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Abstract

We generalize our former localization result about the principal Dirichlet eigenvector of the i.i.d. heavy-tailed random conductance Laplacian to the first k eigenvectors. We overcome the complication that the higher eigenvectors have fluctuating signs by invoking the Bauer-Fike theorem to show that the kth eigenvector is close to the principal eigenvector of an auxiliary spectral problem.

1 Introduction

Let us consider the random conductance Laplacian $\mathcal{L}^w$ acting on real-valued functions $f \in \ell^2(\mathbb{Z}^d)$ as

$$(\mathcal{L}^w f)(x) = \sum_{y\colon |x-y|_1=1} w_{xy}(f(y) - f(x)) \qquad (x \in \mathbb{Z}^d) \tag{1.1}$$

with positive independent and identically distributed random conductances w_{xy}. As usual, we further assume that the operator $\mathcal{L}^w$ is self-adjoint, i.e. $w_{xy} = w_{yx}$. Our goal is to describe the almost-sure behavior of the solution to the spectral problem

$$\begin{aligned} -\mathcal{L}^w\psi &= \lambda\psi \quad \text{on } B_n = [-n,n]^2 \cap \mathbb{Z}^d\,, \\ \psi &= 0 \qquad \text{else.} \end{aligned} \tag{1.2}$$

as the box size n tends to infinity. This means that we are interested in the Dirichlet eigenfunctions and eigenvalues of the operator $-\mathcal{L}^w$ in the box B_n with zero Dirichlet conditions.

In the recent paper [Fle16], we have shown that if $\gamma := \sup\{q \geq 0\colon \mathbb{E}[w^{-q}] < \infty\} < 1/4$ and certain regularity assumptions apply, then the principal Dirichlet eigenvector $\psi_1^{(n)}$ of Problem (1.2) concentrates in a single site as n tends to infinity. To be more precise, let $\pi_z = \sum_{x\colon x\sim z} w_{xz}$ be the local speed measure, i.e., the inverse mean waiting time of the random walk generated by $\mathcal{L}^w$. Then the principal Dirichlet eigenvector $\psi_1^{(n)}$ approaches the δ-function in the site $z_{(1,n)}$ that minimizes the local speed measure π over the box B_n. Furthermore, the principal Dirichlet eigenvalue $\lambda_1^{(n)}$ is asymptotically equivalent to the minimum $\pi_{1,B_n} = \min_{z\in B_n} \pi_z$.

If, on the other hand, $\gamma > 1/4$, then the authors of [FHS17] have proved that the top of the Dirichlet spectrum of $\mathcal{L}^w$ homogenizes. The spectrum of the random conductance Laplacian thus displays a dichotomy between a localized and a homgenized phase.

In the present paper we generalize our findings for $\gamma < 1/4$ to the first k Dirichlet eigenvectors and eigenvalues. More precisely, we show that the kth Dirichlet eigenvector $\psi_k^{(n)}$ concentrates in the site that attains the kth minimum of π. Consequently, the kth Dirichlet eigenvalue $\lambda_k^{(n)}$ is asymptotically equivalent to the kth minimum of π. If the conductances vary regularly at zero with positive index, then despite the dependence structure of the random field $\{\pi_x\}_{x\in\mathbb{Z}^d}$, this kth minimum converges weakly as if $\{\pi_x\}_{x\in\mathbb{Z}^d}$ was an independent field, see the proof of Corollary 2.3. It follows that, in this case, the properly rescaled kth eigenvalue $\lambda_k^{(n)}$ converges in distribution to a non-degenerate random variable. This relates to a similar result in dimension $d = 1$, see [Fag12, Theorem 2.5(i)].

Note that the only reason why we have not generalized our findings to the first k eigenvectors in [Fle16], is that in [Fle16, Lemma 5.6] we rely on the property that the principal Dirichlet eigenvector does not change its sign, according to the Perron-Frobenius theorem. This is no longer true for the higher order eigenvectors. To overcome this difficulty, we now approximate the first k eigenvectors to (1.2) by auxiliary principal eigenvectors using the Bauer-Fike theorem, see Lemma 3.14.

Our results for the random conductance Laplacian compare well to similar results of the random Schrödinger operator $\Delta + \xi$ with random potential $\xi\colon \mathbb{Z}^d \to \mathbb{R}$, see [BK16] and [Ast16, Ch. 6]. To keep the present paper as short as possible, we refer the reader to our first article [Fle16] for more heuristics and references. However, we kept the present paper mostly self-contained.

Model and main objects

We consider the lattice with vertex set $\mathbb{Z}^d$ ($d \geq 2$) and edge set $\mathfrak{E}_d = \{\{x,y\} : x,y \in \mathbb{Z}^d, |x-y|_1 = 1\}$. If two sites $x,y \in \mathbb{Z}^d$ are neighbors according to $\mathfrak{E}_d$, we also write $x \sim y$. To each edge $e \in \mathfrak{E}_d$ we assign a positive random variable w_e. In analogy to a d-dimensional resistor network, we call these random weights w_e *conductances*. We take $(\Omega, \mathcal{F}) = \big((0,\infty)^{\mathfrak{E}_d}, \mathcal{B}((0,\infty))^{\otimes \mathfrak{E}_d}\big)$ as the underlying measurable space and assume that an environment $\boldsymbol{w} = (w_e)_{e\in\mathfrak{E}_d} \in \Omega$ is a family of i.i.d. positive random variables with law $\mathbb{P}$. We denote the expectation with respect to $\mathbb{P}$ by $\mathbb{E}$.

If e is the edge between the sites $x,y \in \mathbb{Z}^d$, we also write w_{xy} or $w_{x,y}$ instead of w_e. Note that by definition of the edge set $\mathfrak{E}_d$, the edges are undirected, whence $w_{xy} = w_{yx}$. If we want to refer to an arbitrary copy of the conductances in general, we simply write w, i.e., for a set $A \in \mathcal{B}((0,\infty))$, the expression $\mathbb{P}[w \in A]$ equals $\mathbb{P}[w_e \in A]$ for an arbitrary edge e.

We call

$$F\colon [0,\infty) \to [0,1]\colon u \mapsto \mathbb{P}[w \leq u] \tag{1.3}$$

the distribution function of the conductances.

For an arbitrary $k \in \mathbb{N}$, our goal is to study the behavior of the first k Dirichlet eigenvalues $\lambda_1^{(n)} \leq \ldots \leq \lambda_k^{(n)}$ and eigenvectors $\psi_1^{(n)}, \ldots, \psi_k^{(n)}$ of the sign-inverted generator $-\mathcal{L}_{\boldsymbol{w}}$ in the ball

$$B_n := \big\{x \in \mathbb{Z}^d\colon |x|_\infty \leq n\big\} = [-n,n]^d \cap \mathbb{Z}^d \tag{1.4}$$

with zero Dirichlet conditions at the boundary.

For a subset $A \subset \mathbb{Z}^d$ we define the function space

$$\ell^2(A) := \left\{ f\colon \mathbb{Z}^d \to \mathbb{R} \text{ such that supp } f \subseteq A \text{ and } \sum_{x\in A} f(x)^2 < \infty \right\} \subset \ell^2(\mathbb{Z}^d)\,, \tag{1.5}$$

where we let “supp f” denote the support of the function f. Accordingly, for functions $f_1, f_2 \in \ell^2(\mathbb{Z}^d)$ we define the scalar product

$$\langle f_1, f_2\rangle_{\ell^2(A)} = \sum_{x\in A} f_1(x) f_2(x)\,.$$

For a real-valued function $f \in \ell^2(\mathbb{Z}^d)$ let us define the Dirichlet energy $\mathcal{E}^{\boldsymbol{w}}(f)$ with respect to the operator $-\mathcal{L}_{\boldsymbol{w}}$ by

$$\mathcal{E}^{\boldsymbol{w}}(f) = \langle f, -\mathcal{L}_{\boldsymbol{w}} f \rangle_{\ell^2(\mathbb{Z}^d)} . \tag{1.6}$$

Then, according to the Courant-Fischer theorem, the kth Dirichlet eigenvalue is given by the variational formula

$$\lambda_k^{(n)} = \inf_{\substack{\mathcal{M} \leq \ell^2(B_n),\\ \dim \mathcal{M} = k}} \sup_{\substack{f \in \mathcal{M},\\ \|f\|_2 = 1}} \mathcal{E}^{\boldsymbol{w}}(f) \tag{1.7}$$

where $\mathcal{M} \leq \ell^2(B_n)$ means that $\mathcal{M}$ is a linear subspace of $\ell^2(B_n)$. Note that $\lambda_k^{(n)} = \mathcal{E}^{\boldsymbol{w}}\big(\psi_k^{(n)}\big)$.

Definition 1.1 (Local speed measure and its order statistics). *We define the local speed measure π by*

$$\pi_z = \sum_{x\colon x \sim z} w_{xz} \qquad (z \in \mathbb{Z}^d) \tag{1.8}$$

and we label the order statistics of the set $\{\pi_z\}_{z \in B_n}$ by

$$\pi_{1,B_n} \leq \pi_{2,B_n} \leq \ldots \leq \pi_{|B_n|,B_n} . \tag{1.9}$$

Furthermore, for $k, n \in \mathbb{N}$ let $z_{(k,n)}$ be the site where π attains its kth minimum over B_n, i.e., $\pi_{z_{(k,n)}} = \pi_{k,B_n}$.

Remark 1.2. *If F is continuous, then $\pi_{1,B_n} < \pi_{2,B_n} < \ldots < \pi_{|B_n|,B_n}$ $\mathbb{P}$-a.s. and therefore the minimizers $z_{(k,n)}$ are $\mathbb{P}$-a.s. unique.*

2 Main result

In what follows we let

$$g : [0,\infty) \to [0,\infty) \colon u \mapsto \sup\left\{s \geq 0 \colon F(s) = u^{-1/2}\right\} . \tag{2.1}$$

Assumption 2.1. *Let F be continuous and vary regularly at zero with index $\gamma \in [0, 1/4)$. Assume that there exists $a^* > 0$ such that $F(ab) \geq bF(a)$ for all $a \leq a^*$ and all $0 \leq b \leq 1$. In the case where $\gamma = 0$, we assume additionally that there exists $\epsilon_1 \in (0,1)$ such that the product $n^{2+\epsilon_1} g(n)$ converges monotonically to zero as n grows to infinity.*

Remark 2.2. *In the case where $\gamma > 0$, it follows that $(1/F(1/s))^2$ varies regularly at infinity with index 2γ. Further, $(1/F(1/s))^2$ diverges as $s \to \infty$. It follows by virtue of [Res87, Prop. 0.8(v)] that $1/g(u) = \inf\{s \geq 0 \colon (1/F(1/s))^2 = u\}$ varies regularly at infinity with index $1/(2\gamma)$ and thus g varies regularly at infinity with index $-1/(2\gamma)$. Since in addition $\gamma < 1/4$, there exists $\epsilon_1 \in (0,1)$ such that $-1/(2\gamma) < -(2+\epsilon_1)$.*

Theorem. *Let $k \in \mathbb{N}$. If Assumption 2.1 holds, then the kth Dirichlet eigenvalue $\lambda_k^{(n)}$ with zero Dirichlet conditions outside the box B_n fulfills*

$$\mathbb{P}\left[\lim_{n\to\infty} \frac{\lambda_k^{(n)}}{\pi_{k,B_n}} = 1\right] = 1 \tag{2.2}$$

and the mass of the kth Dirichlet eigenvector $\psi_k^{(n)}$ asymptotically concentrates in the site $z_{(k,n)}$. More precisely, if $\epsilon_1 > 0$ is as in Assumption 2.1 or Remark 2.2, then $\mathbb{P}$-a.s. for n large enough

$$1 - n^{-\epsilon/8} \le \frac{\lambda_k^{(n)}}{\pi_{k,B_n}} \le 1 \qquad \text{for all } \epsilon < \epsilon_1 \tag{2.3}$$

and

$$\psi_k^{(n)}\big(z_{(k,n)}\big) \ge \sqrt{1 - n^{-\epsilon/4}} \qquad \text{for all } \epsilon < \epsilon_1\,. \tag{2.4}$$

We prove this theorem in Section 4.

Similar to [Fle16, Corollary 1.11], we can now infer the weak convergence of the eigenvalues. Let F_π be the distribution function of the random variable π, i.e., the distribution function of the sum of $2d$ independent copies of the conductance w. Note that since F is continuous, F_π is continuous as well. As in [Fle16, (1.18)], we define

$$h\colon (0,\infty) \to (0,\infty)\colon u \mapsto \inf\left\{s\colon \frac{1}{F_\pi(1/s)} = u\right\}. \tag{2.5}$$

Let F vary regularly at zero with index $\gamma > 0$. Then by virtue of [Fle16, Lemma 5.8], it follows that F_π varies regularly at zero with index $2d\gamma$. It thus follows by virtue of [Res87, Proposition 0.8(v)] that h varies regularly at infinity with index $1/(2d\gamma)$. Therefore there exists a function L^* that varies slowly at infinity such that

$$h(|B_n|) = n^{\frac{1}{2\gamma}} L^*(n)\,. \tag{2.6}$$

Corollary 2.3. *Assume that F fulfills Assumption 2.1 with $\gamma > 0$ and let L^* be as in (2.6). Let $k \in \mathbb{N}$. Then as n tends to infinity, the product $L^*(n) n^{\frac{1}{2\gamma}} \lambda_k^{(n)}$ converges in distribution to a non-degenerate random variable. More precisely,*

$$\lim_{n\to\infty} \mathbb{P}\Big[L^*(n) n^{\frac{1}{2\gamma}} \lambda_k^{(n)} > \zeta\Big] = \exp\big(-\zeta^{2d\gamma}\big) \sum_{j=0}^{k-1} \frac{\zeta^{2d\gamma j}}{j!} \qquad \text{for all } \zeta \in [0,\infty)\,. \tag{2.7}$$

This corollary extends [Fle16, Corollary 1.11] to general $k \in \mathbb{N}$. We prove it at the end of Section 5.

3 Auxiliary spectral problems

Definition 3.1 (Auxiliary lattice and Laplacian)**.** *We define the set*

$$\mathscr{B}_l^{(n)} = B_n \setminus \big\{z_{(1,n)}, \ldots, z_{(l-1,n)}\big\} \tag{3.1}$$

and abbreviate the operator $\mathcal{L}^w$ with zero Dirichlet conditions outside $\mathscr{B}_l^{(n)}$ as $\mathcal{L}^w_{(l,n)}$, i.e., we define

$$\mathcal{L}^w_{(l,n)} := \mathbb{1}_{\mathscr{B}_l^{(n)}}\, \mathcal{L}^w\, \mathbb{1}_{\mathscr{B}_l^{(n)}}\,, \tag{3.2}$$

where the operator $\mathbb{1}_{\mathscr{B}_l^{(n)}}$ is the identity on $\mathscr{B}_l^{(n)}$ and zero otherwise.

Since the operator $-\mathcal{L}^w$ is self-adjoint, the operator $-\mathcal{L}^w_{(l,n)}$ is self-adjoint as well. This justifies the next definition.

Definition 3.2 (Auxiliary eigenvectors and values)**.** *We define the eigenvalues of the operator $-\mathcal{L}^w_{(l,n)}$ restricted to $\ell^2\left(\mathscr{B}_l^{(n)}\right)$ by*

$$\mu_{l,1}^{(n)} \leq \mu_{l,2}^{(n)} \leq \ldots \leq \mu_{l,|\mathscr{B}_l^{(n)}|}^{(n)} \tag{3.3}$$

and its eigenvectors by

$$\phi_{l,1}^{(n)}, \phi_{l,2}^{(n)}, \ldots, \phi_{l,|\mathscr{B}_l^{(n)}|}^{(n)} \in \ell^2\left(\mathscr{B}_l^{(n)}\right) \qquad \textit{with} \quad \left\langle \phi_{l,i}^{(n)}, \phi_{l,j}^{(n)} \right\rangle = \delta_{ij}\,. \tag{3.4}$$

Note that $\mathscr{B}_1^{(n)} = B_n$ and thus $\mu_{1,k}^{(n)} = \lambda_k^{(n)}$ and $\phi_{1,k}^{(n)} = \psi_k^{(n)}$. Moreover the variational formula for the auxiliary eigenvalues reads

$$\mu_{l,m}^{(n)} = \inf_{\substack{\mathcal{M} \leq \ell^2(\mathscr{B}_l^{(n)}),\\ \dim \mathcal{M} = m}} \sup_{\substack{f \in \mathcal{M},\\ \|f\|_2 = 1}} \mathcal{E}^{\boldsymbol{w}}(f)\,. \tag{3.5}$$

Remark 3.3 (Perron-Frobenius)**.** *For a given box B_n the operator $\mathcal{L}^w_{(l,n)}$ can be written as a $(|B_n| - l + 1) \times (|B_n| - l + 1)$-matrix with non-negative entries everywhere except on the diagonal. Since the matrix is finite-dimensional, we can add a multiple of the identity to obtain a non-negative primitive matrix without changing the matrix' spectrum. By the Perron-Frobenius theorem (see e.g. [Sen81, Ch. 1]) it follows that its principal eigenvalue $-\mu_{l,1}^{(n)}$ is simple and we can assume without loss of generality that its principal eigenvector is positive, which implies that $\phi_{l,1}^{(n)}$ is nonnegative.*

Lemma 3.4. *For any $l \in \mathbb{N}$ and $m \in \{1, \ldots, |B_n| - l + 1\}$ the eigenvalue $\mu_{l,m}^{(n)}$ is bounded from above by*

$$\mu_{l,m}^{(n)} \leq \pi_{l+m-1,B_n}\,. \tag{3.6}$$

Proof. We choose

$$\mathcal{M} = \text{span}\left\{\delta_{z_{(l,n)}}, \delta_{z_{(l+1,n)}}, \ldots, \delta_{z_{(l+m-1,n)}}\right\}$$

and insert it as a test space into the variational formula (3.5). □

3.1 Principal eigenvectors

The following lemma is the analogue of [Fle16, Lemma 5.6], where we need the Perron-Frobenius property.

Lemma 3.5. *Let $k \in \mathbb{N}$ and let $y, z \in B_n \cap \mathscr{B}_k^{(n)}$ with $\pi_z < \pi_y$ and $y \nsim z$. Assume that $\phi_{k,1}^{(n)}$ is nonnegative. Further, define $m_y = 2\max_{x\colon x\sim y} \phi_{k,1}^{(n)}(x)$. Then the mass $\phi_{k,1}^{(n)}(y)$ is bounded from above by*

$$\phi_{k,1}^{(n)}(y) \leq \frac{m_y}{1 - \frac{\pi_z}{\pi_y}}\,. \tag{3.7}$$

The proof of this lemma is analogous to the proof of [Fle16, Lemma 5.6] and therefore we omit it here.

For the convenience of the reader, we now repeat some definitions from [Fle16]. For a function $g : (0,\infty) \to (0,\infty)$ and $n \in \mathbb{N}$ we define a percolation environment $\tilde{\boldsymbol{w}}_{g(n)}$ by setting

$$\tilde{w}_{g(n)}(e) := w_e \mathbb{1}_{\{w_e > g(n)\}} \qquad (e \in \mathfrak{E}_d)\,. \tag{3.8}$$

Thus, edges with conductance less than or equal to $g(n)$ are considered to be closed and all others keep their original conductance. With this terminology we can now define the following clusters.

Definition 3.6. *For a fixed function g and a fixed $\epsilon > 0$, let $\mathscr{D}^{(n)}$ be the unique infinite open cluster of the environment $\tilde{\boldsymbol{w}}_{g(n^{1-\epsilon})}$ and let $\mathscr{I}^{(n)} = B_n \backslash \mathscr{D}^{(n)}$ be its set of holes in B_n.*

Definition 3.7. *We call a set $\mathscr{I} \subset \mathbb{Z}^d$* **sparse** *if the set $\mathscr{I}$ does not contain any neighboring sites. Further, a set $\mathscr{I} \subset \mathbb{Z}^d$ is* **b-sparse** *if for any $z \in \mathbb{Z}^d$ the box $B_b(z) := \left\{x \in \mathbb{Z}^d\colon |x-z|_\infty \le b\right\} \subset \mathbb{Z}^d$ contains at most one site of the set $\mathscr{I}$.*

Remark 3.8. *Let $b_1 < b_2$ be natural numbers. If a set $\mathscr{I} \subset \mathbb{Z}^d$ is b_2-sparse, it is also b_1-sparse and sparse.*

Let us collect some facts that we already know about the cluster $\mathscr{D}^{(n)}$ and the set $\mathscr{I}^{(n)}$ from [Fle16].

Remark 3.9. *Let us recall that in Assumption 2.1 we assume that one of the two following cases occurs: $\gamma \in (0,1/4)$ or $\gamma = 0$ and there exists $\epsilon_1 \in (0,1)$ such that the product $n^{2+\epsilon_1} g(n)$ converges monotonically to zero as n grows to infinity. In the case where $\gamma \in (0,1/4)$, we define ϵ_1 as in Remark 2.2.*

In both cases we define $\mathscr{D}^{(n)}$ and $\mathscr{I}^{(n)}$ as in Definition 3.6 with $\epsilon = \epsilon_2 := \frac{7\epsilon_1}{8(2+\epsilon_1)}$. By virtue of [Fle16, Lemma 5.4] and Remark 3.8 we know that for any fixed $b \in \mathbb{N}$ the set $\mathscr{I}^{(n)}$ is b-sparse and therefore sparse $\mathbb{P}$-a.s. for n large enough in the sense of Definition 3.7. Moreover, [Fle16, Lemma 5.4] implies that for any $k \in \mathbb{N}$ we have $\mathbb{P}$-a.s. for n large enough $z_{(1,n)}, \dots, z_{(k+1,n)} \in \mathscr{I}^{(n)}$ and thus $\mathbb{P}$-a.s. for n large enough there is no pair of neighbors among the the sites $z_{(1,n)}, \dots, z_{(k+1,n)}$. Since F is continuous, the sites $z_{(1,n)}, \dots, z_{(k+1,n)}$ are $\mathbb{P}$-a.s. unique.

The next lemma about the principal Dirichlet eigenvector $\phi_{k,1}^{(n)}$ of the auxiliary operator $-\mathcal{L}^w_{(k,n)}$ is very similar to [Fle16, Lemma 5.5]. Indeed, we can nearly copy the proof since the deleted sites $z_{(1,n)}, \dots, z_{(k-1,n)}$ are in $\mathscr{I}^{(n)}$, see Remark 3.9.

Lemma 3.10. *Let the function g be as in (2.1). Assume that there exists $\epsilon_1 \in (0,1)$ such that one of the two cases occurs: g varies regularly at infinity with index $\rho < -(2+\epsilon_1)$ or the product $n^{2+\epsilon_1} g(n)$ converges monotonically to zero as n grows to infinity. Further, let $\epsilon = \epsilon_2 := \frac{7\epsilon_1}{8(2+\epsilon_1)}$ and $\mathscr{D}^{(n)}$ be as in Definition 3.6. Then $\mathbb{P}$-a.s. for n large enough*

$$\left\|\phi_{k,1}^{(n)}\right\|^2_{\ell^2(\mathscr{D}^{(n)})} \le n^{-\epsilon_1/2}\,. \tag{3.9}$$

Proof. The proof follows the lines of the proof of [Fle16, Lemma 5.5] until *right before* (5.8). Here, we then apply Lemma 3.4 to infer that

$$\pi_{k,B_n} \ge \mu_{k,1}^{(n)} = \mathcal{E}^{\boldsymbol{w}}\left(\phi_{k,1}^{(n)}\right).$$

Moreover, by virtue of [Fle16, Lemma 2.6] there exists $c_1 < \infty$ such that $\mathbb{P}$-a.s. for n large enough

$$c_1 g(n^{1-\epsilon_3}) \ge \pi_{k,B_n}$$

with $\epsilon_3 = \epsilon_1(8(2+\epsilon_1))^{-1}$. The rest of the proof follows again the lines of the proof of [Fle16, Lemma 5.5]. □

From Lemma 3.10 to localization in a single site, the main two ingredients are Lemma 3.5 and the following result about the order statistics of $\{\pi_x\}_{x\in B_n}$.

Lemma 3.11 ([Fle16, Lemma 5.10])**.** *Let Assumption 2.1 be true and let $\varepsilon > 0$ and $k \in \mathbb{N}$. Then $\mathbb{P}$-a.s. for n large enough*

$$1 - \frac{\pi_{k,B_n}}{\pi_{k+1,B_n}} > n^{-\varepsilon}. \tag{3.10}$$

The next lemma therefore follows.

Lemma 3.12. *Let $k \in \mathbb{N}$. Under Assumption 2.1, it follows that $\mathbb{P}$-a.s. for n large enough*

$$\phi^{(n)}_{k,1}\big(z_{(k,n)}\big) \geq \sqrt{1 - n^{-\epsilon_1/4}}. \tag{3.11}$$

This implies that $\mathbb{P}$-a.s. for n large enough

$$\mu^{(n)}_{k,1} \geq \big(1 - 2n^{-\epsilon_1/8}\big)\pi_{k,B_n}. \tag{3.12}$$

Proof. In view of Remark 3.9, Lemma 3.5 and the extreme value result Lemma 3.11, the proof of (3.11) is completely analogous to the proof of [Fle16, Theorem 1.8] and thus we omit it here. For (3.12) we observe that since $\mu^{(n)}_{k,1} = \langle \phi^{(n)}_{k,1}, \mathcal{L}^w \phi^{(n)}_{k,1}\rangle$ it follows that $\mathbb{P}$-a.s. for n large enough

$$\mu^{(n)}_{k,1} \geq \sum_{x:\, x\sim z_{(k,n)}} w_{xz_{(k,n)}} \Big(\phi^{(n)}_{k,1}\big(z_{(k,n)}\big) - \phi^{(n)}_{k,1}(x)\Big)^2 \geq \Big(n^{-\epsilon_1/8} - \sqrt{1 - n^{-\epsilon_1/4}}\Big)^2 \pi_{z_{(k,n)}}.$$

□

3.2 Orthogonality of eigenvectors

The next very simple ingredient of our proof is due to the orthogonality of the eigenvectors.

Lemma 3.13. *Let $\varepsilon > 0$, let $j, l, m, n \in \mathbb{N}$ with $j < m$ and let $\phi^{(n)}_{l,j}(z) \geq \sqrt{1 - n^{-\varepsilon/4}}$.*

$$\Big|\phi^{(n)}_{l,m}(z)\Big| \leq n^{-\varepsilon/8}. \tag{3.13}$$

Proof. For $n = 1$ the claim is immediate. For $n \geq 2$ we observe that since the eigenvectors $\phi^{(n)}_{l,j}$ and $\phi^{(n)}_{l,m}$ are orthogonal to each other, it follows that

$$\phi^{(n)}_{l,m}(z) = -\frac{\sum_{x\neq z} \phi^{(n)}_{l,j}(x)\phi^{(n)}_{l,m}(x)}{\phi^{(n)}_{l,j}(z)}.$$

By the Cauchy-Schwarz inequality it follows that for n greater than one

$$\Big(\phi^{(n)}_{l,m}(z)\Big)^2 \leq \frac{\Big(\sum_{x\neq z}\Big(\phi^{(n)}_{l,j}(x)\Big)^2\Big)\Big(1 - \Big(\phi^{(n)}_{l,m}(z)\Big)^2\Big)}{\Big(\phi^{(n)}_{l,j}(z)\Big)^2} \leq \frac{n^{-\varepsilon/4}}{1 - n^{-\varepsilon/4}}\Big(1 - \Big(\phi^{(n)}_{l,m}(z)\Big)^2\Big)$$

where we have also used that the assumption implies that $\sum_{x\neq z}\Big(\phi^{(n)}_{l,j}(x)\Big)^2 \leq n^{-\varepsilon/4}$. The claim follows. □

3.3 Higher eigenvalues and -vectors

We establish the connection to the original eigenvalues and -vectors via the Bauer-Fike theorem [BF60], which we cite below from [JKO94, Lemma 11.2].

Lemma 3.14 ([JKO94, Lemma 11.2])**.** *Let $A\colon H \to H$ be a linear self-adjoint compact operator in a Hilbert space H. Let $\mu \in \mathbb{R}$, and let $u \in H$ be such that $\|u\|_H = 1$ and*

$$\|Au - \mu u\|_H \le \alpha\,, \qquad \alpha > 0\,. \tag{3.14}$$

Then there exists an eigenvalue μ_i of the operator A such that

$$|\mu_i - \mu| \le \alpha\,. \tag{3.15}$$

Moreover, for any $\beta > \alpha$, there exists a vector $\overline{u}$ such that

$$\|u - \overline{u}\|_H \le 2\alpha\beta^{-1}\,, \qquad \|\overline{u}\|_H = 1 \tag{3.16}$$

and $\overline{u}$ is a linear combination of the eigenvectors of operator A corresponding to the eigenvalues from the interval $[\mu - \beta, \mu + \beta]$.

Here comes the first application of Lemma 3.14.

Lemma 3.15. *Let $l \in \mathbb{N}$ and $m \in \{1, \dots, |B_n| - l + 1\}$. Under Assumption 2.1 there exists $i \in \{1, \dots, |B_n| - l + 1\}$ such that*

$$\left|\mu_{l,i}^{(n)} - \mu_{l+m,1}^{(n)}\right| \le n^{-\epsilon_1/4} \cdot \pi_{l+m-1,B_n}\,. \tag{3.17}$$

Proof. We aim to apply Lemma 3.14 with the operator $A = -\mathcal{L}^w_{(l,n)}$, the Hilbert space $H = \ell^2(\mathscr{B}_l^{(n)})$, the value $\mu = \mu_{l+m,1}$ and the vector $u = \phi^{(n)}_{l+m,1}$. First, we note that $\|\phi^{(n)}_{l+m,1}\|_{\ell^2(\mathscr{B}_l^{(n)})} = 1$. Next, we recall that $\phi^{(n)}_{l+m,1}$ is an eigenvector of the operator $-\mathcal{L}^w_{(l+m,n)}$ to the eigenvalue $\mu^{(n)}_{l+m,1}$ and therefore

$$\left\|\mathcal{L}^w_{(l,n)}\phi^{(n)}_{l+m,1} + \mu^{(n)}_{l+m,1}\phi^{(n)}_{l+m,1}\right\|^2_{\ell^2(\mathscr{B}_l^{(n)})} = \sum_{z \in \mathscr{B}_l^{(n)} \setminus \mathscr{B}_{l+m}^{(n)}} \left(\mathcal{L}^w_{(l,n)}\phi^{(n)}_{l+m,1}(z) + \mu^{(n)}_{l+m,1}\phi^{(n)}_{l+m,1}(z)\right)^2,$$

where all other summands vanish. Note that $\mathscr{B}_l^{(n)} \setminus \mathscr{B}_{l+m}^{(n)} = \left\{z_{(l,n)}, \dots z_{(l+m-1,n)}\right\}$ and by definition we have $\phi^{(n)}_{l+m,1}(z) = 0$ for all $z \in \left\{z_{(l,n)}, \dots z_{(l+m-1,n)}\right\}$. It follows that for all $z \in \mathscr{B}_l^{(n)} \setminus \mathscr{B}_{l+m}^{(n)}$ we have

$$\mathcal{L}^w_{(l,n)}\phi^{(n)}_{l+m,1}(z) = \sum_{x\colon x\sim z} w_{xz}\left(\phi^{(n)}_{l+m,1}(x) - \phi^{(n)}_{l+m,1}(z)\right) = \sum_{x\colon x\sim z} w_{xz}\phi^{(n)}_{l+m,1}(x)\,.$$

Since $\pi_{l+m-1,B_n} \ge \pi_{l+m-2,B_n} \ge \dots \ge \pi_{l,B_n}$, it follows that

$$\left\|\mathcal{L}^w_{(l,n)}\phi^{(n)}_{l+m,1} + \mu^{(n)}_{l+m,1}\phi^{(n)}_{l+m,1}\right\|^2_{\ell^2(\mathscr{B}_l^{(n)})} \le \pi^2_{l+m-1,B_n} \sum_{z \in \mathscr{B}_l^{(n)} \setminus \mathscr{B}_{l+m}^{(n)}} \max_{x\colon x\sim z}\left(\phi^{(n)}_{l+m,1}(x)\right)^2.$$

Since by virtue of Remark 3.9 the sites $z_{(1,n)}, \dots, z_{(l+m-1,n)}$ are in $\mathscr{I}^{(n)}$ and are neither neighbors nor do they share a common neighbor $\mathbb{P}$-a.s. for n large enough, it follows that $\mathbb{P}$-a.s. for n large enough

$$\sum_{z \in \mathscr{B}_l^{(n)} \setminus \mathscr{B}_{l+m}^{(n)}} \max_{x\colon x\sim z}\left(\phi^{(n)}_{l+m,1}(x)\right)^2 \le \sum_{x \in \mathscr{D}^{(n)}}\left(\phi^{(n)}_{l+m,1}(x)\right)^2 \le n^{-\epsilon_1/2},$$

where the last bound is due to Lemma 3.10. The claim follows by virtue of Lemma 3.14. □

Here comes the second application of Lemma 3.14.

Lemma 3.16. *Let* $\varepsilon > 0$, $l, m \in \mathbb{N}$. *If Assumption 2.1 holds and* $\mathbb{P}$*-a.s. for* n *large enough*

$$\phi_{l,j}^{(n)}\left(z_{(l+j-1,n)}\right) \geq \sqrt{1-n^{-\varepsilon/4}} \qquad \text{for all } 1 \leq j \leq m\,, \tag{3.18}$$

then $\mathbb{P}$*-a.s. for* n *large enough there exists* $j \in \{1, \dots, |B_n| - l - m + 1\}$ *such that*

$$\left|\mu_{l,m+1}^{(n)} - \mu_{l+m,j}^{(n)}\right| \leq \pi_{l+m-1,B_n}\sqrt{\frac{mn^{-\varepsilon/4}}{1-mn^{-\varepsilon/4}}}\,. \tag{3.19}$$

Proof. We aim to apply Lemma 3.14 with the operator $A = -\mathcal{L}^w_{(l+m,n)}$, the Hilbert space $H = \ell^2(\mathscr{B}^{(n)}_{l+m})$, the value $\mu = \mu^{(n)}_{l,m+1}$ and the vector $u = \phi^{(n)}_{l,m+1}/\|\phi^{(n)}_{l,m+1}\|_{\ell^2(\mathscr{B}^{(n)}_{l+m})}$. First, we note that by definition $\|u\|_{\ell^2(\mathscr{B}^{(n)}_{l+m})} = 1$ and $\mathbb{P}$-a.s. for n large enough

$$\|\phi^{(n)}_{l,m+1}\|^2_{\ell^2(\mathscr{B}^{(n)}_{l+m})} = 1 - \sum_{z \in \mathscr{B}^{(n)}_l \setminus \mathscr{B}^{(n)}_{l+m}} \left(\phi^{(n)}_{l,m+1}(z)\right)^2 \geq 1 - mn^{-\varepsilon/4} \tag{3.20}$$

by virtue of Condition (3.18) and Lemma 3.13.

Next, as we show in detail in (A.1), we can estimate

$$\begin{aligned}\left\|\mathcal{L}^w_{(l+m,n)}\phi^{(n)}_{l,m+1} + \mu^{(n)}_{l,m+1}\phi^{(n)}_{l,m+1}\right\|^2_{\ell^2(\mathscr{B}^{(n)}_{l+m})}& \\ \leq \max_{z \in \mathscr{B}^{(n)}_l \setminus \mathscr{B}^{(n)}_{l+m}} \left(\phi^{(n)}_{l,m+1}(z)\right)^2 &\sum_{x \in B_n}\Bigg(\sum_{\substack{z:\, z\sim x\\ z \in \mathscr{B}^{(n)}_l \setminus \mathscr{B}^{(n)}_{l+m}}} w_{xz}\Bigg)^2 .\end{aligned} \tag{3.21}$$

Since by virtue of Remark 3.9 we have $\mathbb{P}$-a.s. for n large enough

$$\mathscr{B}^{(n)}_l \setminus \mathscr{B}^{(n)}_{l+m} = \{z_{l,n}, \dots, z_{l+m-1,n}\} \subset \mathscr{I}^{(n)}$$

and $\mathscr{I}^{(n)}$ is 1-sparse, it follows that on the RHS of (3.21) for each $x \in B_n$ the sum over all $z \in \mathscr{B}^{(n)}_l \setminus \mathscr{B}^{(n)}_{l+m}$ with $z \sim x$ contains at most one summand. Therefore $\mathbb{P}$-a.s. for n large enough we can pull the square into the inner sum. Then we rearrange both sums and use that for all z we have $\sum_{x:\, x\sim z} w^2_{xz} \leq \pi^2_z$ to infer that $\mathbb{P}$-a.s. for n large enough

$$\left\|\mathcal{L}^w_{(l+m,n)}\phi^{(n)}_{l,m+1} + \mu^{(n)}_{l,m+1}\phi^{(n)}_{l,m+1}\right\|^2_{\ell^2(\mathscr{B}^{(n)}_{l+m})} \leq \max_{z \in \mathscr{B}^{(n)}_l \setminus \mathscr{B}^{(n)}_{l+m}} \left(\phi^{(n)}_{l,m+1}(z)\right)^2 \sum_{z \in \mathscr{B}^{(n)}_l \setminus \mathscr{B}^{(n)}_{l+m}} \pi^2_z\,.$$

By virtue of Lemma 3.13 and Assumption (3.18), for all $z \in \left\{z_{(l,n)}, \dots, z_{(l+m-1,n)}\right\}$ we know that $\mathbb{P}$-a.s. for n large enough

$$\left|\phi^{(n)}_{l,m+1}(z)\right| \leq n^{-\varepsilon/8}\,.$$

Furthermore, $\sum_{z \in \mathscr{B}^{(n)}_l \setminus \mathscr{B}^{(n)}_{l+m}} \pi^2_z \leq m\pi^2_{l+m-1,B_n}$. It follows that $\mathbb{P}$-a.s. for n large enough

$$\left\|\mathcal{L}^w_{(l+m,n)}\phi^{(n)}_{l,m+1} - \mu^{(n)}_{l,m+1}\phi^{(n)}_{l,m+1}\right\|^2_{\ell^2(\mathscr{B}^{(n)}_{l+m})} \leq mn^{-\varepsilon/4}\pi^2_{l+m-1,B_n}\,.$$

Together with (3.20) it follows that $\mathbb{P}$-a.s. for n large enough

$$\left\| \mathcal{L}^{w}_{(l+m,n)} u - \mu^{(n)}_{l,m+1} u \right\|^2_{\ell^2(\mathscr{B}^{(n)}_{l+m})} \leq \frac{mn^{-\varepsilon/4}}{1 - mn^{-\varepsilon/4}} \pi^2_{l+m-1,B_n} \,. \tag{3.22}$$

and therefore the claim follows by virtue of Lemma 3.14. □

Both Lemmas 3.15 and 3.16 imply the following lemma.

Lemma 3.17. *Let $\epsilon \in (0, \epsilon_1)$ and $l, m \in \mathbb{N}$. If Assumption 2.1 holds and $\mathbb{P}$-a.s. for n large enough*

$$\phi^{(n)}_{l,j}\left(z_{(l+j-1,n)}\right) \geq \sqrt{1 - n^{-\epsilon/4}} \qquad \text{for all } 1 \leq j \leq m \,, \tag{3.23}$$

then

$$\mu^{(n)}_{l,m+1} \geq \left(1 - (2 + \sqrt{m}) n^{-\epsilon/8}\right) \pi_{l+m,B_n} \,. \tag{3.24}$$

Proof. Let us first assume that $\mu^{(n)}_{l,m+1} \leq \mu^{(n)}_{l+m,1}$. Due to Assumption (3.23) we can apply Lemma 3.16. Because of the ordering $\mu^{(n)}_{l+m,1} \leq \mu^{(n)}_{l+m,2} \leq \dots$, it follows that Relation (3.19) holds with $j = 1$ and $\varepsilon = \epsilon$. On the other hand, if $\mu^{(n)}_{l,m+1} > \mu^{(n)}_{l+m,1}$, then (3.17) holds with an index $i \leq m + 1$. Let us now argue why (3.17) holds with exactly $i = m + 1$ $\mathbb{P}$-a.s. for n large enough. We assume the contrary, i.e., that $i \leq m$ infinitely often as n tends to infinity. Then (3.17) together with (3.12) implies that

$$\mu^{(n)}_{l,i} \geq \mu^{(n)}_{l+m,1} - n^{-\epsilon_1/4} \pi_{l+m-1,B_n} \geq \left(1 - 2n^{-\epsilon_1/8} - n^{-\epsilon_1/4}\right) \pi_{l+m,B_n}$$

Note that (3.6) implies that $\mu^{(n)}_{l,i} \leq \pi_{l+i-1,B_n}$, which we assumed to be less than or equal to π_{l+m-1,B_n} infinitely often as n tends to infinity. Thus

$$\frac{\pi_{l+m-1,B_n}}{\pi_{l+m,B_n}} \geq 1 - 3n^{-\epsilon_1/8}$$

infinitely often as n tends to infinity. This is a contradiction to Lemma 3.11.

Thus, since $\epsilon < \epsilon_1$, it follows regardless of whether $\mu^{(n)}_{l,m+1} \leq \mu^{(n)}_{l+m,1}$ or $\mu^{(n)}_{l,m+1} > \mu^{(n)}_{l+m,1}$ that $\mathbb{P}$-a.s. for n large enough

$$\left| \mu^{(n)}_{l,m+1} - \mu^{(n)}_{l+m,1} \right| \leq \sqrt{\frac{mn^{-\epsilon/4}}{1 - mn^{-\epsilon/4}}} \pi_{l+m-1,B_n} \leq \sqrt{m} n^{-\epsilon/8} \cdot \pi_{l+m,B_n} \,. \tag{3.25}$$

Therefore $\mathbb{P}$-a.s. for n large enough $\mu^{(n)}_{l,m+1}$ is bounded from below by

$$\mu^{(n)}_{l,m+1} \geq \mu^{(n)}_{l+m,1} - \sqrt{m} n^{-\epsilon/8} \cdot \pi_{l+m,B_n} \overset{(3.12)}{\geq} \left(1 - (2 + \sqrt{m}) n^{-\epsilon/8}\right) \pi_{l+m,B_n} \,. \tag{3.26}$$

□

Now we have the ingredients to prove the main theorem by induction.

4 Proof of the main theorem

By virtue of Lemma 3.4, we already know that

$$\lambda_k^{(n)} \le \pi_{k,B_n} \qquad \text{for all } k \in \mathbb{N}\,.$$

In what follows, we further prove (2.4) and that $\mathbb{P}$-a.s. for n large enough

$$\lambda_k^{(n)} \ge \left(1 - n^{-\epsilon/8}\right)\pi_{k,B_n} \qquad \text{for all } \epsilon < \epsilon_1\,.$$

We prove the claim by induction over k.

Base case: $k = 1$. $\mathbb{P}$-a.s. for n large enough we have

$$\psi_1^{(n)}\left(z_{(1,n)}\right)^2 \ge 1 - n^{-\epsilon_1/4}\,, \tag{4.1}$$

by virtue of [Fle16, Theorem 1.8] and

$$\lambda_1^{(n)} \ge \left(1 - 2n^{-\epsilon_1/8}\right)\pi_{1,B_n} > \left(1 - n^{-\epsilon/8}\right)\pi_{1,B_n} \qquad \text{for all } \epsilon < \epsilon_1 \tag{4.2}$$

by virtue of [Fle16, Equation (5.30)].

Inductive step: $(k-1) \rightsquigarrow k$. Suppose that the claims (2.3) and (2.4) hold for some $k - 1 \in \mathbb{N}$. We now show that this implies that the claims also hold for k instead of $k-1$.

For (2.3) this already follows by Lemma 3.17 with $l = 1$ and $m = k - 1$. Note that here Condition (3.23) holds for all $\epsilon < \epsilon_1$ and therefore (3.24) holds even without the multiplicative constants. For (2.4) we apply the second part of Lemma 3.14: Let $0 < \delta < \epsilon_1/16$ and

$$\beta_k^{(n)} = 2\sqrt{k-1}\,n^{-\delta}\pi_{k,B_n}\,. \tag{4.3}$$

Since $\pi_{k-1,B_n} \le \pi_{k,B_n}$, it follows that $\beta_k^{(n)} > \alpha_k^{(n)}$ with

$$\alpha_k^{(n)} := \sqrt{k-1}\,n^{-\epsilon_1/8}\pi_{k-1,B_n}\,.$$

Therefore Lemma 3.14 and (3.22) with $l = 1$ and $m = k - 1$ imply that there exists a function $\overline{u}\colon \mathbb{Z}^d \to \mathbb{R}$ such that

$$\left\|\psi_k^{(n)} - \overline{u}\right\|_{\ell^2(B_n)} \le \frac{2\sqrt{k-1}\,n^{-\epsilon_1/8}\pi_{k-1,B_n}}{\beta_k^{(n)}} \tag{4.4}$$

where $\overline{u}$ is a linear combination of the eigenvectors $\{\phi_{k,j}\}_{j\ge 1}$ corresponding to the eigenvalues from the interval $\left[\lambda_k^{(n)} - \beta_k^{(n)}, \lambda_k^{(n)} + \beta_k^{(n)}\right]$ of the operator $-\mathcal{L}^w_{(k,n)}$. We now show that $\mathbb{P}$-a.s. for n large enough $\overline{u} = \phi_{k,1}^{(n)}$, i.e., that $\mathbb{P}$-a.s. for n large enough

$$\operatorname{spec} \mathcal{L}^w_{(k,n)} \cap \left[\lambda_k^{(n)} - \beta_k^{(n)}, \lambda_k^{(n)} + \beta_k^{(n)}\right] = \left\{\mu_{k,1}^{(n)}\right\}. \tag{4.5}$$

It suffices to show that $\mathbb{P}$-a.s. for n large enough $\mu_{k,2}^{(n)} > \lambda_k^{(n)} + \beta_k^{(n)}$. We note that Lemma 3.4 implies that

$$\lambda_k^{(n)} + \beta_k^{(n)} \le \Big(1 + 2\sqrt{k-1}\, n^{-\delta}\Big)\pi_{k,B_n}\,. \tag{4.6}$$

By virtue of Lemma 3.11 we have $\mathbb{P}$-a.s. for n large enough $\frac{\pi_{k,B_n}}{\pi_{k+1,B_n}} < 1 - 2\sqrt{k-1}\, n^{-\delta}$, whence it follows that $\mathbb{P}$-a.s. for n large enough

$$\lambda_k^{(n)} + \beta_k^{(n)} < \big(1 - 4(k-1)n^{-2\delta}\big)\pi_{k+1,B_n} \le \mu_{k,2}^{(n)}\,,$$

where the last inequality follows since by the inductive assumption the relation (3.23) holds for all $\epsilon < \epsilon_1$ and therefore (3.24) holds for all $\epsilon < \epsilon_1$ with $l = k$ and $m = 1$. Therefore (4.5) is true.

It follows that for any $0 < \delta < \epsilon_1/16$ we have $\mathbb{P}$-a.s. for n large enough

$$\left|\psi_k^{(n)}(z_{(k,n)}) - \phi_{k,1}^{(n)}(z_{(k,n)})\right| \;\le\; \frac{n^{\delta-\epsilon_1/8}\pi_{k-1,B_n}}{\pi_{k,B_n}} \;<\; n^{\delta-\epsilon_1/8}\,.$$

By virtue of Lemma 3.12, we already know that $\left|\phi_{k,1}^{(n)}(z_{(k,n)})\right| \;\ge\; \sqrt{1-n^{-\epsilon_1/4}}$ $\mathbb{P}$-a.s. for n large enough. It follows that

$$\Big(\psi_k^{(n)}(z_{(k,n)})\Big)^2 \;\ge\; 1 - n^{-\epsilon_1/4} + n^{2\delta-\epsilon_1/4} - 2n^{\delta-\epsilon_1/8} \;\ge\; 1 - 2n^{\delta-\epsilon_1/8}\,.$$

The claim follows since we can choose δ arbitrarily small.

5 Asymptotics of the eigenvalues

The proof of Corollary 2.3 extends the proof of [Fle16, Corollary 1.11], which uses the ideas of [Wat54]. To keep the present paper self-contained, we repeat the initial definitions and statements. We define

$$a_n \;:=\; \Big(n^{\frac{1}{2\gamma}} L^*(n)\Big)^{-1} \;=\; \frac{1}{h(|B_n|)} = \sup\big\{t\colon\, F_\pi(t) = |B_n|^{-1}\big\}$$

with h as in (2.5) and $L^*(n)$ as in (2.6). Then $|B_n| = (\mathbb{P}[\pi_0 \le a_n])^{-1}$ and therefore

$$\lim_{n\to\infty} |B_n|\mathbb{P}[\pi_0 \le a_n\zeta] = \lim_{n\to\infty} \frac{F_\pi(a_n\zeta)}{F_\pi(a_n)} \;=\; \zeta^{2d\gamma} \quad \text{for all } \zeta \ge 0 \tag{5.1}$$

since $a_n \to 0$ as $n \to \infty$ and F_π varies regularly at zero with index $2d\gamma$. We further note that if $\boldsymbol{e}_1 \in \mathbb{Z}^d$ is a neighbor of the origin, then $\mathbb{P}[\{\pi_0 \le a_n\zeta\} \cap \{\pi_{\boldsymbol{e}_1} \le a_n\zeta\}] \le F(a_n\zeta)^{4d-1}$ since for the event $\{\pi_0 \le a_n\zeta\} \cap \{\pi_{\boldsymbol{e}_1} \le a_n\zeta\}$ at least $4d-1$ independent conductances w have to be smaller than or equal to $a_n\zeta$. Since F varies regularly at zero with index γ, it follows that

$$|B_n|\,\mathbb{P}[\{\pi_0 \le a_n\zeta\} \cap \{\pi_{\boldsymbol{e}_1} \le a_n\zeta\}] \to 0 \qquad \text{as } n \to \infty\,. \tag{5.2}$$

We start with the auxiliary Lemma 5.2, for which we need some further definitions. For a set $A \subset \mathbb{Z}^d$ we define $CC(A)$ as the set of connected components of A. Furthermore, we define the outer site boundary of the set A as

$$\partial A := \big\{z \in \mathbb{Z}^d \backslash A\colon\, \exists x \in A \text{ with } x \sim z\big\}\,. \tag{5.3}$$

For the natural numbers $q \le m$ we further define the number

$$C_{m,q}^{(n)}(A) := \big|\{M \subset B_n\backslash(A \cap \partial A)\colon\, |M| = m, |CC(M)| = q\}\big|\,. \tag{5.4}$$

Remark 5.1. *Note that if we fix a $k \in \mathbb{N}$, then as n tends to infinity we have $C^{(n)}_{m,m}(A_n) = |B_n|^m/m! + O(|B_n|^{m-1})$ for all sequences of subsets $A_n \in B_n$ with the constraint $|A_n| = k-1$. Moreover, for $q \le m-1$ there exists a constant $c_q < \infty$ such that for all $n \in \mathbb{N}$ and all sequences of subsets $A_n \subset B_n$ with $|A_n| = k-1$, we have $C^{(n)}_{m,q}(A_n) < c_q |B_n|^q$. Note that this c_q is independent of the specific choice of A_n.*

Lemma 5.2. *For any fixed $k, l \in \mathbb{N}$ the relations (5.1) and (5.2) imply that*

$$\lim_{n\to\infty} \sup_{\substack{A_n \subset B_n,\\ |A_n|=k-1}} \sum_{m=1}^{l} \sum_{q=1}^{m-1} \sum_{\substack{M \subset B_n \setminus (A_n \cap \partial A_n),\\ |M|=m,\\ |CC(M)|=q}} \mathbb{P}\Big[\bigcap_{x\in M} \{\pi_x \le a_n\zeta\}\Big] = 0 \text{ for all } \zeta \ge 0\,. \tag{5.5}$$

Proof. We are summing over sets M with the constraint $|CC(M)| = q < m = |M|$. This means that here all the sets M contain at least one connected component $\mathscr{C}$ with a neighboring pair of sites, i.e., $\mathbb{P}\big[\bigcap_{x\in\mathscr{C}} \{\pi_x \le a_n\zeta\}\big] \le \mathbb{P}[\{\pi_0 \le a_n\zeta\} \cap \{\pi_{e_1} \le a_n\zeta\}]$. Since π_x and π_y are independent if the sites x and y are in two different connected components of M, it follows that

$$\sum_{m=1}^{l} \sum_{q=1}^{m-1} \sum_{\substack{M \subset B_n \setminus (A_n \cap \partial A_n),\\ |M|=m,\\ |CC(M)|=q}} \mathbb{P}\Big[\bigcap_{x\in M} \{\pi_x \le a_n\zeta\}\Big] \\ \le \sum_{m=1}^{l} \sum_{q=1}^{m-1} C^{(n)}_{m,q}(A_n) \mathbb{P}[\pi_0 \le a_n\zeta]^{q-1} \mathbb{P}[\{\pi_0 \le a_n\zeta\} \cap \{\pi_{e_1} \le a_n\zeta\}]\,.$$

By Remark 5.1 there exists a constant $c_q < \infty$ such that $C^{(n)}_{m,q}(A_n) \le c_q |B_n|^q$ for all sequences of subsets $A_n \subset B_n$ with the constraint that $|A_n| = k-1$. Therefore the claim follows by (5.1) and (5.2). □

Proof of Corollary 2.3. Because of the main theorem it remains to show that

$$\lim_{n\to\infty} \mathbb{P}\left[\pi_{k,B_n} > \frac{\zeta}{n^{\frac{1}{2\gamma}} L^*(n)}\right] = \exp\big(-\zeta^{2d\gamma}\big) \sum_{j=0}^{k-1} \frac{\zeta^{2d\gamma j}}{j!} \quad \text{for all } \zeta \ge 0\,. \tag{5.6}$$

The proof extends the proof of [Fle16, Corollary 1.11], where we have already shown that

$$\lim_{n\to\infty} \mathbb{P}\Big[\min_{x\in B_n} \pi_x > a_n\zeta\Big] = \exp\big(-\zeta^{2d\gamma}\big) \qquad \text{for all } \zeta \ge 0 \tag{5.7}$$

by extending the ideas of [Wat54] from $d = 1$ to $d \ge 2$. We will use (5.7) for the inductive base case $k = 1$.

In what follows all the statements hold for all $\zeta \ge 0$. For the inductive step we consider

$$\mathbb{P}[\pi_{k,B_n} > a_n\zeta] = \mathbb{P}[\pi_{k-1,B_n} > a_n\zeta] + \mathbb{P}[\{\pi_{k,B_n} > a_n\zeta\} \cap \{\pi_{k-1,B_n} \le a_n\zeta\}]\,.$$

Let us now assume that the claim (5.6) holds for some $k-1$. It follows that it remains to show that

$$\lim_{n\to\infty} \mathbb{P}[\pi_{k,B_n} > a_n\zeta,\, \pi_{k-1,B_n} \le a_n\zeta] = \frac{\zeta^{2(k-1)d\gamma}}{(k-1)!} \exp\big(-\zeta^{2d\gamma}\big)\,.$$

Let us start with the decomposition

$$\mathbb{P}[\pi_{k,B_n} > a_n\zeta,\, \pi_{k-1,B_n} \le a_n\zeta] = \sum_{\substack{A\subset B_n,\\|A|=k-1}} \mathbb{P}\Bigg[\bigcap_{x\in A}\{\pi_x \le a_n\zeta\} \cap \bigcap_{y\in B_n\setminus A}\{\pi_y > a_n\zeta\}\Bigg]$$
$$= \sum_{\substack{A\subset B_n,\\|A|=k-1}} \mathbb{P}\Bigg[\bigcap_{x\in A}\{\pi_x \le a_n\zeta\} \cap \bigcap_{y\in B_n\setminus(A\cap\partial A)}\{\pi_y > a_n\zeta\}\Bigg]$$
$$- \sum_{\substack{A\subset B_n,\\|A|=k-1}} \mathbb{P}\Bigg[\bigcap_{x\in A}\{\pi_x \le a_n\zeta\} \cap \Bigg(\bigcup_{y\in\partial A}\{\pi_y \le a_n\zeta\}\Bigg)\Bigg]. \tag{5.8}$$

Let us argue that the second term on the above RHS converges to zero. We observe that

$$\sum_{\substack{A\subset B_n,\\|A|=k-1}} \mathbb{P}\Bigg[\bigcap_{x\in A}\{\pi_x \le a_n\zeta\} \cap \Bigg(\bigcup_{y\in\partial A}\{y \le a_n\zeta\}\Bigg)\Bigg]$$
$$\le \sum_{\substack{A\subset B_n,\\|A|=k-1}} \sum_{y\in\partial A} \mathbb{P}\Bigg[\{\pi_y \le a_n\zeta\} \cap \bigcap_{x\in A}\{\pi_x \le a_n\zeta\}\Bigg] \le \sum_{\substack{A\subset B_n,\\|A|=k,\\|CC(A)|\le k-1}} \mathbb{P}\Bigg[\bigcap_{x\in A}\{\pi_x \le a_n\zeta\}\Bigg]$$

which converges to zero by virtue of Lemma 5.2.

Let us now consider the first term on the RHS of (5.8). Since for any $y \in B_n\setminus(A\cap\partial A)$ the random variable π_y is independent of $\{\pi_x\}_{x\in A}$, the first sum on the RHS of (5.8) is

$$\sum_{\substack{A\subset B_n,\\|A|=k-1}} \mathbb{P}\Bigg[\bigcap_{x\in A}\{\pi_x \le a_n\zeta\}\Bigg]\mathbb{P}\Bigg[\min_{y\in B_n\setminus(A\cap\partial A)} \pi_y > a_n\zeta\Bigg]$$
$$\ge \mathbb{P}\Bigg[\min_{y\in B_n} \pi_y > a_n\zeta\Bigg] \sum_{\substack{A\subset B_n,\\|A|=k-1}} \mathbb{P}\Bigg[\bigcap_{x\in A}\{\pi_x \le a_n\zeta\}\Bigg]. \tag{5.9}$$

Due to (5.7), the first factor in the above RHS converges to $\exp\big(-\zeta^{2d\gamma}\big)$. As a part of the proof for (5.7), we have also shown that the second factor converges to $\zeta^{2(k-1)d\gamma}/(k-1)!$. It thus remains to find an upper bound for the LHS of (5.9). Similar to the proof of (5.7), we let l be an even integer and estimate for all sequences of subsets $A_n \subset B_n$ with the constraint $|A_n| = k-1$ that

$$\mathbb{P}\Bigg[\min_{y\in B_n\setminus(A_n\cap\partial A_n)} \pi_y > a_n\zeta\Bigg] \le 1 + \sum_{m=1}^{l}(-1)^m \sum_{\substack{M\subset B_n\setminus(A_n\cap\partial A_n),\\|M|=m}} \mathbb{P}\Bigg[\bigcap_{x\in M}\{\pi_x \le a_n\zeta\}\Bigg]$$
$$= 1 + \sum_{m=1}^{l}(-1)^m \sum_{\substack{M\subset B_n\setminus(A_n\cap\partial A_n),\\|M|=m,\\CC(M)=m}} \mathbb{P}\Bigg[\bigcap_{x\in M}\{\pi_x \le a_n\zeta\}\Bigg]$$
$$+ \sum_{m=1}^{l}\sum_{q=1}^{m-1} \sum_{\substack{M\subset B_n\setminus(A_n\cap\partial A_n),\\|M|=m,\\CC(M)=q}} \mathbb{P}\Bigg[\bigcap_{x\in M}\{\pi_x \le a_n\zeta\}\Bigg]$$

According to Lemma 5.2, the supremum of the last sum on the above RHS taken over all sequences $A_n \subset B_n$ with $|A_n| = k-1$ converges to zero. For the first sum we observe that since $|CC(M)| = |M|$, the set M is sparse and therefore $\{\pi_x\}_{x\in M}$ is a set of independent random variables. It follows that

$$\sum_{m=1}^{l}(-1)^m \sum_{\substack{M\subset B_n\setminus(A_n\cap\partial A_n),\\ |M|=m,\\ CC(M)=m}} \mathbb{P}\Big[\bigcap_{x\in M}\{\pi_x \le a_n\zeta\}\Big] = \sum_{m=1}^{l}(-1)^m C_{m,m}^{(n)}(A)\mathbb{P}[\pi_0 \le a_n\zeta]^m$$

$$= \sum_{m=1}^{l}(-1)^m\big(|B_n|^m/m! + O\big(|B_n|^{m-1}\big)\big)\mathbb{P}[\pi_0 \le a_n\zeta]^m$$

by Remark 5.1. Taking the supremum over all sequences of subsets $A_n \subset B_n$ with the constraint $|A_n| = k-1$, this still converges to $\sum_{m=0}^{l}\zeta^{2d\gamma m}/m!$. Since this holds for every $l \in 2\mathbb{N}$ and we already have the lower bound (5.9), the claim follows. □

A Appendix

For better readability we have shifted a rather lengthy computation in the proof of Lemma 3.16 to this appendix. We start by inserting the definition of the Laplacian, i.e.,

$$\sum_{x\in\mathscr{B}_{l+m}^{(n)}}\Big(\mathcal{L}^w_{(l+m,n)}\phi^{(n)}_{l,m+1}(x) + \mu^{(n)}_{l,m+1}\phi^{(n)}_{l,m+1}(x)\Big)^2$$

$$= \sum_{x\in\mathscr{B}_{l+m}^{(n)}}\Bigg(\sum_{z\colon z\sim x} w_{xz}\Big(\Big(\phi^{(n)}_{l,m+1}\mathbb{1}_{\mathscr{B}^{(n)}_{l+m}}\Big)(z) - \phi^{(n)}_{l,m+1}(x)\Big) + \mu^{(n)}_{l,m+1}\phi^{(n)}_{l,m+1}(x)\Bigg)^2$$

Now we rearrange the terms in order to cancel $\mu^{(n)}_{l,m+1}\phi^{(n)}_{l,m+1}(x)$, i.e.,

$$\text{LHS} = \sum_{x\in\mathscr{B}_{l+m}^{(n)}}\Bigg(\sum_{z\colon z\sim x} w_{xz}\Big(\Big(\phi^{(n)}_{l,m+1}\mathbb{1}_{\mathscr{B}^{(n)}_{l}}\Big)(z) - \phi^{(n)}_{l,m+1}(x)\Big) + \mu^{(n)}_{l,m+1}\phi^{(n)}_{l,m+1}(x)$$
$$- \sum_{z\colon z\sim x} w_{xz}\Big(\phi^{(n)}_{l,m+1}\mathbb{1}_{\mathscr{B}^{(n)}_{l}\setminus\mathscr{B}^{(n)}_{l+m}}\Big)(z)\Bigg)^2,$$

where the first two terms cancel. The last term simplifies to

$$\text{LHS} = \sum_{x\in\mathscr{B}_{l+m}^{(n)}}\Bigg(\sum_{z\in\mathscr{B}^{(n)}_{l}\setminus\mathscr{B}^{(n)}_{l+m}\colon z\sim x} w_{xz}\phi^{(n)}_{l,m+1}(z)\Bigg)^2$$
$$\le \max_{z\in\mathscr{B}^{(n)}_{l}\setminus\mathscr{B}^{(n)}_{l+m}}\Big(\phi^{(n)}_{l,m+1}(z)\Big)^2\sum_{x\in B_n}\Bigg(\sum_{z\in\mathscr{B}^{(n)}_{l}\setminus\mathscr{B}^{(n)}_{l+m}\colon z\sim x} w_{xz}\Bigg)^2. \tag{A.1}$$

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References

[Ast16] A. Astrauskas. From extreme values of i.i.d. random fields to extreme eigenvalues of finite-volume Anderson Hamiltonian. *Probability Surveys*, 13:pp. 156–244, 2016.

[BF60] F. L. Bauer, C. T. Fike. Norms and exclusion theorems. *Numerische Mathematik*, 2(1): pp. 137–141, 1960.

[BK16] M. Biskup, W. König. Eigenvalue Order Statistics for Random Schrödinger Operators with Doubly-Exponential Tails. *Communications in Mathematical Physics*, 341(1):pp. 179–218, 2016.

[Fag12] A. Faggionato. Spectral analysis of 1d nearest-neighbor random walks and applications to subdiffusive trap and barrier models. *Electron. J. Probab.*, 17:no. 15, 1–36, 2012.

[FHS17] F. Flegel, M. Heida, and M. Slowik. Homogenization theory for the random conductance model with degenerate ergodic weights and unbounded-range jumps. *Preprint, available at dx.doi.org/10.20347/WIAS.PREPRINT.2371*, 2017.

[Fle16] F. Flegel. Localization of the principal Dirichlet eigenvector in the heavy-tailed random conductance model. *Preprint, available at dx.doi.org/10.20347/WIAS.PREPRINT.2290*, 2016 (revision: January 19, 2018).

[JKO94] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik. *Homogenization of differential operators and integral functionals*. Springer Berlin Heidelberg, 1994.

[Res87] S. I. Resnick. *Extreme Values, Regular Variation and Point Processes*. Springer New York, 1987.

[Sen81] E. Seneta. *Non-negative Matrices and Markov Chains*. Springer Series in Statistics. Springer-Verlag New York, 1981.

[Wat54] G. S. Watson. Extreme Values in Samples from m-Dependent Stationary Stochastic Processes. *The Annals of Mathematical Statistics*, 25(4):pp. 798–800, 1954.