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Instability of localised buckling modes in a one–dimensional strut model

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Abstract

Stability of localised equilibria arising in a fourth-order partial differential equation modelling struts is investigated. It was shown in Buffoni, Champneys & Toland (1996) that the model exhibits many multi-modal buckling states bifurcating from a primary buckling mode. In this article, using analytical and numerical techniques, the primary mode is shown to be unstable under dead loading in a large range of parameter values, while is likely to be stable under rigid loading for small axial loads. Furthermore, for general reversible or Hamiltonian systems, stability of the multi-modal solutions is established assuming stability of the primary state. As this hypothesis is not satisfied for the buckling mode arising in the strut model, any multi-modal buckling state will be unstable for both loading devices.

1 Introduction

In this article, localised solutions of the fourth-order ordinary differential equation

$$u_{xxxx} + P u_{xx} + u - u^2 = 0, \qquad x \in \mathbb{R}, \tag{1}$$

are investigated. Equation (1) describes equilibrium states of a strut on an elastic foundation with a nonlinear softening restoring force, see, for example, Hunt, Bolt & Thompson (1989). Here, x and u are spatial co-ordinate and vertical displacement, respectively. The parameter P denotes the axial load, while the bending stiffness has been rescaled to unity. The underlying geometry is illustrated in Figure 1. Note that (1) is Hamiltonian with associated energy given by

$$H(u) = \frac{1}{2} P\, u_x^2 + u_x\, u_{xxx} - \frac{1}{2} u_{xx}^2 + \frac{1}{2} u^2 - \frac{1}{3} u^3.$$

Localised solutions h of (1) satisfy the condition

$$\lim_{x \to \pm\infty} h(x) = 0, \tag{2}$$

i.e., they correspond to homoclinic solutions of (1). It was shown by Amick & Toland (1992) that (1,2) has a unique even solution $h(P)$ for each $P \in (-\infty, -2 + \eta)$ for some small $\eta > 0$ to which we refer to as the primary buckling mode. In addition, the solutions $h(P)$ satisfy the following transversality hypothesis:

(H1) h is transversely constructed, i.e., stable and unstable manifolds of the zero equilibrium of (1) intersect transversely at $u = h(0)$ in the zero level set of the energy H.

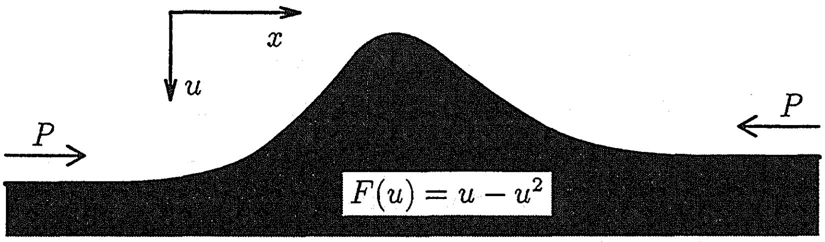

Figure 1: A strut on an asymmetric softening foundation with restoring force F under an axial load P.

There is numerical evidence that the primary buckling mode $h(P)$ persists up to $P = 2$ while still satisfying Hypothesis (H1), see Buffoni et al. (1996) and the references cited therein. Using results of Devaney (1976), Buffoni et al. (1996) proved that for any $P \in (-2, -2+\eta)$, with η sufficiently small, infinitely many buckling modes bifurcate from the primary state. The bifurcating equilibria are multi-modal solutions resembling concatenated, widely spaced copies of the primary state. Numerical simulations suggest that this phenomenon actually occurs for all $P \in (-2, 2)$.

In this article, stability of the localised buckling modes described above is investigated. The total energy of such solutions is given by

$$W(u) = \frac{1}{2}\int_{-\infty}^{\infty} u_{xx}^2(x) - Pu_x^2(x) + u^2(x) - \frac{2}{3}u^3(x)\,dx, \tag{3}$$

see, for instance, Hunt et al. (1989). We shall consider two different loading devices for (1) influencing the notion of stability. Under dead loading, the axial load P is prescribed while the displacement u adjusts according to (1). Under rigid loading, the total displacement

$$I(u) = \int_{-\infty}^{\infty} u_x^2(x)\,dx \tag{4}$$

is fixed while u and the stress parameter P are varying. Thus, accessing stability of an equilibrium under dead loading is equivalent to minimising the total energy $W(u)$ for fixed P, see Thompson & Hunt (1973). This task can be accomplished by verifying positive definiteness of the second variation $\nabla^2 W(h)$ of W given by

$$L(h)\,v := \nabla^2 W(h)\,v = v_{xxxx} + Pv_{xx} + (1-2h)v, \tag{5}$$

at a buckling state h. Note that, on account of translational invariance of (1), $L(h)$ has an eigenvalue at zero. Under rigid loading, we shall minimise $W(u)$ with respect to (u, P) under the additional constraint $I(u) = c$. This amounts to proving positive definiteness of $L(h)$ restricted to the kernel of $\nabla I(h)$.

In this paper, we shall carry out a stability analysis for the localised solutions h of (1) described above under both dead and rigid loading by analytical and numerical techniques. In Section 2, stability of the primary buckling state is investigated. It is shown by analytical means that the primary mode is unstable under dead loading for $P \in (-\infty, -2+\eta)$. Numerical simulations reveal that it is in fact unstable for all $P < 2$. We also consider vibrations of the primary mode governed by the nonlinear wave equation

$$u_{tt} + u_{xxxx} + Pu_{xx} + u - u^2 = 0, \qquad x \in \mathbb{R}, \tag{6}$$

see Lindberg & Florence (1987). For struts under rigid loading, an integral condition for stability is derived analytically. Yet we are not able to verify this condition rigorously. However, using numerical techniques, it is likely that the primary state $(h(P), P)$ is unstable under rigid loading for $P < P_*$ and stable for $P > P_*$, where $P_* \approx 0.8175$. It was shown by Thompson (1979) that, generally speaking, stability under dead loading implies stability under rigid loading but not vice versa, which is consistent with the results presented here. In Section 3, we address the issue of stability of the multi-modal states existing for $P \in (-2, 2)$. As the linearisation at the primary state possesses negative eigenvalues, all multi-modal solutions are unstable on account of results by Alexander, Gardner & Jones (1990). However, we shall still determine the n critical eigenvalues near zero for n-modal buckling states as they can be used to characterise obstacles for the coalescence of multi-modal states. As a matter of fact, the results presented in Section 3 hold for fairly general systems under generic assumptions on the nature of the primary mode. Thus, they may be applied to equations for which the primary mode is stable guaranteeing existence of infinitely many stable multi-modal states. In addition, existence of multi-modal states is shown for $2m$-dimensional equations extending previous results obtained by Devaney (1976), Champneys (1994) and Härterich (1993). Finally, in Section 4, we comment on other equations exhibiting similar phenomena.

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2 Instability of the primary buckling mode

Consider the primary buckling state $h(P)$ known to exist for $P \in (-\infty, -2+\eta)$ and satisfying the transversality hypothesis (H1). We shall assume the following:

(H2) The primary buckling mode $h(P)$ persists for $P \in (-2, 2)$ while obeying (H1).

The linearisation at $h(P)$ is denoted by

$$L_P\, v := L(h(P))\, v = v_{xxxx} + Pv_{xx} + (1 - 2h(P))v, \tag{7}$$

and is regarded as a self-adjoint operator on $H^4(\mathbb{R})$.

2.1 The linearised eigenvalue problem

In this section, spectral properties of L_P are investigated.

Lemma 1 *Consider the operator L_P for $P \in (-\infty, -2+\eta)$. Then the essential spectrum is given by $\sigma_{ess}(L_P) = [1, \infty)$. Furthermore, there exists precisely one negative eigenvalue $\lambda_0(P)$ of L_P. $\lambda_0(P)$ converges to $-\frac{5}{4}$ as P tends to $-\infty$, while the associated eigenfunction $v_0(P)$ approaches $\operatorname{sech}^3(\frac{1}{2}x)$. Let $v_1 = h(P)_x$ be the eigenfunction associated with $\lambda_1 = 0$. Then L_P is strictly positive definite on the orthogonal complement of $\operatorname{span}\{v_0(P), v_1(P)\}$ in $H^4(\mathbb{R})$.*

Proof. As $h(x) \to 0$ for $x \to \pm\infty$ and all P, the claim about the essential spectrum follows immediately. For $P \in (-\infty, -2+\eta)$, the homoclinic orbit $h(P)$ is transverse, see Buffoni et al. (1996). Thus, in this range of parameter values, $\lambda_1 = 0$ is a simple eigenvalue and therefore, due to self-adjointness of L_P, no other eigenvalues can cross the imaginary axis. Hence, it suffices to show that there is a unique negative eigenvalue $\lambda_0(P)$ for P close to $-\infty$. Indeed, L_P is sectorial whence no eigenvalues can escape to infinity. Therefore, the negative eigenvalue $\lambda_0(P)$ persists at least up to $P = -2$ by a continuation argument. We shall exploit a co-ordinate transformation introduced by Amick & Toland (1992) for $P \to -\infty$. Let $-P = \sqrt{\epsilon} + \frac{1}{\sqrt{\epsilon}}$ and $y = \epsilon^{\frac{1}{4}}x$. Then L_P transforms to

$$L_\epsilon\, v = \epsilon v_{yyyy} - (1+\epsilon)v_{yy} + (1 - 2h_\epsilon(y))v. \tag{8}$$

By Amick & Toland (1992), for $\epsilon \to 0$, the continuous family h_ϵ converges in the sup-norm to $h_0(y) := \frac{3}{2}\operatorname{sech}^2(\frac{1}{2}y)$. Consider the eigenvalue problem for the limiting operator L_0 at $\epsilon = 0$

$$-v_{yy} + (1 - 2h_0(y))v = \lambda v, \qquad v \in H^4(\mathbb{R}). \tag{9}$$

Changing co-ordinates according to $z = \tanh(\frac{1}{2}y)$ transforms (9) into a Legendre equation. Using Abramowitz & Stegun (1972, Chapter 8, pp. 332), it is straightforward to calculate the eigenvalues λ_n of (9). In fact, we obtain $\lambda_0 = -\frac{5}{4}$ with eigenfunction $v_0(y) = \operatorname{sech}^3(\frac{1}{2}y)$,

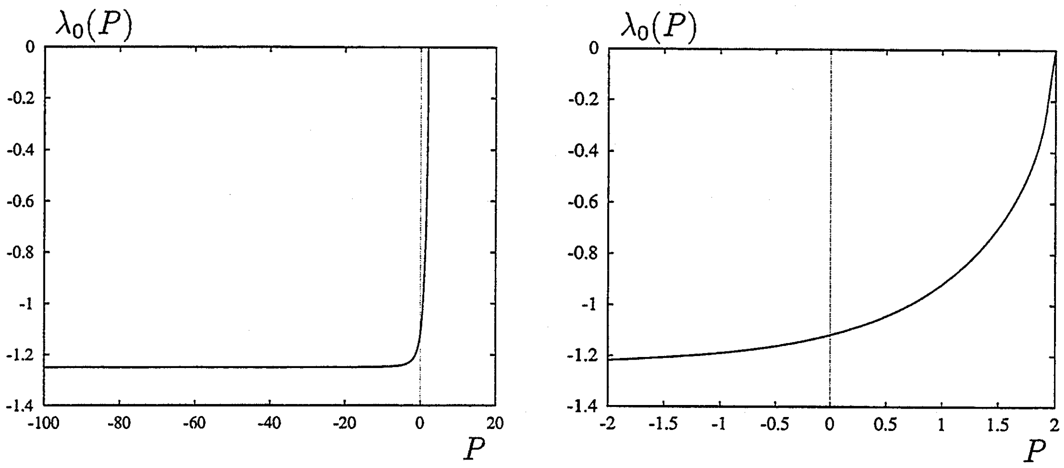

Figure 2: The unstable eigenvalue $\lambda_0(P)$ of the linearisation L_P. For $P = -100$, we have $\lambda_0(P) = -1.24998 \approx -\frac{5}{4}$.

$\lambda_1 = 0$ and $\lambda_2 = \frac{3}{4}$. In particular, L_0 is strictly positive definite on $\text{span}^\perp\{v_0, (h_0)_y\}$. Regarding L_ϵ as a bounded operator from $H^4(\mathbb{R})$ to $L^2(\mathbb{R})$, and using for instance Ljapunov-Schmidt reduction, we see that the eigenvalue λ_0 persists for $\epsilon > 0$ with associated eigenfunction $v_0(\epsilon)$. Furthermore, L_ϵ is strictly positive definite on $\text{span}^\perp\{v_0(\epsilon), (h_\epsilon)_y\}$. Thus the lemma is proved. ∎

Note that the conclusion of Lemma 1 holds for the primary buckling mode $h(P)$ as long as Hypothesis (H2) is met.

We computed the unstable eigenvalue $\lambda_0(P)$ of $L(P)$ using the driver HOMCONT, see Champneys, Kuznetsov & Sandstede (1996) and Champneys, Kuznetsov & Sandstede (1995), for the software package AUTO written by Doedel & Kernévez (1986). Projection boundary conditions with respect to the constant-coefficient operator $v_{xxxx} + Pv_{xx} + v$ are employed for the eigenfunction $v_0(P)$. The initial guess $v_0(y) = \text{sech}^3(\frac{1}{2}y)$, $\lambda_0 = -\frac{5}{4}$ at $P = -100$ corresponding to $\epsilon \approx 1 \cdot 10^{-4}$ has been used. Continuation in P shows that the unstable eigenvalue persists up to $P = 2$, see Figure 2.

2.2 Dead loading

Consider the partial differential equation (6)

$$u_{tt} + u_{xxxx} + Pu_{xx} + u - u^2 = 0, \qquad x \in \mathbb{R}, \tag{10}$$

governing vibrations of the strut under dead loading. We shall rewrite (10) as a first order system

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} v \\ -u_{xxxx} - Pu_{xx} - u + u^2 \end{pmatrix}. \tag{11}$$

The linearisation of (11) at a primary buckling mode $(u, v) = (h(P), 0)$ is given by

$$\mathcal{L}_P = \begin{pmatrix} 0 & \mathrm{id} \\ -\partial_{xxxx} - P\partial_{xx} - \mathrm{id} + 2h & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ -L_P & 0 \end{pmatrix}. \tag{12}$$

On account of Lemma 1, L_P has a simple negative eigenvalue $\lambda_0(P)$, one simple eigenvalue at zero and the rest of the spectrum is bound to the right of the imaginary axis for $P < -2 + \eta$. Thus, for $P < -2 + \eta$, the operator $\mathcal{L}_P$ has precisely two eigenvalues $\pm\sqrt{-\lambda_0(P)}$ on the real axis, a 2×2 Jordan block at zero while the remainder spectrum is located on the imaginary axis. Therefore, we proved the following lemma.

Lemma 2 *The primary buckling mode is linearly unstable with respect to the wave equation (10) under dead loading.*

Remark 1 The conclusion of Lemma 2 holds for the primary state $h(P)$ as long as Hypothesis (H2) is satisfied. It is also valid for parabolic or damped hyperbolic versions of equation (10).

Consider a simply supported strut of length $2l$ described by

$$\begin{aligned} u_{tt} + u_{xxxx} + Pu_{xx} + u - u^2 &= 0, \qquad x \in (-l, l), \\ u(\pm l) = u_{xx}(\pm l) &= 0, \end{aligned} \tag{13}$$

for $P < -2 + \eta$ fixed. Using the techniques of Beyn (1990), it follows that the buckling mode is unstable for (13) as well provided l is larger than some constant $l_0(P)$ depending on P. Indeed, the unstable eigenvalue persists under truncation of the interval.

2.3 Rigid loading

Fix the axial load P_0 and the total displacement $c_0 = I(h(P_0))$ of the primary buckling state $h(P_0)$. Then $(P, h(P))$ satisfies

$$\begin{aligned} u_{xxxx} + Pu_{xx} + u - u^2 &= 0, \\ I(u) = \int_{-\infty}^{\infty} u_x^2(x)\, dx &= c, \end{aligned}$$

for some function $c(P)$ with $c(P_0) = c_0$. Stability of $h = h(P_0)$ is tantamount to proving positive definiteness of the second variation $L(h)$ defined in (5) restricted to the kernel of the gradient $\nabla I(h)$ of the constraint I. The underlying space is $H = H^4(\mathbb{R})$. In other words, we shall verify that

$$\text{(14)} \qquad \langle L(h)u, u\rangle > 0, \qquad u \in \ker \nabla I(h),\ u \notin \ker L(h).$$

In the next lemma, we derive a condition equivalent to (14).

Lemma 3 *Assume that (H2) is met. Then, zero is a simple eigenvalue of* $L(h)$ *and* $\ker L(h) = \operatorname{span} h_x$. *Moreover, (14) holds if and only if*

$$\text{(15)} \qquad c'(P_0) < 0,$$

where $'$ *denotes differentiation with respect to* P *and* $c(P) := I(h(P))$. *In other words, stability of* $h(P)$ *at* $P = P_0$ *under rigid loading is equivalent to* $c(P)$ *being strictly decreasing near* P_0.

Proof. Consider the functional $\nabla I(h) = h_{xx} : H \to \mathbb{R}$ with $H = H^4(\mathbb{R})$ acting according to $\nabla I(h)v = \langle h_{xx}, v\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the L^2-scalar product. On account of the results of the last section, there is a unique negative eigenvalue of $L(h)$. Moreover, zero is a simple eigenvalue with eigenfunction h_x. Note that $h_x \in \ker \nabla I(h)$ as $\langle h_{xx}, h_x\rangle = 0$ by integration by parts. Thus we may restrict attention to the space $X = \operatorname{span}^{\perp} h_x \subset H$. The operator $L(h)$ is self-adjoint and non-singular on X. Hence, $L(h)$ is positive on X if and only if

$$\text{(16)} \qquad \langle L(h)^{-1} h_{xx}, h_{xx}\rangle < 0$$

holds. Indeed, this follows from Alexander, Grillakis, Jones & Sandstede (1995, Proof of Lemma 2), see also Maddocks (1985). On the other hand, $h(P)$ satisfy (1) for all P close to P_0. Differentiating (1) with respect to P, we see that $v = h'(P_0)$ satisfies $L(h)v = -h_{xx}$. Hence, $L(h)^{-1}h_{xx} = -h'(P_0)$ and (16) reads

$$\langle h'(P_0), h_{xx}\rangle > 0.$$

Differentiating $c(P) = I(h(P)) = \int_{-\infty}^{\infty} h_x^2(P)(x)\,dx$ with respect to P and integrating by parts, we conclude that

$$\langle h'(P_0), h_{xx}\rangle = -c'(P_0).$$

Thus (16) and (15) are equivalent. ∎

We were not able to verify condition (15) rigorously. Numerical simulations performed on (1) using HOMCONT suggest that the primary buckling state is stable for $P \in (P_*, 2)$, while being unstable for $P \in (-\infty, P_*)$, see Figure 3. Here, $P_* \approx 0.8175$.

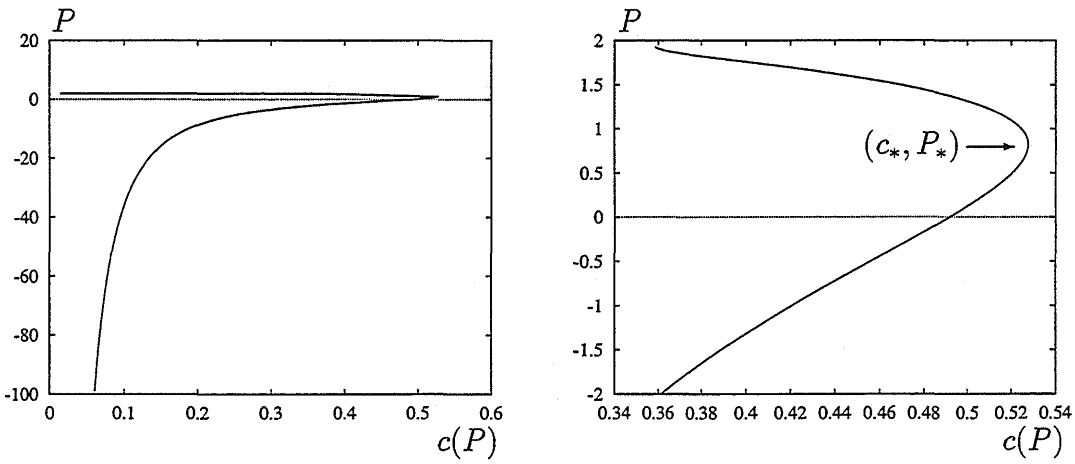

Figure 3: The integral constraint $(c(P), P)$ with $c(P) = I(h(P))$. The function $c(P)$ is increasing for $P < P_*$ and decreasing for $P > P_*$. At $P = P_* = 0.8175$ an exchange of stability occurs.

3 Stability of multi-modal solutions

Consider equation (1) and the eigenvalue problem for (5)

$$\begin{aligned} u_{xxxx} + Pu_{xx} + u - u^2 &= 0, \\ L(u)v = v_{xxxx} + Pv_{xx} + (1-2u)v &= \lambda v. \end{aligned} \tag{17}$$

Assume that a primary non-degenerate homoclinic solution h_1 of (1) has been found. Suppose that h_n is a multi-modal homoclinic orbit resembling n copies of h_1 widely spaced in x. Alexander et al. (1990) proved that close to each eigenvalue of $L(h_1)$ there are n eigenvalues of $L(h_n)$ counted with multiplicity. In particular, there are at least n eigenvalues of $L(h_n)$ close to zero. Indeed, $\lambda = 0$ is an eigenvalue of $L(h_1)$ with eigenfunction $(h_1)_x$. Rewriting (17) as first order system, we obtain

$$U_x = f(U), \tag{18}$$

$$V_x = (Df(U) + \lambda B)\, V, \tag{19}$$

where $U, V \in \mathbb{R}^4$, $f(U) := (U_2, U_3, U_4, -PU_3 - U_1 + U_1^2)$ and $BV := (0,0,0,V_1)$. Section 3.1 is devoted to the calculation of all eigenvalues of (19) evaluated at an n-modal solution of (18) in a neighbourhood of zero. Section 3.2 contains stability results for the strut equation under both dead and rigid loading as well as some remarks on exclusion principles for coalescence of multi-modal states.

3.1 The linearised eigenvalue problem

Consider

$$\dot{u} = f(u), \tag{20}$$

$$\dot{v} = (Df(u) + \lambda B(u))\, v, \tag{21}$$

for $u, v \in \mathbb{R}^{2m}$, $\lambda \in \mathbb{C}$ and $f : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$, $B : \mathbb{R}^{2m} \to \mathbb{R}^{2m\times 2m}$ smooth. We assume that (20) is either reversible or Hamiltonian:

(H3) Suppose that there exists a linear involution $R : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ with $\dim \operatorname{Fix} R = m$ such that $f(Ru) = -Rf(u)$ holds for all u.

(H4) Suppose that there exists a smooth non-degenerate function $H : \mathbb{R}^{2m} \to \mathbb{R}$ such that $f(u) = J\nabla H(u)$ with

$$J = \begin{pmatrix} 0 & \mathrm{id}_m \\ -\mathrm{id}_m & 0 \end{pmatrix}.$$

Let zero be an equilibrium of (20) with complex leading eigenvalues, i.e.,

(H5) There are simple eigenvalues $\pm\alpha \pm i\,\beta$ for some $\alpha, \beta > 0$ contained in $\sigma(Df(0))$. The modulus of the real part of any other eigenvalue of $Df(0)$ is strictly larger than α.

Assume that h_1 is a primary homoclinic solution of (20) converging to zero for $t \to \pm\infty$.

(H6) h_1 is non-degenerate: $T_{h_1(0)}W^s(0) \cap T_{h_1(0)}W^u(0) = \operatorname{span} \dot{h}_1(0)$. If (H3) is satisfied, we assume in addition that h_1 is reversible, i.e., $h_1(t) = -Rh_1(-t)$.

As a consequence of Hypothesis (H6), the equation $\dot{w} = -Df(h_1)^*\, w$ has a unique bounded solution $\psi(t)$, see Sandstede (1996).

Remark 2 Suppose that (H4) is satisfied. Then it is straightforward to check that $\psi(t) = \nabla H(h_1(t))$. In particular, we obtain the asymptotics $\psi(t) = \nabla^2 H(0)\, h_1(t) + \mathrm{O}(|h_1(t)|^2)$.

Remark 2 has very interesting consequences for Hamiltonian flip-bifurcations, see Sandstede, Alexander & Jones (1996).

(H7) If (H3) is satisfied, we assume that the limit $\lim_{t\to\infty} e^{2\alpha t}\, |h_1(t)|\, |\psi(t)| > 0$ is non-zero. If (H4) holds, assume that $\lim_{t\to\infty} e^{2\alpha t}\, |h_1(t)|\, |h_1(-t)| > 0$ is non-zero.

Hypotheses (H6) and (H7) are satisfied for generic systems. Finally, we assume that the Evans function $D(\lambda)$ associated with the primary pulse h_1 is regular at $\lambda = 0$.

(H8) $\int_{-\infty}^{\infty} \langle \psi(t), B(h_1(t))\, \dot{h}_1(t) \rangle \, dt =: M \neq 0$.

We define the sets

$$
\begin{aligned}
\mathcal{R} &= \{\exp(-\tfrac{2\pi\alpha}{\beta} n) \,|\, n \in \mathbb{N}_0\} \cup \{0\} \\
\mathcal{A} &= \{\exp(-\tfrac{\pi\alpha}{\beta} k) \,|\, k \in \mathbb{N}_0\}.
\end{aligned} \tag{22}
$$

Note that $\mathcal{R}$ is a closed metric space.

Definition Suppose that (H3) is satisfied. Then we call a sequence $(x_j)_{j=1,\dots,k}$ *admissible* if and only if $x_{k+1-j} = x_j$ holds for all j. In case (H4) holds, *any* sequence is called admissible.

Theorem 1 *Assume that (H5) to (H8) and either (H3) or (H4) are satisfied. Then there exist a $\delta > 0$ such that for any $n \geq 2$ the following holds.*

For any admissible sequence $a_j^0 \in \mathcal{A}$ for $j = 1, ..., n-1$ with $a_i^0 \in \{1, \exp(-\frac{\pi\alpha}{\beta})\}$ for some i, there exists an $r_0 \in \mathcal{R}$, $r_0 \neq 0$, with the following property:

(i) *There are C^0-functions $a_j(r) \in \mathbb{R}$ for $r \in \mathcal{R}$, $r \leq r_0$, with $a_j(0) = a_j^0$ for $j = 1, ..., n-1$.*

(ii) *For any $r \in \mathcal{R}$ with $0 < r \leq r_0$, there exists an n-modal solution h_n. The return times of h_n to a fixed section transverse to h_1 are given by*

$$L_j(r) = -\frac{1}{\alpha} \ln(a_j(r) r) + \tilde{L} \qquad\qquad j = 1, ..., n-1$$

for some constant $\tilde{L}$. They correspond to the distances of consecutive humps of the n-modal solution.

(iii) *The n-modal orbits satisfying (ii) are unique. If (H3) holds, they are in addition reversible.*

Denote by k_j^0 the natural numbers associated with the $a_j^0 \in \mathcal{A}$ chosen above. Then there are precisely n solutions λ_j of (21) evaluated at h_n in a neighbourhood of zero of size δ, and

(iv) *for $M > 0$ ($M < 0$), we have*

$$
\begin{aligned}
\#\{j \,|\, 1 \leq j \leq n-1,\ \mathrm{Re}\,\lambda_j < 0\} &= \#\{j \,|\, 1 \leq j \leq n-1,\ k_j^0 \text{ is odd (even)}\} \\
\#\{j \,|\, 1 \leq j \leq n-1,\ \mathrm{Re}\,\lambda_j > 0\} &= \#\{j \,|\, 1 \leq j \leq n-1,\ k_j^0 \text{ is even (odd)}\}
\end{aligned}
$$

counted with multiplicity. Moreover, $\lambda_n = 0$ is a simple eigenvalue inevitable due to translational invariance.

Note that the numbers k_j^0 associated with each a_j^0 – and hence with any n-modal state described in Theorem 1 – can be interpreted as the number of half-twists the n-modal state undergoes near the zero equilibrium.

Applying Sandstede (1995, Thm. 1 and 3) provides another way of extending the existence results from four-dimensional to higher-dimensional systems. In addition, Sandstede (1995, Thm. 1) shows that the recurrent dynamics near the primary mode h_1 is confined to a four-dimensional, locally invariant and normally hyperbolic centre manifold W^c_{hom} containing h_1.

The remainder part of the section is devoted to the proof of Theorem 1.

Proposition 1 *Fix $n \geq 2$. Then there exists an n-modal homoclinic orbit h_n of (20) close to h_1 in phase space if and only if*

$$
\begin{array}{lrcll}
 & a_1 \sin\left(-\frac{\beta}{\alpha}\ln(a_1 r)\right) & = & R_1(a,r) & \\
(23) & a_{j-1} \sin\left(-\frac{\beta}{\alpha}\ln(a_{j-1} r)\right) - a_j \sin\left(-\frac{\beta}{\alpha}\ln(a_j r)\right) & = & R_j(a,r) & j = 2,\dots,n-1 \\
 & a_{n-1} \sin\left(-\frac{\beta}{\alpha}\ln(a_{n-1} r)\right) & = & R_n(a,r) &
\end{array}
$$

is satisfied. As a matter of fact, if (H3) holds, it suffices to solve (23) for $j = 1,\dots,[n/2]$, where $[x]$ denotes the largest integer smaller than x; if (H4) is satisfied, it is sufficient to solve (23) for $j = 1,\dots,n-1$. The remainder terms $R_j(a,r)$ are smooth in $a = (a_j)$ for $a_j \in (0,1]$ up to $r = 0$ and

$$(24) \qquad R_j(a,r) = O(r^\gamma), \qquad \frac{d}{da_i}R_j(a,r) = O(r^\gamma)$$

hold for some $\gamma > 0$.

Proof. We shall employ homoclinic Ljapunov-Schmidt reduction. Applying Sandstede (1993, Satz 3), existence of n-modal states is equivalent to solving

$$(25) \qquad \langle\psi(T_j), h_1(-T_j)\rangle - \langle\psi(-T_{j-1}), h_1(T_{j-1})\rangle - R_j((T_j),\mu) = 0$$

for large T_j. In order to derive (23) from (25), we proceed as in Sandstede (1996, Sect. 6) and refer to that article for the details. The statement on the number of equations to be solved for is proved by Sandstede et al. (1996, Lemma 3.1 and 3.2), respectively. Note that in the Hamiltonian case (H4), we have

$$
\begin{aligned}
\langle\psi(t), h_1(-t)\rangle &= \langle\nabla^2 H(0)\, h_1(t), h_1(-t)\rangle + \mathrm{O}(|h_1(t)|^2\,|h_1(-t)|) \\
&= \langle h_1(t), \nabla^2 H(0)\, h_1(-t)\rangle + \mathrm{O}(|h_1(t)|^2\,|h_1(-t)|) \\
&= \langle h_1(t), \psi_1(-t)\rangle + \mathrm{O}(|h_1(t)|\,|h_1(-t)|\,(|h_1(t)| + |h_1(-t)|))
\end{aligned}
$$

by Remark 2. Under Hypothesis (H3), we have $\langle\psi(t), h_1(-t)\rangle = \langle\psi_1(-t), h_1(t)\rangle$ by Sandstede (1996, Lemma 5.3). ∎

Proof of Theorem 1. We shall consider the Hamiltonian case first and comment later on the changes necessary for the reversible case. So, assume that (H4) is met. The proof is similar to Sandstede (1996, Proof of Thm. 3), where saddle-focus bifurcations for generic, non-reversible systems had been studied.

By Proposition 1, we shall solve

$$\begin{array}{lcll} a_1 \sin(-\frac{\beta}{\alpha}\ln(a_1 r)) & = & R_1(a,r) & \\ a_{j-1}\sin(-\frac{\beta}{\alpha}\ln(a_{j-1} r)) - a_j \sin(-\frac{\beta}{\alpha}\ln(a_j r)) & = & R_j(a,r) & j = 2,\dots,n-1. \end{array}$$

Inserting $r = \exp(-\frac{2\pi\alpha}{\beta} n)$ with $n \in \mathbb{N}$, we obtain

$$\begin{array}{lcll} a_1 \sin(-\frac{\beta}{\alpha}\ln a_1) & = & R_1(a,r) & \\ a_{j-1}\sin(-\frac{\beta}{\alpha}\ln a_{j-1}) - a_j \sin(-\frac{\beta}{\alpha}\ln a_j) & = & R_j(a,r) & j = 2,\dots,n-1. \end{array} \tag{26}$$

This allows us to take the limit $r \to 0$ yielding

$$\begin{array}{lcll} a_1 \sin(-\frac{\beta}{\alpha}\ln a_1) & = & 0 & \\ a_{j-1}\sin(-\frac{\beta}{\alpha}\ln a_{j-1}) - a_j \sin(-\frac{\beta}{\alpha}\ln a_j) & = & 0 & j = 2,\dots,n-1. \end{array}$$

However, these equations are satisfied by the chosen sequence $a^0 = (a_j^0)$ with $a_j^0 \in \mathcal{A}$. Moreover, the Jacobian of (26) with respect to a at $r = 0$, $a_j = a_j^0$ is lower-triangular and the entries on the diagonal are given by

$$\left(\sin(-\tfrac{\beta}{\alpha}\ln a_j) - \tfrac{\beta}{\alpha}\cos(-\tfrac{\beta}{\alpha}\ln a_j)\right)\Big|_{a_j = a_j^0 = \exp(-\frac{\pi\alpha}{\beta}k_j^0)} = (-1)^{k_j^0+1}\,\frac{\beta}{\alpha}, \tag{27}$$

and thus are non-zero. Therefore, on account of the differentiability of the remainder terms stated in Proposition 1, an application of the implicit function theorem yields (i)-(iii) of the theorem in the Hamiltonian case.

It remains to prove item (iv), which is concerned with the stability properties of the n-modal orbits. Let h_n be an n-modal solution given by $(a(r), r)$. Then, proceeding as in Section 6 of Sandstede (1996), all eigenvalues $\lambda(r)$ close to zero of (21) evaluated at h_n are given by $\lambda(r) = r\,\nu(r)$ for some continuous function $\nu(r)$ such that $\nu(0)$ is an eigenvalue of the matrix A_0 given by

$$(A_0)_{ij} = \begin{cases} b_j + b_{j-1} & j = i \\ -b_{j-1} & j = i-1 \\ -b_j & j = i+1 \\ 0 & \text{otherwise,} \end{cases}$$

with $b_j = (-1)^{k_j^0} c \operatorname{sign} M$ for some positive constant $c > 0$ and $j = 1, ..., n-1$; we set $b_0 = 0$. Note that A_0 is tridiagonal, symmetric and the sum of entries in each row vanishes. The number of positive and negative eigenvalues for such matrices has been determined in Sandstede (1996, Lemma 5.4). Hence, statement (iv) of Theorem 1 is proved for Hamiltonian systems.

In case (20) is reversible, we shall proceed as above. The only difference is that we solve (26) for $j = 1, ..., [n/2]$ instead for $j = 1, ..., n-1$. This completes the proof of the theorem. $\blacksquare$

3.2 Consequences for the strut model

Buffoni et al. (1996) proved that Hypotheses (H3) to (H6) are satisfied for (1). Hypothesis (H7) is met, too, as (1) is four-dimensional. Finally, Assumption (H8) is true, because zero is a simple eigenvalue of $L(h)$, see Section 2.1. Therefore, Theorem 1 applies to (1). It was proved in Section 2.1 that there exists an unstable eigenvalue of the linearisation $L(h_1)$ of the primary pulse for $P \in [-2, -2+\eta)$. Moreover, the numerical simulations presented in Figure 2 show that the unstable eigenvalue $\lambda_0(P)$ is likely to persist up to $P = 2$. It is a consequence of results of Alexander et al. (1990), see also Sandstede (1996), that the linearisation $L(h_n)$ at a multi-modal state possesses precisely n eigenvalues close to $\lambda_0(P)$. Therefore, all multi-modal solutions of the strut model must be unstable under dead loading by the same arguments as in Lemma 2. Moreover, they will be unstable under rigid loading as well. Indeed, the constraint $I(u)$ can be used to compensate for one of the unstable eigenvalues; however, the other $n-1$ unstable eigenvalues near $\lambda_0(P)$ will make the linearisation restricted to $\ker \nabla I$ indefinite.

Buffoni et al. (1996) observed numerically the coalescence of symmetric and asymmetric multi-modal buckling states, see Figure 4. Generically, coalescence is expected to occur via saddle-node or pitchfork bifurcations in the underlying partial differential equation. Such

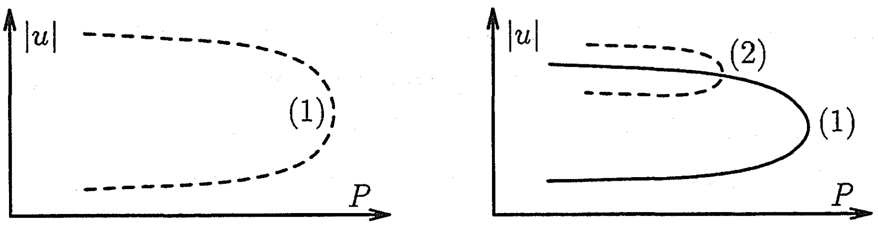

Figure 4: Coalescence of symmetric (solid) and asymmetric (dashed) multi-modal solutions via saddle-node (1) and pitchfork (2) bifurcations, see Buffoni et al. (1996, Fig. 24).

bifurcations are related to an exchange of stability of the contributing n-modal localised solutions. In particular, generically, precisely one eigenvalue should cross the imaginary axis at zero – remember that the operators involved are self-adjoint. Therefore, the index – i.e., the number of unstable eigenvalues – of the involved n-modal solutions should differ by one. Note that we have to take the n unstable eigenvalues near $\lambda_0(P)$ into account. That immediately prevents coalescence of n_1-modal and n_2-modal solutions whenever $n_1 > 2n_2$. Furthermore, observe that the sequence $(k_j^0)_{j=1,\ldots,n-1}$ associated with each n-modal symmetric state is admissible, i.e., $k_j^0 = k_{n-j}^0$ holds for all j. In particular, eigenvalues of n-modal symmetric states come in pairs. Thus, it is likely, that right before two symmetric states do coalesce one eigenvalue has to cross the imaginary axis in a pitchfork bifurcation. No such obstacle exists for asymmetric states. A refined criterion than just counting unstable eigenvalues is to include the symmetry of the corresponding eigenfunctions. Indeed, both, the number of even and odd unstable eigenfunctions, have to coincide for coalescence of two multi-modal states to occur. For each particular sequence (k_j^0), these numbers can be calculated by computing the eigenvectors of the matrix A_0 appearing in the proof of Theorem 1.

4 Discussion

We shall mention that there are several other fourth-order equations for which multi-modal solutions do exist. Consider, for instance,

$$(28) \quad u_{tt} = -(u_{xxxx} + c^2 u_{xx} + e^u - 1) \qquad \text{Suspension bridge,}$$

$$(29) \quad u_t = -(\epsilon u_{xxxx} - u_{xx} - u + u^3) \qquad \text{Extended Fisher-Kolmogorov.}$$

Equation (28) models the dynamics of suspension bridges, see McKenna & Chen (1995), while equation (29) arises in the study of so-called Lifschitz points in phase transitions. Both equations exhibit similar features than (6) and therefore the results of Section 2 and 3 are likely to apply to them as well. For instance, stable solitary waves for (28) have been observed numerically in McKenna & Chen (1995). Therefore, on account of Theorem 1, infinitely many stable multi-modal solitary waves are expected to exist. Indeed, Grillakis, Shatah & Strauss (1987, Thm. 1) can be used to conclude nonlinear stability. Similarly, for equation (29), existence of kinks has been proved by Peletier & Troy (1995*a*, 1995*b*). As these kinks bifurcate from stable kinks of the Nagumo equation at $\epsilon = 0$, they are presumable stable, too. Again, employing Theorem 1, gives the existence of infinitely many stable multi-modal kinks of (29) once its hypotheses are met. Note that the assumptions of that theorem are generic within the class of reversible Hamiltonian equations.

The primary buckling state of (6) bifurcates at $P = 2$ in a 1:1 resonance, see Iooss & Peroueme (1993). It would be challenging to prove (in)stability of the bifurcating modes in this limit. Proceeding this way, the numerical calculations shown in Figure 2 and 3 could be made rigorous.

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