}_r(\tilde{\mathbf{P}}).$$

(B) The streamline $\boldsymbol{\varphi}(t, \mathbf{P}_0)$ through $\mathbf{P}_0$, where $\nabla u(\mathbf{P}_0) \neq 0$ non–tangentially reaches a smoothness point $\tilde{\mathbf{P}}$ of $\partial\Omega_2 = \partial\Omega \backslash \partial\Omega_1$, i. e.

$$\lim_{\substack{t\to t^+_{\mathbf{P}_0} \\ t^+_{\mathbf{P}_0}<+\infty}} \nabla u(\boldsymbol{\varphi}(t, \mathbf{P}_0)) \cdot \boldsymbol{\nu}(\tilde{\mathbf{P}}) \neq 0 \quad \text{or} \quad \lim_{\substack{t\to t^-_{\mathbf{P}_0} \\ t^-_{\mathbf{P}_0}>-\infty}} \nabla u(\boldsymbol{\varphi}(t, \mathbf{P}_0)) \cdot \boldsymbol{\nu}(\tilde{\mathbf{P}}) \neq 0.$$

(C) The streamline $\boldsymbol{\varphi}(t, \mathbf{P}_0)$ through $\mathbf{P}_0$, where $\nabla u(\mathbf{P}_0) \neq 0$ cuts K only at a finite number of smoothness points of K in a non–tangential way.

Under assumption (A), (B) and (C) the system (2.5) is uniquely solvable and the streamlines pass through every K_r $(r = 1 \cdots l)$ at a smoothness point of K_r in a non–tangential way remaining continuous.

We introduce the following subsets of Ω:

$$\Omega_C = \{\mathbf{P}_0 \in \Omega : \nabla u(\mathbf{P}_0) = 0\}$$

$$\Omega^+ = \{\mathbf{P}_0 \in \Omega : \nabla u(\mathbf{P}_0) \neq 0,\ t^+_{\mathbf{P}_0} = +\infty, \text{the streamline through } \mathbf{P}_0 \text{ satisfies (C)}\}$$

$$\Omega^- = \{\mathbf{P}_0 \in \Omega : \nabla u(\mathbf{P}_0) \neq 0,\ t^-_{\mathbf{P}_0} = -\infty, \text{the streamline through } \mathbf{P}_0 \text{ satisfies (C)}\}$$

$$\Omega^+_{\partial\Omega_2} = \{\mathbf{P}_0 \in \Omega : \nabla u(\mathbf{P}_0) \neq 0, t^+_{\mathbf{P}_0} < +\infty, \text{the streamline through } \mathbf{P}_0 \text{ satisfies (B), (C)}\}$$

$$\Omega^-_{\partial\Omega_2} = \{\mathbf{P}_0 \in \Omega : \nabla u(\mathbf{P}_0) \neq 0, t^-_{\mathbf{P}_0} > -\infty, \text{the streamline through } \mathbf{P}_0 \text{ satisfies (B), (C)}\}.$$

Any of these subsets may be empty, Ω_C is the set of the singular points of (2.5). We cite a general L^1–identifiability result obtained by Vainikko and Kunisch ([16]):

Under the assumptions (2.4) and (A) the coefficient a is

L^1–identifiable on the sets $int(\Omega^+)$, $int(\Omega^-)$, $\overline{\Omega^+_{\partial\Omega_2}}$ and $\overline{\Omega^-_{\partial\Omega_2}}$.

In practice the function $u(x,y) \in W^{1,\infty}(\Omega)$ is not arbitrary, but it is obtained from the (unknown) parameter $a(x,y)$ and the function $f(x,y)$ as a solution of the equation (1.1) on Ω with appropriate boundary conditions.

Remark 2.1. *If $u(x,y) = const$ in a subregion $\tilde{\Omega}$ then $\nabla u = \boldsymbol{\theta}$ and (2.3) is satisfied for every constant function a in $\tilde{\Omega}$. Consequently, a is not identifiable in $\tilde{\Omega}$.*

Proposition 2.1. *Let (2.4), (A), (B), (C) be fulfilled and $u(x,y)$ be a solution of the boundary value problem (1.1)–(1.3) where $z(x,y) = const$. Furthermore, we suppose that*

$$meas(\Omega_C) = meas(\partial\Omega^+) = meas(\partial\Omega^+) = 0. \tag{2.6}$$

Then $a(x,y)$ is identifiable on Ω.

Proof. Let $\mathbf{P}_0 \in \Omega$, and assume that $\nabla u(\mathbf{P}_0) \neq 0$. Then we have for the streamline $\boldsymbol{\varphi}(t, \mathbf{P}_0)$ through $\mathbf{P}_0$

$$\frac{du(\boldsymbol{\varphi}(t,\mathbf{P}_0))}{dt} = \mid \nabla u(\boldsymbol{\varphi}(t,\mathbf{P}_0) \mid^2 > 0 \quad \text{for all} \quad t \in (t^-_{\mathbf{P}_0}, t^+_{\mathbf{P}_0})$$

and u increases along this streamline. Because $u(x,y) = const$ on $\partial\Omega_1$ we conclude that $\boldsymbol{\varphi}(t,\mathbf{P}_0)$ contains at most one point of $\partial\Omega_1$. If $\boldsymbol{\varphi}(t,\mathbf{P}_0)$ reaches $\partial\Omega$ at no other point then we have $\mathbf{P}_0 \in \Omega^+$ or $\mathbf{P}_0 \in \Omega^-$. If $\boldsymbol{\varphi}(t,\mathbf{P}_0)$ contains a further point $Q \in \partial\Omega$ then $Q \in \partial\Omega_2$. Because of assumption (B) we have $t^+_{\mathbf{P}_0} < +\infty$ or $t^-_{\mathbf{P}_0} > -\infty$. Hence $\mathbf{P}_0 \in \Omega^+_{\partial\Omega_2}$ or $\mathbf{P}_0 \in \Omega^-_{\partial\Omega_2}$. If $\boldsymbol{\varphi}(t,\mathbf{P}_0)$ leaves Ω in both directions and contains two points of $\partial\Omega_2$ then $\mathbf{P}_0 \in \Omega^+_{\partial\Omega_2} \cap \Omega^-_{\partial\Omega_2}$.

Using the inclusion [16]

$$\Omega \backslash \Omega_C \subset \overline{\Omega^+} \cup \overline{\Omega^-} \cup \overline{\Omega^+_{\partial\Omega_2}} \cup \overline{\Omega^-_{\partial\Omega_2}}$$

(2.6) and the identifiability of a on the sets $int(\Omega^+)$ and $int(\Omega^-)$ we obtain the assertion.

2.3. **Discretization.** The equation (2.1) is discretized by the method of minimal error, which is a special projection method.

Consider finite dimensional subspaces

$$S_h \subset H^1(\Omega, \partial\Omega_1)$$

with the usual admissibility properties and take

$$G_h = \nabla S_h \subset G\,, \quad T^* \nabla S_h \subset L_2(\Omega)$$

as test and trial spaces, respectively. Then from (2.1)

$$\int\limits_{\Omega} a_h \nabla u \cdot \nabla v_h \, dx\, dy = \int\limits_{\Omega} f v_h \, dx\, dy + \int\limits_{\partial\Omega_2} g v_h \, dS \quad \forall v_h \in S_h\,, \tag{2.7}$$

where

$$a_h = \nabla u \cdot \nabla \hat{v}_h\,, \quad \hat{v}_h \in S_h\,,$$

and

$$\|a_h - a\|_{L_2} = \min_{v_h \in S_h} \|\nabla u \cdot \nabla v_h - a\|_{L_2}$$

(method of least error).

Problem (2.7) has a unique solution a_h and $\|a_h - a\|_{L_2} \to 0$ $(h \to 0)$. Here a is a minimal norm solution of (2.1).

2.4. **Implementation.** Let Ω be a polygonal bounded domain and (for a fixed discretization level h) $\mathcal{T}_h$ a regular triangulation, where

$$\overline{\Omega} = \bigcup_{E \in \mathcal{T}_h} \overline{E}.$$

By $\mathcal{N} = \{\mathbf{P}_j\}_{j=1}^n$ we denote the set of all nodes of the triangulation $\mathcal{T}_h$ that do not live on the boundary $\partial\Omega_1$. Moreover, in the finite dimensional subspace $S_h \subset H^1(\Omega, \partial\Omega_1)$ we choose a basis with linear base functions $\{w_j\}_{j=1}^n$ with $w_j = 0$ on $\partial\Omega_1$ and $w_j(\mathbf{P}_i) = \delta_{ij}$, $1 \le i, j \le n$.

We suppose that the coefficient $a(x, y)$ is constant on each element (triangle) $E \in \mathcal{T}_h$ and the discretized coefficient a_h can be represented as the vector

$$\mathbf{a} = (a^E)_{E \in \mathcal{T}_h}.$$

Then for the **direct problem**, where

$$u = \sum_{1 \le j \le n} u_j w_j$$

is asked for, the linear system

$$\sum_{1 \le j \le n} L_{ij}[\mathbf{a}] u_j = d_i, \quad 1 \le i \le n, \tag{2.8}$$

where

$$L_{ij}[\mathbf{a}] \stackrel{def}{=} \sum_E a^E \int_E \nabla w_j \cdot \nabla w_i \, dx\, dy, \quad \mathbf{L}[\mathbf{a}] \stackrel{def}{=} (L_{ij}[\mathbf{a}])_{ij}$$

has to be solved (a^E and d_i are given). The values of u on the boundary $\partial\Omega_1$ are already known as $u(x, y) = 0$ on $\partial\Omega_1$.

For the **inverse problem** the linear system

$$\sum_{1 \le j \le n} M_{ij}[u] c_j = d_j, \quad 1 \le i \le n, \tag{2.9}$$

has to be solved where

$$M_{ij}[u] \stackrel{def}{=} \sum_E \int_E (\nabla u \cdot \nabla w_j)(\nabla u \cdot \nabla w_i) \, dx\, dy$$

(here ∇u and d_i are known). Then a_h will be found from

$$a_h = \sum_{1 \le j \le n} c_j \nabla u \cdot \nabla w_j.$$

In combination with Tikhonov's regularization (2.9) reads as

$$\sum_{1 \le j \le n} (\alpha L_{ij} + M_{ij}[u]) c_j^\alpha = d_j, \quad 1 \le i \le n, \tag{2.10}$$

where

$$L_{ij} \stackrel{def}{=} \sum_E \int_E \nabla w_j \cdot \nabla w_i \, dx \, dy = L_{ij}[\mathbf{1}].$$

It is clear that this method of Vainikko will work well in the case when ∇u has sufficiently good properties. If the matrix $(M_{ij}[u])_{ij}$ in (2.9) is ill–conditioned, Tikhonov's regularization (2.10) with not too small α may produce results. However, if α is chosen too large the computed coefficient

$$a_h^\alpha = \sum_{1\le j\le n} c_j^\alpha \nabla u \cdot \nabla w_j$$

can not be interpreted as a solution to the inverse problem.

Remark 2.2. *The matrices* $\mathbf{L}[\mathbf{a}] = (L_{ij}[\mathbf{a}])_{i,j}$ *and* $\mathbf{M}[u] = (M_{ij}[u])_{i,j}$ *can easily be constructed using the coefficients*

$$L_{ij}^E = \int_E \nabla w_j \cdot \nabla w_i \, dx \, dy.$$

Since $\nabla w_i \, (1 \le i \le n)$ *is constant on each element (triangle)* E*, we have*

$$L_{ij}^E = meas(E) \, \nabla w_j \cdot \nabla w_i,$$

$$L_{ij}[\mathbf{a}] = \sum_E a^E \, L_{ij}^E,$$

$$M_{ji}[u] = M_{ij}[u] = \sum_E \Big(\sum_{\substack{k\in\mathcal{N}\\ 1\le k\le n}} u_k \, L_{ki}^E \Big) \Big(\sum_{\substack{l\in\mathcal{N}\\ 1\le l\le n}} u_l \, L_{lj}^E \Big) \frac{1}{meas(E)}.$$

Remark 2.3. *If the triangle* E *(of the triangulation* $\mathcal{T}_h$ *) has no obtuse angles we have the well-known properties*

$$L_{ij}^E = L_{ji}^E \le 0, \quad i \ne j,$$

$$L_{ii}^E > 0,$$

$$\sum_{\substack{j\in\mathcal{N}\\ 1\le j\le n}} L_{ij}^E = 0 \quad \text{if} \quad \overline{E} \cap \partial\Omega_1 = \emptyset.$$

3. Data preparation

Let $\mathfrak{A}$ be an admissible set of parameters, say

$$\mathfrak{A} = \{\mathbf{a}\,,\ 0 < \alpha^E \le a^E \le \beta^E\}\,, \quad \alpha^E, \beta^E \quad \text{given},$$

and let $\mathbf{a}_0 \in \mathfrak{A}$ be the a priori guess (from geological considerations). Let measurements

$$u_j\,,\ \mathbf{P}_j \in \mathcal{M}$$

(with error ε) be given in a subset $\mathcal{M} \subset \mathcal{N}$. From these measurements a data vector $\bar{\mathbf{u}}$ can be constructed by interpolation or by putting

$$\bar{\mathbf{u}} = (\bar{u}_i)_{\mathbf{P}_i\in\mathcal{N}}\,,\ \bar{u}_i = \begin{cases} u_i\,, & \mathbf{P}_i \in \mathcal{M} \\ u_i[\mathbf{a}_0]\,, & \mathbf{P}_i \in \mathcal{N}\backslash\mathcal{M}\,, \end{cases}$$

where $\mathbf{u}[\mathbf{a}_0] = (\mathbf{L}[\mathbf{a}_0])^{-1}\mathbf{d}$ is the solution of the direct problem

$$\mathbf{L}[\mathbf{a}_0]\mathbf{u} = \mathbf{d} \tag{3.1}$$

and

$$\mathbf{L}[\mathbf{a}_0] = \left(\sum_E a_0^E \int_E \boldsymbol{\nabla} w_j \cdot \boldsymbol{\nabla} w_j \, dx\, dy \right)_{i,j}$$

is the matrix in (2.8) and $\mathbf{d} = (d_i)_i$.

Numerical experiments show the following:

In the case when the set $\mathcal{M}$ of measurement points fills some subdomain $\Omega' \subset \Omega$ then the inversion of the data $\bar{\mathbf{u}}$ by Vainikko's method produces in Ω', i.e. locally, satisfactory results. When there is no such subdomain Ω' and the interpolation $\bar{\mathbf{u}}$ or the guess $\mathbf{a}_0$ are bad, by merely inverting $\bar{\mathbf{u}}$ no inference is possible.

To overcome that difficulty and to exploit the given information concerning $\bar{\mathbf{u}}$ and $\mathbf{a}_0$ in an optimal way the following "data smoothing" procedure is proposed.

We are looking for new data $\hat{\mathbf{u}}$ having a minimal distance from the given data $\bar{\mathbf{u}}$ with the property to satisfy the state equation (3.1) more exactly than $\bar{\mathbf{u}}$. I.e., $\hat{\mathbf{u}}$ is a solution to the constrained minimum problem

$$\|\bar{\mathbf{u}} - \hat{\mathbf{u}}\| = \min_{\{\mathbf{u}, \|\mathbf{L}[\mathbf{a}_0]\mathbf{u} - \mathbf{d}\| \le \delta\}} \|\bar{\mathbf{u}} - \mathbf{u}\| \tag{3.2}$$

where δ is a given tolerance,

$$0 \le \delta \le \|\mathbf{L}[\mathbf{a}_0]\bar{\mathbf{u}} - \mathbf{d}\|,$$

and $\|\cdot\|$ is the usual norm in $\mathbb{R}^n$. The tolerance δ should be chosen as a "measure of confidence" with respect to the data $\bar{\mathbf{u}}$ and the a priori guess $\mathbf{a}_0$: If one trusts more to the guess $\mathbf{a}_0$ than to the data $\bar{\mathbf{u}}$, δ should be small; if the data $\bar{\mathbf{u}}$ are more trustable than the guess $\mathbf{a}_0$, δ should be large.

Concerning the solution $\hat{\mathbf{u}}$ of (3.2) we have the

Theorem 3.1. *The problem* (3.2) *is uniquely solvable with the solution*

$$\hat{\mathbf{u}} = (\mu \mathbf{I} + \mathbf{L}[\mathbf{a}_0]^2)^{-1} (\mu \bar{\mathbf{u}} + \mathbf{L}[\mathbf{a}_0]\mathbf{d}), \tag{3.3}$$

where μ, $0 \le \mu \le \infty$, is unique with the property

$$\delta = \mu \|(\mu \mathbf{I} + \mathbf{L}[\mathbf{a}_0]^2)^{-1} (\mathbf{L}[\mathbf{a}_0]\bar{\mathbf{u}} - \mathbf{d})\|. \tag{3.4}$$

As a function of μ the tolerance δ is monotonously increasing from 0 ($\mu = 0$) to $\|\mathbf{L}[\mathbf{a}_0]\bar{\mathbf{u}} - \mathbf{d}\|$ ($\mu = \infty$) and $\hat{\mathbf{u}}$ changes from the "guessed data" $\mathbf{u}[\mathbf{a}_0] = \mathbf{L}[\mathbf{a}_0]^{-1}\mathbf{d}$ ($\mu = 0$) to the given data $\bar{\mathbf{u}}$ ($\mu = \infty$).

The iterative approach then may run as follows:

(i) Start from the a priori guess $\mathbf{a}_0$.

(ii) Calculate the "guessed data" $\mathbf{L}[\mathbf{a}_0]^{-1}\mathbf{d} = \mathbf{u}[\mathbf{a}_0]$ (solution of a direct problem).

(iii) Choose μ, $0 \le \mu \le \infty$ (where the values $\mu = 0$ and $\mu = \infty$ are not interesting). For given δ the parameter μ can be determined from (3.4) by Newton's method.

(iv) Calculate $\hat{\mathbf{u}}$ from (3.3) (solution of a "regularized" direct problem).

(v) Determine

$$\mathbf{u}_1 = \begin{cases} \bar{\mathbf{u}} & \text{on} \quad \mathcal{M} \\ \hat{\mathbf{u}} & \text{on} \quad \mathcal{N}\backslash\mathcal{M}\,. \end{cases}$$

For $\mathbf{P}_j \in \mathcal{M}$, instead of $\bar{u}_j$, we can take $\tilde{u}_j$ with $|\bar{u}_j - \tilde{u}_j| \le \varepsilon$.

(vi) Invert $\mathbf{u}_1$ by Vainikko's method (2.9) or (2.10), i.e. determine $\tilde{\mathbf{a}}_1$ such that

$$\mathbf{L}[\tilde{\mathbf{a}}_1]\mathbf{u}_1 = \mathbf{d}$$

holds (solution of an inverse problem).

(vii) Determine $\mathbf{a}_1 \in \mathfrak{A}$ from $\tilde{\mathbf{a}}_1$.

(viii) If $c_0 < \|\mathbf{L}[\mathbf{a}_1]\mathbf{u}_1 - \mathbf{d}\| \le \|\mathbf{L}[\mathbf{a}_0]\bar{\mathbf{u}} - \mathbf{d}\| - c_1$, where c_0, c_1 are given (small) positive numbers, go to (i) with $\mathbf{a}_0 = \mathbf{a}_1$, $\bar{\mathbf{u}} = \mathbf{u}_1$, if not: stop.

Concerning the criterion (viii) some explanations are necessary.

In the case of $\tilde{\mathbf{a}}_1 \in \mathfrak{A}$, i.e. $\mathbf{a}_1 = \tilde{\mathbf{a}}_1$, we have

$$\|\mathbf{L}[\mathbf{a}_1]\mathbf{u}_1 - \mathbf{d}\| = 0\,.$$

This means that $\mathbf{a}_1$ is a solution of the identification problem, since $\mathbf{u}_1 = \mathbf{u}[\mathbf{a}_1]$ fits the measurements. If $\tilde{\mathbf{a}}_1$ is near to $\mathbf{a}_1$ (in some sense) then $\|\mathbf{L}[\mathbf{a}_1]\mathbf{u}_1 - \mathbf{d}\| \le c_0$ because of the continuity of the mapping $\mathbf{a} \to \mathbf{u}[\mathbf{a}]$. Then we will accept $\mathbf{a}_1$ as a solution. If $\tilde{\mathbf{a}}_1$ is not too far from $\mathbf{a}_1$ there is hope that at least

$$\|\mathbf{L}[\mathbf{a}_1]\mathbf{u}_1 - \mathbf{d}\| \le \|\mathbf{L}[\mathbf{a}_0]\bar{\mathbf{u}} - \mathbf{d}\| - c_1$$

holds. In this case the pair $(\mathbf{a}_1, \mathbf{u}_1)$ can be considered as "better" than the pair $(\mathbf{a}_0, \bar{\mathbf{u}})$. If at the same time

$$c_0 < \|\mathbf{L}[\mathbf{a}_1]\mathbf{u}_1 - \mathbf{d}\|$$

holds, one will be able to look whether the pair $(\mathbf{a}_1, \mathbf{u}_1)$ can be improved any further. In the case where

$$\|\mathbf{L}[\mathbf{a}_1]\mathbf{u}_1 - \mathbf{d}\| > \|\mathbf{L}[\mathbf{a}_0]\bar{\mathbf{u}} - \mathbf{d}\| - c_1$$

the pair $(\mathbf{a}_0, \bar{\mathbf{u}})$ will be considered as not improvable.

Proof of Theorem 3.1 Put $\mathbf{L} := \mathbf{L}[\mathbf{a}_0]$. It is clear that (3.2) has a solution: Let $\{\mathbf{u}^k\}_k$ be a minimizing sequence, i.e.

$$\|\bar{\mathbf{u}} - \mathbf{u}^k\| \to \inf_{\|\mathbf{L}\mathbf{u} - \mathbf{d}\| \le \delta} \|\bar{\mathbf{u}} - \mathbf{u}\| =: \gamma\,.$$

Then $\|\mathbf{u}^k\| \le C$, for a subsequence $\{\mathbf{u}^{k_r}\}$ $\mathbf{u}^{k_r} \to \tilde{\mathbf{u}}$ as $r \to \infty$ and $\|\bar{\mathbf{u}} - \mathbf{u}^{k_r}\| \to \|\bar{\mathbf{u}} - \tilde{\mathbf{u}}\|$, $\|\mathbf{L}\mathbf{u}^{k_r} - \mathbf{d}\| \to \|\mathbf{L}\tilde{\mathbf{u}} - \mathbf{d}\|$ as $r \to \infty$, $\|\mathbf{L}\tilde{\mathbf{u}} - \mathbf{d}\| \le \delta$, $\|\bar{\mathbf{u}} - \tilde{\mathbf{u}}\| = \gamma$, i.e. $\tilde{\mathbf{u}}$ is a solution of (3.2).

Additionally, if $\tilde{\mathbf{u}}$ solves (3.2) then $\tilde{\mathbf{u}}$ solves

$$\|\bar{\mathbf{u}} - \tilde{\mathbf{u}}\| = \min_{\{\mathbf{u},\, \|\mathbf{L}\mathbf{u} - \mathbf{d}\| = \delta\}} \|\bar{\mathbf{u}} - \mathbf{u}\|\,. \tag{3.5}$$

Indeed, let $\tilde{\mathbf{u}}$ solve (3.2) with $\|\mathbf{L}\tilde{\mathbf{u}} - \mathbf{d}\| = \tilde{\delta} < \delta$. Consider

$$\mathbf{w} := \eta\tilde{\mathbf{u}} + (1-\eta)\bar{\mathbf{u}} \quad \text{for some}\, \eta,\ 0 < \eta < 1\,.$$

Then

$$\|\mathbf{w} - \bar{\mathbf{u}}\| = \eta\|\tilde{\mathbf{u}} - \bar{\mathbf{u}}\| < \|\tilde{\mathbf{u}} - \bar{\mathbf{u}}\|$$

as $\|\tilde{\mathbf{u}} - \bar{\mathbf{u}}\| > 0$. (In the case $\tilde{\mathbf{u}} = \bar{\mathbf{u}}$ the assumption $\delta \leq \|\mathbf{L}\bar{\mathbf{u}} - \mathbf{d}\|$ would be violated.) On the other hand, by directly evaluating

$$\|\mathbf{L}\mathbf{w} - \mathbf{d}\|^2 = \|\mathbf{L}\tilde{\mathbf{u}} - \mathbf{d}\|^2 + (1-\eta)(\overline{c_1} + (1-\eta)\overline{c_2}) \leq \delta^2,$$

if η is sufficiently near to 1. This is a contradiction to the min–property of $\tilde{\mathbf{u}}$.

Now, to solve (3.5), consider the Lagrange function

$$\mathfrak{L}(\mathbf{u}, \alpha) = \|\bar{\mathbf{u}} - \mathbf{u}\|^2 + \alpha(\|\mathbf{L}\mathbf{u} - \mathbf{d}\|^2 - \delta^2).$$

Necessary conditions for a minimum are

$$\frac{\partial \mathfrak{L}}{\partial \mathbf{u}} = 0, \quad \frac{\partial \mathfrak{L}}{\partial \alpha} = 0.$$

We have

$$\begin{aligned} \left(\frac{\partial \mathfrak{L}}{\partial \mathbf{u}}, \boldsymbol{\delta}\mathbf{u}\right) &= \lim_{\lambda \to 0} \frac{1}{\lambda}(\|\mathbf{u} + \lambda\boldsymbol{\delta}\mathbf{u} - \bar{\mathbf{u}}\|^2 - \|\mathbf{u} - \bar{\mathbf{u}}\|^2 + \alpha(\|B(\mathbf{u} + \lambda\boldsymbol{\delta}\mathbf{u}) - \mathbf{d}\|^2 - \|\mathbf{L}\mathbf{u} - \mathbf{d}\|^2)) \\ &= 2((\mathbf{u} - \bar{\mathbf{u}}, \boldsymbol{\delta}\mathbf{u}) + \alpha(\mathbf{L}^T(\mathbf{L}\mathbf{u} - \mathbf{d}), \boldsymbol{\delta}\mathbf{u})). \end{aligned}$$

Then, from the necessary conditions and $\mathbf{L}^T = \mathbf{L}$,

$$\hat{\mathbf{u}} - \bar{\mathbf{u}} + \varrho\mathbf{L}(\mathbf{L}\hat{\mathbf{u}} - \mathbf{d}) = 0, \tag{3.6}$$

$$\delta = \|\mathbf{L}\hat{\mathbf{u}} - \mathbf{d}\|. \tag{3.7}$$

From the equations (3.6), (3.7) the pair $(\hat{\mathbf{u}}, \varrho)$ is uniquely determined, if $\mathbf{d}$, $\bar{\mathbf{u}}$, δ are given. Indeed, (3.6) implies

$$\hat{\mathbf{u}} = (\mathbf{I} + \varrho\mathbf{L}^2)^{-1}(\bar{\mathbf{u}} + \varrho\mathbf{L}\mathbf{d}). \tag{3.8}$$

Since $\mathbf{L}(\mathbf{I} + \varrho\mathbf{L}^2)^{-1} = (\mathbf{I} + \varrho\mathbf{L}^2)^{-1}\mathbf{L}$ and (3.7)

$$\delta = \|(\mathbf{I} + \varrho\mathbf{L}^2)^{-1}(\mathbf{L}\bar{\mathbf{u}} - \mathbf{d})\|. \tag{3.9}$$

In addition, the function

$$\theta(s) = \|(\mathbf{I} + s\mathbf{L}^2)^{-1}(\mathbf{L}\bar{\mathbf{u}} - \mathbf{d})\|^2$$

is strictly decreasing for $0 \leq s < \infty$ and

$$\theta(0) = \|\mathbf{L}\bar{\mathbf{u}} - \mathbf{d}\|^2, \quad \theta(s) \to 0 \quad (s \to \infty).$$

This can be seen by differentiating

$$\begin{aligned} \theta'(s) &= \frac{d}{ds}((\mathbf{I} + s\mathbf{L}^2)^{-1}(\mathbf{L}\bar{\mathbf{u}} - \mathbf{d}),\ (\mathbf{I} + s\mathbf{L}^2)^{-1}(\mathbf{L}\bar{\mathbf{u}} - \mathbf{d})) \\ &= -2((\mathbf{I} + s\mathbf{L}^2)^{-2}\mathbf{L}^2(\mathbf{L}\bar{\mathbf{u}} - \mathbf{d}),\ (\mathbf{I} + s\mathbf{L}^2)^{-1}(\mathbf{L}\bar{\mathbf{u}} - \mathbf{d})). \end{aligned}$$

From the symmetry of $\mathbf{L}$ and positive definiteness of $(\mathbf{I} + s\mathbf{L}^2)^{-1}$

$$\theta'(s) < 0.$$

Then the uniqueness of ϱ follows from (3.9).

The assertion of the theorem then follows by substituting

$$\mu = \varrho^{-1}$$

into (3.8) and (3.9) and easy inferences.

4. Numerical tests

In the following numerical experiments, the effect of the data preparation will be demonstrated. Let us consider a square domain

$$\Omega = \{(x, y) \in \mathbf{R}^2 : 0 < x < 1.1,\ 0 < y < 1.1\}$$

with impermeable upper and lower boundary, homogeneous Dirichlet conditions at the left boundary and inhomogeneous Neumann conditions at the right one. No sources and sinks are considered. The domain Ω is triangulated by 30×30 equidistant nodes.

Data generation: Suppose that we are given (cf. fig. 1)

1. (from a geological a priori information) an a priori guess $\mathbf{a}_0$, i.e., more precisely, a lens $C \supset A$ of diminished (constant) transmissivity $a_{02} = 10^{-3}$ surrounded by an area $\Omega \backslash C$ of (constant) transmissivity $a_{01} = 10^{-5}$;

2. measurements in an area $B' \supset B$ at about 75% of all nodes in B'.

These measurement are simulated by the potential values resulting from an assumed reality, i. e. more precisely, the lenses A and B of transmissivity a_{02} surrounded by an area $\Omega \backslash (A \cup B)$ of transmissivity a_{01}.

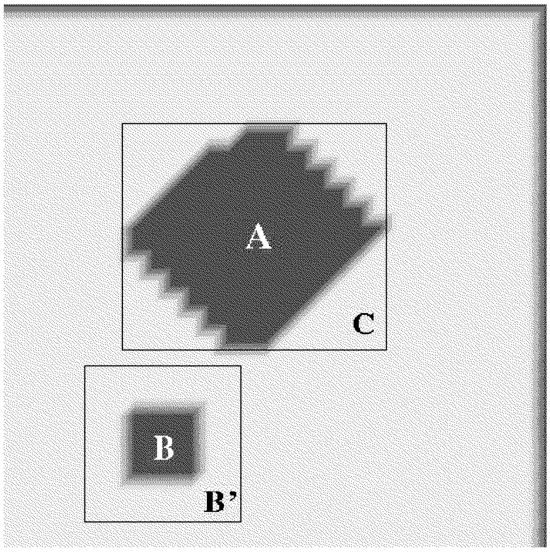

Fig. 1

Then, the data $\bar{\mathbf{u}}$ are composed from these simulated measurements and – on every node where no measurement is given – from potential values $\tilde{\mathbf{u}} = (\mathbf{L}[\mathbf{a}_0])^{-1}\mathbf{d}$ resulting from the a priori guess $\mathbf{a}_0$.

Results: The figures 2 to 5 show the transmissivity gained by

(1) direct inversion of the data $\bar{\mathbf{u}}$ (fig. 2),

(2) inversion after data preparation, $\mu = 10^{-5}$ (fig. 3),

(3) inversion after data preparation, $\mu = 5 \cdot 10^{-6}$ (fig. 4),

(4) inversion after data preparation, $\mu = 10^{-6}$ (fig.5).

Discussion: Generally, in (1) - (4) the lens C is reproduced satisfactorily. This is expected since the data $\tilde{\mathbf{u}}$ are disturbed only in the area B', i. e. disturbances in B'

do not essentially affect the area C. It confirms the above–mentioned local behavior of Vainikko's method. Moreover, fig. 2 shows that the lens B cannot be reconstructed by direct inversion of the data. Fortunately, this can be achieved by additional data preparation according to Section 3. The best reconstruction is obtained for $\mu = 10^{-6}$ (fig. 5), where B has nearly its correct shape, but its mean value is between a_{01} and a_{02}, i. e. much less than its true value a_{02}. The latter fact is not surprising since the used (prepared) data are situated between $\tilde{\mathbf{u}}$ and $\bar{\mathbf{u}}$. The value 10^{-6} for μ seems to be optimal in this context but also $\mu = 5 \cdot 10^{-6}$ (fig. 4) or $\mu = 5 \cdot 10^{-7}$ would be possible. But, from numerical reasons, $\mu \leq 10^{-7}$ is not suitable; in that case the condition number of the matrix $(\mu \mathbf{I} + \mathbf{L}[\mathbf{a}_0])^2$ has appeared as too bad for a calculation.

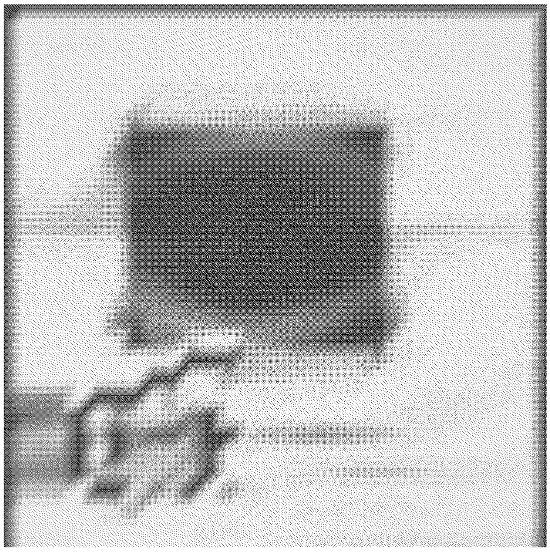

Fig. 2

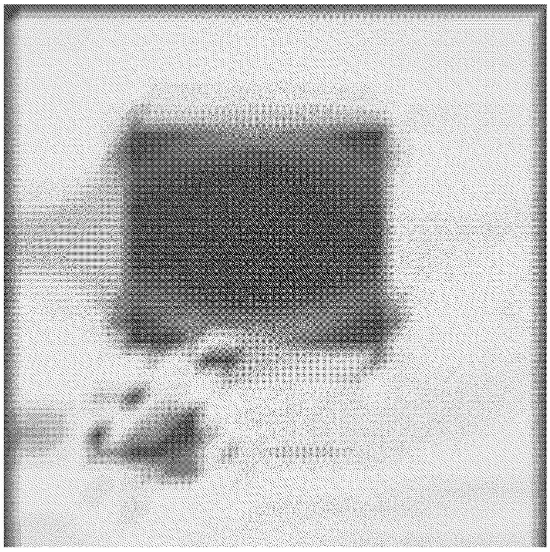

Fig. 3

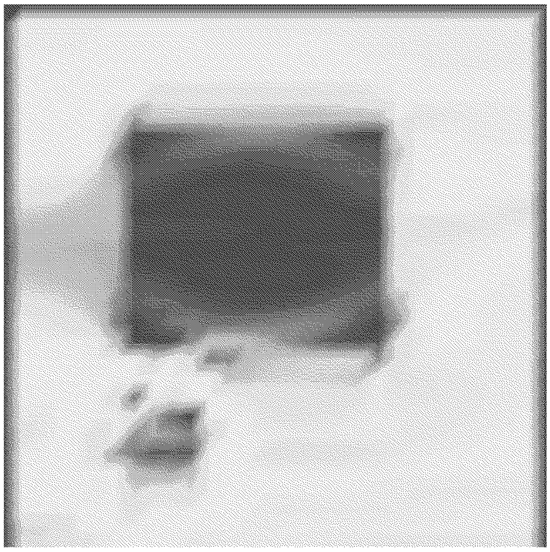

Fig. 4

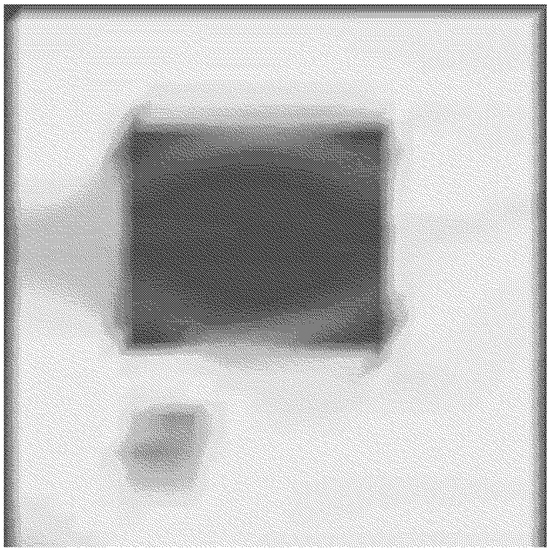

Fig. 5

The following figures 6 to 9 demonstrate how Vainikko's method works for simulated measurements. The a priori guess for the transmissivity is given as shown in figure 6. All parameters were chosen corresponding to a realistic situation. Figures 7 and 8 are obtained from the direct inversion by Vainikko's method using the computed solution of the direct problem (i.e. the computed groundwater level) as measurements. In figure 8 these simulated measurements were disturbed at each grid point by a small random value (1 cm uncertainty of the groundwater level). Figure 9 shows the used grid.

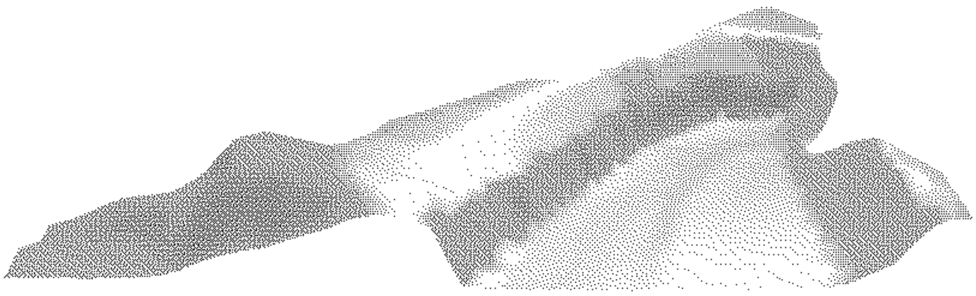

Fig. 6

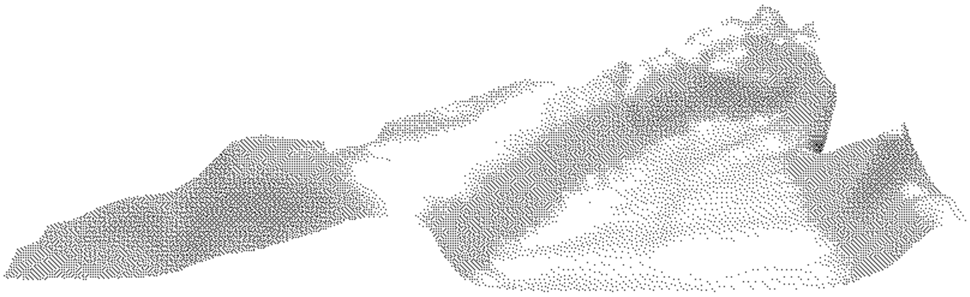

Fig. 7

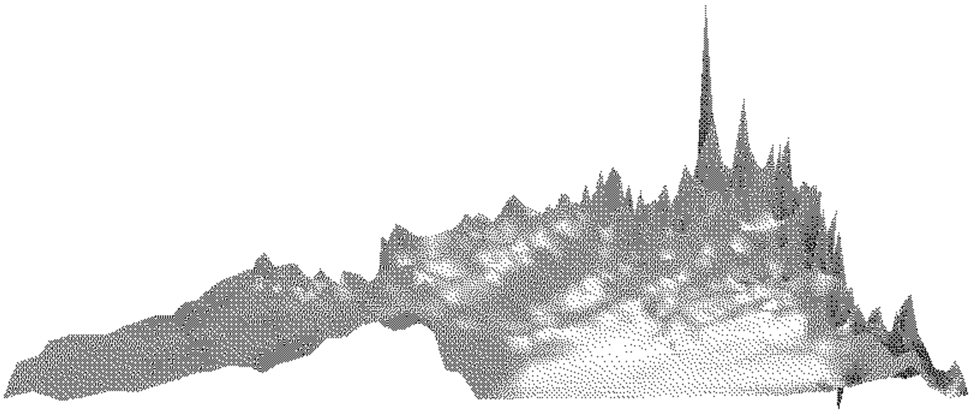

Fig. 8

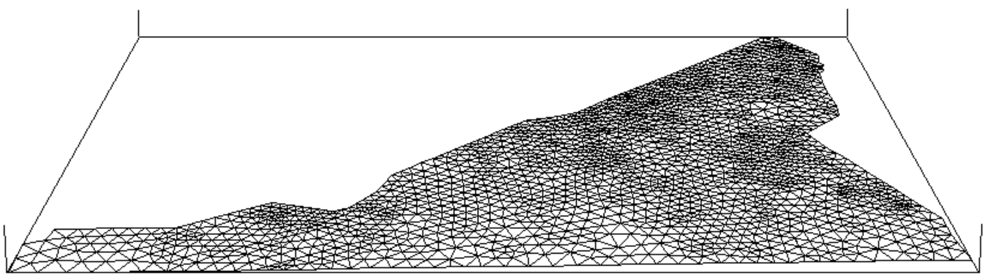

Fig. 9

Acknowledgements. We are grateful to J. Sprekels for useful discussions, to S. Prößdorf for encouragement and to H.–J. Diersch for support and advises from the practical point of view. Furthermore, S. Handrock–Meyer acknowledges partial support through the Deutsche Forschungsgemeinschaft under grant number Pr 336/3–1.

References

[1] Acar, R.: Identification of the coefficient in elliptic equations, SIAM J. Control and Optimization **31** (1993), 1221–1244.

[2] Alessandrini, G.: An identification problem for an elliptic equation in two variables, Annali di Matematica Pura ed Applicata **145** (1986), 265–296.

[3] Gottlieb, J. and Dietrich, P.: Identification of the permeability distribution in soil by hydraulic tomography, Inverse Problems **11** (1995), 353–360.

[4] Guidici, M., Morossi, G., Parravicini, G. and Ponzini, G.: A new method for the identification of distributed transmissivities, to appear in Water Resources Research.

[5] Hoffmann, K.–H. and Sprekels, J.: On the identification of elliptic problems by asymptotic regularization, Numer. Funct. Anal. and Optim. **7** (1984/85), 157–178.

[6] Hoffmann, K.–H. and Sprekels, J.: On the identification of parameters in general variational inequalities by asymptotic regularization, SIAM J. Math. Anal. **17** (1986), 1198–1217.

[7] Ito, K. and Kunisch, K.: A hybrid method combining the Output Least Squares and the Equation Error approach for the estimation of parameters in elliptic systems, SIAM J. Control Optim. **28** (1990), 113–136.

[8] Kravaris, C. and Seinfeld, J.H.: Identification of parameters in distributed systems by regularization, SIAM J. Control Optim. **23** (1985), 217–241.

[9] Lowe, B. and Kohn, R.V.: A variational method for numerically identifying a variable coefficient, in: Proceedings Int. Symp. on Variational Methods in the Geophysical Sciences, Norman, OK, USA (1985).

[10] Michlin, S.G.: Lehrgang der mathematischen Physik, Akademie–Verlag Berlin 1972.

[11] Parker, R.L.: Geophysical inverse theory, Princeton University Press, Princeton 1994.

[12] Parravicini, G., Guidici, M., Morossi, G. and Ponzini, G.: Minimal apriori assignment in a direct method for determining phenomenological coefficients uniquely, to appear in Inverse Probleme.

[13] Richter, G.R.: Numerical identification of a spatially varying diffusion coefficient, Math. Comp. **36** (1981), 375–386.

[14] Sprekels, J.: Identification of parameters in distributed systems: an overview, in: Methods of Operations Research **54**, 163–176, Verlag Anton Hain (1986).

[15] Sun, N.Z.: Inverse Problems in Groundwater Modeling, Kluwer Academic Publishers, Dordrecht 1994.

[16] Vainikko, G. and Kunisch, K.: Identifiability of the transmissivity coefficient in an elliptic boundary value problem, Zeitschrift für Analysis und ihre Anwendungen **12** (1993), 327–341.

[17] Vainikko, G.: Identification of filtration coefficient, A. Tikhonov (Ed.), Ill–Posed Problems in Natural Sciences (1992), 202–213.

[18] Vainikko, G.: On the discretization and regularization of ill-posed problems with noncompact operators, Numer. Funct. Anal. and Optim. **13** (1992), 381–396.