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Asymptotic limits and optimal control for the Cahn–Hilliard system with convection and dynamic boundary conditions

Dedicated to our friend Prof. Dr. Pierluigi Colli

on the occasion of his 60th birthday with best wishes

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Abstract

In this paper, we study initial-boundary value problems for the Cahn–Hilliard system with convection and nonconvex potential, where dynamic boundary conditions are assumed for both the associated order parameter and the corresponding chemical potential. While recent works addressed the case of viscous Cahn–Hilliard systems, the 'pure' nonviscous case is investigated here. In its first part, the paper deals with the asymptotic behavior of the solutions as time approaches infinity. It is shown that the ω-limit of any trajectory can be characterized in terms of stationary solutions, provided the initial data are sufficiently smooth. The second part of the paper deals with the optimal control of the system by the fluid velocity. Results concerning existence and first-order necessary optimality conditions are proved. Here, we have to restrict ourselves to the case of everywhere defined smooth potentials. In both parts of the paper, we start from corresponding known results for the viscous case, derive sufficiently strong estimates that are uniform with respect to the (positive) viscosity parameter, and then let the viscosity tend to zero to establish the sought results for the nonviscous case.

1 Introduction

In the recent paper [18], the following initial-boundary value problem for the Cahn–Hilliard system with convection was studied,

$$\partial_t\rho + \nabla\rho\cdot u - \Delta\mu = 0 \quad \text{and} \quad \tau_\Omega\partial_t\rho - \Delta\rho + f'(\rho) = \mu \quad \text{in } Q_T := \Omega\times(0,T), \tag{1.1}$$

where the unknowns ρ and μ represent the order parameter and the chemical potential, respectively, in a phase separation process taking place in an incompressible fluid contained in a container $\Omega \subset \mathbb{R}^3$. In the above equations, τ_Ω is a nonnegative constant, f' is the derivative of a double-well potential f, and u represents the (given) fluid velocity, which is assumed to satisfy $\operatorname{div} u = 0$ in the bulk and $u\cdot\nu = 0$ on the boundary, where ν denotes the outward unit normal to the boundary $\Gamma := \partial\Omega$. Typical and physically significant examples of f are the so-called *classical regular potential*, the *logarithmic double-well potential*, and the *double obstacle potential*, which are given, in this order, by

$$f_{reg}(r) := \frac{1}{4}\,(r^2-1)^2\,, \quad r\in\mathbb{R}, \tag{1.2}$$

$$f_{log}(r) := \big((1+r)\ln(1+r) + (1-r)\ln(1-r)\big) - c_1 r^2\,, \quad r\in(-1,1), \tag{1.3}$$

$$f_{2obs}(r) := -c_2 r^2 \quad \text{if } |r|\le 1 \quad \text{and} \quad f_{2obs}(r) := +\infty \quad \text{if } |r|>1. \tag{1.4}$$

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Here, the constants c_i in (1.3) and (1.4) satisfy $c_1 > 1$ and $c_2 > 0$, so that f_{log} and f_{2obs} are nonconvex. In cases like (1.4), one has to split f into a nondifferentiable convex part (the indicator function of $[-1,1]$ in the present example) and a smooth perturbation. Accordingly, one has to replace the derivative of the convex part by the subdifferential and interpret the second identity in (1.1) as a differential inclusion.

As far as the conditions on the boundary Γ are concerned, instead of the classical homogeneous Neumann boundary conditions, the dynamic boundary condition for both μ and ρ were considered, namely,

$$\partial_t\rho_\Gamma + \partial_\nu\mu - \Delta_\Gamma\mu_\Gamma = 0 \quad \text{and} \quad \tau_\Gamma\partial_t\rho_\Gamma + \partial_\nu\rho - \Delta_\Gamma\rho_\Gamma + f_\Gamma'(\rho_\Gamma) = \mu_\Gamma \quad \text{on } \Sigma_T := \Gamma\times(0,T), \tag{1.5}$$

where μ_Γ and ρ_Γ are the traces on Σ_T of μ and ρ, respectively; moreover, ∂_ν and Δ_Γ denote the outward normal derivative and the Laplace–Beltrami operator on Γ, τ_Γ is a nonnegative constant, and f_Γ' is the derivative of another potential f_Γ.

The associated total free energy of the phase separation process is the sum of a bulk and a surface contribution and has the form

$$\begin{aligned} &\mathcal{F}_{\rm tot}[\mu(t),\mu_\Gamma(t),\rho(t),\rho_\Gamma(t)] \\ &:= \int_\Omega \Bigl(f(\rho(x,t)) + \frac12 |\nabla\rho(x,t)|^2 - \mu(x,t)\rho(x,t)\Bigr)\,dx \\ &+ \int_\Gamma \Bigl(f_\Gamma(\rho_\Gamma(x,t)) + \frac12 |\nabla_\Gamma\rho_\Gamma(x,t)|^2 - \mu_\Gamma(x,t)\rho_\Gamma(x,t)\Bigr)\,d\Gamma\,, \end{aligned} \tag{1.6}$$

for $t\in[0,T]$. Moreover, we remark that the Cahn–Hilliard type system (1.5) indicates that on the boundary Γ another phase separation process is occurring that is coupled to the one taking place in the bulk. It is worth noting that the total mass of the order parameter is conserved during the separation process; indeed, integrating the first identity in (1.1) for fixed $t\in(0,T]$ over Ω, using the fact that $\operatorname{div} u = 0$ in Ω and $u\cdot\nu = 0$ on Γ, and invoking the first of the boundary conditions (1.5), we readily find that

$$\partial_t\Bigl(\int_\Omega \rho(t) + \int_\Gamma \rho_\Gamma(t)\Bigr) = 0\,. \tag{1.7}$$

The quoted paper [18] was devoted to the study of the initial-boundary value problem obtained by complementing (1.1) and (1.5) with the initial condition $\rho(0)=\rho_0$, where ρ_0 is a given function on Ω. By just assuming that the viscosity coefficients τ_Ω and τ_Γ are nonnegative and that the potentials fulfill suitable assumptions and compatibility conditions, well-posedness and regularity results were established. Moreover, in [21], the study of the longtime behavior was addressed in the viscous case, i.e., if $\tau_\Omega>0$ and $\tau_\Gamma>0$. More precisely, the ω-limit (in a suitable topology) of any trajectory (ρ,ρ_Γ) was characterized in terms of stationary solutions. Finally, in [19], a control problem was studied, the control being the velocity field u, and first-order necessary optimality conditions were derived. Also this study was done in the viscous case.

In the present paper, we extend these results to the pure Cahn–Hilliard system, i.e., to the case when $\tau_\Omega=\tau_\Gamma=0$. For the longtime behavior, this will be done by considering only trajectories that start from smoother initial data, and by assuming some additional summability on the velocity field. For the control problem, we assume that the potentials are of the everywhere regular type, establish the existence of optimal controls, and derive first-order necessary optimality conditions for a given control to be optimal in terms of a proper adjoint problem and a variational inequality. To this end, we start from the results obtained in [19] by taking $\tau_\Omega = \tau_\Gamma = \tau > 0$, and then let τ tend to zero.

Concerning the asymptotic behavior of Cahn–Hilliard type equations, one can find a number of results in several directions. From one side, the study can be either addressed on the convergence of the trajectories or devoted to the existence of attractors. On the other hand, the literature contains studies on several variants of the Cahn–Hilliard equations, which are obtained, e.g., by the coupling with other equations, like heat type equations or fluid dynamics equations, or the changing or adding further terms, like viscosity or memory terms, or the replacing the classical Neumann boundary conditions by other ones, mainly the dynamic boundary conditions in the last years. Without any claim of completeness, by starting from [60], we can quote, e.g., [1, 5, 8, 10, 32, 33, 34, 40, 48, 53, 54] for the first type of issues, and [24, 47, 46, 25, 41, 28, 29, 30, 31, 34, 43, 44, 45, 50] for the existence of global or exponential attractors.

As far as optimal control problems are concerned, there exist numerous recent contributions to general viscous Cahn–Hilliard systems. In this connection, we refer to the papers [9, 12, 16, 17, 11] for the case of standard boundary conditions and to [6, 7, 13, 14, 15, 22, 27] for the case of dynamic boundary conditions. For the 'pure' case $\tau_\Omega = \tau_\Gamma = 0$, there are the works of [52, 37, 58, 59]. Moreover, we refer to the papers [56, 57, 49] that address convective Cahn–Hilliard systems, where in the latter contribution (see also its generalization [26] to nonlocal two-dimensional Cahn–Hilliard/Navier–Stokes systems) the fluid velocity was chosen as the control parameter for the first time in such systems. In connection with Cahn–Hilliard/Navier–Stokes systems, we also mention the works [35, 36, 38, 39, 42]. In this paper, we study the control by the velocity of convective local Cahn–Hilliard systems with dynamic boundary conditions. To the authors' best knowledge, this problem has never been investigated before.

The paper is organized as follows. In the next section, we list our assumptions and notations, recall the properties already known, and state our results on the longtime behavior and the control problem. The corresponding proofs will be given in Sections 3 and 4, respectively.

2 Statement of the problem and results

In this section, we state precise assumptions and notations and present our results. First of all, let the set $\Omega \subset \mathbb{R}^3$ be bounded, connected and smooth. As in the introduction, ν is the outward unit normal vector field on $\Gamma := \partial\Omega$, and ∂_ν and Δ_Γ stand for the corresponding (outward) normal derivative and the Laplace–Beltrami operator, respectively. Furthermore, we denote by ∇_Γ the surface gradient and write $|\Omega|$ and $|\Gamma|$ for the volume of Ω and the area of Γ, respectively. Moreover, we widely use the notations

$$Q_t := \Omega \times (0,t) \quad \text{and} \quad \Sigma_t := \Gamma \times (0,t) \quad \text{for } 0 < t \le +\infty. \tag{2.1}$$

Next, if X is a Banach space, $\|\cdot\|_X$ denotes both its norm and the norm of X^3, and the symbols X^* and $\langle\,\cdot\,,\,\cdot\,\rangle_X$ stand for the dual space of X and the duality pairing between X^* and X. The only exception from the convention for the norms is given by the Lebesgue spaces L^p, for $1 \le p \le \infty$, whose norms are denoted by $\|\cdot\|_p$. Furthermore, we put

$$H := L^2(\Omega)\,, \quad V := H^1(\Omega) \quad \text{and} \quad W := H^2(\Omega), \tag{2.2}$$

$$H_\Gamma := L^2(\Gamma)\,, \quad V_\Gamma := H^1(\Gamma) \quad \text{and} \quad W_\Gamma := H^2(\Gamma), \tag{2.3}$$

$$\mathcal{H} := H \times H_\Gamma\,, \quad \mathcal{V} := \{(v, v_\Gamma) \in V \times V_\Gamma :\ v_\Gamma = v_{|\Gamma}\},$$
$$\text{and} \quad \mathcal{W} := \bigl(W \times W_\Gamma\bigr) \cap \mathcal{V}\,. \tag{2.4}$$

In the following, we will work in the framework of the Hilbert triplet $(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$. We thus have

$$\langle (g, g_\Gamma), (v, v_\Gamma) \rangle_{\mathcal{V}} = \int_\Omega gv + \int_\Gamma g_\Gamma v_\Gamma \quad \text{for every } (g, g_\Gamma) \in \mathcal{H} \text{ and } (v, v_\Gamma) \in \mathcal{V}.$$

Next, we list our assumptions. For our first result, we postulate for the structure of the system the following properties (which are slightly stronger than the analogous ones in [18]):

$$\tau_\Omega \text{ and } \tau_\Gamma \ \text{ are nonnegative real numbers.} \tag{2.5}$$

$$\widehat{\beta},\ \widehat{\beta}_\Gamma : \mathbb{R} \to [0, +\infty] \quad \text{are convex, proper and l.s.c. with} \quad \widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0. \tag{2.6}$$

$$\widehat{\pi},\ \widehat{\pi}_\Gamma : \mathbb{R} \to \mathbb{R} \quad \text{are of class } C^2 \text{ with Lipschitz continuous first derivatives.} \tag{2.7}$$

Moreover, we set

$$f := \widehat{\beta} + \widehat{\pi} \quad \text{and} \quad f_\Gamma := \widehat{\beta}_\Gamma + \widehat{\pi}_\Gamma \tag{2.8}$$

and assume that

$$\text{the functions } f := \widehat{\beta} + \widehat{\pi} \text{ and } f_\Gamma := \widehat{\beta}_\Gamma + \widehat{\pi}_\Gamma \text{ are bounded from below.} \tag{2.9}$$

Thus, compared with [18], we have the additional assumption (2.9). However, we remark that this assumption is fulfilled by all of the potentials (1.2)–(1.4). We set, for convenience,

$$\beta := \partial\widehat{\beta}\,, \quad \beta_\Gamma := \partial\widehat{\beta}_\Gamma\,, \quad \pi := \widehat{\pi}' \quad \text{and} \quad \pi_\Gamma := \widehat{\pi}'_\Gamma, \tag{2.10}$$

and assume that, with some positive constants C and η,

$$D(\beta_\Gamma) \subseteq D(\beta) \quad \text{and} \quad |\beta^\circ(r)| \le \eta|\beta^\circ_\Gamma(r)| + C \quad \text{for every } r \in D(\beta_\Gamma) \tag{2.11}$$

as in [4]. Finally, we assume that

$$\text{at least one of the operators } \beta \text{ and } \beta_\Gamma \text{ is single-valued.} \tag{2.12}$$

In (2.11), the symbols $D(\beta)$ and $D(\beta_\Gamma)$ denote the domains of β and β_Γ, repectively. More generally, we use the notation $D(\mathcal{G})$ for every maximal monotone graph $\mathcal{G}$ in $\mathbb{R} \times \mathbb{R}$, as well as for the corresponding maximal monotone operators induced on L^2 spaces. Moreover, for $r \in D(\mathcal{G})$, $\mathcal{G}^\circ(r)$ stands for the element of $\mathcal{G}(r)$ having minimum modulus, and $\mathcal{G}_\varepsilon$ denotes the Yosida regularization of $\mathcal{G}$ at the level ε (see, e.g., [3, p. 28]).

For the data, we make the following assumptions:

$$u \in (H^1(0, +\infty; L^3(\Omega)))^3. \tag{2.13}$$

$$\operatorname{div} u = 0 \quad \text{in } Q_\infty \quad \text{and} \quad u \cdot \nu = 0 \quad \text{on } \Sigma_\infty. \tag{2.14}$$

$$(\rho_0\,,\ \rho_{0|\Gamma}) \in \mathcal{W}\,, \quad \beta^\circ(\rho_0) \in H \quad \text{and} \quad \beta^\circ_\Gamma(\rho_{0|\Gamma}) \in H_\Gamma. \tag{2.15}$$

$$m_0 := \frac{\int_\Omega \rho_0 + \int_\Gamma \rho_{0|\Gamma}}{|\Omega| + |\Gamma|} \quad \text{belongs to the interior of } D(\beta_\Gamma). \tag{2.16}$$

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Notice that (2.15) implies that $f(\rho_0) \in L^1(\Omega)$ and $f_\Gamma(\rho_{0|\Gamma}) \in L^1(\Gamma)$. Moreover, (2.13) entails that $u \in L^\infty(0,+\infty; L^3(\Omega))$.

Let us come to our notion of solution, which requires a low regularity level. A solution on $(0,T)$ is a triple of pairs $((\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\zeta,\zeta_\Gamma))$ that satisfies

$$(\mu,\mu_\Gamma) \in L^2(0,T;\mathcal{V}), \tag{2.17}$$
$$(\rho,\rho_\Gamma) \in H^1(0,T;\mathcal{V}^*) \cap L^\infty(0,T;\mathcal{V}), \tag{2.18}$$
$$(\zeta,\zeta_\Gamma) \in L^2(0,T;\mathcal{H}). \tag{2.19}$$

However, we write $(\mu,\mu_\Gamma,\rho,\rho_\Gamma,\zeta,\zeta_\Gamma)$ instead of $((\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\zeta,\zeta_\Gamma))$ in order to simplify the notation. As far as the problem under study is concerned, we still state it in a weak form as in [18], by owing to the assumptions (2.14) on u. Namely, we require that

$$\begin{aligned}
&\langle \partial_t(\rho,\rho_\Gamma),(v,v_\Gamma)\rangle_{\mathcal{V}} - \int_\Omega \rho u \cdot \nabla v + \int_\Omega \nabla\mu \cdot \nabla v + \int_\Gamma \nabla_\Gamma \mu_\Gamma \cdot \nabla_\Gamma v_\Gamma = 0 \\
&\quad \text{a.e. in } (0,T) \text{ and for every } (v,v_\Gamma) \in \mathcal{V}, && (2.20)\\
&\tau_\Omega \int_\Omega \partial_t\rho\, v + \tau_\Gamma \int_\Gamma \partial_t\rho_\Gamma\, v_\Gamma + \int_\Omega \nabla\rho\cdot\nabla v + \int_\Gamma \nabla_\Gamma\rho_\Gamma \cdot \nabla_\Gamma v_\Gamma \\
&\quad + \int_\Omega \big(\zeta + \pi(\rho)\big)v + \int_\Gamma \big(\zeta_\Gamma + \pi_\Gamma(\rho_\Gamma)\big)v_\Gamma = \int_\Omega \mu v + \int_\Gamma \mu_\Gamma v_\Gamma \\
&\quad \text{a.e. in } (0,T) \text{ and for every } (v,v_\Gamma) \in \mathcal{V}, && (2.21)\\
&\zeta \in \beta(\rho) \quad \text{a.e. in } Q_T \quad \text{and} \quad \zeta_\Gamma \in \beta_\Gamma(\rho_\Gamma) \quad \text{a.e. on } \Sigma_T, && (2.22)\\
&\rho(0) = \rho_0 \quad \text{a.e. in } \Omega\,. && (2.23)
\end{aligned}$$

The basic well-posedness result is stated below. It was proved in [18, Thm. 2.3] and holds even in a slightly more general situation. In fact, the condition (2.15) can be weakened, and (2.9) is dispensable. Moreover, (2.12) is needed just for the proof of uniqueness.

Theorem 2.1. *Assume that* (2.5)–(2.12) *for the structure and* (2.13)–(2.16) *for the data are fulfilled, and let $T \in (0,+\infty)$ be given. Then the problem* (2.20)–(2.23) *has a unique solution $(\mu,\mu_\Gamma,\rho,\rho_\Gamma,\zeta,\zeta_\Gamma)$ satisfying* (2.17)–(2.19).

We can obviously draw the following consequence:

Corollary 2.2. *Assume that* (2.5)–(2.12) *and* (2.13)–(2.16) *are satisfied. Then there exists a unique 6-tuple $(\mu,\mu_\Gamma,\rho,\rho_\Gamma,\zeta,\zeta_\Gamma)$ defined on $(0,+\infty)$ that fulfills* (2.17)–(2.19) *and solves* (2.20)–(2.23) *for every $T \in (0,+\infty)$.*

We remark that the solution is actually much smoother if both τ_Ω and τ_Γ are strictly positive and the initial datum ρ_0 satisfies further conditions, as it was proved in [18, Thm. 2.6]. In particular, under those assumptions the derivative $\partial_t(\rho,\rho_\Gamma)$ can be split into components, namely, we have the representation

$$\langle \partial_t(\rho,\rho_\Gamma),(v,v_\Gamma)\rangle_{\mathcal{V}} = \int_\Omega \partial_t\rho\, v + \int_\Gamma \partial_t\rho_\Gamma\, v_\Gamma \tag{2.24}$$

for every $(v,v_\Gamma) \in \mathcal{V}$, almost everywhere in $(0,+\infty)$.

At this point, given a solution $(\mu,\mu_\Gamma,\rho,\rho_\Gamma,\zeta,\zeta_\Gamma)$, our first aim to investigate its longtime behavior, namely, the ω-limit (which we simply term ω, for brevity) of the component (ρ,ρ_Γ). We notice that requiring (2.18) for every $T \in (0, +\infty)$ implies that (ρ, ρ_Γ) is a weakly continuous $\mathcal{V}$-valued function, so that the following definition is meaningful. We set

$$\omega := \big\{(\rho^\omega, \rho_\Gamma^\omega) = \lim_{n\to\infty} (\rho, \rho_\Gamma)(t_n) \quad \text{in the weak topology of } \mathcal{V} \\ \text{for some sequence } \{t_n\}_{n\in\mathbb{N}} \text{ such that } t_n \nearrow +\infty\big\} \tag{2.25}$$

and look for the relationship between ω and the set of stationary solutions to the system which is obtained from (2.20)–(2.22) by ignoring the convective term. Indeed, assumption (2.13) implies that $u(t)$ tends to the zero function strongly in $L^3(\Omega)$ as t approaches infinity. It is immediately seen from (2.20) that the components μ and μ_Γ of every stationary solution are spatially constant functions and that the constant values they assume are the same. Therefore, by a stationary solution we mean a quadruplet $(\rho^s, \rho_\Gamma^s, \zeta^s, \zeta_\Gamma^s)$ satisfying, for some $\mu^s \in \mathbb{R}$, the conditions

$$(\rho^s, \rho_\Gamma^s) \in \mathcal{V} \quad \text{and} \quad (\zeta^s, \zeta_\Gamma^s) \in \mathcal{H}, \tag{2.26}$$

$$\int_\Omega \nabla\rho^s \cdot \nabla v + \int_\Gamma \nabla_\Gamma \rho_\Gamma^s \cdot \nabla_\Gamma v_\Gamma + \int_\Omega \big(\zeta^s + \pi(\rho^s)\big) v + \int_\Gamma \big(\zeta_\Gamma^s + \pi_\Gamma(\rho_\Gamma^s)\big) v_\Gamma \\ = \int_\Omega \mu^s v + \int_\Gamma \mu^s v_\Gamma \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}, \tag{2.27}$$

$$\zeta^s \in \beta(\rho^s) \quad \text{a.e. in } \Omega \quad \text{and} \quad \zeta_\Gamma^s \in \beta_\Gamma(\rho_\Gamma^s) \quad \text{a.e. on } \Gamma\,. \tag{2.28}$$

It is not difficult to show that the conditions (2.26)–(2.27) imply that the pair (ρ^s, ρ_Γ^s) belongs to $\mathcal{W}$ and satisfies the boundary value problem

$$-\Delta\rho^s + \zeta^s + \pi(\rho^s) = \mu^s \quad \text{a.e. in } \Omega,$$
$$\partial_\nu \rho^s - \Delta_\Gamma \rho_\Gamma^s + \zeta_\Gamma^s + \pi_\Gamma(\rho_\Gamma^s) = \mu^s \quad \text{a.e. on } \Gamma\,.$$

In [21], the following result was proved:

Theorem 2.3. *Let the assumptions of Corollary 2.2 be satisfied. Moreover, assume that the viscosity coefficients τ_Ω and τ_Γ are strictly positive, and let $(\mu, \mu_\Gamma, \rho, \rho_\Gamma, \zeta, \zeta_\Gamma)$ be the unique global solution on $(0, +\infty)$. Then the ω-limit* (2.25) *is nonempty. Moreover, for every $(\rho^\omega, \rho_\Gamma^\omega) \in \omega$, there exist some $\mu^s \in \mathbb{R}$ and a solution $(\rho^s, \rho_\Gamma^s, \zeta^s, \zeta_\Gamma^s)$ to* (2.26)–(2.28) *such that $(\rho^\omega, \rho_\Gamma^\omega) = (\rho^s, \rho_\Gamma^s)$.*

More precisely, this result holds under slightly weaker assumptions on the velocity field. The first aim of this paper is to extend Theorem 2.3 to the pure case, in which $\tau_\Omega = \tau_\Gamma = 0$, and to the partially viscous situations, i.e., when either $\tau_\Omega > \tau_\Gamma = 0$ or $\tau_\Gamma > \tau_\Omega = 0$. In each of these cases, we need our assumption (2.13) on u and further conditions on the initial datum. Thus, we give a list of properties that could be required on ρ_0. We recall that β_ε and $\beta_{\Gamma,\eta\varepsilon}$ are the Yosida regularizations of β and β_Γ, where η is the constant that appears in (2.11). Moreover, for $v_\Gamma \in V_\Gamma$, we use the symbol v_Γ^h for the harmonic extension of v_Γ to $\overline{\Omega}$, i.e.,

$$v_\Gamma^h \in H^1(\Omega), \quad -\Delta v_\Gamma^h = 0 \quad \text{in } \Omega \quad \text{and} \quad (v_\Gamma^h)_{|\Gamma} = v_\Gamma\,. \tag{2.29}$$

Here are our possible assumptions:

$$\big\|\big(-\Delta\rho_0 + (\beta_\varepsilon + \pi)(\rho_0), \partial_\nu\rho_0 - \Delta_\Gamma \rho_{0|\Gamma} + (\beta_{\Gamma,\eta\varepsilon} + \pi_\Gamma)(\rho_{0|\Gamma})\big)\big\|_{\mathcal{V}} \le C\,, \tag{2.30}$$

$$\big\|\big((w_\Gamma^\varepsilon)^h, w_\Gamma^\varepsilon\big)\big\|_{\mathcal{V}} \le C\,, \quad \text{where} \quad w_\Gamma^\varepsilon := \partial_\nu\rho_0 - \Delta_\Gamma\rho_{0|\Gamma} + (\beta_{\Gamma,\eta\varepsilon} + \pi_\Gamma)(\rho_{0|\Gamma})\,, \tag{2.31}$$

$$\big\|\big(-\Delta\rho_0 + (\beta_\varepsilon + \pi)(\rho_0), (-\Delta\rho_0 + (\beta_\varepsilon + \pi)(\rho_0))_{|\Gamma}\big)\big\|_{\mathcal{V}} \le C\,, \tag{2.32}$$

for some constant C and every $\varepsilon > 0$ small enough. We have chosen to present the above conditions in a unified form, but some of them could be simplified. Some comments are given in the forthcoming Remark 2.5.

Theorem 2.4. *Suppose that the assumptions of Corollary 2.2 are satisfied, and let* $(\mu, \mu_\Gamma, \rho, \rho_\Gamma, \zeta, \zeta_\Gamma)$ *be the unique global solution on* $(0, +\infty)$*. Moreover, assume either* (2.30)*, or* $\tau_\Omega > 0$ *and* (2.31)*, or* $\tau_\Gamma > 0$ *and* (2.32)*. Then the same assertions as in Theorem 2.3 are valid.*

Remark 2.5. The above compatibility assumptions (2.30)–(2.32) seem to be rather restrictive. However, one can find reasonable sufficient conditions for them in the case that the potentials f and f_Γ are either everywhere defined regular potentials, like (1.2), or smooth in $(-1, 1)$ and singular at the end-points ± 1, like the logarithmic potential (1.3). Let us consider the latter case. As for (2.32), by observing that $\|(v_\Gamma)^h\|_V \le C_\Omega \|v_\Gamma\|_{V_\Gamma}$ for every $v_\Gamma \in V_\Gamma$ with C_Ω depending only on Ω, we see that (2.31) is equivalent to the boundedness in V_Γ of the second component. Thus, it is sufficient to assume that $\rho_{0|\Gamma} \in H^2(\Gamma)$ (which follows from (2.15)) and that $\|\rho_{0|\Gamma}\|_\infty < 1$. Moreover, also the condition (2.32) is a consequence of (2.15) if we assume that $\|\rho_0\|_\infty < 1$. So, in fact, only the assumption (2.30) is very restrictive, since it postulates that the second component of the pair appearing there be the trace of the first one. This obviously holds if $\pi(0) = \pi_\Gamma(0) = 0$ and $\rho_0 \in H^3_0(\Omega)$, since (2.6) implies that $\beta_\varepsilon(0) = \beta_{\Gamma,\eta\varepsilon}(0) = 0$. If (2.15) is assumed, a different sufficient condition is the following: f and f_Γ are the same logarithmic type potential on $(-1, 1)$, $\|\rho_0\|_\infty < 1$, and $(\Delta\rho_0, \Delta_\Gamma\rho_{0|\Gamma} - \partial_\nu\rho_0) \in \mathcal{V}$.

The second aim of this paper is to extend the results of [19] regarding the control problem to the pure case $\tau_\Omega = \tau_\Gamma = 0$ on a fixed time interval $[0, T]$. We thus use the simpler notations

$$Q := \Omega \times (0, T) \quad \text{and} \quad \Sigma := \Gamma \times (0, T),$$

omitting the subscript T. The problem addressed in [19] consists of minimizing the cost functional

$$\begin{aligned} \mathcal{J}_{gen}((\mu, \mu_\Gamma), (\rho, \rho_\Gamma), u) &:= \frac{\beta_1}{2} \int_Q |\mu - \widehat{\mu}_Q|^2 + \frac{\beta_2}{2} \int_\Sigma |\mu_\Gamma - \widehat{\mu}_\Sigma|^2 \\ &+ \frac{\beta_3}{2} \int_Q |\rho - \widehat{\rho}_Q|^2 + \frac{\beta_4}{2} \int_\Sigma |\rho - \widehat{\rho}_\Sigma|^2 \\ &+ \frac{\beta_5}{2} \int_\Omega |\rho(T) - \widehat{\rho}_\Omega|^2 + \frac{\beta_6}{2} \int_\Gamma |\rho_\Gamma(T) - \widehat{\rho}_\Gamma|^2 + \frac{\beta_7}{2} \int_Q |u|^2, \end{aligned} \tag{2.33}$$

subject to the state system (2.20)–(2.23) on the time interval $[0, T]$ and to the control constraint $u \in \mathcal{U}_{ad}$, where the convex set $\mathcal{U}_{ad}$ and the related spaces are defined by

$$\mathcal{U}_{ad} := \{u \in \mathcal{X} : \ |u| \le \overline{U} \text{ a.e. in } Q_T, \ \|u\|_{\mathcal{X}} \le R_0\}, \tag{2.34}$$

$$\mathcal{X} := L^2(0, T; Z) \cap (L^\infty(Q))^3 \cap (H^1(0, T; L^3(\Omega)))^3, \tag{2.35}$$

$$Z := \{w \in (L^2(\Omega))^3 : \ \operatorname{div} w = 0 \text{ in } \Omega \text{ and } w \cdot \nu = 0 \text{ on } \Gamma\}. \tag{2.36}$$

In (2.33), the constants β_i, $1 \le i \le 7$, are nonnegative but not all zero, and $\widehat{\mu}_Q$, $\widehat{\mu}_\Sigma$, $\widehat{\rho}_Q$, $\widehat{\rho}_\Sigma$, $\widehat{\rho}_\Omega$, and $\widehat{\rho}_\Gamma$, are prescribed target functions. In (2.34), the function $\overline{U}$ and the constant R_0 are given in such a way that $\mathcal{U}_{ad}$ is nonempty. Moreover, the whole treatement is done in [19] just for the case $\beta_1 = \beta_2 = 0$. We therefore do the same in the present paper from the very beginning, for simplicity, even though such an assumption is not needed immediately. Next, we point out that the results of [19] were only established for potentials f and f_Γ of logarithmic type. Nevertheless, these results also hold true in the case of everywhere defined smooth potentials whenever a uniform L^∞ bound for the component ρ of the solution can be established. Indeed, the form of the logarithmic potential is (besides its obvious physical importance) used there just to ensure that ρ attains its values in a compact subset of the domain $D(\beta) = D(\beta_\Gamma) = (-1,1)$ and that the potentials are smooth in such an interval. Here are our additional assumptions:

$$D(\beta) = D(\beta_\Gamma) = \mathbb{R}, \quad \text{and} \quad \beta, \beta_\Gamma, \pi \text{ and } \pi_\Gamma \text{ are } C^2 \text{ functions.} \tag{2.37}$$

$$\beta_i \ge 0 \quad \text{for } 3 \le i \le 7. \tag{2.38}$$

$$\widehat\rho_Q \in L^2(Q), \quad \widehat\rho_\Sigma \in L^2(\Sigma), \quad \widehat\rho_\Omega \in L^2(\Omega), \quad \text{and} \quad \widehat\rho_\Gamma \in L^2(\Gamma). \tag{2.39}$$

$$\text{The function } \overline{U} \in L^\infty(Q) \text{ and the constant } R_0 > 0 \text{ are such that } \mathcal{U}_{ad} \neq \emptyset. \tag{2.40}$$

We notice that (2.37) allows us to rewrite (2.22) as $\zeta = \beta(\rho)$ and $\zeta_\Gamma = \beta_\Gamma(\rho_\Gamma)$. Therefore, when speaking of a solution, we tacitly assume the validity of these identities and just refer to the quadruplet $(\mu, \mu_\Gamma, \rho, \rho_\Gamma)$. We also remark that (2.37) can be expressed in terms of the potentials f and f_Γ defined in (2.8) by just saying that they are C^3 functions on the whole real line. Moreover, since $\beta_1 = \beta_2 = 0$, we rewrite the cost functional (2.33) in the form

$$\begin{aligned}\mathcal{J}((\rho,\rho_\Gamma),u) := \frac{\beta_3}{2}\int_Q |\rho - \widehat\rho_Q|^2 + \frac{\beta_4}{2}\int_\Sigma |\rho - \widehat\rho_\Sigma|^2 \\ + \frac{\beta_5}{2}\int_\Omega |\rho(T) - \widehat\rho_\Omega|^2 + \frac{\beta_6}{2}\int_\Gamma |\rho_\Gamma(T) - \widehat\rho_\Gamma|^2 + \frac{\beta_7}{2}\int_Q |u|^2 .\end{aligned} \tag{2.41}$$

In the present paper, we can develop a rather complete theory only under strong assumptions on the initial datum ρ_0, for which we require that (2.30) holds, even from the very beginning, for simplicity. We first extend a simplified version of the result of [19, Thm. 4.1] (which required the viscosity coefficients τ_Ω and τ_Γ to be positive) to the pure Cahn–Hilliard system. Indeed, we have the following result.

Theorem 2.6. *Assume* (2.5)–(2.12) *and* (2.37) *on the structure,* (2.13)–(2.16) *and* (2.30) *on the data,* (2.38) *and* (2.39) *on the cost functional, and* (2.40) *on the control box. Then the problem of minimizing the functional* (2.41) *subject to the state system* (2.20)–(2.23) *with* $\tau_\Omega = \tau_\Gamma = 0$ *and to the control constraint* $u \in \mathcal{U}_{ad}$ *has at least one solution.*

The next step is to find necessary conditions for optimality. We first recall the corresponding results of [19]. The main tool is the adjoint problem associated with an optimal control $\overline{u}$ and the corresponding state $(\overline{\mu}, \overline{\mu}_\Gamma, \overline{\rho}, \overline{\rho}_\Gamma)$: find a quadruplet $(p, p_\Gamma, q, q_\Gamma)$ satisfying the regularity requirements

$$(p, p_\Gamma) \in L^2(0,T;\mathcal{V}), \quad (q, q_\Gamma) \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}), \tag{2.42}$$

$$(p + \tau_\Omega q, p_\Gamma + \tau_\Gamma q_\Gamma) \in H^1(0,T;\mathcal{V}^*), \tag{2.43}$$

and solving

$$\begin{aligned} -\langle \partial_t(p + \tau_\Omega q, p_\Gamma + \tau_\Gamma q_\Gamma), (v, v_\Gamma)\rangle_{\mathcal{V}} + \int_\Omega \nabla q \cdot \nabla v + \int_\Gamma \nabla_\Gamma q_\Gamma \cdot \nabla_\Gamma v_\Gamma \\ + \int_\Omega \psi q v + \int_\Gamma \psi_\Gamma q_\Gamma v_\Gamma - \int_\Omega \overline{u} \cdot \nabla p \, v = \int_\Omega \varphi_3 \, v + \int_\Gamma \varphi_4 \, v_\Gamma \\ \text{a.e. in } (0,T) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}, \end{aligned} \tag{2.44}$$

$$\int_\Omega \nabla p \cdot \nabla v + \int_\Gamma \nabla_\Gamma p_\Gamma \cdot \nabla_\Gamma v_\Gamma = \int_\Omega qv + \int_\Gamma q_\Gamma v_\Gamma$$
$$\text{a.e. in } (0,T) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}, \tag{2.45}$$

$$\langle (p + \tau_\Omega q, p_\Gamma + \tau_\Gamma q_\Gamma)(T), (v, v_\Gamma) \rangle_{\mathcal{V}} = \int_\Omega \varphi_5 \, v + \int_\Gamma \varphi_6 v_\Gamma$$
$$\text{for every } (v, v_\Gamma) \in \mathcal{V}, \tag{2.46}$$

where τ_Ω and τ_Γ are positive and the following abbreviations are used:

$$\psi := f''(\overline{\rho}) \quad \text{and} \quad \psi_\Gamma := f''_\Gamma(\overline{\rho}_\Gamma) \tag{2.47}$$
$$\varphi_3 := \beta_3(\overline{\rho} - \widehat{\rho}_Q), \quad \varphi_4 := \beta_4(\overline{\rho}_\Gamma - \widehat{\rho}_\Sigma), \tag{2.48}$$
$$\varphi_5 := \beta_5(\overline{\rho}(T) - \widehat{\rho}_\Omega), \quad \varphi_6 := \beta_6(\overline{\rho}_\Gamma(T) - \widehat{\rho}_\Gamma). \tag{2.49}$$

In the quoted paper (see [19, Thm. 4.4]), an existence result for the above problem was proved under the assumption that τ_Ω and τ_Γ be positive. Moreover, the solution turned out to be unique if $\tau_\Omega = \tau_\Gamma$. In the same paper (see [19, Thm. 4.6]), the following optimality condition was derived:

$$\int_Q \big(\overline{\rho}\, \nabla p + \beta_7 \overline{u}\big) \cdot (v - \overline{u}) \geq 0 \quad \text{for every } v \in \mathcal{U}_{ad}\,. \tag{2.50}$$

In particular, if $\beta_7 > 0$, the optimal control $\overline{u}$ is the L^2-projection of $-\frac{1}{\beta_7}\, \overline{\rho}\, \nabla p$ on $\mathcal{U}_{ad}$. We once more remark that all this was proved in [19] for the case of logarithmic potentials only; but the same analysis can be carried out for everywhere smooth potentials under the assumptions of Theorem 2.6 provided that τ_Ω and τ_Γ are positive.

In the present paper, we prove a weak version of these results in the pure case, i.e., when $\tau_\Omega = \tau_\Gamma = 0$. Indeed, it turns out that the adjoint problem has to be presented in a time-integrated form. To describe it, for every $v \in L^1(Q)$ (and similarly for functions in $L^1(\Sigma)$ or in some product space), we introduce the backward-in-time convolution $1 * v$ by the formula

$$(1 * v)(t) := \int_t^T v(s)\, ds \quad \text{for a.a. } t \in (0,T). \tag{2.51}$$

Then, in the pure case, the adjoint problem associated with an optimal control $\overline{u}$ and the corresponding state $(\overline{\mu}, \overline{\mu}_\Gamma, \overline{\rho}, \overline{\rho}_\Gamma)$ consists in finding a quadruplet $(p, p_\Gamma, q, q_\Gamma)$ satisfying the regularity requirement

$$(p, p_\Gamma) \in L^2(0,T;\mathcal{V}), \tag{2.52}$$
$$(q, q_\Gamma) \in L^2(0,T;\mathcal{H}) \quad \text{with} \quad 1 * (q, q_\Gamma) \in L^\infty(0,T;\mathcal{V}), \tag{2.53}$$

and solving the system (with the notations (2.47)–(2.49))

$$\int_\Omega pv + \int_\Gamma p_\Gamma v_\Gamma + \int_\Omega \nabla(1*q)\cdot\nabla v + \int_\Gamma \nabla_\Gamma(1*q_\Gamma)\cdot\nabla v_\Gamma$$
$$+ \int_\Omega (1*(\psi q))\, v + \int_\Gamma (1*(\psi_\Gamma q_\Gamma)\, v_\Gamma + \int_\Omega (1*(\overline{u}\cdot\nabla p))\, v$$
$$= \int_\Omega (1*\varphi_3)v + \int_\Gamma (1*\varphi_4)v_\Gamma + \int_\Omega \varphi_5 v + \int_\Gamma \varphi_6 v_\Gamma$$

for a.a. $t\in(0,T)$ and for every $(v,v_\Gamma)\in\mathcal{V}$ (2.54)

$$\int_Q \nabla p\cdot\nabla v + \int_\Sigma \nabla_\Gamma p_\Gamma\cdot\nabla_\Gamma v_\Gamma = \int_Q qv + \int_\Sigma q_\Gamma v_\Gamma$$

for a.a. $t\in(0,T)$ and for every $(v,v_\Gamma)\in L^2(0,T;\mathcal{V})$. (2.55)

Here is our result on first-order optimality conditions.

Theorem 2.7. *Suppose that the conditions* (2.5)–(2.12) *and* (2.37) *on the structure,* (2.13)–(2.16) *and* (2.30) *on the data,* (2.38) *and* (2.39) *on the cost functional, and* (2.40) *on the control box, are fulfilled. Moreover, assume that* $\overline{u}$ *and* $(\overline{\mu},\overline{\mu}_\Gamma,\overline{\rho},\overline{\rho}_\Gamma)$ *are an optimal control and the corresponding optimal state, respectively. Then there exists at least one quadruplet* (p,p_Γ,q,q_Γ) *satisfying the conditions* (2.52)–(2.53) *and solving both* (2.54)–(2.55) *and the variational inequality*

$$\int_Q (\overline{\rho}\,\nabla p + \beta_7\overline{u})\cdot(v-\overline{u}) \ge 0 \quad \textit{for every } v\in\mathcal{U}_{ad}. \tag{2.56}$$

In particular, if $\beta_7>0$*, then the optimal control* $\overline{u}$ *is the* L^2*-projection of* $-\frac{1}{\beta_7}\,\overline{\rho}\,\nabla p$ *on* $\mathcal{U}_{ad}$ *.*

Remark 2.8. We cannot prove a uniqueness result for the solution to problem (2.54)–(2.55), unfortunately. However, as will be stated in the forthcoming Remark 4.3, the solution is unique provided it is somewhat smoother. Namely, it is needed that its component (q,q_Γ) belongs to $L^2(0,T;\mathcal{V})$, which we are not able to prove (while the regularity $(p,p_\Gamma)\in L^2(0,T;\mathcal{W})$ we did not require is true and immediately follows from (2.55) and the first (2.53) by applying [18, Lem. 3.1]).

As it has been stated at the end of the introduction, the proofs of our results will be given in Sections 3 and 4. In order to be able to carry out these proofs, we will now introduce some auxiliary tools, namely, the generalized mean value, the related spaces, and the operator $\mathcal{N}$. In doing this, we will be very brief, referring to [18, Sect. 2] for further details. We set

$$\operatorname{mean} g^* := \frac{\langle g^*,(1,1)\rangle_{\mathcal{V}}}{|\Omega|+|\Gamma|} \quad \text{for } g^*\in\mathcal{V}^* \tag{2.57}$$

and observe that

$$\operatorname{mean}(v,v_\Gamma) = \frac{\int_\Omega v + \int_\Gamma v_\Gamma}{|\Omega|+|\Gamma|} \quad \text{if } (v,v_\Gamma)\in\mathcal{H}\,. \tag{2.58}$$

Notice that the constant m_0 appearing in assumption (2.16) is nothing but the mean value $\operatorname{mean}(\rho_0,\rho_{0|\Gamma})$, and that taking $(v,v_\Gamma)=(|\Omega|+|\Gamma|)^{-1}(1,1)$ in (2.21) yields the conservation property for the component (ρ,ρ_Γ) of the solution,

$$\partial_t \operatorname{mean}(\rho,\rho_\Gamma)=0, \quad \text{whence} \quad \operatorname{mean}(\rho,\rho_\Gamma)(t)=m_0 \quad \text{for every } t\in[0,T]. \tag{2.59}$$

We also stress that the function

$$\mathcal{V} \ni (v, v_\Gamma) \mapsto \big(\|\nabla v\|^2_{L^2(0,T;H)} + \|\nabla_\Gamma v_\Gamma\|^2_{L^2(0,T;H_\Gamma)} + |\operatorname{mean}(v, v_\Gamma)|^2 \big)^{1/2} \tag{2.60}$$

yields a Hilbert norm on $\mathcal{V}$ that is equivalent to the natural one. Now, we set

$$\mathcal{V}_{*0} := \{ g^* \in \mathcal{V}^* : \ \operatorname{mean} g^* = 0 \}, \quad \mathcal{H}_0 := \mathcal{H} \cap \mathcal{V}_{*0} \quad \text{and} \quad \mathcal{V}_0 := \mathcal{V} \cap \mathcal{V}_{*0}, \tag{2.61}$$

and notice that the function

$$\mathcal{V}_0 \ni (v, v_\Gamma) \mapsto \|(v, v_\Gamma)\|_{\mathcal{V}_0} := \big(\|\nabla v\|^2_{L^2(0,T;H)} + \|\nabla_\Gamma v_\Gamma\|^2_{L^2(0,T;H_\Gamma)} \big)^{1/2} \tag{2.62}$$

is a Hilbert norm on $\mathcal{V}_0$ which is equivalent to the usual one. Next, we define the operator $\mathcal{N} : \mathcal{V}_{*0} \to \mathcal{V}_0$ (which will be applied to $\mathcal{V}_{*0}$-valued functions as well) as follows: for every element $g^* \in \mathcal{V}_{*0}$, we have that

$$\begin{aligned} &\mathcal{N} g^* = (\mathcal{N}_\Omega g^*, \mathcal{N}_\Gamma g^*) \text{ is the unique pair } (\xi, \xi_\Gamma) \in \mathcal{V}_0 \text{ such that} \\ &\int_\Omega \nabla \xi \cdot \nabla v + \int_\Gamma \nabla_\Gamma \xi_\Gamma \cdot \nabla_\Gamma v_\Gamma = \langle g^*, (v, v_\Gamma) \rangle_{\mathcal{V}} \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}. \end{aligned} \tag{2.63}$$

It turns out that $\mathcal{N}$ is well defined, linear, symmetric, and bijective. Therefore, if we set

$$\|g^*\|_* := \|\mathcal{N} g^*\|_{\mathcal{V}_0}, \quad \text{for } g^* \in \mathcal{V}_{*0}, \tag{2.64}$$

then we obtain a Hilbert norm on $\mathcal{V}_{*0}$ (equivalent to the norm induced by the norm of $\mathcal{V}^*$), and we have that

$$\langle g^*, \mathcal{N} g^* \rangle_{\mathcal{V}} = \|g^*\|^2_* \quad \text{for every } g^* \in \mathcal{V}_{*0}. \tag{2.65}$$

Furthermore, we notice that

$$\langle \partial_t g^*, \mathcal{N} g^* \rangle_{\mathcal{V}} = \frac{1}{2} \frac{d}{dt} \|g^*\|^2_* \quad \text{a.e. in } (0,T), \quad \text{for every } g^* \in H^1(0,T; \mathcal{V}_{*0}). \tag{2.66}$$

It is easy to see that $\mathcal{N} g^*$ belongs to $\mathcal{W}$ whenever $g^* \in \mathcal{H}_0$ and that

$$\|\mathcal{N} g^*\|_{\mathcal{W}} \le C_\Omega \|g^*\|_{\mathcal{H}} \quad \text{for every } g^* \in \mathcal{H}_0, \tag{2.67}$$

where C_Ω depends only on Ω.

Besides the above tools, we will repeatedly use the Young inequality

$$a\, b \le \delta\, a^2 + \frac{1}{4\delta}\, b^2 \quad \text{for all } a, b \in \mathbb{R} \text{ and } \delta > 0, \tag{2.68}$$

as well as Hölder's inequality and the Sobolev inequality

$$\|v\|_p \le C_\Omega \|v\|_V \quad \text{for every } p \in [1,6] \text{ and } v \in V, \tag{2.69}$$

which is related to the continuous embedding $V \subset L^p(\Omega)$ for $p \in [1,6]$ (since Ω is three-dimensional, bounded and smooth). In particular, by also using the equivalent norm (2.60) on $\mathcal{V}$, we have that

$$\|v\|_6^2 \le C_\Omega \big(\|\nabla v\|^2_{L^2(0,T;H)} + \|\nabla_\Gamma v_\Gamma\|^2_{L^2(0,T;H_\Gamma)} + |\operatorname{mean}(v, v_\Gamma)|^2 \big) \tag{2.70}$$

for every $(v, v_\Gamma) \in \mathcal{V}$. In both (2.69) and (2.70), the constant C_Ω depends only on Ω. We also account for the compact embedding $\mathcal{V} \subset \mathcal{H}$ and for the corresponding compactness inequality

$$\|(v, v_\Gamma)\|_{\mathcal{H}}^2 \le \delta \, \|(\nabla v, \nabla_\Gamma v_\Gamma)\|_{\mathcal{H}}^2 + C_{\delta,\Omega} \, \|(v, v_\Gamma)\|_{\mathcal{V}^*}^2 \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}, \tag{2.71}$$

where $\delta > 0$ is arbitrary, and where the constant $C_{\delta,\Omega}$ depends only on Ω and δ.

Finally, as far as constants are concerned, we employ the following general rule: the small-case symbol c stands for different constants which depend only on Ω, the structure of our system and the norms of the data involved in the assumptions (2.13)–(2.16). A notation like c_δ (in particular, with $\delta = T$) allows the constant to depend on the positive parameter δ, in addition. Hence, the meaning of c and c_δ might change from line to line and even within the same chain of inequalities. On the contrary, we mark those constants that we want to refer to by using a different notation (e.g., a capital letter).

3 Longtime behavior

This section is devoted to the proof of Theorem 2.4. However, the procedure we follow is useful for the next section as well, where we prove our results concerning the control problem in the pure case. Here, we fix any global solution $(\mu, \mu_\Gamma, \rho, \rho_\Gamma, \zeta, \zeta_\Gamma)$ once and for all. Our method closely follows the proof of Theorem 2.3 performed in [21]; it thus relies on some global a priori estimates and on the study of the behavior of the a solution on intervals of a fixed length T whose endpoints approach infinity. However, the proof of each crucial estimate will have to be modified.

For brevity, we often proceed formally. In particular, we behave as if the solution were smooth and use the identity

$$\langle \partial_t(\rho, \rho_\Gamma), (v, v_\Gamma) \rangle_{\mathcal{V}} = \int_\Omega \partial_t \rho \, v + \int_\Gamma \partial_t \rho_\Gamma \, v_\Gamma \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}, \tag{3.1}$$

which is not justified, since this representation holds true only in the viscous case $\tau_\Omega > 0$, $\tau_\Gamma > 0$ (see (2.24)). Moreover, we argue as if even the graphs β and β_Γ were everywhere defined smooth functions and often write $\beta(\rho)$ and $\beta_\Gamma(\rho_\Gamma)$ instead of ζ and ζ_Γ, respectively. However, the estimates obtained in this way can be performed rigorously on a proper approximating problem, uniformly with respect to the approximation parameter. The best choice for such an approximating problem could be one of the following: $a)$ the ε-problem which is analogous to the one introduced in [18] and obtained by replacing the graphs β and β_Γ with very smooth functions β_ε and $\beta_{\Gamma,\eta\varepsilon}$ that behave like the Yosida regularizations (i.e., with similar boundedness and convergence properties, like the C^∞ approximations introduced in [33, Sect. 3]); $b)$ the Faedo–Galerkin scheme (depending on the parameter $n \in \mathbb{N}$) used in [18] in order to discretize and then solve the ε-problem: indeed, all of the components of its solution are very smooth.

Now, we start proving some global estimates, assuming that τ_Ω and τ_Γ are nonnegative in order to cover all relevant situations at the same time, if possible. However, we will sometimes be forced to distinguish between the pure case and the partially viscous ones.

First global estimate. We recall the conservation property (2.59), write the equations (2.20) and (2.21) at the time s, and test them by $(\mu, \mu_\Gamma)(s) + \mathcal{N}(\partial_t(\rho, \rho_\Gamma)(s)) \in \mathcal{V}$ and $2\partial_t(\rho, \rho_\Gamma)(s) \in \mathcal{V}$, respectively. Then, we use (2.65) with $g^* = \partial_t(\rho, \rho_\Gamma)(s)$, integrate with respect to s over $(0, t)$ with an arbitrary $t > 0$, sum up and rearrange. We obtain the identity

$$\begin{aligned}
&\int_{Q_t} \partial_t\rho\,\mu + \int_{\Sigma_t} \partial_t\rho_\Gamma\,\mu_\Gamma + \int_{Q_t} |\nabla\mu|^2 + \int_{\Sigma_t} |\nabla_\Gamma\mu_\Gamma|^2 \\
&\quad + \int_0^t \|\partial_t(\rho,\rho_\Gamma)(s)\|_*^2\,ds + \int_{Q_t} \nabla\mu\cdot\nabla\mathcal{N}_\Omega(\partial_t(\rho,\rho_\Gamma)) + \int_{\Sigma_t} \nabla_\Gamma\mu_\Gamma\cdot\nabla_\Gamma\mathcal{N}_\Gamma(\partial_t(\rho,\rho_\Gamma)) \\
&\quad + 2\tau_\Omega\int_{Q_t} |\partial_t\rho|^2 + 2\tau_\Gamma\int_{\Sigma_t} |\partial_t\rho_\Gamma|^2 \\
&\quad + \int_\Omega |\nabla\rho(t)|^2 + \int_\Gamma |\nabla_\Gamma\rho_\Gamma(t)|^2 + 2\int_\Omega f(\rho(t)) + 2\int_\Gamma f_\Gamma(\rho_\Gamma(t)) \\
&= \int_{Q_t} \rho u\cdot\big(\nabla\mu + \nabla\mathcal{N}_\Omega(\partial_t(\rho,\rho_\Gamma))\big) + 2\int_{Q_t} \mu\partial_t\rho + 2\int_{\Sigma_t} \mu_\Gamma\partial_t\rho_\Gamma \\
&\quad + \int_\Omega |\nabla\rho_0|^2 + \int_\Gamma |\nabla_\Gamma\rho_{0|\Gamma}|^2 + 2\int_\Omega f(\rho_0) + 2\int_\Gamma f_\Gamma(\rho_{0|\Gamma}).
\end{aligned}$$

Some integrals cancel out by the definition (2.63) of $\mathcal{N}$, the terms on the left-hand side containing f and f_Γ are bounded from below by (2.9), and the ones on the right-hand side involving the initial values are finite by (2.15). We deal with the convective term by owing to the Young and Hölder inequalities, the Sobolev type inequality (2.70), and the conservation property (2.59). We have

$$\begin{aligned}
&\int_{Q_t} \rho u\cdot\big(\nabla\mu + \nabla\mathcal{N}_\Omega(\partial_t(\rho,\rho_\Gamma))\big) \\
&\le \frac12\int_{Q_t} |\nabla\mu|^2 + \frac12\int_0^t \|\mathcal{N}(\partial_t(\rho,\rho_\Gamma)(s))\|_{\mathcal{V}_0}^2\,ds + \int_0^t \|u(s)\|_3^2\,\|\rho(s)\|_6^2\,ds \\
&\le \frac12\int_{Q_t} |\nabla\mu|^2 + \frac12\int_0^t \|\partial_t(\rho,\rho_\Gamma)(s)\|_*^2\,ds \\
&\quad + c\int_0^t \|u(s)\|_3^2\,\big(\|\nabla\rho(s)\|_{L^2(0,T;H)}^2 + \|\nabla_\Gamma\rho_\Gamma(s)\|_{L^2(0,T;H_\Gamma)}^2 + m_0^2\big)\,ds.
\end{aligned}$$

Since the function $s\mapsto \|u(s)\|_3^2$ belongs to $L^1(0,+\infty)$ by (2.13), we can apply the Gronwall lemma on $(0,+\infty)$ and obtain that

$$\int_{Q_\infty} |\nabla\mu|^2 + \int_{\Sigma_\infty} |\nabla_\Gamma\mu_\Gamma|^2 + \int_0^{+\infty} \|\partial_t(\rho,\rho_\Gamma)(t)\|_*^2\,dt < +\infty\,, \tag{3.2}$$

$$\int_{Q_\infty} |\partial_t\rho|^2 < +\infty \quad \text{if } \tau_\Omega > 0 \quad \text{and} \quad \int_{\Sigma_\infty} |\partial_t\rho_\Gamma|^2 < +\infty \quad \text{if } \tau_\Gamma > 0\,, \tag{3.3}$$

$$f(\rho) \in L^\infty(0,+\infty;L^1(\Omega)) \quad \text{and} \quad f_\Gamma(\rho_\Gamma) \in L^\infty(0,+\infty;L^1(\Gamma))\,, \tag{3.4}$$

as well as $(\nabla\rho,\nabla_\Gamma\rho_\Gamma) \in (L^\infty(0,+\infty;\mathcal{H}))^3$. From this, by accounting for the conservation property (2.59) once more, we conclude that

$$(\rho,\rho_\Gamma) \in L^\infty(0,+\infty;\mathcal{V}). \tag{3.5}$$

Consequence. By using the quadratic growth of $\widehat{\pi}$ and $\widehat{\pi}_\Gamma$, which is implied by the Lipschitz continuity of their derivatives, and combining with (3.4) with (3.5), we deduce that

$$\widehat{\beta}(\rho) \in L^\infty(0,+\infty;L^1(\Omega)) \quad \text{and} \quad \widehat{\beta}_\Gamma(\rho_\Gamma) \in L^\infty(0,+\infty;L^1(\Gamma))\,. \tag{3.6}$$

Second global estimate. We formally differentiate the equations (2.20) and (2.21) with respect to time, behaving as if β and β_Γ were smooth functions and writing $\beta(\rho)$ and $\beta_\Gamma(\rho_\Gamma)$ instead of ζ and ζ_Γ (see (2.22)). We obtain

$$\int_\Omega \partial_t^2\rho\, v + \int_\Gamma \partial_t^2\rho_\Gamma\, v_\Gamma + \int_\Omega \nabla\partial_t\mu\cdot\nabla v + \int_\Gamma \nabla_\Gamma\partial_t\mu_\Gamma\cdot\nabla_\Gamma v_\Gamma$$
$$= \int_\Omega \bigl(\partial_t\rho\, u + \rho\partial_t u\bigr)\cdot\nabla v\,, \tag{3.7}$$
$$\tau_\Omega \int_\Omega \partial_t^2\rho\, v + \tau_\Gamma \int_\Gamma \partial_t^2\rho_\Gamma\, v_\Gamma + \int_\Omega \nabla\partial_t\rho\cdot\nabla v + \int_\Gamma \nabla_\Gamma\partial_t\rho_\Gamma\cdot\nabla_\Gamma v_\Gamma$$
$$\quad + \int_\Omega \beta'(\rho)\partial_t\rho\, v + \int_\Gamma \beta_\Gamma'(\rho_\Gamma)\partial_t\rho_\Gamma\, v_\Gamma$$
$$= \int_\Omega \partial_t\mu\, v + \int_\Gamma \partial_t\mu_\Gamma\, v_\Gamma - \int_\Omega \pi'(\rho)\partial_t\rho\, v - \int_\Gamma \pi_\Gamma'(\rho_\Gamma)\partial_t\rho_\Gamma\, v_\Gamma\,, \tag{3.8}$$

a.e. in $(0,+\infty)$, and for every $(v,v_\Gamma)\in\mathcal{V}$. Recalling that $\partial_t(\rho,\rho_\Gamma)$ is $\mathcal{V}_0$-valued by (2.59), so that $\mathcal{N}\partial_t(\rho,\rho_\Gamma)$ is well defined, we write the above equations at the time s and test them by $\mathcal{N}\partial_t(\rho,\rho_\Gamma)(s)$ and $\partial_t(\rho,\rho_\Gamma)(s)$, respectively. Then we integrate with respect to s over $(0,t)$ with an arbitrary $t>0$ and sum up. It follows that

$$\int_0^t \langle \partial_t^2(\rho,\rho_\Gamma)(s), \mathcal{N}\partial_t(\rho,\rho_\Gamma)(s)\rangle_{\mathcal{V}}\, ds$$
$$\quad + \int_{Q_t} \nabla\partial_t\mu\cdot\nabla\mathcal{N}_\Omega(\partial_t(\rho,\rho_\Gamma)) + \int_{\Sigma_t} \nabla_\Gamma\partial_t\mu_\Gamma\cdot\nabla_\Gamma\mathcal{N}_\Gamma(\partial_t(\rho,\rho_\Gamma))$$
$$\quad + \tau_\Omega\int_{Q_t} \partial_t^2\rho\,\partial_t\rho + \tau_\Gamma\int_{\Sigma_t} \partial_t^2\rho_\Gamma\,\partial_t\rho_\Gamma + \int_{Q_t} |\nabla\partial_t\rho|^2 + \int_{\Sigma_t} |\nabla_\Gamma\partial_t\rho_\Gamma|^2$$
$$\quad + \int_{Q_t} \beta'(\rho)|\partial_t\rho|^2 + \int_{\Sigma_t} \beta_\Gamma'(\rho_\Gamma)|\partial_t\rho_\Gamma|^2$$
$$= \int_{Q_t} \bigl(\partial_t\rho\, u + \rho\partial_t u\bigr)\cdot\nabla\mathcal{N}_\Omega(\partial_t(\rho,\rho_\Gamma))$$
$$\quad + \int_{Q_t} \partial_t\mu\,\partial_t\rho + \int_{\Sigma_t} \partial_t\mu_\Gamma\,\partial_t\rho_\Gamma - \int_{Q_t} \pi'(\rho)|\partial_t\rho|^2 - \int_{\Sigma_t} \pi_\Gamma'(\rho_\Gamma)|\partial_t\rho_\Gamma|^2.$$

The integrals containing $\partial_t\mu$ and $\partial_t\mu_\Gamma$ cancel out by the definition (2.63) of $\mathcal{N}$ (with the choices $g^* = \partial_t(\rho,\rho_\Gamma)(s)$ and $(v,v_\Gamma) = \partial_t(\mu,\mu_\Gamma)(s)$), and the terms involving β' and β_Γ' are nonnegative. Moreover, the last two integrals on the right-hand side can be dealt with by accounting for the Lipschitz continuity of π and π_Γ and the compactness inequality (2.71),

$$-\int_{Q_t} \pi'(\rho)|\partial_t\rho|^2 - \int_{\Sigma_t} \pi_\Gamma'(\rho_\Gamma)|\partial_t\rho_\Gamma|^2$$
$$\le \frac12\int_{Q_t} |\nabla\partial_t\rho|^2 + \frac12\int_{\Sigma_t} |\nabla_\Gamma\partial_t\rho_\Gamma|^2 + c\int_0^t \|\partial_t(\rho,\rho_\Gamma)(s)\|_*^2\, ds\,.$$

Since the last integral is bounded by (3.2), coming back to the previous identity and owing to (2.66) for the first term on the left-hand side, we deduce that

$$\frac{1}{2}\|\partial_t(\rho,\rho_\Gamma)(t)\|_*^2 + \frac{\tau_\Omega}{2}\int_\Omega |\partial_t\rho(t)|^2 + \frac{\tau_\Gamma}{2}\int_\Gamma |\partial_t\rho_\Gamma(t)|^2 + \frac{1}{2}\int_{Q_t}|\nabla\partial_t\rho|^2 + \frac{1}{2}\int_{\Sigma_t}|\nabla_\Gamma\partial_t\rho_\Gamma|^2$$
$$\leq \frac{1}{2}\|\partial_t(\rho,\rho_\Gamma)(0)\|_*^2 + \frac{\tau_\Omega}{2}\int_\Omega |\partial_t\rho(0)|^2 + \frac{\tau_\Gamma}{2}\int_\Gamma |\partial_t\rho_\Gamma(0)|^2$$
$$+ \int_{Q_t}\big(\partial_t\rho\, u + \rho\partial_t u\big)\cdot\nabla\mathcal{N}_\Omega(\partial_t(\rho,\rho_\Gamma)) + c\,. \tag{3.9}$$

Thus, it suffices to estimate the last integral and obtain bounds for the time derivatives evaluated at 0. For the first aim, we use the Hölder, Sobolev and Young inequalities (in particular, (2.70)), the conservation property (2.59), and the already established estimates (3.2) and (3.5). We then have

$$\int_{Q_t}\big(\partial_t\rho\, u + \rho\partial_t u\big)\cdot\nabla\mathcal{N}_\Omega(\partial_t(\rho,\rho_\Gamma))$$
$$\leq \int_0^t \big(\|\partial_t\rho(s)\|_6\,\|u(s)\|_3 + \|\rho(s)\|_6\,\|\partial_t u(s)\|_3\big)\|\nabla\mathcal{N}_\Omega(\partial_t(\rho,\rho_\Gamma)(s))\|_2\,ds$$
$$\leq \frac{1}{4}\Big(\int_{Q_t}|\nabla\partial_t\rho|^2 + \int_{\Sigma_t}|\nabla_\Gamma\partial_t\rho_\Gamma|^2\Big)$$
$$+ c\,\|u\|^2_{L^\infty(0,+\infty;L^3(\Omega))}\int_0^t \|\partial_t(\rho,\rho_\Gamma)(s)\|_*^2\,ds + c\,\|\rho\|^2_{L^\infty(0,+\infty;V)}\int_0^t\|\partial_t u(s)\|_3^2\,ds$$
$$\leq \frac{1}{4}\Big(\int_{Q_t}|\nabla\partial_t\rho|^2 + \int_{\Sigma_t}|\nabla_\Gamma\partial_t\rho_\Gamma|^2\Big) + c\,.$$

In order to find bounds for the initial value of the time derivatives, we have to distinguish between the pure case and the partially viscous ones. Assume first that $\tau_\Omega = \tau_\Gamma = 0$. Then the only initial value that appears on the right-hand side of (3.9) is $\|\partial_t(\rho,\rho_\Gamma)(0)\|_*$, and we can estimate it by accounting for (2.30), which we formally write as

$$\big(-\Delta\rho_0 + (\beta+\pi)(\rho_0), \partial_\nu\rho_0 - \Delta_\Gamma\rho_{0|\Gamma} + (\beta_\Gamma+\pi_\Gamma)(\rho_{0|\Gamma})\big) \in \mathcal{V}\,.$$

We write (2.20) and (2.21) at the time $t = 0$, test them by $(\mu,\mu_\Gamma)(0) + \mathcal{N}(\partial_t(\rho,\rho_\Gamma)(0))$ and $2\partial_t(\rho,\rho_\Gamma)(0)$, respectively, and sum up. Then, we owe to (2.65) with $g^* = \partial_t(\rho,\rho_\Gamma)(0)$, integrate by parts the terms involving $\nabla\rho_0$ and $\nabla_\Gamma\rho_{0|\Gamma}$, and rearrange. We obtain

$$\int_\Omega \partial_t\rho(0)\,\mu(0) + \int_\Gamma \partial_t\rho_\Gamma(0)\,\mu_\Gamma(0) + \int_\Omega|\nabla\mu(0)|^2 + \int_\Gamma|\nabla_\Gamma\mu_\Gamma(0)|^2$$
$$+ \|\partial_t(\rho,\rho_\Gamma)(0)\|_*^2$$
$$+ \int_\Omega \nabla\mu(0)\cdot\nabla\mathcal{N}_\Omega(\partial_t(\rho,\rho_\Gamma)(0)) + \int_\Gamma \nabla_\Gamma\mu_\Gamma(0)\cdot\nabla_\Gamma\mathcal{N}_\Gamma(\partial_t(\rho,\rho_\Gamma)(0))$$
$$= \int_\Omega \rho_0 u(0)\cdot\big(\nabla\mu(0) + \nabla\mathcal{N}_\Omega(\partial_t(\rho,\rho_\Gamma)(0))\big) + 2\int_\Omega \mu(0)\partial_t\rho(0) + 2\int_\Gamma \mu_\Gamma(0)\partial_t\rho_\Gamma(0)$$
$$- 2\int_\Omega\big(-\Delta\rho_0 + (\beta+\pi)(\rho_0)\big)\,\partial_t\rho(0)$$
$$- 2\int_\Gamma\big(\partial_\nu\rho_0 - \Delta_\Gamma\rho_{0|\Gamma} + (\beta_\Gamma+\pi)(\rho_{0|\Gamma})\big)\,\partial_t\rho_\Gamma(0). \tag{3.10}$$

Some integrals cancel each other, either trivially or by the definition of $\mathcal{N}$. In particular, what remains on the right-hand side are just the first and the last two terms, the sum of which is estimated from above by

$$\frac{1}{2}\,\|\partial_t(\rho,\rho_\Gamma)(0)\|_*^2 + c\|\bigl(-\Delta\rho_0 + (\beta+\pi)(\rho_0), \partial_\nu\rho_0 - \Delta_\Gamma\rho_{0|\Gamma} + (\beta_\Gamma+\pi_\Gamma)(\rho_{0|\Gamma})\bigr)\|_{\mathcal{V}}^2,$$

thanks to the Young inequality. Thus, we infer that

$$\int_\Omega |\nabla\mu(0)|^2 + \frac{1}{2}\,\|\partial_t(\rho,\rho_\Gamma)(0)\|_*^2 \le \int_\Omega \rho_0 u(0)\cdot\bigl(\nabla\mu(0) + \nabla\mathcal{N}_\Omega(\partial_t(\rho,\rho_\Gamma)(0))\bigr) + c\,.$$

On the other hand, by the Hölder inequality and due to our assumptions (2.15) and (2.13) on ρ_0 and u, we also have that

$$\begin{aligned}
&\int_\Omega \rho_0 u(0)\cdot\bigl(\nabla\mu(0) + \nabla\mathcal{N}_\Omega(\partial_t(\rho,\rho_\Gamma)(0))\bigr)\\
&\le \frac{1}{2}\int_\Omega |\nabla\mu(0)|^2 + \frac{1}{4}\,\|\nabla\mathcal{N}_\Omega(\partial_t(\rho,\rho_\Gamma)(0))\|_2^2 + c\,\bigl(\|\rho_0\|_6^2 + \|u(0)\|_3^2\bigr)\\
&\le \frac{1}{2}\int_\Omega |\nabla\mu(0)|^2 + \frac{1}{4}\,\|\mathcal{N}(\partial_t(\rho,\rho_\Gamma)(0))\|_{\mathcal{V}_0}^2 + c\\
&\le \frac{1}{2}\int_\Omega |\nabla\mu(0)|^2 + \frac{1}{4}\,\|\partial_t(\rho,\rho_\Gamma)(0)\|_*^2 + c\,.
\end{aligned}$$

By combining this estimate with the above inequality, we obtain the sought bound for $\|\partial_t(\rho,\rho_\Gamma)(0)\|_*$.

If one of the viscosity parameters τ_Ω and τ_Γ is positive, a similar argument applies. Moreover, we can owe to the weaker assumption (2.15) and to either (2.31) or (2.32) (instead of (2.30)) on the initial datum ρ_0, as we show at once. The right-hand side of (3.9) also contains the integrals

$$\frac{\tau_\Omega}{2}\int_\Omega |\partial_t\rho(0)|^2 + \frac{\tau_\Gamma}{2}\int_\Omega |\partial_t\rho_\Gamma(0)|^2\,, \tag{3.11}$$

and one of them is significant and has to be estimated as well. Assume first that $\tau_\Omega > 0$ and $\tau_\Gamma = 0$. Then, testing (2.21) for $t = 0$ by $\partial_t(\rho,\rho_\Gamma)(0)$, as done above, yields the additional term $\tau_\Omega\int_\Omega |\partial_t\rho(0)|^2$ on the left-hand side of (3.10), whence (3.11) can actually be estimated. But the presence of $\tau_\Omega\int_\Omega |\partial_t\rho(0)|^2$ on the left-hand side also helps in estimating the last two terms on the right-hand side of (3.10) by taking advantage of (2.15) and (2.31) instead of the more restrictive (2.30). We formally write (2.31) as

$$(w_\Gamma^h, w_\Gamma) \in \mathcal{V} \quad \text{where} \quad w_\Gamma := \partial_\nu\rho_0 - \Delta_\Gamma\rho_{0|\Gamma} + (\beta_\Gamma+\pi_\Gamma)(\rho_{0|\Gamma}).$$

Since $\|v_\Gamma^h\|_H \le c\|(v_\Gamma^h, v_\Gamma)\|_{\mathcal{V}}$ for every $v_\Gamma \in V_\Gamma$, we have

$$\begin{aligned}
&-2\int_\Omega \bigl(-\Delta\rho_0 + (\beta+\pi)(\rho_0)\bigr)\,\partial_t\rho(0) - 2\int_\Gamma \bigl(\partial_\nu\rho_0 - \Delta_\Gamma\rho_{0|\Gamma} + (\beta_\Gamma+\pi_\Gamma)(\rho_{0|\Gamma})\bigr)\,\partial_t\rho_\Gamma(0)\\
&= -2\int_\Omega \bigl[\bigl(-\Delta\rho_0 + (\beta+\pi)(\rho_0)\bigr) - w_\Gamma^h\bigr]\,\partial_t\rho(0) - 2\,\langle\partial_t(\rho,\rho_\Gamma)(0), (w_\Gamma^h, w_\Gamma)\rangle_{\mathcal{V}}\\
&\le \frac{\tau_\Omega}{4}\int_\Omega |\partial_t\rho(0)|^2 + \frac{1}{4}\,\|\partial_t(\rho,\rho_\Gamma)(0)\|_*^2\\
&\quad + c\,\|-\Delta\rho_0 + (\beta+\pi)(\rho_0)\|_H^2 + c\,\|w_\Gamma^h\|_H^2 + c\,\|(w_\Gamma^h, w_\Gamma)\|_{\mathcal{V}}\\
&\le \frac{\tau_\Omega}{4}\int_\Omega |\partial_t\rho(0)|^2 + \frac{1}{4}\,\|\partial_t(\rho,\rho_\Gamma)(0)\|_*^2 + c\,,
\end{aligned}$$

and this leads to the desired estimate.

Assume now that $\tau_\Gamma > 0$ and $\tau_\Omega = 0$. Similarly as before, testing (3.8) by $\partial_t(\rho, \rho_\Gamma)$ yields a nonnegative contribution to the left-hand side and the second term of (3.11) on the right-hand side. But testing (2.21) for $t = 0$ by $\partial_t(\rho, \rho_\Gamma)(0)$ now produces the integral $\tau_\Gamma \int_\Gamma |\partial_t \rho_\Gamma(0)|^2$ on the left-hand side of (3.10) (whence (3.11) can be estimated) and allows us to treat the last two terms of the right-hand side of (3.10) by owing to (2.15) and (2.32). We formally write the latter as

$$\big(-\Delta\rho_0 + (\beta+\pi)(\rho_0), (-\Delta\rho_0 + (\beta+\pi)(\rho_0))_{|\Gamma}\big) \in \mathcal{V}\,.$$

By taking advantage of the inequalities $\|v_\Gamma\|_{H_\Gamma} \le c\,\|(v, v_\Gamma)\|_{\mathcal{V}}$ for every $(v, v_\Gamma) \in \mathcal{V}$, and $\|\partial_\nu v\|_{H_\Gamma} \le c\,\|v\|_W$ for every $v \in W$ (the former is obvious and the latter is well known), we have the estimate

$$\begin{aligned}
&-2\int_\Omega \big(-\Delta\rho_0 + (\beta+\pi)(\rho_0)\big)\,\partial_t\rho(0) - 2\int_\Gamma \big(\partial_\nu\rho_0 - \Delta_\Gamma\rho_{0|\Gamma} + (\beta_\Gamma+\pi)(\rho_{0|\Gamma})\big)\,\partial_t\rho_\Gamma(0)\\
&= -2\,\langle \partial_t(\rho,\rho_\Gamma)(0), \big(-\Delta\rho_0 + (\beta+\pi)(\rho_0), (-\Delta\rho_0 + (\beta+\pi)(\rho_0))_{|\Gamma}\big)\rangle_{\mathcal{V}}\\
&\quad + 2\int_\Gamma \big[(-\Delta\rho_0 + (\beta+\pi)(\rho_0))_{|\Gamma} - (-\Delta_\Gamma\rho_{0|\Gamma} + (\beta_\Gamma+\pi_\Gamma)(\rho_{0|\Gamma})) - \partial_\nu\rho_0\big]\partial_t\rho_\Gamma(0)\\
&\le \frac14\,\|\partial_t(\rho,\rho_\Gamma)(0)\|_*^2 + \frac{\tau_\Gamma}{4}\int_\Gamma |\partial_t\rho_\Gamma(0)|^2\\
&\quad + c\,\|\big(-\Delta\rho_0 + (\beta+\pi)(\rho_0), (-\Delta\rho_0 + (\beta+\pi)(\rho_0))_{|\Gamma}\big)\|_{\mathcal{V}}^2\\
&\quad + c\,\|(-\Delta\rho_0 + (\beta+\pi)(\rho_0))_{|\Gamma}\|_{H_\Gamma}^2 + c\,\|-\Delta_\Gamma\rho_{0|\Gamma} + (\beta_\Gamma+\pi_\Gamma)(\rho_{0|\Gamma})\|_{H_\Gamma}^2 + c\,\|\partial_\nu\rho_0\|_{H_\Gamma}^2\\
&\le \frac14\,\|\partial_t(\rho,\rho_\Gamma)(0)\|_*^2 + \frac{\tau_\Gamma}{4}\int_\Gamma |\partial_t\rho_\Gamma(0)|^2 + c\,,
\end{aligned}$$

whence the desired bounds for the initial values of the time derivatives follow.

Now, coming back to (3.9), and taking these estimates into account, we conclude that

$$\partial_t(\rho,\rho_\Gamma) \in L^\infty(0,+\infty;\mathcal{V}^*)\,, \tag{3.12}$$

$$\partial_t\rho \in L^\infty(0,+\infty;H) \quad \text{if } \tau_\Omega > 0 \quad \text{and} \quad \partial_t\rho_\Gamma \in L^\infty(0,+\infty;H_\Gamma) \quad \text{if } \tau_\Gamma > 0\,. \tag{3.13}$$

As a by-product, we also obtain the less important result that $\partial_t(\rho,\rho_\Gamma) \in L^2(0,+\infty;\mathcal{V})$.

In the rest of the section, we only consider the pure case $\tau_\Omega = \tau_\Gamma = 0$. However, the whole argument works in the partially viscous cases as well. Indeed, testing equations as we do would produce just additional contributions that can be dealt with in a trivial way.

Third global estimate. Once again, we write $\zeta = \beta(\rho)$ and $\zeta_\Gamma = \beta(\rho_\Gamma)$ and first notice that the inclusion $D(\beta_\Gamma) \subseteq D(\beta)$ (see (2.11)) and assumption (2.16) imply that

$$\beta(r)(r-m_0) \ge \delta_0|\beta(r)| - C_0 \quad \text{and} \quad \beta_\Gamma(r)(r-m_0) \ge \delta_0|\beta_\Gamma(r)| - C_0 \tag{3.14}$$

for every r belonging to the respective domains, where δ_0 and C_0 are some positive constants that depend only on β, β_Γ and on the position of m_0 in the interior of $D(\beta_\Gamma)$ and of $D(\beta)$ (see, e.g. [32, p. 908]). Now, we recall the conservation property (2.59) and test (2.20) and (2.21) by $\mathcal{N}(\rho - m_0, \rho_\Gamma - m_0)$ and $(\rho - m_0, \rho_\Gamma - m_0)$, respectively. Then we add, without integrating with respect to time. We obtain, a.e. in $(0,+\infty)$,

$$\begin{aligned}
&\langle \partial_t(\rho,\rho_\Gamma), \mathcal{N}(\rho-m_0,\rho_\Gamma-m_0)\rangle_{\mathcal{V}} \\
&\quad + \int_\Omega \nabla\mu\cdot\nabla\mathcal{N}_\Omega(\rho-m_0,\rho_\Gamma-m_0) + \int_\Gamma \nabla_\Gamma\mu_\Gamma\cdot\nabla_\Gamma\mathcal{N}_\Gamma(\rho-m_0,\rho_\Gamma-m_0) \\
&\quad + \int_\Omega |\nabla\rho|^2 + \int_\Gamma |\nabla_\Gamma\rho_\Gamma|^2 + \int_\Omega \beta(\rho)(\rho-m_0) + \int_\Gamma \beta_\Gamma(\rho_\Gamma)(\rho_\Gamma-m_0) \\
&= \int_\Omega \rho u\cdot\nabla(\mathcal{N}_\Omega(\rho-\rho_0,\rho_\Gamma-\rho_{0|\Gamma})) + \int_\Omega \mu(\rho-m_0) + \int_\Gamma \mu_\Gamma(\rho_\Gamma-m_0) \\
&\quad - \int_\Omega \pi(\rho)(\rho-m_0) - \int_\Gamma \pi_\Gamma(\rho_\Gamma)(\rho_\Gamma-m_0).
\end{aligned}$$

All of the integrals involving μ and μ_Γ cancel out by (2.63). Now, we owe to (3.14), keep just the positive contribution on the left-hand side, and move the other terms to the right-hand side. Next, we account for the Lipschitz continuity of π and π_Γ, the Hölder and Sobolev inequalities, our assumption (2.13) on u, (3.5), and (3.12). It then results that, a.e. in $(0,+\infty)$,

$$\begin{aligned}
&\int_\Omega |\nabla\rho|^2 + \int_\Gamma |\nabla_\Gamma\rho_\Gamma|^2 + \delta_0\int_\Omega |\beta(\rho)| + \delta_0\int_\Gamma |\beta_\Gamma(\rho_\Gamma)| \\
&\leq \|\partial_t(\rho,\rho_\Gamma)\|_* \, \|\mathcal{N}(\rho-m_0,\rho_\Gamma-m_0)\|_{\mathcal{V}_0} + c\,\|(\pi(\rho),\pi_\Gamma(\rho_\Gamma))\|_{\mathcal{H}}\,\|(\rho-m_0,\rho_\Gamma-m_0)\|_{\mathcal{H}} \\
&\quad + \|\rho\|_6\,\|u\|_3\,\|\nabla(\mathcal{N}_\Omega(\rho-\rho_0,\rho_\Gamma-\rho_{0|\Gamma}))\|_2 \\
&\leq \|\partial_t(\rho,\rho_\Gamma)\|_*\,\|(\rho-m_0,\rho_\Gamma-m_0)\|_* + c\,\bigl(\|(\rho,\rho_\Gamma)\|_{\mathcal{H}}^2 + 1\bigr) \\
&\quad + c\,\|\rho\|_V\,\|(\rho-m_0,\rho_\Gamma-m_0)\|_* \,\leq\, c\,.
\end{aligned}$$

We deduce (in particular) that

$$\zeta\in L^\infty(0,+\infty;L^1(\Omega)) \quad \text{and} \quad \zeta_\Gamma\in L^\infty(0,+\infty;L^1(\Gamma)).$$

Now, we test (2.21) by $(1,1)$ to obtain that, a.e. in $(0,+\infty)$,

$$(|\Omega|+|\Gamma|)\,\mathrm{mean}(\mu,\mu_\Gamma) = \int_\Omega \zeta + \int_\Gamma \zeta_\Gamma + \int_\Omega \pi(\rho) + \int_\Gamma \pi_\Gamma(\rho_\Gamma).$$

Thus, we can infer that

$$\mathrm{mean}(\mu,\mu_\Gamma)\in L^\infty(0,+\infty)\,. \tag{3.15}$$

It was already clear from (3.5) that the ω-limit ω is nonempty. Indeed, the weakly continuous $\mathcal{V}$-valued function (ρ,ρ_Γ) is also bounded, so that there exists a sequence $t_n\nearrow+\infty$ such that the sequence $\{(\rho,\rho_\Gamma)(t_n)\}$ is weakly convergent in $\mathcal{V}$. More precisely, any sequence of times that tends to infinity contains a subsequence of this type. Thus, it remains to prove the second part of the statement. To this end, we fix an element $(\rho^\omega,\rho_\Gamma^\omega)\in\omega$ and a corresponding sequence $\{t_n\}$ as in the definition (2.25). We also fix some $T\in(0,+\infty)$, set for a.a. $t\in(0,T)$

$$\begin{aligned}
&\mu^n(t) := \mu(t_n+t), \quad \rho^n(t) := \rho(t_n+t), \quad \zeta^n(t) := \zeta(t_n+t), \\
&\mu_\Gamma^n(t) := \mu_\Gamma(t_n+t), \quad \rho_\Gamma^n(t) := \rho_\Gamma(t_n+t), \quad \zeta_\Gamma^n(t) := \zeta_\Gamma(t_n+t), \\
&u^n(t) := u(t_n+t),
\end{aligned}$$

and notice that (2.13) implies that

$$u^n \to 0 \quad \text{strongly in } L^\infty(0,T;L^3(\Omega)). \tag{3.16}$$

Moreover, it is clear that the 6-tuple $(\mu^n, \mu_\Gamma^n, \rho^n, \rho_\Gamma^n, \zeta^n, \zeta_\Gamma^n)$ satisfies the regularity conditions (2.17)–(2.19) and the equations (2.20)–(2.22) with u replaced by u^n, as well as the initial condition $\rho^n(0) = \rho(t_n)$. In particular, by construction, we have that

$$(\rho^n, \rho_\Gamma^n)(0) \to (\rho^\omega, \rho_\Gamma^\omega) \quad \text{weakly in } \mathcal{V}. \tag{3.17}$$

Furthermore, the global estimates already performed on $(\mu, \mu_\Gamma, \rho, \rho_\Gamma, \zeta, \zeta_\Gamma)$ immediately imply some estimates on $(\mu^n, \mu_\Gamma^n, \rho^n, \rho_\Gamma^n, \zeta^n, \zeta_\Gamma^n)$ that are uniform with respect to n. Here is a list. From (3.5) and (3.12), we infer that

$$\|(\rho^n, \rho_\Gamma^n)\|_{H^1(0,T;\mathcal{H})\cap L^\infty(0,T;\mathcal{V})} \le c\,. \tag{3.18}$$

By virtue of (3.2), we also deduce that

$$(\nabla\mu^n, \nabla_\Gamma\mu_\Gamma^n) \to 0 \quad \text{strongly in } (L^2(0,T;\mathcal{H}))^3, \tag{3.19}$$

$$(\partial_t\rho^n, \partial_t\rho_\Gamma^n) \to 0 \quad \text{strongly in } L^2(0,T;\mathcal{V}_0^*). \tag{3.20}$$

On the other hand, (3.15) yields a uniform estimate on the mean value $\operatorname{mean}(\mu^n, \mu_\Gamma^n)$. By combining this with (3.19), we conclude that

$$\|(\mu^n, \mu_\Gamma^n)\|_{L^2(0,T;\mathcal{V})} \le c_T\,. \tag{3.21}$$

However, the estimates obtained till now are not sufficient to conclude, and further estimates must be proved that ensure some better convergence for $(\mu^n, \mu_\Gamma^n, \rho^n, \rho_\Gamma^n, \zeta^n, \zeta_\Gamma^n)$ on the interval $(0,T)$. Even though the argument is the same as in [21], we repeat it here for the reader's convenience, at least in a short form.

First auxiliary estimate. We test (2.21) by the $\mathcal{V}$-valued function $(\beta(\rho^n), \beta(\rho_\Gamma^n))$ and integrate with respect to time. We have

$$\begin{aligned}
&\int_{Q_t} \beta'(\rho^n)|\nabla\rho^n|^2 + \int_{\Sigma_t} \beta'(\rho_\Gamma^n)|\nabla_\Gamma\rho_\Gamma^n|^2 + \int_{Q_t} |\beta(\rho^n)|^2 + \int_{\Sigma_t} \beta_\Gamma(\rho_\Gamma^n)\,\beta(\rho_\Gamma^n) \\
&= \int_{Q_t} \bigl(\mu^n - \pi(\rho^n)\bigr)\,\beta(\rho^n) + \int_{\Sigma_t} \bigl(\mu_\Gamma^n - \pi_\Gamma(\rho_\Gamma^n)\bigr)\,\beta(\rho_\Gamma^n)\,,
\end{aligned} \tag{3.22}$$

and we see that also the last integral on the left-hand side is essentially nonnegative. Indeed, from (2.11) and (2.6) it follows that

$$\int_{\Sigma_t} \beta_\Gamma(\rho_\Gamma^n)\,\beta(\rho_\Gamma^n) \ge \frac{1}{2\eta}\int_{\Sigma_t} |\beta(\rho_\Gamma^n)|^2 - c_T\,.$$

Thus, just the last integral on the right-hand side needs some treatment. We have

$$\int_{\Sigma_t} \mu_\Gamma^n\,\beta(\rho_\Gamma^n) \le \frac{1}{4\eta}\int_{\Sigma_t} |\beta(\rho_\Gamma^n)|^2 + c\int_{\Sigma_t} |\mu_\Gamma^n|^2 \le \frac{1}{4\eta}\int_{\Sigma_t} |\beta(\rho_\Gamma^n)|^2 + c\,,$$

thanks to (3.21). By combining these inequalities, we conclude that

$$\|\zeta^n\|_{L^2(0,T;H_\Gamma)} \le c_T\,, \tag{3.23}$$

as well as an estimate for $\|\beta(\rho_\Gamma^n)\|_{L^2(0,T;H_\Gamma)}$, as a by-product.

Second auxiliary estimate. We apply [18, Lem. 3.1] a.e. in $(0,+\infty)$ to (2.21), written for $(\mu^n,\mu_\Gamma^n,\rho^n,\rho_\Gamma^n,\zeta^n,\zeta_\Gamma^n)$, in the following form:

$$\begin{aligned}&\int_\Omega \nabla\rho^n\cdot\nabla v+\int_\Gamma \nabla_\Gamma\rho_\Gamma^n\cdot\nabla_\Gamma v_\Gamma+\int_\Gamma \beta_\Gamma(\rho_\Gamma^n)v_\Gamma\\&=\int_\Omega \mu^n v+\int_\Gamma \mu_\Gamma^n v_\Gamma-\tau_\Omega\int_\Omega \partial_t\rho^n\,v-\tau_\Gamma\int_\Gamma \partial_t\rho_\Gamma^n\,v_\Gamma-\int_\Omega\bigl(\zeta^n+\pi(\rho^n)\bigr)v-\int_\Gamma \pi_\Gamma(\rho_\Gamma^n)v_\Gamma.\end{aligned}$$

We obtain, in particular, that

$$\|\beta_\Gamma(\rho_\Gamma^n(t))\|\le c\left(\|(\mu^n,\mu_\Gamma^n)(t)\|_{\mathcal{H}}+\|\partial_t(\rho^n,\rho_\Gamma^n)(t)\|_H+\|(\zeta^n+\pi(\rho^n))(t)\|_H\right)$$

for a.a. $t\in(0,+\infty)$, where c depends only on Ω. By accounting for (3.18), (3.21), and (3.23), we conclude that

$$\|\zeta_\Gamma^n\|_{L^2(0,T;H_\Gamma)}\le c_T\,. \tag{3.24}$$

Limits. Also this step closely follows [21]. Thanks to the estimates (3.18), (3.21), (3.23) and (3.24), we have, for a subsequence which is still labeled by n,

$$(\rho^n,\rho_\Gamma^n)\to(\rho^\infty,\rho_\Gamma^\infty)\qquad\text{weakly-star in } H^1(0,T;\mathcal{H})\cap L^\infty(0,T;\mathcal{V}), \tag{3.25}$$

$$(\mu^n,\mu_\Gamma^n)\to(\mu^\infty,\mu_\Gamma^\infty)\qquad\text{weakly in } L^2(0,T;\mathcal{V}), \tag{3.26}$$

$$(\zeta^n,\zeta_\Gamma^n)\to(\zeta^\infty,\zeta_\Gamma^\infty)\qquad\text{weakly in } L^2(0,T;\mathcal{H}). \tag{3.27}$$

The next step is to show that $(\mu^\infty,\mu_\Gamma^\infty,\rho^\infty,\rho_\Gamma^\infty,\zeta^\infty,\zeta_\Gamma^\infty)$ satisfies (2.20)–(2.22) with $u=0$. Namely, we can derive the integrated version with time-dependent test functions $(v,v_\Gamma)\in L^2(0,T;\mathcal{V})$, as usual. First of all, we notice that $\rho^n u^n$ converges to zero weakly in $L^\infty(0,T;L^2(\Omega))$, since ρ^n converges to ρ^∞ weakly-star in $L^\infty(0,T;L^6(\Omega))$ and (3.16) holds. Next, we derive from (3.25) the strong convergence

$$(\rho^n,\rho_\Gamma^n)\to(\rho^\infty,\rho_\Gamma^\infty)\quad\text{strongly in } C^0([0,T];\mathcal{H}), \tag{3.28}$$

by owing to the compact embedding $\mathcal{V}\subset\mathcal{H}$ and applying, e.g., [51, Sect. 8, Cor. 4]. Hence, we can identify the limit of $(\pi(\rho^n),\pi_\Gamma(\rho_\Gamma^n))$ as $(\pi(\rho^\infty),\pi_\Gamma(\rho_\Gamma^\infty))$ just by Lipschitz continuity. This concludes the proof that (2.20) and (2.21) holds for the limiting 6-tuple in an integrated form, which is equivalent to the pointwise formulation. In order to derive that $\zeta^\infty\in\beta(\rho^\infty)$ and $\zeta_\Gamma^\infty\in\beta_\Gamma(\rho_\Gamma^\infty)$, we combine the weak convergence (3.27) with the strong convergence (3.28) and apply, e.g., [2, Lemma 2.3, p. 38].

Conclusion. It remains to prove that the above limit leads to a stationary solution having the properties specified in the statement. As in [21], we first derive that $(\rho^\infty,\rho_\Gamma^\infty)$ belongs to $L^2(0,T;\mathcal{W})$ and solves the boundary value problem

$$-\Delta\rho^\infty+\zeta^\infty+\pi(\rho^\infty)=\mu^\infty\quad\text{a.e. in } Q_T\,, \tag{3.29}$$

$$\partial_\nu\rho^\infty-\Delta_\Gamma\rho_\Gamma^\infty+\zeta_\Gamma^\infty+\pi_\Gamma(\rho_\Gamma^\infty)=\mu_\Gamma^\infty\quad\text{a.e. on } \Sigma_T\,. \tag{3.30}$$

Clearly, (3.20) implies that $\partial_t(\rho^\infty,\rho_\Gamma^\infty)$ vanishes identically, so that we are dealing with a time-dependent elliptic problem in a variational form. By using well-known estimates from trace theory and from the theory of elliptic equations, and invoking a bootstrap argument (see [21] for more details), we successively draw the following conclusions:

- (3.29) holds in the sense of distributions on Q_T;
- $\Delta\rho^\infty \in L^2(0,T;H)$, so that $\partial_\nu\rho^\infty$ is a well-defined element of $L^2(0,T;H^{-1/2}(\Gamma))$ satisfying the integration–by–parts formula in a generalized sense;
- (3.30) holds in a generalized sense;
- $\Delta_\Gamma\rho_\Gamma^\infty \in L^2(0,T;H^{-1/2}(\Gamma))$, so that $\rho_\Gamma^\infty \in L^2(0,T;H^{3/2}(\Gamma))$;
- $\rho^\infty \in L^2(0,T;W)$;
- $\partial_\nu\rho^\infty \in L^2(0,T;H_\Gamma)$, so that $\Delta_\Gamma\rho_\Gamma^\infty \in L^2(0,T;H_\Gamma)$ and $\rho_\Gamma^\infty \in L^2(0,T;W_\Gamma)$.

At this point, we can conclude the proof by repeating the argument of [21] for the reader's convenience. Since both $\partial_t(\rho^\infty,\rho_\Gamma^\infty)$ and $(\nabla\mu^\infty,\nabla_\Gamma\mu_\Gamma^\infty)$ vanish by (3.19)–(3.20), there exist $(\rho^s,\rho_\Gamma^s)\in\mathcal{V}$ and $\mu_\infty\in L^2(0,T)$ such that

$$(\rho^\infty,\rho_\Gamma^\infty)(x,t)=(\rho^s,\rho_\Gamma^s)(x) \quad \text{and} \quad (\mu^\infty,\mu_\Gamma^\infty)(x,t)=(\mu_\infty(t),\mu_\infty(t)) \quad \text{for a.a. } (x,t)\in Q_T\,.$$

We show that $(\zeta^\infty,\zeta_\Gamma^\infty)$ is time independent as well and that μ_∞ is a constant by accounting for (2.12). Assume that β is single-valued. Then, $\zeta^\infty=\beta(\rho^\infty)$ takes the value $\zeta^s:=\beta(\rho^s)$ at any time. Therefore, (3.29) implies that μ^∞ is time independent as well, so that the function μ_∞ is a constant that we term μ^s. Thus, the right-hand side of (3.30) is the same constant μ^s. As this does not depend on time, the same holds for ζ_Γ, which takes some value $\zeta_\Gamma^s\in H_\Gamma$ a.e. in $(0,T)$. Assume now that β_Γ is single-valued. Then, we first use (3.30) to derive that $\zeta_\Gamma^\infty=\beta_\Gamma(\rho_\Gamma^\infty)$ and μ_Γ^∞ are time independent. In particular, μ_∞ attains some constant value μ^s, so that ζ^∞ is time independent, by comparison in (3.29). Thus, in both cases the quadruplet $(\rho^s,\rho_\Gamma^s,\zeta^s,\zeta_\Gamma^s)$ is a stationary solution corresponding to the value μ^s of the chemical potential. Finally, we have that $(\rho^s,\rho_\Gamma^s)=(\rho^\omega,\rho_\Gamma^\omega)$. Indeed, (3.25) implies weak convergence also in $C^0([0,T];\mathcal{H})$, whence

$$(\rho^n,\rho_\Gamma^n)(0)\to(\rho^\infty,\rho_\Gamma^\infty)(0)=(\rho^s,\rho_\Gamma^s) \quad \text{weakly in } \mathcal{H},$$

and we can compare with (3.17). □

4 The optimal control problem

In this section, we are going to prove the Theorems 2.6 and 2.7. Thus, we fix $T>0$ and consider all problems on the finite time interval $(0,T)$. We adopt the ideas of [20] and take the problem (2.20)–(2.23), written with $\tau_\Omega=\tau_\Gamma=\tau\in(0,1)$, as an approximating problem for the pure case. We thus work with problems of the form

$$\langle\partial_t(\rho^\tau,\rho_\Gamma^\tau),(v,v_\Gamma)\rangle_{\mathcal{V}}-\int_\Omega\rho^\tau u\cdot\nabla v+\int_\Omega\nabla\mu^\tau\cdot\nabla v+\int_\Gamma\nabla_\Gamma\mu_\Gamma^\tau\cdot\nabla_\Gamma v_\Gamma=0$$
$$\text{a.e. in } (0,T) \text{ and for every } (v,v_\Gamma)\in\mathcal{V}, \tag{4.1}$$

$$\tau\int_\Omega\partial_t\rho^\tau\,v+\tau\int_\Gamma\partial_t\rho_\Gamma^\tau\,v_\Gamma+\int_\Omega\nabla\rho^\tau\cdot\nabla v+\int_\Gamma\nabla_\Gamma\rho_\Gamma^\tau\cdot\nabla_\Gamma v_\Gamma$$
$$+\int_\Omega(\beta+\pi)(\rho^\tau)\big)v+\int_\Gamma(\beta_\Gamma+\pi_\Gamma)(\rho_\Gamma^\tau)v_\Gamma=\int_\Omega\mu^\tau v+\int_\Gamma\mu_\Gamma^\tau v_\Gamma$$
$$\text{a.e. in } (0,T) \text{ and for every } (v,v_\Gamma)\in\mathcal{V}, \tag{4.2}$$

$$\rho^\tau(0)=\rho_0, \tag{4.3}$$

for given $u \in \mathcal{U}_{ad}$ and $\tau \in (0,T)$. Let us assume that the assumptions (2.5)–(2.12), (2.15)–(2.16), and (2.30) are fulfilled. Then we can infer from [18, Thm. 2.6] that the problem (4.1)–(4.3) has for every $u \in \mathcal{U}_{ad}$ and every $\tau \in (0,T)$ a unique solution $(\mu^\tau, \mu_\Gamma^\tau, \rho^\tau, \rho_\Gamma^\tau)$ such that

$$(\mu^\tau, \mu_\Gamma^\tau) \in L^\infty(0,T;\mathcal{W}), \quad (\rho^\tau, \rho_\Gamma^\tau) \in W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{W}). \tag{4.4}$$

In particular, we have that $\mu^\tau, \rho^\tau \in L^\infty(Q)$ and $\mu_\Gamma^\tau, \rho_\Gamma^\tau \in L^\infty(\Sigma)$.

We now aim to derive bounds for the solutions that are uniform with respect to $\tau \in (0,1)$ and $u \in \mathcal{U}_{ad}$. In particular, we establish a uniform L^∞ estimate for the solution to (2.20)–(2.23). This was already announced in the Remarks 2.7 and 7.1 of [18], but not proved, essentially. We sketch the derivation here, for the reader's convenience. For some of the steps, we proceed very quickly and refer for the details to the proofs of the estimates of [18, Sect. 6]. It the following, the symbol c denotes constants that are independent of $\tau \in (0,1)$ and $u \in \mathcal{U}_{ad}$. These constants may however depend on the fixed final time T, although we simply write c.

Uniform estimates. Suppose that $u \in \mathcal{U}_{ad}$. The argument used to prove the estimate (3.5) for the solution to (2.20)–(2.23) applies to the system (4.1)–(4.3) as well and yields (see also (3.2) and (3.4))

$$\|\nabla\mu^\tau\|_{(L^2(0,T;H))^3} + \|\nabla_\Gamma\mu_\Gamma^\tau\|_{(L^2(0,T;H_\Gamma))^3} + \|(\rho^\tau, \rho_\Gamma^\tau)\|_{H^1(0,T;\mathcal{V}^*)\cap L^\infty(0,T;\mathcal{V})} \le c\,, \tag{4.5}$$

$$\|f(\rho^\tau)\|_{L^\infty(0,T;L^1(\Omega))} + \|f_\Gamma(\rho_\Gamma^\tau)\|_{L^\infty(0,T;L^1(\Gamma))} \le c\,. \tag{4.6}$$

Then, we can follow the procedure used to derive (3.12) and obtain

$$\|\partial_t(\rho^\tau, \rho_\Gamma^\tau)\|_{L^\infty(0,T;\mathcal{V}^*)} \le c\,, \tag{4.7}$$

since the strong assumption (2.30) on ρ_0 is assumed to hold true. Now, we define $\widehat{m}(t) := \mathrm{mean}(\mu^\tau, \mu_\Gamma^\tau)(t)$ for a.a. $t \in (0,T)$ and test (4.1), written at the time t, by $(\mu^\tau(t) - \widehat{m}(t), \mu_\Gamma^\tau(t) - \widehat{m}(t)) \in \mathcal{V}_0$. We obtain, a.e. in $(0,T)$,

$$\int_\Omega |\nabla\mu^\tau|^2 + \int_\Gamma |\nabla_\Gamma\mu_\Gamma^\tau|^2 = -\langle \partial_t(\rho^\tau, \rho_\Gamma^\tau), (\mu^\tau - \widehat{m}, \mu_\Gamma^\tau - \widehat{m})\rangle_{\mathcal{V}} + \int_\Omega \rho^\tau u \cdot \nabla\mu^\tau.$$

By using the norms (2.62) and (2.64), as well as Young's inequality and the above estimates, we deduce that

$$\int_\Omega |\nabla\mu^\tau|^2 + \int_\Gamma |\nabla_\Gamma\mu_\Gamma^\tau|^2 \le \|\partial_t(\rho^\tau, \rho_\Gamma^\tau)\|_* \, \|(\mu^\tau - \widehat{m}, \mu_\Gamma^\tau - \widehat{m})\|_{\mathcal{V}_0} + \|u\|_\infty \, \|\rho^\tau\|_2 \, \|\nabla\mu^\tau\|_2$$
$$\le \frac{3}{4}\int_\Omega |\nabla\mu^\tau|^2 + \frac{1}{2}\int_\Gamma |\nabla_\Gamma\mu_\Gamma^\tau|^2 + c\,,$$

so that we can infer that

$$\|(\nabla\mu^\tau, \nabla_\Gamma\mu_\Gamma^\tau)\|_{(L^\infty(0,T;\mathcal{H}))^3} \le c, \quad \text{whence} \quad \|(\mu^\tau - \widehat{m}, \mu_\Gamma^\tau - \widehat{m})\|_{L^\infty(0,T;\mathcal{H})} \le c\,.$$

Then, we can come back to the procedure adopted to show the validity of (3.15). It is clear that we can deduce that the mean value $\widehat{m}$ is bounded in $L^\infty(0,T)$. We thus conclude that

$$\|(\mu^\tau, \mu_\Gamma^\tau)\|_{L^\infty(0,T;\mathcal{V})} \le c\,. \tag{4.8}$$

In addition, since (2.11) implies that

$$\beta_\Gamma(r)\beta(r) \ge \frac{1}{2\eta}\,|\beta(r)|^2 - c \quad \text{for every } r \in \mathbb{R}$$

(in fact, this also holds true for the Yosida regularizations β_ε and $\beta_{\Gamma,\eta\varepsilon}$, as it was proved in [4, Lemma 4.4]), we can test (2.21) by $(\beta(\rho^\tau), \beta(\rho^\tau_\Gamma))$ (without integrating in time) and deduce that

$$\|\beta(\rho^\tau)\|_{L^2(0,T;H)} + \|\beta(\rho^\tau_\Gamma)\|_{L^2(0,T;H_\Gamma)} \le c\,.$$

Therefore, we can use the parts $vi)$ and $v)$ of [18, Thm. 2.1] (indeed, this procedure should be performed on the ε-approximating problem whose nonlinearities are Lipschitz continuous and thus yield constants that do not depend on ε) and infer that

$$\|\beta_\Gamma(\rho^\tau_\Gamma)\|_{L^\infty(0,T;H_\Gamma)} \le c \quad \text{and} \quad \|(\rho^\tau, \rho^\tau_\Gamma)\|_{L^\infty(0,T;\mathcal{W})} \le c\,. \tag{4.9}$$

From the latter we conclude that

$$\|(\rho^\tau, \rho^\tau_\Gamma)\|_\infty \le \widehat{R}\,, \tag{4.10}$$

where the constant $\widehat{R}$ which we have marked with a special symbol depends only on the structure of the system and of the control box $\mathcal{U}_{ad}$, as well as on the norms involved in our assumptions on the initial datum. Hence, this estimate is uniform with respect to $\tau \in (0,T)$ and $u \in \mathcal{U}_{ad}$.

We draw some important consequences from (4.10). To this end, we introduce for $\tau \ge 0$ the solution operators

$$\begin{aligned} &\mathcal{S}_\tau : \mathcal{U}_{ad} \to L^2(0,T;\mathcal{V}) \times \big(H^1(0,T;\mathcal{V}^*) \cap L^\infty(0,T;\mathcal{V})\big), \\ &\mathcal{S}^2_\tau : \mathcal{U}_{ad} \to H^1(0,T;\mathcal{V}^*) \cap L^\infty(0,T;\mathcal{V}), \end{aligned} \tag{4.11}$$

which are defined as follows: for a given $u \in \mathcal{U}_{ad}$, the value $\mathcal{S}_\tau(u)$ is the solution $(\mu^\tau, \mu^\tau_\Gamma, \rho^\tau, \rho^\tau_\Gamma)$ to the system (4.1)–(4.3) given by Theorem 2.1, and $\mathcal{S}^2_\tau(u)$ is its second component pair $(\rho^\tau, \rho^\tau_\Gamma)$.

Now let $\tau \in (0,1)$ and $u \in \mathcal{U}_{ad}$ be arbitrary. Then, with $(\rho^\tau, \rho^\tau_\Gamma) = \mathcal{S}^2_\tau(u)$, it follows from (4.10) that

$$\|f^{(j)}(\rho^\tau)\|_\infty + \|f^{(j)}_\Gamma(\rho^\tau_\Gamma)\|_\infty \le c \quad \text{for } 0 \le j \le 3, \tag{4.12}$$

since the potentials are smooth. In particular, the functions

$$\psi^\tau := f''(\rho^\tau) \quad \text{and} \quad \psi^\tau_\Gamma := f''_\Gamma(\rho^\tau_\Gamma), \tag{4.13}$$

are bounded in $L^\infty(Q)$ and $L^\infty(\Gamma)$, respectively, uniformly with respect to $\tau \in (0,T)$ and $u \in \mathcal{U}_{ad}$. Moreover, the functions

$$\begin{aligned} &\varphi^\tau_3 := \beta_3(\rho^\tau - \widehat{\rho}_Q), \quad \varphi^\tau_4 := \beta_4(\rho^\tau_\Gamma - \widehat{\rho}_\Sigma), \\ &\varphi^\tau_5 := \beta_5(\rho^\tau(T) - \widehat{\rho}_\Omega), \quad \text{and} \quad \varphi^\tau_6 := \beta_6(\rho^\tau_\Gamma(T) - \widehat{\rho}_\Gamma), \end{aligned} \tag{4.14}$$

are bounded in $L^2(Q)$, $L^2(\Sigma)$, $L^2(\Omega)$, and $L^2(\Gamma)$, respectively, uniformly with respect to $\tau \in (0,T)$ and $u \in \mathcal{U}_{ad}$. We also remark that

$$\text{the functions } f^{(j)} \text{ and } f^{(j)}_\Gamma \text{ are Lipschitz continuous on } [-\widehat{R}, \widehat{R}] \text{ for } 0 \le j \le 2. \tag{4.15}$$

After these preparations, we are now ready to go on with our project for the proof of our results. The following approximation result resembles Theorem [20, Thm. 3.1], which was established for the case of logarithmic potentials and $\tau_\Omega > 0$, $\tau_\Gamma > 0$.

Theorem 4.1. *Suppose that the assumptions* (2.6)–(2.11)*,* (2.15)*,* (2.30)*, and* (2.37)–(2.40)*, are fulfilled, and assume that sequences* $\{\tau_n\} \subset (0,1)$ *and* $\{u^{\tau_n}\} \subset \mathcal{U}_{ad}$ *are given such that* $\tau_n \searrow 0$ *and* $u^{\tau_n} \to u$ *weakly-star in* $\mathcal{X}$ *for some* $u \in \mathcal{U}_{ad}$*. Then, with* $(\mu^{\tau_n}, \mu_\Gamma^{\tau_n}, \rho^{\tau_n}, \rho_\Gamma^{\tau_n}) = \mathcal{S}_{\tau_n}(u^{\tau_n})$ *and* $(\mu, \mu_\Gamma, \rho, \rho_\Gamma) = \mathcal{S}_0(u)$*, we have that*

$$(\mu^{\tau_n}, \mu_\Gamma^{\tau_n}) \to (\mu, \mu_\Gamma) \quad \textit{weakly-star in } L^\infty(0,T;\mathcal{V}), \tag{4.16}$$

$$(\rho^{\tau_n}, \rho_\Gamma^{\tau_n}) \to (\rho, \rho_\Gamma) \quad \textit{weakly-star in } W^{1,\infty}(0,T;\mathcal{V}^*) \cap L^\infty(0,T;\mathcal{W}) \quad \textit{and strongly in } C^0(\overline{Q}) \times C^0(\overline{\Sigma}). \tag{4.17}$$

Moreover, it holds that

$$\mathcal{J}(\mathcal{S}_0^2(u), u) \le \liminf_{n\to\infty} \mathcal{J}((\rho^{\tau_n}, \rho_\Gamma^{\tau_n}), u^{\tau_n}) \tag{4.18}$$

$$\mathcal{J}(\mathcal{S}_0^2(v), v) = \lim_{n\to\infty} \mathcal{J}(\mathcal{S}_{\tau_n}^2(v), v) \quad \textit{for every } v \in \mathcal{U}_{ad}. \tag{4.19}$$

Proof: Let $\{\tau_n\} \subset (0,1)$ be any sequence such that $\tau_n \searrow 0$ as $n \to \infty$, and suppose that $\{u^{\tau_n}\} \subset \mathcal{U}_{ad}$ converges weakly-star in $\mathcal{X}$ to some $u \in \mathcal{U}_{ad}$. By virtue of the global estimates (4.7)–(4.9), there are some subsequence of $\{\tau_n\}$, which is again indexed by n, and two pairs $(\mu, \mu_\Gamma), (\rho, \rho_\Gamma)$ such that (4.16) and the first convergence result of (4.17) hold true. It then follows from standard compact embedding results (cf. [51, Sect. 8, Cor. 4]) that

$$\rho^{\tau_n} \to \rho \quad \text{strongly in } L^2(0,T;V) \cap C^0(\overline{Q}), \tag{4.20}$$

which also implies that

$$\rho_\Gamma^{\tau_n} \to \rho_\Gamma \quad \text{strongly in } C^0(\overline{\Sigma}). \tag{4.21}$$

In particular, $(\rho(0), \rho_\Gamma(0)) = (\rho_0, \rho_{0|\Gamma})$ and $\rho_\Gamma = \rho_{|\Sigma}$. In addition, we obviously have that

$$\beta(\rho^{\tau_n}) \to \beta(\rho), \quad \pi(\rho^{\tau_n}) \to \pi(\rho), \quad \text{both strongly in } C^0(\overline{Q}), \tag{4.22}$$

$$\beta_\Gamma(\rho_\Gamma^{\tau_n}) \to \beta_\Gamma(\rho_\Gamma), \quad \pi_\Gamma(\rho_\Gamma^{\tau_n}) \to \pi_\Gamma(\rho_\Gamma), \quad \text{both strongly in } C^0(\overline{\Sigma}). \tag{4.23}$$

Moreover, it is easily verified that, at least weakly in $L^1(Q)$,

$$\nabla\rho^{\tau_n} \cdot u^{\tau_n} \to \nabla\rho \cdot u\,. \tag{4.24}$$

Combining the above convergence results, we may pass to the limit as $n \to \infty$ in the equations (4.1)–(4.3) (written for $\tau = \tau_n$ and $u = u^{\tau_n}$) to find that $(\mu, \mu_\Gamma, \rho, \rho_\Gamma)$ and u satisfy the equations (2.20)–(2.23) for $\tau_\Omega = \tau_\Gamma = 0$ with $\zeta = \beta(\rho)$ and $\zeta_\Gamma = \beta_\Gamma(\rho_\Gamma)$. Owing to the uniqueness result stated in Theorem 2.1, we therefore have that $(\mu, \mu_\Gamma, \rho, \rho_\Gamma) = \mathcal{S}_0(u)$, and since the limit is unique, the convergence properties (4.16) and (4.17) hold true for the entire sequences.

It remains to show the validity of (4.18) and (4.19). In view of (4.17), the inequality (4.18) is an immediate consequence of the weak and weak-star sequential semicontinuity properties of the cost functional $\mathcal{J}$. To establish the identity (4.19), let $v \in \mathcal{U}_{ad}$ be arbitrary and put $(\rho^{\tau_n}, \rho_\Gamma^{\tau_n}) = \mathcal{S}_{\tau_n}^2(v)$, for $n \in \mathbb{N}$. Taking Theorem 2.1 into account, and arguing as in the first part of this proof, we can conclude that $\{\mathcal{S}_{\tau_n}^2(v)\}$ converges to $(\rho, \rho_\Gamma) = \mathcal{S}_0^2(v)$ in the sense of (4.17). In particular,

$$\mathcal{S}_{\tau_n}^2(v) \to \mathcal{S}_0^2(v) \quad \text{strongly in } C^0(\overline{Q}) \times C^0(\overline{\Sigma}).$$

As the cost functional $\mathcal{J}$ is obviously continuous in the variables (ρ, ρ_Γ) with respect to the strong topology of $C^0(\overline{Q}) \times C^0(\overline{\Sigma})$, we thus conclude that (4.19) is valid. □

As a corollary of Theorem 4.1, we can prove Theorem 2.6.

Proof of Theorem 2.6: At first, we observe that all the assumptions for an application of the arguments employed in the proof of Theorem 4.1 are fulfilled. We now conclude in two steps.

STEP 1:

We first consider the problem of minimizing the cost functional $\mathcal{J}$ subject to $u \in \mathcal{U}_{ad}$ and to the state system (2.20)–(2.23) for fixed $\tau_\Omega = \tau_\Gamma = \tau > 0$. We claim that this optimal control problem, which we denote by $(\mathcal{P}_\tau)$, admits at least one optimal pair for every $\tau > 0$. Indeed, let $\tau > 0$ be fixed, and let $((\mu_n, \mu_{n_\Gamma}, \rho_n, \rho_{n_\Gamma}), u_n)$, $n \in \mathbb{N}$, be a minimizing sequence for $(\mathcal{P}_\tau)$, that is, we assume that we have $(\mu_n, \mu_{n_\Gamma}, \rho_n, \rho_{n_\Gamma}) = \mathcal{S}_\tau(u_n)$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \mathcal{J}((\rho_n, \rho_{n_\Gamma}), u_n) \,=\, \inf_{v\in\mathcal{U}_{ad}} J(\mathcal{S}^2_\tau(v), v) \,=:\, \sigma \,\geq\, 0\,.$$

Then, by the same token as in the proof of Theorem 4.1, there are $(\mu, \mu_\Gamma, \rho, \rho_\Gamma)$ and $u \in \mathcal{U}_{ad}$ with $(\mu, \mu_\Gamma, \rho, \rho_\Gamma) = \mathcal{S}_\tau(u)$ such that (cf. (4.16) and (4.17))

$$(\mu_n, \mu_{n_\Gamma}) \to (\mu, \mu_\Gamma) \quad \text{weakly-star in } L^\infty(0,T;\mathcal{V}),$$
$$(\rho_n, \rho_{n_\Gamma}) \to (\rho, \rho_\Gamma) \quad \text{strongly in } C^0(\overline{Q}) \times C^0(\overline{\Sigma}).$$

The sequential lower semicontinuity properties of $\mathcal{J}$ then imply that $((\mu, \mu_\Gamma, \rho, \rho_\Gamma), u)$ is optimal for $(\mathcal{P}_\tau)$, which proves the claim.

STEP 2:

We now pick an arbitrary sequence $\{\tau_n\}$ such that $\tau_n \searrow 0$ as $n \to \infty$. Then, as has been shown in Step 1, the optimal control problem ($\mathcal{P}_{\tau_n}$) has for every $n \in \mathbb{N}$ an optimal pair $((\mu^{\tau_n}, \mu_\Gamma^{\tau_n}, \rho^{\tau_n}, \rho_\Gamma^{\tau_n}), u^{\tau_n})$, where $u^{\tau_n} \in \mathcal{U}_{ad}$ and $(\rho^{\tau_n}, \rho_\Gamma^{\tau_n}) = \mathcal{S}^2_{\tau_n}(u^{\tau_n})$. Since $\mathcal{U}_{ad}$ is a bounded subset of $\mathcal{X}$, we may without loss of generality assume that $u^{\tau_n} \to u$ weakly-star in $\mathcal{X}$ for some $u \in \mathcal{U}_{ad}$. From Theorem 4.1 we infer that with $(\mu, \mu_\Gamma, \rho, \rho_\Gamma) = \mathcal{S}_0(u)$ the convergence properties (4.16) and (4.17) are valid, as well as (4.18). Invoking the optimality of $((\mu^{\tau_n}, \mu_\Gamma^{\tau_n}, \rho^{\tau_n}, \rho_\Gamma^{\tau_n}), u^{\tau_n})$ for ($\mathcal{P}_{\tau_n}$) and (4.19), we then find, for every $v \in \mathcal{U}_{ad}$, that

$$\begin{aligned} \mathcal{J}((\rho, \rho_\Gamma), u) \,&=\, \mathcal{J}(\mathcal{S}^2_0(u), u) \,\leq\, \liminf_{n\to\infty} \mathcal{J}(\mathcal{S}^2_{\tau_n}(u^{\tau_n}), u^{\tau_n}) \\ &\leq\, \liminf_{n\to\infty} \mathcal{J}(\mathcal{S}^2_{\tau_n}(v), v) \,=\, \lim_{n\to\infty} \mathcal{J}(\mathcal{S}^2_{\tau_n}(v), v) \,=\, \mathcal{J}(\mathcal{S}^2_0(v), v), \end{aligned} \tag{4.25}$$

which yields that u is an optimal control for the control problem for $\tau = 0$ with the associate state $(\mu, \mu_\Gamma, \rho, \rho_\Gamma)$. The assertion is thus proved. □

We remark at this place that the existence result of Theorem 2.6 can also be proved using the same direct argument as in Step 1 of the above proof. We have chosen to employ Theorem 4.1, here, since it shows that, for small $\tau > 0$, optimal pairs for $(\mathcal{P}_\tau)$ are likely to be 'close' to optimal pairs for the case $\tau = 0$. However, the result does not yield any information on whether every solution to the optimal control problem for $\tau = 0$ can be approximated by a sequence of solutions to the problems $(\mathcal{P}_{\tau_n})$. Unfortunately, we are not able to prove such a general 'global' result. Instead, we can only give a 'local' answer for every individual optimizer of the control problem for $\tau = 0$. For this purpose, we employ a trick due to Barbu [2]. To this end, let $\bar{u} \in \mathcal{U}_{ad}$ be an arbitrary optimal control for $\tau = 0$, and let $(\bar{\mu}, \bar{\mu}_\Gamma, \bar{\rho}, \bar{\rho}_\Gamma)$ be the associated solution to the state system (2.20)–(2.23) for $\tau_\Omega = \tau_\Gamma = 0$. In particular, $(\bar{\rho}, \bar{\rho}_\Gamma) = \mathcal{S}_0^2(\bar{u})$. We associate with this optimal control the *adapted cost functional*

$$\widetilde{\mathcal{J}}((\rho, \rho_\Gamma), u) := \mathcal{J}((\rho, \rho_\Gamma), u) + \frac{1}{2} \|u - \bar{u}\|^2_{(L^2(Q))^3} \tag{4.26}$$

and, for every $\tau \in (0, 1)$, a corresponding *adapted optimal control problem*,

$(\widetilde{\mathcal{P}}_\tau)$ Minimize $\widetilde{\mathcal{J}}((\rho, \rho_\Gamma), u)$ for $u \in \mathcal{U}_{ad}$, subject to the condition that (4.1)–(4.3) be satisfied.

With the same direct argument as in Step 1 of the proof of Theorem 2.6 (which needs no repetition, here), we can show that under the assumptions of Theorem 2.6 the optimal control problem $(\widetilde{\mathcal{P}}_\tau)$ admits for every $\tau \in (0, 1)$ a solution. The next result is an analogue of Theorem [20, Thm. 3.4], which has been shown for potentials of logarithmic type and $\tau_\Omega > 0$ and $\tau_\Gamma > 0$:

Theorem 4.2. *Suppose that the assumptions* (2.6)–(2.11)*,* (2.15)*,* (2.30)*, and* (2.37)–(2.40)*, are fulfilled. Moreover, let $\overline{u} \in \mathcal{U}_{ad}$ be an optimal control related to the cost functional* (2.41) *and to the state system* (2.20)–(2.23) *for $\tau_\Omega = \tau_\Gamma = 0$, where $(\overline{\mu}, \overline{\mu}_\Gamma, \overline{\rho}, \overline{\rho}_\Gamma) = \mathcal{S}_0(\overline{u})$ is the corresponding state. If $\{\tau_n\} \subset (0, 1)$ is any sequence with $\tau_n \searrow 0$ as $n \to \infty$, then there exist a subsequence $\{\tau_{n_k}\}$ and, for every k, an optimal control $u^{\tau_{n_k}} \in \mathcal{U}_{ad}$ of the adapted control problem $(\widetilde{\mathcal{P}}_{\tau_{n_k}})$, such that, as $k \to \infty$,*

$$u^{\tau_{n_k}} \to \overline{u} \quad \textit{strongly in } (L^2(Q))^3, \tag{4.27}$$

and such that the convergence properties (4.16)–(4.17) *are satisfied, where $\{\tau_n\}$ and $(\mu, \mu_\Gamma, \rho, \rho_\Gamma)$ are replaced by $\{\tau_{n_k}\}$ and $(\overline{\mu}, \overline{\mu}_\Gamma, \overline{\rho}, \overline{\rho}_\Gamma)$, respectively. Moreover, we have*

$$\lim_{k\to\infty} \widetilde{\mathcal{J}}((\rho^{\tau_{n_k}}, \rho_\Gamma^{\tau_{n_k}}), u^{\tau_{n_k}}) = \mathcal{J}((\overline{\rho}, \overline{\rho}_\Gamma), \overline{u}). \tag{4.28}$$

Proof: Let $\tau_n \searrow 0$ as $n \to \infty$. For any $n \in \mathbb{N}$, we pick an optimal control $u^{\tau_n} \in \mathcal{U}_{ad}$ for the adapted problem $(\widetilde{\mathcal{P}}_{\tau_n})$ and denote by $(\mu^{\tau_n}, \mu_\Gamma^{\tau_n}, \rho^{\tau_n}, \rho_\Gamma^{\tau_n})$ the associated solution to the problem (4.1)–(4.3) for $\tau = \tau_n$ and $u = u^{\tau_n}$. By the boundedness of $\mathcal{U}_{ad}$ in $\mathcal{X}$, there is some subsequence $\{\tau_{n_k}\}$ of $\{\tau_n\}$ such that

$$u^{\tau_{n_k}} \to u \quad \text{weakly-star in } \mathcal{X} \quad \text{as } k \to \infty, \tag{4.29}$$

with some $u \in \mathcal{U}_{ad}$. Thanks to Theorem 4.1, the convergence properties (4.16)–(4.17) hold true, where $(\mu, \mu_\Gamma, \rho, \rho_\Gamma)$ is the unique solution to the state system (2.20)–(2.23) for $\tau_\Omega = \tau_\Gamma = 0$. In particular, $((\rho, \rho_\Gamma), u)$ is admissible for the non-adapted control problem with the cost functional (2.41).

We now aim to prove that $u = \bar{u}$. Once this is shown, then the uniqueness result of Theorem 2.1 yields that also $(\mu, \mu_\Gamma, \rho, \rho_\Gamma) = (\overline{\mu}, \overline{\mu}_\Gamma, \overline{\rho}, \overline{\rho}_\Gamma)$, which implies that the properties (4.16)–(4.17) are satisfied, where $(\mu, \mu_\Gamma, \rho, \rho_\Gamma)$ is replaced by $(\bar{\mu}, \bar{\mu}_\Gamma, \bar{\rho}, \bar{\rho}_\Gamma)$.

Now observe that, owing to the weak sequential lower semicontinuity of $\widetilde{\mathcal{J}}$, and in view of the optimality property of $((\bar\rho, \bar\rho_\Gamma), \overline{u})$,

$$\liminf_{k\to\infty} \widetilde{\mathcal{J}}((\rho^{\tau_{n_k}}, \rho_\Gamma^{\tau_{n_k}}), u^{\tau_{n_k}}) \geq \mathcal{J}((\rho, \rho_\Gamma), u) + \frac{1}{2} \|u - \bar u\|^2_{(L^2(Q))^3}$$
$$\geq \mathcal{J}((\bar\rho, \bar\rho_\Gamma), \bar u) + \frac{1}{2} \|u - \bar u\|^2_{(L^2(Q))^3} . \tag{4.30}$$

On the other hand, the optimality property of $((\rho^{\tau_{n_k}}, \rho_\Gamma^{\tau_{n_k}}), u^{\tau_{n_k}})$ for problem $(\widetilde{\mathcal{P}}_{\tau_{n_k}})$ yields that for any $k \in \mathbb{N}$ we have

$$\widetilde{\mathcal{J}}((\rho^{\tau_{n_k}}, \rho_\Gamma^{\tau_{n_k}}), u^{\tau_{n_k}}) = \widetilde{\mathcal{J}}(\mathcal{S}^2_{\tau_{n_k}}(u^{\tau_{n_k}}), u^{\tau_{n_k}}) \leq \widetilde{\mathcal{J}}(\mathcal{S}^2_{\tau_{n_k}}(\bar u), \bar u) , \tag{4.31}$$

whence, taking the limit superior as $k \to \infty$ on both sides and invoking (4.19) in Theorem 4.1,

$$\limsup_{k\to\infty} \widetilde{\mathcal{J}}((\rho^{\tau_{n_k}}, \rho_\Gamma^{\tau_{n_k}}), u^{\tau_{n_k}})$$
$$\leq \widetilde{\mathcal{J}}(\mathcal{S}^2_0(\bar u), \bar u) = \widetilde{\mathcal{J}}((\bar\rho, \bar\rho_\Gamma), \bar u) = \mathcal{J}((\bar\rho, \bar\rho_\Gamma), \bar u) . \tag{4.32}$$

Combining (4.30) with (4.32), we have thus shown that $\frac{1}{2} \|u - \bar u\|^2_{(L^2(Q))^3} = 0$, so that $u = \bar u$ and thus also $(\mu, \mu_\Gamma, \rho, \rho_\Gamma) = (\bar\mu, \bar\mu_\Gamma, \bar\rho, \bar\rho_\Gamma)$. Moreover, (4.30) and (4.32) also imply that

$$\mathcal{J}((\bar\rho, \bar\rho_\Gamma), \bar u) = \widetilde{\mathcal{J}}((\bar\rho, \bar\rho_\Gamma), \bar u) = \liminf_{k\to\infty} \widetilde{\mathcal{J}}((\rho^{\tau_{n_k}}, \rho_\Gamma^{\tau_{n_k}}), u^{\tau_{n_k}})$$
$$= \limsup_{k\to\infty} \widetilde{\mathcal{J}}((\rho^{\tau_{n_k}}, \rho_\Gamma^{\tau_{n_k}}), u^{\tau_{n_k}}) = \lim_{k\to\infty} \widetilde{\mathcal{J}}((\rho^{\tau_{n_k}}, \rho_\Gamma^{\tau_{n_k}}), u^{\tau_{n_k}}) , \tag{4.33}$$

which proves (4.27) and, at the same time, also (4.28). This concludes the proof of the assertion. □

In order to prove Theorem 2.7, we need estimates that are uniform with respect to $\tau \in (0, 1)$ for the solution $(q^\tau, q_\Gamma^\tau, p^\tau, p_\Gamma^\tau)$ to the approximating adjoint system corresponding to a velocity field $\bar u^\tau \in \mathcal{U}_{ad}$ and to the associated solution $(\mu^\tau, \mu_\Gamma^\tau, \rho^\tau, \rho_\Gamma^\tau) = \mathcal{S}_\tau(\bar u^\tau)$ to the state system (4.1)–(4.3) with $u = \bar u^\tau$. The approximating adjoint system reads

$$-\langle \partial_t(p^\tau + \tau q^\tau, p_\Gamma^\tau + \tau q_\Gamma^\tau), (v, v_\Gamma)\rangle_{\mathcal{V}} + \int_\Omega \nabla q^\tau \cdot \nabla v + \int_\Gamma \nabla_\Gamma q_\Gamma^\tau \cdot \nabla_\Gamma v_\Gamma$$
$$+ \int_\Omega \psi^\tau q^\tau v + \int_\Gamma \psi_\Gamma^\tau q_\Gamma^\tau v_\Gamma - \int_\Omega \overline{u}^\tau \cdot \nabla p^\tau \, v = \int_\Omega \varphi_3^\tau \, v + \int_\Gamma \varphi_4^\tau \, v_\Gamma$$
$$\text{a.e. in } (0,T) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}, \tag{4.34}$$

$$\int_\Omega \nabla p^\tau \cdot \nabla v + \int_\Gamma \nabla_\Gamma p_\Gamma^\tau \cdot \nabla_\Gamma v_\Gamma = \int_\Omega q^\tau v + \int_\Gamma q_\Gamma^\tau v_\Gamma$$
$$\text{a.e. in } (0,T) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}, \tag{4.35}$$

$$\langle (p^\tau + \tau q^\tau, p_\Gamma^\tau + \tau q_\Gamma^\tau)(T), (v, v_\Gamma)\rangle_{\mathcal{V}} = \int_\Omega \varphi_5^\tau \, v + \int_\Gamma \varphi_6^\tau \, v_\Gamma$$
$$\text{for every } (v, v_\Gamma) \in \mathcal{V}, \tag{4.36}$$

with the functions defined in (4.13) and (4.14). The velocity field $\overline{u}^\tau$ appearing in the above system could be any element of $\mathcal{U}_{ad}$, in principle. However, we will use (4.34)–(4.36) only when this field is an optimal control for the control problem associated with the adapted cost functional (4.26) and to the τ-state system (4.1)–(4.3).

The basic idea is the following: we integrate equation (4.34) with respect to time over the interval (t,T), where t is arbitrary in $[0,T)$, and use the definition (2.51). Then, we account for the final condition (4.36) and rearrange. We then obtain the identity

$$\begin{aligned}
&\int_\Omega (p^\tau+\tau q^\tau)v+\int_\Gamma (p_\Gamma^\tau+\tau q_\Gamma^\tau)v_\Gamma+\int_\Omega \nabla(1*q^\tau)\cdot\nabla v+\int_\Gamma \nabla_\Gamma(1*q_\Gamma^\tau)\cdot\nabla_\Gamma v_\Gamma\\
&=-\int_\Omega (1*(\psi^\tau q^\tau))v-\int_\Gamma (1*(\psi_\Gamma^\tau q_\Gamma^\tau))v_\Gamma+\int_\Omega (1*(\overline{u}^\tau\cdot\nabla p^\tau))\,v\\
&\quad+\int_\Omega (1*\varphi_3^\tau)\,v+\int_\Gamma (1*\varphi_4^\tau)\,v_\Gamma+\int_\Omega \varphi_5^\tau v+\int_\Gamma \varphi_6^\tau v_\Gamma\,. \qquad (4.37)
\end{aligned}$$

This equality holds at any time and for every $(v,v_\Gamma)\in\mathcal{V}$. In the sequel, we use the notations

$$Q^t:=\Omega\times(t,T) \quad\text{and}\quad \Sigma^t:=\Gamma\times(t,T) \qquad (4.38)$$

for $t\in[0,T)$.

Basic estimate. For a future use, we notice the following property of the convolution (2.51): if $a\in L^\infty(Q)$, $b\in L^2(Q)$, $a_\Gamma\in L^\infty(\Sigma)$ and $b_\Gamma\in L^2(\Sigma)$, then we have, for every $t\in[0,T)$,

$$\int_{Q^t}|1*(ab)|^2\le T\,\|a\|_\infty^2\int_t^T\Big(\int_{Q^s}|b|^2\Big)\,ds\,, \qquad (4.39)$$

$$\int_{\Sigma^t}|1*(a_\Gamma b_\Gamma)|^2\le T\,\|a_\Gamma\|_\infty^2\int_t^T\Big(\int_{S^s}|b_\Gamma|^2\Big)\,ds\,. \qquad (4.40)$$

We write (4.37), (4.35), and (4.34) at the time s and test them by

$$(q^\tau,q_\Gamma^\tau)(s),\quad (p^\tau,p_\Gamma^\tau)(s)-(q^\tau,q_\Gamma^\tau)(s)\quad\text{and}\quad (p^\tau+\tau q^\tau,p_\Gamma^\tau+\tau q_\Gamma^\tau)(s),$$

respectively. Then, we integrate over (t,T) with respect to s, where $t\in[0,T)$ is arbitrary, obtaining the identities

$$\begin{aligned}
&\int_{Q^t}(p^\tau+\tau q^\tau)q^\tau+\int_{\Sigma^t}(p_\Gamma^\tau+\tau q_\Gamma^\tau)\,q_\Gamma^\tau+\int_{Q^t}\nabla(1*q^\tau)\cdot\nabla q^\tau+\int_{\Sigma^t}\nabla_\Gamma(1*q_\Gamma^\tau)\cdot\nabla_\Gamma q_\Gamma^\tau\\
&=-\int_{Q^t}(1*(\psi^\tau q^\tau))q^\tau-\int_{\Sigma^t}(1*(\psi_\Gamma^\tau q_\Gamma^\tau))\,q_\Gamma^\tau+\int_{Q^t}(1*(\overline{u}^\tau\cdot\nabla p^\tau))\,q^\tau\\
&\quad+\int_{Q^t}(1*\varphi_3^\tau)\,q^\tau+\int_{\Sigma^t}(1*\varphi_4^\tau)\,q_\Gamma^\tau+\int_{Q^t}\varphi_5^\tau\,q^\tau+\int_{\Sigma^t}\varphi_6^\tau\,q_\Gamma^\tau\,,
\end{aligned}$$

$$\begin{aligned}
&\int_{Q^t}\nabla p^\tau\cdot\big(\nabla p^\tau-\nabla q^\tau\big)+\int_{\Sigma^t}\nabla_\Gamma p_\Gamma^\tau\cdot\big(\nabla_\Gamma p_\Gamma^\tau-\nabla_\Gamma q_\Gamma^\tau\big)\\
&=\int_{Q^t}q^\tau(p^\tau-q^\tau)+\int_{\Sigma^t}q_\Gamma^\tau\,(p_\Gamma^\tau-q_\Gamma^\tau)\,,
\end{aligned}$$

$$
\begin{aligned}
&- \int_{Q^t} \partial_t(p^\tau + \tau q^\tau)\,(p^\tau + \tau q^\tau) - \int_{\Sigma^t} \partial_t(p_\Gamma^\tau + \tau q_\Gamma^\tau)\,(p_\Gamma^\tau + \tau q_\Gamma^\tau) \\
&\quad + \int_{Q^t} \bigl(\nabla q^\tau \cdot (\nabla p^\tau + \tau \nabla q^\tau)\bigr) + \int_{\Sigma^t} \bigl(\nabla_\Gamma q_\Gamma^\tau \cdot (\nabla_\Gamma p_\Gamma^\tau + \tau \nabla_\Gamma q_\Gamma^\tau)\bigr) \\
&\quad + \int_{Q^t} \psi^\tau q^\tau (p^\tau + \tau q^\tau) + \int_{\Sigma^t} \psi_\Gamma^\tau\, q_\Gamma^\tau\,(p_\Gamma^\tau + \tau q_\Gamma^\tau) - \int_{Q^t} \overline{u}^\tau \cdot \nabla p^\tau\,(p^\tau + \tau q^\tau) \\
&= \int_{Q^t} \varphi_3^\tau\,(p^\tau + \tau q^\tau) + \int_{\Sigma^t} \varphi_4^\tau\,(p_\Gamma^\tau + \tau q_\Gamma^\tau).
\end{aligned}
$$

Now, we add these equalities to each other and account for (4.36). Since several terms cancel out, we obtain

$$
\begin{aligned}
&\tau \int_{Q^t} |q^\tau|^2 + \tau \int_{\Sigma^t} |q_\Gamma^\tau|^2 + \frac{1}{2}\int_\Omega |\nabla(1 * q^\tau)(t)|^2 + \frac{1}{2}\int_\Gamma |\nabla_\Gamma(1 * q_\Gamma^\tau)(t)|^2 \\
&\quad + \int_{Q^t} |\nabla p^\tau|^2 + \int_{\Sigma^t} |\nabla_\Gamma p_\Gamma^\tau|^2 + \frac{1}{2}\int_\Omega |(p^\tau + \tau q^\tau)(t)|^2 + \frac{1}{2}\int_\Gamma |(p_\Gamma^\tau + \tau q_\Gamma^\tau)(t)|^2 \\
&\quad + \int_{Q^t} |q^\tau|^2 + \int_{\Sigma^t} |q_\Gamma^\tau|^2 + \tau \int_{Q^t} |\nabla q^\tau|^2 + \tau \int_{\Sigma^t} |\nabla_\Gamma q_\Gamma^\tau|^2 \\
&= \frac{1}{2}\int_\Omega |\varphi_5^\tau|^2 + \frac{1}{2}\int_\Gamma |\varphi_6^\tau|^2 \\
&\quad - \int_{Q^t} (1 * (\psi^\tau q^\tau))\, q^\tau - \int_{\Sigma^t} (1 * (\psi_\Gamma^\tau\, q_\Gamma^\tau))\, q_\Gamma^\tau + \int_{Q^t} (1 * (\overline{u}^\tau \cdot \nabla p^\tau))\, q^\tau \\
&\quad + \int_{Q^t} (1 * \varphi_3^\tau)\, q^\tau + \int_{\Sigma^t} (1 * \varphi_4^\tau)\, q_\Gamma^\tau + \int_{Q^t} \varphi_5^\tau\, q^\tau + \int_{\Sigma^t} \varphi_6^\tau\, q_\Gamma^\tau \\
&\quad - \int_{Q^t} \psi^\tau q^\tau (p^\tau + \tau q^\tau) - \int_{\Sigma^t} \psi_\Gamma^\tau\, q_\Gamma^\tau\,(p_\Gamma^\tau + \tau q_\Gamma^\tau) + \int_{Q^t} \overline{u}^\tau \cdot \nabla p^\tau\,(p^\tau + \tau q^\tau) \\
&\quad + \int_{Q^t} \varphi_3^\tau\,(p^\tau + \tau q^\tau) + \int_{\Sigma^t} \varphi_4^\tau\,(p_\Gamma^\tau + \tau q_\Gamma^\tau).
\end{aligned}
$$

Every term on the left-hand side is nonnegative. By simply using Young's inequality and (4.39)–(4.40), we see that the right-hand side is estimated from above by

$$
\begin{aligned}
&\frac{1}{2}\int_{Q^t} |q^\tau|^2 + \frac{1}{2}\int_{\Sigma^t} |q_\Gamma^\tau|^2 + \frac{1}{2}\int_{Q^t} |\nabla p^\tau|^2 \\
&\quad + c\,\|\psi^\tau\|_\infty^2 \int_{Q^t} |1 * q^\tau|^2 + c\,\|\psi_\Gamma^\tau\|_\infty^2 \int_{\Sigma^t} |1 * q_\Gamma^\tau|^2 + c\,\|\overline{u}^\tau\|_\infty^2 \int_{Q^t} |1 * \nabla p^\tau|^2 \\
&\quad + c\,\bigl(\|\psi^\tau\|_\infty^2 + \|\overline{u}^\tau\|_\infty^2 + \|\varphi_3^\tau\|_2^2\bigr) \int_{Q^t} |p^\tau + \tau q^\tau|^2 \\
&\quad + c\,\bigl(\|\psi_\Gamma^\tau\|_\infty^2 + \|\varphi_4^\tau\|_2^2\bigr) \int_{\Sigma^t} |p_\Gamma^\tau + \tau q_\Gamma^\tau|^2 + c \sum_{i=3}^{6} \|\varphi_i^\tau\|_2^2\,,
\end{aligned}
$$

since $\|1 * a\|_2 \le c\,\|a\|_2$ for every L^2 function a. By recalling the uniform boundedness properties with respect to $\tau \in (0,1)$ of the controls $\overline{u}^\tau$ and of the functions defined in (4.13) and (4.14), we conclude from Gronwall's lemma that

$$
\begin{aligned}
&\|(\nabla p^\tau, \nabla_\Gamma p_\Gamma^\tau)\|_{(L^2(0,T;\mathcal{H}))^3} + \|(q^\tau, q_\Gamma^\tau)\|_{L^2(0,T;\mathcal{H})} \\
&\quad + \|1 * (\nabla q^\tau, \nabla_\Gamma q_\Gamma^\tau)\|_{(L^\infty(0,T;\mathcal{H}))^3} + \|p^\tau + \tau q^\tau\|_{L^\infty(0,T;\mathcal{H})} \le c\,.
\end{aligned} \tag{4.41}
$$

Moreover, as $\|(p^\tau, p_\Gamma^\tau)\|_{L^2(0,T;H)} \le c\,(\|p^\tau + \tau q^\tau\|_{L^\infty(0,T;\mathcal{H})} + \|(q^\tau, q_\Gamma^\tau)\|_{L^2(0,T;\mathcal{H})})$, we also have that

$$\|(p^\tau, p_\Gamma^\tau)\|_{L^2(0,T;\mathcal{V})} \le c\,. \tag{4.42}$$

Proof of Theorem 2.7: Let an optimal control $\overline{u} \in \mathcal{U}_{ad}$ for the original control problem with $\tau = 0$ be given and $(\bar\mu, \bar\mu_\Gamma, \bar\rho, \bar\rho_\Gamma) = \mathcal{S}_0(\bar u)$ be the associated state. By virtue of Theorem 4.2, we can pick a sequence $\{\tau_n\} \subset (0,1)$ with $\tau_n \searrow 0$ and, for any $n \in \mathbb{N}$, an optimal control $u^{\tau_n} \in \mathcal{U}_{ad}$, with associated solution $(\mu^{\tau_n}, \mu_\Gamma^{\tau_n}, \rho^{\tau_n}, \rho_\Gamma^{\tau_n}) = \mathcal{S}_{\tau_n}(u^{\tau_n})$ to the τ_n-system (4.1)–(4.3) for the adapted control problem $(\widetilde{\mathcal{P}}_{\tau_n})$ related to the cost functional (4.26), such that the following convergence properties hold true (see (4.16),(4.17), and (4.27)):

$$\begin{aligned}
&u^{\tau_n} \to \overline{u} \quad \text{strongly in } (L^2(Q))^3,\\
&(\mu^{\tau_n}, \mu_\Gamma^{\tau_n}) \to (\bar\mu, \bar\mu_\Gamma) \quad \text{weakly-star in } L^2(0,T;\mathcal{V}),\\
&(\rho^{\tau_n}, \rho_\Gamma^{\tau_n}) \to (\bar\rho, \bar\rho_\Gamma) \quad \text{weakly-star in } W^{1,\infty}(0,T;\mathcal{V}^*) \cap L^\infty(0,T;\mathcal{W})\\
&\qquad\qquad\qquad\qquad\quad \text{and strongly in } C^0(\overline{Q}) \times C^0(\overline{\Sigma}).
\end{aligned}$$

But then also

$$\begin{aligned}
&\psi^{\tau_n} = f''(\rho^{\tau_n}) \;\to\; f''(\bar\rho) = \psi \quad \text{strongly in } C^0(\overline{Q}),\\
&\psi_\Gamma^{\tau_n} = f_\Gamma''(\rho_\Gamma^{\tau_n}) \;\to\; f_\Gamma''(\bar\rho_\Gamma) = \psi_\Gamma \quad \text{strongly in } C^0(\overline{\Sigma})\,,
\end{aligned}$$

and, likewise (recall (4.14)),

$$\begin{aligned}
&\varphi_3^{\tau_n} \to \varphi_3 \quad \text{strongly in } C^0(\overline{Q}), \qquad \varphi_4^{\tau_n} \to \varphi_4 \quad \text{strongly in } C^0(\overline{\Sigma}),\\
&\varphi_5^{\tau_n} \to \varphi_5 \quad \text{strongly in } C^0(\overline{\Omega}), \qquad \varphi_6^{\tau_n} \to \varphi_6 \quad \text{strongly in } C^0(\Gamma).
\end{aligned}$$

Now observe that the dependence of the adapted and the non-adapted cost functionals on the state variables is the same. Therefore, nothing changes in the construction of the corresponding adjoint system, which are both given by (4.34)–(4.36). In particular, the basic estimates (4.41) and (4.42) are valid for the adjoint variables, and we thus may assume without loss of generality that there are $(p, p_\Gamma, q, q_\Gamma)$ such that

$$\begin{aligned}
&(p^{\tau_n}, p_\Gamma^{\tau_n}) \to (p, p_\Gamma) \quad \text{weakly in } L^2(0,T;\mathcal{V}),\\
&(q^{\tau_n}, q_\Gamma^{\tau_n}) \to (q, q_\Gamma) \quad \text{weakly in } L^2(0,T;\mathcal{H}),\\
&1 * (\nabla q^{\tau_n}, \nabla_\Gamma q_\Gamma^{\tau_n}) \to 1 * (\nabla q, \nabla_\Gamma q_\Gamma) \quad \text{weakly-star in } (L^\infty(0,T;\mathcal{H}))^3.
\end{aligned}$$

We thus have the convergence properties

$$\begin{aligned}
&\psi^{\tau_n}\, q^{\tau_n} \;\to\; \psi\, q \quad \text{weakly in } L^2(Q), \qquad \psi_\Gamma^{\tau_n}\, q_\Gamma^{\tau_n} \;\to\; \psi_\Gamma\, q_\Gamma \quad \text{weakly in } L^2(\Sigma),\\
&1 * (u^{\tau_n} \cdot \nabla p^{\tau_n}) \;\to\; 1 * (\bar u \cdot \nabla p) \quad \text{weakly in } L^1(Q).
\end{aligned}$$

Therefore, we may pass to the limit as $n \to \infty$ in the equations (4.37) and (4.35), written for $\tau = \tau_n$, $n \in \mathbb{N}$, to conclude that the quadruplet $(p, p_\Gamma, q, q_\Gamma)$ has the regularity properties (2.52)–(2.53) and solves the system (2.54)–(2.55).

It remains to show the validity of the variational inequality (2.56). In this regard, we observe that the variational inequalities for $(\mathcal{P}_{\tau_n})$ and $(\widetilde{\mathcal{P}}_{\tau_n})$ differ. Namely, for the adapted control problem it takes the form

$$\int_Q \big(\rho^{\tau_n}\,\nabla p^{\tau_n} + \beta_7\,u^{\tau_n} + (u^{\tau_n} - \overline{u})\big) \cdot (v - u^{\tau_n}) \geq 0 \quad \text{for every } v \in \mathcal{U}_{ad}\,. \tag{4.43}$$

Therefore, passage to the limit as $n \to \infty$ in (4.43), invoking the above convergence properties, yields the validity of (2.56). This concludes the proof of the assertion. $\square$

Remark 4.3. Coming back to (4.41), it is clear that the constant c is proportional to the sum $\sum_{i=3}^{6}\|\varphi_i^\tau\|_2$ through a constant that depends only on the structure of the state system, the control box and the initial datum. In particular, if each φ_i^τ vanishes, the inequality implies that its left-hand side vanishes as well. Since the problem is linear, this is a uniqueness result for the inhomogeneous problem associated to generic φ_i's. Therefore, if the same procedure were correct for $\tau = 0$, we would obtain a uniqueness result for the corresponding adjoint problem. Coming back to the beginning of the procedure that led to (4.41), we see that what is needed is the validity of the original equation (4.34) with $\tau = 0$, in addition to (2.54)–(2.55). On the other hand, this is a consequence of (2.54) provided that $(q, q_\Gamma) \in L^2(0,T;\mathcal{V})$. Indeed, in this case one is allowed to differentiate (2.54). In conclusion, we then would have a uniqueness result for the solution $(p, p_\Gamma, q, q_\Gamma)$ to the adjoint problem (2.54)–(2.55) whose component (q, q_Γ) is smoother than required.

References

[1] H. Abels, M. Wilke, Convergence to equilibrium for the Cahn–Hilliard equation with a logarithmic free energy, *Nonlinear Anal.* **67** (2007), 3176-3193.

[2] V. Barbu, “Nonlinear Differential Equations of Monotone Type in Banach Spaces”, Springer, London, New York, 2010.

[3] H. Brezis, “Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert”, North-Holland Math. Stud. **5**,North-Holland, Amsterdam, 1973.

[4] L. Calatroni, P. Colli, Global solution to the Allen–Cahn equation with singular potentials and dynamic boundary conditions, *Nonlinear Anal.* **79** (2013), 12-27.

[5] R. Chill, E. Fašangová, J. Prüss, Convergence to steady state of solutions of the Cahn–Hilliard and Caginalp equations with dynamic boundary conditions, *Math. Nachr.* **279** (2006), 448-1462.

[6] P. Colli, M. H. Farshbaf-Shaker, G. Gilardi, J. Sprekels, Optimal boundary control of a viscous Cahn–Hilliard system with dynamic boundary condition and double obstacle potentials, *SIAM J. Control Optim.* **53** (2015), 2696-2721.

[7] P. Colli, M. H. Farshbaf-Shaker, G. Gilardi, J. Sprekels, Second-order analysis of a boundary control problem for the viscous Cahn–Hilliard equation with dynamic boundary conditions, *Ann. Acad. Rom. Sci. Math. Appl.* **7** (2015), 41-66.

[8] P. Colli, G. Gilardi, P. Laurençot, A. Novick-Cohen, Uniqueness and long-time behaviour for the conserved phase-field system with memory, *Discrete Contin. Dynam. Systems* **5** (1999), 375-390.

[9] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, Distributed optimal control of a nonstandard system of phase field equations, *Contin. Mech. Thermodyn.* **24** (2012), 437-459.

[10] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, Well-posedness and long-time behaviour for a nonstandard viscous Cahn–Hilliard system, *SIAM J. Appl. Math.* **71** (2011), 1849-1870.

[11] P. Colli, G. Gilardi, E. Rocca, J. Sprekels, Optimal distributed control of a diffuse interface model of tumor growth, *Nonlinearity* **30** (2017), 2518-2546.

[12] P. Colli, G. Gilardi, J. Sprekels, Analysis and optimal boundary control of a nonstandard system of phase field equations, *Milan J. Math.* **80** (2012), 119-149.

[13] P. Colli, G. Gilardi, J. Sprekels, A boundary control problem for the viscous Cahn–Hilliard equation with dynamic boundary conditions, *Appl. Math. Optim.* **72** (2016), 195-225.

[14] P. Colli, G. Gilardi, J. Sprekels, A boundary control problem for the pure Cahn–Hilliard equation with dynamic boundary conditions, *Adv. Nonlinear Anal.* **4** (2015), 311-325.

[15] P. Colli, G. Gilardi, J. Sprekels, Global existence for a nonstandard viscous Cahn–Hilliard system with dynamic boundary condition, *SIAM J. Math. Anal.* **49** (2017) 1732-1760.

[16] P. Colli, G. Gilardi, J. Sprekels, Distributed optimal control of a nonstandard nonlocal phase field system, *AIMS Math.* **1** (2016), 246-281.

[17] P. Colli, G. Gilardi, J. Sprekels, Distributed optimal control of a nonstandard nonlocal phase field system with double obstacle potential, *Evol. Equ. Control Theory* **6** (2017), 35-58.

[18] P. Colli, G. Gilardi, J. Sprekels, On a Cahn–Hilliard system with convection and dynamic boundary conditions, *Ann. Mat. Pura. Appl. (4)*, Online First 2018, DOI 10.1007/s10231-018-0732-1.

[19] P. Colli, G. Gilardi, J. Sprekels, Optimal velocity control of a viscous Cahn–Hilliard system with convection and dynamic boundary conditions, *SIAM J. Control. Optim.*, to appear 2018.

[20] P. Colli, G. Gilardi, J. Sprekels, Optimal velocity control of a convective Cahn–Hilliard system with double obstacles and dynamic boundary conditions: a ‘deep quench’ approach, Preprint arxiv:1709.03892 [math.AP] (2017), pp. 1-30.

[21] P. Colli, G. Gilardi, J. Sprekels, On the longtime behavior of a viscous Cahn–Hilliard system with convection and dynamic boundary conditions, Preprint arxiv:1803.04318 [math.AP] (2018), pp. 1-20.

[22] P. Colli, J. Sprekels, Optimal boundary control of a nonstandard Cahn–Hilliard system with dynamic boundary condition and double obstacle inclusions, in: “Solvability, Regularity, and Optimal Control of Boundary Value Problems for PDEs” (P. Colli, A. Favini, E. Rocca, G. Schimperna, J. Sprekels, eds.), pp. 151-182, Springer INdAM Series **22**, Springer, Milan 2017.

[23] N. Duan, X. Zhao, Optimal control for the multi-dimensional viscous Cahn–Hilliard equation, *Electronic J. Differ. Equations 2015*, Paper No. 165, 13 pp.

[24] M. Efendiev, A. Miranville, S. Zelik, Exponential attractors for a singularly perturbed Cahn–Hilliard system. *Math. Nachr.* **272** (2004), 11-31.

[25] M. Efendiev, H. Gajewski, S. Zelik, The finite dimensional attractor for a 4th order system of the Cahn–Hilliard type with a supercritical nonlinearity, *Adv. Differential Equations* **7** (2002), 1073-1100.

[26] S. Frigeri, E. Rocca, J. Sprekels, Optimal distributed control of a nonlocal Cahn–Hilliard/Navier–Stokes system in two dimensions. *SIAM J. Control. Optim.* **54** (2016), 221-250.

[27] T. Fukao, N. Yamazaki, A boundary control problem for the equation and dynamic boundary condition of Cahn–Hilliard type, in: "Solvability, Regularity, and Optimal Control of Boundary Value Problems for PDEs" (P. Colli, A. Favini, E. Rocca, G. Schimperna, J. Sprekels, eds.), pp. 255-280, Springer INdAM Series **22**, Springer, Milan 2017.

[28] C. G. Gal, Exponential attractors for a Cahn–Hilliard model in bounded domains with permeable walls, *Electron. J. Differential Equations* **2006** (2006), 1-23.

[29] C. Gal, Well-posedness and long time behavior of the non-isothermal viscous Cahn–Hilliard equation with dynamic boundary conditions, *Dyn. Partial Differ. Equ.* **5** (2008), 39-67.

[30] C. G. Gal, M. Grasselli, Asymptotic behavior of a Cahn–Hilliard–Navier–Stokes system in 2D, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27** (2010), 401-436.

[31] C. G. Gal, M. Grasselli, Instability of two-phase flows: A lower bound on the dimension of the global attractor of the Cahn–Hilliard–Navier–Stokes system, *Physica D: Nonlinear Phenomena* **240** (2011), 629-635.

[32] G. Gilardi, A. Miranville, and G. Schimperna, On the Cahn–Hilliard equation with irregular potentials and dynamic boundary conditions, *Commun. Pure Appl. Anal.* **8** (2009), 881-912.

[33] G. Gilardi, E. Rocca, Well posedness and long time behaviour for a singular phase field system of conserved type IMA J. Appl. Math. **72** (2007) 498-530.

[34] M. Grasselli, H. Petzeltová, G. Schimperna, Asymptotic behavior of a nonisothermal viscous Cahn–Hilliard equation with inertial term, *J. Differential Equations* **239** (2007), 38-60.

[35] M. Hintermüller, M. Hinze, C. Kahle, T. Keil, A goal-oriented dual-weighted adaptive finite element approach for the optimal control of a nonsmooth Cahn–Hilliard–Navier–Stokes system, WIAS Preprint No. 2311, Berlin, 2016.

[36] M. Hintermüller, T. Keil, D. Wegner, Optimal control of a semidiscrete Cahn–Hilliard–Navier–Stokes system with non-matched fluid densities, *SIAM J. Control Optim.* **55** (2018), 1954-1989.

[37] M. Hintermüller, D. Wegner, Distributed optimal control of the Cahn–Hilliard system including the case of a double-obstacle homogeneous free energy density, *SIAM J. Control Optim.* **50** (2012), 388-418.

[38] M. Hintermüller, D. Wegner, Optimal control of a semidiscrete Cahn–Hilliard–Navier–Stokes system, *SIAM J. Control Optim.* **52** (2014), 747–772.

[39] M. Hintermüller, D. Wegner, Distributed and boundary control problems for the semidiscrete Cahn–Hilliard/Navier–Stokes system with nonsmooth Ginzburg–Landau energies. *Topological Optimization and Optimal Transport, Radon Series on Computational and Applied Mathematics* **17** (2017), 40-63.

[40] J. Jiang, H. Wu, S. Zheng, Well-posedness and long-time behavior of a non-autonomous Cahn–Hilliard–Darcy system with mass source modeling tumor growth *J. Differential Equations* **259** (2015), 3032-3077.

[41] D. Li, C. Zhong, Global attractor for the Cahn–Hilliard system with fast growing nonlinearity, *J. Differential Equations* **149** (1998), 191-210.

[42] T. Tachim Medjo, Optimal control of a Cahn–Hilliard–Navier–Stokes model with state constraints, *J. Convex Anal.* **22** (2015), 1135-1172.

[43] A. Miranville, Asymptotic behavior of the Cahn–Hilliard–Oono equation, *J. Appl. Anal. Comput.* **1** (2011), 523-536.

[44] A. Miranville, Asymptotic behavior of a generalized Cahn–Hilliard equation with a proliferation term, *Appl. Anal.* **92** (2013), 1308-1321.

[45] A. Miranville, Long-time behavior of some models of Cahn–Hilliard equations in deformable continua, *Nonlinear Anal.* **2** (2001), 273-304.

[46] A. Miranville, S. Zelik, Robust exponential attractors for Cahn–Hilliard type equations with singular potentials, *Math. Meth. Appl. Sci.* **27** (2004), 545-582.

[47] A. Miranville, S. Zelik, Exponential attractors for the Cahn–Hilliard equation with dynamic boundary conditions, *Math. Models Appl. Sci.* **28** (2005), 709-735.

[48] J. Prüss, V. Vergara, R. Zacher, Well-posedness and long-time behaviour for the non-isothermal Cahn–Hilliard equation with memory, *Discrete Contin. Dyn. Syst. Series A* **26** (2010), 625-647.

[49] E. Rocca, J. Sprekels, Optimal distributed control of a nonlocal convective Cahn–Hilliard equation by the velocity in three dimensions, *SIAM J. Control Optim.* **53** (2015), 1654-1680.

[50] A. Segatti, On the hyperbolic relaxation of the Cahn–Hilliard equation in 3D: approximation and long time behaviour, *Math. Models Methods Appl. Sci.* **17** (2007), 411-437.

[51] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl. (4)* **146**, (1987), 65-96.

[52] Q.-F. Wang, S.-i. Nakagiri, Weak solutions of Cahn–Hilliard equations having forcing terms and optimal control problems, Mathematical models in functional equations (Japanese) (Kyoto, 1999), Surikaisekikenkyusho Kokyuroku No. 1128 (2000), 172-180.

[53] X.-M. Wang, H. Wu, Long-time behavior for the Hele–Shaw–Cahn–Hilliard system, *Asymptot. Anal.* **78** (2012), 217-245.

[54] H. Wu, S. Zheng, Convergence to equilibrium for the Cahn–Hilliard equation with dynamic boundary conditions, *J. Differential Equations* **204** (2004), 511-531.

[55] X. Zhao, C. Liu, On the existence of global attractor for 3D viscous Cahn–Hilliard equation, *Acta Appl. Math.* **138** (2015), 199-212.

[56] X. Zhao, C. Liu, Optimal control of the convective Cahn–Hilliard equation, *Appl. Anal.* **92** (2013), 1028-1045.

[57] X. Zhao, C. Liu, Optimal control for the convective Cahn–Hilliard equation in 2D case, *Appl. Math. Optim.* **70** (2014), 61-82.

[58] J. Zheng, Time optimal controls of the Cahn–Hilliard equation with internal control, *Optimal Control Appl. Methods* **60** (2015), 566-582.

[59] J. Zheng, Y. Wang, Optimal control problem for Cahn–Hilliard equations with state constraints, *J. Dyn. Control Syst.* **21** (2015), 257-272.

[60] S. Zheng, Asymptotic behavior of solution to the Cahn–Hilliard equation, *Appl. Anal.* **23** (1986), 165-184.