Existence and Stability of N-Pulses on Optical Fibers with Phase-Sensitive Amplifiers

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Abstract

The propagation of pulses in optical communication systems in which attenuation is compensated by phase-sensitive amplifiers is investigated. A central issue is whether optical fibers are capable of carrying several pieces of information at the same time. In this paper, multiple pulses are shown to exist for a fourth-order nonlinear diffusion model due to Kutz and co-workers [10]. Moreover, criteria are derived for determining which of these pulses are stable. The pulses arise in a reversible orbit-flip, a homoclinic bifurcation investigated here for the first time. Numerical simulations are used to study multiple pulses far away from the actual bifurcation point. They confirm that properties of the multiple pulses including their stability are surprisingly well predicted by the analysis carried out near the bifurcation.

1 Introduction

In recent years pulse propagation in optical fibers has attracted much interest. For long-distance communication systems, compensating for the attenuation of pulses inherent in the fiber is an important issue. One effective approach is to use Erbium-doped amplifiers, see [9] or [13]. As an alternative, Kutz and co-workers [10] have recently proposed the use of periodically spaced phase-sensitive amplifiers. Each such amplifier exhibits an associated reference phase. The part of the signal in phase with this reference phase is amplified, while the out-of-phase component is attenuated, see [10] for the details. It was shown in [10] that the dynamics of the in-phase component U of the pulse amplitude under the influence of phase-sensitive amplifiers is governed by the fourth-order equation

$$\begin{aligned}(1.1)\qquad \frac{\partial U}{\partial \zeta} &+ \frac{1}{4}\frac{\partial^4 U}{\partial T^4} + \left(\left(3 - \frac{\tanh \Gamma l}{\Gamma l}\right)U^2 - \frac{\kappa}{2}\right)\frac{\partial^2 U}{\partial T^2} + \\ &+ 3\left(2 - \frac{\tanh \Gamma l}{\Gamma l}\right)U\left(\frac{\partial U}{\partial T}\right)^2 + \left(\frac{\kappa}{4} - \Delta\alpha\right)U - \kappa U^3 + U^5 = 0.\end{aligned}$$

Here, $\zeta \in \mathbb{R}$ measures the distance along the fiber on a length scale which is large compared to both distance of the amplifiers and dispersion length. The variable $T \in \mathbb{R}$ is time in a frame moving with the group velocity of light in the optical fiber. Furthermore, κ is related to the reference phase associated with each amplifier. The parameter $\Delta\alpha$ measures the amount of over-amplification, that is, the amount of energy remaining after compensating for the loss in the fiber. Finally, Γl is the product of linear loss rate Γ in the fiber and the distance l of the amplifiers. It is useful to introduce new variables

$$a = \sqrt{\kappa}, \qquad \eta = \pm\sqrt{\kappa \pm 2\sqrt{\Delta\alpha}}, \qquad \sigma = 1 - \frac{\tanh \Gamma l}{\Gamma l}.$$

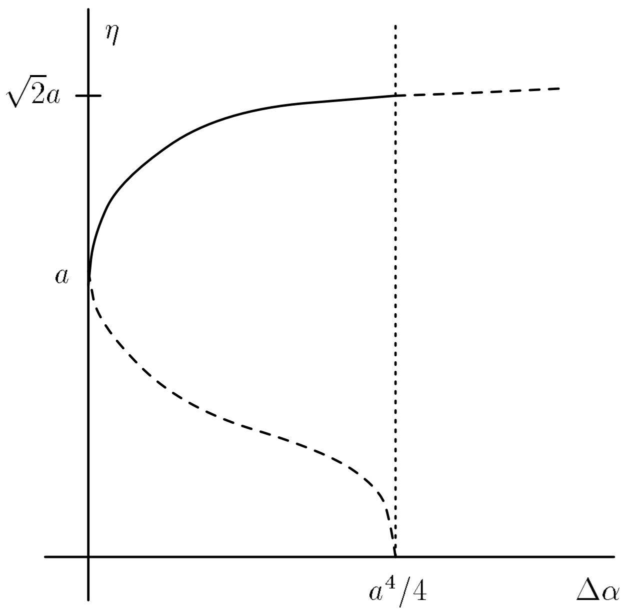

Figure 1: The bifurcation diagram. Dotted lines correspond to unstable $R(T)$, while solid lines correspond to stable $R(T)$.

Note that $\sigma = \sigma(\Gamma l)$ is a monotone increasing diffeomorphism of the interval $(0,\infty)$ onto $(0,1)$. Thus it is possible to use the parameter σ instead of Γl. For $\sigma = 0$, equation (1.1) can be factored into the product of two second-order operators. It is straightforward to calculate that (1.1) then becomes

$$\frac{\partial U}{\partial \zeta} + \left(\frac{\partial^2}{\partial T^2} + 2U^2 - (2a^2 - \eta^2)\right)\left(\frac{\partial^2}{\partial T^2} + 2U^2 - \eta^2\right)U + 4\sigma\left(3U\left(\frac{\partial U}{\partial T}\right)^2 + U^2\frac{\partial^2 U}{\partial T^2}\right) = 0. \tag{1.2}$$

There is a one-parameter family of pulse solutions

$$R(T) = \eta \operatorname{sech} \eta T \tag{1.3}$$

of equation (1.2) for $\sigma = 0$. Stability of these steady states is a critical issue as only stable pulses are expected to be physically realizable. Kath and Kutz [11] proved that, close to the turning point $\Delta\alpha = 0$, the upper branch $\eta > 0$ is stable while the lower branch $\eta < 0$ is unstable. For $\eta > \sqrt{2}a$ a radiation instability occurs. In [2] it has been shown that the upper branch is in fact stable for all $\Delta\alpha \in (0, a^4/4)$, see Figure 1.

The issue addressed in this article is whether the fiber is capable of carrying multiple pulses. These solutions represent the propagation of several pieces of information along the fiber. Besides the existence of multiple pulses, it is important to determine their stability relative to equation (1.2). We shall consider this problem as a bifurcation problem for the steady-state equation

$$\left(\frac{\partial^2}{\partial T^2} + 2U^2 - (2a^2 - \eta^2)\right)\left(\frac{\partial^2}{\partial T^2} + 2U^2 - \eta^2\right)U + 4\sigma\left(3U\left(\frac{\partial U}{\partial T}\right)^2 + U^2\frac{\partial^2 U}{\partial T^2}\right) = 0 \tag{1.4}$$

near the primary pulses $R(T)$ for positive σ close to zero and $|\eta| \in (0, \sqrt{2}a)$. In the original parameters, this corresponds to either small distances of consecutive amplifiers or a small linear loss rate along the optical fiber.

There are two important features of (1.4) we shall exploit. These are time-reversibility, that is, $U(-T)$ is a solution whenever $U(T)$ is, and the $\mathbb{Z}_2$-symmetry $U \to -U$. For generic reversible systems, multiple pulses are expected if the eigenvalues at the zero equilibrium are complex, see [8] or [3]. However, the eigenvalues of (1.4) at $U = 0$ are real

$$\lambda_{1,2} = \pm\sqrt{2a^2 - \eta^2} \qquad\qquad \lambda_{3,4} = \pm\eta \tag{1.5}$$

for $|\eta| \in (0, \sqrt{2}a)$. Thus multiple pulses for (1.4) are expected to occur at codimension-one bifurcation points. Surprisingly, for $\sigma = 0$, the stable pulses on the upper branch are degenerate. Indeed, comparing (1.3) and (1.5), they converge with the larger exponential rate η to zero. It is well known that for non-reversible systems satisfying a non-degeneracy condition this so-called orbit-flip bifurcation leads to the existence of multiple pulses, see [21] and [15].

In this article, we investigate the orbit-flip bifurcation for arbitrary reversible systems and apply the resulting theory to equation (1.4). The method employed is due to Lin [12] and was further extended in [15]. First, under non-degeneracy assumptions, it is shown that many multiple pulses do bifurcate which follow the figure-of-eight formed by $R(T)$ and $-R(T)$. In fact, denote R and $-R$ by up and down, respectively, see Figure 4, then any symmetric or anti-symmetric sequence of ups and downs is realized by a multiple pulse following R and $-R$ in the same order. Here, symmetric (anti-symmetric) refers to reading the sequence backwards (and swapping up and down). In particular, up-up-... as well as up-down-up-... pulses of any length exist. This general result is then applied to equation (1.4). It is shown that all non-degeneracy hypotheses are actually met by equation (1.4)! In particular, the multiple pulses described above occur for (1.4) for $\sigma > 0$, which is the physically relevant parameter regime. As in [2], we shall make use of the decomposition of the nonlinear operator into two second-order operators. This fact as well as formula (1.3) for the primary pulse affords an actual calculation of explicit solutions for the linearized equation along the primary pulses and the associated adjoint equations.

Stability of multiple pulses amounts to proving stability of the primary pulse R and computing N critical eigenvalues near zero, see [1]. Note that stability of the primary pulse has been proved in [2]. Here, the critical eigenvalues are calculated applying the results in [19]. It turns out that only pulses of a certain form are stable. Indeed, for a multiple pulse to be stable it must have every up piece followed by a down, and every down followed by an up. Thus, in particular, all up-up-... and down-down-... pulses are unstable, whereas up-down-up-down-... and down-up-down-... pulses are stable.

We shall mention that a reversible orbit-flip bifurcation was observed numerically by [4] in a fourth-order Hamiltonian system for gravity-capillary water waves. However, Hypothesis (H5)(iii) cannot be satisfied for Hamiltonian systems on account of [17, Remark 2]. Nonetheless, the basic strategy developed herein should be applicable to Hamiltonian orbit-flip bifurcations as well. This is work in progress and will appear elsewhere.

The paper is organized as follows. In Section 2, the main results on reversible orbit-flip bifurcations and the associated eigenvalue problems are given for general systems. These results are proved in Section 3 and applied to equation (1.2) in Section 4. Finally, Section 5 contains numerical simulations for equation (1.2).

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2 The reversible orbit-flip bifurcation

Consider the ordinary differential equation

$$\dot{u} = f(u, \mu) \qquad\qquad (u, \mu) \in \mathbb{R}^{2n} \times \mathbb{R}, \tag{2.1}$$

where f is a smooth nonlinearity such that $f(0, \mu) = 0$ for all $\mu \in \mathbb{R}$. Throughout, we assume that zero is a hyperbolic equilibrium of (2.1) for all μ. We shall impose the following hypotheses on (2.1).

Suppose that (2.1) admits a homoclinic solution $h(t)$ for $\mu = 0$ converging to zero.

(H1) The solution $h(t)$ solves (2.1) for $\mu = 0$ and satisfies

$$\begin{aligned} &\text{(i)} \qquad \lim_{t \to \pm\infty} h(t) = 0 \\ &\text{(ii)} \qquad T_{h(0)} W^s(0) \cap T_{h(0)} W^u(0) = \mathbb{R}\dot{h}(0). \end{aligned}$$

We are going to assume that (2.1) is time-reversible, that is

(H2) There exists a linear operator $R : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with $R^2 = \mathrm{id}$ and $\dim \operatorname{Fix} R = n$ such that $f(Ru, \mu) = -Rf(u, \mu)$ holds for all $(u, \mu) \in \mathbb{R}^{2n} \times \mathbb{R}$. We may assume that R is an isometry. Finally, assume that $h(0) \in \operatorname{Fix}(R)$.

We call a solution $u(t)$ time-reversible or symmetric with respect to R if $u(0) \in \operatorname{Fix} R$. Any symmetric solution u satisfies

$$u(-t) = Ru(t) \qquad t \in \mathbb{R}. \tag{2.2}$$

The next hypothesis is not needed for most of our results. However, if it is met, the results can be strengthens a lot. Moreover, based on the conclusions drawn from the main results under the assumption that (H3) is true, numerical simulations can be used to check whether Hypothesis (H3) is actually satisfied for equation (1.4). Thus, let us suppose that equation (2.1) is in addition conservative, that is admits a first integral.

(H3) Let $H : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\langle \nabla H(u, \mu), f(u, \mu) \rangle = 0$ for all $(u, \mu) \in \mathbb{R}^{2n} \times \mathbb{R}$ with $\nabla = \nabla_u$. Moreover, assume that $\nabla H(h(0), 0) \neq 0$.

Hypothesis (H2) implies that the spectrum of $D_u f(0, \mu)$ is symmetric with respect to the imaginary axis, see for instance [20]. We assume that the spectrum decomposes as follows.

(H4) The spectrum of the equilibrium $u = 0$ is given by

$$\sigma(D_u f(0, \mu)) = \sigma^s \cup \{\pm\alpha^u(\mu), \pm\alpha^{uu}(\mu)\} \cup \sigma^u, \quad 0 < \alpha^u(\mu) < \alpha^{uu}(\mu)$$

with $\operatorname{Re} \sigma^s < -\alpha^r < -\alpha^{uu}(\mu)$, $\operatorname{Re} \sigma^u > \alpha^r > \alpha^{uu}(\mu)$ for all μ. Moreover, $\pm\alpha^u(\mu)$ and $\pm\alpha^{uu}(\mu)$ are simple eigenvalues. We denote the spectral projections onto the leading simple eigenvalues $\pm\alpha^u(\mu)$ by Q^s and Q^u. Moreover, the spectral projection onto σ^s is denoted by Q^r.

Moreover, (H2) does imply that there exist a smooth one-parameter family $h_\mu(t)$ of homoclinic solutions for μ close to zero satisfying (H1) such that $h_0(t) = h(t)$, see [20]. Owing to (H1)(ii), there exists a unique, up to constant multiples, bounded solution $\psi(t)$ of the adjoint variational equation

$$\dot{w} = -D_u f(h(t), 0)^* \, w.$$

In fact,

$$\psi(t) \perp T_{h(t)} W^u(0) + T_{h(t)} W^s(0) \tag{2.3}$$

holds. The definition of an orbit-flip bifurcation occuring at parameter value $\mu = 0$ now reads

(H5) Assume that $h(t) \in W^{ss}(0)$ and the limits

$$\begin{array}{lllll} \text{(i)} & \lim_{t\to\infty} h(t)\, e^{\alpha^{uu} t} & = & v_{ss} \\ \text{(ii)} & \lim_{t\to\infty} \frac{d}{d\mu} h_\mu(t)|_{\mu=0}\, e^{\alpha^u t} & = & v_s \\ \text{(iii)} & \lim_{t\to\infty} \psi(-t)\, e^{\alpha^u t} & = & w_s \end{array}$$

exist and are non-zero. Then the limit

$$\text{(iv)} \qquad \lim_{t\to\infty} \langle \psi(-t), h(t) \rangle\, e^{2\alpha^{uu} t} \;=\; S \neq 0$$

exists, too, and we assume $S \neq 0$.

It is a consequence of (H5) that v_s and w_s are left and right eigenvectors of $D_u f(0,0)$ associated with the eigenvalue $-\alpha^u$ while v_{ss} is an eigenvector of $D_u f(0,0)$ corresponding to $-\alpha^{uu}$, see [15, Lemma 1.7]. Hence, (H5) states that $h_\mu(t)$ switches through the strong stable manifold with non-zero speed as μ passes through zero, see Figure 2. We define

$$\tilde{S} := -\frac{S}{\langle w_s, v_s \rangle} \neq 0,$$

using Hypothesis (H5).

As we are interested in systems derived from nonlinear Schrödinger equations, we are going to assume the existence of an additional symmetry of (2.1).

(H6) Assume that (2.1) is equivariant with respect to $\mathbb{Z}_2 = \{\mathrm{id}, -\mathrm{id}\} = \{1, -1\}$, that is

$$f(-u, \mu) = -f(u, \mu) \qquad\qquad (u, \mu) \in \mathbb{R}^{2n} \times \mathbb{R}.$$

Thus, the homoclinic solutions $h(t)$ and $-h(t)$ form a figure-of-eight in phase space, see Figure 3. Note that as $-\mathrm{id}$ and R commute the nonlinearity is time-reversible with respect to $-R$, too. We call a homoclinic solution $q(t)$ symmetric if either $q(0) \in \mathrm{Fix}(R)$ or $q(0) \in \mathrm{Fix}(-R)$ holds after a suitable choice of $t = 0$.

Remark Similar results to the ones given below hold for more general Z_2-symmetries. In fact, it is only necessary that f is equivariant with respect to a linear involution $\kappa : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ commuting with R and satisfying $\kappa|_{\mathrm{span}\{v_s, v_{ss}\}} = -\mathrm{id}$ or else $\kappa|_{\mathrm{span}\{v_s, v_{ss}\}} = \mathrm{id}$ and $h(0) \notin \mathrm{Fix}\, \kappa$.

The solutions we shall describe are so-called N-pulses. These are homoclinic solutions staying in a small tubular neighborhood of the figure-of-eight configuration and intersecting an appropriate section Σ N-times. Here, $\Sigma = \Sigma_{\mathrm{id}} \cup \Sigma_{-\mathrm{id}}$ where Σ_θ is a section transverse

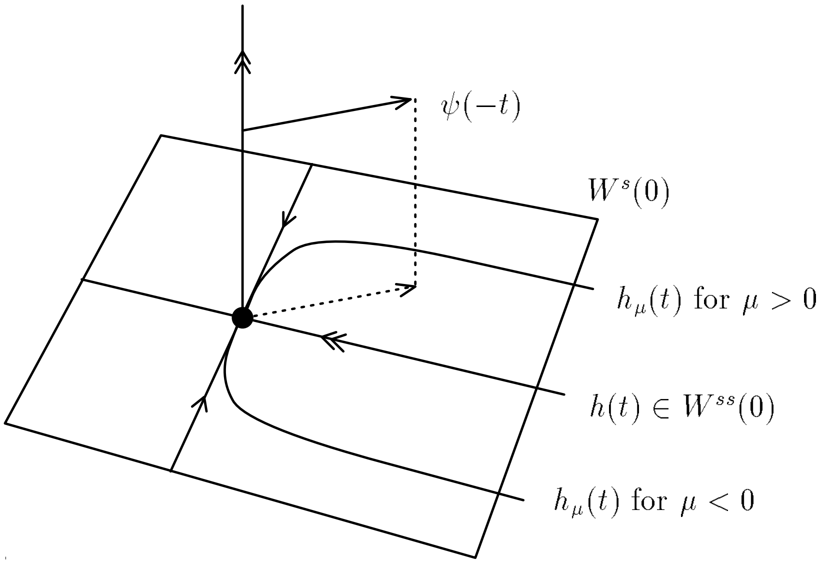

Figure 2: The homoclinic solution passes through the strong stable manifold with non-zero speed.

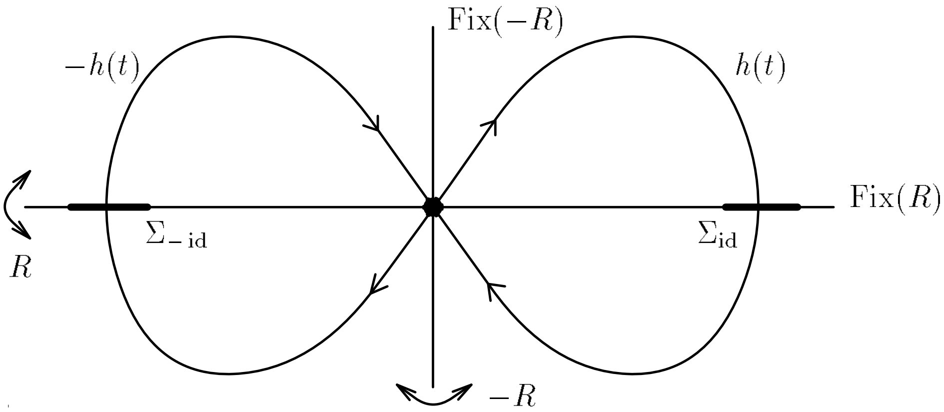

Figure 3: The figure-of-eight configuration.

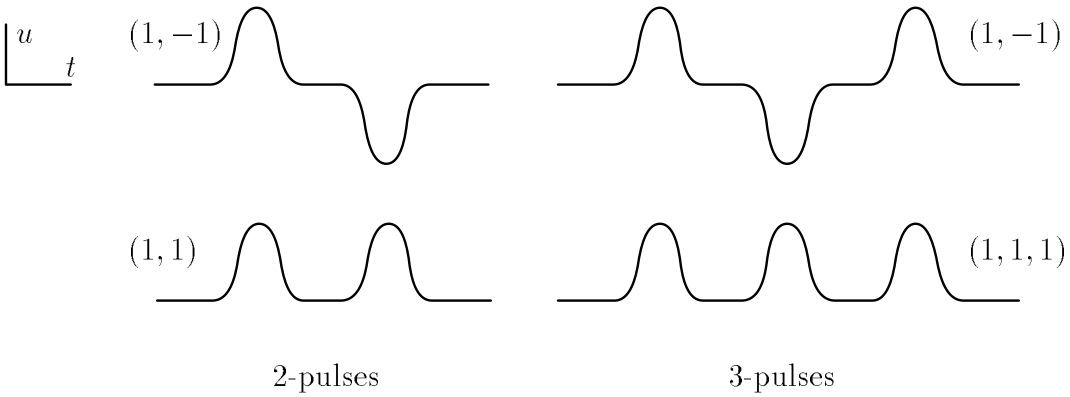

Figure 4: N-pulses $h_\gamma(t)$ for different sequences γ.

to the vector field placed at $\theta h(0)$ for $\theta \in \mathbb{Z}_2$, see Figure 3. We may assume that both sections are invariant under R. The shape of an N-pulse depends on the order in which it intersects the sections Σ_θ for $\theta \in \mathbb{Z}_2$, that is, the order in which it follows either $h(t)$ or $-h(t)$.

Definition We call a sequence $\gamma = (\gamma_j)_{j=1,\dots,N}$ with $\gamma_j \in \mathbb{Z}_2 = \{\mathrm{id}, -\mathrm{id}\}$ admissible if either $\gamma_{N+1-j} = \gamma_j$ for all j or $\gamma_{N+1-j} = -\gamma_j$ for all j.

Then we have the following theorem.

Theorem 1 *Suppose that (H1), (H2), (H4), (H5) and (H6) are satisfied. Then, for any $N > 1$ there exists a $\delta_N > 0$ such that the following holds.*

Choose any sequence $(\gamma_j)_{j=1,\dots,N}$ of length N. If (H3) holds as well, it can be chosen arbitrarily, otherwise it must be admissible.

Then, for any μ with $|\mu| < \delta_N$ and $\operatorname{sign}\mu = \tilde{S}$, there exist an N-pulse $h_\gamma(t)$ such that $h_\gamma(t)$ intersects the sections Σ_θ precisely in the order given by $\theta = \gamma_j$ for $j = 1, \dots, N$, see Figure 4. The distances of consecutive humps are approximately given by

$$L := \frac{\ln|\mu|}{\alpha^u - \alpha^{uu}} + L_0$$

for some constant $L_0 \in \mathbb{R}$.

If (H3) holds, then the N-pulses described above are unique. They are symmetric if and only if the sequence is admissible.

Otherwise, the N-pulses described above are symmetric. Pulses corresponding to sequences $\gamma = (\mathrm{id}, \dots, \mathrm{id})$ and $\gamma = (\mathrm{id}, -\mathrm{id}, \mathrm{id}, -\mathrm{id}, \dots)$ are unique. If there exists any other N-pulse then the corresponding sequence is either not admissible or at least one distance of consecutive humps is given approximately by $(1+\beta)T$ for some $\beta > 0$.

Note that Theorem 1 is valid without Hypothesis (H6). Then the only sequences guaranteed are (id, ..., id) and, in particular, all N-pulses are unique.

Next, stability of the N-pulses is addressed. By [1], this amounts to computing N critical eigenvalues of the PDE linearization at each particular N-pulse. Assuming that the steady-state equation becomes equation (2.1) after rewriting it as a first order system, the eigenvalue problem of the PDE linearization evaluated at an N-pulse h_γ becomes

$$\dot{v} = \Big(D_u f(h_\gamma(t), \mu) + \lambda B\Big)\, v,$$

where B is an $n \times n$-matrix arising when rewriting the PDE as a first order system. The issue is to calculate bounded solutions v, i.e. eigenfunctions, and the associated values of λ near zero. Thus, more general, we shall describe all bounded solutions $v \in C^1(\mathbb{R}, \mathbb{C}^{2n})$ of the equation

$$\dot{v} = \Big(D_u f(h_\gamma(t), \mu) + \lambda B(t)\Big)\, v \tag{2.4}$$

for $\lambda \in U_\delta(0) \subset \mathbb{C}$, where h_γ denotes the N-pulse described in Theorem 1 for a given (admissible) sequence γ of length N and existing for parameter value μ. Here, B is a bounded, continuous and matrix-valued function. Equation (2.4) is a generalized eigenvalue problem of the form

$$Lv = \lambda B v.$$

Generalized eigenfunctions of (2.4) corresponding to an eigenvalue λ are functions v_i satisfying

$$Lv_i = \lambda B v_i + B v_{i-1}$$

with $v_0 = 0$. The algebraic multiplicity of eigenvalues can be defined in the usual way. We assume a non-degeneracy assumption with respect to λ.

(H7) Suppose that the Melnikov integral

$$M := \int_{-\infty}^{\infty} \langle \psi(t), B(t)\, \dot{h}(t) \rangle \, dt \neq 0$$

is non-zero.

The next theorem describes the set of $\lambda \in U_\delta(0) \subset \mathbb{C}$ for $\delta > 0$ small for which (2.4) possesses a bounded solution v.

Theorem 2 *Suppose that (H1), (H2), (H4), (H5) and (H6) are satisfied. Choose $N > 1$, a sequence γ of length N and μ with $|\mu| < \delta_N$ and* $\operatorname{sign} \mu = \tilde{S}$.

If (H3) holds, the sequence γ can be chosen arbitrarily, otherwise it must be admissible. Let h_γ be the corresponding solution described in Theorem 1.

Then there exist precisely N solutions $(\lambda_j, v_j) \in \mathbb{C} \times C^1(\mathbb{R}, \mathbb{C}^{2n})$ of (2.4) with $|\lambda| < \delta$ counted with multiplicity. Moreover, we have

$$\begin{aligned}
\#\{i \,|\, \operatorname{sign} \operatorname{Re} \lambda_i = \operatorname{sign} MS\} &= \#\{i \,|\, \gamma_i \neq \gamma_{i+1},\, 1 \leq i < N\} \\
\#\{i \,|\, \operatorname{sign} \operatorname{Re} \lambda_i = -\operatorname{sign} MS\} &= \#\{i \,|\, \gamma_i = \gamma_{i+1},\, 1 \leq i < N\} \\
\#\{i \,|\, \lambda_i = 0\} &= 1
\end{aligned}$$

and

$$|\lambda_j| \leq K_N \, |\mu|^{\alpha^{uu}/(\alpha^{uu}-\alpha^{u})}$$

as $\mu \to 0$ for some constant K_N. Here, S and M are defined in Hypotheses (H5) and (H7), respectively.

If there is another N-pulse h_γ of (2.1), there exist solutions (λ, v) of (2.4) with $\operatorname{Re}\lambda > 0$ *as well as solutions with* $\operatorname{Re}\lambda < 0$.

Theorem 2 is valid without imposing Hypothesis (H6). Again, the only sequences guaranteed are (id, ..., id) and the real part of all solutions (λ, v) is either positive or negative independent of N. Note that N-pulses are unique in this case.

3 Proofs of Theorem 1 and 2

We shall employ a Lyapunov-Schmidt reduction for proving the theorems. Existence is done using Lin's method, see [12] and [15, 16], while stability will follow from [19].

Throughout, we assume that (H1), (H4), (H5) and (H6) are satisfied. Moreover, the vector field is assumed to be time-reversible, see Hypothesis (H2), and we consider also the case where it is in addition conservative, that is satisfies (H3).

Convention. Throughout this section, we use the convention that the ranges of the indices i and j are $i = 1, ..., N$ and $j = 1, ..., N-1$ as long as stated otherwise. Also, we denote several, possibly different positive constants by $K > 0$. Finally, $[x]$ denotes the largest integer smaller than x for $x \in \mathbb{R}$.

3.1 Existence

3.1.1 Homoclinic Lyapunov-Schmidt reduction

By Hypothesis (H2), there exist a unique family $h_\mu(t)$ of homoclinic solutions for $\mu \in \mathbb{R}$ small, see [20]. They are symmetric as well. Then the adjoint equation

$$\dot{w} = D_u f(h_\mu(t), \mu)^* \, w$$

possesses a unique bounded solution $\psi_\mu(t)$. We are interested in solutions following the primary pulses $h(t)$ and $-h(t)$ in a given order $\gamma = (\gamma_i)_{i=1,\ldots,N}$ with $\gamma_i \in \mathbb{Z}_2$. At this point, the sequence γ is not assumed to be admissible. It will turn out that solutions with the above properties can be distinguished by the time-of-flight $2T_j$ needed for the jth loop in between the consecutive sections Σ_{γ_j} and $\Sigma_{\gamma_{j+1}}$. Thus, we seek solutions $u_i^\pm(t)$ such that

$$
\begin{array}{lrcll}
 & \dot{u}_i^- & = & f(u_i^-, \mu) & \text{for } t \in (-T_{i-1}, 0) \\
 & \dot{u}_i^+ & = & f(u_i^+, \mu) & \text{for } t \in (0, T_i) \\
(3.1) & u_i^\pm(0) & \in & \Sigma_{\gamma_i} & \\
 & u_i^+(0) & = & u_i^-(0) & \\
 & u_j^+(T_j) & = & u_{j+1}^-(-T_j), &
\end{array}
$$

with $|u_i^\pm(t) - \gamma_i h(t)|$ small. Here, as we look for homoclinic solutions, we set $T_0 = T_N = \infty$, that is $u_1^-(t) \in W^u(0)$ and $u_N^+(t) \in W^s(0)$. Lin's method reduces (3.1) to a system of N equations. In fact, it is concerned with the weaker problem

$$
(3.2)\quad
\begin{array}{clcll}
(i) & \dot{u}_i^\pm & = & f(u_i^\pm, \mu) & \text{for } t \in (-T_{i-1}, 0) \text{ or } t \in (0, T_i) \\
(ii) & u_i^\pm(0) & \in & \Sigma_{\gamma_i} & \\
(iii) & u_i^-(0) - u_i^+(0) & \in & \operatorname{span} \gamma_i \, \psi_\mu(0) & \\
(iv) & u_j^+(T_j) & = & u_{j+1}^-(-T_j), &
\end{array}
$$

that is, it allows for jumps of the solutions $u_i^+(t)$ and $u_i^-(t)$ at $t = 0$ in the one-dimensional subspace

$$\operatorname{span} \gamma_i \, \psi_\mu(0) = T_{\gamma_i h_\mu(0)} W^u(0, \mu) + T_{\gamma_i h_\mu(0)} W^s(0, \mu).$$

We have the following proposition.

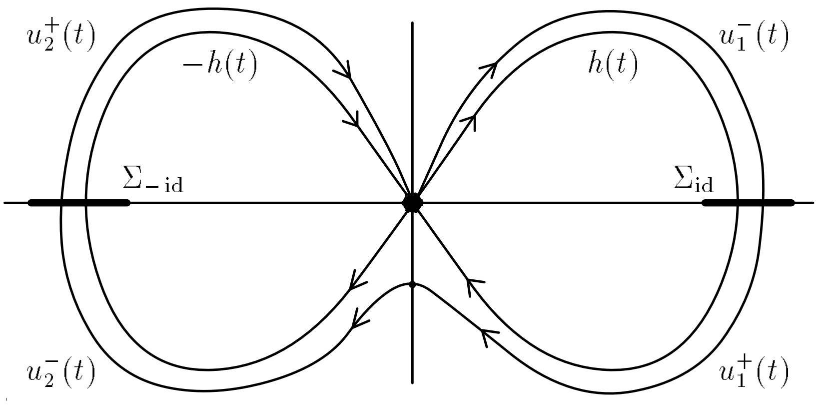

Figure 5: The definition of the functions $u_i(t)$ associated with a 2-pulse.

Proposition 1 ([12],[15]) *There exists a $\delta > 0$ such that system (3.2) has a unique solution $(u_i^\pm)$ for any given sequence $T = (T_j)$ with $T_j > 1/\delta$. Moreover, (3.1) has a solution if and only if*

$$\xi(T,\mu) = \Big(\langle \gamma_i\, \psi_\mu(0), u_i^-(0) - u_i^+(0) \rangle \Big)_i = 0.$$

Then the solution is given by $(u_i^\pm)$.

Let us consider time-reversible systems next. Symmetry of pulses imposes constraints on the possible sequences γ as well as on the return times T. Indeed, both sequences have to be admissible where admissibility of sequences of return times $T = (T_j)$ is defined by

Definition We call a sequence $T = (T_j)_{j=1,\dots,N-1}$ admissible if $T_{N-j} = T_j$ for all j. Any admissible sequence (T_j) is determined by its arbitrarily chosen entries T_j for $j = 1, \dots, [N/2]$.

Then we have the following corollary to Proposition 1.

Lemma 3.1 *Assume that (H2) holds. Then the unique solution (u_i) of (3.2) associated with sequences γ and T is time-reversible with respect to either R or $-R$ if and only if γ and T are both admissible. In that case, the jumps ξ satisfy*

$$\xi_{N+1-i}(T,\mu) = -\xi_i(T,\mu)$$

for all i.

Proof. The argument here is similar to the one given in [20] for 1-homoclinic solutions. It is clear that admissibility of γ and T is necessary for symmetry of (u_i). Let us therefore suppose that γ and T are admissible. By definition, there exists $\theta \in \mathbb{Z}_2$ such that $\gamma_{N+1-i} = \theta\gamma_i$ for all i. Consider the family

$$\begin{aligned} v_i^-(t) &:= R\,\theta\, u_{N+1-i}^+(-t) & \qquad & t \in (-T_{N+1-i}, 0) = (-T_{i-1}, 0) \\ v_i^+(t) &:= R\,\theta\, u_{N+1-i}^-(-t) & & t \in (0, T_{N-i}) = (0, T_i). \end{aligned}$$

Note that we already used admissibility of T. We show next that the family $(v_i^\pm)$ solves (3.2), too, whence by uniqueness we obtain $u = v$ and u is symmetric with respect to $R\,\theta$. First, v solves the ordinary differential equation as

$$\begin{aligned} \dot v_i^\pm &= -R\,\theta\, \dot u_{N+1-i}^\pm = -R\,\theta\, f(u_{N+1-i}^\pm, \mu) \\ &= f(R\,\theta\, u_{N+1-i}^\pm, \mu) = f(v_i^\pm, \mu) \end{aligned}$$

by (H2) and (H6). Moreover,

$$v_i^{\pm}(0) = \theta\, R\, u_{N+1-i}^{\mp}(0) \in \Sigma_{\theta\,\gamma_{N+1-i}} = \Sigma_{\gamma_i}$$

using $\theta^2 = \mathrm{id}$, $R\,\theta = \theta\, R$ and admissibility of γ as well as the definition of θ. Next, we have

$$\begin{aligned} v_i^-(0) - v_i^+(0) &= \theta\, R\,(u_{N+1-i}^+(0) - u_{N+1-i}^-(0)) \\ &\in \operatorname{span} \theta\, R\, \gamma_{N+1-i} \psi_\mu(0) = \operatorname{span} \gamma_i\, R\, \psi_\mu(0) = \operatorname{span} \gamma_i\, \psi_\mu(0) \end{aligned}$$

as $R\,\psi_\mu(0) = \psi_\mu(0)$. By the same token, (3.2)(iv) is satisfied. Indeed,

$$\begin{aligned} v_j^+(T_j) - v_{j+1}^-(-T_j) &= \theta\, R\, u_{N+1-j}^-(-T_j) - \theta\, R\, u_{N-j}^+(T_j) \\ &= \theta\, R\, u_{N+1-j}^-(-T_{N-j}) - \theta\, R\, u_{N+1-j}^+(T_{N-j}) = 0. \end{aligned}$$

Therefore, we conclude $u_i^{\pm} = v_i^{\pm}$ for all i by uniqueness. Finally, we obtain

$$\begin{aligned} \xi_i(T,\mu) &= \langle \gamma_i\, \psi_\mu(0), u_i^-(0) - u_i^+(0) \rangle = \langle \gamma_i\, \psi_\mu(0), v_i^-(0) - v_i^+(0) \rangle \\ &= \langle \gamma_i\, \psi_\mu(0), \theta\, R\,(u_{N+1-i}^+(0) - u_{N+1-i}^-(0)) \rangle \\ &= \langle \theta\, R\, \gamma_i\, \psi_\mu(0), u_{N+1-i}^+(0) - u_{N+1-i}^-(0) \rangle \\ &= \langle \gamma_{N+1-i}\, \psi_\mu(0), u_{N+1-i}^+(0) - u_{N+1-i}^-(0) \rangle \\ &= -\xi_{N+1-i}(T,\mu) \end{aligned}$$

using $u = v$, admissibility of γ and $R\,\psi_\mu(0) = \psi_\mu(0)$. ∎

Thus, as far as symmetric solutions are concerned, it suffices to solve the system

$$\xi_i(T,\mu) = 0 \qquad\qquad i = 1, ..., \left[\frac{N}{2}\right],$$

for admissible sequences γ and T. Indeed, Lemma 3.1 shows that the remaining jumps are zero as well. Note that the variables are μ and T_j for $j = 1, ..., [N/2]$.

Next, consider conservative systems. Then we have the following corollary to Proposition 1.

Lemma 3.2 *Assume that (H3) holds. Suppose that, for given sequences γ and T, we have*

$$\xi_k(T,\mu) = 0 \qquad\qquad k = 1, ..., N-1.$$

Then $\xi_N(T,\mu) = 0$ vanishes as well.

Proof. Again, the argument is similar to the one given in [20] for 1-homoclinic solutions. As $T_0 = \infty$ and $\xi_k(T,\mu) = 0$ for $k = 1, ..., N-1$, the functions $u_k^{\pm}(t)$ with $k = 1, ..., N-1$ and $u_N^-(t)$ do form a solution contained in the unstable manifold of zero. In particular, as H is a conserved quantity, we obtain

$$H(u_N^-(0),\mu) - H(u_N^+(0),\mu) = 0. \tag{3.3}$$

On the other hand, we have

$$u_N^-(0) - u_N^+(0) = \xi_N \nabla H(\gamma_N h_\mu(0),\mu) \tag{3.4}$$

by normalizing $|\nabla H(\gamma_N h_\mu(0),\mu)| = 1$. Using Taylor expansion for the difference (3.3) together with (3.4), is it easy to see that $\xi_N = 0$ must vanish, too. ■

Therefore, for conservative systems, it suffices to solve the first N–1 bifurcation equations as then the last one vanishes automatically. Note that Lemma 3.2 is valid without additional time-reversibility.

3.1.2 Deriving the bifurcation equations

By rescaling time, we may assume that the leading eigenvalue $\alpha^u(\mu) = \alpha^u$ is independent of μ. We cite the following theorem.

Theorem 3 ([16]) *Assume (H1), (H2) and (H6). Choose any sequences $\gamma = (\gamma_i)$ and $T = (T_j)$ with $T_j > 1/\delta$. Then the jumps $\xi = (\xi_i(T,\mu))$ are given by*

$$\xi_i(T,\mu) \;=\; \langle \gamma_i \psi_\mu(-T_{i-1}), \gamma_{i-1} h_\mu(T_{i-1})\rangle - \langle \gamma_i \psi_\mu(T_i), \gamma_{i+1} h_\mu(-T_i)\rangle + R_i(T,\mu),$$

where the remainder term satisfies

$$\begin{aligned} |R_i(T,\mu)| \;=\; & O\Big(e^{-\alpha^u T_{i-1}}\,(|h_\mu(-T_{i-1})| + |h_\mu(T_{i-1})|)^2 + e^{-\alpha^u T_i}\,(|h_\mu(-T_i)| + |h_\mu(T_i)|)^2 + \\ & (e^{-2\alpha^u T_{i-1}} + e^{-2\alpha^u T_i})\,\hat{R}\Big) \end{aligned}$$

with

$$\begin{aligned} |\hat{R}| \;\le\; & \sup\nolimits_{k=1,\ldots,N} \Big(e^{-\alpha^u T_k}\,(|Q^u h_\mu(-T_k)| + |Q^s h_\mu(T_k)|) + \\ & e^{-\alpha^{uu}(\mu) T_k}\,(|h_\mu(-T_k)| + |h_\mu(T_k)|) + e^{-\alpha^u T_k}\,(|h_\mu(-T_k)|^2 + |h_\mu(T_k)|^2)\Big). \end{aligned}$$

The derivatives of the remainder terms with respect to T_k and μ can be estimated by

$$
\begin{aligned}
|D_{T_{i-1}} R_i(T,\mu)| &= O\Big(e^{-\alpha^u T_{i-1}}\,(|h_\mu(-T_{i-1})| + |h_\mu(T_{i-1})|)^2 + e^{-2\alpha^u T_{i-1}}\,\hat{R}\Big)\\
|D_{T_i} R_i(T,\mu)| &= O\Big(e^{-\alpha^u T_i}\,(|h_\mu(-T_i)| + |h_\mu(T_i)|)^2 + e^{-2\alpha^u T_i}\,\hat{R}\Big)\\
|D_{T_k} R_i(T,\mu)| &= O\Big((e^{-2\alpha^u T_{i-1}} + e^{-2\alpha^u T_i})\,(e^{-\alpha^u T_k}\,(|Q^u h_\mu(-T_k)| + |Q^s h_\mu(T_k)|) +\\
&\qquad e^{-\alpha^{uu}(\mu) T_k}\,(|h_\mu(-T_k)| + |h_\mu(T_k)|) + e^{-\alpha^u T_k}\,(|h_\mu(-T_k)|^2 + |h_\mu(T_k)|^2) +\\
&\qquad e^{-2\alpha^u T_k}\,\hat{R})\Big) \qquad \textit{for } k \neq i, i+1\\
|D_\mu R_i(T,\mu)| &\leq \hat{R}
\end{aligned}
$$

uniformly in T and μ.

We shall introduce new variables by defining

$$a_j\, r := e^{-2\alpha^u T_j} \tag{3.5}$$

with $0 \leq a_j \leq K$ and $0 \leq r \leq r_0$. Let

$$\alpha(\mu) = \frac{\alpha^{uu}(\mu)}{\alpha^u} - 1 > 0$$

and $\alpha_0 = \alpha(0)$.

Lemma 3.3 *Assume that the hypotheses of Theorem 1 are satisfied. In the new variables (a_j, r, μ) the jumps read*

$$
\begin{aligned}
\xi_1(a,r,\mu) &= \mu\, a_1\, r - \tilde{S}\, a_1^{1+\alpha(\mu)}\, r^{1+\alpha(\mu)} + O\Big(|a_1|\,(r^{1+\alpha(\mu)}\, r^\beta + r\,|\mu|\,(|\mu| + r^\beta))\Big)\\
\xi_i(a,r,\mu) &= \mu\, a_{i-1}\, r - \tilde{S}\, a_{i-1}^{1+\alpha(\mu)}\, r^{1+\alpha(\mu)} - \delta_i\,(\mu\, a_i\, r + \tilde{S}\, a_i^{1+\alpha(\mu)}\, r^{1+\alpha(\mu)}) +\\
&\qquad O\Big((|a_{i-1}| + |a_i|)\,(r^{1+\alpha(\mu)}\, r^\beta + r\,|\mu|\,(|\mu| + r^\beta))\Big)\\
\xi_N(a,r,\mu) &= \mu\, a_{N-1}\, r - \tilde{S}\, a_{N-1}^{1+\alpha(\mu)}\, r^{1+\alpha(\mu)} + O\Big(|a_{N-1}|\,(r^{1+\alpha(\mu)}\, r^\beta + r\,|\mu|\,(|\mu| + r^\beta))\Big)
\end{aligned}
$$

for $i = 2, ..., N-1$ up to a non-zero constant factor. Here,

$$\beta = \min\Big\{\frac{1}{2}, 2\,(\alpha^r - \alpha^{uu}), \alpha\Big\}.$$

The constants δ_i are defined by

$$\delta_i = \gamma_{i-1}\,\gamma_{i+1} \in \{\pm 1\}$$

for $i = 2, ..., N-1$. The remainder terms are C^1 in (a,μ) for $a_j > 0$. The bifurcation equations can be extended to $a_j < 0$ in a Lipschitz continuous way.

Proof. We have

$$(3.6) \qquad \begin{aligned} h_\mu(t) &= \mu\, e^{-\alpha^u t}\, v_s + e^{-\alpha^{uu}(\mu)t}\, v_{ss} + v_r(t) + \\ &\quad \mathrm{O}\Big(|\mu|\,(|\mu|\, e^{-\alpha^u t} + e^{-2\alpha^u t} + e^{-\alpha^{uu}(\mu)t}) + e^{-2\alpha^{uu}(\mu)t}\Big) \end{aligned}$$

due to Hypothesis (H5) and [15][Lemma 1.7] with $v_r(t) \in \mathrm{R}Q^r$ and

$$|v_r(t)| = \mathrm{O}(e^{-\alpha^r t}).$$

Similarly, using [15][Lemma 1.8], we obtain

$$(3.7) \qquad \psi_\mu(-t) = e^{-\alpha^u t}\, w_s(\mu) + e^{-\alpha^{uu}(\mu)t}\, w_{ss}(\mu) + w_r(t) + \mathrm{O}(e^{-(\alpha^u+\alpha^{uu}(\mu))t})$$

with $w_r(-t) \in \mathrm{R}(Q^r)^*$ and

$$|w_r(-t)| = \mathrm{O}(e^{-\alpha^r t}).$$

Therefore, for $\theta \in \mathbb{Z}_2$, the scalar product appearing in Theorem 3 is given by

$$\begin{aligned} &\langle \psi_\mu(-t), \theta h_\mu(t)\rangle \\ &\quad = \mu\, \langle w_s(\mu), \theta v_s\rangle\, e^{-2\alpha^u t} + \langle w_{ss}(\mu), \theta v_{ss}\rangle\, e^{-2\alpha^{uu}(\mu)t} + \mathrm{O}\Big(e^{-2\alpha^{uu}(\mu)t}\,(e^{-\alpha^u t} + e^{-2(\alpha^r-\alpha^{uu})t}) + \\ &\quad\quad |\mu|\, e^{-2\alpha^u t}\,(|\mu|\, e^{-\alpha^u t} + e^{-2\alpha^u t} + e^{-\alpha^{uu}(\mu)t})\Big) \\ &\quad = \mu\, \langle w_s, \theta v_s\rangle\, e^{-2\alpha^u t} + \langle w_{ss}, \theta v_{ss}\rangle\, e^{-2\alpha^{uu}(\mu)t} + \mathrm{O}\Big(e^{-2\alpha^{uu}(\mu)t}\,(e^{-\alpha^u t} + e^{-2(\alpha^r-\alpha^{uu})t}) + \\ &\quad\quad |\mu|\,(|\mu|\, e^{-2\alpha^u t} + e^{-2\alpha^{uu}(\mu)t}) + |\mu|\, e^{-2\alpha^u t}\,(e^{-\alpha^{uu}(\mu)t} + e^{-2\alpha^u t})\Big) \\ &\quad = \theta\, \langle w_s, v_s\rangle\,(\mu\, e^{-2\alpha^u t} - \tilde{S}\, e^{-2\alpha^{uu}(\mu)t}) + \mathrm{O}\Big(e^{-2\alpha^{uu}(\mu)t}\,(e^{-\alpha^u t} + e^{-2(\alpha^r-\alpha^{uu})t}) + \\ &\quad\quad |\mu|\,(|\mu|\, e^{-2\alpha^u t} + e^{-2\alpha^{uu}(\mu)t}) + |\mu|\, e^{-2\alpha^u t}\,(e^{-\alpha^{uu}(\mu)t} + e^{-2\alpha^u t})\Big). \end{aligned}$$

Note that $\operatorname{sign} S = \operatorname{sign}\langle w_{ss}, v_{ss}\rangle$. The estimates are valid for derivatives with respect to t, too. Thus, using the new variables (a, r) we obtain

$$\begin{aligned} \langle \psi_\mu(-T_i), \theta h_\mu(T_i)\rangle &= \theta\, \langle w_s, v_s\rangle\,(\mu\, a_i\, r - \tilde{S}\, a_i^{1+\alpha(\mu)}\, r^{1+\alpha(\mu)}) + \\ &\quad \mathrm{O}\Big(|a_i|^{1+\alpha(\mu)}\, r^{1+\alpha(\mu)}\, r^\beta + |\mu|\, |a_i|\, r\,(|\mu| + r^\beta)\Big) \end{aligned}$$

with

$$\beta = \min\Big\{\frac{1}{2}, 2\,(\alpha^r - \alpha^{uu}), \alpha\Big\}.$$

The analogous formula holds for $\langle \psi_\mu(T_i), \theta h_\mu(-T_i)\rangle$. Moreover, using the formula for $h_\mu(t)$ obtained above, we see that

$$\hat{R} = \sup_{k=1,\ldots,N} (|\mu|\, e^{-2\alpha^u T_k} + e^{-2\alpha^{uu}(\mu)T_k}) = \mathrm{O}(|\mu|\, r + r^{1+\alpha(\mu)}).$$

Thus the remainder term reads

$$|R_i(T,\mu)| \;=\; \mathrm{O}\Big(r^{\frac{3}{2}}\,(|\mu|^2\,(|a_{i-1}|+|a_i|)+r^{\alpha(\mu)}\,(|a_{i-1}|^{1+\alpha(\mu)}+|a_i|^{1+\alpha(\mu)}))+ (|a_{i-1}|+|a_i|)\,r^2\,(|\mu|+r^{\alpha(\mu)})\Big).$$

Substituting the above into the expressions for the jumps stated in the previous theorem, results in

$$\begin{aligned}
\xi_i(a,r,\mu) \;=\;& \mu\,a_{i-1}\,r-\tilde S\,a_{i-1}^{1+\alpha(\mu)}\,r^{1+\alpha(\mu)}-\delta_i\,(\mu\,a_i\,r-\tilde S\,a_i^{1+\alpha(\mu)}\,r^{1+\alpha(\mu)})+\\
&\mathrm{O}\Big(|a_{i-1}|^{1+\alpha(\mu)}\,r^{1+\alpha(\mu)}\,r^{\beta}+|\mu|\,|a_{i-1}|\,r\,(|\mu|+r^{\beta})\Big)+\\
&\mathrm{O}\Big(|a_i|^{1+\alpha(\mu)}\,r^{1+\alpha(\mu)}\,r^{\beta}+|\mu|\,|a_i|\,r\,(|\mu|+r^{\beta})\Big)+\\
&\mathrm{O}\Big(r^{\frac{3}{2}}\,(|\mu|^2\,(|a_{i-1}|+|a_i|)+r^{\alpha(\mu)}\,(|a_{i-1}|^{1+\alpha(\mu)}+|a_i|^{1+\alpha(\mu)}))+\\
&(|a_{i-1}|+|a_i|)\,r^2\,(|\mu|+r^{\alpha(\mu)})\Big)\\
\;=\;& \mu\,a_{i-1}\,r-\tilde S\,a_{i-1}^{1+\alpha(\mu)}\,r^{1+\alpha(\mu)}-\delta_i\,(\mu\,a_i\,r-\tilde S\,a_i^{1+\alpha(\mu)}\,r^{1+\alpha(\mu)})+\\
&\mathrm{O}\Big((|a_{i-1}|+|a_i|)\,(r^{1+\alpha(\mu)}\,r^{\beta}+r\,|\mu|\,(|\mu|+r^{\beta}))\Big).
\end{aligned}$$

As we are only interested in zeroes of the bifurcation equations, we have omitted the non-zero and constant factor

$$\langle w_s, v_s\rangle\,\gamma_i\,\gamma_{i-1}$$

in front of ξ_i. Owing to the chain rule and the estimates in Theorem 3, the remainder term is differentiable in a_j up to $a_j=0$. By [15, 16] the bifurcation equations can then be extended to negative values of a_j in a Lipschitz-continuous way. ∎

Owing to the ambiguity in the variables (a_j,r), we have

$$\xi(a_j,r)=0 \quad\Longleftrightarrow\quad \xi(c\,a_j,r/c)=0 \tag{3.8}$$

for arbitrary constants $c\in\mathbb{R}$.

3.1.3 Solving the bifurcation equations

According to Lemma 3.3, we have to solve the system

$$\begin{aligned}
\xi_i(a,r,\mu) \;=\;& \mu\,a_{i-1}\,r-\tilde S\,a_{i-1}^{1+\alpha(\mu)}\,r^{1+\alpha(\mu)}-\delta_i\,(\mu\,a_i\,r-\tilde S\,a_i^{1+\alpha(\mu)}\,r^{1+\alpha(\mu)})+\\
&\mathrm{O}\Big((|a_{i-1}|+|a_i|)\,(r^{1+\alpha(\mu)}\,r^{\beta}+r\,|\mu|\,(|\mu|+r^{\beta}))\Big)
\end{aligned} \tag{3.9}$$

for $i=1,...,N$ with $a_0=a_{N+1}=0$. This suggests the scaling

$$\mu=\tilde S\,r^{\alpha_0}. \tag{3.10}$$

Substituting (3.10) into (3.9) yields

$$\hat{\xi}_i(a,r) = a_{i-1} - a_{i-1}^{1+\alpha(\mu)} - \delta_i\,(a_i - a_i^{1+\alpha(\mu)}) + \mathrm{O}\Big((|a_{i-1}| + |a_i|)\,r^{\beta}\Big) \tag{3.11}$$

after factorizing the term $\tilde{S}\,r^{1+\alpha_0}$. Notice that

$$|1 - r^{\alpha(\mu)-\alpha_0}| = |1 - r^{\alpha(\mu(r))-\alpha(0)}| \le C_\beta\,r^\beta$$

for any $\beta < \alpha_0$. It suffices to solve (3.11). Indeed, we have

Lemma 3.4 *Any non-trivial solution (a, r, μ) of (3.9) is given by a corresponding solution of (3.11).*

Proof. Consider (3.9)

$$\mu\,a_1\,r - \tilde{S}\,a_1^{1+\alpha(\mu)}\,r^{1+\alpha(\mu)} + \mathrm{O}\Big(|a_1|\,(r^{1+\alpha(\mu)}\,r^\beta + r\,|\mu|\,(|\mu| + r^\beta))\Big) = 0$$

for $i = 1$. Factorizing $a_1\,r$ – which is allowed as we are interested in non-trivial solutions – gives

$$\mu - \tilde{S}\,a_1^{\alpha(\mu)}\,r^{\alpha(\mu)} + \mathrm{O}\Big(r^{\alpha(\mu)}\,r^\beta + |\mu|\,(|\mu| + r^\beta)\Big) = 0.$$

This can be solved with respect to μ yielding

$$|\mu| \le K\,r^{\alpha(\mu(a_1,r))} \le K\,r^{\alpha_0}.$$

Therefore, introducing a new variable b by

$$\mu = b\,r^{\alpha_0} \qquad\qquad |b| \le K_1 \tag{3.12}$$

captures all solutions of (3.9) for some K_1. By the ambiguity in the variables, we have

$$\xi(a_j, b, r) = 0 \quad\Longleftrightarrow\quad \xi(c\,a_j, c^{\alpha_0}\,b, r/c) = 0, \tag{3.13}$$

see (3.8). Uniqueness can therefore be enforced by requiring that

$$|b| + \sum_{j=1}^{N-1} |a_j|^{\alpha_0} = 1, \tag{3.14}$$

which is equivalent to fix c on the group orbits of solutions described by (3.13). Substituting (3.12) into (3.9) and factorizing $r^{1+\alpha_0}$ shows that

$$|b| > K_2 > 0$$

must hold for any solution of (3.9) and (3.14). Otherwise, there would exist solutions with arbitrarily small $|b|$ which implies that the corresponding values for a_j are small as well contradicting (3.14). Thus solutions are confined to the region

$$|b| + \sum_{j=1}^{N-1} |a_j|^{\alpha_0} = 1, \qquad |b| > K_2 > 0, \qquad 0 < r < r_0.$$

Therefore, it is possible to use c in order to scale solutions to the form

$$(a_j, b, r) = (a_j, 1, r)$$

by the above a priori bounds and the equivalence (3.13). ∎

By the lemma, it suffices to solve (3.11)

$$\hat{\xi}_i(a, r) = a_{i-1} - a_{i-1}^{1+\alpha(\mu)} - \delta_i\,(a_i - a_i^{1+\alpha(\mu)}) + \mathrm{O}((|a_{i-1}| + |a_i|)\, r^{\beta}).$$

Then we have the following existence result.

Lemma 3.5 *Assume either (H3) or admissibility of the sequence γ. Then there exists a solution of (3.11) following the figure-of-eight as prescribed by the sequence γ. The distances of consecutive humps are given by*

$$\frac{\ln|\mu| - \ln|\tilde{S}|}{\alpha^u - \alpha^{uu}} + O(|\mu|^{\delta})$$

for some $\delta > 0$.

Proof. First suppose (H3) is not satisfied and γ is admissible. By Lemma 3.1, it suffices to solve the first $[N/2]$ equations

$$\begin{aligned} 0 &= a_1 - a_1^{1+\alpha(\mu)} + \mathrm{O}(|a_1|\, r^{\beta}) \\ 0 &= a_{i-1} - a_{i-1}^{1+\alpha(\mu)} - \delta_i\,(a_i - a_i^{1+\alpha(\mu)}) + \mathrm{O}((|a_{i-1}| + |a_i|)\, r^{\beta}) \qquad i = 2, ..., [\tfrac{N}{2}] \end{aligned}$$

in the variables a_j for $j = 1, ..., [N/2]$. For $r = 0$, any sequence $a_j \in \{0, 1\}$ yield zeroes of the bifurcation equations. For existence, we choose the sequence $a_j = 1$ for all j at $r = 0$. Applying the implicit function theorem at the point $r = 0$ and $a_j = 1$ for $j = 1, ..., [N/2]$ with respect to the variables a_j proves the lemma in the reversible case as $\alpha_0 > 0$.

The conservative case is proved similarly observing that we only need to solve the first $N-1$ equations in the variables r and a_j for $j = 1, ..., N-1$ by Lemma 3.2. This is done as in the reversible case.

It follows from the above arguments that the solutions satisfy

$$a_j = 1 + \mathrm{O}(r^\beta). \tag{3.15}$$

Due to (3.5) and (3.10), the return times T_j are therefore given by

$$T_j = -\frac{1}{2\alpha^u}\ln(a_j\, r) = -\frac{1}{2\alpha^u\,\alpha_0}\ln(1+\mathrm{O}(|\mu|^\delta))\,(\ln|\mu| - \ln|\tilde{S}|) = \frac{\ln|\mu| - \ln|\tilde{S}|}{2(\alpha^u - \alpha^{uu})} + \mathrm{O}(|\mu|^\delta).$$

Here, $0 < \delta < \beta/\alpha_0$. Observing that, by definition, the distances are twice the return times, proves the lemma. ∎

It remains to prove uniqueness.

Lemma 3.6 *Assume (H3) holds. Then there exists no other N-pulses than the ones derived in Lemma 3.5.*

If (H3) does not hold, any other N-pulse existing for parameter value $\mu = \tilde{S}\, r^{\alpha_0}$ either satisfies $a_k = O(r^\beta)$ for some index k or else the associated sequence γ is not admissible. In any case, there does exist an index k such that a_k, a_{k+1} and a_{k+1} are close to one and $\gamma_k\,\gamma_{k+1} = 1$ as well as $\gamma_{k+1}\,\gamma_{k+2} = -1$ (or vice versa) hold.

The N-pulses with sequences $(\mathrm{id}, ..., \mathrm{id})$ *or* $(\mathrm{id}, -\mathrm{id}, \mathrm{id}, ...)$ *are unique.*

Proof. First suppose that (H3) is satisfied and consider n-pulses. Solving the first $n-1$ equations of (3.11) using the Lipschitz inverse function theorem near any sequence $a_j^0 \in \{0,1\}$ for $j = 1, ..., n-1$ and $r = 0$ results in a unique solution $(a_j(r), r)$ with $a_j(0) = a_j^0$. If $a_j^0 = 1$ for all j, we obtain the same solution as in Lemma 3.5. Assume that $a_k^0 = 0$ and $a_j^0 = 1$ for $j \neq k$. Then we claim that $a_k(r) = 0$ for all r. Indeed, solving (3.11) for $N = k-1$ and $N = n-k$ with initial sequences given by a_j^0 for $j < k$ and $j > k$, respectively, yields the solutions described in Lemma 3.5. Setting $a_k^0 = 0$ decouples the equations for $j < k$ and $j > k$ from each other. Thus, $a_k(r) = 0$ and $a_j(r)$ given by the k- and $(n-k)$-pulses for $j < k$ and $j > k$, respectively, solves (3.11). The case that $a_j^0 = 0$ for several indices is proved similarly.

Next, assume that the vector field is only reversible. Notice that any solution (a_j, r) of (3.11) for r close to zero satisfies

$$a_j = a_j^0 + \mathrm{O}(r^\beta) \qquad\qquad a_j^0 \in \{0,1\}.$$

This proves the statement about the distances of consecutive humps of non-unique multiple pulses as pulses associated with admissible sequences and $a_j^0 = 1$ for all j are described in Lemma 3.5.

Consider the sequences $(\mathrm{id},...,\mathrm{id})$ and $(\mathrm{id},-\mathrm{id},\mathrm{id},...)$. The statement about uniqueness of multiple pulses associated with these sequences follows as in the conservative case once it is observed that any subsequence of these two sequences is again admissible and leads therefore to solutions of (3.11).

Finally, decode the initial values a_j^0 and the sequence γ into one sequence

$$\tilde{a}_j^0 = \begin{cases} \pm 1 & \text{for} \quad a_j^0 = 1 \text{ and } \gamma_j = \pm 1 \\ 0 & \text{for} \quad a_j^0 = 0. \end{cases}$$

Then we claim that there exist an index k such that $\tilde{a}_k = \tilde{a}_{k+1} = -\tilde{a}_{k+2} \neq 0$ (or vice versa) hold provided the N-pulse is not described by Lemma 3.5. Otherwise, all subsequences enclosed by $\tilde{a}_k^0 = 0$ would be of the above form $(\mathrm{id},...\mathrm{id})$ or $(\mathrm{id},-\mathrm{id},\mathrm{id},...)$ leading to a contradiction to the above uniqueness result. ∎

This finishes the proof of Theorem 1.

3.2 Stability

Choose any sequence γ_i and assume that there exists an N-pulse with return times T_j for the parameter value μ. Choose (a_j, r) according to (3.5) and (3.10). We define

$$\delta_j := \gamma_j\, \gamma_{j+1} \in \{\pm 1\}$$

for $j = 1, ..., N-1$. Let

$$A = r^{1+\alpha_0} \begin{pmatrix} d_1 & -d_1 & & & \\ -d_1 & d_1+d_2 & -d_2 & & \\ & -d_2 & d_2+d_3 & -d_3 & \\ & & \ddots & \ddots & \\ & & & -d_{N-1} & d_{N-1} \end{pmatrix} \tag{3.16}$$

with

$$d_j = -S\, \delta_j\, (\alpha^{uu}\, a_j^{1+\alpha_0} - \alpha^u\, a_j).$$

Then the following lemma holds.

Lemma 3.7 *Under the assumptions of Theorem 2, λ is an eigenvalue of (2.4) with $|\lambda|$ small if and only if λ solves*

$$\det(A - M\,\lambda + R(\lambda)) = 0, \tag{3.17}$$

and the respective multiplicities coincide. The remainder term is analytic in λ *and satisfies*

$$R(\lambda) = O(r^{1+\alpha_0+\beta} + |\lambda|\,(|\lambda| + r^{\beta}))$$

for some $\beta > 0$.

Proof. We shall apply [19, Theorem 2] for proving the lemma. However, we need to use the more refined estimates provided in [19, Lemma A.1].

Using the a priori estimate (3.10) and equation (3.6), it follows that

$$\sup_{t \geq \min T_j} |h_\mu(t)| = \mathrm{O}(r^{\frac{1+\alpha_0}{2}}).$$

Thus, in the notation of [19], we obtain the estimates

$$\begin{aligned} |G| + |D| + |p_2(T)| &= \mathrm{O}(r^{\frac{1+\alpha_0}{2}}) \\ |Q_0 D| &= \mathrm{O}(r^{\alpha_0+\frac{1}{2}}) \\ |p_1(T)| &= \mathrm{O}(r^{\frac{\alpha_0}{2}}) \end{aligned}$$

for the functions arising in [19, Lemma A.1]. Here, we have used [19, Lemma 3.1] and (3.6) for the first two equations, while the last estimate follows from [15, Lemma 1.1]. Still in the notation of [19], we therefore get

$$S = (e^{-\alpha^u T}\, p_1(T) + p_2(T) + |G| + e^{-\alpha^{uu} T})\,|D| + e^{-2\alpha^u T}\,|Q_0 D| = \mathrm{O}(r^{1+\alpha_0}).$$

Notice that the additional factor $e^{-\alpha^u T}$ in front of $p_1(T)$ is justified as the third estimate in [19, (A.7)] is not sharp. Thus we end up with the estimate

$$R(\lambda) = \mathrm{O}(r^{1+\alpha_0+\beta} + |\lambda|\,(|\lambda| + r^{\beta}))$$

for some $\beta > 0$ for the remainder term arising in [19, Lemma A.1]. By [16], we can replace the scalar products arising in [19, Lemma A.1] with

$$\langle \psi_\mu(-T_j), \dot{h}_\mu(T_j) \rangle \qquad \text{and} \qquad \langle \psi_\mu(T_j), \dot{h}_\mu(-T_j) \rangle$$

up to an error of the order $\mathrm{O}(r^{1+\alpha_0+\beta})$. By [19, Lemma 5.3], the matrix A arising in [19, Theorem 2] is symmetric. Hence it suffices to compute the scalar product

$$\begin{aligned} \langle \psi_\mu(-T_j), \dot{h}_\mu(T_j) \rangle &= \tilde{S}\,\langle w_s, v_s \rangle\, \delta_j\,(-\alpha^u\, a_j + \alpha^{uu}\, a_j^{1+\alpha_0})\, r^{1+\alpha_0} + \mathrm{O}(r^{1+\alpha_0+\beta}) \\ &= -S\,\delta_j\,(\alpha^{uu}\, a_j^{1+\alpha_0} - \alpha^u\, a_j)\, r^{1+\alpha_0} + \mathrm{O}(r^{1+\alpha_0+\beta}) \end{aligned}$$

using (3.6) and (3.7). This proves the lemma. ∎

It remains to solve (3.17). By [19, Lemmata 5.1 and 5.2] it is sufficient to compute the eigenvalues of the matrix A_0 given by

$$A_0 = \begin{pmatrix} d_1 & -d_1 & & & \\ -d_1 & d_1+d_2 & -d_2 & & \\ & -d_2 & d_2+d_3 & -d_3 & \\ & & \ddots & \ddots & \\ & & & -d_{N-1} & d_{N-1} \end{pmatrix} \tag{3.18}$$

with

$$d_j = -S\,\delta_j\,(\alpha^{uu}\,a_j^{1+\alpha_0} - \alpha^u\,a_j).$$

By [19, Lemma 5.4], the spectrum of A_0 is given by

$$\begin{aligned} \#(\sigma(A_0)\cap\mathbb{R}^-) &= \#\{j\,|\,d_j<0\} \\ \#(\sigma(A_0)\cap\mathbb{R}^+) &= \#\{j\,|\,d_j>0\} \\ \#(\sigma(A_0)\cap\{0\}) &= \#\{j\,|\,d_j=0\}+1 \end{aligned}$$

counting multiplicity.

Now, consider the N-pulses described in Theorem 1. From (3.15) we conclude that $a_j = 1 + \mathrm{O}(r^\beta)$ whence

$$\begin{aligned} \#(\sigma(A_0)\cap\mathbb{R}^-) &= \#\{j\,|\,S\,\delta_j>0\} \\ \#(\sigma(A_0)\cap\mathbb{R}^+) &= \#\{j\,|\,S\,\delta_j<0\} \\ \#(\sigma(A_0)\cap\{0\}) &= 1 \end{aligned}$$

as $\alpha^{uu} > \alpha^u$. Therefore, by [19, Lemmata 5.1 and 5.2], the solutions λ_i of (3.17) have signs given by

$$\begin{aligned} \#\{i\,|\,\mathrm{sign}\,\mathrm{Re}\,\lambda_i<0\} &= \#\{j\,|\,M\,S\,\delta_j>0\} \\ \#\{i\,|\,\mathrm{sign}\,\mathrm{Re}\,\lambda_i>0\} &= \#\{j\,|\,M\,S\,\delta_j<0\} \\ \#\{i\,|\,\lambda_i=0\} &= 1, \end{aligned}$$

which coincides with the statement of Theorem 2. The scaling of the eigenvalues follows from

$$|\lambda| \le K\,r^{1+\alpha_0} \le K\,\mu^{\frac{1+\alpha_0}{\alpha_0}}.$$

Finally, suppose that there exists an N-pulse not described by Theorem 1. Then, by Lemma 3.6, there exists at least one index k with a_k, a_{k+1} and a_{k+2} close to one and $\gamma_k\,\gamma_{k+1} = 1$ as well $\gamma_{k+1}\,\gamma_{k+2} = -1$. Invoking [19, Lemmata 5.1 and 5.2] proves the last part of Theorem 2.

4 Existence and stability of N-pulses in the PSA equation

We consider the fourth-order partial differential equation

$$(4.1)\quad \frac{\partial U}{\partial \zeta}+\Big(\frac{\partial^2}{\partial T^2}+2U^2-(2a^2-\eta^2)\Big)\Big(\frac{\partial^2}{\partial T^2}+2U^2-\eta^2\Big)U+4\sigma\Big(3U\Big(\frac{\partial U}{\partial T}\Big)^2+U^2\frac{\partial^2 U}{\partial T^2}\Big)=0$$

for $\zeta \geq 0$, $T \in \mathbb{R}$ and $U \in \mathbb{R}$. The parameter η is chosen in the interval $\eta \in (a, \sqrt{2}a)$. Equation (4.1) is equivariant with respect to

$$(4.2)\quad \begin{array}{rcccc} \kappa & : & U(T) & \mapsto & -U(T) \\ R & : & U(T) & \mapsto & U(-T) \end{array}$$

acting on $L^2(\mathbb{R})$. Moreover, it admits the steady-state solution

$$U(T) = \eta \operatorname{sech} \eta T$$

for $\sigma = 0$, which is contained in $\operatorname{Fix} R$.

We shall rewrite (4.1) according to

$$(4.3)\quad \frac{\partial U}{\partial \zeta} + \Phi(U, \sigma) = 0.$$

It generates a semiflow on the space $L^2(\mathbb{R})$, see [2]. A steady-state U of (4.1) or (4.3) is stable provided the spectrum of the linearized operator

$$(4.4)\quad L(U) = D_U \Phi(U, \sigma)$$

is contained in the open right half plane bounded away from the imaginary axis with the exception of a simple eigenvalue at zero which is inevitable due to translational invariance. Note that the physicists' notation is used here.

We are interested in solutions looking like N concatenated copies of U and $-U$. The definition of admissible sequences for the actual symmetry group (4.2) reads

Definition We call a sequence $\gamma = (\gamma_j)_{j=1,\ldots,N}$ with $\gamma_j \in \{\pm 1\}$ admissible if either $\gamma_{N+1-j} = \gamma_j$ for all j or $\gamma_{N+1-j} = -\gamma_j$ for all j.

Then we have the following theorem.

Theorem 4 *Fix $\eta \in (a, \sqrt{2}a)$. Then for any $N > 1$ there exists a $\sigma_N > 0$ such that the following holds. Choose any admissible sequence $(\gamma_j)_{j=1,\ldots,N}$ of length N. Then for any σ*

with $0 < \sigma < \sigma_N$ there exists a steady-state $U_\gamma(T)$ of (4.1) looking like the concatenation of $\gamma_j U$ with $j = 1, ..., N$. The solutions described above are symmetric $U_\gamma(T) = U_\gamma(-T)$. The linearization $L(U_\gamma)$ of (4.1) or (4.3) at U_γ possesses a unique simple eigenvalue at zero on the imaginary axis and precisely

$$\#\{i \mid \gamma_i = \gamma_{i+1},\, 1 \le i < N\}$$

eigenvalues with negative real part. In particular, the steady-states U_γ for sequences γ with

$$\gamma_j = -\gamma_{j+1} \qquad\qquad j = 1, ..., N-1$$

are stable. The distance of consecutive humps is approximately given by

$$L := \frac{\ln \sigma}{\sqrt{2a^2 - \eta^2} - \eta} + L_0(a, \eta)$$

for some constant $L_0 \in \mathbb{R}$.

Moreover, any other N-pulse is either asymmetric (that is neither even nor odd) or at least one distance of consecutive humps is of the order $(1 + \beta)\, L$ for some $\beta > 0$. In any case, these additional N-pulses are unstable. Moreover, the pulses with sequences $(\mathrm{id}, ..., \mathrm{id})$ *or* $(\mathrm{id}, \kappa, \mathrm{id}, \kappa, ...)$ *are unique.*

The remainder part of this section is devoted to the proof of Theorem 4. We shall use Theorem 1 and 2 to proof the above result. To this end, we introduce new variables. Then the solutions needed in Hypotheses (H5) and (H7) are computed using the fourth-order equation. Here, the special structure of (4.1) for $\sigma = 0$ is going to be used. Finally, in order to verify the assumptions, we rewrite the fourth-order equation as a first-order system.

Note that it not known whether equation (4.1) admits a first integral. Numerical simulations indicate that it does not. Indeed, once (4.1) has a first integral, N-pulses for any given sequence must bifurcate. However, efforts to compute these N-pulses numerically failed while we were successful in computing N-pulses associated with admissible sequences, see Section 5. Thus, we shall apply Theorem 1 and 2 without using Hypothesis (H3). Observe that, on account of [17, Remark 2], (4.1) cannot be Hamiltonian once Hypothesis (H5)(iii) is satisfied.

4.1 The fourth-order equation

In this section, we calculate the solutions of the linearization of (4.1) needed for the verification of the hypotheses of Theorem 1. We shall use the fourth-order equation for that purpose. First, new variables are introduced as follows.

$$
(4.5)\qquad \begin{aligned} \vartheta &= \frac{\eta}{\sqrt{2a^2-\eta^2}} \qquad & t &= \sqrt{2a^2-\eta^2}\,T \\ \tilde{U} &= \frac{1}{\sqrt{2a^2-\eta^2}}\,U \qquad & \xi &= (2a^2-\eta^2)^2\,\zeta. \end{aligned}
$$

Then (4.1) reads

$$
(4.6)\quad \frac{\partial U}{\partial \xi} + \Big(\frac{\partial^2}{\partial t^2} + 2U^2 - 1\Big)\Big(\frac{\partial^2}{\partial t^2} + 2U^2 - \vartheta^2\Big)U + 4\sigma\Big(3U\Big(\frac{\partial U}{\partial t}\Big)^2 + U^2\frac{\partial^2 U}{\partial t^2}\Big) = 0
$$

where, with an abuse of notation, we have omitted the tilde. Here, ϑ ranges in $\vartheta \in (1,\infty)$ as $\eta \in (a, \sqrt{2}a)$.

The steady-state equation of (4.6) is given by

$$
(4.7)\qquad \Big(\frac{\partial^2}{\partial t^2} + 2U^2 - 1\Big)\Big(\frac{\partial^2}{\partial t^2} + 2U^2 - \vartheta^2\Big)U + 4\sigma\Big(3U\Big(\frac{\partial U}{\partial t}\Big)^2 + U^2\frac{\partial^2 U}{\partial t^2}\Big) = 0.
$$

Moreover, the primary pulse solving (4.7) for $\sigma = 0$ and $\vartheta \in (1,\infty)$ reads

$$
(4.8)\qquad U(t) = \vartheta \operatorname{sech} \vartheta t.
$$

The linearization of (4.7) at U for $\sigma = 0$ is

$$
(4.9)\qquad \Big(\frac{\partial^2}{\partial t^2} + 2U^2 - 1\Big)\Big(\frac{\partial^2}{\partial t^2} + 6U^2 - \vartheta^2\Big)V =: L_-\,L_+\,V = 0,
$$

with the adjoint equation given by

$$
(4.10)\qquad \Big(\frac{\partial^2}{\partial t^2} + 6U^2 - \vartheta^2\Big)\Big(\frac{\partial^2}{\partial t^2} + 2U^2 - 1\Big)W = L_+\,L_-\,W = 0.
$$

Two linearly independent solutions of

$$
L_+\,V = \Big(\frac{\partial^2}{\partial t^2} + 6U^2 - \vartheta^2\Big)V = 0
$$

are given by

$$
(4.11)\qquad \begin{aligned} V_1(t) &= \dot{U}(t) = -\vartheta^2 \operatorname{sech}\vartheta t \tanh \vartheta t \\ V_2(t) &= \frac{1}{2\vartheta^3}\Big(\cosh\vartheta t + 3\vartheta t \sinh\vartheta t \operatorname{sech}^2\vartheta t - 3\operatorname{sech}\vartheta t\Big), \end{aligned}
$$

while the functions

$$
(4.12)\qquad \begin{aligned} W_1(t) &= e^{t}(1 - \vartheta\tanh\vartheta t) \\ W_2(t) &= e^{-t}(1 + \vartheta\tanh\vartheta t), \end{aligned}
$$

solve

$$
L_-\,W = \Big(\frac{\partial^2}{\partial t^2} + 2U^2 - 1\Big)W = 0.
$$

Note that W_1 and W_2 are unbounded and satisfy $W_1(t) = W_2(-t)$. Thus, $V_j(t)$ solve the variational equation (4.9), while $W_j(t)$ solve the associated adjoint equation (4.10) for $j = 1, 2$. It will turn out to be convenient for the upcoming arguments to use the following definition.

Definition We say that $x(t) \doteq_\alpha y(t)$ if $|x(t) - y(t)| = \mathrm{o}(e^{-\alpha t})$ uniformly for $t \to \infty$.

Note that with this notation we have

$$\operatorname{sech} \vartheta t \doteq_\vartheta 2e^{-\vartheta t} \qquad\qquad 1 - \tanh \vartheta t = \mathrm{O}(e^{-2\vartheta t})$$

and therefore

$$(4.13) \qquad \begin{array}{rclcrcl} V_1(t) &=& -2\,\vartheta^2\, e^{-\vartheta|t|}\, \operatorname{sign} t + \mathrm{O}(e^{-2\vartheta|t|}) & \qquad & W_1(t) &\doteq_1& (1 - \vartheta \operatorname{sign} t)\, e^t \\ V_2(t) &=& \dfrac{1}{4\vartheta^3}\, e^{\vartheta|t|} + \mathrm{O}(e^{-\vartheta|t|}) & & W_2(t) &\doteq_1& (1 + \vartheta \operatorname{sign} t)\, e^{-t}. \end{array}$$

First, we calculate the bounded solution of the adjoint equation (4.10), which can be found using variation of parameters

$$(4.14) \qquad \Psi(t) = W_1(t) \int_t^\infty V_1(\tau)\, W_2(\tau)\, d\tau + W_2(t) \int_{-\infty}^t V_1(\tau)\, W_1(\tau)\, d\tau.$$

It is clear that $\Psi(t)$ is bounded. Moreover, $\Psi(t)$ is odd due to

$$\begin{aligned} W_1(-t) \int_{-t}^\infty V_1(\tau)\, W_2(\tau)\, d\tau &= W_2(t) \int_{-t}^\infty V_1(\tau)\, W_2(\tau)\, d\tau = -W_2(t) \int_t^{-\infty} V_1(-s)\, W_2(-s)\, ds \\ &= W_2(t) \int_t^{-\infty} V_1(s)\, W_1(s)\, ds = -W_2(t) \int_{-\infty}^t V_1(s)\, W_1(s)\, ds \end{aligned}$$

and a similar computation for the other term. Owing to the identity

$$\int_t^\infty V_1(\tau)\, W_2(\tau)\, d\tau = -\vartheta^3 \int_t^\infty e^{-\tau} \operatorname{sech}^3 \vartheta\tau\, d\tau - \vartheta^2\, e^{-t} \operatorname{sech} \vartheta t \tanh \vartheta t,$$

see [2, (23),(24)], we obtain

$$\begin{aligned} (4.15) \qquad \Psi(t) &= -2\vartheta^2 \operatorname{sech} \vartheta t \tanh \vartheta t - \vartheta^3\, (1 - \vartheta \tanh \vartheta t)\, e^t \int_t^\infty e^{-\tau} \operatorname{sech}^3 \vartheta\tau\, d\tau \\ &\quad + \vartheta^3\, (1 + \vartheta \tanh \vartheta t)\, e^{-t} \int_{-t}^\infty e^{-\tau} \operatorname{sech}^3 \vartheta\tau\, d\tau. \end{aligned}$$

We shall investigate the asymptotics of $\Psi(-t)$ for $t \to \infty$.

$$\begin{aligned} \Psi(-t) &= 2\vartheta^2 \operatorname{sech} \vartheta t \tanh \vartheta t + \vartheta^3\, (1 - \vartheta \tanh \vartheta t)\, e^t \int_t^\infty e^{-\tau} \operatorname{sech}^3 \vartheta\tau\, d\tau \\ &\quad - \vartheta^3\, (1 + \vartheta \tanh \vartheta t)\, e^{-t} \int_{-t}^\infty e^{-\tau} \operatorname{sech}^3 \vartheta\tau\, d\tau \\ &\doteq_\vartheta 4\vartheta^2\, e^{-\vartheta t} - \vartheta^3\, (1 + \vartheta)\, e^{-t} \int_{-\infty}^\infty e^{-\tau} \operatorname{sech}^3 \vartheta\tau\, d\tau \\ &\doteq_1 -\vartheta^3\, (1 + \vartheta)\, e^{-t} \int_{-\infty}^\infty e^{-\tau} \operatorname{sech}^3 \vartheta\tau\, d\tau \;=:\; J_1\, e^{-t} \end{aligned}$$

yielding

$$(4.16) \qquad \Psi(t) \doteq_\vartheta 4\vartheta^2\, e^{\vartheta t} + J_1\, e^t$$

for $t \to -\infty$. Note that

$$J_1 < 0 \tag{4.17}$$

is negative as the integrand is strictly positive.

Next, we compute the unique bounded solution $U'(t)$ of the inhomogeneous equation

$$\Big(\frac{\partial^2}{\partial t^2} + 2U^2 - 1\Big)\Big(\frac{\partial^2}{\partial t^2} + 6U^2 - \vartheta^2\Big)V = -4\left(3U\Big(\frac{\partial U}{\partial t}\Big)^2 + U^2\frac{\partial^2 U}{\partial t^2}\right), \tag{4.18}$$

that is

$$L_-\, L_+\, U' = G$$

with

$$G := -4\left(3U\Big(\frac{\partial U}{\partial t}\Big)^2 + U^2\frac{\partial^2 U}{\partial t^2}\right) = -4\,\vartheta^5\,(4 - 5\,\mathrm{sech}^2\,\vartheta t)\,\mathrm{sech}^3\,\vartheta t. \tag{4.19}$$

Solving (4.18) is equivalent to

$$\begin{aligned} L_- W &= G \\ L_+ U' &= W. \end{aligned}$$

As the fundamental solutions of L_+ and L_- are given in (4.11) and (4.12), respectively, we obtain

$$W(t) = \frac{1}{2(\vartheta^2-1)}\Big(W_1(t)\int_t^\infty G(\tau)\,W_2(\tau)\,d\tau + W_2(t)\int_{-\infty}^t G(\tau)\,W_1(\tau)\,d\tau\Big) \tag{4.20}$$

$$U'(t) = V_1(t)\int_0^t W(\tau)\,V_2(\tau)\,d\tau + V_2(t)\int_t^\infty W(\tau)\,V_1(\tau)\,d\tau. \tag{4.21}$$

We shall determine the asymptotics of $U'(t)$. Using the expansions (4.13), we obtain

$$W(t) = \frac{1}{2(\vartheta-1)}\,e^{-t}\int_{-\infty}^\infty G(\tau)\,W_1(\tau)\,d\tau + \mathrm{O}(e^{-3\vartheta t}) = \frac{1}{2(\vartheta-1)}\,J_2\,e^{-t} + \mathrm{O}(e^{-3\vartheta t}), \tag{4.22}$$

for $t \to \infty$ with J_2 given by

$$J_2 = \int_{-\infty}^\infty G(\tau)\,W_1(\tau)\,d\tau.$$

We calculate J_2 by means of the Residue Theorem

$$\begin{aligned} J_2 &= \int_{-\infty}^\infty G(t)\,W_1(t)\,dt \\ &= -\int_{-\infty}^\infty 4\vartheta^5\,(4 - 5\,\mathrm{sech}^2\,\vartheta t)\,\mathrm{sech}^3\,\vartheta t\,e^t\,(1 - \vartheta\tanh\vartheta t)\,dt \\ &= -\int_{-\infty}^\infty 4\nu^{-5}\,(4 - 5\,\mathrm{sech}^2\,t)\,\mathrm{sech}^3\,t\,e^{\nu t}\,(\nu - \tanh t)\,dt \end{aligned}$$

for $\nu = 1/\vartheta$. Owing to the identity $\frac{d}{d\nu}(e^{\nu t}\,\mathrm{sech}\,t) = e^{\nu t}\,\mathrm{sech}\,t\,(\nu - \tanh t)$, we get

$$J_2 = 4\nu^{-5}\int_{-\infty}^\infty \frac{d}{d\nu}\Big((4 - 5\,\mathrm{sech}^2\,t)\,\mathrm{sech}^2\,t\Big)\,\mathrm{sech}\,t\,e^{\nu t}\,dt$$

$$
\begin{aligned}
&= 16\nu^{-5} \int_{-\infty}^{\infty} (5\,\mathrm{sech}^2\, t - 2)\,\mathrm{sech}^3\, t\, \tanh t\, e^{\nu t}\, dt \\
&= 16\nu^{-5} \int_0^{\infty} \left(5\left(\frac{2y}{1+y^2}\right)^2 - 2\right)\left(\frac{2y}{1+y^2}\right)^3 \frac{y^2-1}{(1+y^2)^3}\, y^\nu\, \frac{1}{y}\, dy \\
&= -256\nu^{-5} \int_0^{\infty} \frac{(y^4-8y^2+1)\, y^2\,(y^2-1)}{(1+y^2)^6}\, y^\nu\, dy \\
&=: -256\nu^{-5} \int_0^{\infty} g(y)\, dy.
\end{aligned}
$$

By [14, ch.14.2.2,p.404], the integral equals

$$
J_2 = -256\nu^{-5}\, \frac{2\pi i}{1-e^{2\pi\nu i}} \sum_{y\neq 0} \mathrm{res}_y\, g(y)
$$

provided $\nu \in (0,1)$ and the branch $y^\nu = \exp(\nu \log y + i\,\nu\,\arg y)$ with $\arg y \in (0, 2\pi)$ is used. Note that $\nu \in (0,1)$ as $1/\nu = \vartheta \in (1,\infty)$. The poles of g are $\pm i$ and its residues at these points are given by

$$
\mathrm{res}_{\pm i}\, g = \frac{1}{5!}\,\frac{d^5}{dx^5}(x^6\, g(x \pm i))|_{x=0}.
$$

Computing the derivatives yields

$$
\begin{aligned}
\mathrm{res}_i\, g &= \tfrac{1}{384}\,\nu\,(\nu^2-1)^2\, i^\nu\, i &&= \tfrac{1}{384}\,\nu\,(\nu^2-1)^2\, i\, e^{\nu\pi i/2} \\
\mathrm{res}_{-i}\, g &= -\tfrac{1}{384}\,\nu\,(\nu^2-1)^2\,(-i)^\nu\, i &&= -\tfrac{1}{384}\,\nu\,(\nu^2-1)^2\, i\, e^{3\nu\pi i/2}.
\end{aligned}
$$

Hence we obtain

$$
J_2 = 256\nu^{-5}\, 2\pi\, \frac{1}{384}\,\nu\,(\nu^2-1)^2\, \frac{e^{\nu\pi i/2} - e^{3\nu\pi i/2}}{1-e^{2\pi\nu i}} = \frac{4\pi(\nu^2-1)^2}{3\nu^4 \cos\frac{\pi\nu}{2}}. \tag{4.23}
$$

In particular,

$$
J_2 > 0 \tag{4.24}
$$

for $\nu \in (0,1)$ which corresponds to $\vartheta \in (1,\infty)$. With the asymptotics (4.22)

$$
W(t) = \frac{1}{2(\vartheta-1)}\, J_2\, e^{-t} + \mathrm{O}(e^{-3\vartheta t})
$$

of $W(t)$ at hand, we investigate $U'(t)$ for $t \to \infty$. The first integral in (4.13) can be estimated by

$$
\begin{aligned}
&U_1'(t) := V_1(t) \int_0^t W(\tau)\, V_2(\tau)\, d\tau \\
&\quad= -2\vartheta^2\, e^{-\vartheta t}\,(1+\mathrm{O}(e^{-\vartheta t})) \int_0^t \left(\frac{1}{2(\vartheta-1)}\, J_2\, e^{-\tau} + \mathrm{O}(e^{-\vartheta\tau})\right)\left(\frac{1}{4\vartheta^3}\, e^{\vartheta\tau} + \mathrm{O}(e^{-\vartheta\tau})\right) d\tau \\
&\quad= -2\vartheta^2\, e^{-\vartheta t}\,(1+\mathrm{O}(e^{-\vartheta t})) \int_0^t \left(\frac{1}{2(\vartheta-1)}\,\frac{1}{4\vartheta^3}\, J_2\, e^{(\vartheta-1)\tau} + \mathrm{O}(1)\right) d\tau \\
&\quad\doteq_1 -\frac{1}{4\vartheta\,(\vartheta-1)^2}\, J_2\, e^{-t},
\end{aligned}
$$

while the second one yields

$$
\begin{aligned}
U_2'(t) &:= V_2(t) \int_t^\infty W(\tau)\, V_1(\tau)\, d\tau \\
&= \Big(\frac{1}{4\vartheta^3}\, e^{\vartheta t} + \mathrm{O}(e^{-\vartheta t})\Big) \int_t^\infty \Big(\frac{1}{2(\vartheta-1)}\, J_2\, e^{-\tau} + \mathrm{O}(e^{-\vartheta\tau})\Big) \Big(-\, 2\vartheta^2\, e^{-\vartheta t} + \mathrm{O}(e^{-2\vartheta t})\Big)\, d\tau \\
&= \Big(-\, \frac{1}{4\vartheta\,(\vartheta-1)}\, J_2\, e^{\vartheta t} + \mathrm{O}(e^{-\vartheta t})\Big) \int_t^\infty \Big(e^{-(1+\vartheta)\tau} + \mathrm{O}(e^{-2\vartheta t})\Big)\, d\tau \\
&\doteq_1 -\frac{1}{4\vartheta\,(\vartheta^2-1)}\, J_2\, e^{-t}.
\end{aligned}
$$

Summarizing, we obtain

$$
\begin{aligned}
U'(t) &= U_1'(t) + U_2'(t) \\
&\doteq_1 -\frac{1}{4\vartheta\,(\vartheta-1)^2}\, J_2\, e^{-t} - \frac{1}{4\vartheta\,(\vartheta^2-1)}\, J_2\, e^{-t} \\
&\doteq_1 -\frac{1}{2(\vartheta-1)^2\,(\vartheta+1)}\, J_2\, e^{-t}.
\end{aligned}
\tag{4.25}
$$

4.2 Verifying the assumptions of Theorem 1

Having calculated the solutions for the fourth-order equation, we shall interpret the results for the associated first-order system. Rewriting (4.7) yields

$$
\dot u = \begin{pmatrix} \dot u_1 \\ \dot u_2 \\ \dot u_3 \\ \dot u_4 \end{pmatrix} = \begin{pmatrix} u_2 \\ -2u_1^3 + \vartheta^2 u_1 + u_3 \\ u_4 \\ (1-2u_1^2)\, u_3 - 4\sigma\Big(u_1^2(-2u_1^3 + \vartheta^2 u_1 + u_3) + 3u_1 u_2^2\Big) \end{pmatrix} = f(u,\sigma),
\tag{4.26}
$$

where $u_1 = U$. Throughout, we use the convention that capital letters correspond to solutions of the fourth-order equation while small letters correspond to the associated first-order system. The symmetries R and κ of (4.1) defined in (4.2) translate into

$$
\begin{array}{lcccl}
\kappa & : & u & \mapsto & -u \qquad \mathbb{Z}_2\text{-equivariance} \\
R & : & (u_1,u_2,u_3,u_4) & \mapsto & (u_1,-u_2,u_3,-u_4) \qquad \text{time-reversibility}
\end{array}
\tag{4.27}
$$

of equation (4.26). The linearization of (4.26) at

$$
u(t) = (U(t), \dot U(t), 0, 0) = (\vartheta \operatorname{sech} \vartheta t, -\vartheta^2 \operatorname{sech} \vartheta t \tanh \vartheta t, 0, 0)
\tag{4.28}
$$

equals

$$
\dot v = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \vartheta^2 - 6u_1^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1-2u_1^2 & 0 \end{pmatrix} v,
\tag{4.29}
$$

as $u_3 = \ddot{u}_1 + 2u_1^3 - \vartheta^2 u_1 = 0$ and thus $u_4 = 0$ as well. Moreover, the adjoint equation reads

$$\dot{w} = \begin{pmatrix} 0 & 6u_1^2 - \vartheta^2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2u_1^2 - 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} w. \tag{4.30}$$

The relation between solutions of (4.9) and (4.29) as well as (4.10) and (4.30) is as follows

$$\begin{array}{rclcrcl} v & = & (V, \dot{V}, \ddot{V} + (6U^2 - \vartheta^2)V, v_4) & \qquad & v_4 & = & \dot{v}_3 \\ w & = & (w_1, \ddot{W} + (2U^2 - 1)W, -\dot{W}, W) & & w_1 & = & -\dot{w}_2. \end{array} \tag{4.31}$$

The stable eigenvalues and eigenvectors of the linearization of (4.26) at the equilibrium $u = 0$ are given by

$$\begin{array}{rclcrcl} -\alpha^u & = & -1 & \qquad & e_s & = & (1, -1, 1 - \vartheta^2, -1 + \vartheta^2) \\ -\alpha^{uu} & = & -\vartheta & & e_{ss} & = & (1, -\vartheta, 0, 0) \end{array} \tag{4.32}$$

for all σ.

We shall verify Hypotheses (H1) to (H6) except for (H3) of Section 2 for equation (4.26) with

$$h(t) := u(t) = (U(t), \dot{U}(t), 0, 0) \qquad\qquad \mu := \sigma, \tag{4.33}$$

see (4.28).

Assumption (H1)(i) is obviously satisfied. It was proved in [2] that the pulse U is stable with respect to the underlying PDE. Therefore, (H1)(ii) is satisfied. Otherwise, the eigenvalue $\lambda = 0$ of the PDE linearization would possess geometric multiplicity two contradicting stability. Note that (H2) and (H6) are satisfied with R and κ as in (4.27). The same is true for Hypothesis (H4) due to (4.32).

It remains to show (H5) and determine the bifurcation direction in order to conclude the existence part of Theorem 4.

Consider Hypothesis (H5). Due to (4.28), we have

$$\begin{aligned} u(t) &= (U(t), \dot{U}(t), 0, 0) \\ &= (\vartheta \operatorname{sech} \vartheta t, -\vartheta^2 \operatorname{sech} \vartheta t \tanh \vartheta t, 0, 0) \\ &= 2\vartheta\, e^{-\vartheta t} (1, -\vartheta, 0, 0) + \mathrm{O}(e^{-2\vartheta t}) \\ &\doteq_\vartheta 2\vartheta\, e^{-\vartheta t}\, e_{ss} \end{aligned} \tag{4.34}$$

proving that

$$\lim_{t \to \infty} u(t)\, e^{\vartheta t} = 2\vartheta\, e_{ss} =: v_{ss} \neq 0, \tag{4.35}$$

whence (H5)(i) is satisfied.

According to (4.31), the bounded solution $\psi(t)$ of (4.30) is given by

$$\begin{aligned}\psi &= \Big(-\frac{d}{dt}(\ddot{\Psi}+(2U^2-1)\Psi), \ddot{\Psi}+(2U^2-1)\Psi, -\dot{\Psi}, \Psi\Big)\\ &\doteq_\vartheta \Big(\frac{d^3}{dt^3}\Psi-\dot{\Psi}, \ddot{\Psi}-\Psi, -\dot{\Psi}, \Psi\Big)\end{aligned}$$

as $U \doteq_\vartheta 2\vartheta\, e^{-\vartheta t}$. Thus, owing to (4.16), we obtain

$$\begin{aligned}\psi(-t) &\doteq_\vartheta (-4\vartheta^3\,(\vartheta^2-1)\,e^{-\vartheta t}, 4\vartheta^2\,(\vartheta^2-1)\,e^{-\vartheta t}, -4\vartheta^3\,e^{-\vartheta t}-J_1\,e^{-t},\\ &\qquad 4\vartheta^2\,e^{-\vartheta t}+J_1\,e^{-t})\\ &\doteq_\vartheta 4\vartheta^2\,e^{-\vartheta t}\,(\vartheta\,(1-\vartheta^2), \vartheta^2-1, -\vartheta, 1)+J_1\,e^{-t}\,(0,0,-1,1).\end{aligned} \tag{4.36}$$

Therefore, we have accomplished (H5)(iii) as

$$\lim_{t\to\infty}\psi(-t)\,e^t = J_1\,(0,0,-1,1) = w_s \neq 0 \tag{4.37}$$

owing to $J_1 < 0$.

The scalar product appearing in (H5)(iv) reads

$$\begin{aligned}\langle\psi(-t), u(t)\rangle &= \Big\langle 4\vartheta^2\,e^{-\vartheta t}\,(\vartheta\,(1-\vartheta^2), \vartheta^2-1, -\vartheta, 1)+J_1\,e^{-t}\,(0,0,-1,1)+\mathrm{o}(e^{-\vartheta t}),\\ &\qquad 2\vartheta\,e^{-\vartheta t}\,(1,-\vartheta,0,0)+\mathrm{O}(e^{-2\vartheta t})\Big\rangle\\ &\doteq_{2\vartheta} 8\vartheta^3\,e^{-2\vartheta t}\Big\langle(\vartheta\,(1-\vartheta^2), \vartheta^2-1, -\vartheta, 1), (1,-\vartheta,0,0)\Big\rangle\\ &\doteq_{2\vartheta} 16\vartheta^4\,(1-\vartheta^2)\,e^{-2\vartheta t}\end{aligned}$$

for $t\to\infty$ using (4.34) and (4.36), whence

$$\lim_{t\to\infty}\langle\psi(-t), u(t)\rangle\,e^{2\vartheta t} = -16\vartheta^4\,(\vartheta^2-1) = S < 0. \tag{4.38}$$

Thus, Hypothesis (H5)(iv) is satisfied as well and it remains to verify (H5)(ii).

Due to Hypothesis (H1)(ii), there exist a smooth family $h_\sigma(t)$ of homoclinic solutions for (4.26) with $h_0 = u$. The derivative

$$u'(t) := \frac{d}{d\sigma}h_\sigma(t)\Big|_{\sigma=0}$$

is the unique bounded solution of the inhomogeneous variational equation

$$\dot{v} = D_u f(u,0)\,v + D_\sigma f(u,0)$$

with

$$D_\sigma f(u(t),0) = -4\Big(0,0,0,u_1^2(-2u_1^3+\vartheta^2 u_1+u_3)+3u_1u_2^2\Big),$$

see (4.26). Thus, $u'(t)$ coincides with the solution $U'(t)$ computed previously and being transformed according to (4.31). Therefore, using the expansion (4.25), we have

$$
\begin{aligned}
(4.39) \qquad u'(t) \;&\doteq_1\; \Big(U', \dot U', \ddot U' - \vartheta^2 U', \frac{d^3}{dt^3}U' - \vartheta^2 \dot U'\Big) \\
&\doteq_1\; -\frac{1}{2(\vartheta-1)^2\,(\vartheta+1)}\, J_2\, e^{-t}\,(1,-1,1-\vartheta^2,-1+\vartheta^2).
\end{aligned}
$$

In particular,

$$
(4.40) \qquad \lim_{t\to\infty} u'(t)\, e^t = -\frac{1}{2(\vartheta-1)^2\,(\vartheta+1)}\, J_2\, e_s = v_s \neq 0,
$$

whence (H5)(ii) is satisfied.

Finally, we determine the sign of σ for which the bifurcating N-pulses exist. According to Theorem 1, they do bifurcate for

$$
\begin{aligned}
\operatorname{sign}\sigma &= -\operatorname{sign}\Big(S\,\langle w_s, v_s\rangle\Big) \\
&= -\operatorname{sign}\Big(S\,J_1\Big(-\frac{1}{2(\vartheta-1)^2\,(\vartheta+1)}\,J_2\Big)\Big\langle(0,0,-1,1),(1,-1,1-\vartheta^2,-1+\vartheta^2)\Big\rangle\Big) \\
&= \operatorname{sign}\Big(S\,J_1\,J_2\,(\vartheta^2-1)\Big) \\
&= 1,
\end{aligned}
$$

where we have substituted w_s and v_s from (4.37) and (4.40), respectively, and used $S<0$, $J_1<0$ and $J_2>0$, see (4.38), (4.17) and (4.24), together with $\vartheta>1$.

Hence, the proof of the existence part of Theorem 4 is complete.

4.3 Stability of the bifurcating pulses

It remains to prove the statements about the stability of the bifurcating pulses. It was proved in [2] that the underlying primary pulse U is stable with respect to equation (4.1)

$$
\frac{\partial U}{\partial \zeta} + \Phi(U,\sigma) = 0
$$

for $\sigma = 0$ and all $\vartheta > 1$. In other words, the spectrum of the operator

$$
L(U) = D_U\Phi(U,0),
$$

evaluated at the primary pulse is bounded to the right of the imaginary axis except for a simple eigenvalue at zero. Notice that the operator $L(U)$ is sectorial for any U. Thus, the spectrum of $L(U_N)$ with U_N being an N-pulse is bounded to right of the imaginary axis except for N eigenvalues close to zero. Indeed, this follows from the general theory developed in [1]. Therefore, it suffices to calculate these N critical eigenvalues, that is solutions of

$$L(U_N)\,V = \lambda\,V \tag{4.41}$$

for $\lambda \in \mathbb{C}$ close to zero and $V \in L^2(\mathbb{R})$. Writing (4.41) as a first-order system – noticing that it is an ordinary differential equation – yields

$$\dot{v} = D_u f(u_N(t), 0)\, v + \lambda\, B\, v,$$

where v and V are related via (4.31) and the matrix B is

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, it suffices to show Hypothesis (H7) and apply Theorem 2 in order to decide which of the bifurcating multiple pulses are stable and which not.

The sign of the Melnikov integral M has been calculated in [2, eqns. (21)–(30)] for $\vartheta > 1$ and is given by

$$M = \int_{-\infty}^{\infty} \langle \psi(t), B\,\dot{u}(t)\rangle\, dt = \int_{-\infty}^{\infty} \Psi(t)\,\dot{U}(t)\, dt < 0.$$

Thus, according to (4.38) and the above computation, sign $MS = 1$. As we have computed eigenvalues of the linearized operator L in (4.4), the stability part of Theorem 4 follows from Theorem 2. ∎

5 Numerical Simulations

In the last section, we proved the existence of N-pulses U_γ for equation (1.1) for $\Gamma l > 0$ close to zero. Here, γ is an arbitrary admissible sequence. In fact, the multiple-pulse solutions are bifurcating at $\Gamma l = 0$ from the primary pulse $U(T) = \eta \operatorname{sech} \eta T$, where $\eta \in (\sqrt{\kappa}, \sqrt{2\kappa})$ is arbitrary. As Γl is related to the distance of the amplifiers along the fiber, one is interested in the shape of the N-pulses for O(1)-values of Γl or σ. In this section, we are investigating the steady-state equation

$$\frac{1}{4}U'''' + ((2+\sigma)U^2 - \frac{\kappa}{2})\,U'' + 3(1+\sigma)\,U\,(U')^2 + (\frac{\kappa^2}{4} - \Delta\alpha)\,U - \kappa\,U^3 + U^5 = 0 \tag{5.1}$$

by numerical techniques. Here, prime means derivative with respect to T. Throughout, we set $\kappa = 1$ and regard (5.1) as a two-parameter problem in the parameters

$$(\Delta\alpha, \sigma) \in (0, 0.25) \times (0, 1). \tag{5.2}$$

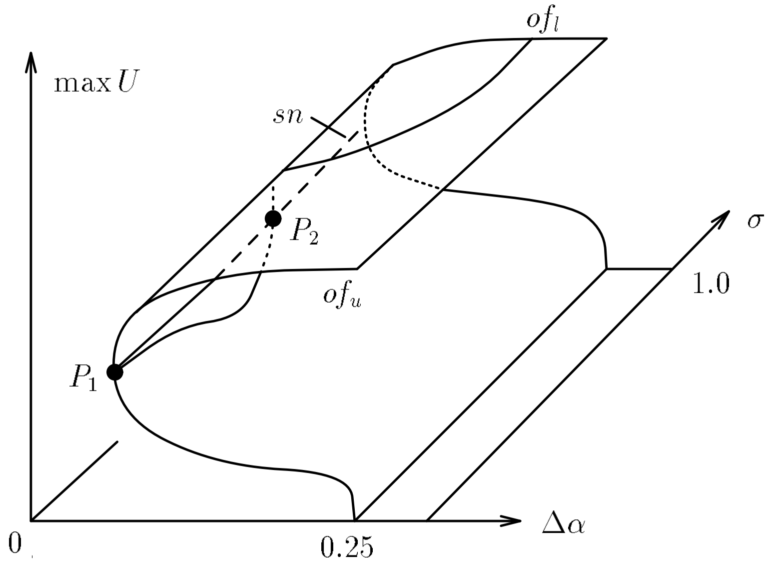

Figure 6: A schematic picture in $(\Delta\alpha, \sigma, \max U) \in \mathbb{R}^3$. The surface plotted correspond to loci at which the primary pulse exists. The curve of saddle-nodes is denoted by sn. Pulses on the upper part of the surface are stable, while those on the lower part are unstable. Along the curves of_u and of_l, orbit-flip bifurcations take place. Note that part of the curve of_l is located at the lower unstable piece of the surface. The degenerate point $(\Delta\alpha, \sigma) = 0$ is denoted by P_1, while the point P_2 is the intersection of sn and of_l.

Note that the variables $(\Delta\alpha, \kappa, \Gamma l)$ and (a, η, σ) are related via

$$\begin{aligned} a &= \sqrt{\kappa} & \eta &= \sqrt{\kappa - 2\sqrt{\Delta\alpha}} \\ \sigma &= 1 - \frac{\tanh \Gamma l}{\Gamma l}, \end{aligned}$$

whence $a = 1$ and $\Gamma l \in (0, \infty)$.

The numerical results have been obtained using a reversible version of the driver HomCont, see [6] and [5], for the software package Auto86 [7]. This driver allows for the continuation of homoclinic solutions as well as for the detection and accurate location of bifurcation points. In particular, it detects reversible orbit-flip bifurcations using an algorithm investigated in [18].

Saddle-node bifurcations of pulses are not computed using the procedure for the computation of limit points of boundary value problems installed within Auto86, but are instead located directly by using the adjoint variational equation. This seems to be much more efficient than the built-in procedure mentioned above.

The N-pulses have been computed in the following way. First, starting data are provided using the primary pulse and periodic orbits computed at the bifurcation point $\sigma = 0$. These pieces of solutions are concatenated in an obvious manner to create an approximation of an N-pulse. Using Newton's method followed by usual continuation within HomCont yields N-pulses. A detailed analysis of this algorithm will appear elsewhere.

As mentioned above, we regard (5.1) as a two-parameter problem in $(\Delta\alpha, \sigma)$. Notice that the point $P_1 := (\Delta\alpha, \sigma) = 0$ is a bifurcation point of (at least) codimension two as we have

$$\dim\left(T_{h(0)}W^s(0) \cap T_{h(0)}W^u(0)\right) = 2$$

and $\alpha^u = \alpha^{uu}$, see (4.12) for $\eta = a$ and (4.32). It corresponds to a degenerate saddle-node of the primary pulse at which an exchange of stability for the 1-pulse takes place, see [11] or [2].

As proved in the last section, a curve of generic orbit-flips emanates from the degenerate point $(\Delta\alpha, \sigma) = 0$. It is given by $\sigma = 0$ and we denote it by $of_u = \{(\Delta\alpha, 0)\} \subset \mathbb{R}^2$ – the upper orbit-flip curve.

The saddle-node at $(\Delta\alpha, \sigma) = 0$ can be continued numerically in the parameters $(\Delta\alpha, \sigma) \in \mathbb{R}^2$ yielding a curve sn shown in Figure 7. It approaches the line $\Delta\alpha = 0.25$ such that the σ tends to infinity and crosses the line $\sigma = 1$, that is $\Gamma l = \infty$, at $\Delta\alpha = 0.088279$.

There is another orbit-flip curve of_l emanating from P_1. That one is computed numerically and depicted in Figure 8. It is located on the lower part of the surface at which the primary pulse exists and hits the saddle-node curve sn at the point $P_2 = (\Delta\alpha, \sigma) = (0.047053, 0.654274)$. Beyond that point the pulses undergoing the flip lie on the upper sheet. The flip curve of_l crosses the line $\Gamma l = \infty$ at $\Delta\alpha = 0.095341$. At $\sigma = 1.875000$ the line $\Delta\alpha = 0.25$ is reached.

Figure 6 contains a schematic picture of the curves computed so far.

As a reasonable value of Γl is $\Gamma l = 1$ – which is the value used in [10] for their computations – we compute and continue the N-pulses bifurcating from the analytically known curve of_u in the parameter Γl up to $\Gamma l = 1$ for $\Delta\alpha = 0.1$. Pictures of theses solutions are given in 11 and 12. In Figure 10, the stable double pulses are shown for various values of σ. The computation shows that the double pulses last at the second orbit-flip curve of_l.

Finally, stability of the double pulses is investigated by solving the partial differential equation (1.2) numerically for $\Gamma l = 1$. For that purpose, equation (1.2) has been discretized in space using central differences. The resulting ODE has been solved numerically using the code LIMEXS written by Nowak and Zugck (ZIB Berlin). Projection-boundary conditions are used requesting solutions to be contained in the stable and unstable eigenspaces of the trivial solution. The time-interval is $[-17.0, 17.0]$.

Conclusion. The numerical results suggest the following global picture. In the region $(\Delta\alpha, \sigma) \in (0, 0.25) \times (0, 1)$, the pulses considered above can be distinguished by the number of humps. There is a surface of 1-pulses folded at the saddle-node curve sn with a stable upper and an unstable lower sheet. There are two curves of orbit-flips of_u and of_l contained in the surface. The first one – of_u – consists of the upper sheet intersected with the σ-axis, the other one – of_l – emanates at $(\Delta\alpha, \sigma) = 0$ and lasts at the point $(\Delta\alpha, \sigma) = (0.25, 1.875000)$. The curve of_l is contained in the unstable lower sheet until the point $P_2 = (\Delta\alpha, \sigma) = (0.047053, 0.654274)$ at which it switches onto the upper stable sheet. In other words, the 1-pulses on of_l are first unstable and change stability at the point P_2 of intersection of sn and of_l. The N-pulses emanating at of_u lasts until the second flip of_l. However, they have to change stability for $\Delta\alpha \leq 0.047053$, presumable by a cascade of saddle-node or Hopf bifurcations.

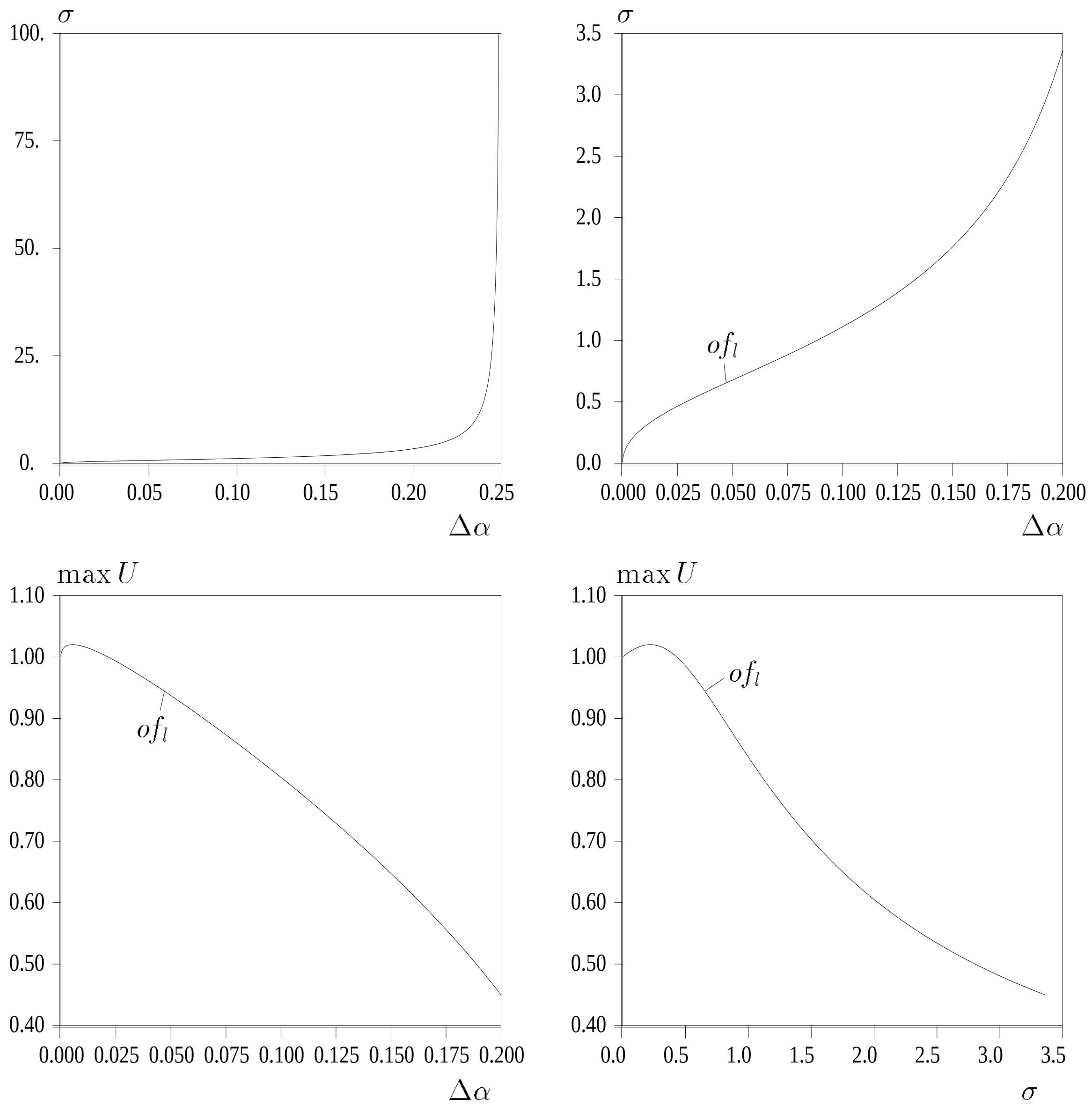

Figure 7: The curve sn of saddle-nodes of the primary pulse is shown in different projections and parameter regions. Here, the label of_l denotes the point of intersection with the curve of_l.

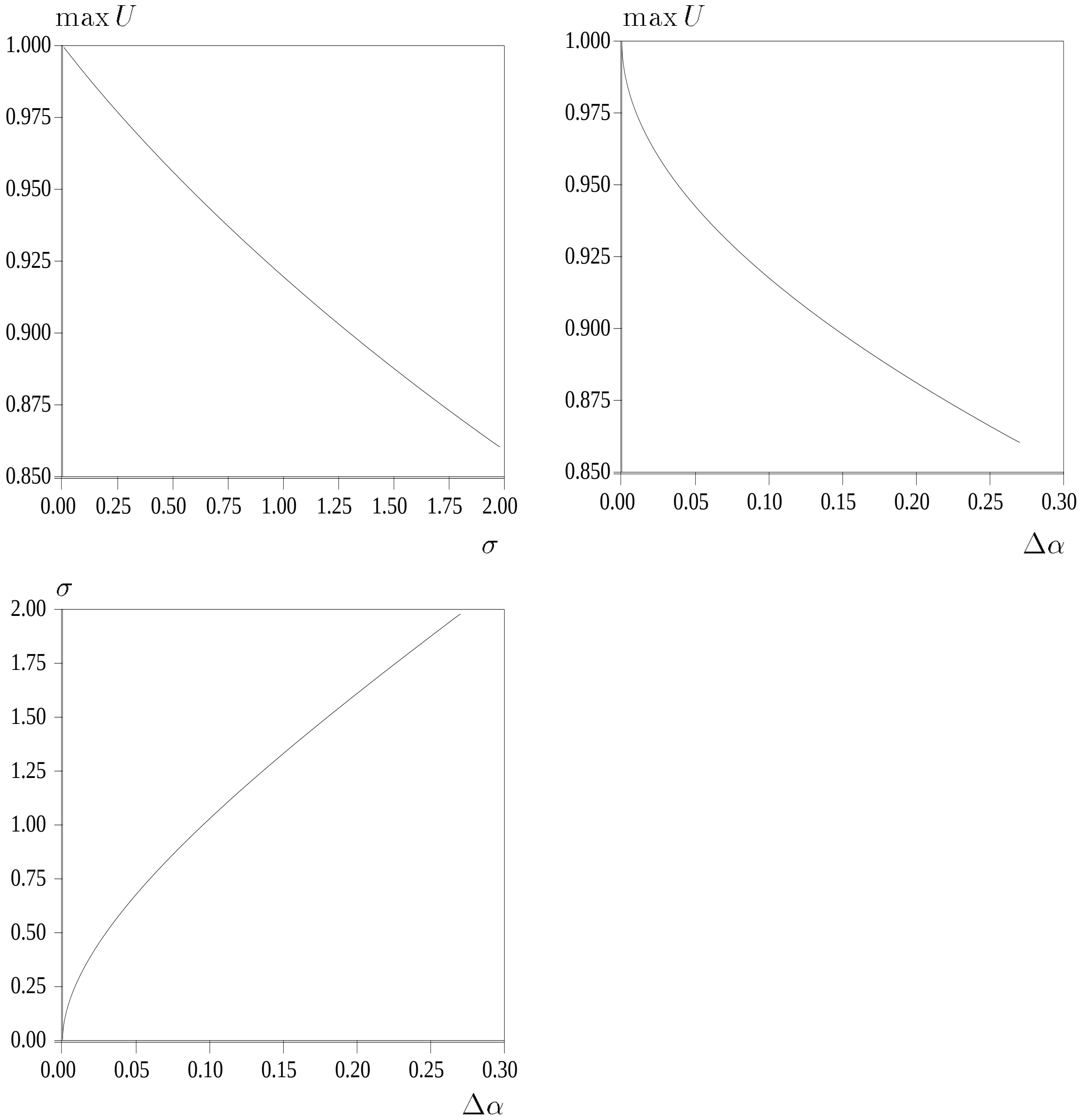

Figure 8: The lower curve of_l of primary pulses undergoing an orbit-flip bifurcation is shown in different projections.

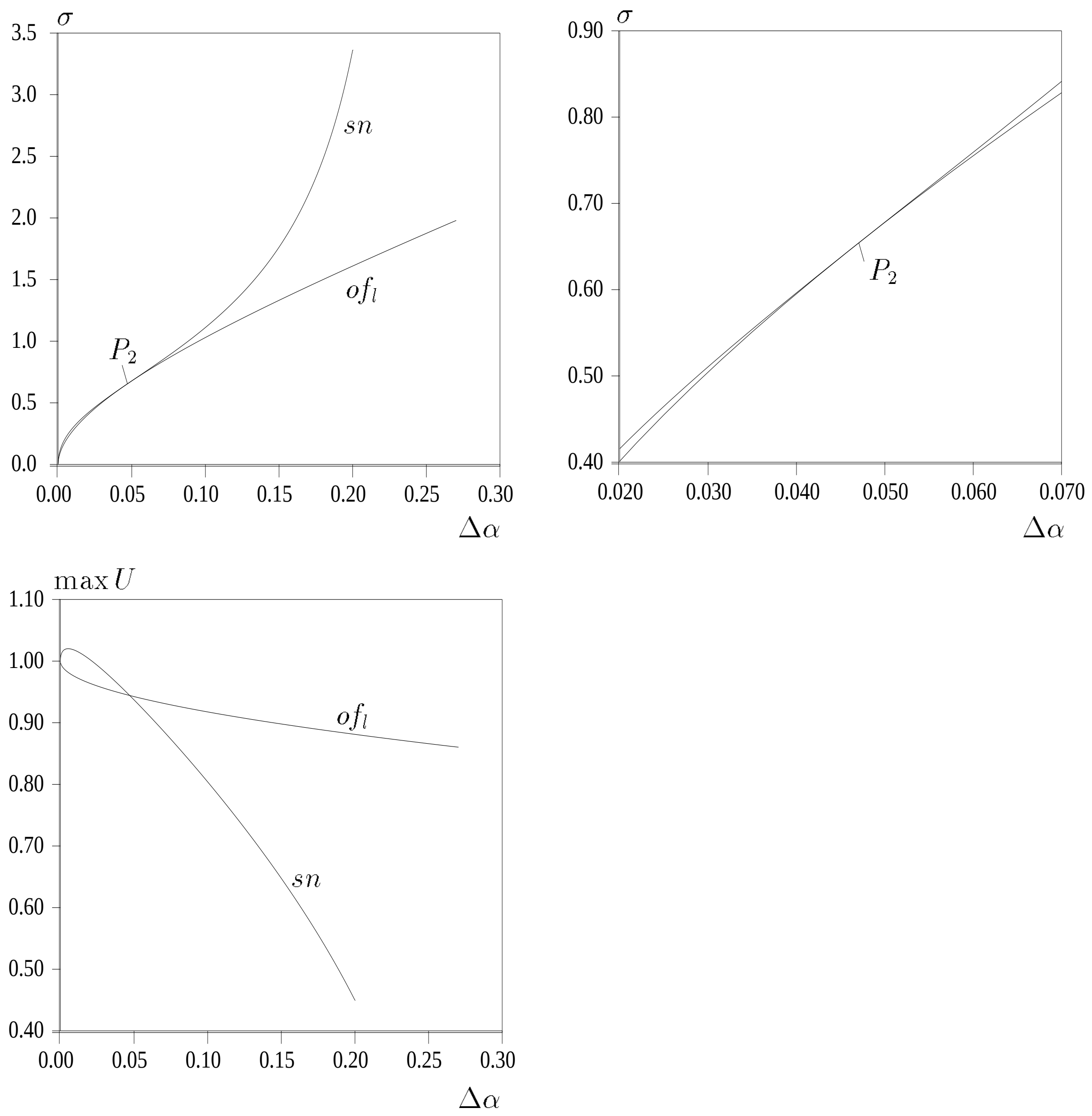

Figure 9: Here both curves sn and of_l are depicted in the same diagrams for comparison. The upper right figure shows an enlarged portion of the upper left one around the point P_2 of intersection of the curves. Note that of_l is always to the right of sn. That both curves do intersect becomes clear from the lower left figure.

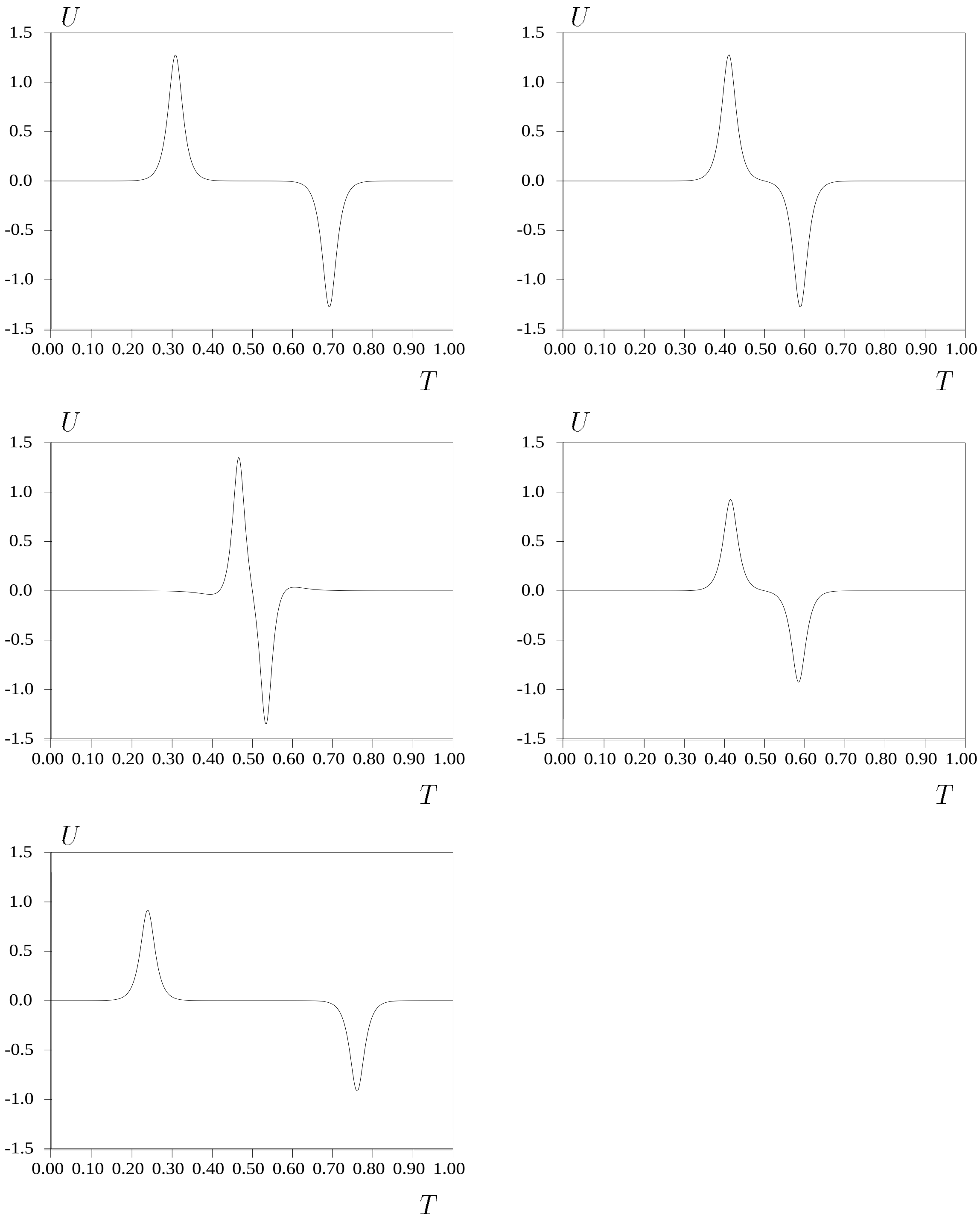

Figure 10: The stable double pulses continued in the parameter σ for $\Delta\alpha = 0.1$ fixed. Here from left to right and top to bottom, $\sigma = 4 \cdot 10^{-6}, 4 \cdot 10^{-3}, 0.238, 1.02, 1.02999$, where $\sigma = 0, 1.03$ correspond to the orbit-flip points of_u and of_l, respectively.

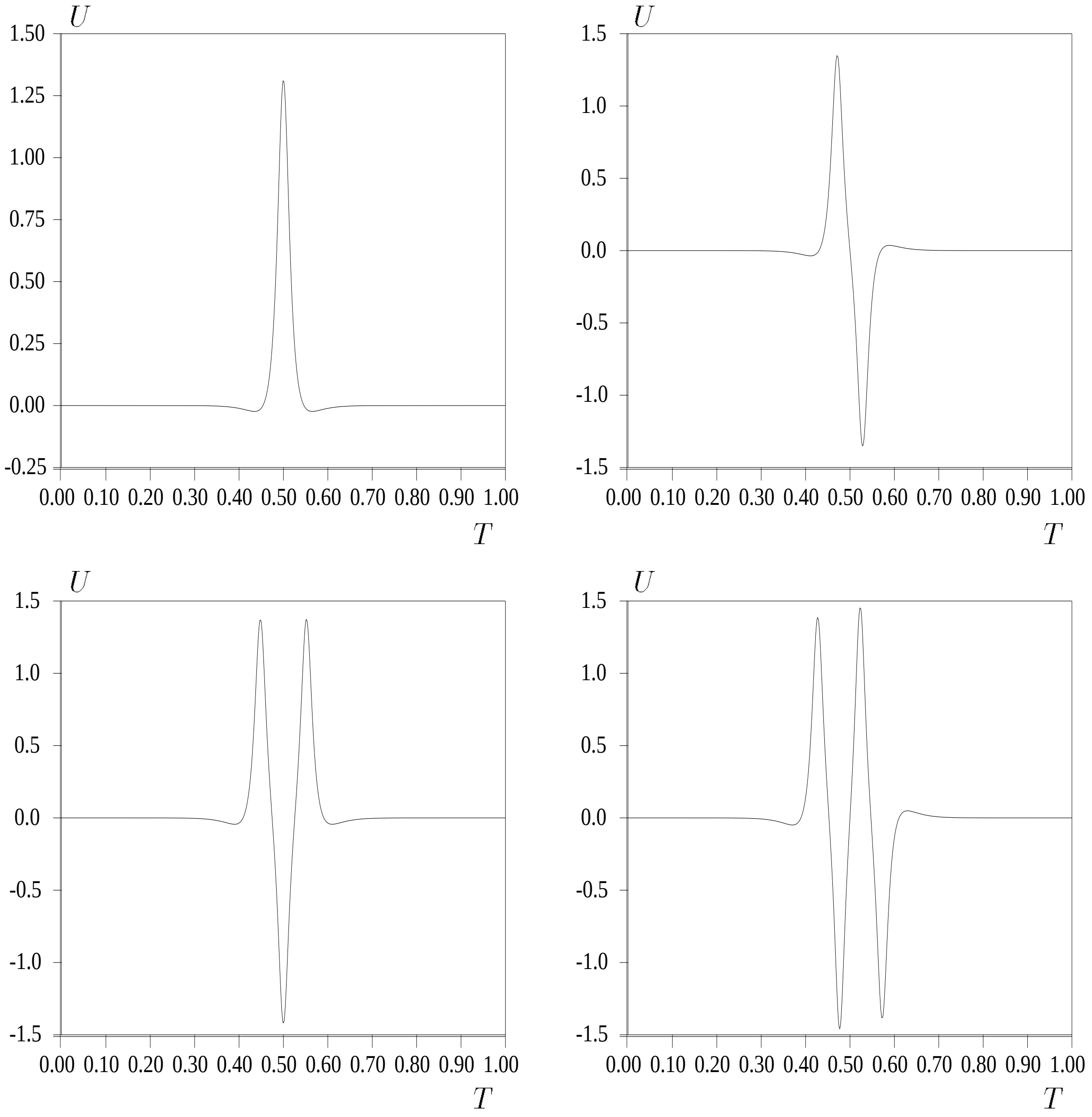

Figure 11: The stable N-pulses for $N = 1, 2, 3, 4$ bifurcating from the upper orbit-flip curve of_u are shown for $\Delta\alpha = 0.1$ and $\Gamma l = 1$. The corresponding sequences γ are $\gamma = (\mathrm{id})$, (id, κ), $(\mathrm{id}, \kappa, \mathrm{id})$ and $(\mathrm{id}, \kappa, \mathrm{id}, \kappa)$.

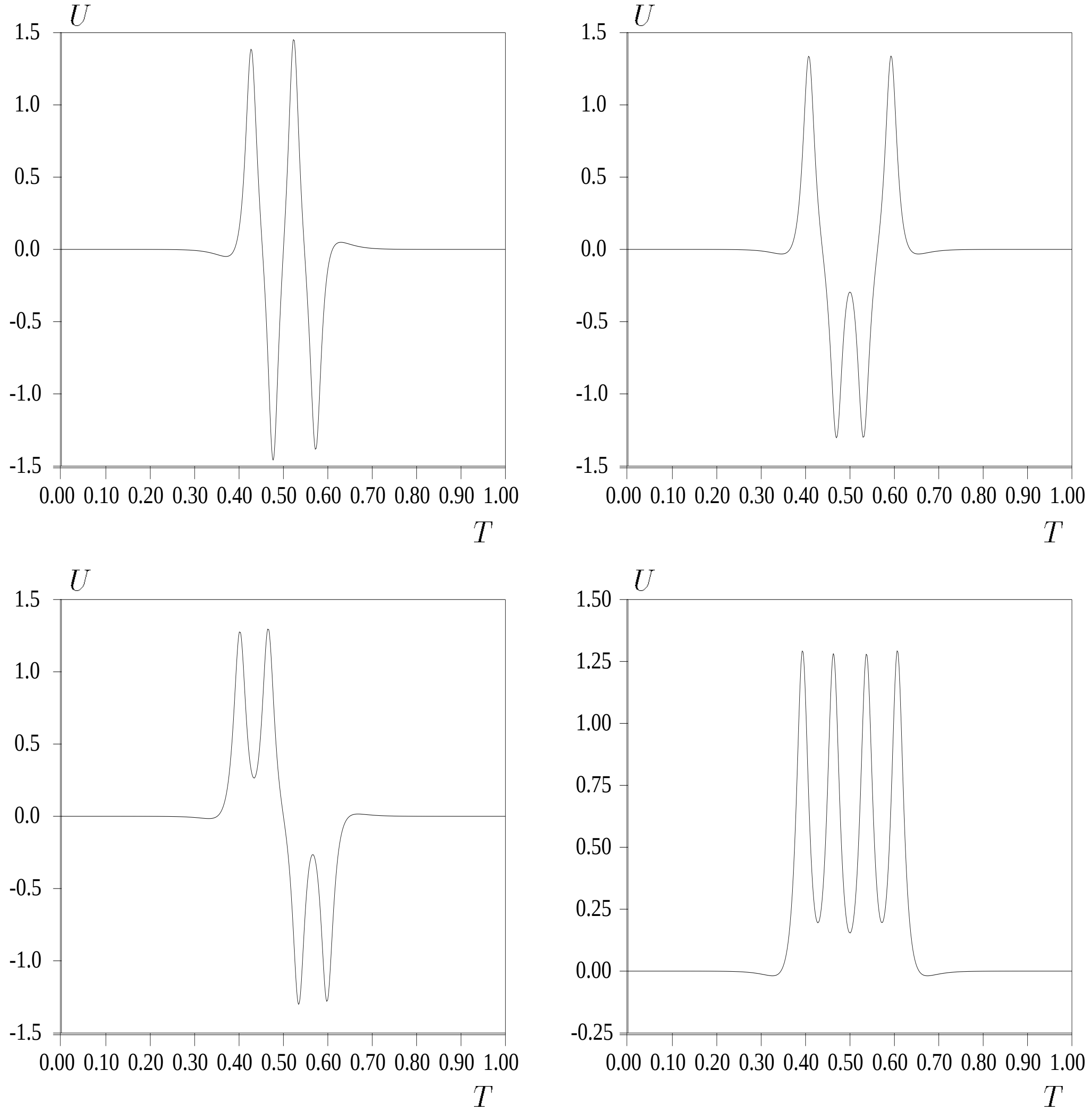

Figure 12: The 4-pulses for different sequences γ with $\Delta\alpha = 0.1$ and $\Gamma l = 1$, that is $\sigma = 0.238$. From left to right and top to bottom, the sequences are $\gamma = (\mathrm{id}, \kappa, \mathrm{id}, \kappa)$, $(\mathrm{id}, \kappa.\kappa, \mathrm{id})$, $(\mathrm{id}, \mathrm{id}, \kappa, \kappa)$ and $(\mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id})$. The number of unstable eigenvalues increases from zero to three.

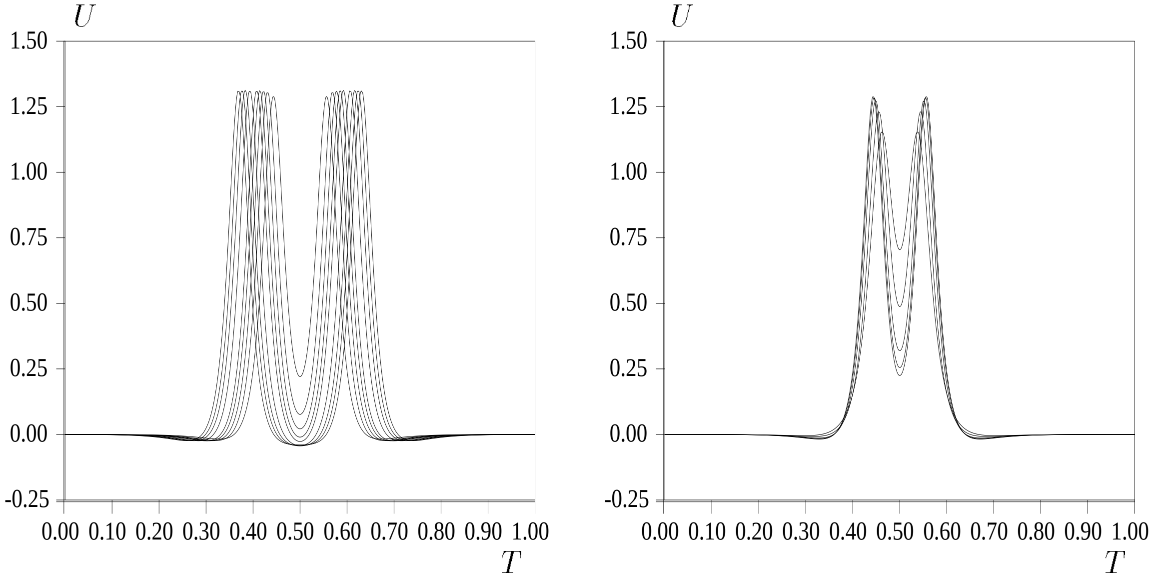

Figure 13: The unstable manifold of the 2-pulse with sequence $(\mathrm{id}, \mathrm{id})$ for $\Gamma l = 1$. In the left plot, the humps move apart from each other; in the right one, the minimum between the two humps moves upward and eventually the solution converges to the stable 1-pulse.

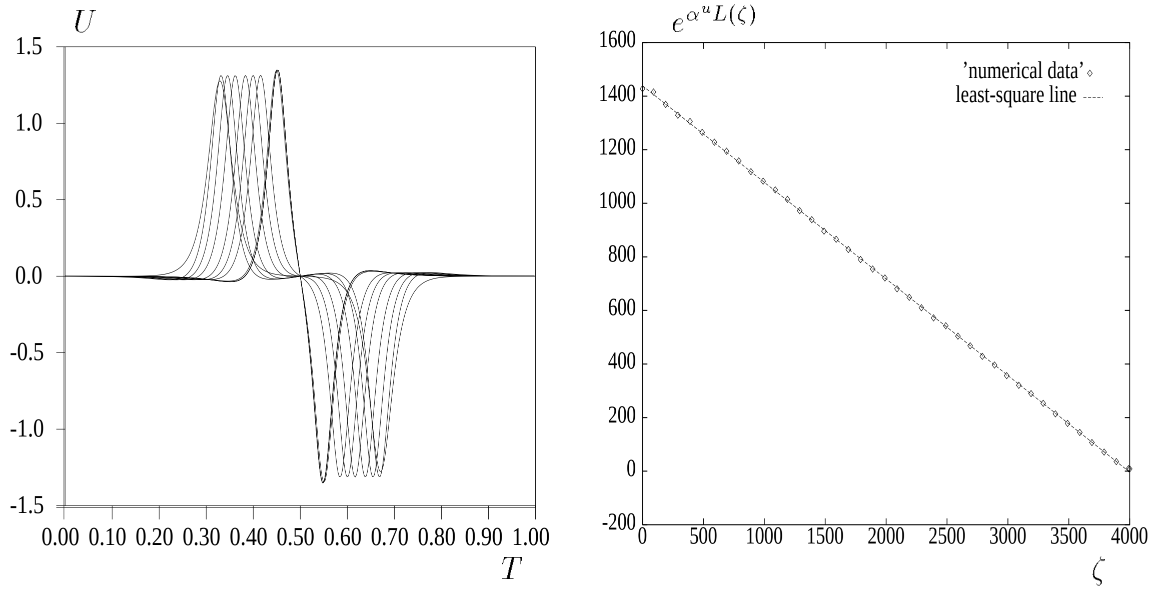

Figure 14: The 2-pulse with sequence (id, κ) is stable for $\Gamma l = 1$. In the left picture, it is shown, how a solution with two widely separated humps converges towards the 2-pulse. The spatial evolution of the distance $L(\zeta)$ of the two humps is shown in the right plot in a suitable scaling.

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