Eigenvector localization in the heavy-tailed random conductance model

Franziska Flegel

submitted: January 19, 2018

No. 2472
Berlin 2018

2010 Mathematics Subject Classification. 47B80, 47A75, 60K37.

Key words and phrases. Random conductance model; Dirichlet spectrum; eigenfunction localization; heavy tails; extreme value analysis.
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Abstract
We generalize our former localization result about the principal Dirichlet eigenvector of the i.i.d. heavy-tailed random conductance Laplacian to the first $k$ eigenvectors. We overcome the complication that the higher eigenvectors have fluctuating signs by invoking the Bauer-Fike theorem to show that the $k$th eigenvector is close to the principal eigenvector of an auxiliary spectral problem.

1 Introduction

Let us consider the random conductance Laplacian $L^w$ acting on real-valued functions $f \in L^2(\mathbb{Z}^d)$ as
\[
(L^w f)(x) = \sum_{y : |x-y|=1} w_{xy}(f(y) - f(x)) \quad (x \in \mathbb{Z}^d)
\] (1.1)
with positive independent and identically distributed random conductances $w_{xy}$. As usual, we further assume that the operator $L^w$ is self-adjoint, i.e. $w_{xy} = w_{yx}$. Our goal is to describe the almost-sure behavior of the solution to the spectral problem
\[
-L^w \psi = \lambda \psi \quad \text{on } B_n = [-n,n]^2 \cap \mathbb{Z}^d,
\]
\[
\psi = 0 \quad \text{else.}
\] (1.2)
as the box size $n$ tends to infinity. This means that we are interested in the Dirichlet eigenfunctions and eigenvalues of the operator $-L^w$ in the box $B_n$ with zero Dirichlet conditions.

In the recent paper [Fle16], we have shown that if $\gamma := \sup \{q \geq 0 : \mathbb{E}[w^{-q}] < \infty \} < 1/4$ and certain regularity assumptions apply, then the principal Dirichlet eigenvector $\psi_1^{(n)}$ of Problem (1.2) concentrates in a single site as $n$ tends to infinity. To be more precise, let $\pi_z = \sum_{x : x \sim z} w_{xz}$ be the local speed measure, i.e., the inverse mean waiting time of the random walk generated by $L^w$. Then the principal Dirichlet eigenvector $\psi_1^{(n)}$ approaches the $\delta$-function in the site $z_{(1,n)}$ that minimizes the local speed measure $\pi$ over the box $B_n$. Furthermore, the principal Dirichlet eigenvalue $\lambda_1^{(n)}$ is asymptotically equivalent to the minimum $\pi_{1,B_n} = \min_{z \in B_n} \pi_z$.

If, on the other hand, $\gamma > 1/4$, then the authors of [FHS17] have proved that the top of the Dirichlet spectrum of $L^w$ homogenizes. The spectrum of the random conductance Laplacian thus displays a dichotomy between a localized and a homogenized phase.

In the present paper we generalize our findings for $\gamma < 1/4$ to the first $k$ Dirichlet eigenvectors and eigenvalues. More precisely, we show that the $k$th Dirichlet eigenvector $\psi_k^{(n)}$ concentrates in the site that attains the $k$th minimum of $\pi$. Consequently, the $k$th Dirichlet eigenvalue $\lambda_k^{(n)}$ is asymptotically...
equivalent to the $k$th minimum of $\pi$. If the conductances vary regularly at zero with positive index, then despite the dependence structure of the random field $\{\pi_x\}_{x \in \mathbb{Z}^d}$, this $k$th minimum converges weakly as if $\{\pi_x\}_{x \in \mathbb{Z}^d}$ was an independent field, see the proof of Corollary 2.3. It follows that, in this case, the properly rescaled $k$th eigenvalue $\lambda_k^{(n)}$ converges in distribution to a non-degenerate random variable. This relates to a similar result in dimension $d = 1$, see [Fag12, Theorem 2.5(i)].

Note that the only reason why we have not generalized our findings to the first $k$ eigenvectors in [Fle16], is that in [Fle16, Lemma 5.6] we rely on the property that the principal Dirichlet eigenvector does not change its sign, according to the Perron-Frobenius theorem. This is no longer true for the higher order eigenvectors. To overcome this difficulty, we now approximate the first $k$ eigenvectors to (1.2) by auxiliary principal eigenvectors using the Bauer-Fike theorem, see Lemma 3.14.

Our results for the random conductance Laplacian compare well to similar results of the random Schrödinger operator $\Delta + \xi$ with random potential $\xi: \mathbb{Z}^d \to \mathbb{R}$, see [BK16] and [Ast16, Ch. 6]. To keep the present paper as short as possible, we refer the reader to our first article [Fle16] for more heuristics and references. However, we kept the present paper mostly self-contained.

**Model and main objects**

We consider the lattice with vertex set $\mathbb{Z}^d$ ($d \geq 2$) and edge set $\mathcal{E}_d = \{\{x, y\} : x, y \in \mathbb{Z}^d, \|x - y\|_1 = 1\}$. If two sites $x, y \in \mathbb{Z}^d$ are neighbors according to $\mathcal{E}_d$, we also write $x \sim y$. To each edge $e \in \mathcal{E}_d$ we assign a positive random variable $w_e$. In analogy to a $d$-dimensional resistor network, we call these random weights $w_e$ conductances. We take $(\Omega, F) = \left((0, \infty)^{\mathcal{E}_d}, B((0, \infty)^{\mathcal{E}_d})\right)$ as the underlying measurable space and assume that an environment $w = (w_e)_{e \in \mathcal{E}_d} \in \Omega$ is a family of i.i.d. positive random variables with law $\mathbb{P}$. We denote the expectation with respect to $\mathbb{P}$ by $\mathbb{E}$.

If $e$ is the edge between the sites $x, y \in \mathbb{Z}^d$, we also write $w_{xy}$ or $w_{yx}$ instead of $w_e$. Note that by definition of the edge set $\mathcal{E}_d$, the edges are undirected, whence $w_{xy} = w_{yx}$. If we want to refer to an arbitrary copy of the conductances in general, we simply write $w$, i.e., for a set $A \in B((0, \infty))$, the expression $\mathbb{P}[w \in A]$ equals $\mathbb{P}[w_e \in A]$ for an arbitrary edge $e$.

We call

$$F: [0, \infty) \to [0, 1]: u \mapsto \mathbb{P}[w \leq u]$$

(1.3)

the distribution function of the conductances.

For an arbitrary $k \in \mathbb{N}$, our goal is to study the behavior of the first $k$ Dirichlet eigenvalues $\lambda_1^{(n)} \leq \ldots \leq \lambda_k^{(n)}$ and eigenvectors $\psi_1^{(n)}, \ldots, \psi_k^{(n)}$ of the sign-inverted generator $-\mathcal{L}_w$ in the ball

$$B_n := \left\{x \in \mathbb{Z}^d : \|x\|_\infty \leq n\right\} = [-n, n]^d \cap \mathbb{Z}^d$$

(1.4)

with zero Dirichlet conditions at the boundary.

For a subset $A \subset \mathbb{Z}^d$ we define the function space

$$\ell^2(A) := \left\{f: \mathbb{Z}^d \to \mathbb{R} \text{ such that } \text{supp } f \subseteq A \text{ and } \sum_{x \in A} f(x)^2 < \infty\right\} \subset \ell^2(\mathbb{Z}^d),$$

(1.5)

where we let “supp $f$” denote the support of the function $f$. Accordingly, for functions $f_1, f_2 \in \ell^2(\mathbb{Z}^d)$ we define the scalar product

$$\langle f_1, f_2 \rangle_{\ell^2(A)} = \sum_{x \in A} f_1(x) f_2(x).$$

DOI 10.20347/WIAS.PREPRINT.2472 Berlin 2018
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For a real-valued function \( f \in \ell^2(\mathbb{Z}^d) \) let us define the Dirichlet energy \( \mathcal{E}^w(f) \) with respect to the operator \(-\mathcal{L}_w\) by

\[
\mathcal{E}^w(f) = \langle f, -\mathcal{L}_w f \rangle_{\ell^2(\mathbb{Z}^d)}.
\]

(1.6)

Then, according to the Courant-Fischer theorem, the \( k \)th Dirichlet eigenvalue is given by the variational formula

\[
\lambda_k^{(n)} = \inf_{\mathcal{M} \subseteq \ell^2(B_n), \dim \mathcal{M} = k} \sup_{f \in \mathcal{M}, \|f\|_2 = 1} \mathcal{E}^w(f)
\]

(1.7)

where \( \mathcal{M} \subseteq \ell^2(B_n) \) means that \( \mathcal{M} \) is a linear subspace of \( \ell^2(B_n) \). Note that \( \lambda_k^{(n)} = \mathcal{E}^w(\psi_k^{(n)}) \).

**Definition 1.1** (Local speed measure and its order statistics). *We define the local speed measure \( \pi \) by

\[
\pi_z = \sum_{x : x \sim z} w_{xz} \quad (z \in \mathbb{Z}^d)
\]

(1.8)

and we label the order statistics of the set \( \{\pi_z\}_{z \in B_n} \) by

\[
\pi_{1,B_n} \leq \pi_{2,B_n} \leq \ldots \leq \pi_{|B_n|,B_n}.
\]

(1.9)

Furthermore, for \( k, n \in \mathbb{N} \) let \( z_{(k,n)} \) be the site where \( \pi \) attains its \( k \)th minimum over \( B_n \), i.e., \( \pi_{z_{(k,n)}} = \pi_{k,B_n} \).

**Remark 1.2.** If \( F \) is continuous, then \( \pi_{1,B_n} < \pi_{2,B_n} < \ldots < \pi_{|B_n|,B_n} \) \( \mathbb{P} \)-a.s. and therefore the minimizers \( z_{(k,n)} \) are \( \mathbb{P} \)-a.s. unique.

## 2 Main result

In what follows we let

\[
g : [0, \infty) \to [0, \infty) : u \mapsto \sup \{ s \geq 0 : F(s) = u^{-1/2} \}.
\]

(2.1)

**Assumption 2.1.** Let \( F \) be continuous and vary regularly at zero with index \( \gamma \in [0, 1/4) \). Assume that there exists \( a^* > 0 \) such that \( F(ab) \geq bF(a) \) for all \( a \leq a^* \) and all \( 0 \leq b \leq 1 \). In the case where \( \gamma = 0 \), we assume additionally that there exists \( \epsilon_1 \in (0,1) \) such that the product \( n^{2+\epsilon_1}g(n) \) converges monotonically to zero as \( n \) grows to infinity.

**Remark 2.2.** In the case where \( \gamma > 0 \), it follows that \( (1/F(1/s))^2 \) varies regularly at infinity with index \( 2\gamma \). Further, \( (1/F(1/s))^2 \) diverges as \( s \to \infty \). It follows by virtue of [Res87, Prop. 0.8(v)] that \( 1/g(u) = \inf \{ s \geq 0 : (1/F(1/s))^2 = u \} \) varies regularly at infinity with index \( 1/(2\gamma) \) and thus \( g \) varies regularly at infinity with index \( -1/(2\gamma) \). Since in addition \( \gamma < 1/4 \), there exists \( \epsilon_1 \in (0,1) \) such that \( -1/(2\gamma) < - (2 + \epsilon_1) \).

**Theorem.** Let \( k \in \mathbb{N} \). If Assumption 2.1 holds, then the \( k \)th Dirichlet eigenvalue \( \lambda_k^{(n)} \) with zero Dirichlet conditions outside the box \( B_n \) fulfills

\[
\mathbb{P} \left[ \lim_{n \to \infty} \frac{\lambda_k^{(n)}}{\pi_{k,B_n}} = 1 \right] = 1
\]

(2.2)
and the mass of the \( k \)th Dirichlet eigenvector \( \psi_k^{(n)} \) asymptotically concentrates in the site \( z_{(k,n)} \). More precisely, if \( \epsilon_1 > 0 \) is as in Assumption 2.1 or Remark 2.2, then \( \mathbb{P} \)-a.s. for \( n \) large enough

\[
1 - n^{-\epsilon/8} \leq \frac{\lambda_k^{(n)}}{\pi_k;B_n} \leq 1 \quad \text{for all } \epsilon < \epsilon_1
\]  

(2.3)

and

\[
\psi_k^{(n)}(z_{(k,n)}) \geq \sqrt{1 - n^{-\epsilon/4}} \quad \text{for all } \epsilon < \epsilon_1.
\]  

(2.4)

We prove this theorem in Section 4.

Similar to [Fle16, Corollary 1.11], we can now infer the weak convergence of the eigenvalues. Let \( F \) be the distribution function of the random variable \( \pi \), i.e., the distribution function of the sum of \( 2d \) independent copies of the conductance \( w \). Note that since \( F \) is continuous, \( F \) is continuous as well. As in [Fle16, (1.18)], we define

\[
h: (0, \infty) \rightarrow (0, \infty): u \mapsto \inf\left\{ s: \frac{1}{F_\pi(1/s)} = u \right\}.
\]  

(2.5)

Let \( F \) vary regularly at zero with index \( \gamma > 0 \). Then by virtue of [Fle16, Lemma 5.8], it follows that \( F_\pi \) varies regularly at zero with index \( 2\gamma \). It thus follows by virtue of [Res87, Proposition 0.8(v)] that \( h \) varies regularly at infinity with index \( 1/(2\gamma) \). Therefore there exists a function \( L^* \) that varies slowly at infinity such that

\[
h(|B_n|) = n^{\frac{1}{2\gamma}} L^*(n).
\]  

(2.6)

Corollary 2.3. Assume that \( F \) fulfills Assumption 2.1 with \( \gamma > 0 \) and let \( L^* \) be as in (2.6). Let \( k \in \mathbb{N} \). Then as \( n \) tends to infinity, the product \( L^*(n) n^{\frac{1}{2\gamma}} \lambda_k^{(n)} \) converges in distribution to a non-degenerate random variable. More precisely,

\[
\lim_{n \rightarrow \infty} \mathbb{P}\left[ L^*(n) n^{\frac{1}{2\gamma}} \lambda_k^{(n)} > \zeta \right] = \exp(-\zeta^{2\gamma}) \sum_{j=0}^{k-1} \frac{\zeta^{2\gamma j}}{j!} \quad \text{for all } \zeta \in [0, \infty).
\]  

(2.7)

This corollary extends [Fle16, Corollary 1.11] to general \( k \in \mathbb{N} \). We prove it at the end of Section 5.

3 Auxiliary spectral problems

Definition 3.1 (Auxiliary lattice and Laplacian). We define the set

\[
\mathcal{B}_{(n)}^{(l)} = B_n \setminus \{ z_{(1,n)}, \ldots, z_{(l-1,n)} \}
\]  

(3.1)

and abbreviate the operator \( L^w \) with zero Dirichlet conditions outside \( \mathcal{B}_{(n)}^{(l)} \) as \( L^w_{(l,n)} \), i.e., we define

\[
L^w_{(l,n)} := 1_{\mathcal{B}_{(n)}^{(l)}} L^w 1_{\mathcal{B}_{(n)}^{(l)}},
\]  

(3.2)

where the operator \( 1_{\mathcal{B}_{(n)}^{(l)}} \) is the identity on \( \mathcal{B}_{(n)}^{(l)} \) and zero otherwise.
Since the operator \(-L\) is self-adjoint, the operator \(-L_{l,n}\) is self-adjoint as well. This justifies the next definition.

**Definition 3.2 (Auxiliary eigenvectors and values).** We define the eigenvalues of the operator \(-L_{l,n}\) restricted to \(\ell^2(B_{l,n})\) by

\[
\mu_{l,1}^{(n)} \leq \mu_{l,2}^{(n)} \leq \cdots \leq \mu_{l,|B_{l,n}|}^{(n)}
\]  

(3.3)

and its eigenvectors by

\[
\phi_{l,1}^{(n)}, \phi_{l,2}^{(n)}, \ldots, \phi_{l,|B_{l,n}|}^{(n)} \in \ell^2(B_{l,n}) \quad \text{with} \quad \langle \phi_{l,i}^{(n)}, \phi_{l,j}^{(n)} \rangle = \delta_{ij}.
\]  

(3.4)

Note that \(B_{l,n} = B_n\) and thus \(\mu_{l,1}^{(n)} = \lambda_k^{(n)}\) and \(\phi_{l,1}^{(n)} = \psi_k^{(n)}\). Moreover the variational formula for the auxiliary eigenvalues reads

\[
\mu_{l,m}^{(n)} = \inf_{M \leq \ell^2(B_{l,n})}, \sup_{\dim M = m, \|f\|_2 = 1} E_{l,n}(f).
\]  

(3.5)

**Remark 3.3 (Perron-Frobenius).** For a given box \(B_n\) the operator \(L_{l,n}\) can be written as a \((|B_n| - l + 1) \times (|B_n| - l + 1)\)-matrix with non-negative entries everywhere except on the diagonal. Since the matrix is finite-dimensional, we can add a multiple of the identity to obtain a non-negative primitive matrix without changing the matrix' spectrum. By the Perron-Frobenius theorem (see e.g. [Sen81, Ch. 1]) it follows that its principal eigenvalue \(-\mu_{l,1}^{(n)}\) is simple and we can assume without loss of generality that its principal eigenvector is positive, which implies that \(\phi_{l,1}^{(n)}\) is nonnegative.

**Lemma 3.4.** For any \(l \in \mathbb{N}\) and \(m \in \{1, \ldots, |B_n| - l + 1\}\) the eigenvalue \(\mu_{l,m}^{(n)}\) is bounded from above by

\[
\mu_{l,m}^{(n)} \leq \pi_{l+m-1,B_n}.
\]  

(3.6)

**Proof.** We choose

\[
\mathcal{M} = \text{span} \left\{ \delta_{z_{l,n}}, \delta_{z_{l+1,n}}, \ldots, \delta_{z_{l+m-1,n}} \right\}
\]

and insert it as a test space into the variational formula (3.5).

\(\square\)

### 3.1 Principal eigenvectors

The following lemma is the analogue of [Fle16, Lemma 5.6], where we need the Perron-Frobenius property.

**Lemma 3.5.** Let \(k \in \mathbb{N}\) and let \(y, z \in B_n \cap \mathcal{B}_{k,n}^{(n)}\) with \(\pi_z < \pi_y\) and \(y \sim z\). Assume that \(\phi_{k,1}^{(n)}\) is nonnegative. Further, define \(m_y = 2 \max_{x: x \sim y} \phi_{k,1}^{(n)}(x)\). Then the mass \(\phi_{k,1}^{(n)}(y)\) is bounded from above by

\[
\phi_{k,1}^{(n)}(y) \leq \frac{m_y}{1 - \frac{m_y}{\pi_y}}.
\]  

(3.7)
The proof of this lemma is analogous to the proof of [Fle16, Lemma 5.6] and therefore we omit it here.

For the convenience of the reader, we now repeat some definitions from [Fle16]. For a function \( g : (0, \infty) \to (0, \infty) \) and \( n \in \mathbb{N} \) we define a percolation environment \( \tilde{\omega}_{g(n)} \) by setting
\[
\tilde{\omega}_{g(n)}(e) := w_e 1_{\{w_e > g(n)\}} \quad (e \in \mathcal{E}_d).
\] (3.8)

Thus, edges with conductance less than or equal to \( g(n) \) are considered to be closed and all others keep their original conductance. With this terminology we can now define the following clusters.

**Definition 3.6.** For a fixed function \( g \) and a fixed \( \epsilon > 0 \), let \( \mathcal{G}^{(n)} \) be the unique infinite open cluster of the environment \( \tilde{\omega}_{g(n-\epsilon)} \) and let \( \mathcal{G}^{(n)} = B_n \setminus \mathcal{G}^{(n)} \) be its set of holes in \( B_n \).

**Definition 3.7.** We call a set \( \mathcal{I} \subset \mathbb{Z}^d \) sparse if the set \( \mathcal{I} \) does not contain any neighboring sites. Further, a set \( \mathcal{I} \subset \mathbb{Z}^d \) is \( b \)-sparse if for any \( z \in \mathbb{Z}^d \) the box \( B_{\epsilon}(z) := \{ x \in \mathbb{Z}^d : |x - z|_\infty \leq b \} \subset \mathbb{Z}^d \) contains at most one site of the set \( \mathcal{I} \).

**Remark 3.8.** Let \( b_1 < b_2 \) be natural numbers. If a set \( \mathcal{I} \subset \mathbb{Z}^d \) is \( b_2 \)-sparse, it is also \( b_1 \)-sparse and sparse.

Let us collect some facts that we already know about the cluster \( \mathcal{G}^{(n)} \) and the set \( \mathcal{G}^{(n)} \) from [Fle16].

**Remark 3.9.** Let us recall that in Assumption 2.1 we assume that one of the two following cases occurs: \( \gamma \in (0, 1/4) \) or \( \gamma = 0 \) and there exists \( \epsilon_1 \in (0, 1) \) such that the product \( n^{2+\epsilon_1} g(n) \) converges monotonically to zero as \( n \) grows to infinity. In the case where \( \gamma \in (0, 1/4) \), we define \( \epsilon_1 \) as in Remark 2.2.

In both cases we define \( \mathcal{G}^{(n)} \) and \( \mathcal{G}^{(n)} \) as in Definition 3.6 with \( \epsilon = \epsilon_2 := \frac{\pi_1}{8(2+\epsilon_1)} \). By virtue of [Fle16, Lemma 5.4] and Remark 3.8 we know that for any fixed \( b \in \mathbb{N} \) the set \( \mathcal{G}^{(n)} \) is \( b \)-sparse and therefore \( \mathbb{P} \)-a.s. for \( n \) large enough in the sense of Definition 3.7. Moreover, [Fle16, Lemma 5.4] implies that for any \( k \in \mathbb{N} \) we have \( \mathbb{P} \)-a.s. for \( n \) large enough: \( z_{(1,n)}, \ldots, z_{(k+1,n)} \in \mathcal{G}^{(n)} \) and thus \( \mathbb{P} \)-a.s. for \( n \) large enough there is no pair of neighbors among the sites \( z_{(1,n)}, \ldots, z_{(k+1,n)} \). Since \( F \) is continuous, the sites are \( \mathbb{P} \)-a.s. unique.

The next lemma about the principal Dirichlet eigenvector \( \phi_{k,1}^{(n)} \) of the auxiliary operator \( -\mathcal{L}_{k,n}^w \) is very similar to [Fle16, Lemma 5.5]. Indeed, we can nearly copy the proof since the deleted sites \( z_{(1,n)}, \ldots, z_{(k-1,n)} \) are in \( \mathcal{G}^{(n)} \), see Remark 3.9.

**Lemma 3.10.** Let the function \( g \) be as in (2.1). Assume that there exists \( \epsilon_1 \in (0, 1) \) such that one of the two cases occurs: \( g \) varies regularly at infinity with index \( p < -2+\epsilon_1 \) or the product \( n^{2+\epsilon_1} g(n) \) converges monotonically to zero as \( n \) grows to infinity. Further, let \( \epsilon = \epsilon_2 := \frac{\pi_1}{8(2+\epsilon_1)} \) and \( \mathcal{G}^{(n)} \) be as in Definition 3.6. Then \( \mathbb{P} \)-a.s. for \( n \) large enough
\[
\|\phi_{k,1}^{(n)}\|_{L^2(\mathcal{G}^{(n)})}^2 \leq n^{-\epsilon_1/2}.
\] (3.9)

**Proof.** The proof follows the lines of the proof of [Fle16, Lemma 5.5] until right before (5.8). Here, we then apply Lemma 3.4 to infer that
\[
\pi_{k,B_n} \geq \lambda_{k,1}^{(n)} = \mathcal{E}^w(\phi_{k,1}^{(n)}).
\]

Moreover, by virtue of [Fle16, Lemma 2.6] there exists \( c_1 \) such that \( \mathbb{P} \)-a.s. for \( n \) large enough
\[
c_1 g(n^{1-\epsilon_3}) \geq \pi_{k,B_n}
\]
with \( \epsilon_3 = \epsilon_1(8(2+\epsilon_1))^{-1} \). The rest of the proof follows again the lines of the proof of [Fle16, Lemma 5.5].

\[\square\]
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From Lemma 3.10 to localization in a single site, the main two ingredients are Lemma 3.5 and the following result about the order statistics of \( \{ \pi_x \}_{x \in B_n} \).

**Lemma 3.11** ([Fle16, Lemma 5.10]). Let Assumption 2.1 be true and let \( \varepsilon > 0 \) and \( k \in \mathbb{N} \). Then \( \mathbb{P} \)-a.s. for \( n \) large enough

\[
1 - \frac{\pi_{k,B_n}}{\pi_{k+1,B_n}} > n^{-\varepsilon}.
\]  

(3.10)

The next lemma therefore follows.

**Lemma 3.12.** Let \( k \in \mathbb{N} \). Under Assumption 2.1, it follows that \( \mathbb{P} \)-a.s. for \( n \) large enough

\[
\phi^{(n)}_{k,1}(z_{(k,n)}) \geq \sqrt{1 - n^{-\varepsilon/4}}.
\]  

(3.11)

This implies that \( \mathbb{P} \)-a.s. for \( n \) large enough

\[
\mu^{(n)}_{k,1} \geq (1 - 2n^{-\varepsilon/8}) \pi_{k,B_n}.
\]  

(3.12)

**Proof.** In view of Remark 3.9, Lemma 3.5 and the extreme value result Lemma 3.11, the proof of (3.11) is completely analogous to the proof of [Fle16, Theorem 1.8] and thus we omit it here. For (3.12) we observe that since \( \mu^{(n)}_{k,1} = \langle \phi^{(n)}_{k,1}, L^w \phi^{(n)}_{k,1} \rangle \) it follows that \( \mathbb{P} \)-a.s. for \( n \) large enough

\[
\mu^{(n)}_{k,1} \geq \sum_{x : x \sim z_{(k,n)}} w_{xz_{(k,n)}} \left( \phi^{(n)}_{k,1}(z_{(k,n)}) - \phi^{(n)}_{k,1}(x) \right)^2 \geq \left( n^{-\varepsilon/8} - \sqrt{1 - n^{-\varepsilon/4}} \right)^2 \pi_{z_{(k,n)}}.
\]

3.2 Orthogonality of eigenvectors

The next very simple ingredient of our proof is due to the orthogonality of the eigenvectors.

**Lemma 3.13.** Let \( \varepsilon > 0 \), let \( j, l, m, n \in \mathbb{N} \) with \( j < m \) and let \( \phi^{(n)}_{l,j}(z) \geq \sqrt{1 - n^{-\varepsilon/4}} \).

\[
\left| \phi^{(n)}_{l,m}(z) \right| \leq n^{-\varepsilon/8}.
\]  

(3.13)

**Proof.** For \( n = 1 \) the claim is immediate. For \( n \geq 2 \) we observe that since the eigenvectors \( \phi^{(n)}_{l,j} \) and \( \phi^{(n)}_{l,m} \) are orthogonal to each other, it follows that

\[
\phi^{(n)}_{l,m}(z) = -\frac{\sum_{x \neq z} \phi^{(n)}_{l,j}(x) \phi^{(n)}_{l,m}(x)}{\phi^{(n)}_{l,j}(z)}.
\]

By the Cauchy-Schwarz inequality it follows that for \( n \) greater than one

\[
\left( \phi^{(n)}_{l,m}(z) \right)^2 \leq \frac{\sum_{x \neq z} \left( \phi^{(n)}_{l,j}(x) \right)^2 \left( 1 - \left( \phi^{(n)}_{l,m}(z) \right)^2 \right)}{\left( \phi^{(n)}_{l,j}(z) \right)^2 \left( 1 - \left( \phi^{(n)}_{l,m}(z) \right)^2 \right)} \leq \frac{n^{-\varepsilon/4}}{1 - n^{-\varepsilon/4}} \left( 1 - \left( \phi^{(n)}_{l,m}(z) \right)^2 \right)
\]

where we have also used that the assumption implies that \( \sum_{x \neq z} \left( \phi^{(n)}_{l,j}(x) \right)^2 \leq n^{-\varepsilon/4} \). The claim follows.

\[\square\]
3.3 Higher eigenvalues and -vectors

We establish the connection to the original eigenvalues and -vectors via the Bauer-Fike theorem [BF60], which we cite below from [JKO94, Lemma 11.2].

**Lemma 3.14 ([JKO94, Lemma 11.2]).** Let $A : H \to H$ be a linear self-adjoint compact operator in a Hilbert space $H$. Let $\mu \in \mathbb{R}$, and let $u \in H$ be such that $\|u\|_H = 1$ and
\[
\|Au - \mu u\|_H \leq \alpha, \quad \alpha > 0.
\] (3.14)

Then there exists an eigenvalue $\mu_i$ of the operator $A$ such that
\[
|\mu_i - \mu| \leq \alpha.
\] (3.15)

Moreover, for any $\beta > \alpha$, there exists a vector $\pi$ such that
\[
\|u - \pi\|_H \leq 2\alpha\beta^{-1}, \quad \|\pi\|_H = 1
\] (3.16)

and $\pi$ is a linear combination of the eigenvectors of operator $A$ corresponding to the eigenvalues from the interval $[\mu - \beta, \mu + \beta]$.

Here comes the first application of Lemma 3.14.

**Lemma 3.15.** Let $l \in \mathbb{N}$ and $m \in \{1, \ldots, |B_n| - l + 1\}$. Under Assumption 2.1 there exists $i \in \{1, \ldots, |B_n| - l + 1\}$ such that
\[
\left|\mu^{(n)}_{l,i} - \mu^{(n)}_{l+1,1}\right| \leq n^{-\epsilon_1/4} \cdot \pi_{l+1,m-1,B_n}.
\] (3.17)

**Proof.** We aim to apply Lemma 3.14 with the operator $A = -\mathcal{L}_{l,(n)}^{w}$, the Hilbert space $H = l^2(\mathcal{B}_{l}^{(n)})$, the value $\mu = \mu_{l+1,1}$ and the vector $u = \phi^{(n)}_{l+1,1}$. First, we note that $\|\phi^{(n)}_{l+1,1}\|_{l^2(\mathcal{B}_{l}^{(n)})} = 1$. Next, we recall that $\phi^{(n)}_{l+1,1}$ is an eigenvector of the operator $-\mathcal{L}_{l,(n)}^{w}$ to the eigenvalue $\mu^{(n)}_{l+1,1}$ and therefore
\[
\left\|\mathcal{L}_{l,(n)}^{w}\phi^{(n)}_{l+1,1} + \mu^{(n)}_{l+1,1}\phi^{(n)}_{l+1,1}\right\|_{l^2(\mathcal{B}_{l}^{(n)})} = \sum_{z \in \mathcal{B}_{l}^{(n)} \setminus \mathcal{B}_{l+1,m}^{(n)}} \left(\mathcal{L}_{l,(n)}^{w}\phi^{(n)}_{l+1,1}(z) + \mu^{(n)}_{l+1,1}\phi^{(n)}_{l+1,1}(z)\right)^2,
\]

where all other summands vanish. Note that $\mathcal{B}_{l}^{(n)} \setminus \mathcal{B}_{l+1,m}^{(n)} = \{z(l) \ldots z(l+1-1)\}$ and by definition we have $\phi^{(n)}_{l+1,1}(z) = 0$ for all $z \in \{z(l) \ldots z(l+1-1)\}$. It follows that for all $z \in \mathcal{B}_{l}^{(n)} \setminus \mathcal{B}_{l+1,m}^{(n)}$ we have
\[
\mathcal{L}_{l,(n)}^{w}\phi^{(n)}_{l+1,1}(z) = \sum_{x : x \sim z} w_{xz} \left(\phi^{(n)}_{l+1,1}(x) - \phi^{(n)}_{l+1,1}(z)\right) = \sum_{x : x \sim z} w_{xz} \phi^{(n)}_{l+1,1}(x).
\]

Since $\pi_{l+1,m-1,B_n} \geq \pi_{l+1,m-2,B_n} \geq \ldots \geq \pi_{l,B_n}$, it follows that
\[
\left\|\mathcal{L}_{l,(n)}^{w}\phi^{(n)}_{l+1,1} + \mu^{(n)}_{l+1,1}\phi^{(n)}_{l+1,1}\right\|_{l^2(\mathcal{B}_{l}^{(n)})}^2 \leq \pi_{l+1,m-1,B_n} \sum_{z \in \mathcal{B}_{l}^{(n)} \setminus \mathcal{B}_{l+1,m}^{(n)}} \max_{x : x \sim z} \left(\phi^{(n)}_{l+1,1}(x)\right)^2.
\]

Since by virtue of Remark 3.9 the sites $z(l), \ldots, z(l+1-1)$ are in $\mathcal{F}^{(n)}$ and are neither neighbors nor do they share a common neighbor $\mathbb{P}$-a.s. for $n$ large enough, it follows that $\mathbb{P}$-a.s. for $n$ large enough
\[
\sum_{z \in \mathcal{F}^{(n)} \setminus \mathcal{B}_{l+1,m}^{(n)}} \max_{x : x \sim z} \left(\phi^{(n)}_{l+1,1}(x)\right)^2 \leq \sum_{x \in \mathcal{B}_{l}^{(n)}} \left(\phi^{(n)}_{l+1,1}(x)\right)^2 \leq n^{-\epsilon_1/2},
\]

where the last bound is due to Lemma 3.10. The claim follows by virtue of Lemma 3.14. \qed
Here comes the second application of Lemma 3.14.

**Lemma 3.16.** Let $\varepsilon > 0$, $l$, $m \in \mathbb{N}$. If Assumption 2.1 holds and $\mathbb{P}$-a.s. for $n$ large enough
\begin{equation}
\phi_{l,m}^{(n)}(z_{(l+1-n)}, \ldots, z_{(l+1-m)}) \geq \sqrt{1 - n^{-\varepsilon/4}} \tag{3.18}
\end{equation}
for all $1 \leq j \leq m$,
then $\mathbb{P}$-a.s. for $n$ large enough there exists $j \in \{1, \ldots, |B_n| - l - m + 1\}$ such that
\begin{equation}
\frac{\|\phi_{l,m}^{(n)}(z_{(l+1-n)}, \ldots, z_{(l+1-m)})\|_2^2}{\|\phi_{l,m+1}^{(n)}(z_{(l+1-n)}, \ldots, z_{(l+1-m)})\|_2} \geq 1 - mn^{-\varepsilon/4} \quad \text{for all } B_n \subset \mathbb{R}^d.
\end{equation}

**Proof.** We aim to apply Lemma 3.14 with the operator $A = -\mathcal{L}_x^{w}$, the Hilbert space $H = l^2(\mathcal{A}^{(n)}_{l+m})$, the value $\mu = \phi_{l,m+1}^{(n)}$ and the vector $u = \phi_{l,m+1}^{(n)}$. First, we note that by definition $\|u\|_{l^2(\mathcal{A}^{(n)}_{l+m})} = 1$ and $\mathbb{P}$-a.s. for $n$ large enough
\begin{equation}
\frac{\|\phi_{l,m+1}^{(n)}\|_2^2}{\|\phi_{l,m}^{(n)}\|_2} = 1 - \sum_{z \in \mathcal{A}^{(n)}_{l+m} \setminus \mathcal{A}^{(n)}_{l+m}} (\phi_{l,m+1}^{(n)}(z))^2 \geq 1 - mn^{-\varepsilon/4} \tag{3.20}
\end{equation}
by virtue of Condition (3.18) and Lemma 3.13.

Next, as we show in detail in (A.1), we can estimate
\begin{equation}
\left\| \mathcal{L}_x^{w}(\phi_{l,m}^{(n)} + 2\mu_{l,m+1}^{(n)}\phi_{l,m+1}^{(n)}) \right\|_{l^2(\mathcal{A}^{(n)}_{l+m})}^2 \leq \max_{z \in \mathcal{A}^{(n)}_{l+m} \setminus \mathcal{A}^{(n)}_{l+m}} (\phi_{l,m+1}^{(n)}(z))^2 \sum_{x \in B_n} \left( \sum_{z : z \sim x \in \mathcal{A}^{(n)}_{l+m}} w_{xz} \right)^2. \tag{3.21}
\end{equation}

Since by virtue of Remark 3.9 we have $\mathbb{P}$-a.s. for $n$ large enough
\begin{equation}
\mathcal{A}^{(n)}_{l+m} \setminus \mathcal{A}^{(n)}_{l+m} = \{z_{l,n}, \ldots, z_{l+m-1,n}\} \subset \mathcal{A}^{(n)}_{l,m}
\end{equation}
and $\mathcal{A}^{(n)}_{l,m}$ is 1-sparse, it follows that on the RHS of (3.21) for each $x \in B_n$ the sum over all $z \in \mathcal{A}^{(n)}_{l+m} \setminus \mathcal{A}^{(n)}_{l+m}$ with $z \sim x$ contains at most one summand. Therefore $\mathbb{P}$-a.s. for $n$ large enough we can pull the square into the inner sum. Then we rearrange both sums and use that for all $z$ we have
\begin{equation}
\sum_{x : x \sim z \in B_n} w_{xz}^2 \leq \pi_z^2.
\end{equation}

By virtue of Lemma 3.13 and Assumption (3.18), for all $z \in \{z_{l,n}, \ldots, z_{l+m-1,n}\}$ we know that $\mathbb{P}$-a.s. for $n$ large enough
\begin{equation}
\frac{\phi_{l,m+1}^{(n)}(z)}{\phi_{l,m}^{(n)}(z)} \leq n^{-\varepsilon/8}.
\end{equation}

Furthermore, $\sum_{z \in \mathcal{A}^{(n)}_{l+m} \setminus \mathcal{A}^{(n)}_{l+m}} \pi_z^2 \leq m\pi_{l+m-1,B_n}^2$. It follows that $\mathbb{P}$-a.s. for $n$ large enough
\begin{equation}
\left\| \mathcal{L}_x^{w}(\phi_{l,m}^{(n)} + 2\mu_{l,m+1}^{(n)}\phi_{l,m+1}^{(n)}) \right\|_{l^2(\mathcal{A}^{(n)}_{l+m})}^2 \leq mn^{-\varepsilon/4} \pi_{l+m-1,B_n}^2.
\end{equation}
Together with (3.20) it follows that $\mathbb{P}$-a.s. for $n$ large enough
\[
\left\| \mathcal{L}^w_{(l+m,n)}u - \mu^{(n)}_{l,m+1} u \right\|_{L^2(\mu^{(n)}_{l+m})}^2 \leq \frac{mn^{-\epsilon/4}}{1 - mn^{-\epsilon/4}} \pi_{l+m-1,B_n}^2.
\] (3.22)
and therefore the claim follows by virtue of Lemma 3.14.

Both Lemmas 3.15 and 3.16 imply the following lemma.

**Lemma 3.17.** Let $\epsilon \in (0, \epsilon_1)$ and $l, m \in \mathbb{N}$. If Assumption 2.1 holds and $\mathbb{P}$-a.s. for $n$ large enough
\[
\phi_{l,j}^{(n)}(z_{(l+j-1,n)}) \geq \sqrt{1 - n^{-\epsilon/4}} \quad \text{for all } 1 \leq j \leq m,
\] (3.23)
then
\[
\mu^{(n)}_{l,m+1} \geq \left( 1 - (2 + \sqrt{m})n^{-\epsilon/8} \right) \pi_{l+m,B_n}.
\] (3.24)

**Proof.** Let us first assume that $\mu^{(n)}_{l,m+1} \leq \mu^{(n)}_{l,m+1}$. Due to Assumption (3.23) we can apply Lemma 3.16. Because of the ordering $\mu^{(n)}_{l,m+1} \leq \mu^{(n)}_{l,m+2} \leq \ldots$, it follows that Relation (3.19) holds with $j = 1$ and $\epsilon = \epsilon$. On the other hand, if $\mu^{(n)}_{l,m+1} > \mu^{(n)}_{l,m+1}$, then (3.17) holds with an index $i \leq m + 1$. Let us now argue why (3.17) holds with exactly $i = m + 1 \mathbb{P}$-a.s. for $n$ large enough. We assume the contrary, i.e., that $i \leq m$ infinitely often as $n$ tends to infinity. Then (3.17) together with (3.12) implies that
\[
\mu^{(n)}_{l,i} \geq \mu^{(n)}_{l,m+1} - n^{-\epsilon_1/4} \pi_{l+m-1,B_n} \geq \left( 1 - 2n^{-\epsilon_1/8} - n^{-\epsilon_1/4} \right) \pi_{l+m,B_n}
\]
Note that (3.6) implies that $\mu^{(n)}_{l,i} \leq \pi_{l+i-1,B_n}$, which we assumed to be less than or equal to $\pi_{l+m-1,B_n}$ infinitely often as $n$ tends to infinity. Thus
\[
\frac{\pi_{l+m-1,B_n}}{\pi_{l+m,B_n}} \geq 1 - 3n^{-\epsilon_1/8}
\]
infinity often as $n$ tends to infinity. This is a contradiction to Lemma 3.11.

Thus, since $\epsilon < \epsilon_1$, it follows regardless of whether $\mu^{(n)}_{l,m+1} \leq \mu^{(n)}_{l,m+1}$ or $\mu^{(n)}_{l,m+1} > \mu^{(n)}_{l,m+1}$ that $\mathbb{P}$-a.s. for $n$ large enough
\[
\left| \mu^{(n)}_{l,m+1} - \mu^{(n)}_{l,m+1} \right| \leq \sqrt{\frac{mn^{-\epsilon/4}}{1 - mn^{-\epsilon/4}} \pi_{l+m-1,B_n}} \leq \sqrt{mn^{-\epsilon/8}} \cdot \pi_{l+m,B_n}.
\] (3.25)
Therefore $\mathbb{P}$-a.s. for $n$ large enough $\mu^{(n)}_{l,m+1}$ is bounded from below by
\[
\mu^{(n)}_{l,m+1} \geq \mu^{(n)}_{l,m+1} - \sqrt{mn^{-\epsilon/8}} \cdot \pi_{l+m,B_n} \geq \left( 1 - (2 + \sqrt{m})n^{-\epsilon/8} \right) \pi_{l+m,B_n}.
\] (3.26)

Now we have the ingredients to prove the main theorem by induction.
4 Proof of the main theorem

By virtue of Lemma 3.4, we already know that

$$\lambda_k^{(n)} \leq \pi_{k,B_n}$$

for all $k \in \mathbb{N}$. In what follows, we further prove (2.4) and that $\mathbb{P}$-a.s. for $n$ large enough

$$\lambda_k^{(n)} \geq \left(1 - n^{-\epsilon/8}\right)\pi_{k,B_n}$$

for all $\epsilon < \epsilon_1$.

We prove the claim by induction over $k$.

**Base case: $k = 1$.** $\mathbb{P}$-a.s. for $n$ large enough we have

$$\psi_1^{(n)}(z_{(1,n)})^2 \geq 1 - n^{-\epsilon_1/4},$$

(4.1)

by virtue of [Fle16, Theorem 1.8] and

$$\lambda_1^{(n)} \geq \left(1 - 2n^{-\epsilon_1/8}\right)\pi_{1,B_n} > \left(1 - n^{-\epsilon/8}\right)\pi_{1,B_n}$$

for all $\epsilon < \epsilon_1$.

(4.2)

by virtue of [Fle16, Equation (5.30)].

**Inductive step: $(k - 1) \rightarrow k$.** Suppose that the claims (2.3) and (2.4) hold for some $k - 1 \in \mathbb{N}$. We now show that this implies that the claims also hold for $k$ instead of $k - 1$.

For (2.3) this already follows by Lemma 3.17 with $l = 1$ and $m = k - 1$. Note that here Condition (3.23) holds for all $\epsilon < \epsilon_1$ and therefore (3.24) holds even without the multiplicative constants. For (2.4) we apply the second part of Lemma 3.14: Let $0 < \delta < \epsilon_1/16$ and

$$\beta_k^{(n)} = 2\sqrt{k-1} n^{-\delta} \pi_{k,B_n}.$$ (4.3)

Since $\pi_{k-1,B_n} \leq \pi_{k,B_n}$, it follows that $\beta_k^{(n)} > \alpha_k^{(n)}$ with

$$\alpha_k^{(n)} := \sqrt{k-1} n^{-\epsilon_1/8} \pi_{k-1,B_n}.$$ (4.4)

Therefore Lemma 3.14 and (3.22) with $l = 1$ and $m = k - 1$ imply that there exists a function $\overline{u}: \mathbb{Z}^d \to \mathbb{R}$ such that

$$\left\|\psi_k^{(n)} - \overline{u}\right\|_{L^2(B_n)} \leq \frac{2\sqrt{k-1} n^{-\epsilon_1/8} \pi_{k-1,B_n}}{\beta_k^{(n)}}$$ (4.5)

where $\overline{u}$ is a linear combination of the eigenvectors $\{\phi_{k,j}\}_{j \geq 1}$ corresponding to the eigenvalues from the interval $[\lambda_k^{(n)} - \beta_k^{(n)}, \lambda_k^{(n)} + \beta_k^{(n)}]$ of the operator $-\mathcal{L}_{w(k,n)}^w$. We now show that $\mathbb{P}$-a.s. for $n$ large enough $\overline{u} = \phi_{k,1}$, i.e., that $\mathbb{P}$-a.s. for $n$ large enough

$$\text{spec } \mathcal{L}_{w(k,n)}^w \cap [\lambda_k^{(n)} - \beta_k^{(n)}, \lambda_k^{(n)} + \beta_k^{(n)}] = \{\mu_{k,1}^{(n)}\}.$$ (4.6)
It suffices to show that \( P \)-a.s. for \( n \) large enough \( \mu_{k,2}^{(n)} > \lambda_k^{(n)} + \beta_k^{(n)} \). We note that Lemma 3.1 implies that

\[
\lambda_k^{(n)} + \beta_k^{(n)} \leq \left(1 + 2\sqrt{k - 1} n^{-\delta}\right) \pi_{k,B_n}.
\]

(4.6)

By virtue of Lemma 3.11 we have \( P \)-a.s. for \( n \) large enough \( \frac{\pi_{k,a_n}}{\pi_{k+1,B_n}} < 1 - 2\sqrt{k - 1} n^{-\delta} \), whence it follows that \( P \)-a.s. for \( n \) large enough

\[
\lambda_k^{(n)} + \beta_k^{(n)} < \left(1 - 4(k - 1)n^{-2\delta}\right) \pi_{k+1,B_n} \leq \mu_{k,2}^{(n)},
\]

where the last inequality follows since by the inductive assumption the relation (3.23) holds for all \( \epsilon < \epsilon_1 \) and therefore (3.24) holds for all \( \epsilon < \epsilon_1 \) with \( l = k \) and \( m = 1 \). Therefore (4.5) is true.

It follows that for any \( 0 < \delta < \epsilon_1/16 \) we have \( P \)-a.s. for \( n \) large enough

\[
\left| \psi_k^{(n)}(z_{(k,n)}) - \phi_{k,1}^{(n)}(z_{(k,n)}) \right| \leq \frac{n^{2\epsilon/8} \pi_{k-1,B_n}}{\pi_{k,B_n}} < n^{2\epsilon/8}.
\]

By virtue of Lemma 3.12, we already know that \( \left| \phi_{k,1}^{(n)}(z_{(k,n)}) \right| \geq \sqrt{1 - n^{-1/4}} \) \( P \)-a.s. for \( n \) large enough. It follows that

\[
\left( \psi_k^{(n)}(z_{(k,n)}) \right)^2 \geq 1 - n^{-1/4} + n^{2\epsilon/4} - 2n^{2\epsilon/8} \geq 1 - 2n^{2\epsilon/8}.
\]

The claim follows since we can choose \( \delta \) arbitrarily small.

5 Asymptotics of the eigenvalues

The proof of Corollary 2.3 extends the proof of [Fle16, Corollary 1.11], which uses the ideas of [Wat54]. To keep the present paper self-contained, we repeat the initial definitions and statements. We define

\[
a_n := \left(n^{\frac{1}{d}} L^*(n)\right)^{-1} = \frac{1}{h(|B_n|)} = \sup \{t: F_\pi(t) = |B_n|^{-1}\}
\]

with \( h \) as in (2.5) and \( L^*(n) \) as in (2.6). Then \( |B_n| = (P[\pi_0 \leq a_n])^{-1} \) and therefore

\[
\lim_{n \to \infty} |B_n| P[\pi_0 \leq a_n\zeta] = \lim_{n \to \infty} \frac{F_\pi(a_n\zeta)}{F_\pi(a_n)} = \zeta^{2d\gamma} \text{ for all } \zeta \geq 0
\]

(5.1)

since \( a_n \to 0 \) as \( n \to \infty \) and \( F_\pi \) varies regularly at zero with index \( 2d\gamma \). We further note that if \( e_1 \in \mathbb{Z}^d \) is a neighbor of the origin, then \( P[\{\pi_0 \leq a_n\zeta\} \cap \{\pi_{e_1} \leq a_n\zeta\}] \leq F(a_n\zeta)^{4d-1} \) since for the event \( \{\pi_0 \leq a_n\zeta\} \cap \{\pi_{e_1} \leq a_n\zeta\} \) at least \( 4d - 1 \) independent conductances \( w \) have to be smaller than or equal to \( a_n\zeta \). Since \( F \) varies regularly at zero with index \( \gamma \), it follows that

\[
|B_n| P[\{\pi_0 \leq a_n\zeta\} \cap \{\pi_{e_1} \leq a_n\zeta\}] \to 0 \quad \text{as } n \to \infty.
\]

(5.2)

We start with the auxiliary Lemma 5.2, for which we need some further definitions. For a set \( A \subset \mathbb{Z}^d \) we define \( CC(A) \) as the set of connected components of \( A \). Furthermore, we define the outer site boundary of the set \( A \) as

\[
\partial A := \{z \in \mathbb{Z}^d \setminus A: \exists x \in A \text{ with } x \sim z\}.
\]

(5.3)

For the natural numbers \( q \leq m \) we further define the number

\[
C_{m,q}^{(n)}(A) := \left|\{M \subset B_n \setminus (A \cap \partial A): |M| = m, |CC(M)| = q\}\right|.
\]

(5.4)
Remark 5.1. Note that if we fix a $k \in \mathbb{N}$, then as $n$ tends to infinity we have $C^{(n)}_{m,m}(A_n) = |B_n|^m / m! + O(|B_n|^{m-1})$ for all sequences of subsets $A_n \subset B_n$ with the constraint $|A_n| = k - 1$. Moreover, for $q \leq m - 1$ there exists a constant $c_q < \infty$ such that for all $n \in \mathbb{N}$ and all sequences of subsets $A_n \subset B_n$ with $|A_n| = k - 1$, we have $C^{(n)}_{m,q}(A_n) < c_q |B_n|^q$. Note that this $c_q$ is independent of the specific choice of $A_n$.

Lemma 5.2. For any fixed $k, l \in \mathbb{N}$ the relations (5.1) and (5.2) imply that

$$\lim_{n \to \infty} \sup_{A_n \subset B_n, |A_n| = k-1} \sum_{l=1}^{m-1} \sum_{q=1}^{m-1} \sum_{M \subset B_n \backslash \{A_n \cup \partial A_n\}, |M| = m, |CC(M)| = q} \mathbb{P}\left[ \bigcap_{x \in M} \{x \in A_n\} \right] = 0 \text{ for all } \zeta \geq 0.$$  

Proof. We are summing over sets $M$ with the constraint $|CC(M)| = q < m = |M|$. This means that here all the sets $M$ contain at least one connected component $C$ with a neighboring pair of sites, i.e., $\mathbb{P}\left[ \bigcap_{x \in C} \{x \in A_n\} \right] \leq \mathbb{P}\left\{ \pi_0 \leq a_n \zeta \right\} \mathbb{P}\left\{ \pi_{e_1} \leq a_n \zeta \right\}$. Since $\pi_x$ and $\pi_y$ are independent if the sites $x$ and $y$ are in two different connected components of $M$, it follows that

$$\lim_{n \to \infty} \mathbb{P}\left[ \bigcap_{x \in M} \{x \in A_n\} \right] \leq \sum_{l=1}^{m-1} \sum_{q=1}^{m-1} C^{(n)}_{m,q}(A_n) \mathbb{P}\{\pi_0 \leq a_n \zeta\}^{q-1} \mathbb{P}\left\{ \pi_0 \leq a_n \zeta \right\} \mathbb{P}\left\{ \pi_{e_1} \leq a_n \zeta \right\}.$$  

By Remark 5.1 there exists a constant $c_q < \infty$ such that $C^{(n)}_{m,q}(A_n) \leq c_q |B_n|^q$ for all sequences of subsets $A_n \subset B_n$ with the constraint that $|A_n| = k - 1$. Therefore the claim follows by (5.1) and (5.2).

Proof of Corollary 2.3. Because of the main theorem it remains to show that

$$\lim_{n \to \infty} \mathbb{P}\left[ \pi_{k,B_n} > \frac{1}{n^{1/2}} L^*(n) \right] = \exp\left(-\zeta^{2d}\right) \sum_{j=0}^{k-1} \frac{\zeta^{2d r_j}}{j!} \text{ for all } \zeta \geq 0.$$  

The proof extends the proof of [Fle16, Corollary 1.11], where we have already shown that

$$\lim_{n \to \infty} \mathbb{P}\left[ \min_{x \in B_n} \pi_x > a_n \zeta \right] = \exp\left(-\zeta^{2d}\right) \text{ for all } \zeta \geq 0$$

by extending the ideas of [Wat54] from $d = 1$ to $d \geq 2$. We will use (5.7) for the inductive base case $k = 1$.

In what follows all the statements hold for all $\zeta \geq 0$. For the inductive step we consider

$$\mathbb{P}[\pi_{k,B_n} > a_n \zeta] = \mathbb{P}[\pi_{k-1,B_n} > a_n \zeta] + \mathbb{P}\{\pi_{k,B_n} > a_n \zeta\} \min_{x \in B_n} \{\pi_{k-1,B_n} \leq a_n \zeta\}.$$  

Let us now assume that the claim (5.6) holds for some $k - 1$. It follows that it remains to show that

$$\lim_{n \to \infty} \mathbb{P}[\pi_{k,B_n} > a_n \zeta, \pi_{k-1,B_n} \leq a_n \zeta] = \frac{\zeta^{2(k-1)d}}{(k-1)!} \exp\left(-\zeta^{2d}\right).$$
Let us start with the decomposition

\[
\mathbb{P}[\pi_{k,B_n} > a_n \zeta, \pi_{k-1,B_n} \leq a_n \zeta] = \sum_{A \subset B_n, |A| = k-1} \mathbb{P}\left[ \bigcap_{x \in A} \{\pi_x \leq a_n \zeta\} \cap \bigcup_{y \in B_n \setminus (A \cap \partial A)} \{\pi_y > a_n \zeta\}\right]
\]

\[
= \sum_{A \subset B_n, |A| = k-1} \mathbb{P}\left[ \bigcap_{x \in A} \{\pi_x \leq a_n \zeta\} \cap \bigcap_{y \in B_n \setminus (A \cap \partial A)} \{\pi_y > a_n \zeta\}\right]
\]

\[
- \sum_{A \subset B_n, |A| = k-1} \mathbb{P}\left[ \bigcap_{x \in A} \{\pi_x \leq a_n \zeta\} \cap \bigcup_{y \in \partial A} \{\pi_y \leq a_n \zeta\}\right].
\tag{5.8}
\]

Let us argue that the second term on the above RHS converges to zero. We observe that

\[
\sum_{A \subset B_n, |A| = k-1} \mathbb{P}\left[ \bigcap_{x \in A} \{\pi_x \leq a_n \zeta\} \cap \left( \bigcup_{y \in \partial A} \{y \leq a_n \zeta\}\right)\right]
\]

\[
\leq \sum_{A \subset B_n, |A| = k-1} \sum_{y \in \partial A} \mathbb{P}\left[ \{\pi_x \leq a_n \zeta\} \cap \{\pi_y \leq a_n \zeta\} \right] \leq \sum_{A \subset B_n, |A| = k-1} \mathbb{P}\left[ \bigcap_{x \in A} \{\pi_x \leq a_n \zeta\}\right]
\]

which converges to zero by virtue of Lemma 5.2.

Let us now consider the first term on the RHS of (5.8). Since for any \(y \in B_n \setminus (A \cap \partial A)\) the random variable \(\pi_y\) is independent of \(\{\pi_x\}_{x \in A}\), the first sum on the RHS of (5.8) is

\[
\sum_{A \subset B_n, |A| = k-1} \mathbb{P}\left[ \bigcap_{x \in A} \{\pi_x \leq a_n \zeta\}\right] \mathbb{P}\left[ \min_{y \in B_n \setminus (A \cap \partial A)} \pi_y > a_n \zeta\right]
\]

\[
\geq \mathbb{P}\left[ \min_{y \in B_n \setminus (A \cap \partial A)} \pi_y > a_n \zeta\right] \sum_{A \subset B_n, |A| = k-1} \mathbb{P}\left[ \bigcap_{x \in A} \{\pi_x \leq a_n \zeta\}\right].
\tag{5.9}
\]

Due to (5.7), the first factor in the above RHS converges to \(e^{-\zeta^2 d^2 y}\). As a part of the proof for (5.7), we have also shown that the second factor converges to \(\zeta^{2(k-1)d^2} / (k-1)!\). It thus remains to find an upper bound for the LHS of (5.9). Similar to the proof of (5.7), we let \(l\) be an even integer and estimate for all sequences of subsets \(A_n \subset B_n\) with the constraint \(|A_n| = k - 1\) that

\[
\mathbb{P}\left[ \min_{y \in B_n \setminus (A_n \cap \partial A_n)} \pi_y > a_n \zeta\right] \leq 1 + \sum_{m=1}^{l} (-1)^m \sum_{M \subset B_n \setminus (A_n \cap \partial A_n), |M| = m} \mathbb{P}\left[ \bigcap_{x \in M} \{\pi_x \leq a_n \zeta\}\right]
\]

\[
= 1 + \sum_{m=1}^{l} (-1)^m \sum_{M \subset B_n \setminus (A_n \cap \partial A_n), |M| = m, \text{CC}(M) = m} \mathbb{P}\left[ \bigcap_{x \in M} \{\pi_x \leq a_n \zeta\}\right]
\]

\[
+ \sum_{m=1}^{l} \sum_{q=1}^{m-1} \sum_{M \subset B_n \setminus (A_n \cap \partial A_n), |M| = m, \text{CC}(M) = q} \mathbb{P}\left[ \bigcap_{x \in M} \{\pi_x \leq a_n \zeta\}\right]
\]
According to Lemma 5.2, the supremum of the last sum on the above RHS taken over all sequences $A_n \subset B_n$ with $|A_n| = k - 1$ converges to zero. For the first sum we observe that since $|CC(M)| = |M|$, the set $M$ is sparse and therefore $\{\pi_x\}_{x \in M}$ is a set of independent random variables. It follows that

$$\sum_{m=1}^{l} (-1)^m \sum_{M \subset B_n \setminus \{A_n \cap \partial A_n\}, \quad CC(M) = m} \mathbb{P}\left( \bigcap_{x \in M} \{\pi_x \leq a_n \xi\} \right) = \sum_{m=1}^{l} (-1)^m C_{m,m}^{(n)} (A) \mathbb{P}[\pi_0 \leq a_n \xi]^m$$

$$= \sum_{m=1}^{l} (-1)^m \left( |B_n|^m / m! + O\left(|B_n|^m \right) \right) \mathbb{P}[\pi_0 \leq a_n \xi]^m$$

by Remark 5.1. Taking the supremum over all sequences of subsets $A_n \subset B_n$ with the constraint $|A_n| = k - 1$, this still converges to $\sum_{m=0}^{l} \xi^{2d} \gamma^m / m!$. Since this holds for every $l \in 2\mathbb{N}$ and we already have the lower bound (5.9), the claim follows.

\[\square\]

A Appendix

For better readability we have shifted a rather lengthy computation in the proof of Lemma 3.16 to this appendix. We start by inserting the definition of the Laplacian, i.e.,

$$\sum_{x \in \mathcal{B}_{l+m}^{(n)}} \left( \mathcal{L}_{l+m}^{(n)} \phi_{l,m+1}^{(n)} (x) + \mu_{l,m+1}^{(n)} \phi_{l,m+1}^{(n)} (x) \right)^2$$

$$= \sum_{x \in \mathcal{B}_{l+m}^{(n)}} \left( \sum_{z: z \sim x} w_{xz} \left( \phi_{l,m+1}^{(n)} (z) - \phi_{l,m+1}^{(n)} (x) \right) + \mu_{l,m+1}^{(n)} \phi_{l,m+1}^{(n)} (x) \right)^2$$

Now we rearrange the terms in order to cancel $\mu_{l,m+1}^{(n)} \phi_{l,m+1}^{(n)} (x)$, i.e.,

$$\text{LHS} = \sum_{x \in \mathcal{B}_{l+m}^{(n)}} \left( \sum_{z: z \sim x} w_{xz} \left( \phi_{l,m+1}^{(n)} (z) - \phi_{l,m+1}^{(n)} (x) \right) + \mu_{l,m+1}^{(n)} \phi_{l,m+1}^{(n)} (x) \right.$$ \[A.1\]

$$\left. - \sum_{z: z \sim x} w_{xz} \phi_{l,m+1}^{(n)} (z)^2 \right),$$

where the first two terms cancel. The last term simplifies to

$$\text{LHS} = \sum_{x \in \mathcal{B}_{l+m}^{(n)}} \left( \sum_{z \in \mathcal{B}_{l+m}^{(n)} \setminus \mathcal{B}_{l+m}^{(n)}: z \sim x} w_{xz} \phi_{l,m+1}^{(n)} (z) \right)^2$$

$$\leq \max_{z \in \mathcal{B}_{l+m}^{(n)} \setminus \mathcal{B}_{l+m}^{(n)}} \left( \phi_{l,m+1}^{(n)} (z) \right)^2 \sum_{x \in B_n} \left( \sum_{z \in \mathcal{B}_{l+m}^{(n)} \setminus \mathcal{B}_{l+m}^{(n)}: z \sim x} w_{xz} \right)^2. \quad (A.1)$$

Acknowledgement. I am grateful to Wolfgang König for his very useful suggestions.
References


