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Abstract

We investigate a probabilistic model for routeing in a multihop ad-hoc communication network, where each user sends a message to the base station. Messages travel in hops via the other users, used as relays. Their trajectories are chosen at random according to a Gibbs distribution that favours trajectories with low interference, measured in terms of sum of the signal-to-interference ratios for all the hops, and collections of trajectories with little total congestion, measured in terms of the number of pairs of hops arriving at each relay. This model was introduced in our earlier paper [KT17], where we expressed, in the high-density limit, the distribution of the optimal trajectories as the minimizer of a characteristic variational formula.

In the present work, in the special case in which congestion is not penalized, we derive qualitative properties of this minimizer. We encounter and quantify emerging typical pictures in analytic terms in three extreme regimes. We analyze the typical number of hops and the typical length of a hop, and the deviation of the trajectory from the straight line in two regimes, (1) in the limit of a large communication area and large distances, and (2) in the limit of a strong interference weight. In both regimes, the typical trajectory turns out to quickly approach a straight line, in regime (1) with equally-sized hops. Surprisingly, in regime (1), the typical length of a hop diverges logarithmically as the distance of the transmitter to the base station diverges. We further analyze the local and global repulsive effect of (3) a densely populated area on the trajectories. Our findings are illustrated by numerical examples. We also discuss a game-theoretic relation of our Gibbsian model with a joint optimization of message trajectories opposite to a selfish optimization, in case congestion is also penalized.

1 Introduction

In this work, we continue our research [KT17] on a spatial Gibbsian model for message routeing in a multi-hop ad-hoc network. In [KT17] we prepared for an analysis of the qualitative properties of the model by deriving simplifying formulas that describe the situation in a densely populated area in the sense of a law of large numbers. In the present work, we carry out this analysis and describe a number of characteristic properties of the message trajectories. In particular, we are interested in the counterplay between probabilistic properties like *entropy* and energetic properties like *interference* and *congestion* and how it develops geometric properties like number and lengths of the hops or straightness of the trajectories. Our goal is to detect some rules of thumbs in the relationships between all these quantities in asymptotic regimes in which they become particularly pronounced, like large areas and long trajectories, strong influence of interference, or local regions with a particularly high population. While [KT17] used mainly probabilistic methods, the present paper entirely employs analytic tools.

1.1 The main features of the model

Let us introduce the reader to the nature of our telecommunication model. The communication area is a bounded set in $\mathbb{R}^d$, and it has a unique base station at the origin o. Many users are randomly distributed according to some measure. Each user sends out a message to the base station along a random multi-hop trajectory that uses other users as relays and has at most $k_{\max}$ steps. The model that we are interested in is a joint distribution of all these message trajectories, conditional on the (random) locations of the users. It is based on a *Gibbsian ansatz*: the *a priori* distribution

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is uniform (i.e., each message chooses first a hop number k and then a k-hop trajectory, both uniformly at random, and trajectories of different users are independent), and there are two exponential weight terms that punish *interference* and *congestion*, respectively. More precisely, the first one weights the interference of each of the hops, measured in terms of the well-known *signal-to-interference ratio (SIR)*, while the other term punishes the entire trajectory family for high total congestion in the system, measured as constant times the sum of the number of pairs of incoming hops at the relays. Note that the SIR term is linear in the number of hops, hence this number is upper bounded by some geometric random variable and thus almost surely finite, even without an artificial upper bound by $k_{\max}$.

The highest probability is attached to those trajectory families that realize the best compromise between entropy (i.e., probability) and energy (i.e., interference plus congestion). In Section 2.4.2, we give a thorough discussion of the motivation for studying such a type of model, in particular the randomness of the trajectories.

The interesting feature is that the total punishment is given to the entire system in terms of a probability weight, in the spirit of a "common-welfare" mechanism, instead of selfish routeing optimization. In Section 6, we give a game-theoretic discussion of the two weight terms in the exponent in the light of traffic theory; more precisely we ask the question under what circumstances the optimization of these two terms can be called selfish or non-selfish. In Section 2.4.2, we also make a connection between this optimization and our model from the viewpoint of stochastic algorithms for an experimental realization of the optimum.

The idea of an optimal compromise between entropy and energy is most clearly realized in a certain limiting sense in [KT17, Theorem 1.4], which will be the starting point of the present paper and will be summarized in Section 2 below. There, we carried out the limit of a high density of users, and we derived a kind of law of large numbers for the "typical" trajectory distribution, i.e., the joint trajectory distribution that has the highest probability under the Gibbs measure. This was done using large-deviation theory, and the answer came with a characteristic *variational formula* in [KT17, Theorem 1.2], whose minimizer(s) describe(s) that object. Roughly speaking, the variational formula is of the form "minimize the sum of entropy and energy among all admissible trajectory families".

1.2 Goals

Our goal in the present paper is to understand the global effects that are induced in the Gibbsian system exclusively by entropy and energy into geometric properties of the joint behaviour of the trajectory family. We will be working in an analytical way; our goal is to reveal macroscopic phenomena in important settings and regimes. As our model depends on various parameters (size and form of the communication area, density of users, choice of the SIR term, strength of interference and congestion weighting, etc.), this can be done rigorously only in certain *limiting* regimes. We are interested in the most important and obvious geometric properties: number and lengths of the hops, and the spatial shape of the trajectory.

In the following three limiting regimes we will encounter particularly clear pictures:

(1) large communication area and long distances (and large hop numbers),

(2) strong interference punishment, and

(3) high local density of users on a subset of the communication area.

In regimes (1) and (2), we expect that the typical trajectories approach straight lines, and in (1) there is an additional question about the typical length of a hop and the number of hops. Here, we would like to understand how the quality of service becomes bad in a large telecommunication area and how many and how large steps the messages would like to make if the artificial constraint by $k_{\max}$ is dropped.

However, the regime (3) and our questions here are of a different nature: We would like to understand if the presence of a subarea with a particularly high population density has a significant (positive or negative) impact on the effective use of the relaying system: on one hand, the trajectories have more available relays in such an area, but on the other hand, the interference achieves high values there. It is a competition between entropy and energy of a particular type that we want to understand.

Let us announce that we are going to work on these questions only in the case where only interference is penalized, but not congestion. This is due to the case that the minimizer(s) are characterized in [KT17, Proposition 1.3] in a way that is enormously implicit and cumbersome in general, but reduces, if the congestion term is dropped, to relatively simple formulas that are amenable for analytical investigations [KT17, Proposition 1.5]. In particular, we know only here that the minimizer is unique. Therefore, we decided to analyze the limiting regimes (1)–(3) under the assumption that only interference is penalized. We believe that the main qualitative properties persist to the case where also congestion is penalized, as this is purely combinatorial and not spatial. In Section 6 we will discuss non-selfishness and other game-theoretic properties of the Gibbsian model in the presence of both terms.

1.3 Our findings

In regimes (1)–(2), we will see that the typical trajectory follows a straight line with exponential decay of probabilities of macroscopic deviations from the straight line. Moreover, in the regime (1) we will also find simple formulas for the asymptotic number of hops and the average length of a hop, which turns out to be the same for each hop of the trajectory.

However, in regime (3), we encounter different effects. First we see the following global effect on the total number of relaying hops in the entire system: If the communication area is small (in the sense that all the interferences in the system do not vary much), then the total number of relaying hops vanishes exponentially fast in the diverging parameter of the dense population, regardless of the choice of the densely populated subset. In some cases, we also detect a local effect on the relaying hops if the densely populated subset is very small: We demonstrate that a certain neighbourhood of that subset is definitely unfavourable for relaying hops for practically all the other users, a very clear effect coming from the high interference of the densely populated area, which expels the trajectories away.

Some of our results are more or less expected, and the main value of our work is the explicit characterization of the quantities and the derivation of exponential bounds for deviations. However, one of our most strinking findings is that, in the regime (1), the typical hop length diverges as a logarithmic function of the distance between the transmitter and the base station, and hence the typical number of hops is sub-linear in the distance. This effect seems to come from the fact that *a priori*, i.e., before switching on the interference weight, every message trajectory of a given length has the same weight, even very ridiculous ones that have long spatial detours, e.g., many loops.

We formulated our results in quite simple settings, by putting the communication area equal to a ball and the user density equal to the Lebesgue measure, but it is clear that they can be extended into various directions with respect to more complex shapes and/or user distributions.

Based on our explicit formulas, we also provide some simulations at the end, in Section 7. They illustrate that most of the effects that we derived analytically in limiting settings, i.e., for large values of the parameters, already appear in a very pronounced way for quite moderate values of the parameters.

1.4 Related literature

The quality of service in highly dense relay-augmented ad-hoc networks has received particular interest in the last years. A multi-hop network with users distributed according to a Poisson point process, the intensity of which tends to infinity, was investigated in [HJKP15]. Using large-deviation methods, this paper derives the asymptotic behaviour of rare frustration events such as many users having an unlikely bad quality of service for an unusually long period of time. [HJP16] also describes frustration probabilities in a network, where relays have a bounded capacity, and users become frustrated when their connection to a relay is refused because it is already occupied; see also [HJ17].

One difference between these works and the model of the present paper introduced in [KT17] is that the latter one uses a notion of quality of service for the entire system rather than for single transmissions. In particular, trajectories with bad SIR are *a priori* not excluded. [KT17] defines a random mechanism for choosing the message trajectories of all users, given the user locations, and its results hold almost surely with respect to all users. For these results, users need not form a Poisson point process, and they can even be located deterministically [KT17, Section 1.7.4]. This is also a difference from [HJKP15], [HJP16] and [HJ17], where user locations are not fixed and their randomness is (at least partially) responsible for unlikely frustration events.

For literature remarks on the notion and use of SIR, in particular for multiple hops, and about the interference penalization term see Section 2.4.1 below.

In Section 2.4.2, we discuss the use of Gibbs sampling for an experimental realization of our Gibbs distribution. Gibbs sampling was used for telecommunication networks, e.g., in [CBK16] for optimal placement of contents in a cellular network, and in [BC12] for power control and for associating users to base stations. These Monte Carlo Markov chain methods are used to decrease some kind of cost in the system via a random mechanism, with no easily implementable deterministic methods being available. Our Gibbsian model also has this property, at least if congestion is penalized as well, and approximating our Gibbs distribution by Markov chain methods such as the Gibbs sampler is imaginable. However, the main focus of our research is the high-density limit, unlike for [CBK16] and [BC12].

1.5 Organization of this paper

In the fundamental Section 2, we present our Gibbsian model and the results of [KT17] that are relevant for the investigations of the current paper. In particular, in Section 2.3, we comment on the objects of our study, the "typical trajectory", and Section 2.4 contains discussions about modeling questions such as our motivation for the Gibbsian ansatz, the choice of the interference penalization term and possible extensions of the model.

Each of the following three sections is devoted to one of our three theoretical investigations, which form the core of this paper, i.e., the analysis of the large-distance limit (1) in Section 3, the limit of strong interference punishment (2) in Section 4 and limit of high local density of users (3) in Section 5. Each of these sections gives the question, the results, the proofs and a discussion in the respective setting.

Section 6 discusses the relevance and properties of our Gibbsian model and the related optimization problem in the light of game-theoretic considerations in traffic theory.

Finally, Section 7 gives numerical plots and studies about qualitative properties of our model.

2 The Gibbsian model and its behaviour in the high-density limit

In this section, we introduce the Gibbsian model of [KT17] and its properties in the limit of high density of users that are most relevant for the analysis of the current paper. We present the model in Section 2.1, describe its behaviour in the high-density limit in Section 2.2, comment on the notion of the typical trajectory sent out by a user in Section 2.3 and provide motivations and discussions about our setting in Section 2.4.

2.1 The Gibbsian model

We introduce the model that we study in the present paper. This model was introduced in [KT17, Section 1.2.4]; it is a special case of the general model of [KT17].

For any $n \in \mathbb{N}$ and for any measurable subset V of $\mathbb{R}^n$, let $\mathcal{M}(V)$ denote the set of all finite nonnegative Borel measures on V.

The model is defined as follows, on $\mathbb{R}^d$ with $d \in \mathbb{N}$ fixed. Let $W \subset \mathbb{R}^d$ be compact, the territory of the telecommunication system, containing the origin o of $\mathbb{R}^d$. Let $\mu \in \mathcal{M}(W)$ be an absolutely continuous measure on W with $\mu(W) > 0$. For $\lambda > 0$, we let $X^\lambda = (X_i)_{i \in I^\lambda} = (X_i)_{i=1}^{N(\lambda)}$ be a Poisson point process in W with intensity measure $\lambda\mu$, such that the *empirical* measure of X^λ normalized by $1/\lambda$,

$$L_\lambda = \frac{1}{\lambda} \sum_{i \in I^\lambda} \delta_{X_i}, \tag{2.1}$$

converges to μ almost surely as $\lambda \to \infty$. This condition is satisfied e.g. if $\lambda \mapsto X^\lambda$ is increasing; for further details see [KT17, Section 1.7.4].

Now we introduce message trajectories. For any $i \in I^\lambda$, we call a vector of the form

$$S^i = (S^i_{-1} = K_i, S^i_0 = X_i, S^i_1 \in X^\lambda, \dots, S^i_{K_i - 1} \in X^\lambda, S^i_{K_i} = o) \in \mathbb{N} \times \Big(\bigcup_{k \in \mathbb{N}} W^k \Big) \times \{o\}, \tag{2.2}$$

a *message trajectory* from X_i to o with K_i hops. That is, S^i starts from X_i and ends in o after a random number K_i of hops from user to user $\in X^\lambda$. Hence, the users also serve as relays. We fix $k_{\max} \in \mathbb{N}$ and write $\mathcal{S}^i_{k_{\max}}(X^\lambda)$ for the set of all possible realizations of the random variable S^i with $K_i \leq k_{\max}$ hops. Hence, elements of $\mathcal{S}^i_{k_{\max}}(X^\lambda)$ satisfy $s^i_{-1} \in \{1, \dots, k_{\max}\}$, $s^i_0 = X_i$ and $s^i_{s^i_{-1}} = o$. We write $\mathcal{S}_{k_{\max}}(X^\lambda) = \prod_{i \in I^\lambda} \mathcal{S}^i_{k_{\max}}(X^\lambda)$ and $[k] = \{1, \dots, k\}$ for $k \in \mathbb{N}$. Given $i \in I^\lambda$, we consider each trajectory S^i in (2.2) as an $S^i_{k_{\max}}(X^\lambda)$-valued random variable.

With this definition of message trajectories, we only consider *uplink communication*, i.e., users transmitting messages to the base station, such as in [KT17]. As we have mentioned in [KT17, Section 1.2.4], the *downlink*, i.e., the reversed direction, works very similarly, and all results of [KT17] have an analogue for the downlink with an analogous proof. We are certain that the same applies to the results of the current paper, and we abstain from spelling out the downlink case.

Next, interference is introduced as follows. We choose a *path-loss function*, which describes the propagation of signal strength over distance. This is a monotone decreasing, continuous function $\ell\colon [0, \infty) \to (0, \infty)$. A typical choice is ℓ corresponding to isotropic antennas with ideal Hertzian propagation, i.e. $\ell(r) = \min\{1, r^{-\alpha}\}$, for some $\alpha > 0$ (see e.g. [GT08, Section II.]). We write $\ell_{\max} = \max_{x,y \in W} \ell(|x - y|)$ and $\ell_{\min} = \min_{x,y \in W} \ell(|x - y|)$ for the maximal and the minimal path-loss values in the system, respectively.The *signal-to-interference ratio (SIR)* of a transmission from $X_i \in X^\lambda$ to $x \in W$ in the presence of the users in X^λ is defined [HJKP15] as

$$\mathrm{SIR}(X_i, x, X^\lambda) = \frac{\ell(|X_i - x|)}{\frac{1}{\lambda} \sum_{j \in I^\lambda} \ell(|X_j - x|)}. \tag{2.3}$$

The denominator of the r.h.s of (2.3) is the *interference*. See Section 2.4.1 for more details about the notion of SIR and the normalization term $1/\lambda$ in the interference.

If one wants to optimize the joint routeing of many messages in a multihop telecommunication network, the first question that arises is what cost function should be minimized. According to [BC12, Section II.A] and [SPW07], a routeing of trajectories is optimal w.r.t. interference if it minimizes the sum of the inverses of the SIR values over all hops of all messages. For more details, we refer the reader to Section 2.4.1.

Now, given a trajectory configuration $s = (s^i)_{i \in I^\lambda} \in \mathcal{S}_{k_{\max}}(X^\lambda)$, we put

$$\mathfrak{S}(s) = \sum_{i \in I^\lambda} \sum_{l=1}^{s^i_{-1}} \mathrm{SIR}(s^i_{l-1}, s^i_l, X^\lambda)^{-1}. \tag{2.4}$$

Next, congestion is defined as follows. We put

$$m_i(s) = \sum_{j \in I^\lambda} \sum_{l=1}^{s^j_{-1}-1} \mathbb{1}\{s^j_l = s^i_0\}, \quad i \in I^\lambda, \tag{2.5}$$

as the number of incoming hops into the user (relay) $s^i_0 = X_i$ of any of the trajectories. For $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$ we define

$$\mathfrak{M}(s) = \sum_{i \in I^\lambda} m_i(s)(m_i(s) - 1). \tag{2.6}$$

Note that $m_i(s)(m_i(s)-1)$ is the number of pairs of hops arriving at the relay $X_i = s^i_0$, and $m_i(s)(m_i(s)-1)$ if $m_i(s) \in \{0,1\}$, i.e., it only penalizes multiple hops arriving at the same relay.

The central object of study of [KT17] is a Gibbs distribution on the set of collections of trajectories as follows. For any $s = (s^i)_{i \in I^\lambda} \in \mathcal{S}_{k_{\max}}(X^\lambda)$ put

$$\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}(s) := \frac{1}{Z^{\gamma,\beta}_\lambda(X^\lambda)} \Big(\prod_{i \in I^\lambda} \frac{1}{N(\lambda)^{s^i_{-1}-1}} \Big) \exp\Big\{ -\gamma \mathfrak{S}(s) - \beta \mathfrak{M}(s) \Big\}, \tag{2.7}$$

where $\gamma > 0$ is a parameter. This is the Gibbs distribution with a uniform and independent reference measure (see [KT17, Section 1.2.2] for more details), subject to an exponential weight with the SIR term in (2.4). Here

$$Z^{\gamma,\beta}_\lambda(X^\lambda) = \sum_{r \in \mathcal{S}_{k_{\max}}(X^\lambda)} \Big(\prod_{i \in I^\lambda} \frac{1}{N(\lambda)^{r^i_{-1}-1}} \Big) \exp\Big\{ -\gamma \mathfrak{S}(r) - \beta \mathfrak{M}(r) \Big\} \tag{2.8}$$

is the normalizing constant, which is referred to as *partition function*. Note that $\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}(\cdot)$ is random conditional on X^λ, and it is a probability measure on $\mathcal{S}_{k_{\max}}(X^\lambda)$.

[KT17] derived the properties of this system in the high-density limit $\lambda \to \infty$. Due to the discontinuity of the congestion term in this limit, the case $\gamma, \beta > 0$ is substantially more involved than the case of no congestion penalization $\gamma > 0, \beta = 0$. As we already mentioned in Section 1.2, for the main part of the present paper, we will concentrate on the case $\beta = 0$ in which congestion is not penalized. The case $\beta > 0$ will occur again in the game-theoretic investigations of Section 6 and in the discussion of Section 2.4.2 about the relation of our model to Monte Carlo Markov chains.

2.2 The limiting behaviour of the telecommunication system

In this section, we summarize those results of [KT17] about the behaviour of the model described in Section 2.1 in the high-density limit $\lambda \to \infty$ that are relevant for the investigations of the present paper. These assertions will allow us to derive variational characterization of qualitative properties of the network, such as the typical number, length and shape of the message trajectories, in limits of some of the parameters tending to infinity. We consider only the case $\beta = 0$.

For $k \in \mathbb{N}$, elements of the product space $W^k = W^{\{0,1,\dots,k-1\}}$ are denoted as $(x_0, \dots, x_{k-1})$. For $l = 0, \dots, k-1$, the l-th marginal of a measure $\nu_k \in \mathcal{M}(W^k)$ is denoted by $\pi_l \nu_k \in \mathcal{M}(W)$, i.e., $\pi_l \nu_k(A) = \nu_k(W^{\{0,\dots,l-1\}} \times A \times W^{\{l+1,\dots,k-1\}})$ for any Borel set A of W.

Indeed, for fixed $k \in [k_{\max}]$ and for a trajectory family $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$, we define

$$R_{\lambda,k}(s) = \frac{1}{\lambda} \sum_{i \in I^\lambda} \delta_{(s^i_0,\dots,s^i_{k-1})} \mathbb{1}\{s^i_{-1} = k\}, \tag{2.9}$$

the empirical measures of all the k-hop trajectories, which is an element of $\mathcal{M}(W^k)$. Since each user sends out exactly one message, we have for any $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$

$$\sum_{k=1}^{k_{\max}} \pi_0 R_{\lambda,k}(s) = L_\lambda. \tag{2.10}$$

This assumption can be relaxed, see Section 2.4.3 for a discussion about this.

Note that (2.4) can be expressed in terms of $(R_{\lambda,k}(s))_{k\in[k_{\max}]}$ as follows

$$\mathfrak{S}(s) = \sum_{k=1}^{k_{\max}} \int_W R_{\lambda,k}(s)(\mathrm{d}x_0,\dots,\mathrm{d}x_{k-1}) \sum_{l=1}^{k} \frac{\int_W \ell(|y-x_l|) L_\lambda(\mathrm{d}y)}{\ell(|x_{l-1}-x_l|)}, \quad x_k = o. \tag{2.11}$$

Since $L_\lambda \Rightarrow \mu$ as $\lambda \to \infty$ and (2.10) holds for any $\lambda > 0$, for $S = (S^i)_{i\in I^\lambda}$, subsequential limits of $(R_{\lambda,k}((S))_{k\in[k_{\max}]}$ in the coordinatewise weak topology are easily seen to have the form $\Sigma = (\nu_k)_{k\in[k_{\max}]}$ with $\nu_k \in \mathcal{M}(W^k)$, satisfying

$$\sum_{k=1}^{k_{\max}} \pi_0 \nu_k = \mu. \tag{2.12}$$

For such Σ, in [KT17, Section 1.6] we have defined the following continuous analogue of (2.11)

$$\mathrm{S}(\Sigma) = \sum_{k=1}^{k_{\max}} \int_W \nu_k(\mathrm{d}x_0,\dots,\mathrm{d}x_{k-1}) \sum_{l=1}^{k} g(x_{l-1},x_l), \quad x_k = o,$$

with

$$g(x,y) = \frac{\int_W \mu(\mathrm{d}z)\ell(|z-y|)}{\ell(|x-y|)}, \tag{2.13}$$

moreover the following *entropy term* that describes counting complexity in the limit $\lambda \to \infty$:

$$\mathrm{J}(\Sigma) = \sum_{k=1}^{k_{\max}} \int_{W^k} \mathrm{d}\nu_k \log \frac{\mathrm{d}\nu_k}{\mathrm{d}\mu^{\otimes k}} + \log \mu(W) \sum_{k=1}^{k_{\max}} (k-1)\nu_k(W) \in [0,\infty], \tag{2.14}$$

with the understanding that $0\log 0 = 0\log(0/0) = 0$ and $\mathrm{J}(\Sigma) = \infty$ whenever $\nu_k \not\ll \mu^{\otimes k}$ for some k.

The key result [KT17, Proposition 1.5, parts (3), (4)] about the limiting behaviour of the telecommunication system that we will use this paper is the following.

Proposition 2.1 (Law of large numbers for the empirical measures)**.** *Let $\gamma > 0$ and $k_{\max} \in \mathbb{N}$. Then, almost surely w.r.t. X^λ, as $\lambda \to \infty$, the distribution of $\Sigma_\lambda(S) = (R_{\lambda,k}(S))_{k\in[k_{\max}]}$ under $\mathrm{P}^{\gamma,0}_{\lambda,X^\lambda}$ converges coordinatewise weakly to the unique minimizer of the variational formula*

$$\inf_{\Sigma=(\nu_k)_{k=1}^{k_{\max}} :\ \sum_{k=1}^{k_{\max}} \pi_0\nu_k = \mu} \Big(\mathrm{J}(\Sigma) + \gamma \mathrm{S}(\Sigma)\Big). \tag{2.15}$$

For $k_{\max} > 1$, the minimizer is given as $\Sigma = (\nu_k)_{k=1}^{k_{\max}}$, where

$$\nu_k(\mathrm{d}x_0,\dots,\mathrm{d}x_{k-1}) = \mu(\mathrm{d}x_0) A(x_0) \prod_{l=1}^{k-1} \frac{\mu(\mathrm{d}x_l)}{\mu(W)} \mathrm{e}^{-\gamma \sum_{l=1}^k g(x_{l-1},x_l)}, \quad x_k = o,\ k \in [k_{\max}], \tag{2.16}$$

where the normalizating function A is defined as

$$\frac{1}{A(x_0)} = \sum_{k=1}^{k_{\max}} \int_{W^{k-1}} \prod_{l=1}^{k-1} \frac{\mu(\mathrm{d}x_l)}{\mu(W)} \mathrm{e}^{-\gamma \sum_{l=1}^k g(x_{l-1},x_l)}, \qquad x_0 \in W \tag{2.17}$$

so that (2.12) *holds.*

In case $k_{\max} = 1$, it is easy to see that the unique minimizer of (2.15) is $\Sigma = (\nu_1)$ with $\nu_1 = \mu$, as this is the unique Σ satisfying (2.12).

Note that the variational formula (2.15) has indeed the form of "entropy plus energy", as anticipated before. In the minimizer (2.16), the starting points of the k-hop message trajectories, $k \in [k_{\max}]$, are chosen according to the measure $\mu(\mathrm{d}x_0)A(x_0)$ and all relaying steps according to the measure $\mu(\mathrm{d}x_l)/\mu(W)$, $l \in [k-1]$, exponentially weighted by the limiting SIR penalization term $\gamma \sum_{l=1}^k g(x_{l-1}, x_l)$.

We refer the reader to [KT17, Section 1.7] for further details about the limiting behaviour of the system. The proof of Proposition 2.1 is carried out in [KT17, Section 5].

Using the exponential decay of the summands in the denominator of (2.17) in k, it is easy to see that the measures ν_k in (2.16) are also well-defined if $k_{\max} = \infty$. However, the proof techniques of [KT17] do not allow us to generalize Proposition 2.1 to the case $k_{\max} = \infty$ or $k_{\max}$ being a function of λ and tending to infinity as $\lambda \to \infty$.

2.3 Interpretation of the limiting trajectory distribution

It is the purpose of the present paper to make further qualitative assertions about the "typical" trajectory from a given transmission site $x_0 \in W$ to the origin, after having taken the high-density limit $\lambda \to \infty$. First we need to think about what quantity we should look at and what properties of the system are reflected in it.

A definition of the "typical" trajectory as a random variable is not immediate, due to the nature of this setting. One possible definition would be something like the random variable S^{i_0} with $i_0 \in I_\lambda$ such that $S_0^{i_0}$ is the Poisson point that is closest to x_0. Another one would be the sum of the (random) empirical measure $R_\lambda(S) = \sum_{k\in[k_{\max}]} R_{\lambda,k}(S)$ on all trajectory families such that $S_0^i \in B_\varepsilon(x_0)$, properly normalized with the L_λ-mass of the ε-ball $B_\varepsilon(x_0)$ around ε (cf. (2.10)).

However, since we want to start from the limit as $\lambda \to \infty$, we will consider $\Sigma = (\nu_k)_{k\in[k_{\max}]}$ instead, the minimizer introduced in Proposition 2.1. Therefore, for fixed $x_0 \in W$, we will in this paper concentrate on the probability measure on $\bigcup_{k\in[k_{\max}]}(\{k\} \times W^{k-1})$ given by its density

$$T_{x_0}(k, x_1, \dots, x_{k-1}) = \frac{\nu_k(\mathrm{d}x_0, \mathrm{d}x_1, \dots, \mathrm{d}x_{k-1})}{\big(\sum_{k=1}^{k_{\max}} \pi_0\nu_k(\mathrm{d}x_0)\big)\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_{k-1})} = \frac{\nu_k(\mathrm{d}x_0, \mathrm{d}x_1, \dots, \mathrm{d}x_{k-1})}{\mu(\mathrm{d}x_0)\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_{k-1})}, \qquad (2.18)$$

w.r.t. $\sum_{k\in[k_{\max}]}(\delta_k \otimes \mu^{\otimes(k-1)})$. This measure carries rightfully the interpretation of the distribution of the "typical" trajectory from x_0 to the origin, after the limit $\lambda \to \infty$ has been taken. This is the main object of our study in the present paper. We normalized T_{x_0} in such a way that $\sum_{k\in[k_{\max}]} \int_{W^{k-1}} T_{x_0}(k, x_1, \dots, x_{k-1})\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_{k-1}) = 1$. According to Proposition 2.1,

$$T_{x_0}(k, x_1, \dots, x_{k-1}) = A(x_0)\mu(W)^{-(k-1)} \prod_{l=1}^{k-1} \mathrm{e}^{-\gamma \sum_{l=1}^k g(x_{l-1}, x_l)}, \qquad (2.19)$$

where we recall (2.13). We will use the convention that the 0th coordinate of T_{x_0} is the one corresponding to k and the lth is the one corresponding to x_l, for $l \in \{1, \dots, k-1\}$. This way, the marginal $\pi_0 T_{x_0}$ is a measure on $[k_{\max}]$.

We note that also the measure $M = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l\nu_k$ carries interesting information about the system. Indeed, in [KT17, Section 1.3] is was explained that $M(\mathrm{d}x)$ is the density of the *number of incoming messages* at a position $x \in W$, the typical number of incoming messages of a user at x is Poisson distributed with parameter $M(\mathrm{d}x)/\mu(\mathrm{d}x)$, and the total mass $M(W)$ is the amount of relaying hops in the entire system, with the understanding that it is zero if every message steps directly into o without any relaying hop. Part of our analysis will also be devoted explicitly to M, see Section 5.

2.4 Discussion and motivation

In this section, we explain our motivation for several aspects of the model and for the questions that we address. In Section 2.4.1 we interpret the SIR-related quantities of the model, in Section 2.4.2 we argue about the relevance of our Gibbsian ansatz, and in Section 2.4.3 we explain possible extensions of the model via allowing users to send no or multiple messages.

2.4.1 The SIR term

In this section, we discuss the SIR-related quantities of our model. We comment on the relevance of our choice of the SIR penalization term $\mathfrak{S}(s)$ in (2.4), explain the conventional definition of SIR and its relation to our understanding of SIR, discuss about the continuity of the path-loss function at 0 and sketch more realistic notions of SIR than the one (2.3).

The SIR term $\mathfrak{S}(s)$ in (2.4) quantifies the quality of the transmission of the messages when using the trajectories s^i from X_i to o. The choice of the *reciprocals* of the SIRs comes from the fact that the *bandwidth* used for a transmission is defined [SPW07] as

$$\frac{R}{\log_2(1+\mathrm{SIR}(\cdot))}, \tag{2.20}$$

where R is the data transmission rate, and SIR is defined as in (2.3) without the factor of $1/\lambda$ in the denominator of (2.3). This quantity is of order $1/\lambda$ for λ large, under the assumption that $L_\lambda \Rightarrow \mu$. In the high-density setting $\lambda \to \infty$ that we study, (2.20) can be approached well by (a constant times) the reciprocals of the SIR, since $\log(1+x) \sim x$ as $x \to 0$. [SPW07, Section 3] suggests that in case of multi-hop communication, the used bandwidth equals the sum of the used bandwidth values corresponding to the individual hops, which explains our choice of the sum over l in (2.4).

Note that the conventional definition of interference of a transmission from X_i to x is $\sum_{j \in I^\lambda \setminus \{i\}} \ell(|X_j - x|)$, in contrast to our definition in (2.3), where we added a factor of $\frac{1}{\lambda}$, following [HJKP15, Section 1]. According to this convention, we should say “total received powerïnstead of “interference", cf. [KB14, Section II.]. As we are interested in the limit $\lambda \to \infty$, where it makes no difference whether or not we add $\frac{1}{\lambda}\ell(|X_i - x|)$ to the denominator, we will stick to our notions “SIRänd “interference". For the same reason, our model does not include noise. However, note also our additional factor of $1/\lambda$, which we think is appropriate, at least mathematically, to our setting, in which we consider the high-density limit $\lambda \to \infty$. We actually scale the “usual” SIR by the density parameter. Indeed, in order to cope with an enormous number of messages in a system with one base station and a fixed bandwidth, one can either distribute the messages over a longer time stretch or decompose the messages into many smaller ones. The factor of $1/\lambda$ is a crude approximation of a combination of these two strategies.

The assumption that the path-loss function ℓ is continuous at 0 comes from [GT08, HJKP15] and is unlike the works [GK00, KB14], which make mathematical use of the perfect scaling of the path-loss function $\ell(r) = r^{-\alpha}$, which is for this reason one of the standard choices. However, for small r, this is an unrealistic choice, cf. [GK00, Section I.A], [GT08, Section I.].

We note that the notion of interference can be made more realistic according to [GK00, Section I.A] via introducing time dependence in our model. E.g., one introduces $k_{\max}$ discrete time slots, and for $l \in [k_{\max}]$, the lth hop of any message trajectory is assumed to happen at time l. Then, the interference of a transmission at time l is obtained from the starting points of all hops that happen at the same time. The SIR is defined analogously to (2.3) but with this notion of interference, which depends on the entire message trajectories rather than only on the users. Time-dependent versions of our model can be set up in various ways; for example, one could allow for messages standing still or for a longer time horizon and users transmitting multiple messages. The new notion of SIR comes with significant changes in the behaviour of the system in the high-density limit, and we decided to defer such investigations to a later work.

2.4.2 Why a Gibbsian ansatz?

Let us comment on the relevance of our ansatz of the model as a Gibbsian probability measure.

In a mathematical description of a telecommunication system, one typically requires that the SIR be larger than a given threshold $\tau > 0$, in order that the signal can be successfully transmitted. However, our model is designed in the spirit of a common wealth approach, where we do not want to consider any single message, but the total quality of transmission in the entire system. This quantity is the sum of all the reciprocal values of the SIRs of all the (hops of the) messages, which we explained in Section 2.4.1. It is exponentially weighted with a negative factor, which "softly" keeps all the SIRs at high values on an average.

One can also modify our Gibbs distribution in such a way that trajectory families exhibiting hops with SIR $\mathrm{SIR}(s^i_{l-1}, s^i_l, X^\lambda)$ less than or equal to τ have probability zero, simply by changing the SIR penalization value (2.4) to ∞ for such families, similarly to [BC12, Section III.A]. For τ large enough, almost surely, the modified model is well-posed for all $\lambda > 0$ sufficiently large. This means a change from the penalization function $x \mapsto \gamma/x$ (applied to $\mathrm{SIR}(s^i_{l-1}, s^i_l, X^\lambda)$) into the function $x \mapsto \infty \times \mathbb{1}_{[0,\tau]}(x)$. We expect that an analogue of Proposition 2.1 in [KT17, Section 5] is valid, but additional topological problems have to be addressed.

One of our motivations is to explore the physical effect of the punishment of the joint probability of the random paths, which are *a priori* randomly picked with equal probability: Does the (soft) requirement of a good transmission quality force the trajectories already to choose geometrically the shortest route? What step sizes do they choose? We would like to understand the interplay between entropy and SIR-energy and the result coming out of this by optimizing their relation.

Another motivation for us to study this model is the fact that one can use it for experimentally produce optimal routeings in a given wireless telecomunication system by making explicit simulations. Here we connect up with the theory of finding the (deterministic) optimal message routeing in a given graph whose bonds are equipped with weights that express the transmission quality. Here we consider the complete graph, where every two vertices (user locations) are connected with each other, and the weights are the reciprocals of the SIR-values along that bond. The quality of a routeing is then the sum of the weights along the trajectory, precisely as in (2.4). This optimization problem searches for the best routeing of all the messages that are to be delivered to the origin, i.e., for the minimum of $\mathfrak{S}(s)$ over s. It becomes much more interesting and relevant if also the congestion term is considered, i.e., if the term $\gamma\mathfrak{S}(s)+\beta\mathfrak{M}(s)$ is optimized. See Section 6 for a discussion of this problem in game-theoretic terms.

We think that our model is a good starting point for a numerical realization of this optimum, using a stochastic algorithm in the spirit of the famous *simulated annealing algorithm*, see [H02, Section 13], based on running a *Monte Carlo Markov chain algorithm*. In order to do this, one first has to determine an explicit Markov chain on the set of trajectory families that has our measure $\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}$ in (2.7) as its invariant distribution (in the best case, satisfying the detailed balance condition). Afterwards, one needs to determine a suitable *cooling strategy*, i.e., a recipe how to choose the parameters β and γ diverging to infinity as a function of the number of Markovian steps carried out so far. We believe that adapting classical methods such as the Gibbs sampler or the Metropolis algorithm [H02, Section 7] will turn out to be appropriate for this purpose. Then one would have to characterize the speed of convergence of such a chain to equilibrium, which is an interesting problem on its own.

2.4.3 Extensions: sending no or multiple messages

One easily sees from the proofs in [KT17, Sections 2–5] that Proposition 2.1 can be extended to the situation where users send no message or multiple messages. This models the standard situation in which large messages are cut into many smaller ones, who independently find their ways through the system.

For this, we have to enlarge the trajectory probability space: to each user $X_i \in X^\lambda$, we attach the number $P_i \in \mathbb{N}_0$ of transmitted messages, and for each $j \in \{1, \dots, P_i\}$, there is an independent trajectory $X_i \to o$. The empirical trajectory measure $R_{\lambda,k}(\cdot)$ must be augmented by these trajectories. The main additional assumption then is that $\sum_{k=1}^{k_{\max}} \pi_0 R_{\lambda,k}(\cdot)$ converges to some measure $\mu_0 \in \mathcal{M}(W)$ with $0 \neq \mu_0 \ll \mu$.

The SIR term also has to be changed. The number P_i can be interpreted as a *signal power* of the user X_i. Thus, according to [BB09, Sections 2.3.1, 5.1], the SIR of his transmission of a message to $x \in W$ should be defined as follows

$$\mathrm{SIR}((X_i, P_i),\ x,\ (X_j, P_j)_{j\in I^\lambda}) = \frac{\ell(|X_i - x|)P_i}{\frac{1}{\lambda}\sum_{j\in I^\lambda} \ell(|X_j - x|)P_j}.$$

One could also incorporate (possibly random) sizes of the messages, which would require an additional enlargement of the trajectory space.

3 Large communication areas with large transmitter–receiver distances

This section is devoted to the analysis of the highly dense telecommunication system described in Section 2.2 in regime (1), i.e., in the limit of a large communication area coupled with a large distance of the user from the base station. In Section 3.1, we present our main results. Section 3.2 discusses these results, and Section 3.3 includes their proofs.

3.1 The typical number, length and direction of hops in a large-distance limit

In this section, the main object of interest is the shape of the optimal trajectory from a certain site to the origin, in particular the typical spatial length of any of the hops, the number of hops and the spatial progress of the trajectory, in particular whether or not it runs along the straight line or how strongly it deviates from it. We will answer these questions for the special choice that W is a closed ball around the origin, μ is the Lebesgue measure on W, and the path-loss function ℓ corresponds to ideal Hertzian propagation so that $b = \int_{\mathbb{R}^d} \ell(|x|)\mathrm{d}x < \infty$, that is, $\ell(r) = \min\{1, r^{-\alpha}\}$ for some $\alpha > d$.

Furthermore, in order to obtain a pronounced picture and to make a strong assertion, we will have to assume that the starting site of our trajectory is far away from the origin. In such a setting, it is plausible to expect that as the radius of the ball tends to infinity, a proportion of users that tends to one takes the same order of magnitude of number of hops. This also gives information about the typical size and direction of each hop, already in large but still compact communication areas.

We will see that this setting exhibits the surprising property that the typical number of hops diverges to infinity as the distance of the user x_0 from o tends to infinity, however, in a sublinear way, more precisely, like the distance divided by a power of its logarithm. Second, using the asymptotics of the value of this largest summand, one can conclude about the typical size of the hops and about how much they deviate from the straight line between the transmitter and the receiver o. In our specific setting, we will be able to give precise and explicit asymptotics for all these effects encountered.

Let us now become more precise. We denote the radius of the communication area $W = \overline{B_r(o)}$ by r, and we recall that $k_{\max}$ is the maximal hop number. We consider the limit of large r and large $k_{\max}$. We consider one user placed at $x_0 \in W$ with a distance to the origin $|x_0| = r_0$ being large, such that $r > r_0$, but $r \asymp r_0$. Then one can say that x_0 is a "typical" location of a user in W, chosen uniformly at random.

In our first result, Theorem 3.1, we examine the "typical" number of hops of a trajectory from x_0 to o as a random variable under the marginal distribution $\pi_0 T_{x_0}$ on $\mathbb{N}$. According to (2.19), in the present setting, this is given by

$$\pi_0 T_{x_0}(k) = A(x_0) a_k(x_0) \qquad \text{where } a_k(x_0) = \int_{(B_r(o))^{k-1}} \prod_{l=1}^{k-1} \Big(\omega_d r^{-d}\, \mathrm{e}^{-\gamma g(x_{l-1}, x_l)}\, \mathrm{d}x_l\Big), \qquad x_k = o, \tag{3.1}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$, and we recall that $g(x_{l-1},x_l) = \frac{\int_W \mathrm{d}y\,\ell(|y-x_l|)}{\ell(|x_{l-1}-x_l|)}$. Note that $a_k(x_0)$ does not depend on whether $k \leq k_{\max}$ or $k > k_{\max}$, so it will be our first task to find its asymptotics without any reference to $k_{\max}$. Interestingly, we encounter a large-deviation principle on a quite surprising scale.

Theorem 3.1 (Large deviations for the hop number)**.** *Fix $t \in (0,\infty)$. Then, in the limit that $r_0 \to \infty$ with $r > r_0 = |x_0| \asymp r$, for any choice of $r_0 \mapsto k(r_0) \in \mathbb{N}$,*

$$\frac{1}{r_0 \log^{1-1/\alpha} r_0} \log a_{k(r_0)}(x_0) \begin{cases} = -\big(dt + b\gamma t^{1-\alpha}\big) + o(1), \\ \leq -b\gamma t^{1-\alpha} + o(1), \\ \leq -dt + o(1), \end{cases} \quad \text{if} \quad \frac{k(r_0)}{r_0 \log^{-1/\alpha} r_0} \begin{cases} \to t, \\ \leq t + o(1) \\ \geq t + o(1), \end{cases} \tag{3.2}$$

where we recall that $b = \int_{\mathbb{R}^d} \mathrm{d}y\,\ell(|y|)$.

The upper bounds in the second line of (3.2) follow from the convexity of $1/\ell(|\cdot|)$ and a comparison between the functionals $(x,y) \mapsto g(x,y)$ and $(x,y) \mapsto b/\ell(|x-y|)$. Theorem 3.1 says that $a_k(x_0)$ satisfies, with $k(r_0) \asymp r_0 \log^{-1/\alpha}(r_0)$, a large-deviation principle on the scale $r_0 \log^{1-1/\alpha} r_0$ with explicit rate function $t \mapsto dt + b\gamma t^{1-\alpha}$. It is easily seen that this rate function has a unique minimizer:

$$\min_{t\in(0,\infty)} \big(dt + b\gamma t^{1-\alpha}\big) = dt^* + b\gamma (t^*)^{1-\alpha} = \frac{(b\gamma)^{1/\alpha}}{(\alpha-1)d}\Big[d + \big((\alpha-1)d\big)^{1/\alpha}\Big], \quad \text{with } t^* = \Big(\frac{b\gamma(\alpha-1)}{d}\Big)^{1/\alpha}. \tag{3.3}$$

As a consequence, we have the following kind of law of large numbers.

Corollary 3.2. *In the limit $r_0 \to \infty$ with $r > r_0 = |x_0| \asymp r$, any maximizer $k^*(r_0)$ of $\mathbb{N} \ni k \mapsto a_k(x_0)$ satisfies*

$$k^*(r_0) \sim t^* \frac{r_0}{\log^{1/\alpha} r_0}. \tag{3.4}$$

Further, if $k_{\max} \geq k^*(r_0)$ for at least one such maximizer for all sufficiently large $r_0 > 0$, then we have

$$\frac{1}{r_0 \log^{1-1/\alpha} r_0} \log \frac{1}{A(x_0)} \to -(dt^* + b\gamma (t^*)^{1-\alpha}). \tag{3.5}$$

If $k_{\max}$ is smaller than all the minimizers, then the asymptotics of $A(x_0)$ depend on those of $a_{k_{\max}}(x_0)$ rather than on $a_{k^*(r_0)}(r_0)$, and (3.5) has to be adapted accordingly. We note that (3.5) requires only a lower bound on $k_{\max}$, and in Corollary 3.2, $k_{\max}$ could be equal to $+\infty$ for each r_0. (3.5) says that the asymptotic logarithmic behaviour of $1/A(x_0)$ on scale $r_0 \log^{1-1/\alpha} r_0$ coincides with the one of the single maximal summand $a_{k^*(r_0)}(r_0)$. Formulated in terms of the marginal distribution $\pi_0 T_{x_0}$ of T_{x_0} on the length k of the path from x_0 to o, since the behaviour of the Lebesgue measure restricted to $\overline{B_r(o)}$ is subexponential in r_0 in the large-distance limit that we are considering, we have that

$$\pi_0 T_{x_0}\Big([t^* - \varepsilon, t^* + \varepsilon]^{\mathrm{c}} \frac{r_0}{\log^{1/\alpha}(r_0)}\Big)$$

tends to zero exponentially fast on the scale $r_0 \log^{1-1/\alpha} r_0$. In Section 3.2.1 we give an explanation of how these scales come about.

In the proof of the lower bound of (3.4), the consideration of a uniform step distribution was sufficient, i.e., $t^* r_0 / \log^{1/\alpha} r_0$ straight steps directed from x_0 to o with size $r_0/k(r_0) \sim \frac{1}{t^*} \log^{1/\alpha} r_0$ each. We now show, again in terms of a large-deviation estimate on the scale $r_0 \log^{1-1/\alpha} r_0$, that macroscopic deviations from this optimal step size on the scale $\log^{1/\alpha} r_0$ have extremely small probability.

Proposition 3.3. *For $\varepsilon, \delta > 0$ and $k \in \mathbb{N}$, let*

$$D_{\varepsilon,\delta}(k, x_0) = \Big\{(x_1, \dots, x_{k-1}) \in B_r(o)^{k-1} \colon \exists I \subseteq [k-1] \colon \#I \geq \delta k, \\ \frac{1}{\#I} \sum_{l \in I} \frac{|x_{l-1} - x_l| - \big||x_{l-1}| - |x_l|\big|}{\log^{1/\alpha} r_0} > \varepsilon\Big\}, \qquad x_k = o. \tag{3.6}$$

Then, in the limit $r_0 \to \infty$ with $r > r_0 = |x_0| \asymp r$, for $k(r_0) \sim t^* r_0 / \log^{1/\alpha} r_0$,

$$\limsup \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log T_{x_0}\big(k(r_0), D_{\varepsilon,\delta}(k(r_0), x_0)\big) < 0. \tag{3.7}$$

In words, the probability that there are $\asymp k(r_0)$ steps $x_l - x_{l-1}$ in the trajectory of relays such that their average step length $\frac{1}{\#I} \sum_{i=1}^{\#I} |x_{l_i} - x_{l_i - 1}|$ deviates, for any index set I of cardinality $\asymp k(r_0)$, from the optimal step length $\frac{1}{t^*} \log^{1/\alpha} r_0 \approx r_0 / k^*(r_0)$ on that scale, decays exponentially fast to zero on the scale $r_0 \log^{1-1/\alpha} r_0$.

The proofs of Theorem 3.1, Corollary 3.2 and Proposition 3.3 are carried out in Sections 3.3.1, 3.3.2 and 3.3.3, respectively. A discussion about their relevance and an explanation of the results is found in Section 3.2.1.

We remarked in [KT17, Section 1.2.4] that the downlink scenario (i.e., messages are transmitted from o to all the users instead of the other way around) can be handled in an analogous way, as it concerns the high-density limit of the Gibbsian model. We are also sure that the results of the present section have an analogue for this setting; we abstain from spelling out the details.

Certainly, our results of this section hold for much more general communication areas W, not only for balls. Essential for our approach is only that a – in every space dimension diverging – neighbourhood of the straight line between x_0 and o is contained in W in the limit considered. The parameter d appearing in the rate function goes back to our assumption that the volume of W grows like the d-th power of r; however, other powers than d in $[1, d]$ are also possible by putting other geometric assumptions on W.

3.2 Discussion of Section 3

This section discusses the relevance, extensions and possible alternative proofs of the results of Section 3.1. In Section 3.2.1 we interpret our large-distance limit, in Section 3.2.2 we explain how the choice of the path-loss function influences our results and in Section 3.2.3 we discuss the possibility of alternative large-deviation approaches.

3.2.1 Discussion about the large-distance limit

In Section 3.1, we consider the typical trajectory in a large homogeneous multi-hop communication system with one base station in the area W, after the high-density limit has been taken. According to the basic rules in this system, virtually every step in the area W is homogeneously admitted (even those that do not bring the message any closer to the base station or even further away), but an exponential interference weight is given to the joint configuration of all the trajectories. It may appear somewhat absurd to consider a limit of large area, large distances and many steps, since with an increasing number of hops the technical difficulties and annoying side-effects become larger, but our work is meant to reveal the basic effects coming from such a setting, in particular the effect from the interference punishment, and our result in terms of a large-deviation principle gives also bounds on deviations from the extreme regime in a certain way.

Since the interference term in particular gives small weights to large steps, it may be expected that the typical trajectory turns out to follow a straight line with all the steps being of the same size, but it may also come as a surprise that the typical step size diverges like a power of the logarithm of the distance. The reason for this is the fact that *a* *priori* all the steps (within the area) are admitted and that, in the distribution T_{x_0} of the typical trajectory, a very small weight term $1/|W|$ for each step appears. This favours a small number of steps. The best compromise between this effect and the interference effect turns out to be on a logarithmic scale.

One could think of a model in which the search for the next hop is done only in a neighbourhood of the current location, which would presumably lead to a removal of the small weight term $1/|W|$ per hop and finally to a number of hops that is linear in the distance to the origin, but this would make the decay of the path-loss function ℓ obsolete and describes a fundamentally different technical organization of the telecommunication system. Such an organization is found e.g. in the continuum percolation setting of [YCG11], where the optimal number of hops turns out to be asymptotically linear in the distance from the user to the origin in a large-distance limit. Further, [YCG11, Theorem 2.1] claims that the probability of having trajectories of a significantly unusual length decays exponentially fast, which can be seen as an analogue of our Proposition 3.3.

If one wants to study the large-area limit, another idea might be to pick the intensity measure μ of the user locations as a finite measure on $\mathbb{R}^d$, e.g., a probability measure, with positive density throughout $\mathbb{R}^d$. However, in areas that are far out and with very small density, the consideration of taking first the limit $\lambda \to \infty$ of a high density and afterwards the limit of a low density makes no sense.

3.2.2 The role of the choice of the path-loss function

We derived our large-distance statements for the path-loss function $\ell(r) = \min\{1, r^{-\alpha}\}$ for $\alpha > d$, since this ℓ is thought to describe propagation of signal strength realistically, see e.g. [BB09, GT08, HJKP15]. However, following the proofs of the results of Section 3.1 in Section 3.3, we see that analogues of these results hold whenever the path-loss function ℓ has the following two properties: $\int_{\mathbb{R}^d} \ell(|x|)\mathrm{d}x < \infty$ and $1/\ell$ is convex. If ℓ satisfies these assumptions, then in our large-distance limit, in the optimal strategy (cf. Section 3.3.1), the user takes $\sim \mathrm{const.} \times k(r_0)$ hops, where $r_0 \mapsto k(r_0)$ satisfies

$$\log(r_0) \sim \ell\Big(\frac{r_0}{k(r_0)}\Big). \tag{3.8}$$

This shows that the optimal scale depends only on the tail behaviour of ℓ. Thus, for example, the results of Section 3.1 also hold for the path-loss function $\ell(r) = (K + r)^{-\alpha}$, $K > 0$, $\alpha > d$. In general, (3.8) shows that under the two above assumptions on ℓ, the optimal scale diverges to ∞ and is sublinear. The faster ℓ decays, the slower $r_0/k(r_0)$ grows. E.g., if $\ell(r) = \mathrm{e}^{-\alpha r}$ for some $\alpha > 0$, then the decisive scale is $k(r_0) \asymp r_0/\log\log r_0$.

3.2.3 Alternative large-deviation approaches

The explicit form of the trajectory distribution T_{x_0} in (2.19) seems to suggest a Markovian approach, combined with a large-deviation argument for an exponential functional of the Markov chain. One might think that a large-deviation principle for the empirical measure L_k of the k steps $x_l - x_{l-1}$ could be the core of a proof, possibly after some spatial rescaling and under conditioning on having a fixed integral of the identity with respect to L_k. The main reasons why such an argument does not work are the following. The state space and the transition kernel of the chain depend on W and on x_0 in a particularly irregular way: they induce two different scales in the interaction of the chain and therefore also change the scale of the probabilities in a non-standard way. Another problem, which is not only technical, is that the integration area for each step is unbounded in the limit $W \uparrow \mathbb{R}^d$, and the steps are integrated with respect to the Lebesgue measure. We found no way to make this route work.

3.3 Proof of the results of Section 3.1

We prove Theorem 3.1, Corollary 3.2 and Proposition 3.3 in Sections 3.3.1, 3.3.2 and 3.3.3, respectively.

All these three results tell about the limit $r_0 \to \infty$ with $r > r_0 \asymp r$, where $x_0 \in W = \overline{B_r(o)}$ has Euclidean norm $|x_0| = r_0$. Throughout this section, we will use the notation $\lim_{r,r_0}$ for this limit.

3.3.1 Proof of Theorem 3.1

We start with the lower bound in the first line of (3.2). Let us first consider $k(r_0)$ satisfying just $k(r_0) = o(r_0)$. Recall (3.1). We obtain a lower bound for $a_{k(r_0)}(r_0)$ by restricting the x_l-integral to the ball with radius one around $\frac{(k(r_0)-l)}{k(r_0)}x_0$ for $l = 1, \dots, k(r_0) - 1$. Then $|x_0|/k(r_0) - 2 \le |x_{l-1} - x_l| \le |x_0|/k(r_0) + 2$ for $l = 1, \dots, k(r_0)$. Note that $g(x_{l-1}, x_l) \le b/\ell(|x_{l-1} - x_l|)$, where we recall that $b = \int_{\mathbb{R}^d} \mathrm{d}y\, \ell(|y|)$. Hence, for any $\varepsilon \in (0, 1)$, eventually, $g(x_{l-1}, x_l) \le b|x_{l-1} - x_l|^\alpha \le (1+\varepsilon)br_0^\alpha/k(r_0)^\alpha$, since the latter goes to infinity in our current situation. This gives

$$a_{k(r_0)}(x_0) \ge (\omega_d r^d)^{-k(r_0)+1} \mathrm{e}^{-\gamma bk(r_0)(1+\varepsilon)(r_0/k(r_0))^\alpha} \ge \mathrm{e}^{-(d+\varepsilon)k(r_0)\log r_0 - \gamma bk(r_0)(1+\varepsilon)(r_0/k(r_0))^\alpha},$$

where the second inequality holds eventually, since $r_0 \asymp r$. Now an elementary optimization on $k(r_0)$ shows that $k(r_0) \asymp r_0 \log^{-1/\alpha} r_0$ is the decisive scale. Then, taking $k(r_0) \sim tr_0 \log^{-1/\alpha} r_0$ for some $t \in (0, \infty)$, carrying out the limit and making $\varepsilon \downarrow 0$ afterwards, we have

$$\liminf_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log a_{k(r_0)}(x_0) \ge -\Big(dt + \gamma bt^{1-\alpha}\Big),$$

which is the lower bound in the first line of (3.2).

Next, we verify the second line of (3.2). We actually show more, namely that if $k(r_0) \le \frac{1}{2}r_0$ for all r_0, then

$$\limsup_{r,r_0} \frac{a_{k(r_0)}}{\mathrm{e}^{-b\gamma r_0^\alpha k(r_0)^{1-\alpha}}} \le 1. \tag{3.9}$$

This will imply the second line of (3.2). Indeed, let $t > 0$ and let $r_0 \mapsto k(r_0)$ be such that $k(r_0) \le (t + o(1))r_0/\log^{1/\alpha} r_0$. Then, by (3.9)

$$\begin{aligned}\limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log a_{k(r_0)}(x_0) &\le \limsup_{r,r_0} \frac{-b\gamma k(r_0)^{1-\alpha} r_0^\alpha}{r_0 \log^{1-1/\alpha} r_0} \\ &= \limsup_{r,r_0} -b\gamma \Big(\frac{k(r_0)\log^{1/\alpha}(r_0)}{r_0}\Big)^{1-\alpha} \le -b\gamma t^{1-\alpha},\end{aligned} \tag{3.10}$$

as wanted.

In order to verify (3.9), we will first concentrate on the function $(x, y) \mapsto b/\ell(|x - y|)$ instead of $(x, y) \mapsto g(x, y)$. For any $k \in \mathbb{N}$ and any $x_0, \dots, x_k \in \overline{B_r(o)}$ satisfying $x_k = o$, using Jensen's inequality for the convex function $1/\ell(|\cdot|)$ and some elementary estimates, we obtain

$$\begin{aligned}\frac{1}{k}\sum_{l=1}^k \frac{1}{\ell(|x_{l-1} - x_l|)} &\ge \frac{1}{\ell\Big(\frac{1}{k}\sum_{l=1}^k |x_{l-1} - x_l|\Big)} \ge \Big(\frac{1}{k}\sum_{l=1}^k |x_{l-1} - x_l|\Big)^\alpha \ge \Big(\frac{\big|\sum_{l=1}^k (x_{l-1} - x_l)\big|}{k}\Big)^\alpha \\ &= \Big(\frac{|x_0 - x_k|}{k}\Big)^\alpha = \Big(\frac{r_0}{k}\Big)^\alpha\end{aligned} \tag{3.11}$$

Thus the version of $a_k(r_0)$ with $b/\ell(|x - y|)$ instead of $g(x, y)$ can be estimated as follows.

$$\int_{\overline{B_r(o)}^{k-1}} \Big(\prod_{l=1}^{k-1} \frac{\mathrm{d}x_l}{\mathrm{Leb}(\overline{B_r(o)})}\Big) \mathrm{e}^{-\gamma\sum_{l=1}^k \frac{b}{\ell(|x_{l-1}-x_l|)}} \le \int_{\overline{B_r(o)}^{k-1}} \Big(\prod_{l=1}^{k-1} \frac{\mathrm{d}x_l}{\mathrm{Leb}(\overline{B_r(o)})}\Big) \mathrm{e}^{-\gamma bk^{1-\alpha}r_0^\alpha} = \mathrm{e}^{-\gamma bk^{1-\alpha}r_0^\alpha}, \tag{3.12}$$

which implies (3.9) for this version.

We now conclude (3.9), for any $k(r_0) \in \mathbb{N}$ satisfying $k(r_0) \leq \frac{1}{2}r_0$. For this, we need to approximate the numerator $\fint_W \ell(|y-x_l|)\,\mathrm{d}y$ by b for sufficiently many l, more precisely to derive, for any $\varepsilon > 0$, a bound of the form

$$\sum_{l=1}^{k(r_0)} g(x_{l-1}, x_l) \geq (1-\varepsilon)^\alpha (b-\varepsilon) k(r_0)^{1-\alpha} r_0^\alpha \tag{3.13}$$

eventually in our limit. Indeed, then, using this in the definition of $a_{k(r_0)}(x_0)$ just as in (3.12), carrying out our limit will imply (3.9), after letting $\varepsilon \downarrow 0$.

Now we derive (3.13). Let us now define an auxiliary function $s\colon (0,\infty) \to (0,\infty)$ such that $r - s(r) \to \infty$ and $0 < r - s(r) = o(r)$ in our limit. Fix $\varepsilon \in (0, \frac{1}{4})$. The idea is to pick r so large that

$$\Big| \fint_{B_r(o)} \ell(|y-x|)\,\mathrm{d}y - b \Big| \leq \varepsilon, \qquad \forall x \in \overline{B_{s(r)}(o)}. \tag{3.14}$$

Let us assume that we are given a trajectory $(x_0, x_1, \dots, x_{k(r_0)-1}, x_{k(r_0)} = o)$ with $k(r_0) \leq \frac{1}{2}r_0$. Let us define the index of the last step outside $B_{s(r)}(o)$:

$$k_0(r_0) = \begin{cases} \max\{l \in \{0,1,\dots,k(r_0)-1\} \colon |x_l| \geq s(r)\}, & \text{if } \exists l\{0,1,\dots,k(r_0)-1\} \colon |x_l| \geq s(r)\}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

Let $r_0 > 0$ be so large that $s(r) > (1-\varepsilon)r$ and (3.14) holds. Then we have

$$\sum_{l=1}^{k(r_0)} g(x_{l-1}, x_l) \geq \sum_{l=k_0(r_0)+1}^{k(r_0)} g(x_{l-1}, x_l) \geq \sum_{l=k_0(r_0)+1}^{k(r_0)} \frac{b-\varepsilon}{\ell(|x_{l-1}-x_l|)} \tag{3.16}$$

$$\geq \frac{(b-\varepsilon)(k(r_0)-k_0(r_0))}{\ell\Big(\frac{1}{k(r_0)-k_0(r_0)} \sum_{l=k_0(r_0)+1}^{k(r_0)} |x_{l-1}-x_l|\Big)} \tag{3.17}$$

$$\geq \frac{(b-\varepsilon)(k(r_0)-k_0(r_0))}{\ell\Big(\frac{1}{k(r_0)-k_0(r_0)}(1-\varepsilon)r_0\Big)} \tag{3.18}$$

$$\geq (1-\varepsilon)^\alpha (b-\varepsilon)(k(r_0)-k_0(r_0))^{1-\alpha} r_0^\alpha \geq (1-\varepsilon)^\alpha (b-\varepsilon)(k(r_0))^{1-\alpha} r_0^\alpha. \tag{3.19}$$

Here in (3.16) we used the fact that $x_{k_0(r_0)}, \dots, x_{k(r_0)-1}, x_{k(r_0)}$ lie in $B_{s(r)}(o)$ and therefore, for the numerator of each $g(x_{l-1}, x_l)$ with $l > k_0(r_0)$, (3.14) can be applied. Next, (3.17) is an application of Jensen's inequality for $1/\ell(|\cdot|)$, and (3.18) uses the following fact. Either $k_0(r_0) = 0$, in which case

$$\sum_{l=k_0(r_0)+1}^{k(r_0)} |x_{l-1}-x_l| \geq \sum_{l=k_0(r_0)+1}^{k(r_0)} (|x_{l-1}| - |x_l|) \geq |x_0| = r_0 \geq 2k(r_0) \geq k(r_0) - k_0(r_0), \tag{3.20}$$

or $k_0(r_0) > 0$, and thus

$$\sum_{l=k_0(r_0)+1}^{k(r_0)} |x_{l-1}-x_l| \geq \sum_{l=k_0(r_0)+1}^{k(r_0)} (|x_{l-1}| - |x_l|) \geq s(r) \geq (1-\varepsilon)r > (1-\varepsilon)r_0 \geq k(r_0) \geq k(r_0) - k_0(r_0). \tag{3.21}$$

In both cases, the argument in $\ell(|\cdot|)$ is ≥ 1, and we can write the term in terms of the α-norm and the first step in (3.19) also follows. Hence, we have derived (3.13).

As for the third line of (3.2), we have the following. Note that for any $x \in \overline{B_r(o)}$, we have

$$\int_{B_r(o)} \ell(|y - re_1|)\,\mathrm{d}y \le \int_{B_r(o)} \ell(|y-x|)\,\mathrm{d}y,$$

where $e_1 = (1, 0, \dots, 0)$ is the first unit vector of $\mathbb{R}^d$.

Let us introduce the quantity $b_0 = \lim_{r\to\infty} \int_{B_r(o)} \ell(|y-re_1|)\,\mathrm{d}y = \sup_{r\in(0,\infty)} \int_{B_r(o)} \ell(|y-re_1|)\,\mathrm{d}y \in (0, b)$. Now, for any $k\colon (0,\infty) \to \mathbb{N}$, in our limit,

$$\begin{aligned}
\mathrm{Leb}(B_r(o))^{-(k(r_0)-1)} a_{k(r_0)}(x_0) &= \int_{(B_r(o))^{k(r_0)-1}} \Big(\prod_{l=1}^{k(r_0)-1} \mathrm{d}x_l\Big)\, \mathrm{e}^{-\gamma \sum_{l=1}^{k(r_0)} \frac{\int_{B_r(o)} \ell(|y-x_l|)\,\mathrm{d}y}{\ell(|x_{l-1}-x_l|)}} \\
&\le \int_{(\mathbb{R}^d)^{k(r_0)-1}} \prod_{l=1}^{k(r_0)-1} \Big(\mathrm{d}x_l\, \mathrm{e}^{-\gamma \frac{b_0-o(1)}{\ell(|x_{l-1}-x_l|)}}\Big) \\
&\le \Big(\int_{\mathbb{R}^d} \mathrm{e}^{-\gamma \frac{b_0-o(1)}{\ell(|y|)}}\,\mathrm{d}y\Big)^{k(r_0)-1} = O(1)^{k(r_0)} = \exp\big(o(k(r_0)\log r_0)\big),
\end{aligned} \tag{3.22}$$

where the first step in the last line follows from an elementary substitution and a reversion of the order of integration. Now, recall that in our limit $r \asymp r_0$. If $t > 0$ and $k(r_0) \ge (t + o(1)) r_0/\log^{1/\alpha} r_0$, we have that

$$\mathrm{Leb}(B_r(o))^{-(k(r_0)-1)} = \exp(-(dt + o(1))k(r_0)\log r_0)) = \exp\big(-(dt+o(1))r_0 \log^{1-1/\alpha} r_0\big).$$

This implies the third line of (3.2).

Next, we shall combine our arguments from the proofs of the upper bounds in the second line and in the third line of (3.2) in order to obtain the upper bound in the first line of (3.2). Indeed, for $t > 0$ and $k(r_0) \sim t r_0/\log^{1/\alpha} r_0$ and $\varepsilon > 0$, let us write $g(x_{l-1}, x_l) = \varepsilon g(x_{l-1}, x_l) + (1-\varepsilon) g(x_{l-1}, x_l)$, estimate the first term like in (3.22) and the second term with the help of (3.13). This gives eventually

$$\begin{aligned}
a_{k(r_0)}(x_0) &\le \int_{W^{k(r_0)-1}} \Big(\prod_{l=1}^{k(r_0)-1} \frac{\mathrm{d}x_l}{\mathrm{Leb}(B_r(o))}\Big)\, \mathrm{e}^{-\varepsilon\gamma \sum_{l=1}^{k(r_0)} \frac{b_0-o(1)}{\ell(|x_{l-1}-x_l|)}}\, \mathrm{e}^{-(1-\varepsilon)(1-\varepsilon)^\alpha (b-\varepsilon)\gamma t^{1-\alpha} r_0 \log^{1-1/\alpha} r_0} \\
&\le \exp\Big(-(dt-\varepsilon) r_0 \log^{1-1/\alpha} r_0 - (1-\varepsilon)^{\alpha+1}\gamma(b-\varepsilon) t^{1-\alpha} r_0 \log^{1-1/\alpha} r_0\Big).
\end{aligned} \tag{3.23}$$

Carrying out our limit and letting $\varepsilon \downarrow 0$ implies the upper bound in the first line of (3.2). This finishes the proof of Theorem 3.1. □

3.3.2 Proof of Corollary 3.2

The equality (3.4) follows immediately from the three lines of (3.2). As for (3.5), let $k^*(r_0)$ be the smallest maximizer of $k \mapsto a_k(x_0)$, and let $r_0 \mapsto k_{\max}(r_0)$ satisfy the assumption of the corollary, i.e., $k_{\max}(r_0) \ge k^*(r_0)$. The lower bound easily follows from (3.2) by estimating $1/A(x_0)$ from below by the single summand $a_{k^*(r_0)(x_0)}$ and using (3.4). As for an upper bound, we first write

$$\begin{aligned}
&\limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log \frac{1}{A(x_0)} \le \limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log\Big(\sum_{k=1}^{\lfloor \frac12 r_0\rfloor} a_k(x_0) + \sum_{k=\lfloor \frac12 r_0 \rfloor+1}^{\infty} a_k(x_0)\Big) \\
&= \max\Big\{\limsup_{r,r_0} \frac{1}{\frac12 r_0 \log^{1-1/\alpha} r_0} \log\Big(\sum_{k=1}^{\lfloor \frac12 r_0\rfloor} a_k(x_0)\Big), \limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log\Big(\sum_{k=\lfloor \frac12 r_0\rfloor+1}^{\infty} a_k(x_0)\Big)\Big\}.
\end{aligned}$$

Then the bound (3.22) implies that there exists a constant $D>0$ such that we have

$$\sum_{k=\lfloor\frac{1}{2}r_0\rfloor+1}^{\infty} a_k(x_0) \le \sum_{k=\lfloor\frac{1}{2}r_0\rfloor+1}^{\infty} (Dr_0^d)^{-k} = \frac{(Dr_0^d)^{-\lfloor\frac{1}{2}r_0\rfloor+1}}{1-\frac{1}{Dr_0^d}} \le \exp(-(\tfrac{1}{2}-o(1))r_0\log r_0),$$

wherefore

$$\limsup_{r,r_0} \frac{1}{r_0\log^{1-1/\alpha} r_0} \log\Big(\sum_{k=\lfloor\frac{1}{2}r_0\rfloor+1}^{\infty} a_k(x_0)\Big) = -\infty.$$

Moreover, the lower bound on $r_0 \mapsto k_{\max}(r_0)$ assumed in Corollary 3.2 and the first line of (3.2) together yield

$$\begin{aligned}
&\limsup_{r,r_0} \frac{1}{r_0\log^{1-1/\alpha} r_0} \log\Big(\sum_{k=1}^{\lfloor\frac{1}{2}r_0\rfloor} a_k(x_0)\Big) \\
&= \limsup_{r,r_0} \frac{1}{r_0\log^{1-1/\alpha} r_0} \log(\lfloor r_0/2\rfloor) + \limsup_{r,r_0} \frac{1}{r_0\log^{1-1/\alpha} r_0} \log a_{k^*}(x_0) = -(dt^*+\gamma b{t^*}^{1-\alpha}),
\end{aligned}$$

where we recall that $t^*=(b\gamma(\alpha-1)/d)^{1/\alpha}$ is the unique minimizer of $t\mapsto dt+t^{1-\alpha}$ on $(0,\infty)$, cf. (3.3). This implies the upper bound between the leftmost and the rightmost side of (3.5). □

3.3.3 Proof of Proposition 3.3

Let $\varepsilon,\delta>0$ be fixed. First, let us note that by the definition of T_{x_0} and the fact that the behaviour of the Lebesgue measure restricted to $\overline{B_r(o)}$ is subexponential in our limit, (3.7) is equivalent to

$$\begin{aligned}
&\limsup_{r,r_0} \frac{1}{r_0\log^{1-1/\alpha} r_0} \log\int_{D_{\varepsilon,\delta}(k(r_0),x_0)} \Big(\prod_{l=1}^{k(r_0)-1} \frac{\mathrm{d}x_l}{\mathrm{Leb}(B_r(o))}\Big)\,\mathrm{e}^{-\gamma\sum_{l=1}^{k(r_0)} g(x_{l-1},x_l)} \\
&< \limsup_{r,r_0} \frac{1}{r_0\log^{1-1/\alpha} r_0} \log a_{k(r_0)}(x_0) = -\big(dt^*+b\gamma{t^*}^{1-\alpha}\big),
\end{aligned} \tag{3.24}$$

with $k(r_0)\sim t^* r_0\log^{-1/\alpha} r_0$ and $x_{k(r_0)}=o$, where in the last step we used the first line of (3.2). For this, it suffices to show that there exists $\varepsilon'>0$ such that for any choice of $x_0\mapsto(x_1,\dots,x_{k(r_0)-1})=(x_1(x_0),\dots,x_{k(r_0)-1}(x_0))\in D_{\varepsilon,\delta}(k(r_0),x_0))$ writing $I=I(x_0,x_1,\dots,x_{k(r_0)-1})$ as in (3.6), we have

$$\liminf_{r,r_0} \frac{\sum_{l=1}^{k(r_0)} g(x_{l-1},x_l)}{k(r_0)\log r_0} = \liminf_{r,r_0} \frac{\sum_{l=1}^{k(r_0)} g(x_{l-1},x_l)}{t^* r_0\log^{1-1/\alpha} r_0} \ge b{t^*}^{-\alpha}+\varepsilon'. \tag{3.25}$$

Indeed, then one can argue analogously to (3.23) to conclude the first inequality in (3.24).

Now we prove (3.25). Similarly to the proof of the second line of (3.2), we will first replace the functional $(x,y)\mapsto g(x,y)$ by $(x,y)\mapsto\frac{b}{\ell(|x-y|)}$ everywhere.

We have, first using Jensen's inequality for the convex function $|\cdot|^\alpha$, then the definition of $D_{\varepsilon,\delta}(k(r_0),x_0)$ together with the fact that $\alpha>1$,

$$\begin{aligned}
\frac{1}{\#I}\sum_{l\in I}|x_l-x_{l-1}|^\alpha &\ge \Big(\frac{1}{\#I}\sum_{l\in I}|x_l-x_{l-1}|\Big)^\alpha \ge \Big(\frac{1}{\#I}\sum_{l\in I}\big||x_l|-|x_{l-1}|\big|\big)+\varepsilon\log^{1/\alpha} r_0\Big)^\alpha \\
&\ge \Big(\frac{1}{\#I}\sum_{l\in I}\big||x_l|-|x_{l-1}|\big|\Big)^\alpha+(\varepsilon\log^{1/\alpha} r_0)^\alpha.
\end{aligned} \tag{3.26}$$

Similarly, by Jensen's inequality and the triangle inequality,

$$\frac{\sum_{l\in[k(r_0)]\setminus I}|x_l-x_{l-1}|^\alpha}{k(r_0)-\#I}\ge\Big(\frac{1}{k(r_0)-\#I}\sum_{l\in[k(r_0)]\setminus I}|x_l-x_{l-1}|\Big)^\alpha\ge\sum_{l\in[k(r_0)]\setminus I}\Big(\frac{1}{k(r_0)-\#I}\big||x_l|-|x_{l-1}|\big|\Big)^\alpha.$$

Hence, more applications of Jensen's inequality yield

$$\begin{aligned}
&\frac{1}{k(r_0)}\sum_{l\in[k(r_0)]}|x_l-x_{l-1}|^\alpha\\
&=\frac{\#I}{k(r_0)}\frac{1}{\#I}\sum_{l\in I}|x_l-x_{l-1}|^\alpha+\frac{k(r_0)-\#I}{k(r_0)}\frac{1}{k(r_0)-\#I}\sum_{l\in[k(r_0)]\setminus I}|x_l-x_{l-1}|^\alpha\\
&\ge\frac{\#I}{k(r_0)}\Big(\frac{1}{\#I}\sum_{l\in I}\big||x_l|-|x_{l-1}|\big|\Big)^\alpha+\frac{k(r_0)-\#I}{k(r_0)}\Big(\frac{\sum_{l\in[k(r_0)]\setminus I}\big||x_l|-|x_{l-1}|\big|}{k(r_0)-\#I}\Big)^\alpha+\frac{\#I(\varepsilon\log^{1/\alpha}r_0)^\alpha}{k(r_0)}\\
&\ge\Big(\frac{1}{k(r_0)}\sum_{l\in[k(r_0)]}\big||x_{l-1}|-|x_l|\big|\Big)^\alpha+\delta\varepsilon^\alpha\log r_0\ge\Big(\frac{r_0}{k(r_0)}\Big)^\alpha+\delta\varepsilon^\alpha\log r_0\\
&=(t^{*-\alpha}+\delta\varepsilon^\alpha)\log r_0,
\end{aligned}\tag{3.27}$$

where in the penultimate step we used that $\#I\ge\delta k(r_0)$.

Now, we turn to $\ell(|\cdot|)$ instead of $|\cdot|^{-\alpha}$. Hence, we have to distinguish $|\cdot|\le1$ and $|\cdot|>1$. Let us define $I'=I'(x_0,(x_1,\dots,x_{k(r_0)-1}))\subseteq[k(r_0)]$ as the set of $l\in[k(r_0)]$ such that $|x_l-x_{l-1}|\le1$. Without loss of generality, I' is not empty. Then, after passing to a subsequence, if needed, we have that $\#I'\sim\delta'k(r_0)$ for some $\delta'\in[0,1]$. Thus,

$$\Big|\frac{1}{\#I'}\sum_{l\in I'}\frac{|x_{l-1}-x_l|-\big||x_{l-1}|-|x_l|\big|}{\log^{1/\alpha}r_0}\Big|=O(1/\log^{1/\alpha}r_0)=o(1).\tag{3.28}$$

Let us assume for a moment that $I\cap I'=\varnothing$ and $\delta'<1$. Splitting into I' and $[k(r_0)]\setminus I'$, we obtain

$$\begin{aligned}
\frac{1}{k(r_0)}\sum_{l\in[k(r_0)]}\frac{1}{\ell(|x_l-x_{l-1}|)}&\ge\frac{1}{k(r_0)}\Big(O(\#I')+\sum_{l\in[k(r_0)]\setminus I'}|x_l-x_{l-1}|^\alpha\Big)\\
&\ge\delta'-o(1)+\frac{1-\delta'-o(1)}{k(r_0)-\#I'}\sum_{l\in[k(r_0)]\setminus I'}|x_l-x_{l-1}|^\alpha.
\end{aligned}\tag{3.29}$$

We want to apply to the last term a lower bound analogous to (3.27), i.e., for the sum over $[k(r_0)]\setminus I'$ instead of $[k(r_0)]$. For this, we need that the sum of the $||x_{l-1}|-|x_l||$ satisfies a lower bound against $\approx r_0$. Using that $I\cap I'=\varnothing$, we indeed see this as follows:

$$\sum_{l\in[k(r_0)]\setminus I'}\big||x_{l-1}|-|x_l|\big|\ge-(\delta'+o(1))k(r_0)+\sum_{l\in[k(r_0)]}\big||x_{l-1}|-|x_l|\big|\ge r_0(1-o(1)).$$

Now, making a computation analogous to (3.27) for the right-hand side of (3.29), we obtain in our limit

$$\begin{aligned}
\frac{1}{k(r_0)}\sum_{l\in[k(r_0)]}\frac{1}{\ell(|x_l-x_{l-1}|)}&\ge\delta'-o(1)+(1-\delta'-o(1))\Big[\Big(\frac{r_0}{\#([k(r_0)]\setminus I')}\Big)^\alpha+\frac{\delta}{1-\delta'}\varepsilon^\alpha\log r_0\Big]\\
&\ge\Big((1-\delta')^{1-\alpha}t^{*-\alpha}+\delta\varepsilon^\alpha-o(1)\Big)\log r_0\ge(t^{*-\alpha}+\delta\varepsilon^\alpha-o(1))\log r_0.
\end{aligned}\tag{3.30}$$

The case $I \cap I' \neq \varnothing$ can be handled analogously as long as $\delta' < 1$. Indeed, in this case, (3.28) implies that $\liminf_{r,r_0} \#(I \setminus I')/k(r_0)$ and $\liminf_{r,r_0} \frac{1}{k(r_0)} \sum_{l \in I \setminus I'} (|x_{l-1} - x_l| - ||x_{l-1} - x_l||)$ are positive. Thus, a lower estimate on $\frac{1}{k(r_0)} \sum_{l \in [k(r_0)]} (|x_{l-1} - x_l| - ||x_{l-1} - x_l||)$ can still be obtained analogously to (3.29), and we observe that this lower bound tends to infinity as $\delta' \uparrow 1$.

Hence, we have in any case that (3.30) holds with $\delta\varepsilon^\alpha$ replaced by some positive number. From this, (3.25) follows for $(x, y) \mapsto g(x, y)$ replaced by $(x, y) \mapsto \frac{b}{\ell(|x-y|)}$ for some $\varepsilon' > 0$.

In order to conclude (3.25), now we will proceed similarly to the proof of the second part of (3.2), that is, we use uniform convergence of the interferences to b within $B_r(o)$ away from the boundary. Let us recall the auxiliary function s and the index $k_0(r_0)$ at (3.15). We essentially show that either a non-negligible part of the deviations from the straight line induced by the definition of $D_{\varepsilon,\delta}(k(r_0), x_0)$ takes place after the $k_0(r_0)$th hop, or the first $k_0(r_0)$ hops have a very high SIR penalization value, and in both cases, (3.25) holds.

For each x_0 with $|x_0| = r_0$, let us choose $(x_1(x_0), \dots, x_{k(r_0)-1}(x_0)) \in D_{\varepsilon,\delta}(k(r_0), x_0)$. We use the notation $\tau(r_0) = \tau(x_0, x_1(x_0), \dots, x_{k(r_0)-1}(x_0))$ for $\tau(r_0) = \frac{k_0(r_0)}{k(r_0)}$. Let us further write $I(r_0) = I(x_0, x_1(x_0), \dots, x_{k(r_0)-1}(x_0))$ for a choice of a set I according to (3.6). According to (3.30), without loss of generality we can assume that $I' = I'(x_0, x_1(x_0), \dots, x_{k(r_0)-1}(x_0)) = \varnothing$ for all x_0 considered.

In our limit, $\int_{B_r(o)} \ell(|z - y|) \mathrm{d}z = b - o(1)$ uniformly in $y \in \overline{B_{s(r)}(o)}$. Thus, in case $\tau(r_0) = 0$, (3.30) implies that (3.25) holds with some ε'. Hence, in order to conclude (3.25), we can assume that $\tau(r_0) \neq 0$ eventually in our limit. Further, by our assumptions on the function s, for any $\varepsilon'' > 0$, eventually $s(r) > (1 - \varepsilon'')r_0$. Now, on the one hand, since $x_l(x_0) \in \overline{B_{s(r)}(o)}$ for all $l > k_0(r_0)$, similarly to (3.30), the convexity of $1/\ell(|\cdot|)$ implies the following

$$\begin{aligned} \frac{1}{k(r_0)} \sum_{l \in [k(r_0)]} g(x_{l-1}, x_l) &\geq \frac{1}{k(r_0)} \sum_{l=k_0(r_0)+1}^{k(r_0)} g(x_{l-1}, x_l) \geq \frac{1}{k(r_0)} \sum_{l=k_0(r_0)+1}^{k(r_0)} \frac{b - o(1)}{\ell(|x_{l-1} - x_l|)} \\ &\geq \kappa(\varepsilon'')(1 - \tau(r_0))^{1-\alpha}(b - o(1)) t^{*-\alpha} \log r_0 \end{aligned} \tag{3.31}$$

for some function $\kappa\colon [0, 1] \to \mathbb{R}$ with $\lim_{\varrho \downarrow 0} \kappa(\varrho) = 1$. Now, taking first our limit and then $\varepsilon'' \downarrow 0$, we see that if $\liminf_{r,r_0} \tau(r_0) > 0$, then the proof of our goal (3.25) is finished. Now assume that $\liminf_{r,r_0} \tau(r_0) = 0$. After passing to a subsequence, we can assume that $\lim_{r,r_0} \tau(r_0) = 0$.

Let us first consider the case that $\liminf_{r,r_0} \frac{1}{r_0} \sum_{l=k_0(r_0)+1}^{k(r_0)} (|x_{l-1}| - |x_l|) \geq 1$ (observe that the total sum over all $l \in [k(r_0)]$ is telescoping and hence equal to r_0) and

$$\liminf_{r,r_0} \frac{1}{\#I(r_0)} \sum_{l \in I(r_0)\colon l > k_0(r_0)} \frac{|x_{l-1} - x_l| - \big||x_{l-1}| - |x_l|\big|}{\log^{1/\alpha} r_0} > 0. \tag{3.32}$$

Then one can employ an estimate analogous to (3.27) in order to conclude (3.25). Next, we investigate the case that $\limsup_{r,r_0} \frac{1}{r_0} \sum_{l=k_0(r_0)+1}^{k(r_0)} (|x_{l-1}| - |x_l|) < 1$. Then we have

$$\liminf_{r,r_0} \frac{1}{r_0} \sum_{l=1}^{k_0(r_0)} \big||x_{l-1}| - |x_l|\big| \geq \liminf_{r,r_0} \frac{1}{r_0} \sum_{l=1}^{k_0(r_0)} (|x_{l-1}| - |x_l|) > \varepsilon'''$$

for some $\varepsilon''' > 0$. Thus, using that $\int_{B_r(o)} \ell(|z - y|) \mathrm{d}z \geq b_0 - o(1)$ uniformly for $y \in \overline{B_r(o)}$ in our limit (where b_0 was defined before (3.22)), a convexity argument similar to (3.27) yields

$$\begin{aligned} \liminf_{r,r_0} \frac{1}{k(r_0) \log r_0} \sum_{l \in [k(r_0)]} g(x_{l-1}, x_l) &\geq \liminf_{r,r_0} \frac{1}{t^{*\alpha} k(r_0)^{1-\alpha} r_0^\alpha} \sum_{l=1}^{k_0(r_0)} g(x_{l-1}, x_l) \\ &\geq \liminf_{r,r_0} \varepsilon'''^\alpha t^{*-\alpha}(b_0 - o(1)) \Big(\frac{k_0(r_0)}{k(r_0)}\Big)^{1-\alpha} = \infty. \end{aligned} \tag{3.33}$$

Hence, in order to finish the proof of (3.25), it remains to consider the case that $\liminf_{r,r_0} \frac{1}{r_0}\sum_{l=k_0(r_0)+1}^{k(r_0)}(|x_{l-1}|-|x_l|) \geq 1$ but (3.32) fails. After passing to a subsequence, we can assume that $\lim_{r,r_0} w(r_0) \geq \varepsilon$, where we put

$$w(r_0) = \frac{1}{\#I(r_0)} \sum_{l\in I(r_0)\cap[k_0(r_0)]} \frac{|x_{l-1}-x_l| - \big||x_{l-1}|-|x_l|\big|}{\log^{1/\alpha} r_0}.$$

Using also that $\#I(r_0) \geq \delta k(r_0) \sim \delta t^* r_0 \log^{1/\alpha} r_0$, we have

$$\varepsilon - o(1) \leq \frac{1}{\#I(r_0)} \sum_{l\in I(r_0)\cap[k_0(r_0)]} \frac{|x_{l-1}-x_l|}{\log^{1/\alpha} r_0} \leq \Big(\frac{1}{\delta t^*}+o(1)\Big) \sum_{l\in I(r_0)\cap[k_0(r_0)]} \frac{|x_{l-1}-x_l|}{r_0}.$$

Thus, a convexity argument similar to (3.27) implies

$$\begin{aligned}
&\liminf_{r,r_0} \frac{1}{k(r_0)\log r_0} \sum_{l\in[k(r_0)]} g(x_{l-1},x_l) \geq \liminf_{r,r_0} \frac{1}{k(r_0)\log r_0} \sum_{l\in I(r_0)\cap[k_0(r_0)]} g(x_{l-1},x_l)\\
&\geq \liminf_{r,r_0} \frac{1}{k(r_0)\log r_0}(b_0-o(1))\Big(\frac{r_0\delta t^*\varepsilon}{\#(I(r_0)\cap[k_0(r_0)])}\Big)^{\alpha}\#(I(r_0)\cap[k_0(r_0)])\\
&\geq \liminf_{r,r_0} \tau(r_0)^{1-\alpha} b_0(\delta\varepsilon)^{\alpha} = \infty.
\end{aligned}$$

This shows that (3.25) holds with a suitable choice of $\varepsilon' > 0$. □

4 Strong punishment for the interference

This section is devoted to regime (2), i.e., the limit of strong penalization of interference. Our main result corresponding to this, Proposition 4.1, is stated in Section 4.1 and proven in Section 4.2.

4.1 Strong interference punishment makes message trajectories straight

Proposition 3.3 shows that in the large-distance limit, with μ being the Lebesgue measure in a large ball W, the typical message trajectory from the transmitter x_0 to $x_k = o$ under T_{x_0} does not deviate much from the straight line with high probability. In this proposition, $|x_0|$, $k = k(|x_0|)$ and the radius of W are assumed to tend to infinity in a certain way. From an application point of view, it is also desirable to see a similar effect for a fixed compact communication area W, a fixed starting site x_0 and a fixed upper bound $k_{\max} \in \mathbb{N}$ on the hop number. One way to find such an effect is to consider the limit of a large SIR penalization parameter γ. It is easily seen from (2.19) that this limit should entirely be determined by the minimizer of $W^{k-1} \ni (x_1,\dots,x_{k-1}) \mapsto \sum_{l=1}^k g(x_{l-1},x_l)$. Our next result gives criteria under which this minimizer follows a straight line and we have exponential estimates for deviations of trajectories from that.

Let us consider the case where W is a closed ball $\overline{B_r(o)}$, $r > 0$, the path-loss function ℓ is strictly monotone decreasing (but satisfies the original condition that it is continuous and positive on $[0,\infty)$). A typical choice [BB09, Section 22.1.2] is $\ell(r) = (1+r)^{-\alpha}$. Further, let us assume that the intensity measure is rotationally invariant, i.e., $\mu \circ O^{-1} = \mu$ for any orthogonal $d\times d$ matrix O. Under these conditions, we conclude that any minimizer of $W^{k-1} \ni (x_1,\dots,x_{k-1}) \mapsto \sum_{l=1}^k g(x_{l-1},x_l)$ is of the form $x_l = c_l x_0$ for $l = 1,\dots,k-1$ with positive constants $1 > c_1 > \dots > c_{k-1} > 0$. Moreover, the total probability mass carried by trajectories deviating from the straight line segment between the transmitter and o at least by some fixed positive quantity decays exponentially fast as $\gamma \to \infty$.

More precisely, writing $[[x,y]] = \{\alpha x + (1-\alpha)y | \alpha \in \mathbb{R}\}$ for the line through $x,y \in \mathbb{R}^d$, we state the following.

Proposition 4.1. *Let $r > 0, W = \overline{B_r(o)}, k_{\max} \geq 2, \ell$ and μ be fixed. Let us assume that ℓ is strictly monotone decreasing and μ is rotationally invariant.*

1 *For $x_0 \in W$, let us write*

$$\mathfrak{m}_{k_{\max}}(x_0) = \min_{k \in [k_{\max}]} \min_{x_1,\dots,x_{k-1} \in W} \sum_{l=1}^{k} g(x_{l-1}, x_l), \qquad x_k = o.$$

Then, for any minimizer $k \in [k_{\max}]$ and $x_1, \dots, x_{k-1}$, there exist $1 > c_1 > \dots > c_{k-1} > 0$ such that $x_l = c_l x_0$ for all $l \in [k-1]$.

2 *For $k \in [k_{\max}]$ and $\varepsilon > 0$, let us define*

$$D_k^\varepsilon(x_0) = \{(x_1, \dots, x_{k-1}) \in W^{k-1} \mid \exists l \in \{1, \dots, k-1\} \colon \operatorname{dist}(x_l, [[x_0, o]]) > \varepsilon\}. \tag{4.1}$$

Then, writing $T_{x_0}^\gamma = T_{x_0}$ for the measure in (2.19) *corresponding to γ, we have*

$$\sup_{x_0 \in W} \sup_{k \in [k_{\max}]} \limsup_{\gamma \to \infty} \frac{1}{\gamma} \log \sup_{(x_1,\dots,x_{k-1}) \in D_k^\varepsilon(x_0)} T_{x_0}^\gamma(k, x_1, \dots, x_{k-1}) < 0. \tag{4.2}$$

The proof of the first part of this proposition is based on simple geometric arguments, while the proof of the second part additionally uses the Laplace method. Note that in the first part, a minimizer always exists because W is compact and g is continuous. The proof is carried out in Section 4.2.

4.2 Proof of Proposition 4.1

Throughout the proof, given any number of hops $k \in [k_{\max}]$, we will always assume that $x_k = o$.

We start with proving part (1). Let us fix $x_0 \in \overline{B_r(o)}$. The fact that $(x, y) \mapsto g(x, y)$ is bounded away from 0 implies that for $x_0 = o$, $\mathfrak{m}_{k_{\max}}(x_0)$ is uniquely attained by the 1-hop trajectory from x_0 to $x_1 = o$. Thus, we can assume that $x_0 \neq o$.

Let now $k \in [k_{\max}]$ and $(x_1, \dots, x_{k-1}) \in \overline{B_r(o)}^{k-1}$. Let us assume that $\sum_{l=1}^k g(x_{l-1}, x_l) = \mathfrak{m}_{k_{\max}}(x_0)$. We show that there are $1 > c_1 > \dots > c_{k-1} > 0$ such that $x_j = c_j x_0$ for all $j \in [k-1]$, proceeding in the following steps.

(i) Let $\mathcal{H}$ denote the closed half-space of $\mathbb{R}^d$ that contains x_0 and whose boundary is orthogonal to the vector from x_0 to o and contains o. Then $(x_1, \dots, x_{k-1}) \in \mathcal{H}^{k-1}$.

(ii) $(x_1, \dots, x_{k-1}) \in (\mathcal{H} \cap [x_0, o])^{k-1}$, where we write $[x, y] = \{\alpha x + (1-\alpha) y \colon \alpha \in [0, 1]\}$ for the closed segment between $x, y \in \mathbb{R}^d$.

(iii) $|x_0| > |x_1| > \dots > |x_{k-1}| > 0$.

We prove these claims respectively as follows.

(i) Assume that the assertion does not hold, then let us define another trajectory $(x_1', \dots, x_{k-1}') \in \mathcal{H}^{k-1}$ via $x_l' = x_l$ if $x_l \in \mathcal{H}$ and x_l' being the image of x_l under reflection across the boundary hyperplane of $\mathcal{H}$ otherwise, for all $l \in [k-1]$. The rotation invariance of μ and W, combined with $|x_l| = |x_l'|$, implies that

$$\int_W \mu(\mathrm{d}y) \ell(|x_l - y|) = \int_W \mu(\mathrm{d}y) \ell(|x_l' - y|), \qquad l \in [k_{\max}]. \tag{4.3}$$

But, since $|x_{l-1} - x_l| \geq |x_{l-1}' - x_l'|$ and ℓ is strictly decreasing,

$$\ell(|x_{l-1} - x_l|) \leq \ell(|x_{l-1}' - x_l'|), \tag{4.4}$$

where equality holds if and only if x_{l-1}, x_l are both in $\mathcal{H}$ or both in $\mathbb{R}^d \setminus \mathcal{H}$. We conclude that $\sum_{l=1}^k g(x_{l-1}, x_l) > \sum_{l=1}^k g(x_{l-1}', x_l')$, which contradicts $(x_1, \dots, x_{k-1})$ being the minimizer in (4.1).

(ii) The case $d = 1$ is trivial. Let us consider the case $d \geq 2$. Assume $(x_1, \ldots, x_{k-1}) \in \mathcal{H}^{k-1}$. Let us define another trajectory $(x'_1, \ldots, x'_{k-1}) \in (\mathcal{H} \cap [x_0, o])^{k-1}$ such that for all $l \in [k-1]$, x'_l satisfies $|x'_l| = |x_l|$ and $x'_l \in [x_0, o]$. That is, $x'_l = x_0|x_l|/|x_0|$. Then, the radial symmetry of μ implies that (4.3) holds. Furthermore, the fact that ℓ is strictly decreasing but $|x_{l-1} - x_l| \geq |x'_{l-1} - x'_l|$ implies that also (4.4) is true in this case, where equality holds if and only if $x_l = x'_l$ for all $l \in [k-1]$, i.e., if $x_l \in [x_0, o]$ for all $l \in [k-1]$.

(iii) Let $(x_1, \ldots, x_{k-1}) \in [x_0, o]^{k-1}$. In the following argument, we cancel in this trajctory all steps that increase the distance to o, and we show that the sum of the SIR terms gets smaller by this. Indeed, let us define $i_0 = 0$ and $i_j = \inf\{l \in [k] \colon |x_l| < |x_{i_{j-1}}|\}$, $j = 1, \ldots, k$. Let m be the largest index j such that $i_j < \infty$, then it is clear that $1 \leq m \leq k$ since $x_0 \neq o$. Now, let us define an m-hop trajectory with relay sequence $(y_1, \ldots, y_{m-1}) = (x_{i_1}, \ldots, x_{i_{m-1}})$, writing $y_0 = x_0$ and $y_m = o$. Let us further define $\varepsilon' = \min_{x,y \in \overline{B_r(o)}} g(x, y) > 0$. Then, since for any $j \in [m-1]$ we have that $|x_{i_j-1} - x_{i_j}| \geq |x_{i_{j-1}} - x_{i_j}|$, we conclude that

$$\sum_{j=1}^{m} g(y_{j-1}, y_j) = \sum_{j=1}^{m} g(x_{i_{j-1}}, x_{i_j}) \leq \sum_{j=1}^{m} g(x_{i_j-1}, x_{i_j}) \leq \sum_{l=1}^{k} g(x_{l-1}, x_l) - (k-m)\varepsilon'.$$

Thus, $(x_1, \ldots, x_{k-1})$ can only minimize (4.1) if $k = m$, that is, if $|x_0| > |x_1| > \ldots > |x_{k-1}| > 0$.

This finishes the proof of part (1) of Proposition 4.1.

As for part (2), we note that the case $d = 1$ is trivial since $D^\varepsilon_k(x_0) = \varnothing$ for all $x_0 \in \overline{B_r(o)}$. Throughout the rest of the proof, let $d \geq 2$. First, we fix $x_0 \in \overline{B_r(o)}$ and $k \in [k_{\max}]$, and we verify that

$$\limsup_{\gamma\to\infty} \frac{1}{\gamma} \log \sup_{(x_1,\ldots,x_{k-1}) \in D^\varepsilon_k(x_0)} T^\gamma_{x_0}(k, x_1, \ldots, x_{k-1}) < -\kappa \tag{4.5}$$

for some $\kappa > 0$ that neither depends on x_0 nor on k. This will imply (4.2).

Again, it is easy to see that if $x_0 = 0$, then (4.5) holds for some $\kappa > 0$, let us therefore assume that $x_0 \neq o$. We first verify that there exists $\delta = \delta(\varepsilon) > 0$, independent of x_0 and k, such that

$$\mathfrak{m}^\varepsilon_{k_{\max}}(x_0) = \inf_{(x_1,\ldots,x_{k-1}) \in D^\varepsilon_k(x_0)} \sum_{l=1}^{k} g(x_{l-1}, x_l) \geq \mathfrak{m}_{k_{\max}} + \delta(\varepsilon). \tag{4.6}$$

In the construction of $(x_1, \ldots, x_{k-1}) \mapsto (x'_1, \ldots, x'_{k-1})$ in the proof of (i) above, the fact that $\mathrm{dist}(x_l, [[x_0, o]]) = \mathrm{dist}(x'_l, [[x_0, o]])$ for all $l \in [k-1]$ and $k \in [k_{\max}]$ implies that if $(x_1, \ldots, x_{k-1}) \in D^\varepsilon_k(x_0)$, then $(x'_1, \ldots, x'_{k-1}) \in D^\varepsilon_k(x_0) \cap \mathcal{H}^{k-1}$. It follows that the infimum in (4.6) can be realized along sequences of trajectories that have all their relays $x_1, \ldots, x_{k-1}$ in $\mathcal{H}$.

Let now $(x_1, \ldots, x_{k-1}) \in D^\varepsilon_k(x_0) \cap \mathcal{H}^{k-1}$, and consider the construction of $(x_1, \ldots, x_{k-1}) \mapsto (x'_1, \ldots, x'_{k-1})$ in the proof of (ii) above. We observe the following. Since $x_0 \in [x_0, o]$ and $(x_1, \ldots, x_{k-1}) \in D^\varepsilon_k(x_0)$, there exists $l_1 \in [k]$ such that

$$\mathrm{dist}(x_{l_1}, [[x_0, o]]) > \mathrm{dist}(x_{l_1-1}, [[x_0, o]]) + \frac{\varepsilon}{k} \geq \mathrm{dist}(x_{l_1-1}, [[x_0, o]]) + \frac{\varepsilon}{k_{\max}},$$

where each $[[x_0, o]]$ can also be replaced by $[x_0, o]$. One easily sees that this bound holds uniformly in $x_0 \in W$ and $k \in [k_{\max}]$.

Now, the Pythagoras theorem together with the fact that ℓ is strictly monotone decreasing yields that in this case there exists $\delta'(\varepsilon) > 0$ such that $\ell(|x_{l_1-1} - x_{l_1}|) < \ell(|x'_{l_1-1} - x'_{l_1}|) - \delta'(\varepsilon)$. Note that $\delta'(\varepsilon)$ depends only on ℓ, r and ε but not on k or l_1. On the other hand, by the rotational symmetry of μ, the equality (4.3) holds for all $l \in [k]$ for this choice of the relays x_l and x'_l. Therefore, we conclude that there exists a constant $\delta = \delta(\varepsilon) > 0$ such that for all $n \in \mathbb{N}$ we have

$$\sum_{l=1}^{k} g(x_{l-1}, x_l) > \sum_{l=1}^{k} g(x'_{l-1}, x'_l) + \delta(\varepsilon) \geq \mathfrak{m}_{k_{\max}} + \delta(\varepsilon).$$

This implies (4.6), and the construction shows that $\delta(\varepsilon) > 0$ can be chosen independently of x_0 and k.

We now finish the proof of part (2). Let us use the notation $A^\gamma(x_0) = A(x_0)$ for the normalization term in (2.17) corresponding to γ and recall the notation $T^\gamma_{x_0} = T_{x_0}$ from Proposition 4.1. It is clear from the Laplace method [DZ98, Section 4.3] that we have

$$A^\gamma(x_0) = \mathrm{e}^{o(\gamma)+\gamma \mathfrak{m}_{k_{\max}}(x_0)} \quad \text{as } \gamma \to \infty.$$

For any $(x_1, \dots, x_{k-1}) \in D_k^\varepsilon$, using (2.16) and (4.6), we can estimate

$$T^\gamma_{x_0}(k, x_1, \dots, x_{k-1}) = \frac{\nu_k^\gamma(\mathrm{d}x_0, \dots, \mathrm{d}x_{k-1})}{\mu(\mathrm{d}x_0)\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_{k-1})} \leq \mathrm{e}^{o(\gamma)+\gamma\mathfrak{m}_{k_{\max}}(x_0)-\gamma\mathfrak{m}^\varepsilon_{\max}(x_0)} \leq \mathrm{e}^{o(\gamma)-\gamma\delta(\varepsilon)}.$$

We conclude (4.5) (with $\kappa > 0$ being independent of $x_0 \in W$ and $k \in [k_{\max}]$). Thus, part (2) of Proposition 4.1 follows.

5 High local density of users

This section describes the behaviour of the system in regime (3), i.e., in the limit of a high local density of users in a subset of the communication area. In Section 5.1, we explain both global and local aspects of this limit. We formulate a result, Proposition 5.1, about the global aspects, the proof of which is carried out in Section 5.2.

5.1 Global and local relaying behaviour

We consider the following question about the behaviour of our model given by (2.16), assuming always that $k_{\max} \geq 2$.

> Does the density of trajectories increase unboundedly in a densely populated subarea, or do the messages avoid such area for the sake of having lower interference?

In order to give substance to this question, we replace our user density measure μ by

$$\mu^a = \mu + a\mathrm{Leb}|_\Delta \in \mathcal{M}(W), \qquad a \in (0, \infty), \tag{5.1}$$

where $\mathrm{Leb}|_\Delta$ is the Lebesgue measure concentrated on a compact set $\Delta \subseteq W$, seen as a measure on W. We think of Δ as of a set of very high concentration of users and will consider the behaviour of the optimal path trajectory in the limit $a \to \infty$. We will from now on label all objects that depend on μ^a instead of μ with the index a. We will study the measure

$$M^a = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k^a, \tag{5.2}$$

where ν_k^a is defined according to (2.16). It receives the interpretation [KT17, Section 1.3] of the measure of all the incoming messages at a given location (see also Section 2.3). Note that the total mass $M^a(W)$ is zero if all messages go directly to the base station without any relaying hop; hence it is a measure for the total amount of hops. Explicitly, we have

$$M^a(\mathrm{d}x) = \mu^a(\mathrm{d}x) \int_W \mu^a(\mathrm{d}x_0) \frac{\sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \int_{W^{k-2}} \prod_{l' \in [k-1]\setminus\{l\}} \mu^a(\mathrm{d}x_{l'})\, \mathrm{e}^{-\gamma \sum_{l'=1}^{k} g^a(x_{l'-1}, x_{l'})}\Big|_{x_l = x}}{\sum_{k=1}^{k_{\max}} \int_{W^{k-1}} \prod_{l=1}^{k-1} \mu^a(\mathrm{d}x_l) \mathrm{e}^{-\gamma \sum_{l=1}^{k-1} g^a(x_{l-1}, x_l)}}. \tag{5.3}$$

Now we are interested in the behaviour of the measure M^a as $a \to \infty$. Since $(x,y) \mapsto \ell(|x-y|)$ is bounded away from 0 on $W \times W$, we first note that the large-a behaviour of the SIR term is given by

$$\lim_{a\to\infty} \frac{1}{a} g^a(x,y) = \frac{\int_\Delta \mathrm{d}z\, \ell(|y-z|)}{\ell(|x-y|)} =: g_\Delta(x,y), \quad x,y \in W. \tag{5.4}$$

The limiting function g_Δ measures the SIR only in relation with the interference coming from Δ. This ratio will turn out to be decisive and the effective SIR term in the limit $a \to \infty$.

Our first result is that, when the path-loss function $(x,y) \mapsto \ell(|x-y|)$ does not vary much on $W \times W$, the presence of the highly dense area Δ has a strongly repellent effect anywhere in the system and suppresses all the relaying hops; indeed, the total mass of the measure M^a tends to zero as $a \to \infty$.

Proposition 5.1 (Criterion for exponential decay of the amount of relays)**.** *We have*

$$\sup_{x\in W} \limsup_{a\to\infty} \frac{1}{a} \log \frac{M^a(\mathrm{d}x)}{\mathrm{d}x} < 0 \tag{5.5}$$

if and only if

$$\min_{x_0\in W} \Big[\min_{x_1\in W} \big(g_\Delta(x_0,x_1) + g_\Delta(x_1,o) \big) - g_\Delta(x_0,o) \Big] > 0, \tag{5.6}$$

Remark 5.2. (i) The condition (5.5) implies an exponential decay of the total mass of M^a, i.e.,

$$\limsup_{a\to\infty} \frac{1}{a} \log M^a(W) < 0.$$

(ii) Since μ^a is clearly subexponential in $a \to \infty$, (5.5) is equivalent with an exponential decay of the density of M^a with respect to μ^a instead of $\mathrm{Leb}|_W$.

(iii) The condition in (5.6) says that the effective SIR for a two-hop trajectory is uniformly worse than the one of a direct hop to the origin. This criterion involves only one- and two-hop trajectories and is valid even when $k_{\max}$ is much larger than 2.

(iv) Multiplying with two of the three denominators in (5.6) and using that the map $W \times W \ni (x,y) \mapsto \ell(|x-y|)$ is bounded and bounded away from zero, we easily see that (5.6) holds if and only if

$$\min_{x_0,x_1\in W} \Big[\ell(|x_1|) \int_\Delta \ell(|z-x_1|)\,\mathrm{d}z + \ell(|x_0-x_1|) \int_\Delta \ell(|z|)\,\mathrm{d}z - \frac{\ell(|x_1|)\ell(|x_0-x_1|)}{\ell(|x_0|)} \int_\Delta \ell(|z|)\mathrm{d}z \Big] > 0. \tag{5.7}$$

(v) A sufficient criterion for (5.6) to hold is as follows. Let $p \in (0,1]$ be such that $p\ell_{\max} = \ell_{\min}$, where we recall that $\ell_{\max}$ and $\ell_{\min}$ are the maximal and minimal values of $W \times W \ni (x,y) \mapsto \ell(|x-y|)$, respectively. Then, a lower bound for the left-hand side of (5.7) is $\ell_{\max}^2 \mathrm{Leb}(\Delta)(2p^2 - \frac{1}{p})$. This is positive as long as p is larger than $2^{-1/3} \approx 0.794$.

Similarly, an upper bound on the left-hand side of (5.7) in terms of p is $\ell_{\max}^2 \mathrm{Leb}(\Delta)(2-p^3)$, but this is larger than zero for all $p \in (0,1]$, so such a general estimate cannot be used for disproving (5.7) in any case.

The proof of Proposition 5.1 is carried out in Section 5.2.

In our numerical results in Examples 7.1 and 7.2 with $W = \Delta$, the condition (5.6) does not hold.

We give now a discussion of spatial statements saying that the quality of service (SIR penalization with interference coming only from Δ) is significantly worse for messages relaying through a neighbourhood of Δ than through an area sufficiently far away from Δ. For simplicity, we do this only for $k_{\max} = 2$, a very small set Δ and a special choice of the path-loss function. We will give arguments that suggest that, for any large a, it is strictly suboptimal to relay through a neighbourhood of Δ as opposed to circumventing Δ sufficiently far.

Analogously to (5.10)–(5.11), the large-a limit for the mass of all relaying hops from x_0 into a set $A \subset W$ (assumed nice, e.g., being equal to the closure of its interior) and further to o is given by

$$-\lim_{a\to\infty} \frac{1}{a} \log T^a_{x_0}(2, A) = \gamma \Big[\Xi_{x_0}(A) - \min\big\{ g_\Delta(x_0, o), \Xi_{x_0}(W) \big\} \Big], \tag{5.8}$$

where

$$\Xi_{x_0}(A) = \min_{x_1 \in W\cap A} [g_\Delta(x_0, x_1) + g_\Delta(x_1, o)].$$

We want to discuss under what circumstances $\Xi_{x_0}(A)$ is smaller for sets A that are bounded away from Δ than for A being a neighbourhood of Δ. For simplicity, let us do that for $W = \mathbb{R}^d$ and very small sets $\Delta = B_r(y_0)$ with $r \ll 1$ only, i.e., we approximate

$$g_\Delta(x, y) \approx |\Delta| \frac{\ell(|y - y_0|)}{\ell(|y - x|)}, \qquad x, y \in \mathbb{R}^d. \tag{5.9}$$

Hence, we will put $\Delta = \{y_0\}$ and discuss the function

$$f_{x_0,y_0}(\varepsilon) = \min_{x_1\in W\colon |x_1-y_0|=\varepsilon} \Big[\frac{\ell(|x_1 - y_0|)}{\ell(|x_0 - x_1|)} + \frac{\ell(|y_0|)}{\ell(|x_1|)} \Big], \qquad \varepsilon \geq 0.$$

This is an approximation of $\Xi_{x_0}(\partial B_\varepsilon(y_0))$. We will see that, under quite general conditions, $f_{x_0,y_0}(\varepsilon) < f_{x_0,y_0}(0)$ for all $\varepsilon \in [0, \varepsilon_0]$ for some $\varepsilon_0 > 0$. This means that, for all sufficiently large a, the probability weight for trajectories $x_0 \to B_{\varepsilon_0-\delta}(y_0) \to o$ is exponentially smaller than the one for trajectories $x_0 \to B_{\varepsilon_0}(y_0)^{\mathrm{c}} \to o$ for any $\varepsilon_0 > \delta > 0$.

To do this, use the triangle inequality and the monotonicity of ℓ to see that

$$f_{x_0,y_0}(\varepsilon) \leq \widetilde{f}_{x_0,y_0}(\varepsilon) := \frac{\ell(\varepsilon)}{\ell(|x_0 - y_0| + \varepsilon)} + \frac{\ell(|y_0|)}{\ell(|y_0| + \varepsilon)}.$$

Note that $\widetilde{f}_{x_0,y_0}(0) = f_{x_0,y_0}(0)$ and that

$$\widetilde{f}'_{x_0,y_0}(0) = \frac{\ell'(0)}{\ell(|x_0 - y_0|)} - \frac{\ell(0)\ell'(|x_0 - y_0|)}{\ell(|x_0 - y_0|)^2} - \frac{\ell'(|y_0|)}{\ell(|y_0|)}.$$

Note that for the choice $\ell(r) = (1+r)^{-\alpha}$ for some $\alpha > 0$, this is negative as soon as $|x_0 - y_0|(1+|x_0 - y_0|)^{\alpha-1} > (1 + |y_0|)^{-1}$, i.e., as soon as y_0 is sufficiently far away from o, given the distance of the transmission site x_0 from y_0. This proves the announced conclusion that a two-hop transmission from x_0 to the origin is strictly not optimal if the relaying step uses a neighbourhood of y_0; here we used no information about the spatial relation of the three sites x_0, y_0 and o, but the fact that $\ell'(0) < 0$. However, for the path-loss function $\ell(r) = \min\{1, r^{-\alpha}\}$, this argument does not work, since $\widetilde{f}'_{x_0,y_0}(0) > 0$ (because $\ell'(0) = 0$).

5.2 Proof of Proposition 5.1

We analyze the behaviour of the left-hand side of (5.5). Taking the limit $a \to \infty$, we obtain for fixed $x, x_0 \in W$ for the numerator of (5.3)

$$\begin{aligned} &\lim_{a\to\infty} \frac{1}{a} \log \Big[\sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \int_{W^{k-2}} \prod_{l'\in[k-1]\setminus\{l\}} \mu^a(\mathrm{d}x_{l'}) \exp\Big(-\gamma \sum_{l'=1}^{k} g^a(x_{l'-1}, x_{l'})\Big|_{x_l=x} \Big) \Big] \\ &= -\gamma \min_{k\in[k_{\max}]\setminus\{1\}} \min_{l\in[k-1]} \min_{x_1,\dots,x_{l-1},x_{l+1},\dots,x_{k-1}\in W} \sum_{l'=1}^{k} g_\Delta(x_{l'-1}, x_{l'})\Big|_{x_l=x}. \end{aligned} \tag{5.10}$$

On the other hand, for the denominator of (5.3) for x_0 fixed, we have

$$\begin{aligned} &\lim_{a\to\infty}\frac{1}{a}\log\Big[\sum_{k=1}^{k_{\max}}\int_{W^{k-1}}\prod_{l=1}^{k-1}\mu^a(\mathrm{d}x_l)\,\exp\Big(-\gamma\sum_{l=1}^{k-1}g^a(x_{l-1},x_l)\Big)\Big]\\ &\qquad=-\gamma\min_{k\in[k_{\max}]}\min_{x_1,\dots,x_{k-1}\in W}\sum_{l=1}^{k}g_\Delta(x_{l-1},x_l).\end{aligned}\tag{5.11}$$

These two assertions follow from the Laplace method [DZ98, Section 4.3] in a standard way, since the a-dependence of the integrating measure μ^a is clearly subexponential. Hence, we obtain that

$$\begin{aligned}\lim_{a\to\infty}\frac{1}{a}\log M^a(\mathrm{d}x)=-\gamma\min_{x_0\in W}\Big[&\min_{k\in[k_{\max}]\setminus\{1\}}\min_{l\in[k-1]}\min_{x_1,\dots,x_{l-1},x_{l+1},\dots,x_{k-1}\in W}\sum_{l'=1}^{k}g_\Delta(x_{l'-1},x_{l'})\Big|_{x_l=x}\\ &-\min_{k\in[k_{\max}]}\min_{x_1,\dots,x_{k-1}\in W}\sum_{l=1}^{k}g_\Delta(x_{l-1},x_l)\Big].\end{aligned}\tag{5.12}$$

Note that after taking supremum over $x\in W$ on the right-hand side of (5.12), we obtain a negative number if and only if

$$\min_{x_0\in W}\Big[\min_{k\in[k_{\max}]\setminus\{1\}}\min_{x_1,\dots,x_{k-1}\in W}\sum_{l=1}^{k}g_\Delta(x_{l-1},x_l)-g_\Delta(x_0,o)\Big]>0.\tag{5.13}$$

Now, assume that the condition (5.6) does not hold. Then we may pick $x_0',x_1'\in W$ with $(g_\Delta(x_0',x_1')+g_\Delta(x_1',o)-g_\Delta(x_0',o))\le 0$. But this implies that (5.13) is false, as is shown by taking $k=2$, $x_0=x_0'$ and $x=x_1'$. We conclude that (5.5) does not hold.

Conversely, let us assume that (5.5) is not satisfied and let us conclude that (5.6) also does not hold. Using (5.5) and (5.12), we can choose $x_0\in W$, $k\in[k_{\max}]\setminus\{1\}$ and $x_1,\dots,x_{k-1}\in W$ such that

$$\sum_{l=1}^{k}g_\Delta(x_{l-1},x_l)\le g_\Delta(x_0,o),\qquad x_k=o.\tag{5.14}$$

Let k be minimal for x_0 with this property. We show that there exists $x_0',x_1'\in W$ such that $g_\Delta(x_0',x_1')+g_\Delta(x_1',o)\le g_\Delta(x_0',o)$, wherefore (5.6) does not hold. Indeed, if this is not the case for $x_0'=x_{k-2}$ and $x_1'=x_{k-1}$, then we have

$$\sum_{l=1}^{k-2}g_\Delta(x_{l-1},x_l)+g_\Delta(x_{k-2},o)\le\sum_{l=1}^{k}g_\Delta(x_{l-1},x_l)\le g_\Delta(x_0,o)<\sum_{l=1}^{k-2}g_\Delta(x_{l-1},x_l)+g_\Delta(x_{k-2},o),$$

where in the last step we used the minimality of k for x_0. This is a contradiction, and thus (5.6) has been disproven. The proposition follows. □

6 Game-theoretic interpretation of the optimization problem

In Section 2.4.2 we explained how our model that we introduced in Section 2.1 can be employed for obtaining a numerical simulation algorithm for finding minimizer(s) s of $\gamma\mathfrak{S}(s)+\beta\mathfrak{M}(s)$, i.e., including the congestion term. In this section, we give a more thorough discussion of this optimization problem from a game-theoretical point of view. In particular, we explain in which sense our model is selfish or not selfish and give a number of explicit examples for illustration. Note that in the term $\mathfrak{S}(s)$ there is no interaction between the trajectories (only with the users), but in the term $\mathfrak{M}(s)$. We therefore keep both $\beta>0$ and $\gamma>0$ fixed.

Let $X^\lambda = \{X_1, \dots, X_n\}$ be fixed, where $n \in \mathbb{N}$. For the rest of this section, we simplify the notation as follows. We write $\mathcal{S} = \mathcal{S}_{k_{\max}}(X^\lambda)$ and for $i \in [n]$, $\mathcal{S}^i = \mathcal{S}^i_{k_{\max}}(X^\lambda)$. Let now $s = (s^i)_{i=1}^n \in \mathcal{S}$ be a collection of message trajectories. For $i \in [n]$, let us write $k_i(s) = s^i_{-1}$ for the number of hops taken by the ith trajectory s^i sent out from X_i to o. Then, in terms of interference and congestion, the *individual cost* $C_i(s)$ of s^i w.r.t. the entire family s is the individual interference penalization of s^i, together with the congestion penalization at all the relays that s^i uses:

$$C_i(s) = \gamma \sum_{l=1}^{k_i(s)} \mathrm{SIR}^{-1}(s^i_{l-1}, s^i_l, X^\lambda) + \beta \sum_{l=1}^{k_i(s)-1} \sum_{j=1}^{n} (m_j(s) - 1) \mathbb{1}\{s^i_l = j\}. \tag{6.1}$$

The *total cost* of the telecommunication system is defined as

$$C(s) = \sum_{i=1}^n C_i(s) = \gamma \mathfrak{S}(s) + \beta \mathfrak{M}(s) = \gamma \sum_{i=1}^n \sum_{l=1}^{k_i(s)} \mathrm{SIR}^{-1}(s^i_{l-1}, s^i_l, X^\lambda) + \beta \sum_{j=1}^n m_j(s)(m_j(s) - 1).$$

We say that s is *system-optimal* if $C(s) \leq C(s')$ for all $s' \in \mathcal{S}$.

For a collection $s = (s^i)_{i=1}^n$ of trajectories we write $s = s_i(s^i, s^{-i})$, where $s^{-i} = (s^j)_{j \in [n], j \neq i}$. Now, given i and $s^{-i} = (s^j)_{j \neq i}$ with $s^j \in \mathcal{S}^j$ for all $j \neq i$, a *best response* of the ith user for s^{-i} is $u^i \in \mathcal{S}^i$ such that $C_i(s_i(u^i, s^{-i})) \leq C_i(s_i(s^i, s^{-i})) = C_i(s)$ for all $s^i \in \mathcal{S}^i$. We say [NRTV07, Section 1.3.3] that $s = (s^i)_{i=1}^n$ is a *pure Nash equilibrium* if s^i is a best response for $s^{-i} = (s^j)_{j \neq i}$ for all $i \in [n]$.

Claim 6.1. *For $\beta, \gamma, \lambda > 0$, given X^λ (with $n > 0$), a pure Nash equilibrium always exists.*

Proof. The claim follows from the well-known result [NRTV07, Theorem 18.12] that unweighted atomic congestion games always have a pure Nash equilibrium. Indeed, the cost functions C_i, $i \in [n]$, and C are the individual respectively total costs in an unweighted atomic congestion game (atomic instance) [NRTV07, Section 18], which is defined as follows. For each $i \in [n]$, the set of all possible paths $s^i \in \mathcal{S}^i$ of length at most $k_{\max}$ from X_i to o via users in $X_j \in X^\lambda$ without visiting the same X_j twice can be seen as the set of the strategies of the ith user (player) X_i. Each user uses precisely one of its strategies, i.e., the game is unweighted, and each user has a finite number of strategies. Indeed, for the sake of optimization of individual and total costs, we can neglect trajectories with loops since removing any loop from the trajectory of the ith user strictly decreases C_i and does not increase C_j for $j \neq i$, neither C.

The cost function in this game is defined as follows. Each hop from X_i to X_j has a constant cost equal to $\gamma \mathrm{SIR}^{-1}(X_i, X_j, X^\lambda)$, and each used relay X_j has a linear cost equal to $\beta(m_j(s) - 1)$, depending on the trajectory configuration s. This way, by (6.1), the cost of the strategy of X_i corresponding to $s \in \mathcal{S}$ equals $C_i(s)$. Thus, the claim follows. $\square$

Now, if there exists a system-optimal $s \in \mathcal{S}$ such that $C(s) < C(s')$ for all Nash equilibria s', then we call s a *non-selfish optimum*, since there exists $i \in [n]$ such that s^i is not the best response of the ith user for the remaining coordinates of the trajectory collection. Example 6.2 shows a two-dimensional example that has a non-selfish optimum, and Remark 6.4 tells more about the relation of the individual and the total costs.

Example 6.2. Let $d = 2$, $\lambda = 1$ and $k_{\max} = 2$, and let $X^\lambda = X^1 = \{X_1, X_2, X_3\}$, ℓ and $\gamma > 0$ be chosen in the following way. X_1, X_2, X_3 and o, X_2, X_3 are vertices of two equilateral triangles with X_1 being in the interior of the latter triangle, so that $|X_1 - X_2| = |X_1 - X_3|$ and $|X_2| = |X_3|$, so that $\gamma \mathrm{SIR}^{-1}(X_1, o, X^1) = \gamma \mathrm{SIR}^{-1}(X_i, X_1, X^1) = 1$ and $\gamma \mathrm{SIR}^{-1}(X_i, o, X^1) = 1 + q$ for all $i \in \{2, 3\}$ for some $q > 0$ (see Figure 1).

The boundedness of $\ell(|\cdot - \cdot|)$ away from 0 on $W \times W$ implies that for any $\beta > 0$ and $i \in \{2, 3\}$, any $s^i \in \mathcal{S}^i$ that uses some X_j with $j \in \{2, 3\}$ as a relay is suboptimal both w.r.t. total and individual costs. Indeed, leaving out this relay and moving on to the next step of the same trajectory instead decreases $C_i(s)$ without increasing any

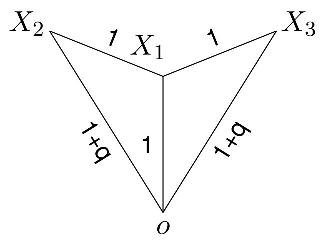

Figure 1: SIR weights per hop in Example 6.2. In the relevant cases, the congestion at X_1 is $\beta y(y-1)$, where y is the number of elements of $\{X_2, X_3\}$ relaying through X_1.

Number of hops of s^2	Number of hops of s^3	$C_2(s)$	$C_3(s)$	$C(s)$
1	1	$2+q$	$2+q$	$5+2q$
2	1	2	$2+q$	$5+q$
2	2	$2+\beta$	$2+\beta$	$5+2\beta$

Table 1: Individual and total costs in standard representatives of the relevant cases in Example 6.2.

$C_m(s)$, $m \neq i$. Using analogous arguments, one easily concludes that in any optimal trajectory and also in any Nash equilibrium, X_1 submits directly to o, and the two users X_2, X_3 use either the direct link to o or the two-hop path via X_1 to o. Table 1 shows the individual costs and the total cost in some standard representatives of these cases.

The positive parameters q and β can be chosen such that the following holds. Given that X_2 uses its two-hop path $X_2 \to X_1 \to o$, the best response of X_3 is to also use its two-hop path $X_3 \to X_1 \to o$ and vice versa, so that both users using their two-hop paths forms the unique Nash equilibrium, but the system optima are the trajectory configurations in which only one of them relays via X_1 and the other one submits directly to o. According to Table 1, this holds if $q > 0$ and $\beta \in (q/2, q)$. Thus, in such cases, the optimum is non-selfish.

Similar effects occur in all dimensions $d \geq 2$, with $d+1$ users $X_1, X_2, \ldots, X_{d+1}$ situated so that $|X_j - X_1| = |X_i - X_1|$ and $|X_1| < |X_i| = |X_j|$ for all $i, j \geq 2$. In such cases, one can choose the parameters in such a way that for all $j \geq 2$, knowing that X_1 transmits directly to o and each X_i, $j \neq i \geq 2$ relays through X_1, the best response of X_j is to use also the relayed link via X_1, but w.r.t. total costs it would be better if X_j transmitted directly to o. Note that if this holds, it may still happen that neither of these two joint strategies is system-optimal. ◇

Remark 6.3. In the setting of our Gibbsian model, Nash equilibria are not necessarily unique. Consider Example 6.2 in the boundary case $\beta = q$. Then one easily checks that the system exhibits three different Nash equilibria, namely the three ones that appear in Table 1. Also for $\beta > q$, there are two Nash equilibria, namely the ones where exactly one of s^2, s^3 transmits directly to o and the other one via X_1, by the symmetry between X_2 and X_3.

Remark 6.4. A situation opposite to Example 6.2 is not possible. I.e., if plugging in an additional relay to a trajectory decreases the total cost, it also decreases the individual cost of the transmitter of that trajectory.

Indeed, consider Figure 2 with $\lambda > 0$, $X_i, X_h \in X^\lambda$ and $x \in X^\lambda \cup \{o\}$, where the direct hop from X_i to x has SIR penalization $p_0 > 0$, while the two-hop path via X_h has SIR penalization $p_1 + p_2$ with $p_1, p_2 > 0$. Now, if $s^{-i} = (s^j)_{j \neq i}$ is given and the number of incoming messages at X_h coming from all trajectories but the one of X_i equals $m \geq 0$, then the direct link from X_i to x has individual cost $p_0 + K$ and the $X_i \to X_h \to o$ relayed link has individual cost $m + p_1 + p_2 + K$ for some $K \geq 0$. On the other hand, the total cost of the configuration with the $X_i \to x$ direct link is $2m + p_1 + p_2 + K'$, and the one with the $X_i \to X_h \to x$ relayed link is $p_0 + K'$ for some $K' \geq 0$. So if plugging in the relay X_h increases the individual cost C_i, then it also increases the total cost C. This implies the claim.

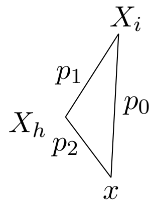

Figure 2: A situation opposite to Example 6.2 is not possible.

7 Numerical studies

In this section, we give numerical illustrations of various properties of the minimizer $(\nu_k)_{k\in[k_{\max}]}$ of (2.16), which describes the limiting empirical trajectory measure according to Proposition 2.1. We consider $k_{\max}=2$, $d=1,2$, ℓ satisfying $\ell(r)\sim r^{-4}$ as $r\to\infty$, W being a ball sufficiently large such that both direct communication and two-hop communication are non-negligible, and μ being the Lebesgue measure on W. We do not consider congestion, i.e., we put $\beta=0$. One of our questions is how strongly the effects that we proved in Sections 4 in the limit $\gamma\to\infty$ emerge. We will see that they are already very pronounced for $\gamma=1$. We will not only look at the areas where one-hop and two-hop communication dominate, respectively, and the approximation of a straight line of the latter trajectories, but we will also see further effects.

First, let us choose $\ell(t)=\min\{1,t^{-4}\}$. Let $W=\overline{B_r(o)}$ be a ball around the origin o. We will pick r so large that the effect of the path-loss function ℓ is strong enough in the sense that we can study areas in W from which a direct hop to o is preferred and areas from which a two-hop trajectory is preferred. We are interested in seeing how sharp the transition between these two areas is. By rotational invariance, we expect that the first area is a centred ball and the second the complement of a ball in W. Hence, we do not lose much when going to $d=1$. We expect the transition close to the point where the interference term gives the transition from optimality of one-step trajectories to two-step trajectories, i.e., at the radius $|x_0|$, where the number

$$g(x_0,o)-\min_{x_1\in W}\big(g(x_0,x_1)+g(x_1,o)\big) \tag{7.1}$$

switches the sign. Let r_0^* denote that point. Our main question is whether already for moderate values of γ, we see a pronounced transition in the measures $\nu_1(\mathrm{d}x_0)$ and $\pi_0\nu_2(\mathrm{d}x_0)$ of the form that $\nu_1(\mathrm{d}x_0)\approx\mu(\mathrm{d}x_0)$ for all x_0 with $|x_0|$ smaller than r_0^* and $\pi_0\nu_2(\mathrm{d}x_0)\approx\mu(\mathrm{d}x_0)$ for all x_0 with $|x_0|$ significantly larger than r_0^*, with a fast change around r_0^*.

In the following one-dimensional numerical example, the answer is yes, already for $\gamma=1$. The plots presented here were created using Wolfram *Mathematica*.

Example 7.1. Let $k_{\max}=2$, $d=1$, $W=[-5,5]=\overline{B_5(o)}\subset\mathbb{R}$, and $\ell(r)=\min\{1,r^{-4}\}$. According to Proposition 2.1, the minimizing measures $\Sigma=(\nu_1,\nu_2)$ are given as follows. With

$$\frac{1}{A(x_0)}=\exp\Big(-\gamma\frac{\int_{-5}^5\ell(|y|)\mathrm{d}y}{\ell(|x_0|)}\Big)+\frac{1}{10}\int_{-5}^5\mathrm{d}x_1\exp\Big(-\gamma\Big(\frac{\int_{-5}^5\ell(|y-x_1|)\mathrm{d}y}{\ell(|x_0-x_1|)}+\frac{\int_{-5}^5\ell(|y|)\mathrm{d}y}{\ell(|x_1|)}\Big)\Big), \tag{7.2}$$

we have

$$\nu_1(\mathrm{d}x_0)=\mathrm{d}x_0A(x_0)\exp\Big(-\gamma\frac{\int_{-5}^5\ell(|y|)\mathrm{d}y}{\ell(|x_0|)}\Big) \tag{7.3}$$

and

$$\nu_2(\mathrm{d}x_0,\mathrm{d}x_1)=\frac{1}{10}\mathrm{d}x_0\mathrm{d}x_1A(x_0)\exp\Big(-\gamma\Big(\frac{\int_{-5}^5\ell(|y-x_1|)\mathrm{d}y}{\ell(|x_0-x_1|)}+\frac{\int_{-5}^5\ell(|y|)\mathrm{d}y}{\ell(|x_1|)}\Big)\Big). \tag{7.4}$$

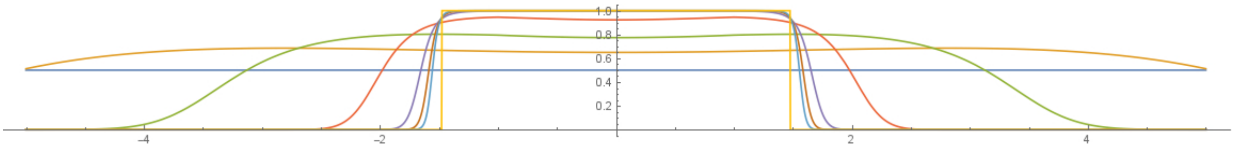

Figure 3: The graphs of $x_0 \mapsto \nu_1(\mathrm{d}x_0)/\mathrm{d}x_0$ as in Example 7.1 for $\gamma = 0, 0.001, 0.01, 0.1, 0.4, 0.7, 1, \infty$.

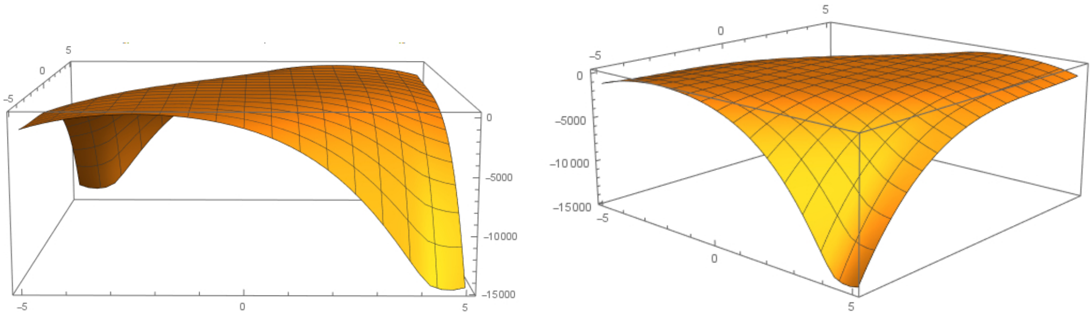

Figure 4: The graphs of $(x_0, x_1) \mapsto \log(\nu_2(\mathrm{d}x_0, \mathrm{d}x_1)/\mathrm{d}x_0\,\mathrm{d}x_1)$ as in Example 7.1 for $\gamma = 1$ from two different views.

All integrals are numerically tractable for $\gamma \in [0,1]$. As seen in Figure 3, already for $\gamma = 1$, the density of ν_1 is very close to the step function with a jump at the point r_0^* where (7.1) switches its sign. Also the density of two-hops paths, $\nu_2(\mathrm{d}x_0, \mathrm{d}x_1)/(\mathrm{d}x_0 \mathrm{d}x_1)$, is extremely small for $|x_0 - x_1|$ large, already for $\gamma = 1$, so that we prefer to plot it on a logarithmic scale, see Figure 4.

Now we ask the question which x_1^* maximizes $x_1 \mapsto \nu_2(\mathrm{d}x_0, \mathrm{d}x_1)/\mathrm{d}x_0\,\mathrm{d}x_1$ for given $x_0 \in W$. In Figure 5, we see one approximate maximizer $x_1^*(x_0)$ mapped to each $x_0 \in W$. For x_0 with $|x_0|$ not exceeding the critical distance $r_0^* \in (1.45, 1.5)$ significantly, the picture is very noisy. Due to multiple approximate maximizers, the numerical plot is even not symmetric to 0, although it is clear that $\nu_2(\mathrm{d}x_0, \mathrm{d}x_1) = \nu_2(-\mathrm{d}x_0, -\mathrm{d}x_1)$ must hold for any $x_0, x_1 \in W$ (cf. [KT17, Section 1.7.3]). When $|x_0|$ becomes large enough so that it leaves the noisy area, the function becomes close to linear in x_0 with slope 1, followed by two symmetrically located breakpoints around $|x_0| \approx 2.5$ and afterwards, the function continues to be approximately linear but with a slope smaller than 1. We observe that in the steeper linear part, only the length of the second hop increases, and the optimal first step always has length equal to 1, which is the maximal distance for which ℓ takes its maximal value $\ell_{\max} = 1$. This eventually ceases to be the optimal strategy for

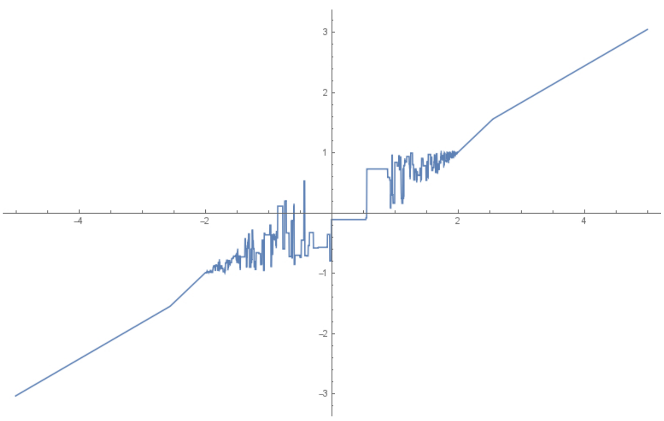

Figure 5: The graph of $x_0 \in W$, mapped to a maximizer $x_1^*(x_0) \in W$ of $x_1 \mapsto \nu_1(\mathrm{d}x_0, \mathrm{d}x_1)/\mathrm{d}x_0\,\mathrm{d}x_1$ in Example 7.1. The function ceases to be noisy shortly after x_0 enters the regime where (7.1) switches sign.

x_0		$x_1^*(x_0)$		(7.1)
x coord.	y coord.	x coord.	y coord.	positive for x_0?
0	0	8.92×10^{-11}	6.80×10^{-11}	no
0.1	0	0.05002	4.44×10^{-13}	no
0.4	0	0.20007	7.37×10^{-13}	no
0.6	0.2	0.30001	0.10003	yes
1	0	0.50010	-5.68×10^{-9}	yes
3	4	1.49893	1.99859	yes
5	0	2.49823	1.94×10^{-8}	yes

Table 2: In Example 7.2, the optimal relay $x_1^*(x_0)$ of transmitter x_0 is close to $x_0/2$. For $|x_0|$ large, using this relay is more favourable than the direct link towards o.

$|x_0| \approx 2.5$, and after this breakpoint, the path-loss causes a slow, continuous increase in the length of the first hop as well. For further details about what we expect instead of the picture shown in Figure 3 for larger γ and/or higher numerical precision in this example, see Example 7.2.

$\diamond$

In the next, two-dimensional example, we illustrate the large-γ limit of Section 4. We choose a rotationally invariant intensity μ and a strictly monotone decreasing path-loss function ℓ. We observe that the density ν_2 concentrates very strongly on the straight line already for $\gamma = 1$.

Example 7.2. We choose $d = 2$, $k_{\max} = 2$, $W = \overline{B_5(o)} \subset \mathbb{R}^2$, $\mu = \mathrm{Leb}|_W$, $\ell(r) = (1+r)^{-4}$ and $\gamma = 1$. Now, the one-hop trajectory measure ν_1 is a measure on $W \subset \mathbb{R}^2$ and the two-hop one ν_2 is a measure on $W^2 \subset \mathbb{R}^4$; they are defined as in (2.16), analogously to the concrete case (7.2)–(7.4), with a suitable adaptation to the new parameters.

We observe that for any user $x_0 \in W$, the map $x_1 \mapsto \nu_2(\mathrm{d}x_0, \mathrm{d}x_1)/\mathrm{d}x_0\,\mathrm{d}x_1$ is maximized in $x_1^*(x_0) \approx x_0/2$ with a very high accuracy, even for x_0 close to o for which the optimal trajectory towards o is the direct one. This implies in particular that $\nu_2(\mathrm{d}x_0, \cdot)$ is strongly concentrated on the straight line $[x_0, o]$ for any $x_0 \in W$. The critical distance $r_0^* = |x_0|$ from o at which (7.1) switches sign is in $(0.4, 0.45)$. We note that, approximately, the same $x_1^*(x_0)$ minimizes the SIR penalization term on the right-hand side of (7.1). Thus, in this example with $\gamma = 1$, the qualitative behaviour of the system is already close to the one described in Section 4 for large γ. Table 7.2 shows $x_1^*(x_0)$ as a function of x_0.

Note that in this example $1/\ell(|\cdot|)$ is convex, and for two-hop trajectories of the form $W \ni x_0 \to x_0/2 \to o$, the interference at $x_0/2$ is almost the same as the one at o (at o it is about 0.970 and at x_0 with $|x_0| = 2.5$ it is about 0.937). Consequently, optimizing the SIR penalty over 2-hop trajectories is almost the same as optimizing $1/\ell$ over the same trajectories, and the latter optimization clearly leads to an optimal trajectory with two equal-sized hops. Similar properties also hold for ℓ in the setting of Example 7.1. However, the constant part of the path-loss function makes the SIR landscape much more disordered, at least when it comes to the numerical approximations such as the one in Figure (5). For larger γ and/or better numerical precision, we expect that also in Example 7.1, $\nu_k(\mathrm{d}x_0, \cdot)$ concentrates around $x_0/2$.

The properties of ℓ (and $\mu = \mathrm{Leb}$ on the rotationally symmetric W) in this second example have a strong regularizing effect on the trajectories; otherwise, the cutoff phenomenon in $\nu_1(\mathrm{d}x_0)$ around the values of x_0 satisfying $|x_0| = r_0^*$ is less strong for $\gamma = 1$ than in Example 7.1. Indeed, on the one hand, for $|x_0|$ small, the proportion of one-hop trajectories $\nu_1(\mathrm{d}x_0)/\mu(\mathrm{d}x_0)$ is further away from 1; indeed, even for $x_0 = o$, a non-negligible amount 0,18% of the messages takes a two-hop trajectory, and for $|x_0| = 0.4 < r_0^*$, already 2,82%. On the other hand, for $|x_0| = 0.5 > r_0^*$, still only 8,67% of the messages goes via two hops, and for $|x_0| = 0.8$, still only 94,86%. In comparison, in the setting of Example 7.1, we have $1.45 < r_0^* < 1.5$, and for $|x_0| = 1.4$, already only 0.07% of the messages takes a two-hop path and for $|x_0| = 0$, less than 0.01%. For $|x_0| = 1.5$, already 11,91%, and for $|x_0| = 1.7$, an overwhelming proportion 99.83%. ◇

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