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# Geometric properties of cones with applications on the Hellinger–Kantorovich space, and a new distance on the space of probability measures


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# Geometric properties of cones with applications on the Hellinger–Kantorovich space, and a new distance on the space of probability measures

Vaios Laschos, Alexander Mielke

**Abstract**

By studying general geometric properties of cone spaces, we prove the existence of a distance $\mathsf{SHK}_{\alpha,\beta}$ on the space of Probability measures that turns the Hellinger–Kantorovich space $(\mathcal{M}(X), \mathsf{HK}_{\alpha,\beta})$ into a cone space over the space of probabilities measures $(\mathcal{P}(X), \mathsf{SHK}_{\alpha,\beta})$. Here we exploit a natural two-parameter scaling property of the Hellinger-Kantorovich distance $\mathsf{HK}_{\alpha,\beta}$. For $\mathsf{SHK}_{\alpha,\beta}$ we obtain a full characterization of the geodesics. We also provide new geometric properties for $\mathsf{HK}_{\alpha,\beta}$, including a two parameter rescaling and reparametrization of the geodesics, local-angle condition and some partial $K$-semiconcavity of the squared distance, that it will be used in a future paper to prove existence of gradient flows.

## 1 Introduction

In [LMS16, LMS17], and independently in [KMV16] and [CP*15b, CP*15a], a new family of distances $\mathsf{HK}_{\alpha,\beta}$ on the space $\mathcal{M}(X)$ of arbitrary nonnegative and finite measures was introduced, where $(X, \mathsf{d}_X)$ is a geodesic, Polish space. This new family of Hellinger–Kantorovich distances generalize both the Kantorovich–Wasserstein distance (for $\alpha = 1$ and $\beta = 0$) and the Hellinger-Kakutani distance (for $\alpha = 0$ and $\beta = 1$), allowing for both transportation and creation/annihilation of mass, which is organized in a jointly optimal fashion depending on the ratio of the parameters $\alpha$ and $\beta$.

The origin of our work stems from the observation in [LMS16, Prop. 19] that the total mass $m(s) = \int_X 1 \mathrm{d}\mu(s)$ of a constant-speed geodesic $[0,1] \ni s \mapsto \mu(s) \in \mathcal{M}(X)$ is a quadratic function in $s$, viz.

$$m(s) = (1{-}s)m(0) + sm(1) - s(1{-}s)\frac{4}{\beta}\mathsf{HK}^2_{\alpha,\beta}(\mu(0), \mu(1)). \tag{1.1}$$

We will show here that this formula is already a consequence of a simpler scaling property, that **fully characterizes cone spaces**, which in the case of $\mathsf{HK}^2_{\alpha,\beta}$, takes the form

$$\mathsf{HK}^2_{\alpha,\beta}(r_0^2\mu_0, r_1^2\mu_1) = r_0 r_1 \mathsf{HK}^2_{\alpha,\beta}(\mu_0, \mu_1) + (r_0^2{-}r_0r_1)\frac{4}{\beta}\mu_0(X) + (r_1^2{-}r_0r_1)\frac{4}{\beta}\mu_1(X). \tag{1.2}$$

The property is proved independently in Theorem 3.3 based on the characterization of $\mathsf{HK}^2_{\alpha,\beta}$ via the logarithmic-entropy functional $\mathscr{LET}_\ell$, cf. Theorem 3.1.

This suggests to write arbitrary measures $\mu \in \mathcal{M}(X) \setminus \{0\}$ as

$$\mu = r^2\nu \quad \text{with} \quad [\nu, r] \in \mathcal{P}(X) \times (0, \infty), \quad \text{where } r = \sqrt{\mu(X)},\ \nu = \frac{1}{r^2}\mu, \tag{1.3}$$

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and $\mathcal{P}(X)$ denotes the probability measures. Thus, the set $\mathcal{M}(X)$ can be interpreted as a cone over $\mathcal{P}(X)$ in the sense of Section 2, and the Hellinger–Kantorovich distance has the form

$$\mathsf{HK}^2_{\alpha,\beta}(r_0^2\nu_0, r_1^2\nu_1) = \frac{4}{\beta}\Big(r_0^2 + r_1^2 - 2r_0r_1\cos\big(\mathsf{SHK}_{\alpha,\beta}(\nu_0,\nu_1)\big)\Big),$$

where the so-called *spherical Hellinger–Kantorovich distance* on $\mathcal{P}(X)$ is simply defined by

$$\mathsf{SHK}_{\alpha,\beta}(\nu_0,\nu_1) = \arccos\left(1 - \frac{\frac{\beta}{4}\mathsf{HK}^2_{\alpha,\beta}(\nu_0,\nu_1)}{2}\right).$$

One main result is that $\mathsf{SHK}_{\alpha,\beta}$ is indeed a distance on the space of probability measures, such that the Hellinger–Kantorovich space $(\mathcal{M}(X), \mathsf{HK}_{\alpha,\beta})$ is indeed a cone space over the space of probability measures, namely $(\mathcal{P}(X), \mathsf{SHK}_{\alpha,\beta})$. This distance is a generalization of the spherical Hellinger distance, also called "Fisher-Rao distance" or "Bhattacharya distance 1" in [DeD09, Sec. 7.2+Sec. 14.2], in a similar way that the Hellinger-Kantorovich distance is a generalization of the Hellinger distance.

The fact that $\mathsf{SHK}_{\alpha,\beta}$ satisfies the triangle inequality will be derived in the abstract Section 2 for general distances $\mathsf{d}_\mathcal{C}$ satisfying a scaling property as in (1.2). We work on the cone $(\mathcal{C}, \mathsf{d}_\mathcal{C})$ over a general space $(\mathcal{X}, \mathsf{d}_\mathcal{X})$, and the sole additional assumption we need is that the distance $\mathsf{d}_\mathcal{C}$ is bounded on the set $\{\,[x,1] \,:\, x\in\mathcal{X}\,\} \subset \mathcal{C}$ by the constant 2, see Theorem 2.2. The latter bound follows easily for the Hellinger-Kantorovich distance from

$$\frac{\beta}{4}\mathsf{HK}^2_{\alpha,\beta}(\nu_0,\nu_1) \le \frac{\beta}{4}\left(\frac{4}{\beta}\nu_0(X) + \frac{4}{\beta}\nu_1(X)\right) = 2 \le 4.$$

In Sections 2.2 to 2.4 we consider the case that $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ is a geodesic space and that $\mathsf{d}_\mathcal{C}$ is given by

$$\mathsf{d}^2_\mathcal{C}([x_0,r_0],[x_1,r_1]) = r_0^2 + r_1^2 - 2r_0r_1\cos_\pi(\mathsf{d}_\mathcal{X}(x_0,x_1)), \tag{1.4}$$

where $\cos_a(b) = \cos(\min\{a,b\})$. In Sections 2.3 and 2.4 we show how geodesics in $(\mathcal{C}, \mathsf{d}_\mathcal{C})$ between $[x_0,r_0]$ and $[x_1,r_1]$ can be obtained from those between $x_0$ and $x_1$ in $(\mathcal{X}, \mathsf{d}_\mathcal{X})$. Based on this, we discuss how comparison angles and local angles behave when we move between the spherical space $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ and the cone $(\mathcal{C}, \mathsf{d}_\mathcal{C})$. In particular, we discuss the *local angle condition* $m$-LAC, see Definition 2.15 and [Sav07, OPV14] for the usefulness of this in the theory of metric gradient flows. The main observation is that if $\mathsf{d}_\mathcal{X}(x_0,x_i) < \pi$, $\boldsymbol{x}_{0i}$ are constant-speed geodesics in $\mathcal{X}$ connecting $x_0$ with $x_i$, and if $\boldsymbol{z}_{0i}$ are the corresponding geodesics in $\mathcal{C}$ connecting $z_0 = [x_0,r_0]$ and $z_i = [x_i,r_i]$ with $r_0, r_i > 0$, then the upper angles satisfy the relation

$$\begin{aligned}\mathsf{d}_\mathcal{C}(z_0,z_i)\mathsf{d}_\mathcal{C}(z_0,z_j)\cos\big(\measuredangle_{\mathrm{up}}(\boldsymbol{z}_{0i},\boldsymbol{z}_{0j})\big) &= (r_0 - r_i\cos(\mathsf{d}_\mathcal{X}(x_0,x_i)))(r_0 - r_j\cos(\mathsf{d}_\mathcal{X}(x_0,x_j)))\\ &\quad + r_ir_j\sin(\mathsf{d}_\mathcal{X}(x_0,x_i))\sin(\mathsf{d}_\mathcal{X}(x_0,x_j))\cos\big(\measuredangle_{\mathrm{up}}(\boldsymbol{x}_{0i},\boldsymbol{x}_{0j})\big).\end{aligned}$$

Based on this, Theorem 2.21 establishes that the $m$-LAC condition transfers between $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ and $(\mathcal{C}\setminus\{\mathbf{0}\}, \mathsf{d}_\mathcal{C})$.

Section 3 shows that the abstract results apply in the specific case of the Hellinger-Kantorovich space $(\mathcal{M}(X), \mathsf{HK}_{\alpha,\beta})$, which takes the role of $(\mathcal{C}, \mathsf{d}_\mathcal{C})$, which then leads to the spherical space $(\mathcal{P}(X), \mathsf{SHK}_{\alpha,\beta})$. A direct characterization in the sense of [LMS17, Sec. 8.6] of the geodesic curves using a continuity and a Hamilton-Jacobi equation in the latter space is given in Theorem 3.7.

In Section 4.1 we provide additional geometric properties that hold for the Hellinger–Kantorovich distance. Among them, is the local-angle condition, which also holds for the Spherical Hellinger-Kantorovich, and some partial semi-concavity. In [LMS16], it was proved that $K$-semiconcavity, a

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property, which is associated among other things with the existence of gradient flows, does not hold in general. In this article, we prove that on the subsets of measures that have bounded density (both from below and above) with respect to some finite, locally doubling measure $\mathcal{L}$, this property holds for sufficient large $K$ depending only on the bounds and $\mathcal{L}$. This result will be used in a consecutive paper to prove the existence of gradient flows. For this we provide a sharp estimate of the total mass of the calibration measure associated with the optimal entropy-transport problem. This estimate is used in our proofs, but it is also helpful for the numerical approximations of the Hellinger-Kantorovich distance.

To simplify the subsequent notations we use the simple relation $\mathsf{HK}^2_{\alpha,\beta} = \frac{1}{\beta}\mathsf{HK}^2_{\alpha/\beta,1}$, which shows that it suffices to work with a one-parameter family. We set $\mathsf{HK}^2_{\boldsymbol{\ell}} = \mathsf{HK}^2_{1/\boldsymbol{\ell}^2,4}$, which allows us to recover $\mathsf{HK}_{\alpha,\beta}$ via $\mathsf{HK}^2_{\alpha,\beta} = \frac{4}{\beta}\mathsf{HK}^2_{\boldsymbol{\ell}}$ with $\boldsymbol{\ell}^2 = \beta/(4\alpha)$.

# 2 Cones over metric spaces

## 2.1 Background and scaling property

In [Ber83] (see also [ABN86], [BrH99], and [BBI01]), the concept of the cone $\mathfrak{C}$ over a metric space $(\mathfrak{X}, \mathsf{d}_\mathfrak{X})$, is introduced. The cone is the quotient of the product $\mathfrak{X} \times [0,\infty)$, obtained by identifying together all points in $\mathfrak{X} \times \{0\}$ with a point $\mathbf{0}$, called the apex or tip of the cone. The cone $\mathfrak{C}$ is equipped with the distance $\mathsf{d}_\mathfrak{C}$ given in (1.4). In [BBI01], one can find a proof that $\mathsf{d}_\mathfrak{C}$ is a metric distance. The following results exhibits the scaling properties of such cone distances.

**Lemma 2.1 (Cone distances have scaling properties)** *The cone distance* $\mathsf{d}_\mathfrak{C}$ *in* (1.4) *satisfies the scaling property*

$$\forall\, [x_0,r_0],[x_1,r_1] \in \mathfrak{C}: \quad \mathsf{d}^2_\mathfrak{C}([x_0,r_0],[x_1,r_1]) = r_0r_1\mathsf{d}^2_\mathfrak{C}([x_0,1],[x_1,1]) + (r_0{-}r_1)^2. \tag{2.1}$$

*Moreover, any distance* $\mathsf{d}_\mathfrak{C}$ *satisfying* (2.1) *(i.e. without assuming* (1.4) *a priori) satisfies the more general scaling property*

$$\mathsf{d}^2_\mathfrak{C}([x_0,r_0\widetilde r_0],[x_1,r_1\widetilde r_1]) = r_0r_1\mathsf{d}^2_\mathfrak{C}([x_0,r_0],[x_1,r_1]) + (\widetilde r_0^2{-}\widetilde r_0\widetilde r_1)r_0^2 + (\widetilde r_1^2{-}\widetilde r_0\widetilde r_1)r_1^2 \tag{2.2}$$

*for all* $\widetilde r_0$ *and* $\widetilde r_1$.

**Proof:** Statement (2.1) follows by using (1.4) twice, once as it is given, and once with $r_0 = r_1 = 1$, and then eliminating $\cos_\pi(\mathsf{d}_\mathfrak{X}(x_0,x_1))$.

Statement (2.2) follows by using (2.1) twice, once as it is given, and once with $r_0\ r_1$ replaced by $r_0\widetilde r_0$ and $r_1\widetilde r_1$, respectively. After eliminating $\mathsf{d}^2_\mathfrak{C}([x_0,1],[x_1,1])$ the assertion follows. ∎

While we were studying the Hellinger-Kantorovich space, we noticed that the scaling property (2.1) actually fully characterizes a cone space. We have the following general theorem, which allows us to derive the cone distance from the scaling property.

**Theorem 2.2 (Scaling implies cone distance)** *For a metric space* $(\mathfrak{C}, \mathsf{d}_\mathfrak{C})$, *let assume that it exists a set* $\mathfrak{X}$, *that could possibly be identified with a subset of* $\mathfrak{C}$, *and a surjective function* $[\cdot,\cdot] : \mathfrak{X} \times [0,\infty) \to \mathfrak{C}$, *such that the distance* $\mathsf{d}_\mathfrak{C}$ *satisfies* (2.1) *and*

$$\forall x_0 \neq x_1 \in \mathfrak{X}: \quad 0 < \mathsf{d}^2_\mathfrak{C}([x_0,1],[x_1,1]) \leq 4; \tag{2.3}$$

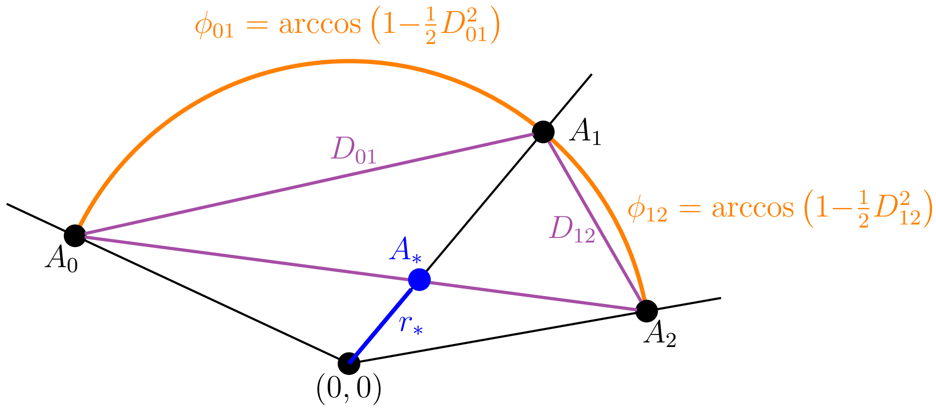


Figure 1: Construction of the optimal radius $r_*$. The points $A_j$ have distance $r_j = 1$ from the origin and thus correspond to $z_j = [x_j, 1]$, which gives $D_{1j} = |\overline{A_1A_j}| = \mathsf{d}_{\mathfrak{C}}(z_1, z_j)$ for $j = 0$ and $2$. The point $A_*$, which corresponds to $z_* = [x_1, r_*]$, is chosen such that $|\overline{A_0A_*}| + |\overline{A_*A_2}| = |\overline{A_0A_2}|$.

*then* $\mathsf{d}_{\mathfrak{X}} : \mathfrak{X}\times\mathfrak{X} \to [0,\infty)$ *given by* $\mathsf{d}_{\mathfrak{X}}(x_0, x_1) = \arccos\Big(1 - \frac{\mathsf{d}^2_{\mathfrak{C}}([x_0,1],[x_1,1])}{2}\Big) \in [0,\pi]$ *is a metric distance on* $\mathfrak{X}$*, and* $(\mathfrak{C}, \mathsf{d}_{\mathfrak{C}})$ *is a metric cone over* $(\mathfrak{X}, \mathsf{d}_{\mathfrak{X}})$*, i.e.* (1.4) *holds.*

**Proof:** Clearly, $\mathsf{d}_{\mathfrak{X}}$ as defind in the assertion is symmetric and positive. Hence, it remains to establish the triangle inequality. Given $x_0, x_1, x_2 \in \mathfrak{X}$, we set

$$D_{ij} = \mathsf{d}_{\mathfrak{C}}([x_i, 1], [x_j, 1]) \;\text{ and }\; \phi_{ij} = \arccos\Big(1 - \frac{D_{ij}^2}{2}\Big), \quad \text{for } i \neq j \in \{0, 1, 2\}.$$

Hence, we have to show $\mathsf{d}_{\mathfrak{X}}(x_0, x_2) = \phi_{02} \le \phi_{01}+\phi_{12} = \mathsf{d}_{\mathfrak{X}}(x_0, x_1)+\mathsf{d}_{\mathfrak{X}}(x_1, x_2)$. If $\phi_{01}+\phi_{12} \ge \pi$ then there is nothing to show. Without loss of generality, we will have $\phi_{01} = \min\{\phi_{01}, \phi_{12}\} < \frac{\pi}{2}$, and $\phi_{01} + \phi_{12} < \pi$. We consider a comparison triangle in $\mathbb{R}^2$, as is depicted in Figure 1. In particular, $A_j$ are chosen on the unit circle such that $\phi_{i,i+1}$ and $D_{i,i+1}$ are the angle (arclength on the unit circle) and the Euclidean distance, respectively, between $A_i$ and $A_{i+1}$. Now, $A_*$ is chosen as the intersection of $\overline{OA_1}$ with the segment $\overline{A_0A_2}$, see Figure 1.

With this choice of $r_*$ we retur to the cone $(\mathfrak{C}, \mathsf{d}_{\mathfrak{C}})$ and let $r_* = |\overline{OA_*}|$ and $z_0 = [x_0, 1], z_1 = [x_1, r_*], z_2 = [x_2, 1] \in \mathfrak{C}$. The scaling property (2.1) for $\mathsf{d}_{\mathfrak{C}}$, gives

$$\mathsf{d}^2_{\mathfrak{C}}([x_0, 1], [x_1, r_*]) = 1 + r_*^2 - 2r_* \cos\phi_{01} = |\overline{A_0A_*}|^2 \text{ and}$$
$$\mathsf{d}^2_{\mathfrak{C}}([x_1, r_*], [x_2, 1]) = 1 + r_*^2 - 2r_* \cos\phi_{12} = |\overline{A_2A_*}|^2.$$

Using the triangle inequality for $\mathsf{d}_{\mathfrak{C}}$, we arrive at

$$\begin{aligned} D_{02}^2 = \mathsf{d}^2_{\mathfrak{C}}([x_0, 1], [x_2, 1]) &\le \big(\mathsf{d}_{\mathfrak{C}}([x_0, 1], [x_1, r_*]) + \mathsf{d}_{\mathfrak{C}}([x_1, r_*], [x_2, 1])\big)^2 \\ &= (|\overline{A_0A_*}| + |\overline{A_*A_2}|)^2 = |\overline{A_0A_2}|^2 = 1 + 1 - 2\cos(\phi_{01}+\phi_{12}). \end{aligned} \tag{2.4}$$

Thus, we conclude that $\phi_{02} = \arccos\Big(1 - \frac{D_{02}^2}{2}\Big) \le \phi_{01}+\phi_{12}$, which is the desired triangle inequality for $\mathsf{d}_{\mathfrak{X}}$, namely $\mathsf{d}_{\mathfrak{X}}(x_0, x_2) \le \mathsf{d}_{\mathfrak{X}}(x_0, x_1) + \mathsf{d}_{\mathfrak{X}}(x_1, x_2)$. Thus, inserting $\mathsf{d}^2_{\mathfrak{C}}([x_0, 1], [x_1, 1]) = 2 - 2\cos(\mathsf{d}_{\mathfrak{X}}(x_0, x_1))$ into (2.1), we have established (1.4), and consequently (2.2) follows as well. ∎

As a first consequence we obtain the following result.

**Corollary 2.3** *Let $\mathcal{X}$ a set, and $\mathcal{C}$ the quotient of the product $\mathcal{X}\times[0,\infty)$, obtained by identifying together all points in $\mathcal{X}\times 0$. If $\mathsf{d}_\mathcal{C}:\mathcal{C}\times\mathcal{C}\to[0,\infty)$ given by* (1.4)*, for some $\mathsf{d}_\mathcal{X}:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ is a metric distance on $\mathcal{C}$, then $\mathsf{d}_\mathcal{X}\wedge\pi$ is a metric distance on $\mathcal{X}$.*

**Proof:** By setting $z_0=[x_0,1]$, and $z_1=[x_1,1]$, we can recover both the positivity and symmetry property. For the proof of the triangle inequality, we just notice that $\mathsf{d}_\mathcal{C}$ satisfies the scaling property, and then the result is an application of Theorem 2.2. ■

From the perspective of $(\mathcal{X},\mathsf{d}_\mathcal{X})$, we call $(\mathcal{C},\mathsf{d}_\mathcal{C})$ the cone space over $\mathcal{X}$; from the perspective of $(\mathcal{C},\mathsf{d}_\mathcal{C})$, we call $(\mathcal{X},\mathsf{d}_\mathcal{X}\wedge\pi)$ the spherical space in $\mathcal{C}$.

## 2.2 Geodesics curves

We first recall the standard definition and hence introduce our notations.

**Definition 2.4** *Let $(\mathcal{X},\mathsf{d}_\mathcal{X})$ be a metric space, and $\boldsymbol{x}:[0,\tau]\to\mathcal{X}$, a continuous mapping. Furthermore, let $\mathcal{T}$ be the set of all partitions $T=\{0=\tau_0\le\cdots\le\tau_{n_T}=\tau\}$ of $[0,\tau]$. Then, the* length of the curve *$\boldsymbol{x}$ is given by $\mathrm{Len}(\boldsymbol{x}):=\sup_{T\in\mathcal{T}}\sum_{i=1}^{n_T}\mathsf{d}_\mathcal{X}(\boldsymbol{x}(\tau_i),\boldsymbol{x}(\tau_{i-1}))$.*

**Definition 2.5** *Let $(\mathcal{X},\mathsf{d}_\mathcal{X})$ be a metric space. We will call $(\mathcal{X},\mathsf{d}_\mathcal{X})$* geodesic*, if and only if for every two points $x_0,x_1$ there exists a continuous mapping $\boldsymbol{x}_{01}:[0,\tau]\to\mathcal{X}$ such that*

$$\boldsymbol{x}_{01}(0)=x_0,\quad \boldsymbol{x}_{01}(\tau)=x_1,\quad\textit{and}\quad \mathsf{d}_\mathcal{X}(x_0,x_1)=\mathrm{Len}(\boldsymbol{x}_{01}).$$

*A function like that will be called a* geodesic curve *or simply a* geodesic*. A geodesic satisfying*

$$\mathsf{d}_\mathcal{X}(\boldsymbol{x}_{01}(t_1),\boldsymbol{x}_{01}(t_2))=C|t_2-t_1|$$

*for some constant $C>0$, will be called a* constant-speed geodesic*. If $C=1$, then the geodesic is called a* unit-speed geodesic*. Finally for $x_0,x_1\in\mathcal{X}$, any geodesic $\boldsymbol{x}_{01}:[0,1]\to\mathcal{X}$, with $\boldsymbol{x}_{01}(0)=x_0$, $\boldsymbol{x}_{01}(1)=x_1$ is called a* geodesic joining $x_0$ *to* $x_1$*. We will denote the set of all such geodesics with $\mathrm{Geod}(x_0,x_1)$, i.e.*

$$\mathrm{Geod}(x_0,x_1):=\{\,\boldsymbol{x}:[0,1]\to\mathcal{X}\mid \boldsymbol{x}(0)=x_0,\ \boldsymbol{x}(1)=x_1,\ \boldsymbol{x}\textit{ is constant-speed geodesic}\}. \tag{2.5}$$

In [BrH99, Chap. I, Prop. 5.10], the following Theorem is proved.

**Theorem 2.6** *Let $(\mathcal{X},\mathsf{d}_\mathcal{X})$ be a geodesic space. Let also $z_0=[x_0,r_0]$ and $z_1=[x_1,r_1]$ be elements of $\mathcal{C}$.*

*1 If $r_0,r_1\in(0,\infty)$ and $\mathsf{d}_\mathcal{X}(x_0,x_1)<\pi$, then there is a bijection between $\mathrm{Geod}(x_0,x_1)$, and $\mathrm{Geod}(z_0,z_1)$.*

*2 In all other cases, $\mathrm{Geod}(z_0,z_1)$ has a unique element.*

As a corollary, we get that $\mathcal{C}$ is geodesic, if and only if $\mathcal{X}$ is geodesic for points of distance less than $\pi$. In the following two Subsections 2.3 and 2.4 we give explicit correspondences in the sense of part 1. of the above theorem for the case of constant-speed geodesics.

## 2.3 Lifting from $\mathcal{X}$ into the cone

In [LMS16], it is proved that the constant-speed geodesics $\boldsymbol{z}_{01}(t)$ connecting $z_0 = [x_0, r_0]$ to $z_1 = [x_1, r_1]$, with $0 < \mathsf{d}_{\mathcal{X}}(x_0, x_1) < \pi$, have the following parametrization

$$\boldsymbol{z}_{01}(t) = [\boldsymbol{x}_{01}(\boldsymbol{\zeta}_{01}(t)), \boldsymbol{r}_{01}(t)], \tag{2.6}$$

where $\boldsymbol{x}_{01}(t)$ is a constant-speed geodesic joining $x_0$ to $x_1$ and where $\boldsymbol{\zeta}_{01}(t)$ and $\boldsymbol{r}_{01}(t)$ are given by

$$\begin{aligned} \boldsymbol{r}_{01}^2(t) &= (1-t)^2 r_0^2 + t^2 r_1^2 + 2t(1-t) r_0 r_1 \cos(\mathsf{d}_{\mathcal{X}}(x_0, x_1)), \\ \boldsymbol{\zeta}_{01}(t) &= \frac{1}{\mathsf{d}_{\mathcal{X}}(x_0, x_1)} \arcsin\left( \frac{t r_1 \sin\big(\mathsf{d}_{\mathcal{X}}(x_0, x_1)\big)}{\boldsymbol{r}_{01}(t)} \right) \\ &= \frac{1}{\mathsf{d}_{\mathcal{X}}(x_0, x_1)} \arccos\left( \frac{(1-t) r_0 + t\, r_1 \cos(\mathsf{d}_{\mathcal{X}}(x_0, x_1))}{\boldsymbol{r}_{01}(t)} \right) \\ &= \frac{1}{\mathsf{d}_{\mathcal{X}}(x_0, x_1)} \arctan\left( \frac{t r_1 \sin\big(\mathsf{d}_{\mathcal{X}}(x_0, x_1)\big)}{(1-t) r_0 + t\, r_1 \cos\big(\mathsf{d}_{\mathcal{X}}(x_0, x_1)\big)} \right). \end{aligned} \tag{2.7}$$

Alternatively if we want the parametrization with respect to $\mathsf{d}_{\mathcal{C}}$, (2.7) becomes

$$\begin{aligned} \boldsymbol{r}_{01}^2(t) &= ((1-t) r_0 + t r_1)^2 - r_0 r_1 t (1-t) \mathsf{d}_{\mathcal{C}}^2([x_0, 1], [x_1, 1]) \\ \boldsymbol{\zeta}_{01}(t) &= \frac{1}{\mathsf{d}_{\mathcal{X}}(x_0, x_1)} \arccos\left( \frac{(1-t) r_0 + t\, r_1 \left(1 - \frac{\mathsf{d}_{\mathcal{C}}^2([x_0,1],[x_1,1])}{2}\right)}{\boldsymbol{r}_{01}(t)} \right). \end{aligned} \tag{2.8}$$

If we differentiate twice the first equation in (2.7), we get

$$(\boldsymbol{r}_{01}^2)''(t) = r_0^2 + r_1^2 - 2 r_0 r_1 \cos(\mathsf{d}_{\mathcal{X}}(x_0, x_1)) = \mathsf{d}_{\mathcal{C}}^2(z_0, z_1),$$

from which we also recover the following formula

$$\boldsymbol{r}_{01}^2(t) = (1-t) r_0^2 + t r_1^2 - t(1-t) \mathsf{d}_{\mathcal{C}}^2(z_0, z_1), \tag{2.9}$$

which later applied to $\mathsf{HK}_{\alpha,\beta}$ will give (1.1). Furthermore (2.9), trivially gives convexity of $\boldsymbol{r}_{01}^2$, i.e.

$$\boldsymbol{r}_{01}^2(t) \le (1-t) r_0^2 + t r_1^2. \tag{2.10}$$

Finally for the case where $\mathsf{d}_{\mathcal{X}}(x_0, x_1) \le \frac{\pi}{2}$, we get

$$\boldsymbol{r}_{01}^2(t) \ge (1-t)^2 r_0^2 + t^2 r_1^2 \ge \frac{1}{2} \min\{r_0^2, r_1^2\}. \tag{2.11}$$

## 2.4 Projecting from cone to $\mathcal{X}$

We are now going to provide the inverse parametrization of the geodesics in $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$, with respect to the geodesics in $(\mathcal{C}, \mathsf{d}_{\mathcal{C}})$.

**Theorem 2.7** *For* $x_0, x_1 \in \mathfrak{X}$, *with* $0 < \mathsf{d}_{\mathfrak{X}}(x_0, x_1) < \pi$, *and* $r_0, r_1 > 0$ *consider* $\boldsymbol{z}_{01} \in \mathrm{Geod}(z_0, z_1)$, *where* $z_0 = [x_0, r_0], z_1 = [x_1, r_1]$. *Then,*

$$t \mapsto \boldsymbol{x}_{01}(t) = \overline{\boldsymbol{x}}_{01}\big(\boldsymbol{\beta}_{01}(t)\big) \text{ with } \boldsymbol{\beta}_{01}(t) = \frac{r_0 \sin\big(t\mathsf{d}_{\mathfrak{X}}(x_0, x_1)\big)}{r_1 \sin\big((1-t)\mathsf{d}_{\mathfrak{X}}(x_0, x_1)\big) + r_0 \sin\big(t\mathsf{d}_{\mathfrak{X}}(x_0, x_1)\big)} \tag{2.12}$$

*is an element of* $\mathrm{Geod}(x_0, x_1)$. *Furthermore*

$$\boldsymbol{r}_{01}(\boldsymbol{\beta}_{01}(t)) = \frac{r_0 r_1 \sin(\mathsf{d}_{\mathfrak{X}}(x_0, x_1))}{r_1 \sin((1-t)\mathsf{d}_{\mathfrak{X}}(x_0, x_1)) + r_0 \sin(t\mathsf{d}_{\mathfrak{X}}(x_0, x_1))} \tag{2.13}$$

**Proof:** We will start by proving that $\boldsymbol{\beta}_{01}$ in (2.12) is the inverse of $\boldsymbol{\zeta}_{01}$.

By using the third representation in (2.7), we get

$$\tan\left(\boldsymbol{\zeta}_{01}(t)\mathsf{d}_{\mathfrak{X}}(x_0, x_1)\right) = \frac{t r_1 \sin(\mathsf{d}_{\mathfrak{X}}(x_0, x_1))}{(1-t)r_0 + t r_1 \cos(\mathsf{d}_{\mathfrak{X}}(x_0, x_1))}. \tag{2.14}$$

Let $\boldsymbol{\beta}_{01}$ be the inverse of $\boldsymbol{\zeta}_{01}$. By composing every elemet of (2.14) with $\boldsymbol{\beta}_{01}$, we get

$$\tan\left(t\mathsf{d}_{\mathfrak{X}}(x_0, x_1)\right) = \frac{\boldsymbol{\beta}_{01}(t) r_1 \sin\left(\mathsf{d}_{\mathfrak{X}}(x_0, x_1)\right)}{(1-\boldsymbol{\beta}_{01}(t))r_0 + \boldsymbol{\beta}_{01}(t) r_1 \cos(\mathsf{d}_{\mathfrak{X}}(x_0, x_1))},$$

which gives

$$\boldsymbol{\beta}_{01}(t) = \frac{r_0 \tan\left(t\mathsf{d}_{\mathfrak{X}}(x_0, x_1)\right)}{r_1 \sin(\mathsf{d}_{\mathfrak{X}}(x_0, x_1)) + r_0 \tan(t\mathsf{d}_{\mathfrak{X}}(x_0, x_1)) - r_1 \tan(t\mathsf{d}_{\mathfrak{X}}(x_0, x_1)) \cos(\mathsf{d}_{\mathfrak{X}}(x_0, x_1))}.$$

Multiplying both the nominator and denominator with $\cos(t\mathsf{d}_{\mathfrak{X}}(x_0, x_1))$, we get

$$\boldsymbol{\beta}_{01}(t) = \frac{r_0 \sin\left(t\mathsf{d}_{\mathfrak{X}}(x_0, x_1)\right)}{r_1 \sin(\mathsf{d}_{\mathfrak{X}}(x_0, x_1)) \cos(t\mathsf{d}_{\mathfrak{X}}(x_0, x_1)) + \sin(t\mathsf{d}_{\mathfrak{X}}(x_0, x_1))(r_0 - r_1 \cos(\mathsf{d}_{\mathfrak{X}}(x_0, x_1)))}$$

and by an application of $\sin(a)\cos(b) - \cos(a)\sin(b) = \sin(a-b)$, we get (2.12).

Now by using the first representation of (2.7), we get

$$\sin(t\mathsf{d}_{\mathfrak{X}}(x_0, x_1)) = \frac{\boldsymbol{\beta}_{01}(t) r_1 \sin\left(\mathsf{d}_{\mathfrak{X}}(x_0, x_1)\right)}{\boldsymbol{r}_{01}(\boldsymbol{\beta}_{01}(t))},$$

and combining with (2.12) we get (2.13).

To see that $\boldsymbol{x}_{01}(t) = \overline{\boldsymbol{x}}_{01}\big(\boldsymbol{\beta}_{01}(t)\big)$ is indeed a geodesic in $(\mathfrak{X}, \mathsf{d}_{\mathfrak{X}})$, one has either to check the proof of Theorem 2.6 about the bijection between the geodesics in $(\mathfrak{X}, \mathsf{d}_{\mathfrak{X}})$, and those in $(\mathfrak{C}, \mathsf{d}_{\mathfrak{C}})$, or just use

$$\begin{aligned} &\boldsymbol{r}_{01}^2(\boldsymbol{\beta}_{01}(t)) + \boldsymbol{r}_{01}^2(\boldsymbol{\beta}_{01}(s)) - 2\boldsymbol{r}_{01}(\boldsymbol{\beta}_{01}(t))\boldsymbol{r}_{01}(\boldsymbol{\beta}_{01}(s)) \cos(\mathsf{d}_{\mathfrak{X}}(\overline{\boldsymbol{x}}_{01}(\boldsymbol{\beta}_{01}(t)), \overline{\boldsymbol{x}}_{01}(\boldsymbol{\beta}_{01}(s)))) \\ &\quad = \mathsf{d}_{\mathfrak{C}}^2(\boldsymbol{z}_{01}(\boldsymbol{\beta}_{01}(t)), \boldsymbol{z}_{01}(\boldsymbol{\beta}_{01}(s))) = (\boldsymbol{\beta}_{01}(t) - \boldsymbol{\beta}_{01}(s))^2 \mathsf{d}_{\mathfrak{C}}(z_0, z_1), \end{aligned} \tag{2.15}$$

and solve for $\mathsf{d}_{\mathfrak{X}}(\boldsymbol{x}_{01}(t), \boldsymbol{x}_{01}(s)) = \mathsf{d}_{\mathfrak{X}}(\overline{\boldsymbol{x}}_{01}(\boldsymbol{\beta}_{01}(t)), \overline{\boldsymbol{x}}_{01}(\boldsymbol{\beta}_{01}(s))).$ ■

Finally, we are now interested in the scaling properties of constant-speed geodesics on $\mathfrak{C}$ is we simple change the radius of $z_j = [x_j, r_j]$ into $r_j \widetilde{r}_j$. We will show that the constant-speed geodesic curves behave nicely under the two-parameter rescaling. In the sequel, for $z = [x, r] \in \mathfrak{C}$, and $\widetilde{r} > 0$, we denote with $\widetilde{r}z$, the element $[x, r\widetilde{r}] \in \mathfrak{C}$.

**Proposition 2.8** *For $z_0 = [x_0, r_0], z_1 = [x_1, r_1] \in \mathfrak{C}$ and $\widetilde{r}_0, \widetilde{r}_1 \geq 0$, we have that if $\boldsymbol{z}_{01}(\cdot) = [\overline{\boldsymbol{x}}_{01}(\cdot), \boldsymbol{r}_{01}(\cdot)]$ belongs in $\mathrm{Geod}(z_0, z_1)$, then $\widetilde{\boldsymbol{z}}_{01}(\cdot) = A_{01}(\cdot)\boldsymbol{z}_{01}(B_{01}(\cdot))$, with*

$$A_{01}(t) = \widetilde{r}_0 + (\widetilde{r}_1 - \widetilde{r}_0)\, t \ \text{ and } B_{01}(t) = \frac{\widetilde{r}_1 t}{A_{01}(t)}, \tag{2.16}$$

*is an element of $\mathrm{Geod}(\widetilde{r}_0 z_0, \widetilde{r}_1 z_1)$.*

**Proof:** We first observe $\widetilde{\boldsymbol{z}}_{01}(0) = \widetilde{r}_0 z_0$ and $\widetilde{\boldsymbol{z}}_{01}(1) = \widetilde{r}_1 z_1$, because $A_{01}(0) = \widetilde{r}_0$ and $A_{01}(1) = \widetilde{r}_1$. Thus, to check that $t \mapsto \overline{\boldsymbol{z}}_{01}(t)$ is a geodesic it suffices to show

$$\mathsf{d}_{\mathfrak{C}}(\overline{\boldsymbol{z}}_{01}(0), \overline{\boldsymbol{z}}_{01}(t)) = t\, \mathsf{d}_{\mathfrak{C}}(\overline{\boldsymbol{z}}_{01}(0), \overline{\boldsymbol{z}}_{01}(1)) = t\, \mathsf{d}_{\mathfrak{C}}(r_0 z_0, r_1 z_1),$$

i.e. $\overline{\boldsymbol{z}}_{01}$ is a constant-speed geodesic. However, using (2.9), we first observe

$$\boldsymbol{r}_{01}^2(B_{01}(t)) = (1 - B_{01}(t)) r_0^2 + B_{01}(t) r_1^2 - B_{01}(t)(1 - B_{01}(t)) \mathsf{d}_{\mathfrak{C}}^2(z_0, z_1). \tag{2.17}$$

With this, the abbreviation $a_t = A_{01}(t)$, and the relations $B_{01}(t) = \frac{\widetilde{r}_1 t}{a_t}$ and $1 - B_{01}(t) = \frac{\widetilde{r}_0(1-t)}{a_t}$ we obtain

$$\begin{aligned}
&\mathsf{d}_{\mathfrak{C}}^2(\overline{\boldsymbol{z}}_{01}(0), \overline{\boldsymbol{z}}_{01}(t)) = \mathsf{d}_{\mathfrak{C}}^2\big(\widetilde{r}_0 z_0, a_t \boldsymbol{z}_{01}(B_{01}(t))\big) \\
&\overset{(2.1)}{=} \widetilde{r}_0 a_t \mathsf{d}_{\mathfrak{C}}^2(z_0, \boldsymbol{z}_{01}(B_{01}(t))) + \widetilde{r}_0(\widetilde{r}_0 - a_t) r_0^2 + a_t(a_t - \widetilde{r}_0) \boldsymbol{r}_{01}^2(B_{01}(t)) \\
&\underset{(2.17)}{\overset{\boldsymbol{z}_{01}\text{ is geod.}}{=}} \widetilde{r}_0 a_t \tfrac{\widetilde{r}_1^2 t^2}{a_t^2} \mathsf{d}_{\mathfrak{C}}^2(z_0, z_1) + \widetilde{r}_0(\widetilde{r}_0 - a_t) r_0^2 + a_t(a_t - \widetilde{r}_0)\big(\tfrac{\widetilde{r}_0(1-t)}{a_t} r_0^2 + \tfrac{\widetilde{r}_1 t}{a_t} r_1^2 - \tfrac{\widetilde{r}_0 \widetilde{r}_1 t(1-t)}{a_t^2} \mathsf{d}_{\mathfrak{C}}^2(z_0, z_1)\big) \\
&\overset{*}{=} \widetilde{r}_0 \widetilde{r}_1 t^2 \mathsf{d}_{\mathfrak{C}}^2(z_0, z_1) + (\widetilde{r}_0^2 - \widetilde{r}_0 \widetilde{r}_1) t^2 r_0^2 + (\widetilde{r}_1^2 - \widetilde{r}_0 \widetilde{r}_1) t^2 r_1^2 \overset{(2.1)}{=} t^2 \mathsf{d}_{\mathfrak{C}}^2(\widetilde{r}_0 z_0, \widetilde{r}_1 z_1) = t^2 \mathsf{d}_{\mathfrak{C}}^2(\overline{z}_0, \overline{z}_1),
\end{aligned}$$

where in $\overset{*}{=}$ we simply used the definition of $a_s = A_{01}(s)$. Thus, the assertion is shown. ■

## 2.5 Comparison and local angles

We now introduce comparison angles, see e.g. [Stu99, BBI01, AKP], that are used to study notions of curvature and their properties, and subsequentially be utilized to generate gradient flows on metric spaces, cf. [Oht09, AKP, Sav07, OPV14]. Since we relate the space $(\mathfrak{X}, \mathsf{d}_{\mathfrak{X}})$ with the cone $(\mathfrak{C}, \mathsf{d}_{\mathfrak{C}})$, we will see in the next subsection (cf. the proof of Theorem 2.21)) that it is natural to use comparison angles $\widetilde{\measuredangle}_\kappa$ for different $\kappa$ on these to two spaces.

**Definition 2.9 (Comparison angles)** *Let $(\mathfrak{X}, \mathsf{d}_{\mathfrak{X}})$ be a metric space and $x_0, x_1, x_2 \in \mathfrak{X}$ with $x_0 \notin \{x_1, x_2\}$. For $\kappa \in \mathbb{R}$ we define $a_\kappa$ via*

$$a_\kappa(x_0; x_1, x_2) := \begin{cases} \dfrac{\mathsf{d}_{\mathfrak{X}}^2(x_0, x_1) + \mathsf{d}_{\mathfrak{X}}^2(x_0, x_2) - \mathsf{d}_{\mathfrak{X}}^2(x_1, x_2)}{2\mathsf{d}_{\mathfrak{X}}(x_0, x_1)\mathsf{d}_{\mathfrak{X}}(x_0, x_2)} & \text{for } \kappa = 0, \\[2ex] \dfrac{\cos(\sqrt{\kappa}\, \mathsf{d}_{\mathfrak{X}}(x_1, x_2)) - \cos(\sqrt{\kappa}\, \mathsf{d}_{\mathfrak{X}}(x_0, x_1)) \cos(\sqrt{\kappa}\, \mathsf{d}_{\mathfrak{X}}(x_0, x_2))}{\sin(\sqrt{\kappa}\, \mathsf{d}_{\mathfrak{X}}(x_0, x_1))\ \sin(\sqrt{\kappa}\, \mathsf{d}_{\mathfrak{X}}(x_0, x_2))} & \text{for } \kappa > 0, \\[2ex] \dfrac{\cosh(\mathsf{k}\, \mathsf{d}_{\mathfrak{X}}(x_0, x_1)) \cosh(\mathsf{k}\, \mathsf{d}_{\mathfrak{X}}(x_0, x_2)) - \cosh(\mathsf{k}\, \mathsf{d}_{\mathfrak{X}}(x_1, x_2))}{\sinh(\mathsf{k}\, \mathsf{d}_{\mathfrak{X}}(x_0, x_1))\ \sinh(\mathsf{k}\, \mathsf{d}_{\mathfrak{X}}(x_0, x_2))} & \text{for } \kappa < 0, \end{cases}$$

*where $\mathsf{k} = \sqrt{-\kappa}$. The $\kappa$-*comparison angle $\widetilde{\measuredangle}_\kappa(x_0; x_1, x_2) \in [0, \pi]$ with vertex $x_0$ *is defined by the formula*

$$\widetilde{\measuredangle}_\kappa(x_0; x_1, x_2) = \arccos(a_\kappa(x_0; x_1, x_2)).$$

From now on, the value of $\kappa$ in the previous definition will be refereed as the choice of model space $\mathbb{M}^2(\kappa)$. This terminology is borrowed from the study of Alexandrov spaces, where the sphere ($\kappa > 0$), the plane ($\kappa = 0$), and the hyberbolic plane ($\kappa < 0$) are used as reference, cf. [Stu99, BBI01, AKP]. Later, our main choice will be $\kappa = 1$ on the spherical space (hence the name) $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ and $\kappa = 0$ on the cone $(\mathcal{C}, \mathsf{d}_\mathcal{C})$.

Let $\boldsymbol{x}_{01}$ and $\boldsymbol{x}_{02}$, be two geodesics in $(\mathcal{X}, \mathsf{d}_\mathcal{X})$, emanating from the same initial point $x_0 := \boldsymbol{x}_{01}(0) = \boldsymbol{x}_{02}(0)$. The following theorem guarantees that the set

$$\boldsymbol{\mathcal{AP}}(\boldsymbol{x}_{01}, \boldsymbol{x}_{02}) := \{\, c \in [-1,1] \mid \exists\, 0 < s_k, t_k \to 0 :\ a_\kappa(x_0; \boldsymbol{x}_{01}(t_k), \boldsymbol{x}_{02}(s_k)) \to c\,\} \tag{2.18}$$

of accumulation points of $a_\kappa(x_0; \boldsymbol{x}_{01}(t), \boldsymbol{x}_{02}(s))$ as $t, s \to 0$ is independent of $\kappa$.

**Proposition 2.10** *Let $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ be a metric space and $\boldsymbol{x}_{01} : [0, \tau_1] \to \mathcal{X}$, $\boldsymbol{x}_{02} : [0, \tau_2] \to \mathcal{X}$ be two unit-speed geodesics, issuing from $x_0 \in \mathcal{X}$. Then, for $\kappa \in \mathbb{R}$ we have*

$$a_0(x_0; \boldsymbol{x}_{01}(t), \boldsymbol{x}_{02}(s)) - a_\kappa(x_0; \boldsymbol{x}_{01}(t), \boldsymbol{x}_{02}(s)) \to 0 \quad \textit{for } t, s \to 0. \tag{2.19}$$

We will provide an analytical proof here. For the reader with a more geometrically oriented mind we suggest the proof in [AKP, Page 52, Lemma 6.3.1], which became known to us after the completion of the article.

**Proof:** We give here details for the case $\kappa = 1$. The other cases work exactly the same. For $(t, s) \in (0, \tau] \times (0, \tau]$ with $\tau < \min\{1/2, \tau_1, \tau_2\}$ we set $c_{t,s} := \mathsf{d}(\boldsymbol{x}_{01}(t), \boldsymbol{x}_{02}(s))$. Using $t = \mathsf{d}(x_0, \boldsymbol{x}_{01}(t))$ and $s = \mathsf{d}(x_0, \boldsymbol{x}_{02}(s))$, the triangle inequality gives $|t-s| \le c_{s,t} \le t+s$. This is equivalent to

$$\exists\, \theta \in [-1,1] : \quad c^2_{t,s} = s^2 + t^2 - 2st\theta,$$

where $\theta$ equals $a_0(x_0; \boldsymbol{x}_{01}(t), \boldsymbol{x}_{02}(s))$. Now, defining the function

$$G(s,t;\theta) = \theta - \frac{\cos\sqrt{s^2+t^2-2st\theta} - \cos(s)\cos(t)}{\sin(s)\ \sin(t)},$$

we see that (2.19) is established if we show $\|G(s,t;\cdot)\|_\infty \to 0$ for $s, t \to 0$, where $\|\cdot\|_\infty$ means the supremum over $\theta \in [-1,1]$. To establish the uniform convergence of $G(s,t;\cdot)$ we decompose $G$ in three parts, namely

$$\begin{aligned}
G(s,t;\theta) &= G_1(s,t;\theta) + G_2(s,t;\theta) + G_3(s,t;\theta) \quad \text{with} \\
G_1(s,t;\theta) &:= \theta - \frac{\sin(st\theta)}{\sin(s)\ \sin(t)} = \Big(1 - \frac{F(s)F(t)}{F(st\theta)}\Big)\frac{\sin(st\theta)}{\sin(s)\sin(t)}, \\
G_2(s,t;\theta) &:= \frac{\sin(st\theta) - \cos\sqrt{s^2+t^2-2st\theta} + \cos\sqrt{s^2+t^2}}{\sin(s)\ \sin(t)}, \\
G_3(s,t;\theta) &:= \frac{\cos(s)\cos(t) - \cos\sqrt{s^2+t^2}}{\sin(s)\ \sin(t)},
\end{aligned}$$

where the function $F(r) = \frac{1}{r}\sin r$ can be analytically extended by $F(0) = 1$.

Using $s, t \le 1/2$ and $|\theta| \le 1$ we easily obtain

$$|G_1(s,t;\theta)| \le 6\big(s+t\big)\frac{st}{(s/2)\ (t/2)} \le 24(s+t) \ \to\ 0 \text{ for } s,t \to 0.$$

For $G_3$ we use that $K(r) = 1 - \cos(\sqrt{r})$ is an analytic function with $K(0) = 0$. Thus, with $\sigma = s^2$ and $\tau = t^2$ we have

$$\begin{aligned}&\big|\cos(s)\cos(t) - \cos\sqrt{s^2{+}t^2}\big| = \big|(1{-}K(\sigma))(1{-}K(\tau)) - 1 + K(\sigma{+}\tau)\big|\\ &\leq \big|K(\sigma) + K(\tau) - K(\sigma{+}\tau) - K(0)\big| + K(\sigma)K(\tau)\\ &\leq \Big|\int_0^\sigma\int_0^\tau K''(\hat\sigma{+}\hat\tau)\,\mathrm{d}\hat\tau\,\mathrm{d}\hat\sigma\Big| + C_1^2\sigma\tau \leq \big(C_2{+}C_1^2\big)\sigma\tau \;=\; \big(C_2{+}C_1^2\big)s^2t^2,\end{aligned}$$

where $C_1$ and $C_2$ are bounds for $|K'(r)|$ and $|K''(r)|$ with $r \in [0, 1/2]$, repesctively. Inserting this into the definition of $G_3$ we find

$$|G_3(s,t;\theta)| \leq \frac{(C_2{+}C_1^2)s^2t^2}{(s/2)\,(t/2)} \;\leq 4\big(C_2{+}C_1^2\big)st \;\to\; 0 \text{ for } s,t \to 0.$$

The estimate for $G_2$ we use $K$ again and rewrite the nominator as

$$\sin(st\theta) + K(s^2{+}t^2) - K(s2{+}t^2{-}2st\theta) = \sin(st\theta) - st\theta + \int_0^1\!\big(1{-}2K'(s^2{+}t^2{-}2st\theta\eta)\big)\mathrm{d}\eta\, st\theta.$$

Using $1 = 2K'(0)$ we can estimate the integral by the bound $C_2$ on $K''$ and obtain

$$|G_2(s,t;\theta)| \leq \frac{|st\theta|^3/6 + 2C_2(t{+}s)^2st|\theta|}{(s/2)\,(t/2)} \leq \frac{4}{6}s^2t^2 + 8C_2(t{+}s)^2 \;\to\; 0 \text{ for } s,t \to 0.$$

With this, the desired uniform convergence $G(s,t;\cdot) \to 0$ is established, and the proof is complete. ■

We are now going to introduce the notion of local angles.

**Definition 2.11 (Local Angles)** *Let $\boldsymbol{x}_{01}$and $\boldsymbol{x}_{02}$ be two geodesics in $\mathfrak{X}$ emanating from the same initial point $x_0 := \boldsymbol{x}_{01}(0) = \boldsymbol{x}_{02}(0)$. The* upper angle $\measuredangle_{\mathrm{up}}(\boldsymbol{x}_{01}, \boldsymbol{x}_{02}) \in [0, \pi]$ *and the* lower angle $\measuredangle_{\mathrm{lo}}(\boldsymbol{x}_{01}, \boldsymbol{x}_{02}) \in [0, \pi]$, *between $\boldsymbol{x}_{01}$ and $\boldsymbol{x}_{02}$ are defined by*

$$\measuredangle_{\mathrm{up}}(\boldsymbol{x}_{01}, \boldsymbol{x}_{02}) := \limsup_{s,t\downarrow 0} \widetilde{\measuredangle}_0(x_0, \boldsymbol{x}_{01}(s), \boldsymbol{x}_{02}(t)) = \arccos\big(\inf \boldsymbol{\mathcal{AP}}(\boldsymbol{x}_{01}, \boldsymbol{x}_{02})\big), \tag{2.20a}$$

$$\measuredangle_{\mathrm{lo}}(\boldsymbol{x}_{01}, \boldsymbol{x}_{02}) := \liminf_{s,t\downarrow 0} \widetilde{\measuredangle}_0(x_0; \boldsymbol{x}_{01}(s), \boldsymbol{x}_{02}(t)) = \arccos\big(\sup \boldsymbol{\mathcal{AP}}(\boldsymbol{x}_{01}, \boldsymbol{x}_{02})\big). \tag{2.20b}$$

*When $\measuredangle_{\mathrm{up}}(\boldsymbol{x}_{01}, \boldsymbol{x}_{02}) = \measuredangle_{\mathrm{lo}}(\boldsymbol{x}_{01}, \boldsymbol{x}_{02})$, we say that the* (local) angle *exists in the strict sense and write $\measuredangle(\boldsymbol{x}_{01}, \boldsymbol{x}_{02})$.*

In the previous definition, we could use any model space $\mathbb{M}^2(\kappa)$, since as we have seen in Proposition 2.10 the set of limit points of $a_\kappa(x_0; \boldsymbol{x}_{01}(t), \boldsymbol{x}_{02}(s))$ as $t, s \to 0$, is independent of $\kappa$. It is also trivial that the above limits are invariant under re-parametrization, and that is why we are mostly going to use constant-speed geodesics for joining points.

## 2.6 Curvature and Local Angle Condition

Curvature is one of the most fundamental geometric properties in geodesic metric spaces, and it has applications in gradient flows (see [Oht09, AKP, Sav07]). There are many equivalent characterizations, see [AKP, BBI01, Ber83] for definitions and exposition. We are going to provide the one that is closer to our results, which was introduced in [Stu99].

**Definition 2.12** *We will say that a geodesic metric space $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ has* curvature not less than $\kappa$ *at a point $x$, if there is a neighborhood $U$ of $x$, such that*

$$\sum_{i,j=1}^{m} b_i b_j a_\kappa(x_0; x_i, x_j) \geq 0 \tag{2.21}$$

*for every $m \in \mathbb{N}$, $x_0, x_1, \dots, x_m$ in $U$, and $b_1, \dots, b_m \in [0, \infty)$. We say that $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ has* curvature not less than $\kappa$ "in the large", *if we can take $U = \mathcal{X}$. We shortly write $\underline{\mathrm{curv}}_\mathcal{X}(x) > \kappa$, if the space $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ has curvature not less than $\kappa$, at $x$. We finally write $\underline{\mathrm{curv}}_\mathcal{X} \geq \kappa$ if the space $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ has curvature not less than $\kappa$, in the large.*

We would like to note at this point that $\underline{\mathrm{curv}}_\mathcal{X}(x) > \kappa$ for every $x \in \mathcal{X}$, does not a-priori imply that $\underline{\mathrm{curv}}_\mathcal{X} > \kappa$, since the second will require for (2.21) to hold for arbitrarily big triangles. However we recall the following beautiful theorem (see [BBI01, Th. 10.3.1]), which we will use at a later point.

**Theorem 2.13 (Toponogov's Theorem)** *If a complete geodesic metric space $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ has curvature not less than $\kappa$ at every point, then it has curvature not less than $\kappa$ in the large, i.e.*

$$(\forall x \in \mathcal{X} : \ \underline{\mathrm{curv}}_\mathcal{X}(x) > \kappa) \Leftrightarrow \underline{\mathrm{curv}}_\mathcal{X} > \kappa$$

Concerning the curvatures of a cone $\mathcal{C}$ and its spherical space $\mathcal{X}$, the following result is well-known.

**Theorem 2.14** *[BBI01, Thm. 4.7.1] Let $(\mathcal{C}, \mathsf{d}_\mathcal{C})$ be a cone over a geodesic metric space $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ , and $\mathbf{0}$ its apex. Then, the following holds:*

(a) $(\forall z \in \mathcal{C} \setminus \{\mathbf{0}\} : \underline{\mathrm{curv}}_\mathcal{C}(z) > 0)$ , *if and only if* $\underline{\mathrm{curv}}_\mathcal{X} \geq 1$.

(b) $\underline{\mathrm{curv}}_\mathcal{C} \geq 0$, *if and only if* $\underline{\mathrm{curv}}_\mathcal{X} \geq 1$ *and no triangle in $\mathcal{X}$ has perimeter greater than $2\pi$ (i.e. for any pairwise different $x_1, x_2, x_3$, we have $\mathsf{d}_\mathcal{X}(x_1, x_2) + \mathsf{d}_\mathcal{X}(x_2, x_3) + \mathsf{d}_\mathcal{X}(x_3, x_1) \leq 2\pi$).*

(c) $\overline{\mathrm{curv}}_\mathcal{C} \leq 0$, *if and only if* $\overline{\mathrm{curv}}_\mathcal{X} \leq 1$.

We will not provide the definition for curvature not more than $\kappa'$, i.e. $\overline{\mathrm{curv}}_\mathcal{X} \leq \kappa'$, since it only has little relevance to our work. The interested reader can see [AKP, BBI01, Ber83] for definitions and exposition. We are later only going to use part (c) of the above theorem to get that every ball in the cone over a model space $\mathbb{M}^2(\kappa)$ with $\kappa \leq 1$, is geodesically convex.

The notion of curvature is not very stable when we take the cone $(\mathcal{C}, \mathsf{d}_\mathcal{C})$ over a space $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ or when constructing the Wasserstein space $(\mathcal{P}_2(\mathcal{X}), \mathsf{W})$ over $(\mathcal{X}, \mathsf{d}_\mathcal{X})$. For the first statement, we recall the previous theorem and see that we need $\underline{\mathrm{curv}}_\mathcal{X} \geq 1$ to achieve $\underline{\mathrm{curv}}_\mathfrak{C} \geq 0$, while any other "lower curvature bound" $\kappa < 1$ for $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ is not enough to guarantee any "lower curvature bound" for $(\mathcal{C}, \mathsf{d}_\mathcal{C})$. For the second statement, we refer to [AGS05], where it is shown that we **need** $\underline{\mathrm{curv}}_\mathcal{X} \geq 0$ to deduce $\underline{\mathrm{curv}}_{\mathcal{P}_2(\mathcal{X})} \geq 0$.

Hence, we are going to investigate a significantlly weaker but much more stable notion than lower curvature, which along with some other geometric properties, is enough enough to prove existence of gradient flows, cf. [OPV14, Part 1, Ch. 6]. The property that we are going to examine is the *Local Angle Condition* (LAC). As it will be shown, LAC is a property that is transferable from $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ to $(\mathcal{C} \setminus \{\mathbf{0}\}, \mathsf{d}_\mathcal{C})$, but is also stable when we move to the Wasserstein and the Hellinger-Kantorovich space $(\mathcal{M}(\mathcal{X}), \mathsf{HK}_{\boldsymbol{\ell}})$ over $(\mathcal{X}, \mathsf{d}_\mathcal{X})$.

**Definition 2.15** *For $m \in \mathbb{N}$, a geodesic metric space $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ satisfies $m$-*LAC at a point $x_0$, *if for every choice of $m$ non-trivial geodesics $\boldsymbol{x}_{0i}$ starting at $x_0$ and positive real numbers $b_i$, $i \in \{1, \dots m\}$, we have*

$$\sum_{i,j=1}^{m} b_i b_j \cos(\sphericalangle_{\mathrm{up}}(\boldsymbol{x}_{0i}, \boldsymbol{x}_{0j})) \geq 0. \tag{2.22}$$

*If $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ satisfies $m$-*LAC *at all points, we say that the space satisfies $m$-*LAC.

We note that $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ satisfying $m$-LAC at a point $x_*$ is a fundamentally weaker notion than having $\underline{\mathrm{curv}}_\mathcal{X}(x_*) \geq \kappa$ for some $\kappa \in \mathbb{R}$. For $m$-LAC, one has to look only at infinitesimal triangles with common vertex $x_*$, while for curvature bounds, one has to look at all triangles in a neighborhood of $x_*$. Furthermore, since the triangles used in the definition of $m$-LAC are arbitrarily small, by application of Proposition 2.10 the dependance on any specific $\kappa$ dissapears. Using loose terminology, one can say that curvature is a second order property, while $m$-LAC is a first order property, and that the later, just captures, in a rough sense, the infinitesimally Euclidean nature of the space along "$m$-dimensional projections" near $x_*$. By using geodesics in (2.21), taking limits, and recalling the fact that angles exist in spaces with curvature not less than a real number (see [BBI01]), one can easily retrieve the following theorem.

**Theorem 2.16** *Let $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ a geodesic metric space and $x$ a point in it. If $\underline{\mathrm{curv}}_\mathcal{X}(x) \geq \kappa$ for some $\kappa \in \mathbb{R}$, then $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ satisfies $m$-LAC at every $x_0$ in a neighborhood $U$ of $x$ and for all $m \in \mathbb{N}$.*

For $m = 1$ and $2$ the condition is trivially satisfied. For $m = 3$, which is the case needed for construction solutions for gradient flows, we have the following equivalent, more geometric characterization.

**Theorem 2.17 ([Sav07, OPV14])** *A geodesic metric space $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ satisfies $3$-LAC at $x_0$, if and only if for all triples of geodesics $\boldsymbol{x}_{01}, \boldsymbol{x}_{02}, \boldsymbol{x}_{03}$ emanating from $x_0$, we have*

$$\sphericalangle_{\mathrm{up}}(\boldsymbol{x}_{01}, \boldsymbol{x}_{02}) + \sphericalangle_{\mathrm{up}}(\boldsymbol{x}_{02}, \boldsymbol{x}_{03}) + \sphericalangle_{\mathrm{up}}(\boldsymbol{x}_{03}, \boldsymbol{x}_{01}) \leq 2\pi.$$

We now provide one of our major abstract results. We will show that $m$-LAC is stable on lifting to cones and projecting to the spherical space inside a cone.

**Theorem 2.18** *Let $(\mathcal{C}, \mathsf{d}_\mathcal{C})$ be the cone over a geodesic metric space $(\mathcal{X}, \mathsf{d}_\mathcal{X})$. Then we have*

(a) *If $(\mathcal{C}, \mathsf{d}_\mathcal{C})$ satisfies $m$-LAC at $z_0 = [x_0, r_0]$ for some $x_0 \in \mathcal{X}$ and $r_0 > 0$, then $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ satisfies $m$-LAC at $x_0$.*

(b) *Conversely if $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ satisfies $m$-LAC at $x_0$, then $z_0 = (x_0, r_0) \in (\mathcal{C}, \mathsf{d}_\mathcal{C})$ also satisfies it for every $r_0 > 0$.*

(c) *$(\mathcal{C}, \mathsf{d}_\mathcal{C})$ satisfies $3$-LAC at the apex $\mathbf{0}$ if and only if $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ has perimeter less than $2\pi$.*

(d) *If $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ has diameter less or equal to $\pi/2$, then $(\mathcal{C}, \mathsf{d}_\mathcal{C})$ satisfies $m$-LAC at $\mathbf{0}$ for all $m \in \mathbb{N}$.*

Before we prove this theorem, we provide some auxiliary lemmas. For notational economy, we again set $\phi_{ij} = \mathsf{d}_\mathcal{X}(x_i, x_j)$ and $D_{ij} = \mathsf{d}_\mathcal{C}(z_i, z_j)$. We will use planar comparison angles (i.e. $\kappa = 0$) for the cone $\mathcal{C}$, and spherical comparison angles ($\kappa = 1$) for the underlying space $\mathcal{X}$ (recall Definition 2.9).

**Lemma 2.19** *Let* $z_0 = [x_0, r_0] \in \mathcal{C} \setminus \{\mathbf{0}\}$, $z_1 = [x_1, r_1]$, $z_2 = [x_2, r_2] \in \mathcal{C}$, *and* $0 < \mathsf{d}_{\mathcal{X}}(x_0, x_i) < \pi$, $i \in \{1, 2\}$. *Let* $\boldsymbol{x}_{0i} \in \mathrm{Geod}(x_0, x_i)$, *for* $i = 1, 2$. *Let also* $\boldsymbol{z}_{0i} = [\overline{\boldsymbol{x}}_{0i}, \boldsymbol{r}_{0i}]$ *be the corresponding constant-speed geodesics in* $\mathcal{C}$. *Then,* $\mathcal{A}_{0,\mathcal{C}}(t,s) := a_0(z_0; \boldsymbol{z}_{01}(t), \boldsymbol{z}_{02}(s))$ *and* $\overline{\mathcal{A}}_{1,\mathcal{X}}(t,s) := a_1(x_0; \overline{\boldsymbol{x}}_{01}(t), \overline{\boldsymbol{x}}_{02}(s))$ *are connected by the relation*

$$\begin{aligned}\mathcal{A}_{0,\mathcal{C}}(t,s) = & \frac{(r_1\cos(\phi_{01}) - r_0)(r_2\cos(\phi_{02}) - r_0)}{D_{01}D_{02}} \\ & + \frac{\sin(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{01}(t)))\sin(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{02}(s)))}{\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{01}(t))\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{02}(s))} \frac{\boldsymbol{r}_{01}(t)\boldsymbol{r}_{02}(s)\boldsymbol{\zeta}_{01}(t)\boldsymbol{\zeta}_{02}(s)\phi_{01}\phi_{02}}{tsD_{01}D_{02}}\overline{\mathcal{A}}_{1,\mathcal{X}}(t,s).\end{aligned} \tag{2.23}$$

**Proof:** By the reparametrization rule(2.7) we have $\overline{\boldsymbol{x}}_{0i}(t) = \boldsymbol{x}_{0i}(\boldsymbol{\zeta}_{0i}(t))$, where

$$\boldsymbol{\zeta}_{0i}(t) = \frac{1}{\phi_{0i}}\arccos\left(\frac{(1-t)r_0 + tr_i\cos(\phi_{0i})}{\boldsymbol{r}_{0i}(t)}\right), \tag{2.24}$$

from which we obtain

$$\begin{aligned}\boldsymbol{r}_{0i}(t)\cos(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{0i}(t))) = \boldsymbol{r}_{0i}(t)\cos(\boldsymbol{\zeta}_{0i}(t)\phi_{0i}) &= (1-t)r_0 + tr_i\cos(\phi_{0i}) \\ &= r_0 + t(r_i\cos(\phi_{0i}) - r_0).\end{aligned} \tag{2.25}$$

On the one hand the definition of the comparison angles $a_1$ on $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$ yields

$$\begin{aligned}\cos(\mathsf{d}_{\mathcal{X}}(\overline{\boldsymbol{x}}_{01}(t), \overline{\boldsymbol{x}}_{02}(s))) = & \cos(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{01}(t)))\cos(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{02}(s))) \\ & + \overline{\mathcal{A}}_{1,\mathcal{X}}(t,s)\sin(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{01}(t)))\sin(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{02}(s))).\end{aligned} \tag{2.26}$$

On the other hand, the definition of $a_0$ on $(\mathcal{C}, \mathsf{d}_{\mathcal{C}})$ and $\mathsf{d}_{\mathcal{C}}(z_0, \boldsymbol{z}_{0j}(t)) = tD_{0j}$ lead to

$$\mathcal{A}_{0,\mathcal{C}}(t,s) = \frac{\mathsf{d}^2_{\mathcal{C}}(z_0, \boldsymbol{z}_{01}(t)) + \mathsf{d}^2_{\mathcal{C}}(z_0, \boldsymbol{z}_{02}(s)) - \mathsf{d}^2_{\mathcal{C}}(\boldsymbol{z}_{01}(t), \boldsymbol{z}_{02}(s))}{2tsD_{01}D_{02}}. \tag{2.27}$$

The nominator of the right-hand side is equal to

$$\begin{aligned} & r_0^2 + \boldsymbol{r}_{01}(t)^2 - 2r_0\boldsymbol{r}_{01}(t)\cos(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{01}(t))) \\ & \quad + r_0^2 + \boldsymbol{r}_{02}(s)^2 - 2r_0\boldsymbol{r}_{02}(s)\cos(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{02}(s))) \\ & \quad - \boldsymbol{r}_{01}(t)^2 - \boldsymbol{r}_{02}(s)^2 + 2\boldsymbol{r}_{01}(t)\boldsymbol{r}_{02}(s)\cos(\mathsf{d}_{\mathcal{X}}(\overline{\boldsymbol{x}}_{01}(t), \overline{\boldsymbol{x}}_{02}(s))) \\ & = \underline{2r_0^2 - 2r_0\boldsymbol{r}_{01}(t)\cos(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{01}(t))) - 2r_0\boldsymbol{r}_{02}(s)\cos(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{02}(s)))} \\ & \quad \underline{+2\boldsymbol{r}_{01}(t)\boldsymbol{r}_{02}(s)\cos(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{01}(t)))\cos(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{02}(s)))} \\ & \quad + 2\boldsymbol{r}_{01}(t)\boldsymbol{r}_{02}(s)\overline{\mathcal{A}}_{1,\mathcal{X}}(t,s)\sin(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{01}(t)))\sin(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{02}(s))).\end{aligned} \tag{2.28}$$

Using (2.25) on the underlined terms on the last sum, we obtain

$$\begin{aligned} & 2r_0^2 - 2r_0\left(r_0 + t(r_1\cos(\phi_{01}) - r_0)\right) - 2r_0\left(r_0 + s(r_2\cos(\phi_{02}) - r_0)\right) \\ & \quad + 2\left(r_0 + t(r_1\cos(\phi_{01}) - r_0)\right)\left(r_0 + s(r_2\cos(\phi_{02}) - r_0)\right) \\ & = 2ts\left(r_1\cos(\phi_{01}) - r_0\right)\left(r_2\cos(\phi_{02}) - r_0\right).\end{aligned}$$

So (2.27) takes the form

$$\begin{aligned}\mathcal{A}_{0,\mathcal{C}}(t,s) = & \frac{(r_1\cos(\phi_{01}) - r_0)(r_2\cos(\phi_{02}) - r_0)}{D_{01}D_{02}} \\ & \quad + \frac{\boldsymbol{r}_{01}(t)\boldsymbol{r}_{02}(s)\sin(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{01}(t)))\sin(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{02}(s)))}{tsD_{01}D_{02}}\overline{\mathcal{A}}_{1,\mathcal{X}}(t,s) \\ = & \frac{(r_1\cos(\phi_{01}) - r_0)(r_2\cos(\phi_{02}) - r_0)}{D_{01}D_{02}} \\ & + \frac{\sin(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{01}(t)))\sin(\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{02}(s)))}{\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{01}(t))\mathsf{d}_{\mathcal{X}}(x_0, \overline{\boldsymbol{x}}_{02}(s))} \frac{\boldsymbol{r}_{01}(t)\boldsymbol{r}_{02}(s)\boldsymbol{\zeta}_{01}(t)\boldsymbol{\zeta}_{02}(s)\phi_{01}\phi_{02}}{tsD_{01}D_{02}}\overline{\mathcal{A}}_{1,\mathcal{X}}(t,s),\end{aligned}$$

which is the desired result (2.23). ■

Since local angles do not depend on the choice of model space $\mathbb{M}^2(\kappa)$, the previous lemma provieds a direct connection between the local angles of geodesics in $(\mathcal{C}, \mathsf{d}_\mathcal{C})$ and the the local angles of the corresponding geodesics in $(\mathcal{X}, \mathsf{d}_\mathcal{X})$.

**Proposition 2.20** *Let* $z_0 = [x_0, r_0] \in \mathcal{C} \setminus \{\mathbf{0}\}$, $z_1 = [x_1, r_1], z_2 = [x_2, r_2] \in \mathcal{C}$ *and* $0 < \mathsf{d}_\mathcal{X}(x_0, x_i) < \pi$ *for* $i \in \{1, 2\}$. *Let* $\boldsymbol{x}_{0i} \in \mathrm{Geod}(x_0, x_i)$ *for* $i = 1, 2$. *Let also* $\boldsymbol{z}_{0i} = [\overline{\boldsymbol{x}}_{0i}, \boldsymbol{r}_{0i}]$ *the corresponding geodesics in* $\mathcal{C}$. *Then,* $\boldsymbol{\mathcal{AP}}(\boldsymbol{x}_{01}, \boldsymbol{x}_{02})$ *and* $\boldsymbol{\mathcal{AP}}(\boldsymbol{z}_{01}, \boldsymbol{z}_{02})$ *(see* (2.18) *for definition) satisfy the relation*

$$\boldsymbol{\mathcal{AP}}(\boldsymbol{z}_{01}, \boldsymbol{z}_{02}) = \frac{(r_0 - r_1 \cos\phi_{01})(r_0 - r_2\cos\phi_{02})}{\mathsf{d}_\mathcal{C}(z_0, z_1)\mathsf{d}_\mathcal{C}(z_0, z_2)} + \frac{r_1 r_2 \sin(\phi_{01})\sin(\phi_{02})}{\mathsf{d}_\mathcal{C}(z_0, z_1)\mathsf{d}_\mathcal{C}(z_0, z_2)} \boldsymbol{\mathcal{AP}}(\boldsymbol{x}_{01}, \boldsymbol{x}_{02}), \tag{2.29}$$

*where* $\phi_{0j} = \mathsf{d}_\mathcal{X}(x_0, x_j)$ *and where the operations between set and real numbers are per element. More specifically we have*

$$\cos\big(\measuredangle_{\mathrm{up}}(\boldsymbol{z}_{01}, \boldsymbol{z}_{02})\big) = \frac{(r_0 - r_1\cos\phi_{01})(r_0 - r_2\cos\phi_{02}) + r_1 r_2 \sin(\phi_{01})\sin(\phi_{02})\cos\big(\measuredangle_{\mathrm{up}}(\boldsymbol{x}_{01}, \boldsymbol{x}_{02})\big)}{\mathsf{d}_\mathcal{C}(z_0, z_1)\mathsf{d}_\mathcal{C}(z_0, z_2)}, \tag{2.30}$$

*and*

$$\cos\big(\measuredangle_{\mathrm{up}}(\boldsymbol{x}_{01}, \boldsymbol{x}_{02})\big) = \frac{\mathsf{d}_\mathcal{C}(z_0, z_1)\mathsf{d}_\mathcal{C}(z_0, z_2)\cos\big(\measuredangle_{\mathrm{up}}(\boldsymbol{z}_{01}, \boldsymbol{z}_{02})\big)}{r_1 r_2 \sin(\phi_{01})\sin(\phi_{02})} - \frac{(r_0 - r_1\cos\phi_{01})(r_0 - r_2\cos\phi_{02})}{r_1 r_2 \sin(\phi_{01})\sin(\phi_{02})}. \tag{2.31}$$

*Furthermore, when* $x_0 = x_1$ *or* $x_0 = x_2$, *formula* (2.30) *holds trivially with the right-hand side of the sum being equal to zero.*

**Proof:** By reparametrization (2.24) we have $\overline{\mathcal{A}}_{0,\mathcal{X}}(t, s) = \mathcal{A}_{0,\mathcal{X}}(\boldsymbol{\zeta}_{01}(t), \boldsymbol{\zeta}_{02}(s))$, therefore $\overline{\mathcal{A}}_{0,\mathcal{X}}(t, s)$ and $\mathcal{A}_{0,\mathcal{X}}(t, s)$ have the same accumulation points. Furthermore, Proposition 2.10 guarantees that $\overline{\mathcal{A}}_{0,\mathcal{X}}(t, s)$ and $\overline{\mathcal{A}}_{1,\mathcal{X}}(t, s) = a_1(x_0; \overline{\boldsymbol{x}}_{01}(t), \overline{\boldsymbol{x}}_{02}(s))$ have the same accumulation points.

Let $\ell$ an accumulation point for $\overline{\mathcal{A}}_{1,\mathcal{X}}(t, s)$ and $t_n, s_n$ sequences that achieve that the limit $\ell$. By using formula (2.23) in Lemma 2.19 and $\lim_{\tau\to 0} \frac{\sin(\tau)}{\tau} = 1$, we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{A}_{0,\mathcal{C}}(t_n, s_n) = {} & \frac{(r_1\cos(\phi_{01}) - r_0)(r_2\cos(\phi_{02}) - r_0)}{D_{01}D_{02}} \\ & + \lim_{n\to\infty} \frac{r^{01}(t_n) r^{02}(s_n) \boldsymbol{\zeta}_{01}(t_n)\boldsymbol{\zeta}_{02}(s_n)\phi_{01}\phi_{02}}{t_n s_n D_{01}D_{02}} \lim_{n\to\infty} \overline{\mathcal{A}}_{1,\mathcal{X}}(t_n, s_n).\end{aligned}$$

Using formula (2.7), we have $\lim_{\epsilon\to 0} \frac{\boldsymbol{\zeta}_{0i}(\epsilon)}{\epsilon} = \frac{r_i \sin(\phi_{0i})}{r_0\phi_{0i}}$ and $\lim_{\epsilon\to 0} \boldsymbol{r}_{0i}(\epsilon) = r_0$, and find

$$\lim_{n\to\infty} \mathcal{A}_{0,\mathcal{C}}(t_n, s_n) = \frac{(r_1\cos(\phi_{01}) - r_0)(r_2\cos(\phi_{02}) - r_0) + r_1 r_2 \sin(\phi_{01})\sin(\phi_{02})\ell}{D_{01}D_{02}}. \tag{2.32}$$

Doing the same for all accumulation points of $\mathcal{A}_{0,\mathcal{C}}(t, s)$, we recover the desired formula (2.29).

The formulas for the upper local angle follow simply the taking the infimum of the sets of accumulation points, see (2.20). ■

We are now ready to establish the main result giving the connection between the local angle condition in $(\mathcal{C}, \mathsf{d}_\mathcal{C})$ and $(\mathcal{X}, \mathsf{d}_\mathcal{X})$, respectively.

**Proof:** [Theorem 2.21]

Since the local angle between geodesics depends only on their behavior in neighborhoods around point $x_0$ or $z_0$ respectively, for this proof we will assume, without any loss of generality, that $\mathsf{d}_{\mathcal{X}}(x_0,x_i) < \pi$.

Part (a): Let now assume that $z_0=[x_0,r_0]\in(\mathcal{C}\setminus\{\mathbf{0}\})$ satisfies $m$-LAC for some $m\in\mathbb{N}$. For $x_0\in\mathcal{X}$, consider $m$ non-trivial constant-speed geodesics $\boldsymbol{x}_{0i}$, connecting $x_0$ to $x_1,\dots,x_m$, respectively. Let $\boldsymbol{x}^\epsilon_{0i}(t)=\boldsymbol{x}_{0i}(\epsilon t)$ be defined on $[0,1]$ and consider the geodesics $\boldsymbol{z}^\epsilon_{0i}$ in $\mathcal{C}$ that corresponds to $\boldsymbol{x}^\epsilon_{0i}$ and $\boldsymbol{r}^\epsilon_{0i}(0)=\boldsymbol{r}^\epsilon_{0i}(1)=r_0$. Let finally $b_1,\dots,b_m\geq 0$. Using $\sphericalangle_{\rm up}(\boldsymbol{x}^\epsilon_{0i},\boldsymbol{x}^\epsilon_{0j})=\sphericalangle_{\rm up}(\boldsymbol{x}_{0i},\boldsymbol{x}_{0j})$ for all $\epsilon\in(0,1)$, applying (2.31) with $r_i=r_0$, and using the simple limits $\lim_{\tau\to0}\frac{\sqrt{2-2\cos(\tau)}}{\sin(\tau)}=1$ and $\lim_{\tau\to0}\frac{1-\cos(\tau)}{\sin(\tau)}=0$, we have

$$\begin{aligned}
&\sum_{i,j=1}^m b_ib_j\cos(\sphericalangle_{\rm up}(\boldsymbol{x}_{0i},\boldsymbol{x}_{0j}))=\lim_{\epsilon\to0}\sum_{i,j=1}^m b_ib_j\cos(\sphericalangle_{\rm up}(\boldsymbol{x}^\epsilon_{0i},\boldsymbol{x}^\epsilon_{0j}))\\
&=\lim_{\epsilon\to0}\sum_{i,j=1}^m b_ib_j\bigg(\frac{\sqrt{2-2\cos\mathsf{d}_{\mathcal{X}}(x_0,\boldsymbol{x}_{0i}(\epsilon))}\sqrt{2-2\cos\mathsf{d}_{\mathcal{X}}(x_0,\boldsymbol{x}_{0j}(\epsilon))}\cos(\sphericalangle_{\rm up}(\boldsymbol{z}^\epsilon_{0i},\boldsymbol{z}^\epsilon_{0j}))}{\sin(\mathsf{d}_{\mathcal{X}}(x_0,\boldsymbol{x}_{0i}(\epsilon)))\sin(\mathsf{d}_{\mathcal{X}}(x_0,\boldsymbol{x}_{0j}(\epsilon)))}\\
&\qquad\qquad-\frac{(\cos(\mathsf{d}_{\mathcal{X}}(x_0,\boldsymbol{x}_{0i}(\epsilon)))-1)(\cos(\mathsf{d}_{\mathcal{X}}(x_0,\boldsymbol{x}_{0j}(\epsilon)))-1)}{\sin(\mathsf{d}_{\mathcal{X}}(x_0,\boldsymbol{x}_{0i}(\epsilon)))\sin(\mathsf{d}_{\mathcal{X}}(x_0,\boldsymbol{x}_{0j}(\epsilon)))}\bigg)\\
&=\lim_{\epsilon\to0}\sum_{i,j=1}^m b_ib_j\cos(\sphericalangle_{\rm up}(\boldsymbol{z}^\epsilon_{0i},\boldsymbol{z}^\epsilon_{0j}))\geq0.
\end{aligned}$$

Part (b): We start by assuming that $x_0\in\mathcal{X}$ satisfies $m$-LAC for some $m\in\mathbb{N}$. Let $z_0=[x_0,r_0]\in\mathcal{C}\setminus\{\mathbf{0}\}$ and $\boldsymbol{z}_{01},\dots,\boldsymbol{z}_{0m}$, $m$ non-trivial constant-speed geodesics connecting $z_0$ to some $z_1,\dots,z_m\in\mathcal{C}$. By applying (2.30), for all $b^{\mathcal{C}}_i\geq0$ we have

$$\begin{aligned}
&\sum_{i,j=1}^m b^{\mathcal{C}}_i b^{\mathcal{C}}_j\cos(\sphericalangle_{\rm up}(\boldsymbol{z}_{0i},\boldsymbol{z}_{0j}))\\
&=\sum_{i,j=1}^m b^{\mathcal{C}}_i b^{\mathcal{C}}_j\frac{(r_i\cos(\phi_{0i})-r_0)(r_j\cos(\phi_{0j})-r_0)+r_ir_j\sin(\phi_{0i})\sin(\phi_{0j})\cos(\sphericalangle_{\rm up}(\boldsymbol{x}_{0i},\boldsymbol{x}_{0j}))}{D_{0i}D_{0j}}\\
&=\left(\sum_i^m b^{\mathcal{C}}_i\frac{(r_i\cos(\phi_{0i})-r_0)}{D_{0i}}\right)^2+\sum_{i,j=1}^m b^{\mathcal{C}}_i b^{\mathcal{C}}_j\frac{r_ir_j\sin(\phi_{0i})\sin(\phi_{0j})}{D_{0i}D_{0j}}\cos(\sphericalangle_{\rm up}(\boldsymbol{x}_{0i},\boldsymbol{x}_{0j})).
\end{aligned}$$

Since $x_0$ satisfies $m$-LAC, the last term is non-negative as we may choose $b^X_j:=b^{\mathcal{C}}_j r/D_{0j}\geq0$ as testvector. As the first term is a square we conclude that $z_0\in(\mathcal{C},\mathsf{d}_{\mathcal{C}})$ satisfies $m$-LAC as well.

For parts (c) and (d) we have to study the geodesics $\boldsymbol{z}_{0i}$ starting at the apex $\mathbf{0}$. For this we just notice that for such geodesics $\boldsymbol{z}_{01},\boldsymbol{z}_{02}$ ending at some $z_1=[x_1,r_1],z_2=[x_2,r_2]$ the angle is equal to $\mathsf{d}_{\mathcal{X}}(x_1,x_2)\wedge\pi$. Therefore by using Definition 2.17, we see that $3$-LAC is satisfied if and only if for every choice of pairwise different points $x_1,x_2,x_3$, we have $\mathsf{d}_{\mathcal{X}}(x_1,x_2)\wedge\pi+\mathsf{d}_{\mathcal{X}}(x_2,x_3)\wedge\pi+\mathsf{d}_{\mathcal{X}}(x_3,x_1)\wedge\pi\leq2\pi$, which by applying the triangule inequality is easy to see that it holds if and only if for every choice of pairwise different points $x_1,x_2,x_3$, we have $\mathsf{d}_{\mathcal{X}}(x_1,x_2)+\mathsf{d}_{\mathcal{X}}(x_2,x_3)+\mathsf{d}_{\mathcal{X}}(x_3,x_1)\leq2\pi$. This shows part (c).

When the diameter is less than $\pi/2$, then all cosines are positive and therefore (2.22) is satisfied trivially for all $m\in\mathbb{N}$. Hence, part (d) is shown as well. ■

We can now recover the following immediate result.

**Corollary 2.21** *Let $(\mathcal{C}, \mathsf{d}_{\mathcal{C}})$ be the cone over a geodesic metric space $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$.*

(a) *If $(\mathcal{C}, \mathsf{d}_{\mathcal{C}})$ satisfies $m$-LAC, then $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$ does too.*

(b) *Conversely if $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$ satisfies $m$-LAC, then $(\mathcal{C}, \mathsf{d}_{\mathcal{C}})$ satisfies it at every point in $\mathcal{C} \setminus \{\mathbf{0}\}$.*

(c) *$(\mathcal{C}, \mathsf{d}_{\mathcal{C}})$ satisfies $3$-LAC, if and only if $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$ satisfies $3$-LAC has perimeter less than $2\pi$.*

(d) *If $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$ has diameter less or equal to $\pi/2$ and satisfies $m$-LAC for some $m$, then $(\mathcal{C}, \mathsf{d}_{\mathcal{C}})$ satisfies $m$-LAC.*

## 2.7 K semiconcavity

Another notion that we are going to introduce is the one of $K$-semiconcavity.

**Definition 2.22** *We say that $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$ satisfies $K$*-semiconcavity along a geodesic $\boldsymbol{x}_{01} \in \mathrm{Geod}(x_0, x_1)$ for some $x_0, x_1 \in \mathcal{X}$, with respect to the "observer" $x_2$, *if $f(t) = \mathsf{d}^2_{\mathcal{X}}(x_2, \boldsymbol{x}_{01}(t)) - Kt^2\mathsf{d}^2_{\mathcal{X}}(x_0, x_1)$ is concave. Furthermore, we say that $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$ satisfies $K$*-semiconcavity on $A \subset \mathcal{X}$ with respect to observers from $B \subset \mathcal{X}$, *if it satisfies $K$*-semiconcavity *along every geodesic $\boldsymbol{x}_{01} \in \mathrm{Geod}(x_0, x_1)$ for every $x_0, x_1 \in A$, and with respect to every observer $x_2 \in B$. When $A = B$, we say that $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$ satisfies $K$*-semiconcavity on $A \subset \mathcal{X}$. *Finally we say that $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$ satisfies $K$*-semiconcavity*, if $A = \mathcal{X}$.*

We would like to remark that in the previous definition, $\boldsymbol{x}_{01}(t)$ for $t \in (0,1)$ doesn't have to belong in $A$. Now, we are going to prove the following Lemma, that is going to be used later to prove $K$-semiconcavity on "important" subsets of $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$, when $X$ has curvature not less than a real number $\kappa$ (trivially including the case where $X$ is a convex subset of $\mathbb{R}^d$).

**Lemma 2.23** *Let $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$ a complete geodesic metric space of curvature not less than some $\kappa \in \mathbb{R}$. Then, for $R_1, R_2 > 0$, and $0 < d < \frac{\pi}{2}$ it exists a $K \in \mathbb{R}$ that depends only on $R_1, R_2, d, \kappa$, such that for every $x \subset \mathcal{X}$, $(\mathfrak{C}, \mathsf{d}_{\mathfrak{C}})$ satisfies $K$-semiconcavity at $B\,(x, d) \times [R_1, R_2]$.*

**Proof:** Let $x_0, x_1, x_2$ three points in $A$, and let also $\boldsymbol{x}_{01}(t)$ for some $t \in (0,1)$. By applying the main result in [Wal36] (Also see [AKP, Exercise 8.1.5]), we can isometrically embed the four points $x_0, x_1, x_2, \boldsymbol{x}_{01}(t)$ in one of the model spaces $\mathbb{M}^2(\tilde{\kappa})$, with $\tilde{\kappa} \geq \kappa$. Of course since $\boldsymbol{x}_{01}(t)$ satisfies $\mathsf{d}_{\mathcal{X}}(x_0, \boldsymbol{x}_{01}(t)) + \mathsf{d}_{\mathcal{X}}(\boldsymbol{x}_{01}(t), x_1) = \mathsf{d}_{\mathcal{X}}(x_0, x_1)$, its image will belong in a geodesic connecting the images of $x_0$, and $x_1$. From that, and the fact that distance between points on a cone depends only on distance between the points on the spherical space, we can conclude that is enough to prove the $K$-semiconcavity on $B\,(x, d) \times [R_1, R_2]$, where $B\,(x, d)$ is a subset of a model space $\mathbb{M}^2(\tilde{\kappa})$, with $\tilde{\kappa} \geq \kappa$, where $K$ should only depend on $\kappa$ (and $d, R_1, R_2$). When $\tilde{\kappa} \geq 1$. then we can apply point (a) of Theorem 2.14 to get that the whole cone has curvature bigger than $0$, in which case we know that $K$-semiconcavity holds globally with $K = 1$ (see [AGS05]). Now let $\tilde{\kappa} < 1$. By applying point $(c)$ in Theorem 2.14, we get that the cone has curvature less than zero in the large. This straightforwardly gives us that every closed ball is a complete geodesically convex set. For $R = \frac{R_2^2}{R_1 \cos(d)}$, we can find $a = 1 - \frac{R_1^2 \cos^2(d)}{R_2^2} < 1$, that depends only on $d, R_1, R_2$, for which $B(x,d) \times [R_1, R_2] \subset \overline{B([x,R], aR)}$. Indeed for $y \in B(x,d), R_1 \le \bar{R} \le R_2$, we have

$$\begin{aligned} \mathsf{d}_{\mathfrak{C}}^2([x,R],[y,\bar{R}]) =& R^2 + \bar{R}^2 - 2\bar{R}R\cos(d) \le \left(\frac{R_2^2}{R_1\cos(d)}\right)^2 + R_2^2 - 2R_1\left(\frac{R_2^2}{R_1\cos(d)}\right)\cos(d) \\ =& \left(\frac{R_2^2}{R_1\cos(d)}\right)^2\left[1 + \frac{R_1^2\cos^2(d)}{R_2^2} - \frac{2R_1^2\cos^2(d)}{R_2^2}\right] = \left[1 - \frac{R_1^2\cos^2(d)}{R_2^2}\right]R^2. \end{aligned} \tag{2.33}$$

Now since the ball $\overline{B([x,R], aR)}$ has a positive distance from the apex, we have that is locally isometric to a smooth manifold with curvature no less than $\bar{\kappa}$ that depends only on $\kappa$ and $a, R$ (and therefore only on $d, R_1, R_2$). By applying Theorem 2.13, we get that the whole ball as a complete geodesic metric space that has curvature no less than $\bar{\kappa}$. Now by applying [Oht09, Lemma 3.3] we get K-semiconcavity with

$$K = 1 + \bar{\kappa}^2 2aR \le 1 + \bar{\kappa}^2 2\left[1 - \frac{R_1^2\cos^2(d)}{R_2^2}\right]\frac{R_2^2}{R_1\cos(d)} \le 1 + \frac{2\bar{\kappa}^2 R_2^2}{R_1\cos(d)},$$

Now since $\bar{\kappa}$, depends only on $\kappa, d, R_1, R_2$, we get what we want. ∎

Before we close this section we are going to remark that $(\mathfrak{X}, \mathsf{d}_{\mathfrak{X}})$ satisfies K-semiconcavity along a geodesic $\boldsymbol{x}_{01} \in \mathrm{Geod}(x_0, x_1)$ for some $x_0, x_1 \in \mathfrak{X}$, with respect to the öbserver"$x_2$, if and only if for every $t_1, t_2 \in [0,1]$ if we set $\tilde{x}_0 = \boldsymbol{x}_{01}(t_1), \tilde{x}_1 = \boldsymbol{x}_{01}(t_2)$, and $\tilde{\boldsymbol{x}}_{01}$ equal to the geodesic $\boldsymbol{x}_{01} \llcorner [t_1, t_2]$, reparametrized, we have

$$\mathsf{d}_{\mathfrak{X}}^2(x_2, \tilde{\boldsymbol{x}}_{01}(t)) + Kt(1-t)\mathsf{d}_{\mathfrak{X}}^2(\tilde{x}_0, \tilde{x}_1) \ge (1-t)\mathsf{d}_{\mathfrak{X}}^2(x_2, \tilde{x}_0) + t\mathsf{d}_{\mathfrak{X}}^2(x_2, \tilde{x}_1). \tag{2.34}$$

# 3 Hellinger–Kantorovich space $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$

In the sequel we are going to work on spaces of measures over some underlying (geodesic) metric space $(X, \mathsf{d}_\mathsf{X})$ and denote the associated cone by $(\mathfrak{C}, \mathsf{d}_{\mathfrak{C}})$. A typical example will be $X = \Omega \subset \mathbb{R}^d$, where $\Omega$ convex, compact and equipped with the Euclidean metric $\mathsf{d}_X(x,y) = |x-y|$. All the abstract theory from above applies to these couples; however, our main interest lies in the case where $(\mathfrak{C}, \mathsf{d}_{\mathfrak{C}})$ is identified with $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$ while the spherical space $(\mathfrak{X}, \mathsf{d}_{\mathfrak{X}})$ will be given in terms of the probability measures $\mathcal{P}(X)$ equipped with the metric $\mathsf{SHK}_{\boldsymbol{\ell}}$, which is still to be constructed.

## 3.1 Notation and preliminaries

For the sequel, let $(X, \mathsf{d}_X)$ be a geodesic, Polish space. We will denote by $\mathcal{M}(X)$ the space of all nonnegative and finite Borel measures on $X$ endowed with the weak topology induced by the duality with the continuous and bounded functions of $\mathrm{C}_b(X)$. The subset of measures with finite quadratic moment will be denoted by $\mathcal{M}_2(X)$. The spaces $\mathcal{P}(X)$ and $\mathcal{P}_2(X)$ are the corresponding subsets of probability measures.

If $\mu \in \mathcal{M}(X)$ and $T : X \to Y$ is a Borel map, $T_\sharp\mu$ will denote the push-forward measure on $\mathcal{M}(Y)$, defined by

$$T_\sharp\mu(B) := \mu(T^{-1}(B)) \quad \text{for every Borel set } B \subset Y. \tag{3.1}$$

We will often denote elements of $X\times X$ by $(x_0,x_1)$ and the canonical projections by $\pi^i:(x_0,x_1)\to x_i$ for $i=0,1$. A transport plan on $X$ is a measure $M_{01}\in\mathcal{M}(X{\times}X)$ with marginals $\mu_i:=\pi^i_\sharp M_{01}$.

Given a couple of measures $\mu_0,\mu_1\in\mathcal{P}_2(X)$ with $\mu_0(X)=\mu_1(X)$, its (quadratic) Kantorovich-Wasserstein distance $\mathsf{W}_{\mathsf{d}_X}$ is defined by

$$\mathsf{W}^2_{\mathsf{d}_X}(\mu_0,\mu_1):=\min\Big\{\iint \mathsf{d}^2_X(x_0,x_1)\,\mathrm{d}M_{01}(x_0,x_1)\,\Big|\,M_{01}\in\mathcal{P}_2(X{\times}X),\ \pi^i_\sharp M_{01}=\mu_i,\ i=0,1\Big\}. \tag{3.2}$$

We refer to [AGS05] for a survey on the Kantorovich–Wasserstein distance and related topics.

## 3.2 The logaritmic-entropy transport formulation

Here we first provide the definition of the $\mathsf{HK}_{\boldsymbol{\ell}}(\mu_0,\mu_1)$ distance in terms of a minimization problem that balances a specific transport problem of measures $\sigma_0\mu_0$ and $\sigma_1\mu_1$ with the relative entropies of $\sigma_j\mu_j$ with respect to $\mu_j$. From this, the fundamental scaling property (1.2) of $\mathsf{HK}_{\boldsymbol{\ell}}$ will follow, see Theorem 3.3.

For the characterization of the Hellinger–Kantorovich distance via the static Logarithmic-Entropy Transport (LET) formulation, we define the logarithmic entropy density $F:[0,\infty[\,\to[0,\infty[$ via $F(r)=r\log r-r+1$ and the cost function $L_{\boldsymbol{\ell}}:[0,\infty[\,\to[0,\infty]$ via $L_{\boldsymbol{\ell}}(R)=-2\log\left(\cos\left(R\boldsymbol{\ell}\right)\right)$ for $R\boldsymbol{\ell}<\frac{\pi}{2}$ and $L_{\boldsymbol{\ell}}\equiv+\infty$ otherwise. For given measures $\mu_0,\mu_1$ the LET functional $\mathcal{LET}_{\boldsymbol{\ell}}(\,\cdot\,;\mu_0,\mu_1):\mathcal{M}(X\times X)\to[0,\infty[$ reads

$$\mathcal{LET}_{\boldsymbol{\ell}}(H_{01};\mu_0,\mu_1):=\int_X F(\sigma_0)\mathrm{d}\mu_0+\int_X F(\sigma_1)\mathrm{d}\mu_1+\iint_{X\times X}L_{\boldsymbol{\ell}}(\mathsf{d}_X(x_0,x_1))\mathrm{d}H_{01} \tag{3.3}$$

with $\eta_i:=(\pi_i)_\sharp H_{01}=\sigma_i\mu_i\ll\mu_i$. With this, the equivalent formulation of the Hellinger–Kantorovich distance as entropy-transport problem reads as follows.

**Theorem 3.1 (LET formulation, [LMS17, Sec. 5])** *For every $\mu_0,\mu_1\in\mathcal{M}(X)$ we have*

$$\mathsf{HK}^2_{\boldsymbol{\ell}}(\mu_0,\mu_1)=\min\big\{\mathcal{LET}_{\boldsymbol{\ell}}(H_{01};\mu_0,\mu_1)\,\big|\,H_{01}\in\mathcal{M}(X\times X),\ (\pi_i)_\sharp H_{01}\ll\mu_i\big\}. \tag{3.4}$$

An optimal transport plan $H_{01}$, which always exists, gives the effective transport of mass. Note, in particular, that only $\eta_i\ll\mu_i$ is required and the cost of a deviation of $\eta_i$ from $\mu_i$ is given by the entropy functionals associated with $F$. Moreover, the cost function $\ell_{\boldsymbol{\ell}}$ is finite in the case $\boldsymbol{\ell}\,\mathsf{d}_X(x_0,x_1)<\frac{\pi}{2}$, which highlights the sharp threshold between transport and pure absorption-generation mentioned earlier.

In general, optimal transport plans $H_{01}\in\mathcal{M}(X\times X)$ are not unique. However, due to the strict convexity of $F$ its marginals $\eta_i$ are unique such that the non-uniqueness of the plan $H_{01}$ is solely a property of the optimal transport problem for the cost $\ell_{\boldsymbol{\ell}}$.

**Theorem 3.2 (Optimality conditions [LMS17, Thm. 6.3])** *For $\mu_0,\mu_1\in\mathcal{M}(X)$ let*

$$A'_i:=\Big\{x\in X:\boldsymbol{\ell}\mathsf{dist}(x,\mathrm{supp}(\mu_{1-i}))<\frac{\pi}{2}\Big\},\quad A''_i:=X\setminus A'_i, \tag{3.5}$$

*with the related decomposition*

$$\mu_i:=\mu'_i+\mu''_i,\quad \mu'_i:=\mu_i\llcorner A'_i,\quad \mu''_i:=\mu_i\llcorner A''_i. \tag{3.6}$$

*(i) A plan $H_{01} \in \mathcal{M}(X \times X)$ is optimal for the logarithmic entropy-transport problem in* (3.4) *for $\mu_0$, $\mu_1 \in \mathcal{M}(X)$ if and only if $\iint \ell_{\boldsymbol{\ell}} \mathrm{d}H_{01} < \infty$ and its marginals $\eta_i$ are absolutely continuous with respect to $\mu_i$ with densities $\sigma_i$, which satisfy (we adopt the convention $0 \cdot \infty = 1$ in* (3.7c)*)*

$$\sigma_i = 0 \quad \textit{on} \quad \operatorname{supp}(\mu_i'') \subset A_i'' \tag{3.7a}$$

$$\sigma_i > 0 \quad \textit{on} \quad X \setminus \operatorname{supp}(\mu_i''), \tag{3.7b}$$

$$\sigma_0(x_0)\sigma_1(x_1) \geq \cos^2_{\pi/2}(\boldsymbol{\ell}\,\mathsf{d}_X(x_0,x_1)) \quad \textit{on} \quad X \times X, \tag{3.7c}$$

$$\sigma_0(x_0)\sigma_1(x_1) = \cos^2_{\pi/2}(\boldsymbol{\ell}\,\mathsf{d}_X(x_0,x_1)) \quad H_{01}\textit{-a.e. on} \quad A_0' \times A_1'. \tag{3.7d}$$

*(ii) Moreover, we have that*

$$\mathsf{HK}^2_{\boldsymbol{\ell}}(\mu_0,\mu_1) = \mathsf{HK}^2_{\boldsymbol{\ell}}(\mu_0',\mu_1') + \mathsf{HK}^2_{\boldsymbol{\ell}}(\mu_0'',\mu_1''), \tag{3.8a}$$

*the couples $(\mu_0,\mu_1)$ and $(\mu_0',\mu_1')$ share the same optimal plans $\boldsymbol{\eta}$, and* (3.8b)

$$\mathsf{HK}^2_{\boldsymbol{\ell}}(\mu_0'',\mu_1'') = \mu_0''(X) + \mu_1''(X) = \mu_0(X \setminus A_0') + \mu_1(X \setminus A_1'). \tag{3.8c}$$

We easily obtain upper bounds on $\mathsf{HK}^2_{\boldsymbol{\ell}}$ by inserting $H_{01} = 0$ into the definition of $\mathcal{LET}_{\boldsymbol{\ell}}$ in (3.4), viz, for $\mu_0, \mu_1 \in \mathcal{M}(X)$ and $\nu_0, \nu_1 \in \mathcal{P}(X)$ we have

$$\mathsf{HK}^2_{\boldsymbol{\ell}}(\mu_0,\mu_1) \leq \mu_0(X) + \mu_1(X) \qquad \text{and} \qquad \mathsf{HK}^2_{\boldsymbol{\ell}}(\nu_0,\nu_1) \leq 2. \tag{3.9}$$

## 3.3 Scaling property of $\mathsf{HK}_{\boldsymbol{\ell}}$ and the definition of $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}})$.

Here we give the basic scaling property of the Hellinger–Kantorovich distance that is the basis of our interpretation of $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$ as a cone space.

**Theorem 3.3 (Scaling property of $\mathsf{HK}_{\boldsymbol{\ell}}$)** *For all $\mu_0, \mu_1 \in \mathcal{M}(X)$ and $r_0, r_1 \geq 0$ we have*

$$\mathsf{HK}^2_{\boldsymbol{\ell}}(r_0^2\mu_0, r_1^2\mu_1) = r_0r_1\mathsf{HK}^2_{\boldsymbol{\ell}}(\mu_0,\mu_1) + (r_0^2 - r_0r_1)\mu_0(X) + (r_1^2 - r_0r_1)\mu_1(X). \tag{3.10}$$

*Evenmore, if $H_{01}$ is an optimal plan for the $\mathcal{LET}_{\boldsymbol{\ell}}$ formulation of $\mathsf{HK}_{\boldsymbol{\ell}}(\mu_0,\mu_1)$, then $H_{01}^{r_0r_1} = r_0r_1H_{01}$ is an optimal plan for $\mathsf{HK}_{\boldsymbol{\ell}}(r_0^2\mu_0, r_1^2\mu_1)$.*

**Proof:** Let $H$ be the minimizer in the definition of $\mathcal{LET}_{\boldsymbol{\ell}}(\cdot;\mu_0,\mu_1)$. We now calculate the scale version $\mathcal{LET}_{\boldsymbol{\ell}}(r_0r_1H_{01}; r_0^2\mu_0, r_1^2\mu_1)$ as an upper estimate for $\inf \mathcal{LET}_{\boldsymbol{\ell}}(\cdot; r_0^2\mu_0, r_1^2\mu_1) = \mathsf{HK}_{\boldsymbol{\ell}}(r_0^2\mu_0, r_1^2\mu_1)^2$. For the relative densities $\sigma_0^{r_0r_1}$ and $\sigma_1^{r_0r_1}$ we calculate

$$\eta_0^{r_0r_1} = r_0r_1\eta_0 = r_0r_1\sigma_0\mu_0 = \frac{r_1}{r_0}\sigma_0\, r_0^2\mu_0 \quad \text{and} \quad \eta_1^{r_0r_1} = r_0r_1\eta_1 = r_0r_1\sigma_1\mu_1 = \frac{r_0}{r_1}\sigma_1\, r_1^2\mu_1,$$

from which we obtain $\sigma_0^{r_0r_1} = \frac{r_1}{r_0}\sigma_0$ and $\sigma_1^{r_0r_1} = \frac{r_0}{r_1}\sigma_1$. To determine $\mathcal{LET}_{\boldsymbol{\ell}}(r_0r_1H_{01}; r_0^2\mu_0, r_1^2\mu_1)$ we first calculate the relative entropy for $\sigma_0^{r_0r_1}$:

$$\begin{aligned}
&\int_X F(\sigma_0^{r_0r_1}(x_0))r_0^2\mu_0(\mathrm{d}x_0) = \int_X \big(\sigma_0^{r_0r_1}(x_0)\log(\sigma_0^{r_0r_1}(x_0)) - \sigma_0^{r_0r_1}(x_0) + 1\big)r_0^2\mu_0(\mathrm{d}x_0) \\
&= \int \left(\frac{r_1}{r_0}\sigma_0(x_0)\log\big(\frac{r_1}{r_0}\sigma_0(x_0)\big) - \frac{r_1}{r_0}\sigma_0(x_0) + 1\right) r_0^2\mu_0(\mathrm{d}x_0) \\
&= \int_X \left(r_0r_1\big(\sigma_0(x_0)\log\sigma_0(x_0) - \sigma_0(x_0) + 1\big) + r_0r_1\log\big(\frac{r_1}{r_0}\big)\sigma_0(x_0) + (r_0^2 - r_0r_1)\right)\mu_0(\mathrm{d}x_0) \\
&= r_0r_1\int_X F(\sigma_0(x_0))\mu_0(\mathrm{d}x_0) + r_0r_1\log\big(\frac{r_1}{r_0}\big)\eta_0(X) + (r_0^2 - r_0r_1)\mu_0(X).
\end{aligned}$$

Adding the corresponding term for $\sigma_1^{r_0 r_1}$ we see that the middle term cancels because we have $\eta_0(X) = \eta_1(X)$, and we arrive at the following upper bound:

$$\begin{aligned}\mathsf{HK}_{\boldsymbol{\ell}}^2(r_0^2\mu_0, r_1^2\mu_1) &\leq \mathscr{LET}_{\boldsymbol{\ell}}(r_0r_1H_{01}; r_0^2\mu_0, r_1^2\mu_1)\\ &= r_0r_1\left(\int_X F(\sigma_0)\mu_0(dx_0) + \int_X F(\sigma_1)\mu_1(dx_1)\right) + (r_0^2{-}r_0r_1)\mu_0(X)\\ &\qquad + (r_1^2{-}r_0r_1)\mu_1(X) + \iint_{X\times X} \ell_{\boldsymbol{\ell}}(\mathsf{d}_X(x_0,x_1))r_0r_1H_{01}(\mathrm{d}x_0\mathrm{d}x_1)\\ &= r_0r_1\mathscr{LET}_{\boldsymbol{\ell}}(H_{01};\mu_0,\mu_1) + (r_0^2{-}r_0r_1)\mu_0(X) + (r_1^2{-}r_0r_1)\mu_1(X)\\ &= r_0r_1\mathsf{HK}_{\boldsymbol{\ell}}^2(\mu_0,\mu_1) + (r_0^2{-}r_0r_1)\mu_0(X) + (r_1^2{-}r_0r_1)\mu_1(X),\end{aligned}$$

where in the last step we used that $H_{01}$ is optimal.

By replacing $r_j$ by $1/r_j$ and $\mu_j$ by $r_j^2\mu_j$ this upper bound also yields

$$\mathsf{HK}_{\boldsymbol{\ell}}^2(\mu_0,\mu_1) \leq \frac{1}{r_0r_1}\mathsf{HK}_{\boldsymbol{\ell}}^2(r_0^2\mu_0, r_1^2\mu_1) + \left(\frac{1}{r_0^2} - \frac{1}{r_0r_1}\right) r_0^2\mu_0(X) + \left(\frac{1}{r_1^2} - \frac{1}{r_0r_1}\right) r_1^2\mu_1(X).$$

Multiplying by $r_0r_1$ and rearranging the terms, we obtain the desired lower bound for $\mathsf{HK}_{\boldsymbol{\ell}}^2(r_0^2\mu_0, r_1^2\mu_1)$, and the scaling relation (3.10) is proved. ∎

The above theory for the Hellinger-Kantorovich distance $\mathsf{HK}_{\boldsymbol{\ell}}$ and the abstract Theorem 2.2 allows us now to introduce a new metric distance on the probability measure $\mathcal{P}(X)$ via

$$\mathsf{SHK}_{\boldsymbol{\ell}}(\nu_0,\nu_1) := \arccos\left(1 - \frac{1}{2}\mathsf{HK}_{\boldsymbol{\ell}}^2(\nu_0,\nu_1)\right) \quad \text{for } \nu_0,\nu_1 \in \mathcal{P}(X), \tag{3.11}$$

where the mass bound (3.9) gives $\mathsf{HK}_{\boldsymbol{\ell}}(\nu_0,\nu_1) \leq \sqrt{2}$, which guarantees that the argument of "arccos" is in the interval $[0,1]$, so that $\mathsf{SHK}_{\boldsymbol{\ell}}$ takes values in $[0,\pi/2]$. The mapping $[\cdot,\cdot] : \mathcal{P}(X) \times [0,\infty) \to \mathcal{M}(X)$ is given via

$$\mathcal{P}(X) \times [0,\infty) \ni (\nu, r) \;\mapsto\; [\nu, r] \mathrel{\widehat{=}} r\nu \in \mathcal{M}(X).$$

The general theory of Section 2 shows that $\mathsf{SHK}_{\boldsymbol{\ell}}$ is indeed a metric and, even more, it is a geodesic metric if $(X, \mathsf{d}_X)$ is a geodesic space. It is shown in [LMS17] that $\mathsf{HK}_{\boldsymbol{\ell}}$ is geodesic and hence our Theorem 2.7 shows that $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}})$ is a geodesic space. We summarize the result as follows.

**Theorem 3.4** *The Hellinger–Kantorovich space $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$ can be identified with the cone over the spherical space $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}})$ in the above sense. Moreover, the latter has diameter less than $\frac{\pi}{2}$.*

## 3.4 Cone space formulation

Amongst the many charaqctierizations of $\mathsf{HK}_{\boldsymbol{\ell}}$ discussed in [LMS17] there is one that connects $\mathsf{HK}_{\boldsymbol{\ell}}$ with the classic Kantorovich–Wasserstein distance on the cone $\mathfrak{C}$ over the base space $(X, \boldsymbol{\ell}\mathsf{d}_X)$ with metric

$$\mathsf{d}_{\mathfrak{C},\boldsymbol{\ell}}^2(z_0,z_1) := r_0^2 + r_1^2 - 2r_0r_1\cos_\pi\left(\boldsymbol{\ell}\mathsf{d}_X(x_0,x_1)\right), \quad z_i = [x_i, r_i], \tag{3.12}$$

where as above $\cos_b(a) = \cos(\min\{b,a\})$. Measures in $\mathcal{M}(X)$ can be "lifted" to measures in $\mathcal{M}(\mathfrak{C})$, e.g. by considering the measure $\mu \otimes \delta_1$ for $\mu \in \mathcal{M}(X)$. On the other, we can define the projection of measures in $\mathcal{M}_2(\mathfrak{C})$ onto measures in $\mathcal{M}(X)$ via

$$\mathfrak{P} : \begin{cases} \mathcal{M}_2(\mathfrak{C}) & \to & \mathcal{M}(X), \\ \lambda & \mapsto & \int_{r=0}^{\infty} r^2\,\lambda(\cdot, \mathrm{d}r). \end{cases}$$

For example, the lift $\lambda = m_0\delta_{\{\mathbf{0}\}} + \mu \otimes \frac{1}{r(\cdot)^2}\delta_{r(\cdot)}$, with $m_0 \geq 0$ and $r : \mathrm{supp}(\mu) \to \,]0,\infty[$ arbitrary, gives $\mathfrak{P}\lambda = \mu$. Now, the cone space formulation of the Hellinger–Kantorovich distance of two measures $\mu_0$, $\mu_1 \in \mathcal{M}(X)$ is given as follows.

**Theorem 3.5 (Optimal transport formulation on the cone)** *For $\mu_0, \mu_1 \in \mathcal{M}(\mathbb{R}^d)$ we have*

$$\mathsf{HK}^2_{\boldsymbol{\ell}}(\mu_0,\mu_1) = \min\Big\{ \mathsf{W}^2_{\mathsf{d}_{\mathfrak{C},\boldsymbol{\ell}}}(\lambda_0,\lambda_1) \,\Big|\, \lambda_i \in \mathcal{P}_2(\mathfrak{C}),\ \mathfrak{P}\lambda_i = \mu_i \Big\} \tag{3.13a}$$

$$= \min\Big\{ \iint_{\mathfrak{C}\times\mathfrak{C}} \mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(z_0,z_1)\mathrm{d}\Lambda_{01}(z_0,z_1) \,\Big|\, \pi^i_\sharp\Lambda_{01} = \lambda_i,\ \textit{and}\ \mathfrak{P}\lambda_i = \mu_i \Big\}. \tag{3.13b}$$

**Remark 3.6** *By [LMS17, Lem. 7.19], we also have*

$$\mathsf{HK}^2_{\boldsymbol{\ell}}(\mu_0,\mu_1) = \min\Big\{ \iint_{\mathfrak{C}\times\mathfrak{C}} \widetilde{\mathsf{d}}^2_{\mathfrak{C},\boldsymbol{\ell}}(z_0,z_1)\mathrm{d}\Lambda_{01}(z_0,z_1) \,\Big|\, \pi^i_\sharp\Lambda_{01} = \lambda_i\ \textit{and}\ \mathfrak{P}\lambda_i = \mu_i \Big\}, \tag{3.14}$$

*where $\widetilde{\mathsf{d}}^2_{\mathfrak{C},\boldsymbol{\ell}}([x_0,r_0],[x_1,r_1]) = r_0^2 + r_1^2 - 2r_0r_1\cos_{\pi/2}(\boldsymbol{\ell}\mathsf{d}_X(x_0,x_1))$ is defined with the earlier cut-off at $\pi/2$ instead of $\pi$ as in* (3.12).

The cone space formulation is reminiscent of classical optimal transport problems. Here, however, the marginals $\lambda_i$ of the transport plan $\Lambda_{01} \in \mathcal{M}(\mathfrak{C}\times\mathfrak{C})$ are not fixed, and it is part of the problem to find an optimal pair of measures $\lambda_i$ satisfying the constraints $\mathfrak{P}\lambda_i = \mu_i$ and having minimal Kantorovich–Wasserstein distance on the cone $(\mathfrak{C}, \mathsf{d}_{\mathfrak{C}})$.

The squared cone distance $\mathsf{d}_{\mathfrak{C}}$ has an important scaling invariance: For an arbitrary Borel function $\theta : \mathfrak{C}^2 \to \,]0,\infty[$ , we define the transformation $\mathrm{prd}_\theta : \mathfrak{C}^2 \to \mathfrak{C}^2$ via

$$\mathrm{prd}_\theta(z_0,z_1) := ([x_0, r_0/\theta(z_0,z_1)]; [x_1, r_1/\theta(z_0,z_1)]),\ \text{where } z_i = [x_i, r_i].$$

Its dilation on measures $\Lambda_{01} \in \mathcal{M}(\mathfrak{C}^2)$ is defined by

$$\mathrm{dil}_\theta(\Lambda_{01}) := (\mathrm{prd}_\theta)_\sharp(\theta^2\Lambda_{01}), \quad \text{whenever } \theta \in \mathrm{L}^2(\mathfrak{C}^2;\Lambda_{01}). \tag{3.15}$$

Using the transformation rule, it is easy to see that

$$\int_{\mathfrak{C}^2} \mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(z_0,z_1)\,\mathrm{d}\Lambda_{01} = \int_{\mathfrak{C}^2} \mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(z_0,z_1)\,\mathrm{d}\big(\mathrm{dil}_\theta(\Lambda_{01})\big). \tag{3.16}$$

## 3.5 Characterization of geodesics in $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}})$.

For $X$ being a closed convex subset of $\mathbb{R}^d$ with the Euclidean distance, we want to show that the goedesic curves can be characterized in terms of a generalized continuity equation and a Hamilton–Jacobi equation. Thus, $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}})$ has pseudo-Riemannian structure that is in complete analogy to that of $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$ or that of the Wasserstein space $(\mathcal{P}(X), \mathsf{W}_2)$.

Indeed, according to [LMS16, Eqn. (5.1)] or [LMS17, Thm. 8.19] all constant-speed geodesics for $\mathsf{HK}_{\boldsymbol{\ell}}$ are given as suitable solutions of the coupled system of equations

$$\partial_t\mu + \frac{1}{\boldsymbol{\ell}^2}\mathrm{div}\big(\mu\nabla\xi\big) = 4\xi\mu, \quad \partial_t\xi + \frac{1}{2\boldsymbol{\ell}^2}|\nabla\xi|^2 + 2\xi^2 = 0. \tag{3.17}$$

Here $\xi = \xi(t,x)$ is the dual potential, which satisfies the Hamilton–Jacobi equation, while the measure $\mu(t) \in \mathcal{M}(X)$ follows the generlized continuity equation with transport via $V = \frac{1}{\boldsymbol{\ell}^2}\nabla\xi$ and growth-decay according to $4\xi$.

We now want to derive the corresponding system for the spherical space $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}})$ by applying Theorem 2.7, which tells us that any geodesic $s \mapsto \nu(s) \in \mathcal{P}(X)$ is a rescaling of the geodesic for $\mathsf{HK}_{\boldsymbol{\ell}}$ connecting $\nu_0$ and $\nu_1$.

**Theorem 3.7 (Equation for geodesics in**$(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}})$**)** *The geodesic curves $s \mapsto \nu(s)$ lying in $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}})$ are given by*

$$\partial_s\nu + \frac{1}{\boldsymbol{\ell}^2}\mathrm{div}\big(\nu\nabla\zeta\big) = 4\big(\zeta - \textstyle\int_X \zeta\,\mathrm{d}\nu\big)\nu, \quad \partial_s\zeta + \frac{1}{2\boldsymbol{\ell}^2}|\nabla\zeta|^2 + 2\big(\zeta - \textstyle\int_X \zeta\,\mathrm{d}\nu\big)^2 = 0, \tag{3.18}$$

*where the equations have to be understood in the sense as described in [LMS17, Sec. 8.6].*

**Proof:** We simply use the result in [LMS17, Thm. 8.19] and transform it as given the abstract projection from the cone $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$ to the spherical space $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}})$, namely by a renormalizing of the mass and a rescaling of the arclength parameter. For this, we use the ansatz

$$\nu(s) = n(s)\mu(\tau(s)) \quad \text{and} \quad \zeta(s,x) = a(s)\xi(\tau(s),x) + b(s),$$

where the functions $n,\ \tau,\ a$, and $b$ have to be chosen suitably as functions of $s$, but will be independent of $x \in X$. In particular, we have

$$\int_X \zeta(s,\cdot)\,\mathrm{d}\nu(s) = b(s) + a(s)\int_X \xi(\tau(s),\cdot)\,\mathrm{d}\nu(s) = b(s) + \frac{a(s)}{n(s)}\int_X \xi(\tau(s),\cdot)\,\mathrm{d}\mu(s). \tag{3.19}$$

Using that $(\mu,\xi)$ solves (3.17), we obtain the relations

$$\partial_s\nu + \frac{\dot\tau}{a\boldsymbol{\ell}^2}\mathrm{div}\big(\nu\nabla\zeta\big) = \Big(\frac{4\dot\tau}{a}(\zeta - b) + \frac{\dot n}{n}\Big)\nu, \quad \partial_s\zeta + \frac{\dot\tau}{a\boldsymbol{\ell}^2}|\nabla\zeta|^2 + \frac{2\dot\tau}{a}(\zeta - b)^2 = \frac{\dot a}{a}(\zeta - b) + \dot b.$$

To keep the transport terms, which involve the spatial derivatives, correct we choose $\tau$ such that $\dot\tau = a$ from now on. As $\nu(s) \in \mathcal{P}(X)$, the term on the right-hand side of the continuity equation must have average $0$, hence we impose

$$4\textstyle\int_X \zeta\,\mathrm{d}\nu = 4b + \dot n/n. \tag{3.20}$$

With this, we can rewrite the Hamilton–Jacobi equation for $\zeta$ in the form

$$\partial_s\zeta + \frac{1}{\boldsymbol{\ell}^2}|\nabla\zeta|^2 + 2\big(\zeta - \textstyle\int_X \zeta\,\mathrm{d}\nu\big)^2 = \big(\frac{\dot a}{a} - \frac{\dot n}{n}\big)\zeta + \dot b - \frac{\dot a}{a}b - 2b^2 + 2\big(\textstyle\int_X \zeta\,\mathrm{d}\nu\big)^2.$$

Choosing further $a = n$ the right-hand side simplifies further, because the term linear in $\zeta$ vanishes and the remaining term is $\dot b + 2(b - \int_X \zeta\,\mathrm{d}\nu)^2$.

Now, we show starting from a solution $(\nu,\zeta)$ of (3.18) we can find a solution $(\mu,\xi)$ of (3.17). We first solve $\dot b + 2(b - \int_X \zeta\,\mathrm{d}\nu)^2 = 0$ with $b(s_0)$ such that (3.19) holds at initial time $s_0$. Then, $a = n$

is determined from (3.20) with $n(s_0) = 1$. Finally, the reparametrization $t = \tau(s)$ is obtained from $\dot\tau(s) = a(s)$ and $\tau(s_0) = t_0$. The inverse direction from a solution $(\mu, \xi)$ of (3.17) to a solution $(\nu, \zeta)$ of (3.18) works similarly. ■

The dual dissipation potential $\mathcal{R}^*$ and the associated Onsager operator $\mathbb{K}$, as described in [Mie11, LiM13, LM*17] for $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}})$ are given formally as

$$\mathcal{R}^*_{\boldsymbol{\ell}}(\nu,\zeta) = \int_X \Big(\frac{1}{2\boldsymbol{\ell}^2}|\nabla\zeta|^2 + 2\big(\zeta - \textstyle\int_X \zeta \,\mathrm{d}\nu\big)^2\Big)\,\mathrm{d}\nu \quad \text{and}$$

$$\mathbb{K}_{\boldsymbol{\ell}}(\hat\nu)\zeta = -\frac{1}{\boldsymbol{\ell}^2}\mathrm{div}\big(\hat\nu\nabla\zeta\big) + 4\hat\nu\,\big(\zeta - \textstyle\int_X \zeta\,\mathrm{d}\nu\big),$$

where in the latter case $\nu$ is assumed to have the density $\hat\nu$ with respect to the Lebesgue measure. Note that $\mathcal{R}^*_{\boldsymbol{\ell}}(\nu,\zeta)$ is no longer affine in $\nu$, but it is still concave, which reflects the fact that the set of geodesic curves connecting two measures $\nu_0$ and $\nu_1 \in \mathcal{P}(X)$ is still convex, a fact which is inherited from $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$.

Thus, a gradient flow for a density $\mathcal{E}(\nu) = \int_X E(\hat\nu)\,\mathrm{d}x$ would formally take the form

$$\partial_t\hat\nu = -\mathbb{K}_{\boldsymbol{\ell}}(\hat\nu)\mathrm{D}\mathcal{E}(\hat\nu) = \frac{1}{\boldsymbol{\ell}^2}\mathrm{div}\big(\hat\nu\nabla(E'(\hat\nu))\big) - 4\hat\nu\Big(E'(\hat\nu) - \textstyle\int_X E'(\hat\nu)\,\hat\nu\,\mathrm{d}x\Big).$$

Existence results for such gradient-flow equations will be studied in a forthcoming paper. The next section provides first steps into this direction.

# 4 Finer properties of the Hellinger–Kantorovich space

In this section we are going to prove that the metric space $(X, \mathsf{d}_X)$ satisfies $m$-LAC, if and only if both $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$ and $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}})$ satisfy $m$-LAC. This result is surprising since the cone $(\mathfrak{C}, \mathsf{d}_{\mathfrak{C}})$, which is intrinsically linked to $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$, does not share this equivalence; however the disturbing role of the apex $\mathfrak{o} \in \mathfrak{C}$ is irrelevant for $\mathsf{HK}_{\boldsymbol{\ell}}$.

We are also going to prove that under some extra assumptions on $(X, \mathsf{d}_X)$, we can find sets $\overline{\mathcal{M}}^{\mathcal{L}}_\delta(X) \subset \mathcal{M}(X)$, where $K$-semiconcavity of $\mathsf{HK}_{\boldsymbol{\ell}}$ holds. As it is was mentioned in Section 2.6 (see [OPV14, Part 1, Ch. 6], [Sav07]), when these two properties hold in a space, and a functional $F$ defined on that space is $\lambda$-convex, then for every point in the space there exists a unique gradient flow with respect to $\mathcal{F}$ starting on that point. In some parallel work, we are aiming to extend that result to cover cases where $K$-seminconcavity holds only on suitable collections of subsets, as long as the functionals $\mathcal{F}$ have the property that starting from any point that belongs in a set in the collection, then any minimizer in the JKO scheme, belongs in an another suitable subset in the class. This way, we are going to provide several examples of gradient flows in $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$. Unfortunately, since $K$-semiconcavity is not a local property, we are currently not aware of a way to prove that it also holds in subsets of $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}})$, and we leave that as an open problem for future research. It remains also an open problem, whether $K$-semiconcavity can be replaced by some other (preferably local) property that is stable when moving from $(X, \mathsf{d}_X)$ to $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$.

## 4.1 Stability of $m$-LAC between $(X, \mathsf{d}_X)$, $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}}(X))$, and $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}}(X))$

We will start by proving that the metric space $(X, \mathsf{d}_X)$ satisfies $m$-LAC, if and only if both $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$ and $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}})$ satisfy it too. The proof of the first is a modification of the proof that if a metric space

$(X, \mathsf{d}_X)$ satisfies $m$-LAC, then the Wasserstein space $(\mathcal{P}_2(X), \mathsf{W}_2)$ over $(X, \mathsf{d}_X)$ also satisfies it, which was kindly communicated to us by Giuseppe Savaré (personal communication, May 2017). Although the cone $(\mathfrak{C}, \mathsf{d}_\mathfrak{C})$, over $(X, \mathsf{d}_X)$ does not necessarily satisfy $m$-LAC because of issues related to the apex (see Theorem 2.21), and therefore one cannot use the argument verbatim, the fact that the minimizers satisfy the optimality condition is enough to provide the desired equivalence as we see below.

**Proposition 4.1** *Let $\mu_0 \in (\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$ such that $(X, \mathsf{d}_X)$, satifies $m$-LAC for $\mu_0$ almost every point $x_0$. $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}}(X))$ satisfies $m$-LAC at $\mu_0$.*

**Proof:** For the proof, we are going to utilize the cone representation introduced in Section 3.4. Let $\boldsymbol{\mu}_{01}, \dots, \boldsymbol{\mu}_{0m}$ be geodesics connecting $\mu_0 \in \mathcal{M}(X)$, with $\mu_i \in \mathcal{M}(X), i = \{1, \dots, m\}$. By an application of [LMS17, Thm. 8.4], we can find geodesics $\boldsymbol{\lambda}_{01}, \dots, \boldsymbol{\lambda}_{0m}$ in $\mathcal{P}(\mathfrak{C})$, such that $\mathfrak{P}\boldsymbol{\lambda}_{0i}(t) = \boldsymbol{\mu}_{0i}(t)$ (the fact that we can have $\boldsymbol{\lambda}_{0i}(0)$ to be equal to some fixed $\lambda_0$ for $i = 1, \dots, m$ is given by [LMS17, Lemma 7.10]). By [Lis06, Thm. 6] we can find optimal geodesic plan $\boldsymbol{\Lambda}_{0\to i} \in \mathcal{P}(C[0,1]; \mathfrak{C})$ in the sense that $(e_t)_\sharp \boldsymbol{\Lambda}_{0\to i} = \boldsymbol{\lambda}_{0i}(t)$. By a refined version of the glueing lemma we can find a plan $\boldsymbol{\Lambda} \in \mathcal{P}((C([0,1]; \mathfrak{C})^m)$, such that $\pi_\sharp^{0\to i}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{0\to i}$. For $\boldsymbol{\Lambda}$-a.e. $\boldsymbol{z} = (\boldsymbol{z}_{01}, \dots, \boldsymbol{z}_{0m})$ we have that $\boldsymbol{z}_{01}, \dots, \boldsymbol{z}_{0m}$ are geodesics and $\boldsymbol{z}_{01}(0) = \cdots = \boldsymbol{z}_{0m}(0)$. We split the measure $\boldsymbol{\Lambda}$ in two parts $\boldsymbol{\Lambda}^{\{\mathbf{0}\}}$ and $\boldsymbol{\Lambda}^{\mathfrak{C}\setminus\{\mathbf{0}\}}$, such that $\boldsymbol{\Lambda}^{\{\mathbf{0}\}}(\boldsymbol{z}_{0i}(0) = \{\mathbf{0}\}) = \boldsymbol{\Lambda}(\boldsymbol{z}_{0i}(0) = \{\mathbf{0}\})$ and $\boldsymbol{\Lambda}^{\mathfrak{C}\setminus\{\mathbf{0}\}}(\boldsymbol{z}_{0i}(0) \neq \{\mathbf{0}\}) = \boldsymbol{\Lambda}(\boldsymbol{z}_{0i}(0) \neq \{\mathbf{0}\})$. For $\boldsymbol{\Lambda}^{\mathfrak{C}\setminus\{\mathbf{0}\}}$ let us set $\theta_{ij}(\boldsymbol{z}) = \sphericalangle_{\mathrm{up}}(\boldsymbol{z}_{0i}, \boldsymbol{z}_{0j})$. Since $m$-LAC is satisfied for $\mu_0-$a.e. $x_0$ in $(X, \mathsf{d}_X)$, by an application of Theorem 2.18, we have that $m$-LAC is satisfied for $(e_t)_\sharp \pi_\sharp^{0\to i}\boldsymbol{\Lambda}^{\mathfrak{C}\setminus\{\mathbf{0}\}}$-a.e. $z_0$ in $(\mathfrak{C}, \mathsf{d}_\mathfrak{C})$, and therefore for $\boldsymbol{\Lambda}^{\mathfrak{C}\setminus\{\mathbf{0}\}}$-a.e $\boldsymbol{z} = (\boldsymbol{z}_{01}, \dots, \boldsymbol{z}_{0m})$. We will assume without any loss of generality that all geodesics have length equal to $a$. By applying Remark 3.6 we get

$$
\begin{aligned}
a^2 \cos \sphericalangle_{\mathrm{up}}(\boldsymbol{\mu}_{0i}, \boldsymbol{\mu}_{0j}) &= \liminf_{s,t\downarrow 0} \frac{1}{2st}\left(\mathsf{HK}^2_{\boldsymbol{\ell}}(\mu_0, \boldsymbol{\mu}_{0i}(t)) + \mathsf{HK}^2_{\boldsymbol{\ell}}(\mu_0, \boldsymbol{\mu}_{0j}(s)) - \mathsf{HK}^2_{\boldsymbol{\ell}}(\boldsymbol{\mu}_{0i}(t), \boldsymbol{\mu}_{0j}(s))\right) \\
&\geq \liminf_{s,t\downarrow 0} \frac{1}{2st}\left(W^2_{\mathsf{d}_{\mathfrak{C},\boldsymbol{\ell}}}(\lambda_0, \boldsymbol{\lambda}_{0i}(t)) + W^2_{\mathsf{d}_{\mathfrak{C},\boldsymbol{\ell}}}(\lambda_0, \boldsymbol{\lambda}_{0j}(s)) - W^2_{\widetilde{\mathsf{d}}_{\mathfrak{C},\boldsymbol{\ell}}}(\boldsymbol{\lambda}_{0i}(t), \boldsymbol{\lambda}_{0j}(s))\right) \\
&\geq \liminf_{s,t\downarrow 0} \frac{1}{2st}\int\left(\mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(z_0, \boldsymbol{z}_{0i}(t)) + \mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(z_0, \boldsymbol{z}_{0j}(s)) - \widetilde{\mathsf{d}}^2_{\mathfrak{C},\boldsymbol{\ell}}(\boldsymbol{z}_{0i}(t), \boldsymbol{z}_{0j}(s))\right)\mathrm{d}\boldsymbol{\Lambda} \\
&\geq \liminf_{s,t\downarrow 0} \frac{1}{2st}\int\left(\mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(\mathbf{0}, \boldsymbol{z}_{0i}(t)) + \mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(\mathbf{0}, \boldsymbol{z}_{0j}(s)) - \widetilde{\mathsf{d}}^2_{\mathfrak{C},\boldsymbol{\ell}}(\boldsymbol{z}_{0i}(t), \boldsymbol{z}_{0j}(s))\right)\mathrm{d}\boldsymbol{\Lambda}^{\{\mathbf{0}\}} \\
&\quad + \liminf_{s,t\downarrow 0} \frac{1}{2st}\int\left(\mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(z_0, \boldsymbol{z}_{0i}(t)) + \mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(z_0, \boldsymbol{z}_{0j}(s)) - \widetilde{\mathsf{d}}^2_{\mathfrak{C},\boldsymbol{\ell}}(\boldsymbol{z}_{0i}(t), \boldsymbol{z}_{0j}(s))\right)\mathrm{d}\boldsymbol{\Lambda}^{\mathfrak{C}\setminus\{\mathbf{0}\}}.
\end{aligned}
$$

The first term in the last sum is strictly positive, and for the second part, we can use $\widetilde{\mathsf{d}}^2_{\mathfrak{C},\boldsymbol{\ell}}(\boldsymbol{z}_{0i}(t), \boldsymbol{z}_{0j}(s)) \leq \mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(\boldsymbol{z}_{0i}(t), \boldsymbol{z}_{0j}(s))$. Therefore, by applying Fatou's lemma we have

$$
\begin{aligned}
&a^2 \cos \sphericalangle_{\mathrm{up}}(\boldsymbol{\mu}_{0i}, \boldsymbol{\mu}_{0j}) \\
&\geq \int \liminf_{s,t\downarrow 0} \frac{1}{2st}\left(\mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(z_0, \boldsymbol{z}_{0i}(t)) + \mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(z_0, \boldsymbol{z}_{0j}(s)) - \mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(\boldsymbol{z}_{0i}(t), \boldsymbol{z}_{0j}(s))\right)\mathrm{d}\boldsymbol{\Lambda}^{\mathfrak{C}\setminus\{\mathbf{0}\}} \\
&\geq \int \cos(\theta_{ij}(\boldsymbol{z}))\boldsymbol{\Lambda}^{\mathfrak{C}\setminus\{\mathbf{0}\}}.
\end{aligned}
$$

Thus, applying part (b) of Theorem 2.21 for every choice of positive $b_i, i = 1, \dots, m$, we find

$$
\sum_{i,j=1}^m \cos(\boldsymbol{\mu}_{0i}, \boldsymbol{\mu}_{0j}) b_i b_j \geq \frac{1}{a^2}\int\left(\sum_{i,j=1}^m \cos(\theta_{ij}(\boldsymbol{z})) b_i b_j\right)\boldsymbol{\Lambda}^{\mathfrak{C}\setminus\{\mathbf{0}\}} \geq 0,
$$

which is the desired result for $\mu_0$. ■

We conclude this subsection with the following main result.

**Theorem 4.2** *$(X, \mathsf{d}_X)$ satisfies $m$-LAC, if and only if $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$ satisfies $m$-LAC, if and only if $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}})$ satisfies $m$-LAC.*

**Proof:**

$((X, \mathsf{d}_X) \Rightarrow (\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}}))$: It is a straightforward application of Proposition 4.1.

$((\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}}) \Rightarrow (X, \mathsf{d}_X))$: We just use Dirac measures, and the fact that geodesics stay within the set of Dirac measures.

$((\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}}) \Leftrightarrow (\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}}))$: The proof is a straightforward application of Theorem 2.21 part (d), using that $(\mathcal{P}(X), \mathsf{SHK}_{\boldsymbol{\ell}}))$ has diameter less than $\pi/2$ (see Theorem 3.4.) ■

## 4.2 $K$-semiconcavity on sets of measures with doubling properties

Here we are going to provide results related to $K$-semiconcavity. We will start with a general lemma that gives an estimate for the total mass of the minimizer in $\mathcal{LET}(\cdot; \mu_0, \mu_1)$ (see Theorem 3.1). By $\mathcal{B}(X)$ we denote the collection of all Borel sets in $(X, \mathsf{d}_X)$.

**Lemma 4.3** *Let $\mu_0,\ \mu_1 \in \mathcal{M}(X)$, and let $H_{01}$ be a minimizer for $\mathcal{LET}_{\boldsymbol{\ell}}(\cdot; \mu_0, \mu_1)$, then*

$$H_{01}(X \times X) \leq \sqrt{\mu_0'(X)\mu_1'(X)} \leq \sqrt{\mu_0(X)\mu_1(X)}, \tag{4.1}$$

*where $(\mu_0', \mu_1')$ is the reduced couple of $(\mu_0, \mu_1)$. Furthermore, we have*

$$H_{01}(A \times X) \leq \sqrt{\mu_0'(A)\mu_1'\left(A_{\frac{\pi}{2\boldsymbol{\ell}}}\right)} \quad \text{for all } A \in \mathcal{B}(X), \tag{4.2}$$

*where $A_b = \{y \in X \mid \forall x \in A:\ \ \mathsf{d}_X(x, y) \leq b\}$. Finally, if $X \subset \mathbb{R}^d$ and $\mu_0,\ \mu_1 \ll \mathcal{L}$, and $T : X \to X$ is a function whose graph is the support of $H_{01}$ (such a function exists by [LMS17, Theorem 6.6]), then*

$$H_{01}(A \times T(A)) \leq \sqrt{\mu_0'(A)\mu_1'(T(A))} \quad \text{for all } A \in \mathcal{B}(X). \tag{4.3}$$

**Proof:** By (3.8c), $(\mu_0, \mu_1)$ and $(\mu_0', \mu_1')$, share the same optimal plans. Let $\sigma_i$ be the optimal densities $\frac{d\boldsymbol{\eta}_i}{d\mu_i'}$. Then, the optimality condition (3.7d), which is valid in the support of $H_{01}$, gives

$$\begin{aligned}
H_{01}^2(X \times X) &= \left(\int_{A_0' \times A_1'} 1 \mathrm{d}H_{01}\right)^2 \overset{\text{(3.7d)}}{=} \left(\int_{A_0' \times A_1'} \frac{\cos\left(\boldsymbol{\ell}\mathsf{d}_X(x_0, x_1)\right)}{\sqrt{\sigma_0(x_0)\sigma_1(x_1)}} \mathrm{d}H_{01}\right)^2 \\
&\overset{\cos\leq 1}{\leq} \left(\int_{A_0' \times A_1'} \frac{1}{\sqrt{\sigma_0(x_0)\sigma_1(x_1)}} \mathrm{d}H_{01}\right)^2 \overset{\text{C-S}}{\leq} \left(\int_{A_0' \times A_1'} \frac{1}{\sigma_0(x_0)} \mathrm{d}H_{01}\right)\left(\int_{A_0' \times A_1'} \frac{1}{\sigma_1(x_1)} \mathrm{d}H_{01}\right) \\
&= \int_{A_0'} \frac{1}{\sigma_0} \mathrm{d}\eta_0 \int_{A_1'} \frac{1}{\sigma_1} \mathrm{d}\eta_1 = \int_{A_0'} \mathrm{d}\mu_0 \int_{A_1'} \mathrm{d}\mu_1 \ = \ \mu_0'(X)\, \mu_1'(X).
\end{aligned}$$

For showing (4.2) we define

$$\sigma_{1,A} = \frac{\mathrm{d}H_{01}(A \times \cdot)}{\mathrm{d}\mu_1'} \quad \text{and} \quad \sigma_1 = \frac{dH_{01}(X \times \cdot)}{\mathrm{d}\mu_1'}.$$

such that $0 \leq \sigma_{1,A} \leq \sigma_1$. We define two measures $\widetilde{\mu}_1'$ and $\overline{\mu}_1'$ via

$$\widetilde{\mu}_1'(B) = \int_B \frac{\sigma_{1,A}(x_1)}{\sigma_1(x_1)} \mu_1'(\mathrm{d}x_1) \quad \text{and} \quad \overline{\mu}_1'(B) = \mu_1'(B) - \widetilde{\mu}_1'(B) \text{ for all } B \in \mathcal{B}(X). \tag{4.4}$$

We have that $(H_{01}) \llcorner (A \times X)$, is a plan between $(\mu_0') \llcorner A$ and $\widetilde{\mu}_1'$. Similarly we have that $(H_{01}) \llcorner ((X \backslash A) \times X)$ is a plan between $(\mu_0') \llcorner (X \backslash A)$ and $\overline{\mu}_1'$. Also it is straightforward to see that the sum of the cost of the two plans is equal to the cost of $H_{01}$, therefore these plans must be both optimal. Now applying the first part, i.e. (4.1), w we have

$$\begin{aligned} H_{01}(A \times X) &= \big(H_{01} \llcorner (A \times X)\big)(X \times X) \leq \sqrt{\mu_0'(A)\widetilde{\mu}_1'(X)} \\ &\leq \sqrt{\mu_0'(A)\widetilde{\mu}_1' \left(A_{\frac{\pi}{2\ell}}\right)} \leq \sqrt{\mu_0'(A)\mu_1' \left(A_{\frac{\pi}{2\ell}}\right)}, \end{aligned}$$

which is the desired result (4.2).

Finally, if $H_{01}$ is an optimal plan for $\mu_0', \mu_1'$, and $T : X \to X$ is a function whose graph is the support of $H_{01}$, then $H_{01} \llcorner (A \times T(A)) = H_{01} \llcorner (A \times X)$ is an optimal plan between $\mu_0' \llcorner A$ and $\tilde{\mu}_1' \llcorner T(A) = \tilde{\mu}_1'$, where $\tilde{\mu}_1'$ is defined as in (4.4). Now by applying the same argument as before, we have

$$\begin{aligned} H_{01}(A \times T(A)) &= \big(H_{01} \llcorner (A \times T(A))\big)(X \times X) = \big(H_{01} \llcorner (A \times X)\big)(X \times X) \leq \sqrt{\mu_0'(A)\widetilde{\mu}_1'(X)} \\ &\leq \sqrt{\mu_0'(A)(\widetilde{\mu}_1' \llcorner T(A))(X)} \leq \sqrt{\mu_0'(A)\widetilde{\mu}_1'(T(A))} \leq \sqrt{\mu_0'(A)\mu_1' \, (T(A))}, \end{aligned}$$

■

Before we proceed with the main result of this subsection, we are going to provide some definitions and extra notation. In the following we use the notation $B(x, r)$ for metric balls in $(X, \mathsf{d}_X)$ and possibly in over metric spaces.

**Definition 4.4 (Doubling metric space)** *A metric space $(X, \mathsf{d}_X)$ is called* doubling*, if for every $\mathfrak{r}_2 \geq \mathfrak{r}_1 > 0$, there exists a constant $C(\mathfrak{r}_2/\mathfrak{r}_1) \geq 1$, that depends only on the ratio, such that every ball of radius $\mathfrak{r}_2$ can be covered by $C(\mathfrak{r}_2/\mathfrak{r}_1)$ balls of radius $\mathfrak{r}_1$.*

**Definition 4.5 (Doubling measure on metric space)** *In $(X, \mathsf{d}_X)$, a Borel measure $\mathcal{L}$ is called* doubling *if for every $\mathfrak{r}_2 \geq \mathfrak{r}_1 > 0$, it exists a constant $\bar{C}(\mathfrak{r}_2/\mathfrak{r}_1) \geq 1$, that depends only on the ratio, that for every $x \in X$, we have $\mathcal{L}(B(x, \mathfrak{r}_2)) \leq \bar{C}(\mathfrak{r}_2/\mathfrak{r}_1)\mathcal{L}(B(x, \mathfrak{r}_1))$.*

In [HK*15, Hei01] one can find more information on doubling spaces and measures; for a collection of examples we refer to [CoW77]. The existence of a doubling measure in every complete doubling metric space is provided in [Hei01, Thm. 13.3].

We are mostly interested in $X = \mathbb{R}^d$ or $X = \Omega$, where $\Omega$ is a compact subset of $\mathbb{R}^d$ with Lipschitz boundary, in which case the Lebesgue measure is doubling. We are also interested in manifolds of finite dimension and lower bounds on the Ricci curvature, where the volume measure is doubling, see [Stu06b, Stu06a].

**Definition 4.6 (Locally doubling measure)** *In a metric space $(X, \mathsf{d}_X)$, a Borel measure $\mathcal{L}$ is called* locally doubling, *if for every $M > 0$ and $M \geq \mathfrak{r}_2 \geq \mathfrak{r}_1 > 0$ there exists a constant $\bar{C}_M(\mathfrak{r}_2/\mathfrak{r}_1) \geq 1$ that depends only on the ratio $\mathfrak{r}_2/\mathfrak{r}_1$ and on the upper bound $M$ such that for every $x \in X$ we have $\mathcal{L}(B(x, \mathfrak{r}_2)) \leq \bar{C}_M(\mathfrak{r}_2/\mathfrak{r}_1)\mathcal{L}(B(x, \mathfrak{r}_1))$.*

Since for our result it is easier to work with finite reference measures, we provide the following useful lemma, where we exchange the global doubling property with finiteness of the reference measure.

**Lemma 4.7** *For every doubling measure $\widetilde{\mathcal{L}}$ we can find a finite locally doubling measure $\mathcal{L}$ that is equivalent to $\widetilde{\mathcal{L}}$ (i.e. $\widetilde{\mathcal{L}} \ll \mathcal{L}$ and $\mathcal{L} \ll \widetilde{\mathcal{L}}$ ).*

**Proof:** For some point $x_\mathfrak{a} \in X$, we define $\mathcal{L}(dx) = \frac{1}{(1+\bar{C}(2))^{2\mathsf{d}(x_\mathfrak{a},x)}}\widetilde{\mathcal{L}}(dx)$. For the finiteness of $\mathcal{L}$, we observe that

$$\begin{aligned}\mathcal{L}(X) &= \sum_{i=0}^{\infty}\int_{\overline{B(x_\mathfrak{a},i+1)}\setminus B(x_\mathfrak{a},i)}\frac{1}{(1+\bar{C}(2))^{2\mathsf{d}(x_\mathfrak{a},x)}}\widetilde{\mathcal{L}}(\mathrm{d}x) \leq \sum_{i=0}^{\infty}\int_{\overline{B(x_\mathfrak{a},i+1)}}\frac{1}{(1+\bar{C}(2))^{2i}}\widetilde{\mathcal{L}}(\mathrm{d}x)\\ &\leq \sum_{i=0}^{\infty}\frac{\mathcal{L}(\overline{B(x_\mathfrak{a},i+1)})}{(1+\bar{C}(2))^{2i}} \leq \mathcal{L}(B(x_\mathfrak{a},1))\sum_{i=0}^{\infty}\frac{\bar{C}(2)^{i+2}}{(1+\bar{C}(2))^{2i}} < \infty,\end{aligned} \tag{4.5}$$

where $\bar{C}(2)$ is the doubling constant for $\mathcal{L}$. We also have

$$\frac{\mathcal{L}B(x,\mathfrak{r}_2)}{\mathcal{L}B(x,\mathfrak{r}_1)} \leq \frac{\tilde{\mathcal{L}}B(x,\mathfrak{r}_2)}{\tilde{\mathcal{L}}B(x,\mathfrak{r}_1)}\frac{(1+\bar{C}(2))^{2(\mathsf{d}(x_\mathfrak{a},x)+\mathfrak{r}_1)}}{(1+\bar{C}(2))^{2(\mathsf{d}(x_\mathfrak{a},x)-\mathfrak{r}_2)}} \leq \bar{C}(\mathfrak{r}_2/\mathfrak{r}_1)(1+\bar{C}(2))^{2(\mathfrak{r}_1+\mathfrak{r}_2)}.$$

Therefore for $M > 0$, we conclude that $\mathcal{L}$ is locally doubling with constant $\bar{C}_M(\mathfrak{r}_2/\mathfrak{r}_1) := \bar{C}(\mathfrak{r}_2/\mathfrak{r}_1)(1+\bar{C}(2))^{6M}$, which proves the result. ∎

For a finite, locally doubling measure $\mathcal{L}$ and $\delta \in (0, 1)$ we define the set

$$\overline{\mathcal{M}}_\delta^{\mathcal{L}}(X) = \left\{\mu \in \mathcal{M}(X) : \mu \ll \mathcal{L},\ \delta \leq \frac{d\mu}{d\mathcal{L}}(x) \leq \frac{1}{\delta},\ \text{for } \mathcal{L}\text{-a.e. } x \in X\right\}. \tag{4.6}$$

For positive numbers $\mathsf{d}_1, \mathsf{d}_2$, we also define

$$\widetilde{\mathcal{M}}_{\mathsf{d}_1,\mathsf{d}_2}^{\mathcal{L}}(X) = \left\{\mu \in \mathcal{M}(X) : \forall x \in X :\ \mathsf{d}_2 \leq \frac{\mu\left(B\left(x,\mathsf{d}_1\right)\right)}{\mathcal{L}(B\left(x,\mathsf{d}_1\right))} \leq \frac{1}{\mathsf{d}_2}\right\}. \tag{4.7}$$

It is straightforward to see that $\overline{\mathcal{M}}_\delta^{\mathcal{L}}(X) \subset \widetilde{\mathcal{M}}_{\mathsf{d}_1,\delta}^{\mathcal{L}}(X)$. Furthermore all elements in $\overline{\mathcal{M}}_\delta^{\mathcal{L}}(X)$ have total mass bounded by $\frac{1}{\delta}\mathcal{L}(X)$. The reason that we are using these two sets instead of just of one of them is that neither is geodesically closed in $(\mathcal{M}(X), \mathsf{HK}_{\boldsymbol{\ell}})$. However, as will be proved later, for each $\delta > 0$ we can find $\tilde{d}_1, \tilde{d}_2 > 0$ such that for every $\mu_0, \mu_1 \in \overline{\mathcal{M}}_\delta^{\mathcal{L}}(X)$ we have $\boldsymbol{\mu}_{01}(t) \in \widetilde{\mathcal{M}}_{\mathsf{d}_1,\mathsf{d}_2}^{\mathcal{L}}(X)$ for all $t \in [0, 1]$.

**Theorem 4.8 ($K$-semiconcavity)** *Let $(X, \mathsf{d}_X)$ be doubling and has curvature no less than $\kappa$, $\mathcal{L}$ be a finite, locally doubling measure, and $\overline{\mathcal{M}}_\delta^{\mathcal{L}}(X)$ as in* (4.6)*. Then, for every $\delta > 0$ there exists $K$ such that for all $\mu_0, \mu_1, \mu_2 \in \overline{\mathcal{M}}_\delta^{\mathcal{L}}(X)$, the function*

$$[0, 1] \ni t \to \mathsf{HK}_{\boldsymbol{\ell}}^2(\mu_2, \boldsymbol{\mu}_{01}(t)) - Kt^2\mathsf{HK}_{\boldsymbol{\ell}}^2(\mu_0, \mu_1), \tag{4.8}$$

*is concave, where $\boldsymbol{\mu}_{01} \in \mathrm{Geod}(\mu_0, \mu_1)$.*

The result is based on two facts. The first one is Lemma 2.23, i.e that for $R_1, R_2 > 0$, and $0 < d < \frac{\pi}{2}$ it exists a $K \in \mathbb{R}$ that depends only on $R_1, R_2, d, \kappa$, such that for every $x \subset \mathfrak{X}$, $(\mathfrak{C}, \mathsf{d}_{\mathfrak{C}})$ satisfies $K$-semiconcavity at $B(x, d) \times [R_1, R_2]$. The second is that when two measures, are "uniform" enough, and have bounded densities with respect to each other, then the transport happens in distances less than some $d$ with $d < \pi/2$, and also the densities with respect to the optimal plan are bounded. The result is established on the basis of several intermediate results.

**Lemma 4.9** *Let $(X, \mathsf{d}_X)$ be doubling, $\mathcal{L}$ a finite locally doubling measure, and $\widetilde{\mathcal{M}}^{\mathcal{L}}_{\mathsf{d}_1,\mathsf{d}_2}(X)$, as in (4.7) for $0 < \mathsf{d}_1 < \frac{\pi}{2\boldsymbol{\ell}}$ and $\mathsf{d}_2 > 0$. Then, there exists $0 < C_{\min} \leq C_{\max}$, such that for every $\mu_0, \mu_1 \in \widetilde{\mathcal{M}}^{\mathcal{L}}_{\mathsf{d}_1,\mathsf{d}_2}(X)$ and any optimal plan $H_{01}$ for $\mathcal{LET}_{\boldsymbol{\ell}}(\cdot; \mu_0, \mu_1)$ we have*

$$C_{\min} \leq \sigma_i(x_i) \leq C_{\max}, \quad \eta_i\text{-a.e.} \tag{4.9}$$

*where $\eta_i = \pi^i_{\#} H_{01} = \sigma_i \mu_i$ for $i = 0, 1$. Furthermore, any transportation happens in distances strictly less than some $\frac{\pi}{2\boldsymbol{\ell}}$, i.e. it exists $d < \frac{\pi}{2}$ that depends only on $\mathsf{d}_1, \mathsf{d}_2$, such that $\boldsymbol{\ell}\mathsf{d}_X(x_0, x_1) \leq d$ for $H_{01}$ almost every $(x_0, x_1)$.*

**Proof:** By the optimality conditions, we know that there exist sets $A_0, A_1$ with $\mu_0(X \backslash A_0) = \eta_0(X \backslash A_0) = \mu_1(X \backslash A_1) = \eta_1(X \backslash A_1) = 0$, such that

$$\sigma_0(x_0)\sigma_1(x_1) \geq \cos^2_{\frac{\pi}{2}}\left(\boldsymbol{\ell}\mathsf{d}_X(x_0, x_1)\right) \quad \text{in} \quad A_0 \times A_1. \tag{4.10}$$

By dividing with $\sigma_1(x_1)$ and integrating with respect to $\mu_0$ on $B(x_1, \mathsf{d}_1)$, we obtain

$$\eta_0(B(x_1, \mathsf{d}_1)) \geq \frac{\cos^2_{\frac{\pi}{2}}(\boldsymbol{\ell}\mathsf{d}_1)}{\sigma_1(x_1)}\mu_0(B(x_1, \mathsf{d}_1)) \geq \frac{\cos^2_{\frac{\pi}{2}}(\boldsymbol{\ell}\mathsf{d}_1)}{\sigma_1(x_1)}\mathsf{d}_2\mathcal{L}(B(x_1, \mathsf{d}_1)), \tag{4.11}$$

for every $x_1 \in A_1$. Using Lemma 4.3 we find

$$\begin{aligned}
\eta_0(B(x_1, \mathsf{d}_1)) &\leq \sqrt{\mu_0(B(x_1, \mathsf{d}_1))\mu_1(B(x_1, \mathsf{d}_1)_{\frac{\pi}{2\boldsymbol{\ell}}})} \\
&\leq \sqrt{\mu_0(B(x_1, \mathsf{d}_1))}\sqrt{C\left(\left(\frac{\pi}{2\boldsymbol{\ell}}+\mathsf{d}_1\right)/\mathsf{d}_1\right) \sup_{y \in B_{\frac{\pi}{2\boldsymbol{\ell}}}(x_1, \mathsf{d}_1)} \mu_1(B(y, \mathsf{d}_1))} \\
&\leq \frac{1}{\mathsf{d}_2}\sqrt{\mathcal{L}(B(x_1, \mathsf{d}_1))}\sqrt{C\left(\left(\frac{\pi}{2\boldsymbol{\ell}}+\mathsf{d}_1\right)/\mathsf{d}_1\right) \sup_{y \in B_{\frac{\pi}{2\boldsymbol{\ell}}}(x_1, \mathsf{d}_1)} \mathcal{L}(B(y, \mathsf{d}_1))} \\
&\leq \frac{1}{\mathsf{d}_2}\sqrt{\mathcal{L}(B(x_1, \mathsf{d}_1))}\sqrt{C\left(\left(\frac{\pi}{2\boldsymbol{\ell}}+\mathsf{d}_1\right)/\mathsf{d}_1\right)\mathcal{L}(B(x_1, \mathsf{d}_1)_{\frac{\pi}{2\boldsymbol{\ell}}})} \\
&\leq \frac{1}{\mathsf{d}_2}\mathcal{L}(B(x_1, \mathsf{d}_1))\sqrt{C\left(\left(\frac{\pi}{2\boldsymbol{\ell}}+\mathsf{d}_1\right)/\mathsf{d}_1\right)\bar{C}_{\frac{\pi}{\boldsymbol{\ell}}}\left(\left(\frac{\pi}{2\boldsymbol{\ell}}+\mathsf{d}_1\right)/\mathsf{d}_1\right)},
\end{aligned} \tag{4.12}$$

where the constant $C\left(\left(\frac{\pi}{2\boldsymbol{\ell}}+\mathsf{d}_1\right)/\mathsf{d}_1\right)$ is as in the definition of doubling metric spaces to cover a set of radius $\frac{\pi}{2\boldsymbol{\ell}} + \mathsf{d}_1$ by balls of radius $\mathsf{d}_1$, and $\bar{C}_{\frac{\pi}{\boldsymbol{\ell}}}\left(\left(\frac{\pi}{2\boldsymbol{\ell}}+\mathsf{d}_1\right)/\mathsf{d}_1\right)$ is the doubling measure constant for radius less than $\frac{\pi}{\boldsymbol{\ell}}$. We set $\widetilde{C} = \sqrt{C\left(\left(\frac{\pi}{2\boldsymbol{\ell}}+\mathsf{d}_1\right)/\mathsf{d}_1\right)\bar{C}_{\frac{\pi}{\boldsymbol{\ell}}}\left(\left(\frac{\pi}{2\boldsymbol{\ell}}+\mathsf{d}_1\right)/\mathsf{d}_1\right)}$, and by combining (4.11) and (4.12), we derive the lower bound

$$\sigma_1(x_1) \geq \cos^2_{\frac{\pi}{2}}(\boldsymbol{\ell}\mathsf{d}_1)\, d_2^2/\widetilde{C} \quad \text{in} \quad A_1. \tag{4.13}$$

Now, by the second optimality condition we have

$$\sigma_0(x_0) = \frac{\cos^2_{\frac{\pi}{2}}(\boldsymbol{\ell}\mathsf{d}_X(x_0,x_1))}{\sigma_1(x_1)} \le \frac{\widetilde{C}}{\cos^2_{\frac{\pi}{2}}(\boldsymbol{\ell}\mathsf{d}_1)\, d_2^2}, \quad H_{01}\text{-a.e. in} \quad A_0 \times A_1. \tag{4.14}$$

By interchaning the roles of $\sigma_0$ and $\sigma_1$ and combining all the inequalities we arrive at

$$C_{\min} := \frac{\cos^2_{\frac{\pi}{2}}(\boldsymbol{\ell}\mathsf{d}_1)\, d_2^2}{\widetilde{C}} \le \sigma_i(x_i) \le \frac{\widetilde{C}}{\cos^2_{\frac{\pi}{2}}(\boldsymbol{\ell}\mathsf{d}_1)\, d_2^2} =: C_{\max}, \quad \boldsymbol{\eta}_i\text{-a.e. in} \quad A_i,$$

which is the desired result.

Now by visiting the second optimality condition one more time, we get that $\cos^2_{\frac{\pi}{2}}(\boldsymbol{\ell}\mathsf{d}_X(x_0,x_1))$ is bounded from below by a positive constant that depends only on the bounds of $\sigma_i$, for $H_{01}$-a.e. $(x_0,x_1)$. Therefore by continuity of the cosine, we get that it exists $d < \frac{\pi}{2}$ such that for every $\mu_0,\mu_1 \in \widetilde{\mathcal{M}}^{\mathcal{L}}_{\mathsf{d}_1,\mathsf{d}_2}(X)$, we have $\boldsymbol{\ell}\mathsf{d}_X(x_0,x_1) \le d$, for $H_{01}$-a.e. $(x_0,x_1)$.

■

The next result shows that the geodesic closure of $\overline{\mathcal{M}}^{\mathcal{L}}_{\delta}(X)$ is contained in $\widetilde{\mathcal{M}}^{\mathcal{L}}_{\mathsf{d}_1,\mathsf{d}_2}(X)$ for suitably chosen $\mathsf{d}_1, \mathsf{d}_2$.

**Lemma 4.10** *Let $(X,\mathsf{d}_X)$ be doubling, $\mathcal{L}$ be a finite locally doubling measure, and $\overline{\mathcal{M}}^{\mathcal{L}}_{\delta}(X)$ be as in* (4.6)*. Then, for each $\delta > 0$ there exist $\mathsf{d}_1 \in (0,\frac{\pi}{2\boldsymbol{\ell}})$ and $\mathsf{d}_2 > 0$ such that any constant-speed geodesic $\boldsymbol{\mu}_{01}$ connecting $\mu_0$ to $\mu_1$ with $\mu_0,\mu_1 \in \overline{\mathcal{M}}^{\mathcal{L}}_{\delta}(X)$ satisfies $\boldsymbol{\mu}_{01}(t) \in \widetilde{\mathcal{M}}^{\mathcal{L}}_{\mathsf{d}_1,\mathsf{d}_2}(X)$ for all $t \in [0,1]$.*

**Proof:** It is straightforward to see that $\overline{\mathcal{M}}^{\mathcal{L}}_{\delta}(X)$ is a subset of some $\widetilde{\mathcal{M}}^{\mathcal{L}}_{\min\{\frac{\pi}{4\boldsymbol{\ell}},\frac{1}{2}\},\delta}(X)$. Therefore, by Lemma 4.9, we have that it exists $\tilde{d}$ that depends only on $\delta$ with $\tilde{d} < \frac{\pi}{2}$, for which we have that for $H_{01}$-a.e., it holds that $\boldsymbol{\ell}\mathsf{d}_X(x_0,x_1) \le \tilde{d} < \frac{\pi}{2}$. Let $\Lambda_{01}$, be the optimal plan in the cone definition, and $\boldsymbol{\Lambda}_{0\to1}$ the occurring plan on the geodesics. For $x_0 \in X$, we have

$$\boldsymbol{\mu}_{01}\left(t; B\left(x_0, \frac{\pi+2\tilde{d}}{4\boldsymbol{\ell}}\right)\right) \ge \mathfrak{P}\left[(e_t)_\sharp(\boldsymbol{\Lambda}_{0\to1}) \llcorner \left\{\boldsymbol{x}_{01}(0) \in B\left(x_0, \frac{\pi-2\tilde{d}}{4\boldsymbol{\ell}}\right)\right\}\right](X), \tag{4.15}$$

since all points in $B\left(x_0,\frac{\pi-2\tilde{d}}{4\boldsymbol{\ell}}\right)$, will be transfered at most distance $\frac{\tilde{d}}{\boldsymbol{\ell}}$. Therefore will remain in a ball of radius $B\left(x_0,\frac{\pi+2\tilde{d}}{4\boldsymbol{\ell}}\right)$. Now $\tilde{\boldsymbol{\mu}}_{01}(t) = \mathfrak{P}\left[(e_t)_\sharp(\boldsymbol{\Lambda}_{0\to1}) \llcorner \left\{\boldsymbol{x}_{01}(0) \in B\left(x_0,\frac{\pi-2\tilde{d}}{4\boldsymbol{\ell}}\right)\right\}\right]$ is a geodesic starting from $\mu_0 \llcorner B\left(x_0,\frac{\pi-2\tilde{d}}{4\boldsymbol{\ell}}\right)$. Let $\tilde{m}(t) = (\tilde{\boldsymbol{\mu}}_{01}(t))(X)$. By (2.11) and recalling (1.3) we get

$$\tilde{m}(t) \ge (1-t)^2\tilde{m}(0) + t^2\tilde{m}(1),$$

which in turn for $t \in \left[0, \frac{1}{2}\right]$, gives

$$\begin{aligned}
&\mathfrak{P}\left[(e_t)_\sharp\left(\boldsymbol{\Lambda}_{0\to 1}\right)\llcorner\left\{\boldsymbol{x}_{01}(0)\in B\left(x_0,\frac{\pi-2\tilde{d}}{4\boldsymbol{\ell}}\right)\right\}\right](X)\\
&\geq (1-t)^2\mathfrak{P}\left[(e_0)_\sharp\left(\boldsymbol{\Lambda}_{0\to 1}\right)\llcorner\left\{\boldsymbol{x}_{01}(0)\in B\left(x_0,\frac{\pi-2\tilde{d}}{4\boldsymbol{\ell}}\right)\right\}\right](X)\\
&\geq (1-t)^2\mu_0\left(B\left(x_0,\frac{\pi-2\tilde{d}}{4\boldsymbol{\ell}}\right)\right)\geq \frac{1}{4}\mu_0\left(B\left(x_0,\frac{\pi-2\tilde{d}}{4\boldsymbol{\ell}}\right)\right)\geq \frac{\delta}{4}\mathcal{L}\left(B\left(x_0,\frac{\pi-2\tilde{d}}{4\boldsymbol{\ell}}\right)\right)\\
&\geq \frac{\delta}{4\widetilde{C}_M\left(\left(\frac{\pi+2\tilde{d}}{4\boldsymbol{\ell}}\right)/\left(\frac{\pi-2\tilde{d}}{4\boldsymbol{\ell}}\right)\right)}\mathcal{L}\left(B\left(x_0,\frac{\pi+2\tilde{d}}{4\boldsymbol{\ell}}\right)\right).
\end{aligned} \tag{4.16}$$

Combining 4.15 and 4.16, we get that

$$\frac{\boldsymbol{\mu}_{01}\left(t;B\left(x_0,\frac{\pi+2\tilde{d}}{4\boldsymbol{\ell}}\right)\right)}{\mathcal{L}\left(B\left(x_0,\frac{\pi+2\tilde{d}}{4\boldsymbol{\ell}}\right)\right)}\geq \frac{\delta}{4\widetilde{C}_M\left(\left(\frac{\pi+2\tilde{d}}{4\boldsymbol{\ell}}\right)/\left(\frac{\pi-2\tilde{d}}{4\boldsymbol{\ell}}\right)\right)} \tag{4.17}$$

We work in the same manner with the roles of $\mu_0,$ and $\mu_1$ being inversed to recover the same estimate on the interval $[1/2,1]$, and this way we retrieve the lower bound with $\mathsf{d}_1 = \frac{\pi+2\tilde{d}}{4\boldsymbol{\ell}}$, and $\mathsf{d}_2 = \frac{\delta}{4\widetilde{C}_M\left(\left(\frac{\pi+2\tilde{d}}{4\boldsymbol{\ell}}\right)/\left(\frac{\pi-2\tilde{d}}{4\boldsymbol{\ell}}\right)\right)}$. In a similar manner by utilizing (2.10), we retrieve an upper bound. ■

**Lemma 4.11** *Let $(X,\mathsf{d}_X)$ be doubling, $\mathcal{L}$ a finite locally doubling measure, and $\widetilde{\mathcal{M}}^{\mathcal{L}}_{\mathsf{d}_1,\mathsf{d}_2}(X)$ as in* (4.7)*. There exist $R_{\min}, R_{\max} > 0$ that depend on $\mathsf{d}_1,\mathsf{d}_2,$ such that for $\mu_0,\mu_1,$ with $\boldsymbol{\mu}_{01}(t)\in \widetilde{\mathcal{M}}^{\mathcal{L}}_{\mathsf{d}_1,\mathsf{d}_2}(X)$, and $\mu_2\in\widetilde{\mathcal{M}}^{\mathcal{L}}_{\mathsf{d}_1,\mathsf{d}_2}(X)$, we can find measures $\lambda_0,\lambda_1,\lambda_2,\lambda_t\in\mathcal{P}_2(\mathfrak{C}[R_{\min},R_{\max}])$ with*

$$\mathfrak{P}\lambda_i=\mu_i,\quad \mathfrak{P}\lambda_t=\boldsymbol{\mu}_{01}(t),\quad \mathsf{W}_{\mathsf{d}_{\mathfrak{C},\boldsymbol{\ell}}}(\lambda_i,\lambda_t)=\mathsf{HK}_{\boldsymbol{\ell}}(\mu_i,\boldsymbol{\mu}_{01}(t)),\ i=0,1,2.$$

**Proof:** For $i=0,1,2,$ let $H_{ti}$ be the optimal plan in the definition of $\mathcal{LET}_{\boldsymbol{\ell}}(\cdot;\mu_i,\boldsymbol{\mu}(t))$, and $\sigma_i^{ti},\sigma_t^{ti}$ the densities of $\eta_i^{ti},\eta_t^{ti}$ with respect to $\mu_i,\mu_t$. Let now the plans

$$\Lambda_{ti}(\mathrm{d}z_i,\mathrm{d}z_t)=\delta_{\sqrt{\sigma_i^{ti}(x_i)}}(\mathrm{d}r_i)\delta_{\sqrt{\sigma_t^{ti}(x_t)}}(\mathrm{d}r_i)H_{ti}(\mathrm{d}x_i,\mathrm{d}x_t).$$

For $i=0,1,2,$ we take $\theta^{ti}([z_t,z_i])=\sqrt{\frac{\sigma_t^{ti}(x_t)}{\sigma_t^{t0}(x_t)}}$, and we define $\tilde{\Lambda}_{ti}=\mathrm{dil}_{\theta^{ti}}(\Lambda^{ti})$. Finally we set $\lambda_i=\pi^i_\sharp\tilde{\Lambda}_{ti}$ for $i=0,1,2.$ It is straightforward to see that $r_i=\sqrt{\frac{\sigma_t^{ti}(x_t)}{\sigma_t^{t0}(x_t)}}\sqrt{\sigma_i^{ti}(x_i)}$ for $\lambda_i$-a.e. $z_i=[x_i,r_i]$, with $i=0,1,2.$ By Lemma 4.9, we now obtain

$$R_{\min}:=\frac{C_{\min}}{\sqrt{C_{\max}}}\leq r_i\leq\frac{C_{\max}}{\sqrt{C_{\min}}}=:R_{\max}\quad\text{for}\quad\lambda_i\text{-a.e.}\quad z_i=[x_i,r_i],\quad\text{for}\quad i=0,1,2.$$

This proves the the claim that all $\lambda_i$ are supported in $\mathfrak{C}[R_{\min},R_{\max}]$. ■

Now we are able to conclude the proof of the main result.

**Proof:** [Proof of Theorem 4.8] By Lemma 4.10 there exists $0 < \mathsf{d}_1 < \frac{\pi}{2\boldsymbol{\ell}}$ and $0 < \mathsf{d}_2$ such that every geodesic $\boldsymbol{\mu}_{01}$ connecting $\mu_0, \mu_1 \in \overline{\mathcal{M}}_\delta^{\mathcal{L}}(X)$ satisfies $\boldsymbol{\mu}_{01}(t) \in \widetilde{\mathcal{M}}^{\mathcal{L}}_{\mathsf{d}_1,\mathsf{d}_2}(X)$ for all $t \in [0,1]$. We also have $\mu_2 \in \widetilde{\mathcal{M}}^{\mathcal{L}}_{\mathsf{d}_1,\mathsf{d}_2}(X) \supset \overline{\mathcal{M}}_\delta^{\mathcal{L}}(X)$. We would like to utilize the equivalent definition of $K-$semiconcavity given in (2.34), therefore we will just take $\tilde{\mu}_0 = \boldsymbol{\mu}_{01}(t_1), \tilde{\mu}_1 = \boldsymbol{\mu}_{01}(t_2)$, for $t_1, t_2 \in [0,1]$, and for $\tilde{\boldsymbol{\mu}}_{01}$ we just re-parametrize $(\boldsymbol{\mu}_{01} \llcorner [t_1, t_2])$. By Lemma 4.11, there exists $R_{\min}, R_{\max}$ that depend on $\mathsf{d}_1, \mathsf{d}_2$, and therefore on $\delta$, such that for every $\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2 \in \widetilde{\mathcal{M}}^{\mathcal{L}}_{\mathsf{d}_1,\mathsf{d}_2}(X)$ and $0 < t < 1$ we can find measures $\lambda_0, \lambda_1, \lambda_2, \lambda_t \in \mathcal{P}_2(\mathfrak{C}[R_{\min}, R_{\max}])$ with

$$\mathfrak{P}\lambda_i = \tilde{\mu}_i, \quad \mathfrak{P}\lambda_t = \tilde{\boldsymbol{\mu}}_{01}(t), \text{ and } \mathsf{W}_{\mathsf{d}_{\mathfrak{C},\boldsymbol{\ell}}}(\lambda_i, \lambda_t) = \mathsf{HK}_{\boldsymbol{\ell}}(\tilde{\mu}_i, \tilde{\boldsymbol{\mu}}_{01}(t)), \; i = 0,1,2. \tag{4.18}$$

Using the geodesic property of $\tilde{\boldsymbol{\mu}}_{01}$ yields

$$\begin{aligned} \mathsf{W}_{\mathsf{d}_{\mathfrak{C},\boldsymbol{\ell}}}(\lambda_0, \lambda_t) + \mathsf{W}_{\mathsf{d}_{\mathfrak{C},\boldsymbol{\ell}}}(\lambda_1, \lambda_t) &= \mathsf{HK}_{\boldsymbol{\ell}}(\mu_0, \tilde{\boldsymbol{\mu}}_{01}(t)) + \mathsf{HK}_{\boldsymbol{\ell}}(\mu_1, \tilde{\boldsymbol{\mu}}_{01}(t)) \\ &= \mathsf{HK}_{\boldsymbol{\ell}}(\tilde{\mu}_0, \tilde{\mu}_1) \leq \mathsf{W}_{\mathsf{d}_{\mathfrak{C},\boldsymbol{\ell}}}(\lambda_0, \lambda_1). \end{aligned}$$

Hence, it is straightforward to see that there exists a geodesic $\boldsymbol{\lambda}_{01}$ connecting $\lambda_0, \lambda_1$, such that $\boldsymbol{\lambda}_{01}(t) = \lambda_t$. Furthermore, by [Lis06, Thm. 6] there is a plan $\boldsymbol{\Lambda}_{0\to1}$ on the geodesics such that $\Lambda_{ts} := (e_t, e_s)_\sharp \boldsymbol{\Lambda}_{0\to1}$ is an optimal plan between $\boldsymbol{\lambda}(t)$ and $\boldsymbol{\lambda}(s)$. Now, by using a gluing lemma, we can find a plan $\boldsymbol{\Lambda}_{2t}^{0\to1}$ in $\mathcal{P}((C[0,1];\mathfrak{C}) \times \mathfrak{C})$, such that $\Lambda_{01} = (e_0, e_1)_\sharp \left(\pi_\sharp^{0\to1} \boldsymbol{\Lambda}_{2t}^{0\to1}\right)$, and $(e_t(\pi^{0\to1}) \times I)_\sharp \boldsymbol{\Lambda}_{2t}^{0\to1}$ is an optimal plan for $\mathsf{W}_{\mathsf{d}_{\mathfrak{C},\boldsymbol{\ell}}}(\lambda_2, \boldsymbol{\lambda}_{01}(t))$. Finally by applying the last part of Lemma 4.9, we get the existence of a $d < \frac{\pi}{2}$ such that $\boldsymbol{\ell}\mathsf{d}_X(x_2, x_t) < d$ for $(e_t(\pi^{0\to1}) \times I)_\sharp \boldsymbol{\Lambda}_{2t}^{0\to1}$ almost every $(z_2, z_t)$, similarly $\boldsymbol{\ell}\mathsf{d}_X(x_0, x_1) < d$ for $\Lambda_{01}$ almost every $[z_0, z_1]$. Therefore, for $\boldsymbol{\Lambda}_{2t}^{0\to1}$ almost every $(z_2, \boldsymbol{z}(\cdot, z_0, z_1))$, where $\boldsymbol{z}(\cdot, z_0, z_1)$ is a geodesic connecting $z_0, z_1$, we have $x_0, x_1, x_2, \bar{\boldsymbol{x}}(t, z_0, z_1) \in B\left(\bar{\boldsymbol{x}}(t, z_0, z_1), d\right)$. By Lemma 2.23 we get a $K$ such that

$$\mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(z_2, \boldsymbol{z}(t, z_0, z_1)) + Kt(1-t)\mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(z_0, z_1) \geq (1-t)\mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(z_2, z_0) + t\,\mathsf{d}^2_{\mathfrak{C},\boldsymbol{\ell}}(z_2, z_1), \tag{4.19}$$

for $\boldsymbol{\Lambda}_{2t}^{0\to1}$ almost every $(z_2, \boldsymbol{z}(\cdot, z_0, z_1))$. By integrating with respect to $\boldsymbol{\Lambda}_{2t}^{0\to1}$, we find

$$\mathsf{W}^2_{\mathsf{d}_{\mathfrak{C},\boldsymbol{\ell}}}(\lambda_2, \boldsymbol{\lambda}_{01}(t)) + Kt(1-t)\mathsf{W}^2_{\mathsf{d}_{\mathfrak{C},\boldsymbol{\ell}}}(\lambda_0, \lambda_1) \geq (1-t)\mathsf{W}^2_{\mathsf{d}_{\mathfrak{C},\boldsymbol{\ell}}}(\lambda_2, \lambda_0) + t\,\mathsf{W}^2_{\mathsf{d}_{\mathfrak{C},\boldsymbol{\ell}}}(\lambda_2, \lambda_1). \tag{4.20}$$

Using (4.18) we find the desired semiconcavity (4.8). ■

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