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Analysis and simulations for a phase-field fracture model at finite strains based on modified invariants

Marita Thomas, Carola Bilgen, Kerstin Weinberg

Abstract

Phase-field models have already been proven to predict complex fracture patterns in two and three dimensions for brittle fracture at small strains. In this paper we discuss a model for phase-field fracture at finite deformations in more detail. Among the identification of crack location and projection of crack growth the numerical stability is one of the main challenges in solid mechanics. We here present a phase-field model at finite strains, which takes into account the anisotropy of damage by applying an anisotropic split and the modified invariants of the right Cauchy-Green strain tensor. We introduce a suitable weak notion of solution that also allows for a spatial and temporal discretization of the model. In this framework we study the existence of solutions and we show that the time-discrete solutions converge in a weak sense to a solution of the time-continuous formulation of the model. Numerical examples in two and three space dimensions are carried out in the range of validity of the analytical results.

1 Introduction

In solid mechanics one of the main challenges is the prediction of crack growth and fragmentation patterns. Regarding the modeling side complicated structures and non-regular behavior of cracks turn numerical simulations into a difficult task. The classical brittle fracture approach of Griffith and Irwin [Gri21, Irw58] is based on an energy minimization setting for the whole structure. Let us consider a solid with domain $\mathcal{B}_0 \subset \mathbb{R}^3$ and boundary $\partial\mathcal{B}_0 \equiv \Gamma \subset \mathbb{R}^2$ deforming within a time interval $t \in [0, T]$. Each crack that is located in a solid forms a new surface $\Gamma(t)$ of a priori unknown position which needs to be identified. Therefore, the total potential energy of a homogeneous but cracking solid is composed of its bulk energy with a Helmholtz free energy density Ψ and of surface energy contributions due to growing cracks:

$$E = \int_{\mathcal{B}_0} \Psi \,\mathrm{d}X + \int_{\Gamma(t)} \mathcal{G}_c \,\mathrm{d}\Gamma \tag{1.1}$$

For rate-independent brittle fracture the material's resistance to cracking, the fracture toughness $\mathcal{G}_c$, corresponds to Griffith energy release rate: Rate-independent crack growth sets in as soon as the energy release rate reaches the critical value $\mathcal{G}_c$. The evolution law encoded therein at time t amounts to a minimization of the energy (1.1) under the constraint that $\Gamma(\underline{t}) \subset \Gamma(t)$ for all $\underline{t} < t \in [0, T]$.

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However, the minimization of the energy functional (1.1) is a challenging task because of the moving boundary $\Gamma(t)$. Several sophisticated discretization techniques exist, e.g. cohesive zone models [XN94, OP99, RS03], eroded finite elements [MP03, SSBP03] or eigenfracture strategies [SL09, PO12] to name some of them. Another approach to such moving boundary problems is a diffuse-interface approach, which approximates the 2D crack surface by a 3D damaged volume. These types of phase-field models for fracture have gained much attention in recent literature, cf. e.g. [HL04, KKL01, MHW10, VdB13, BVS$^+$12]. The main idea of this ansatz is to mark the damage state of the body by a continuous order parameter $s : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$, which evolves in space and time.

From a mathematical point of view, the phase-field fracture model, we shall investigate in this work is a modification of the Ambrosio-Tortorelli functional [AT90], which can be used to model rate-independent volume damage, and which we here augment by a viscous dissipation potential for the variable s. The modifications made, allow it to consider the evolution problem at finite strains using polyconvex energy densities and they take into account the anisotropy of damage, meaning that damage only increases under tensile loadings but not under compression. For this, it will be important to consider the stored energy density as a function of the *modified principal invariants of the right Cauchy-Green strain tensor*, and based on this, to introduce an *anisotropic split* of the energy density, which we shall explain in Section 2. This model is analyzed in detail in Section 3. We introduce a suitable notion of solution and in this setting we prove the existence of solutions using a time-discrete scheme via a minimizing movement approach. In particular, we show that solutions constructed with a staggered time-discrete scheme converge in a weak sense to a solution of the time-continuous formulation of the problem. This convergence result is confirmed within a series of numerical examples in Section 4, where we also provide further details on the spatial and temporal discretization. We demonstrate a simple but typical problem of a mode-I-tension test in two and three dimensions to study different influencing factors.

2 The phase-field fracture approach

In this section the focus is set on the phase-field approach for fracture to overcome the difficulty of moving boundaries. The phase-field is introduced as an additional parameter which is by definition a continuous field. Thus, the moving crack boundaries are 'smeared' over a small but finite length. The order parameter $s : \mathcal{B}_0 \times [0, T] \to \mathbb{R}$ with $s \in [0, 1]$ characterizes the state of the material, whereby $s = 1$ indicates the unbroken state and $s = 0$ the broken state. The surface integral in (1.1) is approximated by a regularization using a crack density function $\gamma : \mathbb{R} \times \mathbb{R}^3 \to [0, \infty)$

$$\int_{\Gamma(t)} \mathrm{d}\Gamma \approx \int_{\mathcal{B}_0} \gamma(s(t, X), \nabla s(t, X))\, \mathrm{d}X\,. \tag{2.1}$$

This approximation (2.1) is inserted into (1.1) so that the total potential energy reads

$$E = \int_{\mathcal{B}_0} (\Psi + \mathcal{G}_c \gamma)\, \mathrm{d}X\,. \tag{2.2}$$

The crack density function is by definition only different from zero along cracks. Typically a second-order phase-field approach is defined as:

$$\gamma(s, \nabla s) := \frac{1}{2l_c}(1-s)^2 + \frac{l_c}{2}|\nabla s|^2 \tag{2.3}$$

with the fixed parameter $l_c \in (0,1)$ which is a measure for the width of the diffuse interface zone, see Fig. 1. Furthermore, the length-scale parameter l_c weights the influence of the linear and the gradient

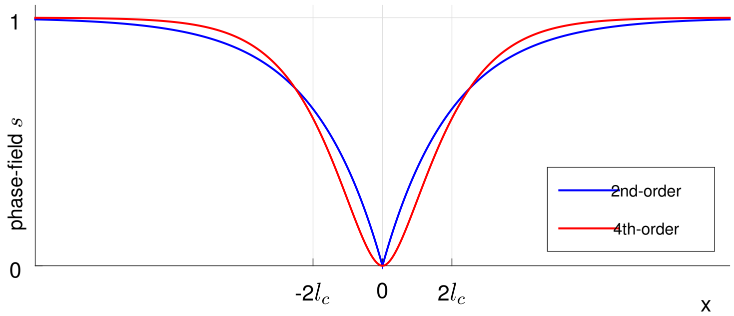

Figure 1: Uniaxial model with a crack at $x = 0$ and with a continuous phase-field $s \in [0,1]$; phase-field approximation for a second order and a fourth order approximation of crack density function γ.

term whereby the gradient enforces the regularization of the sharp interface. The insertion of (2.3) in (2.2) leads to a potential which is related to the work of Ambrosio and Tortorelli [AT90].

2.1 Governing Equations

The elastic boundary value problem is based on the balance of linear momentum

$$\text{div}(\boldsymbol{P}) + \bar{b} = \rho_0 \ddot{\boldsymbol{\varphi}} \quad \text{in } \mathcal{B}_0 \tag{2.4}$$

where $\boldsymbol{P}$ is the first Piola-Kirchhoff stress tensor, ρ_0 the mass density, $\bar{b}, \bar{t}$ the prescribed body force and traction, and $\ddot{\boldsymbol{\varphi}}$ the material acceleration. In this contribution we assume no body forces and investigate a quasi-static setting where the inertia term in (2.4) will be neglected. Therefore, the following evolution equation for the mechanical field is applied:

$$\text{div}(\boldsymbol{P}) = 0 \quad \text{in } \mathcal{B}_0. \tag{2.5}$$

The solid boundary is divided into displacement and traction boundaries $\partial\mathcal{B}_0^{\mathrm{D}}$ and $\partial\mathcal{B}_0^{\mathrm{N}}$ with $\overline{\partial\mathcal{B}_0^{\mathrm{D}}} \cup \overline{\partial\mathcal{B}_0^{\mathrm{N}}} = \partial\mathcal{B}_0$, $\partial\mathcal{B}_0^{\mathrm{D}} \cap \partial\mathcal{B}_0^{\mathrm{N}} = \emptyset$, and

$$\boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}} \quad \text{on } \partial\mathcal{B}_0^{\mathrm{D}}, \tag{2.6}$$

$$\boldsymbol{P} \cdot n = \bar{t} \quad \text{on } \partial\mathcal{B}_0^{\mathrm{N}}. \tag{2.7}$$

Furthermore, the phase-field evolution equation is stated in general form as

$$\dot{s} = MY, \tag{2.8}$$

where the parameter M denotes the kinematic mobility and Y summarizes all driving forces which typically represent a competition of bulk material and surface forces, cf. [WDS+16]:

$$Y = -\delta_s(\Psi + \mathcal{G}_c\gamma) = -(\delta_s\Psi + \mathcal{G}_c\delta_s\gamma) \tag{2.9}$$

In particular, the phase-field model is based on the crack density function (2.3) and the driving force for crack growth that consists of components of the free energy. Finally, the governing balance equations for the phase-field fracture are summarized in Table 1.

Table 1: Balance equations for a cracking solid subjected to prescribed deformations $\bar{\boldsymbol{\varphi}}$ on $\partial\mathcal{B}_0^{\mathrm{D}}$ and tractions $\bar{\boldsymbol{t}}$ on $\partial\mathcal{B}_0^{\mathrm{N}}$ with $\partial\mathcal{B}_0^{\mathrm{D}} \cup \partial\mathcal{B}_0^{\mathrm{N}} = \partial\mathcal{B}_0$.

Balance of linear momentum:	$\mathrm{div}(\boldsymbol{P}) = 0$	on $\mathcal{B}_0$
Boundary conditions:	$\boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}}$	on $\partial\mathcal{B}_0^{\mathrm{D}}$
	$\boldsymbol{P}\cdot\vec{n} = \bar{\boldsymbol{t}}$	on $\partial\mathcal{B}_0^{\mathrm{N}}$
Phase-field equation:	$M\left(\frac{\partial\Psi}{\partial s} - \frac{\mathcal{G}_c}{l_c}(1-s) - \mathcal{G}_c l_c \triangle s\right) = \dot{s}$	on $\mathcal{B}_0$
Boundary conditions:	$s = 1$	on $\partial\mathcal{B}_0^{\mathrm{D}}$
	$\nabla s \cdot \vec{n} = 0$	on $\partial\mathcal{B}_0^{\mathrm{N}}$

Here it has to be mentioned that our analytical results also account for the unidirectionality of the damage evolution, i.e., that damage of the material can only increase during the evolution, but not heal. This can be done by reformulating equation (2.8) for the damage evolution as follows: Under the assumption that the kinematic mobility is strictly positive we can equivalently rewrite (2.8) as $M^{-1}\dot{s} = Y$. We then introduce (the density of) the quadratic dissipation potential

$$V_\alpha(\dot{s}) := \frac{M^{-1}}{2}|\dot{s}|^2 + \alpha I_{(-\infty,0]}(\dot{s}) \tag{2.10}$$

with $I_{(-\infty,0]} : \mathbb{R} \to \{0,\infty\}$ the characteristic function of the negative real line, i.e., $I_{(-\infty,0]}(z) = 0$ if $z \in (-\infty,0]$ and $I_{(-\infty,0]}(z) = \infty$ otherwise. In this way, the time derivative $\dot{s}$ is forced to take its values in $(-\infty,0]$. According to the definition $s = 1$ for the unbroken and $s = 0$ for the completely broken state of the material, it thus ensures that the damage of the material can only increase in time, which means the unidirectionality of the damage evolution. Since the unidirectionality constraint is not incorporated in the numerical simulations presented in Sect. 4, we use the prefactor $\alpha = \mathrm{const} \geq 0$ to indicate that we switch this constraint on or off, so that we can consider two different types of models: A model with $\alpha = \mathrm{const} > 0$, where the unidirectionality constraint is active, and a model where $\alpha = 0$, where unidirectionality is not incorporated (i.e. $0\cdot\infty = 0$) and which is used in the simulations. Our analytical results cover both cases $\alpha = \mathrm{const} > 0$ and $\alpha = 0$. Using (2.10) we see that the evolution equation (2.8) for $\alpha = 0$ is given by

$$\mathrm{D}_{\dot{s}}V_\alpha(\dot{s}(t)) = Y(t) \quad \text{in } \mathcal{B}_0\,. \tag{2.11a}$$

In the case $\alpha = \mathrm{const} > 0$ the dissipation potential V_α is non-smooth for $\dot{s} = 0$. Then equation (2.11a) is replaced by the subdifferential inclusion

$$\partial_{\dot{s}}V_\alpha(\dot{s}(t)) \ni Y(t) \quad \text{in } \mathcal{B}_0 \tag{2.11b}$$

featuring the multivalued subdifferential of the convex potential V_α. More precisely,

$$\partial_{\dot{s}} V_\alpha(\dot{s}(t)) = M^{-1}\dot{s} + \partial_{\dot{s}} I_{(-\infty,0]}(\dot{s})\,, \quad \text{where } \partial_{\dot{s}} I_{(-\infty,0]}(\dot{s}) = \begin{cases} \{0\} & \text{if } \dot{s} \in (-\infty, 0)\,, \\ \mathbb{R}_+ & \text{if } \dot{s} = 0\,, \\ \emptyset & \text{if } \dot{s} \in (0, \infty)\,. \end{cases} \tag{2.11c}$$

In Section 3 we will introduce a suitable weak formulation for the evolution problems given by two cases (2.11a) & (2.11b).

2.2 Finite elasticity and the anisotropy of damage

In this work we set the focus on finite strains and we will subsequently introduce a nonlinear material model and its split into compressive and tensile parts, which makes sure that only the latter are responsible for crack growth.
In the finite deformation regime a deformation mapping $\boldsymbol{\varphi} : \mathcal{B}_0 \times [0, T] \to \mathbb{R}^3$ is regarded and the deformation gradient $\boldsymbol{F} : \mathcal{B}_0 \times [0, T] \to \mathbb{R}^{3\times 3}$ is defined as

$$\boldsymbol{F} = \nabla_X \boldsymbol{\varphi} = \frac{\partial \boldsymbol{\varphi}}{\partial X}. \tag{2.12}$$

Concerning the following notation the fields X in capitals refer to the reference configuration. Furthermore, the volume map is given by the determinant of the deformation gradient, denoted as $\det \boldsymbol{F}$. Further assuming hyperelasticity, objectivity, and isotropy of the constitutive law implies that the free energy density equivalently can be written as a function of the principal invariants of the right Cauchy-Green strain tensor $\boldsymbol{C} = \boldsymbol{F}^\top \boldsymbol{F}$, which, in three space dimensions, are given by

$$\iota_1(\boldsymbol{C}) := \operatorname{tr} \boldsymbol{C} = |\boldsymbol{F}|^2,\ \iota_2(\boldsymbol{C}) := \operatorname{tr}(\operatorname{cof} \boldsymbol{C}) = |\operatorname{cof} \boldsymbol{F}|^2,\ \iota_3(\boldsymbol{C}) := \det \boldsymbol{C} = \det \boldsymbol{F}. \tag{2.13}$$

Assuming that $\boldsymbol{C}$ is invertible, the cofactor $\operatorname{cof} \boldsymbol{C}$ can be calculated on the one hand via $\operatorname{cof} \boldsymbol{C} := (\det \boldsymbol{C})\boldsymbol{C}^{-\top}$ the so called Nanson's rule, cf. e.g. [BW08], or on the other hand via the tensor cross product operation, proposed in [BGO15].
Since many materials behave quite differently under bulk and shear loads, it is often convenient for numerical simulations to introduce a multiplicative decomposition of $\boldsymbol{F}$ into a volume-changing and a volume-preserving part:

$$\boldsymbol{F} = (\det \boldsymbol{F})^{1/3} \mathbb{1} \overline{\boldsymbol{F}}, \quad \text{resp. } \overline{\boldsymbol{F}} = (\det \boldsymbol{F})^{-1/3} \mathbb{1} \boldsymbol{F} \tag{2.14}$$

with $(\det \boldsymbol{F})^{1/3} \mathbb{1}$ associated to the volume-changing deformation and $\boldsymbol{F}$ associated to the volume-preserving deformation. This split was originally proposed by [Flo61] and also successfully used by [ST91] and [Ogd97], also in the context of finite strain elastoplasticity [STP85], see also [Hol04, p. 228]. This leads to the set of modified principal invariants

$$U(\boldsymbol{C}) = \iota_3(\boldsymbol{C})^{-1/3} \iota_1(\boldsymbol{C}) = \iota_1(\overline{\boldsymbol{F}}^\top \overline{\boldsymbol{F}}) \text{ and } V(\boldsymbol{C}) = \iota_3(\boldsymbol{C})^{-2/3} \iota_2(\boldsymbol{C}) = \iota_2(\overline{\boldsymbol{F}}^\top \overline{\boldsymbol{F}})\,, \tag{2.15}$$

which in fact guarantee a stress-free reference configuration since $\partial_{\boldsymbol{F}} U(\boldsymbol{F}^\top \boldsymbol{F}) = \partial_{\boldsymbol{F}} V(\boldsymbol{F}^\top \boldsymbol{F}) = 0,$ cf. (3.61). The free energy density Ψ may now be expressed either in terms of the invariants

$(\iota_1(\boldsymbol{C}), \iota_2(\boldsymbol{C}), \iota_3(\boldsymbol{C}))$ or in terms of the modified invariants $(U(\boldsymbol{C}), V(\boldsymbol{C}), \iota_3(\boldsymbol{C}))$ and as we shall see, that the use of the modified invariants is more reasonable.

The form of the free energy density is further specified by taking into account the anisotropy of fracture, i.e., that damage increases only under tensile deformations but not under compression. To incorporate this feature, we first renormalize the stored energy density of the reference configuration. Here we assume that the reference configuration is associated with the state $(\boldsymbol{F}, s) = (\mathbb{1}, 1)$, where $\mathbb{1} \in \mathbb{R}^{3\times 3}$ is the identity matrix and $s = 1$ the undamaged state of the body. We now impose that $W(\mathbb{1}, 1) = 0$. For this, we note that $\iota_1(\mathbb{1}) = U(\mathbb{1}) = \iota_2(\mathbb{1}) = V(\mathbb{1})$. Thus, the condition $W(\mathbb{1}, 1) = 0$ can be achieved if $W(\cdot, 1)$ is composed e.g. of renormalized power terms of the form $(A(\boldsymbol{F}^\top\boldsymbol{F}) - A(\mathbb{1}))^i$ or $(A(\boldsymbol{F}^\top\boldsymbol{F})^i - A(\mathbb{1})^i)$ with $i \geq 1$ and A a placeholder for the invariant functions $\iota_1, \iota_2, \iota_3, U, V$. Moreover, compressive deformations are characterized by $A(\boldsymbol{F}^\top\boldsymbol{F}) < A(\mathbb{1})$, while tensile deformations are associated with $A(\boldsymbol{F}^\top\boldsymbol{F}) \geq A(\mathbb{1})$. Hence, the anisotropy of fracture can be incorporated into the model by making the specific ansatz

$$\Psi = W(\boldsymbol{F}, s) := \beta(s)\widetilde{W}_+(\iota_1(\boldsymbol{F}^\top\boldsymbol{F}), \iota_2(\boldsymbol{F}^\top\boldsymbol{F}), \iota_3(\boldsymbol{F}^\top\boldsymbol{F})) + \widetilde{W}_-(\iota_1(\boldsymbol{F}^\top\boldsymbol{F}), \iota_2(\boldsymbol{F}^\top\boldsymbol{F}), \iota_3(\boldsymbol{F}^\top\boldsymbol{F}))$$

or

$$\Psi = W(\boldsymbol{F}, s) := \beta(s)\overline{W}_+(U(\boldsymbol{F}^\top\boldsymbol{F}), V(\boldsymbol{F}^\top\boldsymbol{F}), \iota_3(\boldsymbol{F}^\top\boldsymbol{F})) + \overline{W}_-(U(\boldsymbol{F}^\top\boldsymbol{F}), V(\boldsymbol{F}^\top\boldsymbol{F}), \iota_3(\boldsymbol{F}^\top\boldsymbol{F}))\,,$$

where

$$\begin{aligned}&\dddot{W}_\pm(I_1, I_2, I_3) = \dddot{W}(m_1^\pm(I_1(\boldsymbol{F}^\top\boldsymbol{F})), m_2^\pm(I_2(\boldsymbol{F}^\top\boldsymbol{F})), m_3^\pm(I_3(\boldsymbol{F}^\top\boldsymbol{F}))) \qquad \text{with}\\ &m_i^\pm(I_i(\boldsymbol{F}^\top\boldsymbol{F})) = \pm\max\{\pm(I_i(\boldsymbol{F}^\top\boldsymbol{F})^{\alpha_i} - I_i(\mathbb{1})^{\alpha_i}), 0\} \ \text{ for } \alpha_i \geq 1\,,\end{aligned} \tag{2.16}$$

$\dddot{W} \in \{\widetilde{W}, \overline{W}\}$, and $(I_1, I_2, I_3) \in \{(\iota_1, \iota_2, \iota_3), (U, V, \iota_3)\}$ and the degradation function β with the specific ansatz

$$\beta : [0,1] \to [\eta, \infty),\ \beta(s) := (\eta + s^2). \tag{2.17}$$

The parameter $\eta > 0$ is a very small value $\eta \ll 1$ to catch numerical instabilities in cases where $s = 0$ for the phase-field. In fact, from a mathematical point of view the introduction of $\eta > 0$ ensures the coercivity of $W(\cdot, s)$ for any $s \in [0, 1]$. Thus, at level $\eta > 0$, only partial damage is modeled, because the material is still able to carry stresses even if $s = 0$. The split with $m_i^\pm$ is explicitely tailored for energy densities of power-law type alike

$$\dddot{W}(I_1(\boldsymbol{F}^\top\boldsymbol{F}), I_2(\boldsymbol{F}^\top\boldsymbol{F}), I_3(\boldsymbol{F}^\top\boldsymbol{F})) = \sum_{i=1}^{l} c_{1i}(I_1^{\alpha_{1i}} - 3^{\alpha_{1i}})^{r_{1i}} + c_{2i}(I_2^{\alpha_{2i}} - 3^{\alpha_{2i}})^{r_{2i}} + c_{3i}(I_3^{\alpha_{3i}} - 1)^{r_{3i}}\,, \tag{2.18}$$

with $r_i \geq 1$, and $\alpha_i \geq 1$. When using a given density $\dddot{W}$ to introduce $\widetilde{W}_\pm(\iota_1, \iota_2, \iota_3)$ and $\overline{W}_\pm(U, V, \iota_3)$ as in (2.16), there must not hold $\widetilde{W}_\pm(\iota_1, \iota_2, \iota_3) = \overline{W}_\pm(U, V, \iota_3)$, since the densities $\widetilde{W}_\pm$ and $\overline{W}_\pm$ use different sets of invariants on which the functions $m_i^\pm$ are applied. Because of premultiplication with $\beta(s)$ the two definitions of Ψ in terms of either $\widetilde{W}$ or $\overline{W}$ are different and lead to different results. In fact, for numerical simulations the use of $\overline{W}$ has turned out to be more stable [HGO+17].

In order to fulfill the assumption of hyperelasticity we also want to make sure that the density $\dddot{W}_\pm$: $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable with respect to the invariants (I_1, I_2, I_3). It can be checked that this holds true for energy densities $\dddot{W}$ of the form (2.18) with $r_i \geq 1$, and $\alpha_i \geq 1$. In this way, when focussing on the case $l = 1$, the densities $\dddot{W}_\pm$ defined in (2.16), are given by

$$\begin{aligned}\dddot{W}_\pm(I_1(\boldsymbol{F}^\top\boldsymbol{F}), I_2(\boldsymbol{F}^\top\boldsymbol{F}), I_3(\boldsymbol{F}^\top\boldsymbol{F})) &= \sum_{k=1}^{3} c_k m_k^\pm (I_k(\boldsymbol{F}^\top\boldsymbol{F}))^{r_k} \\ &= \sum_{k=1}^{3} c_k \big(\pm \max\{\pm(I_k(\boldsymbol{F}^\top\boldsymbol{F})^{\alpha_k} - I_k(\mathbb{1})^{\alpha_k}), 0\}\big)^{r_k} \\ &\text{for } \alpha_k \geq 1, r_k > 1\,,\end{aligned} \tag{2.19}$$

where summation now runs over the 3 invariants. Then it is

$$\partial_{I_j}\dddot{W}_\pm(I_1,I_2,I_3) = \begin{cases} 0 & \text{if } \pm I_j^{\alpha_j} \leq \pm I_j(\mathbb{1})^{\alpha_j}\,, \\ \alpha_j I_j^{\alpha_j-1} c_j r_j (I_j^{\alpha_j} - I_j(\mathbb{1})^{\alpha_j})^{r_j-1} & \text{if } \pm I_j^{\alpha_j} > \pm I_j(\mathbb{1})^{\alpha_j}\,, \end{cases} \tag{2.20}$$

and $\lim_{h\to 0}(\pm(I_j(\mathbb{1}) \pm h)^{\alpha_j} \mp I_j(\mathbb{1})^{\alpha_i})^{r_j-1} = 0$ thanks to $r_j > 1$. Hence, each of the derivatives $\partial_{I_j}\dddot{W}_\pm : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is continuous with $\partial_{I_j}\dddot{W}_\pm(I_1(\mathbb{1}), I_2(\mathbb{1}), I_3(\mathbb{1})) = 0$. This ensures that the reference configuration (associated with the deformation gradient $\boldsymbol{F} = \mathbb{1}$) is stress-free. In particular, for continuously differentiable densities $\dddot{W}_\pm : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, due to the assumption of hyperelasticity and relation (2.16), the first Piola-Kirchhoff stress can be calculated with the chain rule as

$$\begin{aligned}\boldsymbol{P}(\boldsymbol{F},s) &= \partial_{\boldsymbol{F}} W(\boldsymbol{F},s) \\ &= \sum_{i=1}^{3} \big(\beta(s)\partial_{I_i}\dddot{W}_+(I_1(\boldsymbol{F}^\top\boldsymbol{F}), I_2(\boldsymbol{F}^\top\boldsymbol{F}), I_3(\boldsymbol{F}^\top\boldsymbol{F})) \\ &\quad + \partial_{I_i}\dddot{W}_-(I_1(\boldsymbol{F}^\top\boldsymbol{F}), I_2(\boldsymbol{F}^\top\boldsymbol{F}), I_3(\boldsymbol{F}^\top\boldsymbol{F}))\big)\frac{\partial I_i(\boldsymbol{F}^\top\boldsymbol{F})}{\partial \boldsymbol{F}}\end{aligned} \tag{2.21}$$

and $\boldsymbol{P}(\mathbb{1}, s) = 0$ for any $s \in \mathbb{R}$ because $\partial_{I_j}\dddot{W}_\pm(I_1(\mathbb{1}), I_2(\mathbb{1}), I_3(\mathbb{1})) = 0$. In contrast, for $r_i = 1$ in (2.19) one can see that $\overline{W}_\pm$ is no longer continuously differentiable in $I_j(\mathbb{1})$ since the left and the right differential quotient in $I_j(\mathbb{1})$ do not coincide. Then, the choice of the set of invariants has an influence on the continuity properties of the first Piola-Kirchhoff stress. Exemplarily, let us consider the density of a Neo-Hooke material $c(\iota_1(\boldsymbol{F}^\top\boldsymbol{F}) - 3)$ in the split (2.16), i.e., $\alpha_1, r_1 = 1, c_1 = c$ and $c_2 = c_3 = 0$ in (2.19). Using (2.16) we have $\widetilde{W}(\boldsymbol{F}, s) = \beta(s)c(m_1^+(\iota_1(\boldsymbol{F}^\top\boldsymbol{F}) - 3)) + c(m_1^-(\iota_1(\boldsymbol{F}^\top\boldsymbol{F}) - 3))$. For $\boldsymbol{F} \neq \mathbb{1}$, the density $\widetilde{W}(\cdot, s)$ is continuously differentiable so that the first Piola-Kirchhoff tensor takes the form

$$\boldsymbol{P}(\boldsymbol{F},s) = \begin{cases} \beta(s)2\boldsymbol{F} & \text{if } \iota_1(\boldsymbol{F}^\top\boldsymbol{F}) > 3\,, \\ 2\boldsymbol{F} & \text{if } \iota_1(\boldsymbol{F}^\top\boldsymbol{F}) < 3\,, \end{cases} \tag{2.22}$$

and for $\boldsymbol{F} = \mathbb{1}$ the energy density is non-differentiable, since $\max\{\iota_1(\cdot), 3\}$ has a kink in $\mathbb{1}$. But the left and right limits of $\boldsymbol{P}(\boldsymbol{F}, s)$ do exist for $\boldsymbol{F} = \mathbb{1}$ and for $\beta(s) = 1$ these two limits coincide.

However, a discontinuity of $\boldsymbol{P}(\boldsymbol{F},s)$ in $\boldsymbol{F}=\mathbb{1}$ cannot be prevented if $\beta(s)\neq 0$, i.e., when damage occurs. Instead, when using the Neo-Hooke law with the modified first invariant in the split (2.16), i.e., $\overline{W}(\boldsymbol{F},s)=\beta(s)c(m_1^+(U(\boldsymbol{F}^\top\boldsymbol{F})-3))+c(m_1^-(U(\boldsymbol{F}^\top\boldsymbol{F})-3))$, we find that

$$\boldsymbol{P}(\boldsymbol{F},s)=\begin{cases}\beta(s)\partial_{\boldsymbol{F}}U(\boldsymbol{F}^\top\boldsymbol{F})=2\beta(s)\iota_3^{-1/3}(\boldsymbol{F}^\top\boldsymbol{F})\big(\boldsymbol{F}-\frac{1}{3}\iota_1(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^{-\top}\big) & \text{if } U(\boldsymbol{F}^\top\boldsymbol{F})>3\,,\\ \partial_{\boldsymbol{F}}U(\mathbb{1})=0 & \text{if } U(\boldsymbol{F}^\top\boldsymbol{F})=3\,,\\ \partial_{\boldsymbol{F}}U(\boldsymbol{F}^\top\boldsymbol{F})=2\iota_3^{-1/3}(\boldsymbol{F}^\top\boldsymbol{F})\big(\boldsymbol{F}-\frac{1}{3}\iota_1(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^{-\top}\big) & \text{if } U(\boldsymbol{F}^\top\boldsymbol{F})<3\,,\end{cases} \tag{2.23}$$

with $\lim_{|H|\to 0}\partial_{\boldsymbol{F}}U(\mathbb{1}\pm H)=0$ and where the expression $\partial_{\boldsymbol{F}}U(\boldsymbol{F}^\top\boldsymbol{F})$ is determined in (3.61d). In other words, the first Piola-Kirchhoff stress of a Neo-Hooke material with modified invariant $U(\boldsymbol{F}^\top\boldsymbol{F})$ is continuous with $\boldsymbol{P}(\mathbb{1},s)=0$ for any $s\in\mathbb{R}$, which is not the case when the principal invariant ι_1 is used.

For numerical simulations it may be more convenient to introduce the anisotropic split for energy densities of power law type in a slightly different way, as proposed in [HGO+17]. While the split defined in (2.16) & (2.19) is incorporated into the terms of the free energy density, the anisotropic split proposed in [HGO+17] is directly imposed on the modified invariants; more precisely, the *anisotropically splitted* invariants are defined by

$$I_A^\pm(\boldsymbol{F}^\top\boldsymbol{F})=\overline{I}_A^\pm(A(\boldsymbol{F}^\top\boldsymbol{F}))=\pm\max\{\pm(A(\boldsymbol{F}^\top\boldsymbol{F})-A(\mathbb{1})),0\}+A(\mathbb{1}), \tag{2.24}$$

with A a placeholder for the invariants U,V,ι_3. In Fig. 2 the split is visualized for the first invariant.

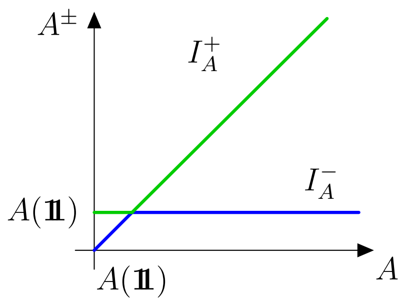

Figure 2: Illustration of the anisotropic split of the invariants $A=U,V,\iota_3$ into tensile and compressive parts with dimension $\mathrm{d}\in\{2,3\}$.

We note that

$$I_A^+(\boldsymbol{F}^\top\boldsymbol{F})+I_A^-(\boldsymbol{F}^\top\boldsymbol{F})=A(\boldsymbol{F}^\top\boldsymbol{F})+A(\mathbb{1})\quad\text{for } A=U,V,\iota_3\,. \tag{2.25}$$

This is due to the fact that the split (2.24) is again tailored to densities of polynomial type as in (2.18). In particular, for $\overline{W}$ of the form (2.18) we can check that $\overline{W}_\pm(U,V,\iota_3)=\overline{W}(I_U^\pm,I_V^\pm,I_{\iota_3}^\pm)$. Therefore, the above discussion about the continuity of the first Piola-Kirchhoff tensor remains valid also when the anisotropic split is formulated in terms of (2.24): Continuity of $\boldsymbol{P}(\cdot,s)$ in $\mathbb{1}$ holds true for exponents $r_{ik}>1$ in (2.18) for both sets $(\iota_1,\iota_2,\iota_3)$ and (U,V,ι_3). For $r_{ik}=1$ the

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densities $\widetilde{W}_{\pm}(\iota_1(\boldsymbol{F}^\top\boldsymbol{F}), \iota_2(\boldsymbol{F}^\top\boldsymbol{F}), \iota_3(\boldsymbol{F}^\top\boldsymbol{F})) = \dddot{W}(I^{\pm}_{\iota 1}(\boldsymbol{F}^\top\boldsymbol{F}), I^{\pm}_{\iota 2}(\boldsymbol{F}^\top\boldsymbol{F}), I^{\pm}_{\iota 3}(\boldsymbol{F}^\top\boldsymbol{F}))$ are non-differentiable in $(\boldsymbol{F}^\top\boldsymbol{F}) = \mathbb{1}$, so that stress $\boldsymbol{P}(\cdot, s)$ has a discontinuity in $\mathbb{1}$, which cannot be prevented for $\beta(s) \neq 1$. Instead, when the modified invariants are used, $\boldsymbol{P}(\cdot, s)$ is continuous with $\boldsymbol{P}(\mathbb{1}, s) = 0$.

To manifest this statement we will consider a simple mode-I-tension test in a loading and unloading regime. We regard a two dimensional plate with a required horizontal notch. The geometry and the related boundary conditions are depicted in Fig. 3. On the lower boundary of the plate the displacements are constrained in horizontal and vertical direction and on the upper side prescribed displacements are applied incrementally which are realized by making use of a Dirichlet boundary condition. The displacements are increased until the time $t = 0.5$ sec is reached - after that the plate is relieved to the reference configuration. Furthermore, the mesh presented in Fig. 3 is on the basis of the hierarchical refinement strategy ([HSD+16]) and consists of 20×20 quadratic B-splines elements before making use of the refinement. After three local refinement levels in total 2656 elements with overall 12288 degrees of freedom are employed for the tension test.

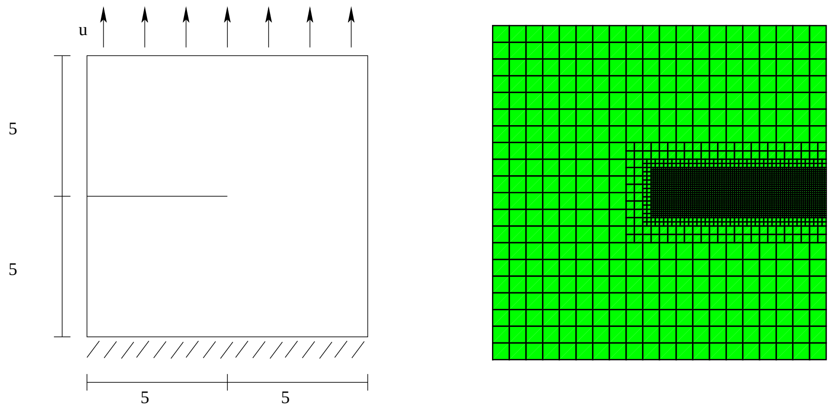

Figure 3: Boundary conditions (left) of a mode-I-tension test and the related mesh based on a hierarchical refinement strategy (right)

The simulation is based on the non-linear Neohookean material model with the proposed anisotropic split (2.24)

$$W(\boldsymbol{F}, s) = \beta(s)\left(\frac{\mu}{2}(I_1^+(\boldsymbol{F}^T\boldsymbol{F}) - d) + \frac{\kappa}{2}(I_3^+(\boldsymbol{F}^T\boldsymbol{F}) - 1)^2\right) + \frac{\mu}{2}(I_1^-(\boldsymbol{F}^T\boldsymbol{F}) - d) + \frac{\kappa}{2}(I_3^-(\boldsymbol{F}^T\boldsymbol{F}) - 1)^2, \quad (2.26)$$

whereas the material parameter are chosen as the shear modulus $\mu = 80.7692 \times 10^9 \ \frac{\mathrm{N}}{\mathrm{m}^2}$, the bulk modulus $\kappa = 121.1538 \times 10^9 \ \frac{\mathrm{N}}{\mathrm{m}^2}$ and the specific fracture energy as $\mathcal{G}_c = 2.7 \times 10^3 \ \frac{\mathrm{J}}{\mathrm{m}^2}$. The process of loading and unloading is presented within the load-deflection curve in Fig. 4. The focus is set on the stresses $\sigma = (\det\boldsymbol{F})^{-1}\boldsymbol{P}\boldsymbol{F}^T$ during the simulation after different time steps, cf. Fig. 5. The snapshots of stresses demonstrate that the configuration is stressfree after unloading again. That means, the stresses are continuous with $\boldsymbol{P}(\mathbb{1}, s) = 0$.

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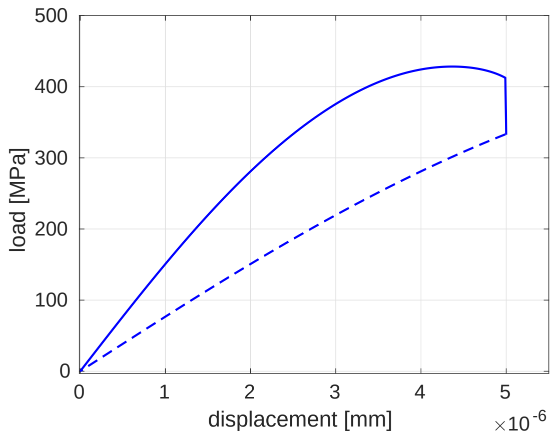

Figure 4: Load-deflection curve of the mode-I-tension test under loading and unloading at different times steps.

Our analysis uses the formulation (2.16) to take into account the anisotropy of damage. It is carried out directly for the density $W(\boldsymbol{F}, s)$, hence it is independent of the set of invariants, but it relies on the assumption that the energy density is continuously differentiable.

3 Analytical Setup, Discretization, and Convergence Result

For the mathematical analysis of the regularized crack model given by the equations from Table 1 we define the free energy functional $\mathcal{E} : [0, T] \times \mathbf{U} \times \mathbf{Z} \to \mathbb{R}$,

$$\mathcal{E}(t, \boldsymbol{\varphi}, s) := \int_{\mathcal{B}_0} \big(\beta(s) W_1(\mathbb{M}\nabla\boldsymbol{\varphi}) + W_2(\mathbb{M}\nabla\boldsymbol{\varphi}) + \gamma(s, \nabla s)\big)\,\mathrm{d}X + \int_{\partial\mathcal{B}_0^{\mathrm{N}}} \boldsymbol{h} \cdot \boldsymbol{\varphi}\,\mathrm{d}\mathcal{H}^2 \tag{3.1}$$

on suitable Banach spaces $\mathbf{U}, \mathbf{Z}$ with $\gamma : \mathbb{R} \times \mathbb{R}^3 \to [0, \infty)$ from (2.3) and $\beta : \mathbb{R} \to [\eta, \infty)$, with $\beta(s) := \eta + s^2$ as in (2.17). Moreover,

$$\mathbb{M} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R}\,, \ \mathbb{M}\boldsymbol{F} := (\boldsymbol{F}, \operatorname{cof} \boldsymbol{F}, \det \boldsymbol{F}) \tag{3.2}$$

maps a 3×3-matrix onto the vector of its minors. We remark that the ansatz $W(\nabla\boldsymbol{\varphi}, s) := \beta(s)W_1(\mathbb{M}\nabla\boldsymbol{\varphi}) + W_2(\mathbb{M}\nabla\boldsymbol{\varphi})$ used in (3.1) is in accordance with the setting proposed in (2.16). In particular, one may choose $W_1 = W_+$ and $W_2 = W_-$ from (2.16). In this way, the anisotropy of damage can be reflected in the model, given that the densities W_i are sufficiently smooth. The assumptions on the densities $W_i : \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R} \to \mathbb{R}$ are specified more precisely in (3.10) in Sec. 3.1. For the existence analysis, carried out in Sections 3.1–3.3, it will be more convenient to regard the densities W_i directly as functions of the minors of gradients, and not as functions of the modified invariants of the right Cauchy-Green tensor as introduced in (2.15). Instead, in Section 3.4 we will translate the assumptions (3.10) imposed on the densities W_i into assumptions for densities $\overline{W}_i$ being functions of the modified invariants.

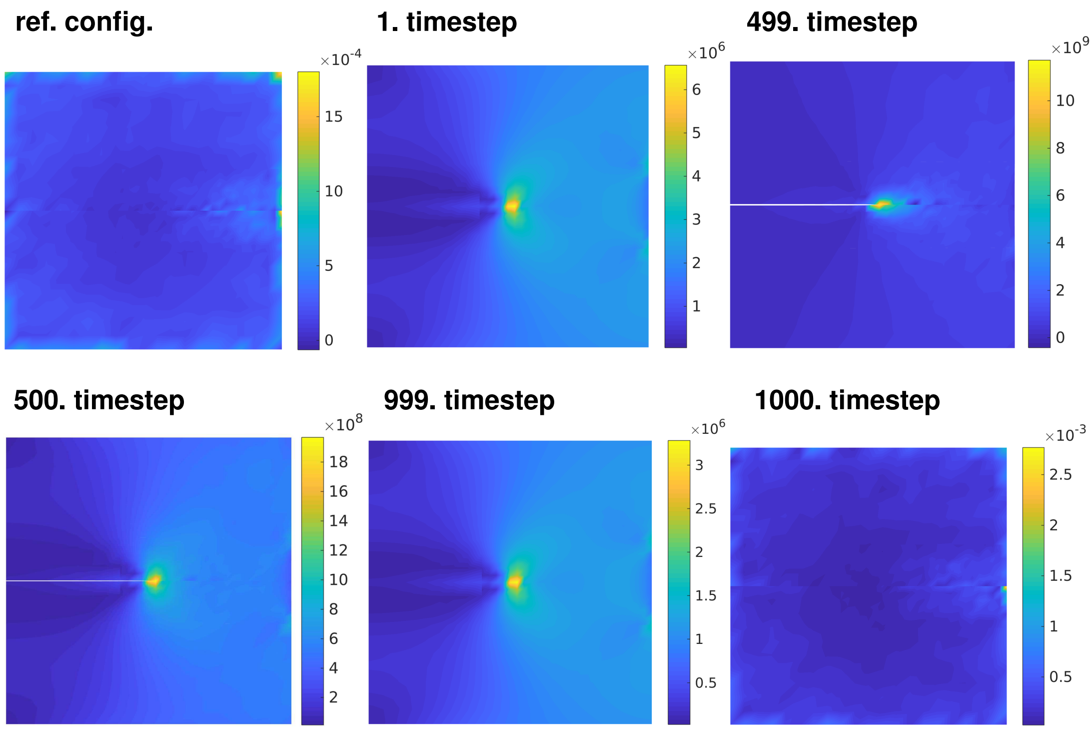

Figure 5: Von Mises stresses of the mode-I-tension test under loading and unloading at different times steps.

In accordance with (2.10) we additionally introduce the viscous dissipation potential $\mathcal{V}_\alpha : \mathbf{Z} \to [0, \infty)$

$$\mathcal{V}_\alpha(\dot{s}) := \int_{\mathcal{B}_0} \big(\frac{M^{-1}}{2}|\dot{s}|^2 + \alpha I_{(-\infty,0]}(\dot{s})\big)\,\mathrm{d}X \tag{3.3}$$

with M^{-1} the inverse of the kinematic mobility M from (2.8) and $I_{(-\infty,0]} : \mathbb{R} \to \{0, \infty\}$ the characteristic function of the the negative real line, i.e., $I_{(-\infty,0]}(z) = 0$ if $z \in (-\infty, 0]$ and $I_{(-\infty,0]}(z) = \infty$ otherwise. This constraint forces the time derivative $\dot{s}$ to take its values in $(-\infty, 0]$. According to the definition $s = 1$ for the unbroken and $s = 0$ for the completely broken state of the material, it thus ensures that the damage of the material can only increase in time, which means the unidirectionality of the damage evolution. With the prefactor $\alpha = \mathrm{const} \geq 0$ we indicate that we switch this constraint on or off, so that we can consider two different types of models: A model with $\alpha = \mathrm{const} > 0$, where the unidirectionality constraint is active, and a model where $\alpha = 0$, where unidirectionality is not incorporated (i.e. $0 \cdot \infty = 0$). The latter case is considered in the phase-field flow rule (2.8) and used for the numerical simulations presented in Section 4.

Notion of solution for the body with damage: The elastic body undergoing damage is thus characterized by a suitable state space $\mathbf{U} \times \mathbf{Z}$, the energy functional $\mathcal{E}$ from (3.1) and the dissipation potential $\mathcal{V}_\alpha$ from (3.3) and we refer to it as the (evolution) system $(\mathbf{U} \times \mathbf{Z}, \mathcal{E}, \mathcal{V}_\alpha)$.

In accordance with Section 2 we will show in Thm. 3.11 that a solution $(\boldsymbol{\varphi}, s) : [0,T] \to \mathbf{U} \times \mathbf{Z}$ of system $(\mathbf{U} \times \mathbf{Z}, \mathcal{E}, \mathcal{V}_\alpha)$ is characterized for a.a. $t \in (0,T)$ by

$$\text{for all } \tilde{\boldsymbol{\varphi}} \in \mathbf{U}: \;\; \mathcal{E}(t, \boldsymbol{\varphi}(t), s(t)) \leq \mathcal{E}(t, \tilde{\boldsymbol{\varphi}}, s(t)), \tag{3.4a}$$

$$\text{if } \alpha = 0: \;\; M^{-1}\dot{s}(t) + \beta'(s(t))W_1(\mathbb{M}\nabla\boldsymbol{\varphi}(t)) + \gamma'(s(t)) - \Delta s(t) = 0 \;\text{ in } \mathbf{X}^*, \tag{3.4b}$$

$$\text{if } \alpha > 0: \;\; M^{-1}\dot{s}(t) + \beta'(s(t))W_1(\mathbb{M}\nabla\boldsymbol{\varphi}(t)) + \gamma'(s(t)) - \Delta s(t) \geq 0 \;\text{ in } (\mathbf{X}_-)^*, \tag{3.4c}$$

together with the energy dissipation estimate:

$$\text{for all } t \in (0,T): \quad \mathcal{E}(t, \boldsymbol{\varphi}(t), s(t)) + \int_0^t \mathcal{V}_\alpha(\dot{s}(\tau))\,\mathrm{d}\tau \leq \mathcal{E}(0, \boldsymbol{\varphi}^0, s^0) + \int_0^t \mathcal{P}(\tau, s(\tau))\,\mathrm{d}\tau, \tag{3.4d}$$

with $\mathcal{V}_\alpha$ from (3.3) and with $\mathcal{P}(\tau, s(\tau)) := \sup\{\partial_\tau\mathcal{E}(\tau, \hat{\boldsymbol{\varphi}}, s(\tau)),\ \hat{\boldsymbol{\varphi}} \in \mathrm{argmin}\mathcal{E}(\tau, \cdot, s(\tau))\}$ as a surrogate for the partial time derivative. This surrogate arises due to the non-uniquess of minimizers for polyconvex energies in the finite strain setting and such a formulation has been proposed in [MRS16] in the context of finite-strain viscoplasticity. Moreover, $\mathbf{X}^*$ in (3.4b) denotes the dual of the Banach space $\mathbf{X}$ and $(\mathbf{X}_-)^*$ in (3.4c) the elements of $\mathbf{X}^*$ restricted to elements of $\mathbf{X}_- := \{v \in \mathbf{X},\ v \leq 0\}$.

Let us discuss formulation (3.4) more in detail. Condition (3.4a) provides the minimality property of the deformation $\boldsymbol{\varphi}$. Under the assumptions of [Bal02, Thm. 2.4], minimality condition (3.4a) is equivalent to (the weak formulation in $\mathbf{U}$ of) its Euler-Lagrange equation, which is the weak formulation of the mechanical force balance (2.5). In this case, (3.4a) yields the weak formulation of (2.5). Instead, the interpretation of (3.4b) & (3.4c) is more involved: On the one hand, to well-define the energy functional (3.1) with the quadratic damage gradient γ and the function β premultiplied to the deformation gradients, solutions s should satisfy $s \in \mathbf{X} := H^1(\mathcal{B}_0) \cap L^\infty(\mathcal{B}_0)$. On the other hand, due to the quadratic growth of $\mathcal{V}_\alpha$ it would be a first choice to understand (3.4b), resp. (3.4c), as an L^2-gradient flow. However, for the driving force of damage $\beta'(s(t))W(\mathbb{M}\nabla\boldsymbol{\varphi}(t)) + \gamma'(s(t)) - \Delta s(t) \in L^2(\Omega)$ cannot be expected, so that $L^2(\Omega)$ is not the right choice for the state space. In fact, we will find in Lemma 3.10 that $\mathrm{D}_s\mathcal{E}(t, \boldsymbol{\varphi}(t), s(t)) := \beta'(s(t))W(\mathbb{M}\nabla\boldsymbol{\varphi}(t)) + \gamma'(s(t)) - \Delta s(t)$ is bounded only in $\mathbf{X}^*$ the dual of the space $\mathbf{X} := H^1(\Omega) \cap L^\infty(\Omega)$.

For $\alpha = 0$ we thus find condition (3.4b), which is the weak formulation of the phase-field equation (4.2). Formulation (3.4c) for $\alpha > 0$, i.e. when the unidirectionality constraint is active, is given in terms of a one-sided variational inequality. Together with the energy-dissipation estimate (3.4d) and (3.4a) it provides a characterization of the solutions for $\alpha > 0$. Such a formulation of the non-smooth damage evolution in terms of a one-sided variational inequality combined with an energy dissipation estimate has been applied in [HK11] in the small-strain setting, and later also e.g. in [Ros17, CL16] in the case of small-strain (visco)plasticity.

Main result and analytical strategy: Our main result, Thm. 3.11, provides the existence of a solution $(\varphi, s) : [0,T] \to \mathbf{U} \times \mathbf{Z}$ of $(\mathbf{U} \times \mathbf{Z}, \mathcal{E}, \mathcal{V}_\alpha)$, which satisfies the governing equations (3.4). Its proof will be carried out in Section 3.3 via a time-discretization. For this, we will consider an equidistant partition of the time interval

$$\Pi_N := \{0 = t_N^0 < t_N^1 < \ldots < t_N^N = T\} \text{ with time-step size } \tau_N := t_N^i - t_N^{i-1} \to \text{ as } N \to \infty. \tag{3.5}$$

In addition, for the analysis, we will also regularize the unidirectionality constraint $I_{(-\infty,0]}$ by its corresponding Yosida approximation, i.e. for each $N \in \mathbb{N}$ we introduce

$$\mathcal{V}_{\alpha N}(\dot{s}) := \int_{\mathcal{B}_0} \big(\frac{M^{-1}}{2}|\dot{s}|^2 + \alpha\frac{N}{2}|(\dot{s})_+|^2\big)\,\mathrm{d}X \tag{3.6}$$

where $(\dot{s})_+ := \max\{0, \dot{s}\}$ is the positive part of $\dot{s}$. Starting out from an admissible inital datum $(\boldsymbol{\varphi}^0, z^0) \in \mathbf{U} \times \mathbf{Z}$, at each time-step $t_N^k \in \Pi_N$ we then alternatingly solve

$$\boldsymbol{\varphi}_N^k \in \operatorname{argmin}_{\tilde{\boldsymbol{\varphi}}\in\mathbf{U}}\mathcal{E}(t_N^k, \tilde{\boldsymbol{\varphi}}, s_N^{k-1}), \tag{3.7a}$$

$$s_N^k \in \operatorname{argmin}_{\tilde{s}\in\mathbf{Z}}\big(\mathcal{E}(t_N^k, \boldsymbol{\varphi}_N^k, \tilde{s}) + \tau_N\mathcal{V}_{\alpha N}(\tfrac{\tilde{s}-s_N^{k-1}}{\tau_N})\big), \tag{3.7b}$$

which is also called a *staggered time-discrete scheme* and used for the simulations in Section 4. Existence of solutions $(\boldsymbol{\varphi}_N^k, s_N^k)$ of (3.7) at each time-step t_N^k will be shown in Prop. 3.9. With the discrete solutions $(\boldsymbol{\varphi}_N^k, s_N^k)_{k=1}^N$ we will construct suitable interpolants with respect to time and show in our main result, Thm. 3.11, that these interpolants approximate a solution of the continuous problem (3.4).

Comparison with other results in literature: Our evolution system $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_\alpha)$ combines the energy functional $\mathcal{E}$ from (3.1), which is a modification of the Ambrosio-Tortorelli functional [AT90], with a quadratic dissipation potential $\mathcal{V}_\alpha$, cf. (3.3), which thus causes a viscous evolution of the phase-field variable. Without this viscous contribution, i.e. $M^{-1} = 0$ in (3.3), the (standard) Ambrosio-Tortorelli functional, combined with the unidirectionality property ensured by $\alpha > 0$, models the rate-independent, unidirectional evolution of the phase-field variable. In this setting, it was shown in [Gia05] that the standard Ambrosio-Tortorelli functional Γ-converges to the Francfort-Marigo model for brittle, Griffith-type fracture, cf. e.g. [BFM08] as the diffuse-interface parameter $l_c \to 0$ in the definition of γ, cf. (2.3). Later, similar approximation results have been obtained in the rate-independent satting at small strains, allowing for the use of the *linearized strain tensor* and for modifications of the energy functional leading to *cohesive or elasto-plastic* fracture models, cf. [DMI13, Iur13, CFI16, FI17, DMOT16, CLO16]. Instead, the limit passage $l_c \to 0$ of an energy functional of Ambrosio-Tortorelli-type in combination with a *viscous evolution of the displacements* was investigated in [BM14].

In this work we do not consider the limit $l_c \to 0$. We rather study for $l_c > 0$ fixed the existence of solutions in the sense of (3.4) for the system $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_\alpha)$ at finite strains for a modified energy functional that takes into account the different evolution behavior of a viscous-type phase-field variable with regard to tensile or compressive loads. Due to this modification, since the energy contribution W_2, which accounts for compression, is not premultiplied by the function β, we do not expect our model to approximate the Francfort-Marigo fracture model. Instead, we understand $(\mathbf{U}\times\mathbf{Z}, \mathcal{E}, \mathcal{V}_\alpha)$ as a very specific model for partial, isotropic damage, which has the property to localize damage in zones of width l_c. This viewpoint on the Ambrosio-Tortorelli model has also been taken in the rate-independent setting e.g. in [KN17], where the alternate minimization scheme (3.4) has been further iterated in each time-step leading to parametrized BV-evolutions af the rate-independent problem, and in [Neg16], where also a viscous regularization has been taken into account.

Other models for partial, isotropic damage allow for more general forms of the function β and the regularization γ. While the very specific properties of the functions β, γ in (3.1) make it possible to show that a solution s satisfies $s(t) \in [0,1]$ a.e. in $\mathcal{B}_0$ for a.e. $t \in (0,T)$, cf. Prop. 3.9, this bound is in general lost for other physically reasonable choices of β and γ. Then, additional indicator terms have to be incorporated into the energy in order in order to ensure that $s \in [0,1]$ for physical and analytical reasons. For the analysis of general models for partial, isotropic damage with a rate-independent damage evolution we refer to the works, e.g. [MR06, TM10, Tho13, FKS10] and to the monography [MR15] for an overview on rate-independent processes. The viscous, rate-dependent counterpart is studied e.g. in the works [HK11, BB08, RR15, HKRR17], also in combination with dynamics, heat transport, and phase separation, and vanishing-viscosity limits from viscous damage models at small strains to rate-indpendent ones are investigated in the series of works [KRZ13, KRZ15, KRZ17].

Our approach to the analysis of system $(\mathbf{U} \times \mathbf{Z}, \mathcal{E}, \mathcal{V}_\alpha)$ extends the ideas used in [Tho10], based on [MM09], for a rate-independent damage model at finite strains, to the present viscous setting by making use of the notion of solution studied in e.g. [HK11] at small strains. The study of the properties of energy densities as functions of the modifed invariants, cf. (2.15), builds on results drawn from the works [Bal77, Bal02, MQY94, CDHL88, SN03, HN03].

3.1 Analytical Setup: Assumptions and Direct Implications

A physically reasonable deformation preserves orientation, which is ensured by

$$\nabla\boldsymbol{\varphi} \in \mathrm{GL}_+(3) = \{\boldsymbol{F} \in \mathbb{R}^{d\times d} \,|\, \det \boldsymbol{F} > 0\}\,.$$

Further natural requirements on the constitutive relations of particular importance are material frame indifference (3.8a) and the non-interpenetration condition (3.8b):

$$\hat{W}(\boldsymbol{RF}) \;=\; \hat{W}(\boldsymbol{F}) \quad \text{for } \boldsymbol{R} \in \mathrm{SO}(3),\ \boldsymbol{F} \in \mathbb{R}^{3\times 3}, \tag{3.8a}$$

$$\begin{cases} \hat{W}(\boldsymbol{F}) = +\infty & \text{for } \det \boldsymbol{F} \le 0, \\ \hat{W}(\boldsymbol{F}) \to +\infty & \text{for } \det \boldsymbol{F} \to 0_+, \end{cases}. \tag{3.8b}$$

However, they are not compatible with convexity, which is a convenient claim in the setting of small strains. To see the incompatibility with convexity consider $\boldsymbol{P}, \boldsymbol{Q} \in \mathrm{SO}(3)$, $\lambda \in (0,1)$, such that $(\lambda\boldsymbol{P} + (1-\lambda)\boldsymbol{Q}) \notin \mathrm{SO}(3)$, which conforms to a strain. Then convexity together with material frame indifference yields the following contradiction:

$$0 < \hat{W}(\lambda\boldsymbol{P} + (1-\lambda)\boldsymbol{Q}) \le \lambda\hat{W}(\boldsymbol{P}) + (1-\lambda)\hat{W}(\boldsymbol{Q}) = \lambda\hat{W}(\mathbb{1}) + (1-\lambda)\hat{W}(\mathbb{1}) = 0.$$

The class of energy densities which fit to these natural requirements and which admit to prove existence are the polyconvex energy densities. They were introduced by J.M. Ball in [Bal77].

Definition 3.1 (Polyconvexity)**.** *The function* $\hat{W} : \mathbb{R}^{3\times3} \to \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ *is called polyconvex if there exists a convex function* $W_\bullet : \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R} \to \mathbb{R}_\infty$*, such that* $\hat{W}(\boldsymbol{F}) = W_\bullet(\mathbb{M}(\boldsymbol{F}))$ *for all* $\boldsymbol{F} \in \mathbb{R}^{3\times3}$*, where*

$$\mathbb{M} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R},\ \mathbb{M}(\boldsymbol{F}) = (\boldsymbol{F}, \operatorname{cof}\boldsymbol{F}, \det\boldsymbol{F}) \tag{3.9}$$

is the function, which maps a matrix to all its minors.

In [Bal77, p. 362] it was established that the polyconvexity of $\hat{W} : \mathbb{R}^{3\times3} \to \mathbb{R}$ implies its quasiconvexity. By C.B. Morrey in [Mor52] it was proven that quasiconvexity is the notion of convexity which is necessary and sufficient for the lower semicontinuity of the corresponding integral functionals, so that quasiconvexity together with other technical assumptions ensures the existence of minimizers. But quasiconvexity does not admit infinitely valued functions, i.e. $\hat{W} : \mathbb{R}^{3\times3} \to \mathbb{R}_\infty$. However in [Bal77, Th. 7.3, p. 376] it was shown that the polyconvexity of the density $\hat{W} : \mathbb{R}^{3\times3} \to \mathbb{R}_\infty$ together with other technical assumptions is sufficient for the existence of minimizers of infinitely valued functionals.

Assumptions on the stored elastic energy density: More precisely, for the stored elastic energy density $W : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$W(\boldsymbol{F}, s) := \beta(s)W_1(\mathbb{M}\boldsymbol{F}) + W_2(\mathbb{M}\boldsymbol{F}) \text{ from (3.1) with } \beta : \mathbb{R} \to [a, \infty) \text{ from (3.10h)} \tag{3.10a}$$

we make the following assumptions:

- **Continuity:** $W(\cdot,\cdot) \in \mathrm{C}^0(\mathbb{R}^{3\times3} \times \mathbb{R}, \mathbb{R})$, (3.10b)
- **Polyconvexity:** $W_1, W_2 : \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R} \to \mathbb{R}$ are convex. (3.10c)
- **Coercivity:** It holds for all $(\boldsymbol{F}, s) \in \mathbb{R}^{3\times3} \times \mathbb{R}$:
  $$c_1|\boldsymbol{F}|^p + c_2|\mathrm{cof}\boldsymbol{F}|^{p_2} + c_3|\mathrm{det}\boldsymbol{F}|^{p_3} - C \le W(\boldsymbol{F}, s) \tag{3.10d}$$
  with given, fixed constants $p, p_1, p_2, c_1, c_2, c_3, C$ satisfying one of the following:
  a) $p > 3, c_1 > 0, p_1, p_2 \ge 1, c_2, c_3, C \ge 0$, or
  b) $p \ge 2, p_2 \ge \frac{p}{p-1}, p_3 > 1, c_1, c_2, c_3 > 0, C \ge 0$, moreover
  b1) $C \ge 0, p \ge 2, c_1 > 0,$ if $W_i(\mathbb{M}\boldsymbol{F}) = W_i(\boldsymbol{F}),\ i = 1, 2$,
  b2) $C \ge 0, p \ge 2, p_2 \ge \frac{p}{p-1}, c_1, c_2 > 0,$ if $W_i(\mathbb{M}\boldsymbol{F}) = W_i(\boldsymbol{F}, \operatorname{cof}\boldsymbol{F}),\ i = 1, 2$, or
  c) $p \ge 2, c_1c_2 > 0, p_2 \ge 3/2, c_3, C \ge 0$.

- **Stress control:** (3.10e)
  For all $s\in\mathbb{R}$ we have $W(\cdot,s)\in \mathrm{C}^1(\mathrm{GL}_+(3),\mathbb{R})$ and there are constants $c>0, \tilde{c}\geq 0$ such that for all $(\boldsymbol{F},s)\in\mathbb{R}^{3\times 3}\times\mathbb{R}$ it holds
  $$|\partial_{\boldsymbol{F}} W(\boldsymbol{F},s)\boldsymbol{F}^\top| \leq c(W(\boldsymbol{F},s)+\tilde{c})\,. \tag{3.10f}$$
- **Uniform continuity of the stresses:**
  There is a modulus of continuity $o:[0,\infty]\to[0,\infty],\ \delta>0$ so that for all $(\boldsymbol{F},s)\in\mathbb{R}^{3\times3}\times\mathbb{R}$ and all $C\in\mathrm{GL}_+(3)$ with $|C-\mathbb{1}|\leq\delta$ we have
  $$|\partial_{\boldsymbol{F}} W(C\boldsymbol{F},s)(C\boldsymbol{F})^\top - \partial_{\boldsymbol{F}} W(\boldsymbol{F},s)\boldsymbol{F}^\top| \leq o(|C-\mathbb{1}|)(W(\boldsymbol{F},s)+\tilde{c})\,. \tag{3.10g}$$
- **Definition of β:** $\beta:\mathbb{R}\to[\eta,\infty),\ \beta(s)=\eta+s^2\,.$ (3.10h)
- **Definition of γ:** $\gamma\in\mathrm{C}^2(\mathbb{R}\times\mathbb{R}^3,\mathbb{R}),\ \gamma(s,\nabla s):=\dfrac{\mathcal{G}_c}{2l_c}(1-s)^2+\frac{\mathcal{G}_c l_c}{2}|\nabla s|^2\,.$ (3.10i)

Herein, assumptions (3.10b)–(3.10d) ensure the existence of minimizers, see [Dac89, p. 182, Thm. 2.10] and the discussion in Remark 3.6 below. In fact, the cases **a), b) & c)** provide three alternative sets of relations for the exponents p,p_2,p_3 building on compactness results for minors of gradients given in [Bal77, Dac89, MQY94], see Remark 3.6. In analytical works on evolution problems for generalized standard materials often (the 2D- or dD-version of) assumption (3.10d) **a)** is used, cf. e.g. [MR06, FM06, MM09, KAZ10, MRS16, MR16], since this ensures the continuity of the deformation and avoids ambiguities in the definition of its minors, see Remark 3.6. Within cases **b) & c)** we slightly weaken these assumptions in accordance with the results of [Bal77, Dac89, MQY94] in order to include energy densities into our analysis, which are popular in engineering. We also refer e.g. to the works [DML10, Laz11], resp. [MS13, NZ14, NT15], where (the 2D- or dD-versions of) assumption (3.10d) **b)** has been applied in the context of rate-independent fracture, resp. for the Γ-limit passage from finite- to small-strain linear elasticity for rate-independent processes in generalized standard materials. Furthermore, assumptions (3.10f) & (3.10g) ensure that a minimizer of (3.4a) satisfies the (weak form of the) corresponding Euler-Lagrange equation (4.1) and, similarly, a control of the power of the energy, cf. forthcoming Prop. 3.8. This will be obtained via the following result, cf. [Bal02, Thm. 2.4 & Lemma 2.5] or [FM06].

Lemma 3.2. *Let* (3.10f) *be satisfied. Then, for $\boldsymbol{\varphi}\in W^{1,p}(\mathcal{B}_0;\mathbb{R}^3)$, here $p\in[1,\infty)$, satisfying a Dirichlet boundary condition and being a minimizer of the energy according to* (3.4a)*, there also holds*

$$\int_{\mathcal{B}_0}\big(\mathrm{D}_{\boldsymbol{F}} W(\mathrm{D}\boldsymbol{y})\mathrm{D}\boldsymbol{y}^\top\big):\mathrm{D}\boldsymbol{\varphi}(\boldsymbol{y})\,\mathrm{d}X=0 \tag{3.11}$$

for all $\boldsymbol{\varphi}\in C^1(\mathbb{R}^3;\mathbb{R}^3)$ such that $\boldsymbol{\varphi}$ and $\mathrm{D}\boldsymbol{\varphi}$ are uniformly bounded and satisfy $\boldsymbol{\varphi}(\boldsymbol{y})=0$ on the Dirichlet boundary in trace sense. Moreover, there is $\delta>0$ so that for all $\boldsymbol{C}\in\mathrm{GL}_+(3)$ with $|\boldsymbol{C}-\mathbb{1}|\leq\delta$ we have

$$W(\boldsymbol{C}\boldsymbol{F},s)+\tilde{c}\leq\tfrac{3}{2}(W(\boldsymbol{F},s)+\tilde{c}) \tag{3.12}$$
$$|\partial_{\boldsymbol{F}} W(\boldsymbol{C}\boldsymbol{F},s)\boldsymbol{F}^\top|\leq 3c\,(W(\boldsymbol{F},s)+\tilde{c})\,. \tag{3.13}$$

Assumptions on the domain, state spaces & given data: As in (3.1) we consider a body with reference configuration $\mathcal{B}_0 \subset \mathbb{R}^3$ consisting of a nonlinearly elastic material, such that

$$\mathcal{B}_0 \subset \mathbb{R}^3 \text{ is a bounded Lipschitz domain, } \partial\mathcal{B}_0^{\mathrm{D}} \subset \partial\mathcal{B}_0 \text{ with } \partial\mathcal{B}_0^{\mathrm{D}} \neq \emptyset,\ \partial\mathcal{B}_0^{\mathrm{N}} := \partial\mathcal{B}_0 \backslash \partial\mathcal{B}_0^{\mathrm{D}}\,. \tag{3.14}$$

This body undergoes a damage process driven by time-dependent exterior forces $\boldsymbol{h}(t)$ located on the Neumann part $\partial\mathcal{B}_0^{\mathrm{N}} \subset \partial\mathcal{B}_0$ of the boundary. Moreover, the body is assumed to be clamped at the remaining part $\partial\mathcal{B}_0^{\mathrm{D}}$ of its boundary, so that the deformation is prescribed there: $\boldsymbol{\varphi}(t) = \boldsymbol{g}(t)$ on $\partial\mathcal{B}_0^{\mathrm{D}}$.

Thus, the set of admissible deformations at time $t \in [0, T]$ is given by

$$\mathbf{U}(t) := \{\boldsymbol{\phi} \in W^{1,p}(\Omega, \mathbb{R}^3) \,|\, \boldsymbol{\phi} = \boldsymbol{g}(t) \text{ on } \partial\mathcal{B}_0^{\mathrm{D}}\} \quad \text{for } p \text{ as in (3.10d)} \tag{3.15}$$

with the weak $W^{1,p}$-topology. By the assumption on p in (3.10d) we have in particular that $p > 3/2$ and thus, in three space dimensions,

$$W^{1,p}(\mathcal{B}_0, \mathbb{R}^3) \Subset L^{p'}(\mathcal{B}_0, \mathbb{R}^3) \text{ compactly, where } p' := \frac{p}{p-1}\,. \tag{3.16}$$

Adapting the ideas of [FM06] from the setting where $p > 3$ to the present setting $p \in [2, \infty)$, we assume that the Dirichlet datum can be extended to $\mathbb{R}^3$ in the following way:

$$\begin{aligned}
&\boldsymbol{g} \in \mathrm{C}^1([0,T] \times \mathbb{R}^3, \mathbb{R}^3),\ \nabla\boldsymbol{g} \in \mathrm{BC}^1([0,T] \times \mathbb{R}^3, \mathrm{Lin}(\mathbb{R}^3, \mathbb{R}^3)),\\
&\nabla^2\boldsymbol{g} \in \mathrm{B}([0,T] \times \mathbb{R}^3, \mathrm{Lin}(\mathbb{R}^{3\times3}, \mathbb{R}^{3\times3})), && \text{(3.17a)}\\
&\text{with } C_{\boldsymbol{g}} := \sup_{t \in [0,T], y \in \mathbb{R}^3} (|\nabla\boldsymbol{g}(t,y)| + |\partial_t\nabla\boldsymbol{g}(t,y)| + |\nabla^2\boldsymbol{g}(t,y)|), && \text{(3.17b)}\\
&|\boldsymbol{g}(t,y)| \le c_{\boldsymbol{g}}(1 + |y|) \quad \text{for all } (t,y) \in [0,T] \times \mathbb{R}^3\,, && \text{(3.17c)}\\
&|(\nabla\boldsymbol{g}(t,y))^{-1}| \le \tilde{C}_{\boldsymbol{g}} \text{ for all } (t,y) \in [0,T] \times \mathbb{R}^3. && \text{(3.17d)}
\end{aligned}$$

For the time-dependent Neumann datum we impose that

$$\boldsymbol{h} \in \mathrm{C}^1([0,T], L^{p'}(\partial\mathcal{B}_0^{\mathrm{N}}, \mathbb{R}^3)) \text{ with } C_{\boldsymbol{h}} := \|\boldsymbol{h}\|_{\mathrm{C}^1([0,T], L^{p'}(\partial\mathcal{B}_0^{\mathrm{N}}, \mathbb{R}^3))}\,. \tag{3.18}$$

To handle the time-dependent Dirichlet conditions one assumes that the deformation is of the form

$$\begin{aligned}
&\boldsymbol{\varphi}(t, X) = \boldsymbol{g}(t, y(X)) \text{ with } y \in \mathbf{Y}, \text{ where} && \text{(3.19)}\\
&\mathbf{Y} := \{y \in W^{1,p}(\Omega, \mathbb{R}^d) \,|\, y = \mathrm{id} \text{ on } \partial\mathcal{B}_0^{\mathrm{D}}\} \quad \text{for } p \text{ as in (3.10d)} && \text{(3.20)}
\end{aligned}$$

with the weak $W^{1,p}$-topology. By the chain rule, this composition leads to a multiplicative split of the deformation gradient:

$$\nabla\boldsymbol{\varphi}(t, x) = \nabla_X\boldsymbol{g}(t, y(X)) = \nabla_y\boldsymbol{g}(t, y(X))\nabla_X y(X) = \nabla\boldsymbol{g}(t, y)\nabla y\,.$$

Furthermore, we introduce the space

$$\mathbf{Y}_0 := \mathbf{Y} - \{\mathrm{id}\}\,. \tag{3.21}$$

Under consideration of (3.1) and the explanations along with (3.4) we choose the set of admissible damage variables $\mathbf{Z}$ in (3.1) and the set of admissible test functions $\mathbf{X}$ in (3.4) as

$$\mathbf{Z} := \{\tilde{s} \in H^1(\mathcal{B}_0),\ \tilde{s} = 1 \text{ on } \partial\mathcal{B}_0^{\mathrm{D}} \text{ in trace sense}\} \tag{3.22a}$$
$$\mathbf{X} := \mathbf{Z} \cap L^\infty(\mathcal{B}_0) \tag{3.22b}$$

equipped with the respective weak topologies. The sets $\mathbf{Y}$ and $\mathbf{Z}$ form the state space $\mathbf{Y} \times \mathbf{Z}$, which is endowed with the weak topology of the product space.

For the closed subspace $\mathbf{Y}_0 \subset W^{1,p}(\mathcal{B}_0, \mathbb{R}^3)$ Friedrich's inequality is available:

Theorem 3.3 (Friedrich's inequality)**.** *Let $\mathcal{B}_0 \subset \mathbb{R}^3$ be a Lipschitz domain with Dirichlet conditions on $\partial\mathcal{B}_0^{\mathrm{D}} \subset \partial\mathcal{B}_0$, where $\partial\mathcal{B}_0^{\mathrm{D}} \neq \emptyset$. Let $1 < p < \infty$. There is a constant $C_F = C_F(\mathcal{B}_0, p)$ such that the following estimate holds for every $y_0 \in \mathbf{Y}_0$:*

$$\|y_0\|_{W^{1,p}(\mathcal{B}_0,\mathbb{R}^3)} \leq C_F \|\nabla y_0\|_{L^p(\mathcal{B}_0,\mathbb{R}^{3\times3})} \ . \tag{3.23}$$

Bounds and convergence properties for time-dependent Dirichlet data: We now provide bounds and convergence properties in the spaces $\mathbf{U}$ and $\mathbf{Y}$, which follow from the relations for the Dirichlet datum (3.17) & (3.19). First of all, the lemma below is a consequence of the growth restriction (3.17c).

Lemma 3.4. *Let* (3.14)*,* (3.17) *as well as* (3.19) *hold. For every $y \in \mathbf{Y}$ and $\boldsymbol{\varphi}(t) = \boldsymbol{g}(t, y)$ it holds $\|\boldsymbol{\varphi}(t)\|^p_{W^{1,p}(\mathcal{B}_0,\mathbb{R}^3)} \leq \hat{C}_{\boldsymbol{g}}(\|y\|^p_{W^{1,p}(\mathcal{B}_0,\mathbb{R}^3)} + 1)$*

Proof. By the growth restriction (3.17c) one directly obtains

$$\|\boldsymbol{\varphi}(t)\|^p_{W^{1,p}(\mathcal{B}_0,\mathbb{R}^3)} \leq 2^{p-1} c^p_{\boldsymbol{g}} (\mathcal{L}^3(\mathcal{B}_0) + \|y\|^p_{L^p(\mathcal{B}_0,\mathbb{R}^3)}) + C^p_{\boldsymbol{g}} \|\nabla y\|^p_{L^p(\mathcal{B}_0,\mathbb{R}^{3\times3})} \,.$$

Hence $\hat{C}_{\boldsymbol{g}} := \max\{2^{p-1} c^p_{\boldsymbol{g}}, C^p_{\boldsymbol{g}}, 2^{p-1} c^p_{\boldsymbol{g}} \mathcal{L}^3(\mathcal{B}_0)\}$. □

Due to the realization of the Dirichlet condition in terms of a superposition, cf. (3.19), thanks to the regularity properties (3.17) of the Dirichlet datum the convergence of a sequence in $\mathbf{Y}$ translates to the convergence of a sequence respecting the Dirichlet condition in $\mathbf{U}$ as follows:

Lemma 3.5. *Let* (3.17)*,* (3.19) *and* (3.20) *hold. Consider a sequence $(y_k)_{k\in\mathbb{N}} \subset \mathbf{Y}$ such that $y_k \rightharpoonup y$ in $\mathbf{Y}$. Then $\boldsymbol{\varphi}_k(t) = \boldsymbol{g}(t, y_k) \rightharpoonup \boldsymbol{g}(t, y) = \boldsymbol{\varphi}(t)$ in $\mathbf{U}(t)$ for all $t \in [0, T]$.*

Proof. We have the compact embeddings $W^{1,p}(\mathcal{B}_0, \mathbb{R}^3) \Subset L^p(\mathcal{B}_0, \mathbb{R}^3)$ and $W^{1,p}(\mathcal{B}_0, \mathbb{R}^3) \Subset L^{p'}(\mathcal{B}_0, \mathbb{R}^3)$ by (3.16) and hence $y_k \to y$ in both in $L^{p'}(\mathcal{B}_0, \mathbb{R}^3)$ and in $L^p(\mathcal{B}_0, \mathbb{R}^3)$. By (3.17a) & (3.17b) we now find $\|\boldsymbol{\varphi}_k(t) - \boldsymbol{\varphi}(t)\|_{L^p(\mathcal{B}_0,\mathbb{R}^3)} \leq \sup_{\tilde{y}\in\mathbb{R}^3} |\nabla \boldsymbol{g}(t, \tilde{y})| \|y_k(t) - y(t)\|_{L^p(\mathcal{B}_0,\mathbb{R}^3)} \leq C_{\boldsymbol{g}} \|y_k(t) - y(t)\|_{L^p(\mathcal{B}_0,\mathbb{R}^3)} \to 0$ as $k \to \infty$. Furthermore, we obtain $\nabla\boldsymbol{\varphi}_k(t) \rightharpoonup \nabla\boldsymbol{\varphi}(t)$ in $L^p(\Omega, \mathbb{R}^{d\times d})$, since $\nabla y_k \rightharpoonup \nabla y$ in $L^p(\Omega, \mathbb{R}^{d\times d})$ and $\nabla_y \boldsymbol{g}(t, y_k) \to \nabla_y \boldsymbol{g}(t, y)$ in $L^{p'}(\mathcal{B}_0, \mathbb{R}^{3\times3})$, which ensues from (3.17b) by $\|\nabla_y \boldsymbol{g}(t, y_k) - \nabla_y \boldsymbol{g}(t, y)\|_{L^{p'}(\mathcal{B}_0,\mathbb{R}^{3\times3})} \leq \sup_{\tilde{y}\in\mathbb{R}^3} |\nabla^2 \boldsymbol{g}(t, \tilde{y})| \|y_k(t) - y(t)\|_{L^{p'}(\mathcal{B}_0,\mathbb{R}^3)} \leq C_{\boldsymbol{g}} \|y_k(t) - y(t)\|_{L^{p'}(\mathcal{B}_0,\mathbb{R}^3)} \to 0$ as $k \to \infty$. □

Compactness of minors of gradients according to (3.10d)**:** We first motivate our sets of coercivity assumptions (3.10d) **a), b) & c)** in a remark.

Remark 3.6 (Results on compactness of minors)**.** *In three space dimensions the minors of a matrix* $\nabla\boldsymbol{\varphi}$ *are given by* $\nabla\boldsymbol{\varphi}$ *itself, its cofactor matrix* $\mathrm{cof}\,\nabla\boldsymbol{\varphi}$*, and its determinant* $\det\nabla\boldsymbol{\varphi}$*. Hereby, the entries of* $\mathrm{cof}\,\nabla\boldsymbol{\varphi}$ *contain products of two derivatives* $\partial_i\boldsymbol{\varphi}_j\partial_k\boldsymbol{\varphi}_l$*, whereas* $\det\nabla\boldsymbol{\varphi}$ *is composed of terms given by products of three derivatives* $\partial_i\boldsymbol{\varphi}_j\partial_k\boldsymbol{\varphi}_l\partial_m\boldsymbol{\varphi}_n$*. In this spirit, it is shown in [Bal77, Cor. 6.2.2] that the map* $\boldsymbol{\varphi}\mapsto\mathrm{cof}\,\nabla\boldsymbol{\varphi}:W^{1,p}(\mathcal{B}_0)\to L^{p/2}(\mathcal{B}_0)$ *is weakly sequentially lower semicontinuous if* $p>2$*, whilst the map* $\boldsymbol{\varphi}\mapsto\det\nabla\boldsymbol{\varphi}:W^{1,p}(\mathcal{B}_0)\to L^{p/3}(\mathcal{B}_0)$ *is weakly sequentially lower semicontinuous if* $p>3$*. As done in [Bal77, p. 370] the functions* $\mathrm{cof}\,\nabla\boldsymbol{\varphi},\det\nabla\boldsymbol{\varphi}$ *can given meaning in the sense of distributions by* defining *suitable distributions denoted as* $\mathrm{Cof}\,\nabla\boldsymbol{\varphi},\mathrm{Det}\,\nabla\boldsymbol{\varphi}$*, and in general* $\mathrm{cof}\,\nabla\boldsymbol{\varphi}$ *and* $\mathrm{Cof}\,\nabla\boldsymbol{\varphi}$*, resp.* $\det\nabla\boldsymbol{\varphi}$ *and* $\mathrm{Det}\,\nabla\boldsymbol{\varphi}$ *must not coincide. We point out the result [Bal77, Thm. 6.2], which relaxes* $p>3$ *to the condition* $p>3/2,\frac{1}{p}+\frac{1}{q}<4/3$ *and ensures: If* $\boldsymbol{\varphi}_k\rightharpoonup\boldsymbol{\varphi}$ *in* $W^{1,p}(\mathcal{B}_0,\mathbb{R}^3)$*, then* $\mathrm{Cof}\,\nabla\boldsymbol{\varphi}_k\rightharpoonup\mathrm{Cof}\,\nabla\boldsymbol{\varphi}$ *and* $\mathrm{Det}\,\nabla\boldsymbol{\varphi}_k\to\mathrm{Det}\,\nabla\boldsymbol{\varphi}$ *in* $\mathcal{D}'(\mathcal{B}_0)$*, i.e., the price is that the* defined *distributional minors cannot be identified with the minors as a function. Yet, as previously discussed, this ambiguity can be ruled out if* $p>3$*. Hence, taking a look at the coercivity assumptions* (3.10d)*, it is therefore sufficient for* $p>3$*, cf.* (3.10d) **a)***, to claim boundedness of the energy density from below only with regard to* $|\nabla\boldsymbol{\varphi}|^p$*; compactness of the other minors then follows by [Bal77, Cor. 6.2.2]. In contrast, in the case of* (3.10d) **b)***,* $p\in[2,3)$ *is possible, so that* $p/3<1$*, and the weak sequential lower semicontinuity of the determinant cannot be concluded directly. Similarly, if* $p=2$ *compactness cannot be concluded for bounded sequences of cofactors. This is why coercivity assumption* (3.10d) **b)***, which is taken from [Dac89, Thm. 2.10, p. 182], requires bounds with exponents larger than one for all of the three minors. In particular, the set of assumptions is designed exactly in such a way that, thanks to [Dac89, Thm. 2.6, Part 5, p. 173], ambiguities between* $\mathrm{cof}\,\nabla\boldsymbol{\varphi}$ *and* $\mathrm{Cof}\,\nabla\boldsymbol{\varphi}$*, resp.* $\det\nabla\boldsymbol{\varphi}$ *and* $\mathrm{Det}\,\nabla\boldsymbol{\varphi}$ *can be ruled out, even though* $p<3$ *is admissible. Moreover, assumptions* (3.10d) **b1) & b2)** *are tailored to the case that the energy density does not depend on* $\det\nabla\boldsymbol{\varphi}$*, resp. neither on* $\mathrm{cof}\,\nabla\boldsymbol{\varphi}$ *nor on* $\det\nabla\boldsymbol{\varphi}$*, so that compactness in these terms is not needed. In this way our analysis also includes e.g. Neo-Hooke and Mooney-Rivlin materials, cf.* (3.64) *for the definition. Finally, assumption* (3.10d) **c)** *builds on the improved compactness result [MQY94, Lemma 4.1] stating that: Let* $p\geq(d-1),p_2>d/(d-1),d$ *space dimension, and* $(\nabla\boldsymbol{\varphi}_n)_n\in W^{1,p}(\mathcal{B}_0,\mathbb{R}^d)$ *such that* $\nabla\boldsymbol{\varphi}_n\rightharpoonup\boldsymbol{\varphi}$ *in* $W^{1,p}(\mathcal{B}_0,\mathbb{R}^d),\mathrm{cof}\,\nabla\boldsymbol{\varphi}_n$ *bounded in* $L^{p_2}(\mathcal{B}_0,\mathbb{R}^{d\times d}),\det\boldsymbol{\varphi}_n>0$*. Then*

$$\mathrm{cof}\,\nabla\boldsymbol{\varphi}_n\rightharpoonup\mathrm{cof}\,\nabla\boldsymbol{\varphi}\quad\text{in } L^{p_2}(\mathcal{B}_0,\mathbb{R}^{d\times d})\text{ and}\tag{3.24a}$$

$$\det\nabla\boldsymbol{\varphi}_n\rightharpoonup\det\nabla\boldsymbol{\varphi}\quad\text{in } L^r(\mathcal{B}_0),\ \text{where } r=\tfrac{p_2(d-1)}{d}\,.\tag{3.24b}$$

Moreover, if $p_2=d/(d-1)$ *and if* $\det\nabla\boldsymbol{\varphi}_n\geq0$ *a.e. in* $\mathcal{B}_0$*, then* (3.24b) *is replaced by*

$$\det\nabla\boldsymbol{\varphi}_n\rightharpoonup\det\nabla\boldsymbol{\varphi}\quad\text{in } L^1(K)\tag{3.24c}$$

for all compact sets $K\subset\mathcal{B}_0$*.*

Proposition 3.7 (Compactness of minors of gradients according to (3.10d))**.** *Consider a sequence* $(\boldsymbol{\varphi}_k)_k\subset\mathbf{U}$ *such that for all* $k\in\mathbb{N}$

$$C\geq c_1\|\nabla\boldsymbol{\varphi}_k\|^p_{L^p(\mathcal{B}_0,\mathbb{R}^{3\times3})}+c_2\|\mathrm{cof}\nabla\boldsymbol{\varphi}_k\|^{p_2}_{L^{p_2}(\mathcal{B}_0,\mathbb{R}^{3\times3})}+c_3\|\det\nabla\boldsymbol{\varphi}_k\|^{p_3}_{L^{p_3}(\mathcal{B}_0,\mathbb{R})}-C\,,\tag{3.25}$$

where the constants $p, p_2, p_3, c_1, c_2, c_2, C$ *match with the conditions of* (3.10d) *either* **a)** *or* **b)**. *Then there exists a not relabeled subsequence* $(\boldsymbol{\varphi}_k)_k \subset \mathbf{U}$ *and* $\boldsymbol{\varphi} \in \mathbf{U}$ *such that* $\boldsymbol{\varphi}_k \rightharpoonup \boldsymbol{\varphi}$ *in* $\mathbf{U}$, $\mathrm{cof}\nabla\boldsymbol{\varphi}_k \rightharpoonup \mathrm{cof}\nabla\boldsymbol{\varphi}$ *in* $L^{p_2}(\mathcal{B}_0, \mathbb{R}^3 \times 3)$, *and* $\det\nabla\boldsymbol{\varphi}_k \rightharpoonup \det\nabla\boldsymbol{\varphi}$ *in* $L^{p_3}(\mathcal{B}_0, \mathbb{R})$. *In case* **b)**, *if* $c_j = 0, j = 2, 3,$ *then* $p_j = p/j$ *in the previous weak convergence statement. In case* **c)**, *for* $c_3 = 0$, $p_3 = \frac{p_2(d-1)}{d}$ *in the previous convergence statement and if* $p_2 = 3/2$, *then* $\det\nabla\boldsymbol{\varphi}_k \rightharpoonup \det\nabla\boldsymbol{\varphi}$ *in* $L^1(K)$ *for any compact set* $K \subset \mathcal{B}_0$.

Proof. The proof of case **b)** can be retrieved from [Dac89, p. 183]. Case **a)** follows with [Bal77, Cor. 6.2.2] and case **c)** is due to [MQY94, Lemma 4.1], cf. (3.24). ☐

Temporal regularity of the energy functional due to assumptions (3.17)**,** (3.18) **and** (3.10)**:** In the following we prove temporal regularity properties of the energy functional, based on assumption (3.17), (3.18) and (3.10). An analogous result was first obtained in [FM06, Lemma 5.5].

Proposition 3.8. *Let* (3.17)*,* (3.18)*, and* (3.10) *be satisfied. Then there exist constants* $c_0 \geq 0, c_1 > 0$ *such that for all* $(t_*, \boldsymbol{g}(t_*, y), s) \in [0, T] \times \mathbf{U} \times \mathbf{Z}$ *with* $\mathcal{E}(t_*, \boldsymbol{g}(t_*, y), s) < \infty$ *it holds:* $\mathcal{E}(\cdot, \boldsymbol{g}(\cdot, y), s) \in \mathrm{C}^1([0, T])$ *with*

$$\partial_t\mathcal{E}(t, q) = \int_{\mathcal{B}_0} \partial_F W(F(t), s)F^\top : G(t) - \langle\dot{\boldsymbol{h}}(t), \boldsymbol{\varphi}(t)\rangle - \langle\boldsymbol{h}(t), \partial_t\boldsymbol{\varphi}(t)\rangle \tag{3.26}$$

for $F(t) := \nabla\boldsymbol{\varphi}(t)$ *and* $G(t) := (\nabla\boldsymbol{g}(t, y))^{-1}\partial_t\nabla\boldsymbol{g}(t, y)$ *and*

$$|\partial_t\mathcal{E}(t, \boldsymbol{g}(t, y), s)| \leq c_1(\mathcal{E}(t, \boldsymbol{g}(t, y), s) + c_0) \quad \textit{for every } t \in [0, T] . \tag{3.27}$$

Moreover, if $\mathcal{E}(t, \boldsymbol{g}(t, y), s) < E$ *for some constant* $E \in \mathbb{R}$, *the following Lipschitz-estimate holds true:*

$$|\mathcal{E}(t, \boldsymbol{g}(t, y), s) - \mathcal{E}(\tau, \boldsymbol{g}(\tau, y), s)| \leq c_E|t - \tau| . \tag{3.28}$$

Proof. **Proof of** (3.26) **&** (3.27)**:** We confine ourselves to prove the existence of $\partial_t\mathcal{E}(\cdot, q)$ and estimate (3.27) in a neighborhood $\mathrm{N}(t_q)$ of $t_q \in [0, T]$. Similarly to the small-strain setting, where an analogouse proof was carried out in [TM10, Thm. 3.7], this is basically done with the mean value theorem of differentiability and the dominated convergence theorem. But the different treatment of the inhomogeneous Dirichlet condition requires different estimates, which will be carried out here. The existence of $\partial_t\mathcal{E}(\cdot, q)$ and the validity of (3.27) on the whole interval $[0, T]$ can then be concluded with the same arguments as in the proof of [TM10, Thm. 3.7].

Since $\partial_t \int_{\Gamma_\mathrm{N}} \boldsymbol{h}(t)\boldsymbol{\varphi}(t)\,\mathrm{d}\mathcal{H}^2$ exists by (3.17) & (3.18), it remains to show the existence of $\partial_t \int_{\mathcal{B}_0} W(\nabla\boldsymbol{\varphi}(t)\,\mathrm{d}X$ in $\mathrm{N}(t_q)$. For this we define for $t \in \mathrm{N}(t_q)$

$$\omega(x, t, \alpha) := \begin{cases} \frac{1}{\alpha}\left(W(\nabla\boldsymbol{\varphi}(t+\alpha), s) - W(\nabla\boldsymbol{\varphi}(t), s)\right) & \text{if } \alpha \neq 0 \\ \partial_F W(\nabla\boldsymbol{\varphi}(t), s)(\nabla\boldsymbol{\varphi}(t))^\top : (\nabla\boldsymbol{g}(t, y))^{-1}\partial_t\nabla\boldsymbol{g}(t, y) & \text{if } \alpha = 0 \end{cases}$$

and we have to show that $\omega(x,t,\cdot) \in \mathrm{C}^0([-\alpha_t,\alpha_t])$ for α_t suitably. By the mean value theorem of differentiability we find $\tilde\alpha = \tilde\alpha(\alpha)$ such that it holds for every $\alpha \in [-\alpha_t,\alpha_t]$

$$\begin{aligned}&\frac{1}{\alpha}\left(W(\nabla\boldsymbol{\varphi}(t{+}\alpha),s){-}W(\nabla\boldsymbol{\varphi}(t),s)\right)\\ &=\partial_F W(\nabla\boldsymbol{\varphi}(t{+}\tilde\alpha),s)(\nabla\boldsymbol{\varphi}(t{+}\tilde\alpha))^\top{:}(\nabla\boldsymbol{g}(t{+}\tilde\alpha,y))^{-1}\partial_t\nabla\boldsymbol{g}(t{+}\tilde\alpha,y)\\ &\quad\to\partial_F W(\nabla\boldsymbol{\varphi}(t),s)(\nabla\boldsymbol{\varphi}(t))^\top{:}(\nabla\boldsymbol{g}(t,y))^{-1}\partial_t\nabla\boldsymbol{g}(t,y)\end{aligned}\tag{3.29}$$

as $\alpha,\tilde\alpha\to 0$ by (3.10f) and (3.17). In order to show that the integrals converge as well, we are going to apply the dominated convergence theorem. For this, we have to construct an integrable majorant for expression (3.29). Again by the mean value theorem of differentiability we first obtain $\hat\alpha$ such that

$$\nabla\boldsymbol{\varphi}(t{+}\tilde\alpha){=}\nabla(\boldsymbol{\varphi}(t){+}\partial_t\boldsymbol{\varphi}(t{+}\hat\alpha)\tilde\alpha){=}\big(\mathbb{1}+\tilde\alpha\partial_t\nabla\boldsymbol{g}(t{+}\hat\alpha,y)(\nabla\boldsymbol{g}(t,y))^{-1}\big)\nabla\boldsymbol{\varphi}(t){=}C(\tilde\alpha)\nabla\boldsymbol{\varphi}(t)$$

with $C(\tilde\alpha)\to\mathbb{1}$ as $\tilde\alpha\to 0$ by (3.17). Hence we conclude by (3.13) and (3.17):

$$\begin{aligned}|(3.29)| &\le \tilde C_g C_g|\partial_F W(C(\tilde\alpha),s)(\nabla\boldsymbol{\varphi}(t))^\top C(\tilde\alpha)^\top|\\ &\le \tilde C_g C_g dc(W(\nabla\boldsymbol{\varphi}(t),s)+\tilde c)(\sqrt{d}{+}\tilde\alpha C_g\tilde C_g)\,.\end{aligned}\tag{3.30}$$

Now, estimate (3.27) is derived under consideration of

$$|\partial_t\mathcal{E}(t,q)| \le \Big|\int_{\mathcal{B}_0}\omega(x,t,0)\Big| + |\langle\dot{\boldsymbol{h}}(t),\boldsymbol{\varphi}(t)\rangle| + |\langle\boldsymbol{h}(t),\partial_t\boldsymbol{\varphi}(t)\rangle|\,.\tag{3.31}$$

In view of (3.17), (3.18), Lemma 3.4, Friedrich's inequality (3.23), Young's inequality and 3.10d we derive for the loading terms in (3.31) an estimate of the form

$$|\langle\dot{\boldsymbol{h}}(t),\boldsymbol{\varphi}(t)\rangle| + |\langle\boldsymbol{h}(t),\partial_t\boldsymbol{\varphi}(t)\rangle| \le A_1\mathcal{E}(t,q)+B_1.$$

For the elastic energy term in (3.31) estimate (3.30) and (3.17), (3.18) lead to

$$\Big|\int_{\mathcal{B}_0}\omega(x,t,0)\Big| \le (3.30) \le A_2\mathcal{E}(t,q)+B_2,$$

so that inequality (3.27) is obtained.

Proof of (3.28)**:** The Lipschitz estimate (3.28) follows by applying the mean value theorem of differentiability and then by making use of the continuity properties of $\boldsymbol{g},\boldsymbol{h}$, cf. (3.17) & (3.18), together with stress control (3.10f) as in (3.29). □

3.2 Analytical Results on the Convergence of Discrete Solutions

In this section we gather and explain all our analytical results.

In a first step we verify the existence of discrete solutions $(\boldsymbol{\varphi}_N^k, s_N^k) \in \mathbf{U}\times\mathbf{Z}$ at each time-step $t_N^k \in \Pi_N$.

Proposition 3.9 (Existence of solutions for the discrete problem (3.7))**.** *Let the assumptions* (3.10)*,* (3.18)*,* (3.17) *&* (3.14) *hold true and* $(\mathbf{U}\times\mathbf{Z},\mathcal{E},\mathcal{V}_{\alpha N})$ *be given by* (3.1)*,* (3.6)*,* (3.15)*,* (3.22a)*. Consider a partition* Π_N *of* $[0,T]$ *as in* (3.5)*. Suppose that* $(y^0,s^0)\in\mathbf{Y}\times\mathbf{Z}$ *is an admissible initial datum. Then, the following statements hold true for each* $t_N^k\in\Pi_N$ *as in* (3.5)*:*

1. *There exists a pair* $(y_N^k,s_N^k)\in\mathbf{Y}\times\mathbf{Z}$ *such that* $(\boldsymbol{\varphi}(t_N^k,y_N^k),s_N^k)\in\mathbf{U}\times\mathbf{Z}$ *is a solution for minimization problem* (3.7)*.*
2. *Let the initial datum* $(y^0,s^0)\in\mathbf{Y}\times\mathbf{Z}$ *such that* $s^0\in[0,1]$ *a.e. on* $\mathcal{B}_0$*. Then, a solution* s_N^k *of* (3.7b) *also satisfies* $s_N^k\in[0,1]$ *a.e. in* $\mathcal{B}_0$ *both for* $\alpha=0$ *and for* $\alpha>0$ *in* (3.6)*, i.e.* $s_N^k\in\mathbf{X}=H^1(\mathcal{B}_0)\cap L^\infty(\mathcal{B}_0)$*.*

Using the discrete solutions $(y_N^k,s_N^k)_{k=1}^N$ obtained in Prop. 3.9 we now introduce piecewise constant left-continuous $(\overline{y}_N,\overline{s}_N)$ (right-continous $(\underline{y}_N,\underline{s}_N)$) piecewise constant interpolants and linear interpolants s_N^ℓ as follows:

$$
\begin{aligned}
(\overline{y}_N(t),\overline{s}_N(t)) &:= (y_N^k,s_N^k) \ \text{ for all } t\in(t_N^{k-1},t_N^k]\,, && (3.32a)\\
(\underline{y}_N(t),\underline{s}_N(t)) &:= (y_N^{k-1},s_N^{k-1}) \ \text{ for all } t\in[t_N^{k-1},t_N^k)\,, && (3.32b)\\
s_N^\ell &:= \tfrac{t-t_N^{k-1}}{\tau_N}s_N^k+\tfrac{t_N^k-t}{\tau_N}s_N^{k-1} \ \text{ for all } t\in[t_N^{k-1},t_N^k) && (3.32c)
\end{aligned}
$$

and accordingly, we set $\overline{\boldsymbol{\varphi}}_N(t):=\overline{\boldsymbol{g}}(t,\overline{y}_N(t))$ and $\underline{\boldsymbol{\varphi}}_N(t):=\underline{\boldsymbol{g}}(t,\underline{y}_N(t))$.

For the interpolants $(\overline{y}_N,\underline{y}_N,\overline{s}_N,\underline{s}_N,s_N^\ell)$ we then verify that they satisfy a discrete version of the governing equations (3.4) and uniform apriori estimates.

Proposition 3.10 (Properties of the interpolants $(\overline{y}_N,\underline{y}_N,\overline{s}_N,\underline{s}_N,s_N^\ell)$)**.** *Let the assumptions of Prop. 3.9 hold true with* $s_N^0\in[0,1]$ *a.e. in* $\mathcal{B}_0$*. Then the interpolants* $(\overline{y}_N,\underline{y}_N,\overline{s_N}_N,\underline{s}_N,s_N^\ell)$ *contructed from solutions* $(y_N^k,s_N^k)_{k=1}^N$ *of problem* (3.7) *via* (3.32) *satisfy uniformly for all* $N\in\mathbb{N}$*:*

$$
\begin{aligned}
&\textit{For all } \tilde{\boldsymbol{\varphi}}\in\mathbf{U}(t):\ \mathcal{E}(t,\overline{\boldsymbol{\varphi}}_N(t),\underline{s}_N(t))\le\mathcal{E}(t,\tilde{\boldsymbol{\varphi}},\underline{s}_N(t))\,, && (3.33a)\\
&\mathrm{D}_s\mathcal{E}(t,\overline{\boldsymbol{\varphi}}_N(t),\overline{s}_N(t))+\tau_N\mathrm{D}\mathcal{V}_{\alpha N}(\dot{s}_N^\ell(t))=0 \ \textit{ in } \mathbf{X}^*,\ \textit{i.e., for all } \tilde{s}\in\mathbf{X}: && (3.33b)\\
&\int_{\mathcal{B}_0}\Big(\big(\beta'(\overline{s}_N(t))W_1(\mathbb{M}\nabla\overline{\boldsymbol{\varphi}}_N(t))\big)-\tfrac{1}{l_c}(1-\overline{s}_N)+M^{-1}\dot{s}_N^\ell+\alpha N(\dot{s}_N^\ell)_+\big)\tilde{s}\\
&\qquad+l_c\nabla\overline{s}_N(t):\nabla\tilde{s}\Big)\,\mathrm{d}X=0\,.
\end{aligned}
$$

In addition, the following energy-dissipation estimate holds true for all $N\in\mathbb{N}$ *and* $t\in[0,T]$*:*

$$
\mathcal{E}(t,\overline{\boldsymbol{\varphi}}_N(t),\overline{s}_N(t))+\int_0^t\mathcal{V}_{\alpha N}(\dot{s}_N^\ell(\tau))\,\mathrm{d}\tau\le\mathcal{E}(0,\boldsymbol{\varphi}_N^0,s_N^0)+\int_0^t\partial_\tau\mathcal{E}(\tau,\boldsymbol{g}(\tau,\underline{y}_N(\tau)),\underline{s}_N(\tau))\,\mathrm{d}\tau\,, \tag{3.34}
$$

where the partial time derivative $\partial_\tau\mathcal{E}(\tau,\boldsymbol{g}(\tau,\underline{y}_N(\tau)),\underline{s}_N(\tau))$ *is given by* (3.26)*.*

Furthermore, there is a constant $C>0$ *such that the following apriori estimates are satisfied uniformly* *for all* $N \in \mathbb{N}$ *and all* $t \in [0,T]$*:*

$$\mathcal{E}(t, \overline{\boldsymbol{\varphi}}_N(t), \overline{s}_N(t)) \le C,\ \mathcal{E}(t, \underline{\boldsymbol{\varphi}}_N(t), \underline{s}_N(t)) \le C,\ \int_0^t \mathcal{V}_{\alpha N}(\dot{s}_N^\ell(\tau))\,\mathrm{d}\tau \le C, \tag{3.35a}$$

$$\|\overline{y}_N(t)\|_{\mathbf{Y}} \le C \ \ \& \ \ \|\underline{y}_N(t)\|_{\mathbf{Y}} \le C\,, \tag{3.35b}$$

$$\|\dot{s}_N^\ell\|_{L^2(0,t;L^2(\mathcal{B}_0))} \le C\,, \tag{3.35c}$$

$$\|\overline{s}_N(t)\|_{\mathbf{Z}} \le C \ \ \& \ \ \|\underline{s}_N(t)\|_{\mathbf{Z}} \le C\,, \tag{3.35d}$$

$$\|\overline{s}_N(t)\|_{L^\infty(\mathcal{B}_0)} \le 1 \ \ \& \ \ \|\underline{s}_N(t)\|_{L^\infty(\mathcal{B}_0)} \le 1\,, \tag{3.35e}$$

$$\|\beta(\overline{s}_N(t))W_1(\mathbb{M}\nabla\overline{\boldsymbol{\varphi}}_N(t))\|_{\mathbf{X}^*} \le C\,. \tag{3.35f}$$

Thanks to the apriori estimates (3.35) we are now in the position to extract a (not relabeled) subsequence $(\overline{y}_N, \underline{y}_N, \overline{s_N}, \underline{s}_N, s_N^\ell)_N$ of the interpolants, which converge to a limit pair (y,s) that satisfies (3.4):

Theorem 3.11 (Convergence of the time-discrete solutions, existence of a solution for (3.4))**.** *Let the assumptions of Prop. 3.10 hold true. Then there is a (not relabeled) subsequence* $(\overline{y}_N, \underline{y}_N, \overline{s_N}, \underline{s}_N, s_N^\ell)_N$ *satisfying* (3.33)–(3.35)*, a function* $\underline{y} : [0,T] \to \mathbf{Y}$*, and a pair* $(y,s) : [0,T] \to \mathbf{Y} \times \mathbf{X}$ *with* $\boldsymbol{\varphi}(t) = \boldsymbol{g}(t, y(t))$ *such that*

$$
\begin{aligned}
s_N^\ell &\rightharpoonup s \text{ in } H^1(0,T;L^2(\mathcal{B}_0))\,, && (3.36a)\\
s_N^\ell, \overline{s}_N, \underline{s}_N &\rightharpoonup s \text{ in } L^2(0,T;L^2(\mathcal{B}_0))\,, && (3.36b)\\
\overline{s}_N(t), \underline{s}_N(t) &\rightharpoonup s(t) \text{ in } \mathbf{X} \text{ for all } t \in [0,T]\,, && (3.36c)\\
\overline{y}_N(t) &\rightharpoonup y(t) \text{ in } \mathbf{Y} \text{ for a.a. } t \in (0,T)\,, && (3.36d)\\
\underline{y}_N(t) &\rightharpoonup \underline{y}(t) \text{ in } \mathbf{Y} \text{ for a.a. } t \in (0,T)\,, && (3.36e)\\
W_i(\mathbb{M}\nabla\overline{\boldsymbol{\varphi}}_N(t)) &\to W_i(\mathbb{M}\nabla\boldsymbol{\varphi}(t)) \text{ in } L^1(\mathcal{B}_0) \text{ for a.a. } t \in (0,T),\ i = 1,2\,. && (3.36f)
\end{aligned}
$$

In particular, the pair $(y,s) : [0,T] \to \mathbf{Y} \times \mathbf{X}$ *satisfies:*

- *For all* $t \in [0,T]$*, for all* $\tilde{\boldsymbol{\varphi}} \in \mathbf{U}$ *:* $\mathcal{E}(t, \boldsymbol{\varphi}(t), s(t)) \le \mathcal{E}(t, \tilde{\boldsymbol{\varphi}}, s(t))\,,$ (3.37a)
- *if* $\alpha = 0$*, for a.a.* $t \in (0,T)$*, for all* $\tilde{s} \in \mathbf{X}$ *:* (3.37b)
$$\int_{\mathcal{B}_0} \Big(\big(\beta'(s(t))W_1(\mathbb{M}\nabla\boldsymbol{\varphi}(t))) - \tfrac{1}{l_c}(1 - s(t)) + M^{-1}\dot{s}(t)\big)\tilde{s} + l_c \nabla s(t) : \nabla\tilde{s} \Big)\,\mathrm{d}X = 0\,,$$
- *if* $\alpha > 0$*, for a.a.* $t \in (0,T)$*, for all* $\tilde{s} \in \mathbf{X}$ *such that* $\tilde{s} \le 0$ *a.e. in* $\mathcal{B}_0$ *:* (3.37c)
$$\int_{\mathcal{B}_0} \Big(\big(\beta'(s(t))W_1(\mathbb{M}\nabla\boldsymbol{\varphi}(t))) - \tfrac{1}{l_c}(1 - s(t)) + M^{-1}\dot{s}(t)\big)\tilde{s} + l_c \nabla s(t) : \nabla\tilde{s} \Big)\,\mathrm{d}X \ge 0\,,$$
- *if* $\alpha > 0$*, then* $\dot{s}(t) \le 0$ *for a.a.* $t \in (0,T)$*, a.e. in* $\mathcal{B}_0$ *,* (3.37d)
- *and the energy dissipation inequality for all* $t \in [0,T]$*:* (3.37e)
$$\mathcal{E}(t, \boldsymbol{\varphi}(t), s(t)) + \int_0^t \mathcal{V}_\alpha(\dot{s}(\tau))\,\mathrm{d}\tau \le \mathcal{E}(0, \boldsymbol{\varphi}^0, s^0) + \int_0^t \mathcal{P}(\tau, s(\tau))\,\mathrm{d}\tau\,,$$

with $\mathcal{V}_\alpha$ *from* (3.3) *and with* $\mathcal{P}(\tau, s(\tau)) := \sup\{\partial_\tau\mathcal{E}(\tau, \hat{\boldsymbol{\varphi}}, s(\tau)),\ \hat{\boldsymbol{\varphi}} \in \mathrm{argmin}\mathcal{E}(\tau, \cdot, s(\tau))\}$ *as a surrogate for the partial time derivative from* (3.26)*.*

3.3 Proofs of Prop. 3.9–Thm. 3.11

3.3.1 Proof of Prop. 3.9

In order to establish the proof of Item 1, we will employ the direct method of the calculus of variations. For this, we will verify the coercivity and the weak sequential lower semicontinuity of the functional $\mathcal{E}(t,\cdot,\cdot)$. To deduce the latter for the polyconvex functional $\mathcal{E}(t,\cdot,\cdot)$ we use the following result on the convergence of minors of gradients, which goes back on [Res67, Bal77], cf. also [MM09]. With this at hand we now establish weak sequential lower semicontinuity and coercivity.

Lemma 3.12. *Let* (3.14)*,* (3.17)*,* (3.18) *as well as* (3.10a)–(3.10d) *hold. Then, for all* $t \in [0,T]$ *the following statements hold true:*

1 $\mathcal{E}(t,\cdot,\cdot)$ *is coercive on* $\mathbf{U}\times\mathbf{Z}$ *for all* $t\in[0,T]$*, in particular, there are constants* $B, C > 0$ *such that for all* $(y,s)\in\mathbf{Y}\times\mathbf{Z}$ *with* $\boldsymbol{\varphi}=\boldsymbol{g}(t,y)$ *it holds:*

$$\mathcal{E}(t,\boldsymbol{\varphi},s) \geq C\big(\|y\|^p_{W^{1,p}(\mathcal{B}_0,\mathbb{R}^3)}+\|\mathrm{cof}\nabla\boldsymbol{\varphi}\|^{p_2}_{L^{p_2}(\mathcal{B}_0,\mathbb{R}^{3\times3})}+\|\det\nabla\boldsymbol{\varphi}\|^{p_3}_{L^{p_3}(\mathcal{B}_0,\mathbb{R})}\big)+\tfrac{l_c}{2}\|s\|^2_{H^1(\mathcal{B}_0)}-B. \tag{3.38}$$

2 $\mathcal{E}(t,\cdot,\cdot):\mathbf{U}\times\mathbf{Z}\to\mathbb{R}$ *has weakly sequentially compact sublevels.*

Proof. **Proof of 1.:** Let $(y_k,s_k)_{k\in\mathbb{N}}\subset\mathbf{Y}\times\mathbf{Z}$. By (2.3), (3.10d), (3.17), (3.18), Young's inequality with $\varepsilon=\big(\frac{c_1 p}{2C_g^pC_F^p}\big)^{\frac1p}$, Lemma 3.4 and Friedrich's inequality it is:

$$\begin{aligned}
&\mathcal{E}(t,\boldsymbol{\varphi}_k,s_k)\\
&\geq\big(c_1\|\nabla\boldsymbol{\varphi}_k(t)\|^p_{L^p(\Omega,\mathbb{R}^{3\times3})}+c_2\|\mathrm{cof}\nabla\boldsymbol{\varphi}_k\|^{p_2}_{L^{p_2}(\mathcal{B}_0,\mathbb{R}^{3\times3})}+c_3\|\det\nabla\boldsymbol{\varphi}_k\|^{p_3}_{L^{p_3}(\mathcal{B}_0,\mathbb{R})}-C\mathcal{L}^3(\mathcal{B}_0)\\
&\quad+\tfrac{l_c}{2}\|s\|^2_{H^1(\mathcal{B}_0)}-\tfrac{1}{2l_c}\mathcal{L}^3(\mathcal{B}_0)-\Big(\tfrac{C_{\boldsymbol{h}}}{p'\varepsilon}\Big)^{p'}-\tfrac{\varepsilon^p}{p}\|\boldsymbol{\varphi}_k(t)\|^p_{W^{1,p}(\mathcal{B}_0,\mathbb{R}^3)}\big)\\
&\geq\big(\tfrac{c_1}{C_g^p}(2^{1-p}\|\nabla(y_k-\mathrm{id})\|^p_{L^p(\mathcal{B}_0,\mathbb{R}^{3\times3})}-3^{\frac1p}\mathcal{L}^3(\mathcal{B}_0))+c_2\|\mathrm{cof}\nabla\boldsymbol{\varphi}_k\|^{p_2}_{L^{p_2}(\mathcal{B}_0,\mathbb{R}^{3\times3})}\\
&\quad+c_3\|\det\nabla\boldsymbol{\varphi}_k\|^{p_3}_{L^{p_3}(\mathcal{B}_0,\mathbb{R})}-C\mathcal{L}^3(\mathcal{B}_0)+\tfrac{l_c}{2}\|s\|^2_{H^1(\mathcal{B}_0)}-\tfrac{\varepsilon^p}{p}(\|y_k\|^p_{W^{1,p}(\mathcal{B}_0,\mathbb{R}^3)}+1)-\tilde{B}\big)\\
&\geq\Big(\tfrac{c_1}{C_g^pC_F^p}-\tfrac{\varepsilon^p}{p}\Big)\|y_k\|^p_{W^{1,p}(\mathcal{B}_0,\mathbb{R}^3)}+c_2\|\mathrm{cof}\nabla\boldsymbol{\varphi}_k\|^{p_2}_{L^{p_2}(\mathcal{B}_0,\mathbb{R}^{3\times3})}+c_3\|\det\nabla\boldsymbol{\varphi}_k\|^{p_3}_{L^{p_3}(\mathcal{B}_0,\mathbb{R})}\\
&\quad+\tfrac{l_c}{2}\|s_k\|^2_{H^1(\mathcal{B}_0)}-B,
\end{aligned}$$

which states (3.38).

Proof of 2.: To establish the weak sequential compactness of the energy sublevels we now consider a sequence $(y_k,s_k)_{k\in\mathbb{N}}\subset\mathbf{Y}\times\mathbf{Z}$ with $\mathcal{E}(t,\boldsymbol{\varphi}_k,s_k)\leq C$ uniformly for all $k\in\mathbb{N}$. Coercivity estimate (3.10d) thus allows us to employ Prop. 3.7, which, by Proposition (3.5), implies the existence of a subsequence $y_k\rightharpoonup y$ in $\mathbf{Y}$, such that $\boldsymbol{\varphi}_k\rightharpoonup\boldsymbol{\varphi}$ in $\mathbf{U}$, $\mathrm{cof}\nabla\boldsymbol{\varphi}_k\rightharpoonup\mathrm{cof}\nabla\boldsymbol{\varphi}$ in $L^{p_2}(\mathcal{B}_0,\mathbb{R}^3\times3)$, and $\det\nabla\boldsymbol{\varphi}_k\rightharpoonup\det\nabla\boldsymbol{\varphi}$ in $L^{p_3}(\mathcal{B}_0,\mathbb{R})$. In addition, coercivity estimate (3.10d) allows us to select fr Moreover, we also find a subsequence $s_k\rightharpoonup s$ in $\mathbf{Z}$. It thus remains to deduce the weak sequential lower semicontinuity of each of the contributions of $\mathcal{E}$.

To establish the weak sequential lower semicontinuity of the functional $\int_{\mathcal{B}_0} \mathcal{G}_c\gamma(\cdot,\cdot)\,\mathrm{d}X : \mathbf{Z}\times\mathbf{Z} \to [0,\infty)$ with γ from (2.3), we first note that $\gamma \in \mathrm{C}^1(\mathbb{R}\times\mathbb{R}^3,\mathbb{R})$ and bounded from below by 0. Moreover, the gradient term is strictly convex and the compact embedding $\mathbf{Z} \Subset L^2(\Omega)$ will ensure that $s_k \to s$ strongly in $L^2(\mathcal{B}_0)$ if $s_k \rightharpoonup s$ in $\mathbf{Z}$. Hence, the weak sequential lower semicontinuity of the integral functional follows by [Dac89, Sec. 3, Thm. 3.4].

Assumption (3.18) and Lemma 3.5 ensure that $\int_{\partial\mathcal{B}_0^{\mathrm{N}}} \boldsymbol{h}(t)\cdot\boldsymbol{g}(t,y_k)\,\mathrm{d}X \to \int_{\partial\mathcal{B}_0^{\mathrm{N}}} \boldsymbol{h}(t)\cdot\boldsymbol{g}(t,y)\,\mathrm{d}X.$ Further taking into account Hypotheses (3.10b)–(3.10d), which state that W is a Carathéodory-function, polyconvex and bounded from below for every $\boldsymbol{F}\in\mathrm{GL}_+(3)$, the weak sequential lower semicontinuity of $\int_{\mathcal{B}_0}\beta(\cdot)W_1(\mathbb{M}(\cdot))+W_2(\mathbb{M}(\cdot))\,\mathrm{d}X$ can be obtained by applying weak lower semicontinuity results for the convex case, cf. [Dac89, Sec. 3, Thm. 3.4]. □

We are now in the position to verify the existence of minimizers for problem (3.4) via the direct method of the calculus of variations.

Proof of Prop. 3.9, Item 1: Asume that $s_N^0\in[0,1]$ a.e. in $\mathcal{B}_0$. Thus, $\mathcal{E}(t_N^1,\cdot,s_N^0):\mathbf{U}\to\mathbb{R}$ is well-defined. We conclude the existence of a minimizer $y_N^1\in\mathbf{Y}$ such that $\boldsymbol{\varphi}_N^1=\boldsymbol{g}(t_N^1,y_n^1)$ via the direct method of the calculus of variations by applying Lemma 3.12 to $\mathcal{E}(t_N^1,\cdot,s_N^0)$. Let $t_N^k\in\Pi_N$ fixed and assume that $s_N^{k-1}\in[0,1]$ (which we will show in Item 2 by induction). Again, $\mathcal{E}(t_N^k,\cdot,s_N^{k-1}):\mathbf{U}\to\mathbb{R}$ is well-defined and we may deduce the existence of a minimizer $y_N^k\in\mathbf{Y}_N$ such that $\boldsymbol{\varphi}_N^k=\boldsymbol{g}(t_N^k,y_n^k)$ via the direct method by applying Lemma 3.12 to $\mathcal{E}(t_N^k,\cdot,s_N^{k-1})$. Similarly, the existence of a minimizer s_N^k follows from Lemma 3.12 applied to the functional $\mathcal{E}(t_N^k,\boldsymbol{\varphi}_N^k,\cdot)+\tau_N\mathcal{V}_{\alpha N}\big(\frac{(\cdot)-s_N^{k-1}}{\tau_N}\big)$. For this, note that $\mathcal{V}_{\alpha N}\big(\frac{(\cdot)-s_N^{k-1}}{\tau_N}\big)$ only contains quadratic, convex lower order terms. ■

Proof of Prop. 3.9, Item 2: We proceed by contradiction. For this, suppose that $s_N^{k-1}\in[0,1]$ a.e. in $\mathcal{B}_0$ but that there exist sets $B^0,B^1\subset\mathcal{B}_0$ with $\mathcal{L}^3(B^0),\mathcal{L}^3(B^1)>0$ such that $s_N^k<0$ a.e. on B^0 and $s_N^k>1$ a.e. on B^1. We test the minimality (3.7b) by $\tilde{s}:=\min\{1,\max\{0,s_N^k\}\}$, which is an admissible testfunction according to [MM79]. In view of (3.10h), (3.10i), (3.1), and (3.6) we thus have

$$\begin{aligned}
\beta(s_N^k)W_1(\mathbb{M}\nabla\boldsymbol{\varphi}_N^k) &\geq \beta(\tilde{s})W_1(\mathbb{M}\nabla\boldsymbol{\varphi}_N^k),\\
(1-s_N^k)^2 &\geq (1-\tilde{s})^2,\\
\int_{\mathcal{B}_0}|\nabla s_N^k|\,\mathrm{d}X &\geq \int_{\mathcal{B}_0}|\nabla\tilde{s}|\,\mathrm{d}X,\\
\mathcal{V}_{\alpha N}\big(\tfrac{s_N^k-s_N^{k-1}}{\tau_N}\big) &\geq \mathcal{V}_{\alpha N}\big(\tfrac{\tilde{s}-s_N^{k-1}}{\tau_N}\big).
\end{aligned}$$

Here, the first inequality follows from the monotonicity of β on $[0,\infty)$ and $(-\infty,0]$. To see the second inequality observe that $(1-s)^2>1=(1-\tilde{s})^2$ for $s<0$ and $(1-s)^2>0=(1-\tilde{s})^2$ for $s>1$. The third inequality follows from [MM79, Sec. 2], which implies that $\nabla\tilde{s}=0$ on $B=B^0\cup B^1$ and $\nabla\tilde{s}=\nabla s_N^k$ on $\mathcal{B}_0\backslash B$. The fourth inequality ensues from

$$\begin{aligned}
&\text{for } s_N^k<0,\, s_N^{k-1}\geq 0: \big(\tfrac{s_N^k-s_N^{k-1}}{\tau_N}\big)^2 \geq \big(\tfrac{-s_N^{k-1}}{\tau_N}\big)^2 \quad\text{and}\quad 0=\big(\tfrac{s_N^k-s_N^{k-1}}{\tau_N}\big)_+^2 \geq \big(\tfrac{-s_N^{k-1}}{\tau_N}\big)_+^2=0,\\
&\text{for } s_N^k>1,\, s_N^{k-1}\leq 1: \big(\tfrac{s_N^k-s_N^{k-1}}{\tau_N}\big)^2 \geq \big(\tfrac{1-s_N^{k-1}}{\tau_N}\big)^2 \quad\text{and}\quad \big(\tfrac{s_N^k-s_N^{k-1}}{\tau_N}\big)_+^2 \geq \big(\tfrac{1-s_N^{k-1}}{\tau_N}\big)_+^2.
\end{aligned}$$

Alltogether, the four relations imply

$$\mathcal{E}(t_N^k, \boldsymbol{\varphi}_N^k, \tilde{s}) + \mathcal{V}_{\alpha N}(\tfrac{\tilde{s}-s_N^{k-1}}{\tau_N}) < \mathcal{E}(t_N^k, \boldsymbol{\varphi}_N^k, s_N^k) + \mathcal{V}_{\alpha N}(\tfrac{s_N^k - s_N^{k-1}}{\tau_N}),$$

which is in contradiction to the minimality of s_N^k and hence $s_N^k \in [0,1]$ a.e. in $\mathcal{B}_0$.

Note that the above contradicition argument holds in particular for $k = 1$ under the assumption that $s_N^0 \in [0,1]$ a.e. in $\mathcal{B}_0$, i.e., in this case we find that $s_N^1 \in [0,1]$ a.e.. Therefore, the above argument allows us to conclude the statement by induction. ■

3.3.2 Proof of Prop. 3.10

We start with the proof of the discrete notion of solution (3.33).

Proof of (3.33a)**:** We observe that a minimizer $\boldsymbol{\varphi}_N^k$ of problem (3.7a) equivalently satisfies for all $\tilde{\boldsymbol{\varphi}} \in \mathbf{U}$

$$\mathcal{E}(t_N^k, \boldsymbol{\varphi}_N^k, s_N^{k-1}) \leq \mathcal{E}(t_N^k, \tilde{\boldsymbol{\varphi}}, s_N^{k-1}). \tag{3.39}$$

Applying the definition of the interpolants (3.32) we find (3.33a).

Proof of (3.33b)**:** We use that a minimizer s_N^k of problem (3.7b) satisfies the corresponding Euler-Lagrange equations for all $\tilde{s} \in \mathbf{X}$:

$$\begin{aligned}\int_{\mathcal{B}_0} \Big(\big(\beta'(s_N^k) W_1(\mathbb{M}\nabla\boldsymbol{\varphi}_N^k(t_N^k)) \big) - \tfrac{1}{l_c}(1 - s_N^k) + M^{-1}(\tfrac{s_N^k - s_N^{k-1}}{\tau_N}) \\ + \alpha N(\tfrac{s_N^k - s_N^{k-1}}{\tau_N})_+ \big) \tilde{s} + l_c \nabla s_N^k : \nabla\tilde{s} \Big) \, \mathrm{d}X = 0 \,.\end{aligned}$$

Again using the definition of the interpolants (3.32) we find (3.33b).

Proof of (3.34)**:** In order to find the energy dissipation estimate we test the minimality of s_N^k in (3.7b) by s_N^{k-1}, exploit the minimality of y_N^k, and add and subtract $\mathcal{E}(t_N^{k-1}, \boldsymbol{\varphi}_N^{k-1}, s_N^{k-1})$,

$$\begin{aligned} &\mathcal{E}(t_N^k, \boldsymbol{\varphi}_N^k, s_N^k) + \tau_N \mathcal{V}_{\alpha N}(\tfrac{s_N^k - s_N^{k-1}}{\tau_N}) \leq \mathcal{E}(t_N^k, \boldsymbol{\varphi}_N^k, s_N^{k-1}) \\ &\leq \mathcal{E}(t_N^k, \boldsymbol{\varphi}_N^{k-1}, s_N^{k-1}) + \mathcal{E}(t_N^{k-1}, \boldsymbol{\varphi}_N^{k-1}, s_N^{k-1}) - \mathcal{E}(t_N^{k-1}, \boldsymbol{\varphi}_N^{k-1}, s_N^{k-1}) \\ &= \mathcal{E}(t_N^{k-1}, \boldsymbol{\varphi}_N^{k-1}, s_N^{k-1}) + \int_{t_N^{k-1}}^{t_N^k} \partial_\tau \mathcal{E}(\tau, \boldsymbol{g}(\tau, y_N^{k-1}), s_N^{k-1}) \, \mathrm{d}\tau \,. \end{aligned} \tag{3.40}$$

Consider now $t \in (t_N^{m-1}, t_N^m]$. Then summing up the above relation over $k \in \{1, \dots, m\}$, results in

$$\mathcal{E}(t_N^m, \boldsymbol{\varphi}_N^m, s_N^m) + \sum_{k=1}^m \tau_N \mathcal{V}_{\alpha N}(\tfrac{s_N^k - s_N^{k-1}}{\tau_N}) \leq \mathcal{E}(t_N^0, \boldsymbol{\varphi}_N^0, s_N^0) + \sum_{k=1}^m \int_{t_N^{k-1}}^{t_N^k} \partial_\tau \mathcal{E}(\tau, \boldsymbol{g}(\tau, y_N^{k-1}), s_N^{k-1}) \, \mathrm{d}\tau \,.$$

Again by the definition of the interpolants and using that $t \in (t_N^{m-1}, t_N^m]$ we see that this relation is equivalent to (3.34).

Proof of (3.35)**:** We now want to exploit the previously obtained discrete estimate (3.40) to deduce the apriori estimates (3.35). To do so, we apply (3.27) under the integral of (3.40). This allows us to apply the classical Gronwall inequality and, following the arguments of e.g. [FM06], one finds for every $m \in \{1, \dots, N\}$

$$\mathcal{E}(t_N^m, \boldsymbol{\varphi}_N^m, s_N^m) + \sum_{k=1}^{m} \mathcal{V}_{\alpha N}\Big(\frac{s_N^k - s_N^{k-1}}{\tau_N}\Big) \le C\,. \tag{3.41}$$

This translates into (3.35a) and, thanks to (3.38), also yields the estimates (3.35b)–(3.35d). Moreover, in view of $s_N^0 \in [0,1]$, estimate (3.35e) is due to Prop. 3.9, Item 2. Now, thanks to the properties (3.10h) of β it is $\beta'(s) = 2s \le \frac{2}{\eta}\beta(s)$ for $s \in [0,1]$. Thus, together with the properties (3.10) of W_1, W_2 and coercivity estimate (3.38) we can verify that there is a constant $c_1, c_2 > 0$ such that for all $\tilde{s} \in \mathbf{X}$ we have

$$\Big| \int_{\mathcal{B}_0} \beta'(\overline{s}_N(t)) W_1(\mathbb{M}\nabla\overline{\boldsymbol{\varphi}}_N(t))\tilde{s}\,\mathrm{d}X \Big| \le \|\tilde{s}\|_{\mathbf{X}} \big(c_1 \mathcal{E}(t, \overline{\boldsymbol{\varphi}}_N(t), \overline{s}_N(t)) + c_2\big)\,.$$

This proves (3.35f). ■

3.3.3 Proof of Theorem 3.11

Proof of convergences (3.36)**:** In view of (3.35c) we find subsequence and a limit $s \in H^1(0,T;L^2(\mathcal{B}_0))$ such that (3.36a) holds true. Similarly, by (3.35d) & (3.35e) we find further subequences and $\overline{s}, \underline{s} \in \mathbf{X}$, such that also $\overline{s} \overset{*}{\rightharpoonup} \overline{s}$ and $\underline{s} \overset{*}{\rightharpoonup} \underline{s}$ in $L^\infty(0,T;\mathbf{X})$. Since $s_N^\ell(t) - \overline{s}_N(t) = (t - t_N^k)\dot{s}_N^\ell(t)$ and $s_N^\ell(t) - \underline{s}_N(t) = (t - t_N^{k-1})\dot{s}_N^\ell(t)$, we deduce from convergence (3.36a) that in fact $s = \overline{s} = \underline{s}$ in $L^\infty(0,T;H^1(\mathcal{B}_0))$. This proves convergences (3.36b) & (3.36c) due to the pointwise bounds in time (3.35d) & (3.35e). Convergences (3.36d) & (3.36e) also follow by the boundedness in $L^\infty(0,T;\mathbf{Y})$ implied by the pointwise in time bounds (3.35b). Here, the limits $y(t)$ and $\underline{y}$ of the two sequences must not coincide.

It remains to verify the convergence of $W_i(\mathbb{M}\nabla\overline{\boldsymbol{\varphi}}_N(t))$, i.e., (3.36f). For this we test the minimality of $\overline{\boldsymbol{\varphi}}_N(t)$ in (3.33a) by the limit $\boldsymbol{\varphi}(t)$ at any time $t \in (0,T)$, where (3.36d) holds true. Based on this we argue that

$$\begin{aligned}
&\limsup_{N\to\infty} \int_{\mathcal{B}_0} \big(\beta(\underline{s}_N(t)) W_1(\mathbb{M}\nabla\overline{\boldsymbol{\varphi}}_N(t)) + W_2(\mathbb{M}\nabla\overline{\boldsymbol{\varphi}}_N(t))\big)\,\mathrm{d}X \\
&\le \int_{\mathcal{B}_0} \big(\beta(s(t)) W_1(\mathbb{M}\nabla\boldsymbol{\varphi}(t)) + W_2(\mathbb{M}\nabla\boldsymbol{\varphi}(t))\big)\,\mathrm{d}X \\
&\qquad + \limsup_{N\to\infty} \int_{\partial\mathcal{B}_0^{\mathrm{N}}} \boldsymbol{h}\cdot(\boldsymbol{\varphi}(t) - \boldsymbol{\varphi}_N(t))\,\mathrm{d}X + \limsup_{N\to\infty} \int_{\mathcal{B}_0} (\beta(\overline{s}_N(t)) - \beta(s(t))) W_1(\mathbb{M}\nabla\boldsymbol{\varphi}(t))\,\mathrm{d}X \\
&= \int_{\mathcal{B}_0} \big(\beta(s(t)) W_1(\mathbb{M}\nabla\boldsymbol{\varphi}(t)) + W_2(\mathbb{M}\nabla\boldsymbol{\varphi}(t))\big)\,\mathrm{d}X\,.
\end{aligned} \tag{3.42}$$

Here, the convergence of the Neumann boundary-term follows by weak strong convergence arguments taking into account (3.36d) and Prop. 3.7. The convergence of the quantity $(\beta(\overline{s}_N(t)) -$

$\beta(s(t)))W_1(\mathbb{M}\nabla\boldsymbol{\varphi}(t)) \to 0$ in $L^1(\mathcal{B}_0)$ ensues by the dominated convergence theorem, using that convergence (3.36c) implies convergence in measure and that $(1+\eta)W_1(\mathbb{M}\nabla\boldsymbol{\varphi}(t))+W_2(\mathbb{M}\nabla\boldsymbol{\varphi}(t)$ provides a majorant. Since all terms are non-negative and $\beta(\cdot)\eta > 0$, estimate (3.42) implies that also

$$\limsup_{N\to\infty}\int_{\mathcal{B}_0}\big(W_1(\mathbb{M}\nabla\overline{\boldsymbol{\varphi}}_N(t))+W_2(\mathbb{M}\nabla\overline{\boldsymbol{\varphi}}_N(t))\big)\,\mathrm{d}X \le \int_{\mathcal{B}_0}\big(W_1(\mathbb{M}\nabla\boldsymbol{\varphi}(t))+W_2(\mathbb{M}\nabla\boldsymbol{\varphi}(t))\big)\,\mathrm{d}X\,. \tag{3.43}$$

On the other hand, we also have by lower semicontinuity

$$\liminf_{N\to\infty}\int_{\mathcal{B}_0}\big(W_1(\mathbb{M}\nabla\overline{\boldsymbol{\varphi}}_N(t))+W_2(\mathbb{M}\nabla\overline{\boldsymbol{\varphi}}_N(t))\big)\,\mathrm{d}X \ge \int_{\mathcal{B}_0}\big(W_1(\mathbb{M}\nabla\boldsymbol{\varphi}(t))+W_2(\mathbb{M}\nabla\boldsymbol{\varphi}(t))\big)\,\mathrm{d}X\,. \tag{3.44}$$

Hence, (3.36f) is proven.

Proof of the minimality condition (3.37a)**:** Thanks to convergence (3.36d) we find by Prop. 3.7 the weak convergences of the corresponding minors. Additionally, convergence (3.36b) yields the strong convergence $\underline{s}_N(t) \to s(t)$ in $L^2(\mathcal{B}_0)$ for all $t \in [0,T]$, which in turn implies convergence in measure. Using that $\big(\beta(\underline{s}_N(t))W_1(\mathbb{M}\nabla\tilde{\boldsymbol{\varphi}}(t)) + W_2(\mathbb{M}\nabla\tilde{\boldsymbol{\varphi}}(t))\big) \le \big(W_1(\mathbb{M}\nabla\tilde{\boldsymbol{\varphi}}(t)) + W_2(\mathbb{M}\nabla\tilde{\boldsymbol{\varphi}}(t))\big)$ we have found a convergent majorant, which allows us to apply the dominated convergence theorem and to pass to the limit on the right-hand side of (3.33a) by continuity. In turn, the limit passage on the left-hand side of (3.33a) is done by weak lower semicontinuity using (3.36d). Observe that (3.36d) only holds on $(0,T)\backslash\mathrm{N}$ with $\mathcal{L}^3(\mathrm{N}) = 0$. We can define $y(t)$ for $t \in \mathrm{N}$ by choosing $y(t) \in \mathbf{Y}$ such that $\boldsymbol{\varphi}(t) = \boldsymbol{g}(t, y(t)) \in \mathrm{argmin}_{\tilde{\boldsymbol{\varphi}}\in\mathbf{U}(t)}\mathcal{E}(t,\tilde{\boldsymbol{\varphi}}, s(t))$. Moreover, it has to be noted that, due to polyconvexity, i.e., the lack of (strict) convexity the uniqueness of minimizers is not guaranteed, so that the definition of $y(t)$ is not unique. This proves (3.37a).

Proof of the evolution equation (3.37b) **for** $\alpha = 0$**:** Let now $\alpha = 0$ and we want to show (3.37b). In view of convergences (3.36), we may apply weak-strong convergence arguments to pass to the limit in (3.33b) as an equality, i.e., we find for a.e. $t \in (0,T)$, for every $\tilde{s} \in \mathbf{X}$

$$\int_{\mathcal{B}_0}\Big(\big(\beta'(s(t))W_1(\mathbb{M}\nabla\boldsymbol{\varphi}(t))(t) - \tfrac{1}{l_c}(1-s(t)) + M^{-1}\dot{s}\big)\tilde{s} + l_c\nabla s(t) : \nabla\tilde{s}\Big)\,\mathrm{d}X = 0\,. \tag{3.45}$$

More precisely, to obtain the first term we apply the dominated convergence theorem arguing that $\beta'(\overline{s}_N(t))W_1(\mathbb{M}\nabla\overline{\boldsymbol{\varphi}}_N(t)))\tilde{s} \to s(t))W_1(\mathbb{M}\nabla\boldsymbol{\varphi}(t)))\tilde{s}$ in measure, thanks to (3.36c) & (3.36f), and that $W_1(\mathbb{M}\nabla\overline{\boldsymbol{\varphi}}_N(t)))|\tilde{s}|$ provides a convergent majorant for $\tilde{s} \in \mathbf{X}$.

Proof of the evolution equation (3.37c) **for** $\alpha > 0$**:** Let now $\alpha > 0$. To show the validity of (3.37c) we observe that $\alpha N(\dot{s}_N^\ell)_+\tilde{s} \le 0$ for every $\tilde{s} \in \mathbf{X}$ with $\tilde{s} \le 0$ a.e. in $\mathcal{B}_0$. Hence, when moving this term to the other side of the equation, we find that (3.33b) can be reformulated as a variational inequality, i.e., for all $\tilde{s} \in \mathbf{X}$ with $\tilde{s} \le 0$ a.e. in $\mathcal{B}_0$:

$$0 \le \int_{\mathcal{B}_0}\Big(\big(\beta'(\overline{s}_N(t))W_1(\mathbb{M}\nabla\overline{\boldsymbol{\varphi}}_N(t)))-\tfrac{1}{l_c}(1-\overline{s}_N)+M^{-1}\dot{s}_N^\ell\big)\tilde{s}+l_c\nabla\overline{s}_N(t) : \nabla\tilde{s}\Big)\,\mathrm{d}X\,. \tag{3.46}$$

We can then pass to the limit on the right-hand side of (3.46) using convergences (3.36) and weak-strong convergence arguments and arguing by dominated convergence for the first term, as in the case $\alpha = 0$.

Proof of nonpositivity (3.37d) **if** $\alpha > 0$**:** From the third bound in (3.35a), we gather that

$$\int_0^T \int_{\mathcal{B}_0} (\dot{s}_N^\ell)_+ \,\mathrm{d}X\,\mathrm{d}t \le \frac{C}{\alpha N} \to 0 \text{ as } N \to \infty\,.$$

By weak lower semicontinuity and convergence (3.36a) we conclude that

$$0 = \liminf_{N\to\infty} \int_0^T \int_{\mathcal{B}_0} (\dot{s}_N^\ell)_+ \,\mathrm{d}X\,\mathrm{d}t \ge \int_0^T \int_{\mathcal{B}_0} (\dot{s})_+ \,\mathrm{d}X\,\mathrm{d}t\,,$$

which implies that $\dot{s} \le 0$ a.e. in $(0,T)$, a.e. in $\mathcal{B}_0$.

Proof of the energy-dissipation estimate (3.37f)**:** Thanks to convergences (3.36) we can pass to the limit on the left-hand side of the dicrete energy-dissipation estimate (3.34) by lower semicontinuity arguments, also using in the case $\alpha > 0$ that $\mathcal{V}_{\alpha N}(\dot{s}_N^\ell(t)) \ge \mathcal{V}_0(\dot{s}_N^\ell(t))$ and $\mathcal{V}_\alpha(\dot{s}(t)) = \mathcal{V}_0(\dot{s}(t))$ since $\dot{s}(t) \le 0$ for a.e. $t \in (0,T)$ by (3.37d). On the right-hand side, the energy at initial time is constant wrt. $N \in \mathbb{N}$ and we only have to take care about the limit passage in the powers of the external loadings. For this, we want to show that

$$\limsup_{N\to\infty} \int_0^t \partial_\tau \mathcal{E}(\tau, \underline{\boldsymbol{\varphi}}_N(\tau), \underline{s}_N(\tau)\,\mathrm{d}\tau \le \int_0^t \mathcal{P}(\tau, s(\tau))\,\mathrm{d}\tau\,, \tag{3.47}$$

where $\mathcal{P}(\tau, s(\tau)) := \sup\{\partial_\tau \mathcal{E}(\tau, \hat{\boldsymbol{\varphi}}, s(\tau)),\ \hat{\boldsymbol{\varphi}} \in \mathrm{argmin}\mathcal{E}(\tau, \cdot, s(\tau))\}$ is introduced as a surrogate for the partial time derivative from (3.26). We can conclude (3.47) if we first show that

$$\underline{\boldsymbol{\varphi}}(\tau) \text{ is a minimizer of } \mathcal{E}(\tau, \cdot, s(\tau)) \tag{3.48}$$

and secondly verify that

$$\int_0^t \partial_\tau \mathcal{E}(\tau, \underline{\boldsymbol{\varphi}}_N(\tau), \underline{s}_N(\tau))\,\mathrm{d}\tau \to \int_0^t \partial_\tau \mathcal{E}(\tau, \underline{\boldsymbol{\varphi}}(\tau), s(\tau))\,\mathrm{d}\tau\,. \tag{3.49}$$

Clearly, these two properties imply (3.47) due to the definition of $\mathcal{P}$. In addition, by the power control estimate (3.27), we see that $\int_0^t \mathcal{P}(\tau, s(\tau))\,\mathrm{d}\tau$ is well-defined and finite.

We now prove statement (3.48). For this, we introduce a further interpolant, i.e.

$$\underline{\underline{s}}_N(t) := s_N^{k-1} \text{ for all } t \in [t_N^k, t_N^{k+1}) \text{ for } k \in \{1,\ldots,N\}, \underline{\underline{s}}_N(t) := s_N^0 \text{ for all } t \in [t_N^0, t_N^1)\,, \tag{3.50}$$

which thus satisfies $\underline{\underline{s}}_N(t) = \underline{s}_N(t-\tau_N) = \overline{s}_N(t-2\tau_N)$ for $t \in [t_N^k, t_N^{k+1})$ and all $k \in \{1,\ldots,N\}$. With similar arguments as for the proof of convergences (3.36b) & (3.36c) we find that

$$\underline{\underline{s}}_N(t) \overset{*}{\rightharpoonup} s(t) \text{ in } L^\infty(0,T;\mathbf{X})\,. \tag{3.51}$$

Using the interpolant $\underline{\underline{s}}_N$, we can rewrite minimizality condition (3.39) for all $\tilde{\boldsymbol{\varphi}} \in \mathbf{U}(\underline{t}_N(t))$ as

$$\mathcal{E}(\underline{t}_N(t), \underline{\boldsymbol{\varphi}}_N(t), \underline{\underline{s}}_N(t)) \le \mathcal{E}(\underline{t}_N(t), \tilde{\boldsymbol{\varphi}}, \underline{\underline{s}}_N(t))\,. \tag{3.52}$$

Using convergences (3.36e) & (3.51) and by repeating the arguments of the proof of minimality condition (3.37a) we conclude (3.48).

We now turn to the proof of the convergence of the powers of the energy (3.47). For this, we will adapt the arguments of [Tho10, Sec. 3] and [FM06, Prop. 3.3] to the present, rate-dependent situation. More precisely, for $\mathcal{I} : [0,T] \times \mathbf{U} \times \mathbf{Z}$, $\mathcal{I}(t,\boldsymbol{\varphi},s) := \int_{\mathcal{B}_0} W(\nabla\boldsymbol{\varphi},s)\,\mathrm{d}X - \int_{\partial\mathcal{B}_0^{\mathrm{N}}} \boldsymbol{h}\cdot\boldsymbol{\varphi}\,\mathrm{d}X$ we shall show in Lemma 3.13 below that

1 It holds $\mathcal{I}(t,\boldsymbol{\varphi}_m,s_m) \to \mathcal{I}(t,\boldsymbol{\varphi},s)$ for every sequence $s_m \rightharpoonup s$ in $\mathbf{X}$ and $\boldsymbol{\varphi}_m \rightharpoonup \boldsymbol{\varphi}$ in $\mathbf{U}$ such that $\boldsymbol{\varphi}_m \in \operatorname{argmin}\{\mathcal{I}(t_m,\tilde{\boldsymbol{\varphi}},s_m),\ \tilde{\boldsymbol{\varphi}} \in \mathbf{U}\}$.

2 For every pair (y,s) such that $\mathcal{E}(0,\boldsymbol{g}(0,y),s) < E$ the derivative $\partial_t\mathcal{E}(\cdot,\boldsymbol{g}(\cdot,y),s) = \partial_t\mathcal{I}(\cdot,\boldsymbol{g}(\cdot,y),s)$ is uniformly continuous.

The lower semicontinuity of $\mathcal{I}(t,\cdot,\cdot)$ in $\mathbf{U}\times\mathbf{Z}$ together with the above Items 1& 2 will allow us to apply [FM06, Prop. 3.3], which then implies that $\partial_t\mathcal{I}(t,\boldsymbol{\varphi}_m,s_m) \to \partial_t\mathcal{I}(t,\boldsymbol{\varphi},s)$. In other words, this result allows us to conclude that $\partial_\tau\mathcal{E}(\tau,\underline{\boldsymbol{\varphi}}(\tau),\underline{s}(\tau)) \to \partial_\tau\mathcal{E}(\tau,\boldsymbol{\varphi}(\tau),s(\tau))$ pointwise in $\tau\in[0,T]$. Using again the power control (3.27) providing an integrable majorant, the dominated convergence theorem yields statement (3.49). Hence the upper energy-dissipation estimate (3.37f) is proven, so that the proof of Thm. 3.11 is concluded. ■

We now provide the following result, which was used for the proof of the upper energy dissipation estimate (3.37f):

Lemma 3.13 (Convergence of the energies and powers)**.** *Let the assumptions of Thm. 3.11 be satisfied and denote* $\mathcal{I} : [0,T] \times \mathbf{U} \times \mathbf{Z}$, $\mathcal{I}(t,\boldsymbol{\varphi},s) := \int_{\mathcal{B}_0} W(\nabla\boldsymbol{\varphi},s)\,\mathrm{d}X - \int_{\partial\mathcal{B}_0^{\mathrm{N}}} \boldsymbol{h}\cdot\boldsymbol{\varphi}\,\mathrm{d}X$*. Then, the following statements hold true:*

1 It holds $\mathcal{I}(t,\boldsymbol{\varphi}_m,s_m) \to \mathcal{I}(t,\boldsymbol{\varphi},s)$ *for every sequence* $s_m \rightharpoonup s$ *in* $\mathbf{X}$ *and* $\boldsymbol{\varphi}_m \rightharpoonup \boldsymbol{\varphi}$ *in* $\mathbf{U}$ *such that* $\boldsymbol{\varphi}_m \in \operatorname{argmin}\{\mathcal{I}(t_m,\tilde{\boldsymbol{\varphi}},s_m),\ \tilde{\boldsymbol{\varphi}} \in \mathbf{U}\}$.

2 For every pair (y,s) *such that* $\mathcal{E}(0,\boldsymbol{g}(0,y),s) < E$ *the derivative* $\partial_t\mathcal{E}(\cdot,\boldsymbol{g}(\cdot,y),s) = \partial_t\mathcal{I}(\cdot,\boldsymbol{g}(\cdot,y),s)$ *is uniformly continuous, i.e., for each* $E,\varepsilon>0$ *there is* $\delta>0$ *such that for all* (y,s) *with* $\mathcal{E}(0,\boldsymbol{g}(0,y),s) < E$ *it is*

$$|\partial_t\mathcal{I}(t,\boldsymbol{g}(\cdot,y),s) - \partial_t\mathcal{I}(\tau,\boldsymbol{g}(\cdot,y),s)| < \varepsilon \ \text{ if } |t-\tau| < \delta\,. \tag{3.53}$$

Proof. We start with the proof Item 1. Consider a sequence $s_m \rightharpoonup s$ in $\mathbf{X}$ with $\|s_m\|_{L^\infty(\mathcal{B}_0)} \le 1$ and $y \in \mathbf{X}$ such that $\mathcal{E}(0,\boldsymbol{g}(0,y),s_1) < E$. Then we find that $\mathcal{I}(t,\boldsymbol{g}(t,y),s_m) \to \mathcal{I}(t,\boldsymbol{g}(t,y),s)$. Note that the convergence of the Neumann boundary terms is due to the assumptions (3.17) & (3.18). Moreover, the convergence of the bulk term follows from the dominated convergence theorem, since $W(t,\boldsymbol{g}(t,y),s_m) \to W(t,\boldsymbol{g}(t,y),s)$ in measure thanks to convergence (3.36c) and since $W(t,\boldsymbol{g}(t,y),1)$ provides an integrable majorant. Moreover, $\boldsymbol{\varphi}_m$ minimizes $\mathcal{I}(t_m,\cdot,s_m)$. Hence, by assumptions (3.17) & (3.18) there is a constant E such that $\mathcal{I}(t,\boldsymbol{\varphi}_m,s_m) < E$ for all $t\in[0,T]$ and, by lower semicontinuity of $\int_{\mathcal{B}_0}(W_1(\cdot)+W_2(\cdot))\,\mathrm{d}X$ also $\mathcal{I}(t,\boldsymbol{\varphi},s_m) < E$ for all $t\in[0,T]$ and $m\in\mathbb{N}$. Thus (3.28) holds and we infer

$$\mathcal{I}(t,\boldsymbol{\varphi}_m,s_m) - c_E|t_m-t| \le \mathcal{I}(t_m,\boldsymbol{\varphi}_m,s_m) \le \mathcal{I}(t_m,\boldsymbol{\varphi},s_m) \le \mathcal{I}(t,\boldsymbol{\varphi},s_m) + c_E|t_m-t| \to \mathcal{I}(t,\boldsymbol{\varphi},s)\,;$$

here the first and the third inequality follow from (3.28) and the second inequality is due to the minimality property of $\boldsymbol{\varphi}_m$ for $\mathcal{I}(t_m,\cdot,s_m)$. We conclude $\mathcal{I}(t_m,\boldsymbol{\varphi}_m,s_m)\to\mathcal{I}(t,\boldsymbol{\varphi},s)$ exploiting the weak sequential lower semicontinuity

$$\begin{aligned}\mathcal{I}(t,\boldsymbol{\varphi},s)&\leq\liminf_{m\to\infty}\big(\mathcal{I}(t,\boldsymbol{\varphi}_m,s_m)-c_E|t_m-t|\big)\leq\liminf_{m\to\infty}\mathcal{I}(t_m,\boldsymbol{\varphi}_m,s_m)\\&\leq\limsup_{m\to\infty}\mathcal{I}(t_m,\boldsymbol{\varphi}_m,s_m)\leq\limsup_{m\to\infty}\mathcal{I}(t,\boldsymbol{\varphi},s_m)\\&\leq\limsup_{m\to\infty}\big(\mathcal{I}(t,\boldsymbol{\varphi},s_m)+c_E|t_m-t|\big)=\mathcal{I}(t,\boldsymbol{\varphi},s)\,.\end{aligned}$$

Hence Item 1 of the Lemma is verified.

We now prove Item 2. Consider (y,s) such that $\mathcal{E}(0,\boldsymbol{g}(0,y),s)<E$. Due to (3.17) and (3.18) we find for every $\tilde{\varepsilon}>0$ a $\tilde{\delta}>0$ such that for all $\tau,t\in[0,T]$ with $|\tau-t|<\tilde{\delta}$ we have $\|g(\tau,y)-g(t,y)\|_{C^1(\mathcal{B}_0,\mathbb{R}^3)}+\|\dot{g}(\tau,y)-\dot{g}(t,y)\|_{C^1(\mathcal{B}_0,\mathbb{R}^3)}<\tilde{\varepsilon}$. Choose now $\varepsilon,E>0$. By estimate (3.38) we obtain for $t=0$:

$$\|y\|_{W^{1,p}(\mathcal{B}_0,\mathbb{R}^3)}\leq\left(\tfrac{\mathcal{E}(0,\boldsymbol{g}(0,y),s)+C_3}{c_3}\right)^{\frac{1}{p}}\leq\left(\tfrac{E+C_3}{c_3}\right)^{\frac{1}{p}}=:\tilde{B}\,.$$

Thanks to the growth control (3.17c) for $\boldsymbol{g}$ his shows that $\boldsymbol{g}(t,y)$ for (y,s) with bounded energy at initial time are uniformly bounded for every $t\in[0,T]$.

Furthermore we estimate

$$\begin{aligned}&|\partial_t\mathcal{I}(t,\boldsymbol{g}(t,y),s)-\partial_t\mathcal{I}(\tau,\boldsymbol{g}(\tau,y),s)|\\&\leq\left|\int_{\mathcal{B}_0}\partial_FW(\nabla\boldsymbol{g}(t,y),s)(\nabla\boldsymbol{g}(t,y))^\top:\nabla(\dot{\boldsymbol{g}}(t,y)-\dot{\boldsymbol{g}}(\tau,y))\,\mathrm{d}X\right| &(3.54)\\&\quad+\left|\int_{\mathcal{B}_0}(\partial_FW(\nabla\boldsymbol{g}(t,y),s)(\nabla\boldsymbol{g}(t,y))^\top-\partial_FW(\nabla\boldsymbol{g}(\tau,y),s))(\nabla\boldsymbol{g}(\tau,y))^\top:\nabla\dot{\boldsymbol{g}}(\tau,y)\,\mathrm{d}X\right| &(3.55)\\&\quad+\left|\int_{\partial\mathcal{B}_0^{\mathrm{N}}}(\dot{\boldsymbol{h}}(t)-\dot{\boldsymbol{h}}(\tau))\cdot\boldsymbol{g}(t,y)\,\mathrm{d}X\right|+\left|\int_{\partial\mathcal{B}_0^{\mathrm{N}}}\dot{\boldsymbol{h}}(\tau)\cdot(\boldsymbol{g}(t,y)-\boldsymbol{g}(\tau,y))\,\mathrm{d}X\right| &(3.56)\\&\quad+\left|\int_{\partial\mathcal{B}_0^{\mathrm{N}}}(\boldsymbol{h}(t)-\boldsymbol{h}(\tau))\cdot\dot{\boldsymbol{g}}(t,y)\,\mathrm{d}X\right|+\left|\int_{\partial\mathcal{B}_0^{\mathrm{N}}}\boldsymbol{h}(\tau)\cdot(\dot{\boldsymbol{g}}(t,y)-\dot{\boldsymbol{g}}(\tau,y))\,\mathrm{d}X\right|, &(3.57)\end{aligned}$$

where, thanks to assumptions (3.17) & (3.18), each of the terms in (3.56) & (3.57) can be estimated from above by $\varepsilon/8$ for $|t-\tau|<\tilde{\delta}_0$ sufficiently small.

In view of coercivity (3.10d), stress control (3.10f) and Lipschitz estimate (3.28) we see that

$$\begin{aligned}(3.54)\quad&\leq\quad\|\partial_FW(\nabla\boldsymbol{g}(t,y),s)(\nabla\boldsymbol{g}(t,y))^\top\|_{L^1(\mathcal{B}_0)}\|\nabla(\dot{\boldsymbol{g}}(t,y)-\dot{\boldsymbol{g}}(\tau,y))\|_{L^\infty(\mathcal{B}_0,\mathbb{R}^{3\times3})}\\&\leq\quad(\mathcal{E}(0,\boldsymbol{g}(0,y),s)+C\mathcal{L}^3(\mathcal{B}_0)+c_ET+c_lB)\|\nabla(\dot{\boldsymbol{g}}(t,y)-\dot{\boldsymbol{g}}(\tau,y))\|_{L^\infty(\mathcal{B}_0,\mathbb{R}^{3\times3})}<\frac{\varepsilon}{4}\,,\end{aligned}$$

if $|t-\tau|<\tilde\delta_1$ is sufficiently small. In view of the continuity of the stresses (3.10g) and the Gronwall estimate we find

$$\text{(3.55)} \leq c_g\omega\big(\|\nabla(\dot{\boldsymbol{g}}(t,y)-\dot{\boldsymbol{g}}(\tau,y))\|_{L^\infty}\big)\big(\|W(\nabla\boldsymbol{g}(t,y),s)\|_{L^1}\big(1+\exp(2cc_g)\big)+C\big) < \frac{\varepsilon}{4}$$

for $|\tau-t|<\tilde\delta_2$ sufficiently small, where we used $C:=(1+\exp(2cc_g)cc_g)\tilde c\mathcal{L}^d(\Omega)$. Hence we obtain (3.55) $<\frac{\varepsilon}{4}$ if $|s-t|<\tilde\delta_2$. Altogether we conclude that $|\partial_t\mathcal{I}(s,q)-\partial_t\mathcal{I}(t,q)|<\varepsilon$ if $|s-t|<\delta:=\min\{\tilde\delta_0,\tilde\delta_1,\tilde\delta_2\}$. □

3.4 Examples of Energy Densities Satisfying Assumptions (3.10)

In this section we discuss well-known constitutive laws in nonlinear elasticity with regard to their admissibility for assumptions (3.10), which are at the core of our existence result. Note that assumptions (3.10) are formulated for an energy density W in dependence of a matrix $\boldsymbol{F}=\nabla\boldsymbol{\varphi}\in\mathbb{R}^{3\times3}$ and its minors. However, by making use of the assumption of material frame indifference and isotropy, many constitutive laws used in engineering are equivalently formulated with respect to invariants of the right Cauchy-Green strain tensor $\boldsymbol{C}:=\boldsymbol{F}^\top\boldsymbol{F}$ or with respect to the modified invariants introduced in Section 2, as it will also be the case in the numerical examples shown in Section 4. Then it is not obvious that constitutive laws given in this way also match with the assumptions (3.10) of our existence theorem. This is why we will now take a closer look at densities given as functions of the invariants $\iota_1(\boldsymbol{C}),\iota_2(\boldsymbol{C}),\iota_3(\boldsymbol{C})$ and at densities given as functions of the modified invariants $U(\boldsymbol{C}),V(\boldsymbol{C}),\iota_3(\boldsymbol{C})$. In the subsequent discussion we will neglect the dependence of the densities on the phase-field parameter; in other words, the densities studied here play the role of W_1 and W_2 in (3.10a). For our further investigations we recall the notation

$$\iota_1(\boldsymbol{C}):=\operatorname{tr}\boldsymbol{C},\ \iota_2(\boldsymbol{C}):=\operatorname{tr}\operatorname{cof}\boldsymbol{C},\ \iota_3(\boldsymbol{C}):=\det\boldsymbol{C},\ \text{ for } \boldsymbol{C}:=\boldsymbol{F}^\top\boldsymbol{F}\,, \tag{3.58a}$$

$$U(\boldsymbol{C})=\iota_3^{-1/3}(\boldsymbol{C})\iota_1(\boldsymbol{C})\,,\ V(\boldsymbol{C})=\iota_3^{-2/3}(\boldsymbol{C})\iota_2(\boldsymbol{C})\,. \tag{3.58b}$$

Here, the expressions ι_i in (3.58a) are the invariants of the right Cauchy-Green strain tensor $\boldsymbol{C}:=\boldsymbol{F}^\top\boldsymbol{F}$ and U,V in (3.58b) denote the modified invariants of $\boldsymbol{C}$. Using the relations

$$|\boldsymbol{F}|^2=\iota_1(\boldsymbol{F}^\top\boldsymbol{F}),\ \iota_2(\boldsymbol{C})=\operatorname{tr}\big(\operatorname{cof}(\boldsymbol{F}^\top\boldsymbol{F})\big)=|\operatorname{cof}\boldsymbol{F}|^2,\ \text{ and } \det(\boldsymbol{F}^\top\boldsymbol{F})=(\det\boldsymbol{F})^2 \tag{3.58c}$$

the modified invariants $U(\boldsymbol{F}^\top\boldsymbol{F}),V(\boldsymbol{F}^\top\boldsymbol{F})$ can also be reformulated directly in terms of $\boldsymbol{F}$, i.e.,

$$\overline{U}(\boldsymbol{F})=\iota_3^{-2/3}(\boldsymbol{F})|\boldsymbol{F}|^2=U(\boldsymbol{C})\,,\ \overline{V}(\boldsymbol{F})=\iota_3^{-4/3}(\boldsymbol{F})|\operatorname{cof}\boldsymbol{F}|^2=V(\boldsymbol{C})\,. \tag{3.58d}$$

As in Section 2 we may also set $\overline{\boldsymbol{F}}:=(\det\boldsymbol{F})^{-1/3}\boldsymbol{F}$ and $\overline{\boldsymbol{H}}:=(\det\boldsymbol{F})^{-2/3}\operatorname{cof}\boldsymbol{F}$ and find that

$$|\overline{\boldsymbol{F}}|^2=\overline{U}(\boldsymbol{F})=U(\boldsymbol{C})\ \text{ and }\ |\overline{\boldsymbol{H}}|^2=\overline{V}(\boldsymbol{F})=V(\boldsymbol{C})\,. \tag{3.58e}$$

In accordance with (3.58), we subsequently assume that we are given densities $W,W_\bullet,\widetilde{W}$, and $\overline{W}$, which satisfy the relation

$$W(\boldsymbol{F})=W_\bullet(\mathbb{M}\boldsymbol{F})=\widetilde{W}(\iota_1(\boldsymbol{F}^\top\boldsymbol{F}),\iota_2(\boldsymbol{F}^\top\boldsymbol{F}),\iota_3(\boldsymbol{F}^\top\boldsymbol{F}))=\overline{W}(U(\boldsymbol{F}^\top\boldsymbol{F}),V(\boldsymbol{F}^\top\boldsymbol{F}),\iota_3(\boldsymbol{F}^\top\boldsymbol{F})) \tag{3.59}$$

for all $\boldsymbol{F} \in \mathbb{R}^{3\times3}$ and $\mathbb{M}\boldsymbol{F} = (\boldsymbol{F}, \operatorname{cof}\boldsymbol{F}, \det\boldsymbol{F})$. In (3.59), the density $\widetilde{W} : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a function of the invariants (3.58a) of the matrix $\boldsymbol{F}^\top\boldsymbol{F}$ and the density $\overline{W} : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a function of the modified invariants (3.58b). The first Piola-Kirchhoff stress tensor is determined as

$$\begin{aligned}
\boldsymbol{P} = \frac{\mathrm{d}W(\boldsymbol{F})}{\mathrm{d}\boldsymbol{F}} &= \frac{\partial\widetilde{W}(\iota_1,\iota_2,\iota_3)}{\partial\iota_1}\frac{\partial\iota_1(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} + \frac{\partial\widetilde{W}(\iota_1,\iota_2,\iota_3)}{\partial\iota_2}\frac{\partial\iota_2(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} \\
&\quad + \frac{\partial\widetilde{W}(\iota_1,\iota_2,\iota_3)}{\partial\iota_3}\frac{\partial\iota_3(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} \\
&= \frac{\partial\overline{W}(U,V,\iota_3)}{\partial U}\frac{\partial U(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} + \frac{\partial\overline{W}(U,V,\iota_3)}{\partial V}\frac{\partial V(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} \\
&\quad + \frac{\partial\overline{W}(U,V,\iota_3)}{\partial\iota_3}\frac{\partial\iota_3(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} ,
\end{aligned} \tag{3.60}$$

with the expressions for the derivatives of the invariants gathered in the next lemma. We point out that (3.62) provides a stress control for the invariant functions alike (3.10f), which in view of (3.60) will be used lateron to formulate sufficient conditions for the densities $\widetilde{W}, \overline{W}$ in order to guarantee the stress control (3.10f) for $\boldsymbol{P}$.

Lemma 3.14 (Derivatives of the invariants)**.** *Let the relations* (3.58) *hold true. For a matrix* $\boldsymbol{F} \in \mathbb{R}^{3\times3}$ *with* $\det\boldsymbol{F} > 0$ *it is*

$$\frac{\partial\iota_1(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} = 2\boldsymbol{F} , \tag{3.61a}$$

$$\frac{\partial\iota_2(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} = 2\big(\iota_2(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^{-\top} - \iota_3(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}\boldsymbol{F}^{-\top}\big) , \tag{3.61b}$$

$$\frac{\partial\iota_3(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} = 2\iota_3(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^{-\top} , \tag{3.61c}$$

$$\frac{\partial U(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} = 2\iota_3^{-1/3}(\boldsymbol{F}^\top\boldsymbol{F})\big(\boldsymbol{F} - \tfrac{1}{3}\iota_1(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^{-\top}\big) , \tag{3.61d}$$

$$\frac{\partial V(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} = 2\iota_3^{-2/3}(\boldsymbol{F}^\top\boldsymbol{F})\big(\tfrac{1}{3}\iota_2(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^{-\top} - \iota_3(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}\boldsymbol{F}^{-\top}\big) . \tag{3.61e}$$

Moreover, let A *be a placeholder for the invariant functions* $\iota_1, \iota_2, \iota_3, U, V : \mathrm{GL}_+(3) \to \mathbb{R}$*. The invariant function* A *satisfies the following stress control estimate:*

$$\big|\partial_{\boldsymbol{F}} A(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^\top\big| \le c_A A(\boldsymbol{F}^\top\boldsymbol{F}) . \tag{3.62}$$

The proof consists of a straight forward but lengthy application of the product and chain rule and we carry it out in detail in Section 3.5. There we also give the proof of the next lemma, which provides continuity estimates for the invariants and their stresses in terms of moduli of continuity multiplied by the invariant, as required in the assumption (3.10g). Lateron, they will be used to deduce similar continuity estimates for the first Piola-Kirchhoff stress.

Lemma 3.15 (Continuity properties of the invariants)**.** *Let A be a placeholder for the invariant functions $\iota_1, \iota_2, \iota_3, U, V : \mathrm{GL}_+(3) \to \mathbb{R}$ from* (3.58a) *&* (3.58b)*. The invariant function A and its derivative $\partial_{\boldsymbol{F}} A : \mathrm{GL}^3_+ \to \mathrm{GL}^3_+$ is continous. Moreover, there exists a modulus of continuity $o : [0,\infty] \to [0,\infty]$, $\delta > 0$, so that for all $\boldsymbol{F} \in \mathbb{R}^{3\times 3}$ and all $C \in \mathrm{GL}_+(3)$ with $|C - \mathbb{1}| \le \delta$ we have*

$$|A((C\boldsymbol{F})^\top (C\boldsymbol{F})) - A(\boldsymbol{F}^\top \boldsymbol{F})| \le o(|C - \mathbb{1}|) A(\boldsymbol{F}^\top \boldsymbol{F}) \,, \tag{3.63a}$$

$$|\partial_{\boldsymbol{F}} A((C\boldsymbol{F})^\top (C\boldsymbol{F}))(C\boldsymbol{F})^\top - \partial_{\boldsymbol{F}} A(\boldsymbol{F}^\top \boldsymbol{F})\boldsymbol{F}^\top| \le o(|C - \mathbb{1}|) A(\boldsymbol{F}^\top \boldsymbol{F}) \,, \tag{3.63b}$$

again with $A \in \{\iota_1, \iota_2, \iota_3, U, V\}$.

3.4.1 Discussion of Well-Known Constitutive Laws

In the following we investigate some material laws, which are widely used in nonlinear elasticity for their admissibility with respect to assumptions (3.10). More precisely, we will take a closer look at the following isochoric energy densities:

$$\text{Neo-Hooke [Riv48]:}\quad \widetilde{W}(\iota_1(\boldsymbol{C}), \iota_2(\boldsymbol{C}), \iota_3(\boldsymbol{C})) := c_1(\iota_1(\boldsymbol{C}) - 3) \,, \tag{3.64a}$$

$$\text{Mooney-Rivlin [Moo40]:}\quad \widetilde{W}(\iota_1(\boldsymbol{C}), \iota_2(\boldsymbol{C}), \iota_3(\boldsymbol{C})) := c_1(\iota_1(\boldsymbol{C}) - 3) + c_2(\iota_2(\boldsymbol{C}) - 3) \,, \tag{3.64b}$$

$$\text{Arruda-Boyce [AB93]:}\quad \widetilde{W}(\iota_1(\boldsymbol{C}), \iota_2(\boldsymbol{C}), \iota_3(\boldsymbol{C})) := \sum_{i=1}^{m} c_i(\iota_1(\boldsymbol{C})^i - 3^i) \text{ with } c_i > 0 \,, \tag{3.64c}$$

$$\begin{aligned}\text{Rivlin [Riv56]:}\quad \widetilde{W}(\iota_1(\boldsymbol{C}), \iota_2(\boldsymbol{C}), \iota_3(\boldsymbol{C})) := &\sum_{i,j,k=0}^{m} c_{ijk}(\iota_1(\boldsymbol{C}) - 3)^i (\iota_2(\boldsymbol{C}) - 3)^j \\ &\times (\iota_3(\boldsymbol{C}) - 1)^k \,,\end{aligned} \tag{3.64d}$$

$$\text{Rivlin-Saunders [RS51]:}\quad \widetilde{W}(\iota_1(\boldsymbol{C}), \iota_2(\boldsymbol{C}), \iota_3(\boldsymbol{C})) := \sum_{i,j=0}^{m} c_{ij}(\iota_1(\boldsymbol{C}) - 3)^i (\iota_2(\boldsymbol{C}) - 3)^j \,, \tag{3.64e}$$

$$\text{Yeoh [YF97]:}\quad \widetilde{W}(\iota_1(\boldsymbol{C}), \iota_2(\boldsymbol{C}), \iota_3(\boldsymbol{C})) := \sum_{i=1}^{3} c_i(\iota_1(\boldsymbol{C}) - 3)^i \,. \tag{3.64f}$$

In [RS51] it is set $m = \infty$ in (3.64e). In particular, the Neo-Hooke and Mooney-Rivlin law can be obtained from the Rivlin-Saunders law, the first by choosing $c_{10} \neq 0$, but $c_{ij} = 0$ for any other combination of $i \in \mathbb{N}_0, j \in \mathbb{N}$, the second by choosing $c_{01} \neq 0, c_{10} \neq 0$, but $c_{ij} = 0$ for any other $i, j \in \mathbb{N}$. As explained in [RS51], the Neo-Hooke and Mooney-Rivlin law can be used as an approximation of (3.64e) valid if the deformations are sufficiently small so that higher order (product) terms are negligibly small. The Arruda-Boyce law originates from a statistical model for rubber taking into account the orientation of the polymer chains. The strain energy function is derived from the inverse Langevin function, cf. e.g. [Tre75, Chapter 6], in terms of a Taylor expansion, and therefore the coefficients $c_i > 0$ in (3.64c) take a very specific form involving the parameters of the polymer. The above material laws (3.64a), (3.64b), (3.64c) & (3.64f) can also be found in [Hol04, Chapter 6.5].

Polyconvexity (3.10c)**:** We refer to the works [Bal77, Cia88, SN03], where the polyconvexity of some of (3.64) and many other constitutive laws, such as e.g. Ogden's materials, has been discussed. We here collect statements on the polyconvexity of the constitutive laws (3.64):

Proposition 3.16 (Polyconvexity of the laws (3.64))**.** *Assume that* $c_i > 0, c_{ij} \geq 0$ *in* (3.64)*. Then the following statements regarding assumption* (3.10c) *hold true:*

1 *The energy density of the Neo-Hooke material* (3.64a) *is polyconvex.*
2 *The energy density of the Mooney-Rivlin material* (3.64b) *is polyconvex.*
3 *The energy density of the Arruda-Boyce material* (3.64c) *is polyconvex.*
4 *None of the densities* (3.64d)*,* (3.64e) *&* (3.64f) *is polyconvex.*
5 *The volumetric density* $\boldsymbol{F} \in \mathbb{R}^{3\times3} \mapsto \iota_3(\boldsymbol{F}^\top \boldsymbol{F})^{-\gamma}$ *for* $\gamma > 0$ *is polyconvex.*

Proof. The **Proof of 1. & 2.** is immediate thanks to the relations (3.58c), which show that the energy terms are quadratic expressions in $\boldsymbol{F}$ and $\operatorname{cof} \boldsymbol{F}$.

Proof of 3.: Once more by (3.58c) we rewrite the density of the Arruda-Boyce model as

$$\widetilde{W}(\iota_1(\boldsymbol{F}^\top \boldsymbol{F}), \iota_2(\boldsymbol{F}^\top \boldsymbol{F}), \iota_3(\boldsymbol{F}^\top \boldsymbol{F})) = \sum_{i=1}^{m} c_i(\iota_1(\boldsymbol{F}^\top \boldsymbol{F})^i - 3^i) = \sum_{i=1}^{3} c_i(|\boldsymbol{F}|^{2i} - 3^i) = W(\boldsymbol{F}) ,$$

and we study the convexity of the function

$$g_i : \mathbb{R}_+ \to \mathbb{R}, \ g_i(x) := (x^{2i} - 3^i) . \tag{3.65}$$

We see that $\mathrm{D}^2 g_i(x) = 2i(2i-1)x^{2i-2} \geq 0$ for all $x \in \mathbb{R}$, given that $i \in \mathbb{N}$. This shows that g_i is convex. Moreover, f_i is non-decreasing. The density $W(\boldsymbol{F}) = \sum_{i=1}^m g_i(|\boldsymbol{F}|)$ is the composition of the convex, non-decreasing function g_i with the convex function $|\cdot| : \mathbb{R}^{3\times3} \to \mathbb{R}$ and hence it is convex.

Proof of 4.: Again by (3.58c) we rewrite the density of the Yeoh model as

$$\widetilde{W}(\iota_1(\boldsymbol{F}^\top \boldsymbol{F}), \iota_2(\boldsymbol{F}^\top \boldsymbol{F}), \iota_3(\boldsymbol{F}^\top \boldsymbol{F})) = \sum_{i=1}^{3} c_i(\iota_1(\boldsymbol{F}^\top \boldsymbol{F}) - 3)^i = \sum_{i=1}^{3} c_i(|\boldsymbol{F}|^2 - 3)^i = W(\boldsymbol{F}) .$$

This expression is also a factor in the Rivlin and in the Rivlin-Saunders model. Convexity of W would equivalent to the positive-definiteness of its Hessian $\mathrm{D}^2_{\boldsymbol{F}} W(\boldsymbol{F})$. To investigate this feature, we first study the convexity of the function

$$f_i : \mathbb{R}_+ \to \mathbb{R}, \ f_i(x) := (x^2 - a)^i \ \text{ with constants } a \geq 0, \ i > 1 . \tag{3.66}$$

We obtain that $\mathrm{D}^2_x f_i(x) = 4i(i-1)x^2(x^2-a)^{i-2} + 2i(x^2-a)^{i-1}$. First let $i = 2$. Then $\mathrm{D}^2_x f_2(x) = 12x^2 - 4a$. We find that $\mathrm{D}^2_x f_2(x) < 0$ for any $x^2 < a/3$. Let now $i = 3$. Then $\mathrm{D}^2_x f_3(x) = 24x^2(x^2-a) + 6(x^2-a)^2$. Again, $\mathrm{D}^2_x f_3(x) < 0$ for any $x^2 \in (a/5, a)$. Hence, the f_i is not convex and therefore the Yeoh model cannot be polyconvex. Since the term (3.66) also occurs in the Rivlin and in the Rivlin-Saunders model, also their polyconvexity is disproved.

Proof of 5.: We study the convexity of the function $f(x) = x^{-2\gamma}$. We calculate that $f''(x) = \gamma(2\gamma + 1)x^{-2(\gamma+1)} > 0$ for all $x > 0$. ∎

Coercivity (3.10d)**:** In view of (3.58c) we have the following immediate results regarding the coercivity of the polyconvex constitutive laws (3.64a)–(3.64c).

Proposition 3.17 (Coercivity of the densities (3.64))**.** *Let $c_i > 0$ in* (3.64)*. Then, the following statements hold true:*

1 *The energy density* (3.64a) *of the Neo-Hooke material satisfies* (3.10d) **b1)** *with $p = 2$.*
2 *The energy density* (3.64b) *of the Mooney-Rivlin material satisfies* (3.10d) **b2)** *with $p = 2$ and $p_2 = 2$.*
3 *The energy density of the Arruda-Boyce material satisfies* (3.10d) **a)** *with $p = 2m$ for $m \geq 2$, and* (3.10d) **b1)** *otherwise.*

Stress control (3.10f) **& continuity of the stresses** (3.10g)**:** In view of relations (3.60)–(3.63) for the derivatives of the invariant functions we are also in the position to make the following statement regarding the assumtions on the stress control (3.10f) and the uniform continuity of the stresses (3.10g):

Proposition 3.18 (Stress control (3.10f) & uniform continuity of the stresses (3.10g))**.** *Let the relations* (3.58)–(3.60) *hold true.*

1 *Assume that there is a constant $K > 0$ such that $\widetilde{W} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies for all $(\iota_1, \iota_2, \iota_3) \in \mathbb{R}^3_+$:*

$$|\partial_{\iota_1}\widetilde{W}(\iota_1, \iota_2, \iota_3)U| + |\partial_{\iota_2}\widetilde{W}(\iota_1, \iota_2\iota_3)V| + |\partial_{\iota_3}\widetilde{W}(\iota_1, \iota_2, \iota_3)\iota_3| \leq K(\widetilde{W}(\iota_1, \iota_2, \iota_3) + 1)\,. \tag{3.67}$$

Then stress control (3.10f) *is true.*

2 *Assume that there is a modulus of continuity $o : [0, \infty] \to [0, \infty]$, $\delta > 0$, so that for all $(\boldsymbol{F}, s) \in \mathbb{R}^{3\times3} \times \mathbb{R}$ and all $C \in \mathrm{GL}_+(3)$ with $|C - \mathbb{1}| \leq \delta$ we have*

$$\begin{aligned}
&\Big|\Big(\tfrac{\partial\widetilde{W}(\iota_1,\iota_2,\iota_3)}{\partial\iota_1}\tfrac{\partial\iota_1((C\boldsymbol{F})^\top(C\boldsymbol{F}))}{\partial(C\boldsymbol{F})} + \tfrac{\partial\widetilde{W}(\iota_1,\iota_2,\iota_3)}{\partial\iota_2}\tfrac{\partial\iota_2((C\boldsymbol{F})^\top(C\boldsymbol{F}))}{\partial(C\boldsymbol{F})} + \tfrac{\partial\widetilde{W}(\iota_1,\iota_2,\iota_3)}{\partial\iota_3}\tfrac{\partial\iota_3((C\boldsymbol{F})^\top(C\boldsymbol{F}))}{\partial(C\boldsymbol{F})}\Big)(C\boldsymbol{F})^\top\\
&\qquad - \Big(\tfrac{\partial\widetilde{W}(\iota_1,\iota_2,\iota_3)}{\partial\iota_1}\tfrac{\partial\iota_1(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} + \tfrac{\partial\widetilde{W}(\iota_1,\iota_2,\iota_3)}{\partial\iota_2}\tfrac{\partial\iota_2(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} + \tfrac{\partial\widetilde{W}(\iota_1,\iota_2,\iota_3)}{\partial\iota_3}\tfrac{\partial\iota_3(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}}\Big)\boldsymbol{F}^\top\Big|\\
&\leq o(|C - \mathbb{1}|)(\widetilde{W}(\iota_1, \iota_2, \iota_3) + \tilde{c}) = o(|C - \mathbb{1}|)(W(\boldsymbol{F}) + \tilde{c})\,.
\end{aligned} \tag{3.68}$$

Then the uniform continuity of the stresses (3.10g) *is true.*

Proof. **Proof of 1.:** From (3.60) we infer that

$$\begin{aligned}
|\partial_{\boldsymbol{F}}W(\boldsymbol{F})\boldsymbol{F}^\top| \leq \big|\partial_{\iota_1}\widetilde{W}(\iota_1, \iota_2, \iota_3)\big|\,|\partial_{\boldsymbol{F}}\iota_1\boldsymbol{F}^\top| + \big|\partial_{\iota_2}\widetilde{W}(\iota_1, \iota_2, \iota_3)\big|\,|\partial_{\boldsymbol{F}}\iota_2\boldsymbol{F}^\top|\\
+ \big|\partial_{\iota_3}\widetilde{W}(\iota_1, \iota_2, \iota_3)\big|\,|\partial_{\boldsymbol{F}}\iota_3\boldsymbol{F}^\top|\,,
\end{aligned}$$

with $\iota_1 = \iota_1(\boldsymbol{F}^\top\boldsymbol{F}), \iota_2 = \iota_2(\boldsymbol{F}^\top\boldsymbol{F})$, and $\iota_3 = \iota_3(\boldsymbol{F}^\top\boldsymbol{F})$. The stress control estimate for the invariant functions (3.62) and assumption (3.67) now allow us to conclude that

$$\begin{aligned}
|\partial_{\boldsymbol{F}}W(\boldsymbol{F})\boldsymbol{F}^\top| &\leq c_{\iota_1}|\partial_U\widetilde{W}(\iota_1, \iota_2, \iota_3)\,\iota_1| + c_{\iota_2}|\partial_V\widetilde{W}(\iota_1, \iota_2, \iota_3)\,\iota_2| + c_{\iota_3}|\partial_{\iota_3}\widetilde{W}(\iota_1, \iota_2, \iota_3)\,\iota_3|\\
&\leq K(\widetilde{W}(\iota_1, \iota_2, \iota_3) + 1) = K(W(\boldsymbol{F}) + 1)\,,
\end{aligned}$$

which is stress control (3.10f).

Proof of 2.: The condition on continuity of the stresses (3.10g) formulated for the density W, directly follows from (3.68) using the relations $W(\boldsymbol{F}) = \widetilde{W}(\iota_1(\boldsymbol{F}^\top\boldsymbol{F}), \iota_2(\boldsymbol{F}^\top\boldsymbol{F}), \iota_3(\boldsymbol{F}^\top\boldsymbol{F}))$ from (3.59) and (3.60) for the first Piola-Kirchhoff stress. ☐

Corollary 3.19 (Energy densities matching with Prop. 3.27)**.** *The densities* (3.64a)–(3.64c) *satisfy conditions* (3.67) *&* (3.68)*.*

Proof. The proof exploits the form of the first Piola-Kirchhoff stress (3.60) and uses the stress control estimate (3.67) and the continuity relation (3.68) for the invariant functions ι_i, $i = 1, 2, 3$. For more details we point ahead to Cor. 3.28, where the calculations are carried out for $\overline{W}$ depending on the modified invariants. ☐

Energy densities satisfying assumptions (3.10)**:** As a result of Proposition 3.16, and Corollaries 3.17 & 3.19 allows us to conclude:

Corollary 3.20 (Energy densities satisfying assumptions (3.10))**.** *The densities* (3.64a)–(3.64c) *satisfy all the assumptions* (3.10)*.*

3.4.2 Assumptions (3.10) and the Modified Invariants

In the following we discuss the Assumptions (3.10) for a stored energy density that is a function of the modified invariants introduced in Section 2. Quite often in literature, the density $\widetilde{W}$ is used and assumed to be a function of the modified invariants $U(\boldsymbol{C}), V(\boldsymbol{C}), \iota_3(\boldsymbol{C})$. In this spirit we will now consider the constitutive laws from (3.64) as functions of $U(\boldsymbol{C}), V(\boldsymbol{C}), \iota_3(\boldsymbol{C})$, i.e.

$$\text{Neo-Hooke [Riv48]:}\quad \overline{W}(U(\boldsymbol{C}), V(\boldsymbol{C}), \iota_3(\boldsymbol{C})) := c_1(U(\boldsymbol{C}) - 3)\,, \tag{3.69a}$$

$$\text{Mooney-Rivlin [Moo40]:}\quad \overline{W}(U(\boldsymbol{C}), V(\boldsymbol{C}), \iota_3(\boldsymbol{C})) := c_1(U(\boldsymbol{C}) - 3) + c_2(V(\boldsymbol{C}) - 3)\,, \tag{3.69b}$$

$$\text{Arruda-Boyce [AB93]:}\quad \overline{W}(U(\boldsymbol{C}), V(\boldsymbol{C}), \iota_3(\boldsymbol{C})) := \sum_{i=1}^{m} c_i(U(\boldsymbol{C})^i - 3^i)\ \text{with} c_i > 0\,, \tag{3.69c}$$

$$\text{Yeoh [YF97]:}\quad \overline{W}(U(\boldsymbol{C}), V(\boldsymbol{C}), \iota_3(\boldsymbol{C})) := \sum_{i=1}^{3} c_i(U(\boldsymbol{C}) - 3)^i\,, \tag{3.69d}$$

and we will investigate how the above isochoric material laws match with assumptions (3.10).

In [CDHL88] the properties of the modified Neo-Hooke and Mooney-Rivlin material (3.69a) & (3.69b) have been analyzed. In particular, it is shown that the term $V(\boldsymbol{F}^\top\boldsymbol{F}) = |\det\boldsymbol{F}|^{-4/3}|\operatorname{cof}\boldsymbol{F}|^2$ is not polyconvex itself, so that the modified Mooney-Rivlin material (3.69b) cannot be polyconvex either. Moreover, in [CDHL88] optimal coercivity properties are derived for functions of modified invariants, which guarantee the validity of Ball's existence result [Bal77, Thm. 6.2], see also the discussion in Remark 3.6.

Polyconvexity (3.10c)**:** Firstly, we refer to the works [SN03, HN03], which investigate the polyconvexity properties of the modified Arruda-Boyce model (3.69c) and of a modified Rivlin-Saunders-type model. Amongst others, they also give the following polyconvexity result:

Lemma 3.21 ([HN03, Cor. 2.3 & Table 3])**.** *Let $\boldsymbol{F}\in\mathbb{R}^{3\times 3}$. Then the following terms are polyconvex:*

$$\boldsymbol{F}\in\mathbb{R}^{3\times3}\mapsto\big(U(\boldsymbol{F}^\top\boldsymbol{F})^i-3^i\big)^k\,,\qquad \text{for } i\geq 1,\ k\geq 1\,, \tag{3.70a}$$

$$\boldsymbol{F}\in\mathbb{R}^{3\times3}\mapsto\big(V(\boldsymbol{F}^\top\boldsymbol{F})^{3i/2}-3^{3i/2}\big)^k\,,\quad \text{for } i\geq 1,\ k\geq 1\,, \tag{3.70b}$$

$$\boldsymbol{F}\in\mathbb{R}^{3\times3}\mapsto\big(\iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{1/2}-1\big)^k\,,\qquad \text{for } k>1\,. \tag{3.70c}$$

In addition, we now gather the following statement on the polyconvexity of further energy terms depending on modified invariants:

Proposition 3.22 (Polyconvexity)**.**

1 *The function $f:\mathbb{R}^{3\times3}\times\mathbb{R}_+\to\mathbb{R},\ f(A,\iota):=\frac{|A|^\alpha}{\iota^\beta}$ is polyconvex if $\beta>0$ and $\alpha\geq\beta+1$.*

2 *The function $\overline{U}:\mathbb{R}^{3\times3}\to\mathbb{R}$, $\overline{U}(\boldsymbol{F})=U(\boldsymbol{F}^\top\boldsymbol{F})=\iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{-1/3}\iota_1(\boldsymbol{F}^\top\boldsymbol{F})$ is polyconvex.*

3 *The function $\overline{V}:\mathbb{R}^{3\times3}\to\mathbb{R}$, $\overline{V}(\boldsymbol{F})=V(\boldsymbol{F}^\top\boldsymbol{F})=\iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{-2/3}\iota_2(\boldsymbol{F}^\top\boldsymbol{F})$ is not polyconvex.*

4 *The energy density of the modified Arruda-Boyce material* (3.69c) *is polyconvex.*

5 *The energy density $\overline{W}_2$ is polyconvex, where*

$$\overline{W}_2(V(\boldsymbol{F}^\top\boldsymbol{F})):=\sum_{i=2}^m c_i(V(\boldsymbol{F}^\top\boldsymbol{F})^i-3^i)\,. \tag{3.71a}$$

6 *The energy density $\overline{W}_3$ is polyconvex, where*

$$\overline{W}_3(\iota_3(\boldsymbol{F}^\top\boldsymbol{F})):=\sum_{i=1}^m c_i(\iota_3(\boldsymbol{F}^\top\boldsymbol{F})^i-1)\,. \tag{3.71b}$$

7 *The energy density of the modified Yeoh material* (3.69d) *is polyconvex.*

Proof. **Proof of 1.:** In order to deduce polyconvexity relations for f we first study the convexity properties of the function $\hat f:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R},\ \hat f(x,y)=\frac{x^\alpha}{y^\beta}$ with $\alpha,\beta>0$. Its Hessian takes the form

$$\mathrm{D}^2\hat f(x,y):=\begin{pmatrix}\alpha(\alpha-1)\frac{x^{\alpha-2}}{y^\beta} & -\alpha\beta\frac{x^{\alpha-1}}{y^{\beta+1}}\\ -\alpha\beta\frac{x^{\alpha-1}}{y^{\beta+1}} & \beta(\beta+1)\frac{x^\alpha}{y^{\beta+2}}\end{pmatrix}$$

and its positive semidefiniteness is given if $\operatorname{tr}\mathrm{D}^2\hat f(x,y)\geq 0$ together with $\det\mathrm{D}^2\hat f(x,y)\geq 0$. It is $\operatorname{tr}\mathrm{D}^2\hat f(x,y)=\alpha(\alpha-1)\frac{x^{\alpha-2}}{y^\beta}+\beta(\beta+1)\frac{x^\alpha}{y^{\beta+2}}$. For $x,y>0$ the condition $\operatorname{tr}\mathrm{D}^2\hat f(x,y)\geq 0$ is equivalent to $\beta(\beta+1)\frac{x^2}{y^2}\geq\alpha(1-\alpha)$, which holds true for any $\alpha\geq 1$ and $\beta>0$.

It is $\det\mathrm{D}^2\hat f(x,y)=\alpha\beta(\alpha-\beta-1)\frac{x^{2\alpha-2}}{y^{2\beta+2}}$. For any $x,y,\alpha,\beta>0$ the condition $\det\mathrm{D}^2\hat f(x,y)\geq 0$ is satisfied if $\alpha\geq\beta+1$. Hence, $\hat f$ is convex if $\beta>0$ and $\alpha\geq\beta+1$. Since $\hat f(\cdot,y)$ is non-decreasing we now conclude the convexity of f:

$$
\begin{aligned}
f(\lambda A+(1-\lambda)B,\lambda y+(1-\lambda)z) &= \hat{f}(|\lambda A+(1-\lambda)B|,\lambda y+(1-\lambda)z)\\
&\leq \hat{f}(\lambda|A|+(1-\lambda)|B|,\lambda y+(1-\lambda)z)\\
&\leq \lambda\hat{f}(|A|,y)+(1-\lambda)\hat{f}(|B|,z)=\lambda f(A,y)+(1-\lambda)f(B,z)\,.
\end{aligned}
$$

Proof of 2.: Since $U(\boldsymbol{F}^\top\boldsymbol{F})=(\det\boldsymbol{F})^{-2/3}|\boldsymbol{F}|^2$, hence $\alpha=2\geq\beta+1=2/3+1$. Thus the findings of Item 1. ensure the polyconvexity of U.

Proof of 3.: It is $V(\boldsymbol{F}^\top\boldsymbol{F})=(\det\boldsymbol{F})^{-4/3}|\operatorname{cof}\boldsymbol{F}|^2$, i.e., $\alpha=2$ and $\beta=4/3$, so that here $\alpha<\beta+1$. In other words, here the exponents do not belong to the regime of Item 1. However, Item 1 just gives a *sufficient* condition on the exponents to ensure polyconvexity. Since polyconvexity implies rank-one convexity, polyconvexity of V is disproved, if we succeed to show that V is not rank-one convex. For this, as done in [HN03, Lemma 2.4] we consider the deformation gradient $\boldsymbol{F}=\operatorname{diag}(0.1,10,t)$, which can be rewritten as $\boldsymbol{F}(t)=\boldsymbol{A}+ta\otimes a$ with $\boldsymbol{A}=\operatorname{diag}(0.1,10,0)$ and $a=(0,0,1)$. We have $f(t)=V(\boldsymbol{F}(t)^\top\boldsymbol{F}(t))=100.1t^{2/3}+t^{-4/3}$ with $f''(t)=-\frac{200.2}{9}t^{-4/3}+\frac{28}{9}t^{-10/3}$. Now, if V were rank-one convex, then f were convex and thus $f''(t)\geq0$ for any $t\in\mathbb{R}$. But, e.g. for $t=1$ we find that $f''(1)=-\frac{200.2}{9}+\frac{28}{9}<0$. Thus, V is not rank-one convex, hence not polyconvex.

Proof of 4.: To verify the polyconvexity of the modified Arruda-Boyce law we check that $i\alpha\geq i\beta+1$ for $\alpha=2$, $\beta=2/3$ and all $i\in\mathbb{N}$. This is indeed true, because $i\geq1>3/4$. Thus, Item 1 ensures polyconvexity.

Proof of 5.: To find that the density $\overline{W}_2(V(\boldsymbol{F}^\top\boldsymbol{F})):=\sum_{i=2}^m c_i(V(\boldsymbol{F}^\top\boldsymbol{F})^i-3^i)$ is polyconvex, we check that $i\alpha\geq i\beta+1$ for $\alpha=2$, $\beta=4/3$ and all $i\in\mathbb{N}$ with $i\geq2$. Indeed, $i\geq2>3/2$ and hence, Item 1 yields polyconvexity. We remark that this result can also be retrieved from (3.70b).

Proof of 6.: The polyconvexity of $\overline{W}_3$ is immediate, since $\overline{W}_3(x)=\sum_{i=1}^m c_i(x^i-1)$ for $x>0$ is composed as the sum of terms being convex in $x>0$.

Proof of 7.: The polyconvexity of the modified Yeoh material (3.69d) directly follows from the polyconvexity of the term (3.70a) with $i=1$ and $k=1,2,3$. □

Remark 3.23. *The results in [CDHL88, Lemma 2.2] show for the specific cases of interest*

$$\overline{U}(\boldsymbol{F})^{q/2}=\frac{|\boldsymbol{F}|^q}{(\det\boldsymbol{F})^{q/3}}\,,\quad \textit{i.e.,}\ \alpha=q,\ \beta=q/3\,, \tag{3.72a}$$

$$\overline{V}(\boldsymbol{F})^{q/2}=\frac{|\operatorname{cof}\boldsymbol{F}|^q}{(\det\boldsymbol{F})^{q/3}}\,,\quad \textit{i.e.,}\ \alpha=q,\ \beta=2q/3\,, \tag{3.72b}$$

that the condition $\alpha\geq\beta+1$ is not only sufficient but even necessary for polyconvexity. More precisely, they show that $q\geq3/2$ in (3.72a) *and $q\geq3$ in* (3.72b) *are necessary conditions.*

Coercivity (3.10d)**:** Coercivity condition (3.10d) is formulated for the density W as a function of $\boldsymbol{F}$. We now transfer (3.10d) into an analogous condition for $\overline{W}$ as a function of U,V,ι_3, resp. $\overline{\boldsymbol{F}},\overline{\boldsymbol{H}},\det\boldsymbol{F}$, cf. (3.58).

Proposition 3.24 (Coercivity (3.10d) for the modified invariants)**.** *Assume that there are constants* $p, p_2, p_3, q, q_2, q_3, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{C}$, *such that*

$$
\begin{aligned}
&q > p \geq 2\,, \quad q_2 > p_2 \geq \tfrac{p}{p-1} \quad \text{and}\\
&\text{either } q_3 \geq \max\left\{p_3, \tfrac{pq}{3(q-p)}, \tfrac{2p_2q_2}{3(q_2-p_2)}\right\} > 1 \ \text{ if } W(\boldsymbol{F}) = \overline{W}(|\overline{\boldsymbol{F}}|, |\overline{\boldsymbol{H}}|, \det \boldsymbol{F})\,, \qquad (3.73a)\\
&\text{or } q_3 \geq \max\left\{p_3, \tfrac{pq}{3(q-p)}\right\} > 1 \ \text{ if } W(\boldsymbol{F}) = \overline{W}(|\overline{\boldsymbol{F}}|, \det \boldsymbol{F})\,, \ \text{ and such that}
\end{aligned}
$$

$$
\begin{aligned}
W(\boldsymbol{F}) &= \overline{W}(U(\boldsymbol{F}^\top\boldsymbol{F}), V(\boldsymbol{F}^\top\boldsymbol{F}), \iota_3(\boldsymbol{F}^\top\boldsymbol{F})) = \overline{W}(|\overline{\boldsymbol{F}}|^2, |\overline{\boldsymbol{H}}|^2, (\det \boldsymbol{F})^2)\\
&\geq \tilde{c}_1|\overline{\boldsymbol{F}}|^q + \tilde{c}_2|\overline{\boldsymbol{H}}|^{q_2} + \tilde{c}_3|\det \boldsymbol{F}|^{q_3} + \tilde{C} \qquad (3.73b)\\
&= \tilde{c}_1|U(\boldsymbol{F}^\top\boldsymbol{F})|^{q/2} + \tilde{c}_2|V(\boldsymbol{F}^\top\boldsymbol{F})|^{q_2/2} + \tilde{c}_3|\iota_3(\boldsymbol{F}^\top\boldsymbol{F})|^{q_3/2} + \tilde{C}\,.
\end{aligned}
$$

Then (3.10d) *is satisfied with some constants* c_1, c_2, c_3, C, *given that* $p, p_2, p_3, c_1, c_2, c_3$ *match with one of the cases* **a), b), c)** *of* (3.10d).

Proof. From (3.58) we recall that $\overline{\boldsymbol{F}} := (\det \boldsymbol{F})^{-1/3}\boldsymbol{F}$ and $\overline{\boldsymbol{H}} := (\det \boldsymbol{F})^{-2/3}\operatorname{cof}\boldsymbol{F}$. Hence, it is

$$
\begin{aligned}
c_1|\boldsymbol{F}|^p &= c_1|(\det \boldsymbol{F})^{-1/3}\boldsymbol{F}|^p|\det \boldsymbol{F}|^{p/3} = c_1|\overline{\boldsymbol{F}}|^p|\det \boldsymbol{F}|^{p/3}\\
&\leq \tfrac{c_1p}{q}|\overline{\boldsymbol{F}}|^q + \tfrac{c_1(q-p)}{q}|\det \boldsymbol{F}|^{pq/(3(q-p))}\\
&= \tfrac{c_1p}{q}|U(\boldsymbol{F}^\top\boldsymbol{F})|^{q/2} + \tfrac{c_1(q-p)}{q}|\iota_3(\boldsymbol{F}^\top\boldsymbol{F})|^{pq/(6(q-p))}
\end{aligned}
$$

thanks to Young's inequality $|xy| \leq \frac{1}{a}|x|^a + \frac{1}{a'}|y|^{a'}$ with the exponents $a = q/p$ and $a' = \frac{a}{a-1} = q/(q-p)$, which imposes the constraint

$$
q > p \geq 2\,. \qquad (3.74a)
$$

With the same reasoning we additionally find that

$$
\begin{aligned}
c_2|\operatorname{cof}\boldsymbol{F}|^{p_2} &= c_2|(\det \boldsymbol{F})^{-2/3}\operatorname{cof}\boldsymbol{F}|^{p_2}|\det \boldsymbol{F}|^{2p_2/3} = c_2|\overline{\boldsymbol{H}}|^{p_2}|\det \boldsymbol{F}|^{2p_2/3}\\
&\leq \tfrac{c_2p_2}{q_2}|\overline{\boldsymbol{H}}|^{q_2} + \tfrac{c_2(q_2-p_2)}{q_2}|\det \boldsymbol{F}|^{2p_2q_2/(3(q_2-p_2))}\\
&= \tfrac{c_2p_2}{q_2}|V(\boldsymbol{F}^\top\boldsymbol{F})|^{q_2/2} + \tfrac{c_2(q_2-p_2)}{q_2}|\iota_3(\boldsymbol{F}^\top\boldsymbol{F})|^{p_2q_2/(3(q_2-p_2))}\,,
\end{aligned}
$$

thanks to Young's inequality $|xy| \leq \frac{1}{a}|x|^a + \frac{1}{a'}|y|^{a'}$ with the exponents $a = q_2/p_2$ and $a' = \frac{a}{a-1} = \frac{q_2}{q_2-p_2}$, which results in the constraint

$$
q_2 > p_2 \geq \tfrac{p}{p-1}\,. \qquad (3.74b)
$$

The determinant terms can be further estimated using that $a^\alpha \leq (a+1)^\alpha \leq (a+1)^\beta \leq 2^{\beta-1}(a^\beta+1)$ for $a > 0$, $0 \leq \alpha \leq \beta$ and $\beta \geq 1$. In this way, we find

$$
\begin{aligned}
&c_3|\det \boldsymbol{F}|^{p_3} + \tfrac{c_1(q-p)}{q}|\det \boldsymbol{F}|^{pq/(3(q-p))} + \tfrac{c_2(q_2-p_2)}{q_2}|\det \boldsymbol{F}|^{2p_2q_2/(3(q_2-p_2))}\\
&\leq c_3\big(|\det \boldsymbol{F}|+1\big)^{p_3} + \tfrac{c_1(q-p)}{q}\big(|\det \boldsymbol{F}|+1\big)^{pq/(3(q-p))} + \tfrac{c_2(q_2-p_2)}{q_2}\big(|\det \boldsymbol{F}|+1\big)^{2p_2q_2/(3(q_2-p_2))}\\
&\leq \big(c_3 + \tfrac{c_1(q-p)}{q} + \tfrac{c_2(q_2-p_2)}{q_2}\big)\big(|\det \boldsymbol{F}|+1\big)^{q_3}\\
&\leq 2^{q_3-1}\big(c_3 + \tfrac{c_1(q-p)}{q} + \tfrac{c_2(q_2-p_2)}{q_2}\big)\big(|\det \boldsymbol{F}|^{q_3}+1\big)\,,
\end{aligned}
$$

given that

$$q_3 \geq \max\left\{p_3, \tfrac{pq}{3(q-p)}, \tfrac{p_2 q_2}{3(q_2-p_2)}\right\} > 1 \text{ if } W(\boldsymbol{F}) = \overline{W}(|\overline{\boldsymbol{F}}|, |\overline{\boldsymbol{H}}|, \det \boldsymbol{F})\,,$$
$$q_3 \geq \max\left\{p_3, \tfrac{pq}{3(q-p)}\right\} > 1 \text{ if } W(\boldsymbol{F}) = \overline{W}(|\overline{\boldsymbol{F}}|, \det \boldsymbol{F})\,. \tag{3.74c}$$

Thus, under the validity of the constraints (3.74), we are in the position to conclude

$$\begin{aligned}
&c_1|\boldsymbol{F}|^p + c_2|\operatorname{cof}\boldsymbol{F}|^{p_2} + c_3|\det\boldsymbol{F}|^{p_3}\\
&\leq \tfrac{c_1 p}{q}|\overline{\boldsymbol{F}}|^q + \tfrac{c_2 p_2}{q_2}|\overline{\boldsymbol{H}}|^{q_2} + 2^{q_3-1}\big(c_3 + \tfrac{c_1(q-p)}{q} + \tfrac{c_2(q_2-p_2)}{q_2}\big)\big(|\det\boldsymbol{F}|^{q_3}+1\big)\\
&= \tfrac{c_1 p}{q}|\iota_1(\boldsymbol{F}^\top\boldsymbol{F})|^{q/2} + \tfrac{c_2 p_2}{q_2}|\iota_2(\boldsymbol{F}^\top\boldsymbol{F})|^{q_2/2} + 2^{q_3-1}\big(c_3 + \tfrac{c_1(q-p)}{q} + \tfrac{c_2(q_2-p_2)}{q_2}\big)\big(|\iota_3(\boldsymbol{F}^\top\boldsymbol{F})|^{q_3/2}+1\big)
\end{aligned}$$

This proves (3.73b) with the constants $\tilde{c}_1 = \frac{c_1 p}{q}$, $\tilde{c}_2 = \frac{c_2 p_2}{q_2}$, $\tilde{c}_3 = \tilde{C} = 2^{q_3-1}\big(c_3 + \frac{c_1(q-p)}{q} + \frac{c_2(q_2-p_2)}{q_2}\big)$. □

Remark 3.25. *If the modified invariants are used, the density W depends on $U(\boldsymbol{F}^\top\boldsymbol{F})$, hence at least on $\boldsymbol{F}$ and on $\det\boldsymbol{F}$. Therefore, the sub-cases* (3.10d) **b1) & b2)** *are irrelevant and the coercivity estimate has to feature the term $|\det\boldsymbol{F}|^{q_3}$ with a suitable power q_3 satisfying* (3.73a). *An analogous observation holds true if the energy density also depends on $V(\boldsymbol{F}^\top\boldsymbol{F})$.*

We further observe that, even if enriched by an additional summand involving the determinant, neither the modified Neo-Hooke material (3.69a) *nor the modified Mooney-Rivlin material* (3.69b) *complies with the modified coercivity condition* (3.73), *since in both cases $q = 2$, which does not allow for $q > p \geq 2$. Yet, according to* (3.73a), *it is possible to find $p > 3/2$ for q_3 large enough. Then our existence proof may be carried out on the basis of [Bal77, Thm. 6.1], at the price of the identification of the distributional minors with the minars as a function, cf. also Remark 3.6.*

It remains to show that some of the energy densities (3.69)–(3.71) match with the modified coercivity condition (3.73). As pointed out in the above Remark 3.25 the presence of a determinant term is of importance. Therefore, we give in Cor. 3.26 some exemplary combinations terms being functions of the modified invariants, but they may be also combined differently.

Corollary 3.26 (Energy densities matching with coercivity condition (3.73))**.** *Consider the densities*

$$\overline{W}(U,\iota_3) = \sum_{i=1}^{2} c_i\big(U-3\big)^i + d(\det\boldsymbol{F}-1)^2 + \frac{h}{\iota_3^\gamma}\,, \tag{3.75a}$$

$$\overline{W}(U,V,\iota_3) = \sum_{i=1}^{3} c_i\big(U-3\big)^i + d(\det\boldsymbol{F}-1)^2 + f\big(V^{3/2}-3^{3/2}\big) + \frac{h}{\iota_3^\gamma}\,, \tag{3.75b}$$

$$\overline{W}(U,V,\iota_3) = \sum_{i=1}^{m_1} c_i\big(U^i-3^i\big) + \sum_{i=2}^{m_2} f_i\big(V^i-3^i\big) + \sum_{i=1}^{m_3} d_i(\iota_3^i-1) + \frac{h}{\iota_3^\gamma}\,, \tag{3.75c}$$

$$\overline{W}(U,V,\iota_3) = c_1(U^{\alpha_1}-3^{\alpha_1})^{r_1} + c_2(V^{3\alpha_2/2}-3^{3\alpha_2/2})^{r_2} + c_3(\iota_3^{\alpha_3}-1)^{r_3} + \frac{h}{\iota_3^\gamma} \tag{3.75d}$$

with $U = U(\boldsymbol{F}^\top\boldsymbol{F})$, $V = V(\boldsymbol{F}^\top\boldsymbol{F})$, and $\iota_3 = \iota_3(\boldsymbol{F}^\top\boldsymbol{F})$. Assume that all the coefficients are positive, i.e., $d, f > 0$, $h \geq 0$ and $c_i, d_i, f_i > 0$ for any $i \in \mathbb{N}$ and that the exponents satisfy $\gamma, \alpha_i, r_i \geq 1$, $r_i \in \mathbb{N}$.

1. *The density* (3.75a) *satisfies* (3.73) *with the exponents $q = 4$ and $q_3 = 2$. Moreover,* (3.10d) **b)** *is satisfied with the exponents $p = 12/5$ and $p_3 = 2$.*
2. *The density* (3.75b) *satisfies* (3.73) *with the exponents $q = 6$, $q_2 = 3$, and $q_3 = 2$. Indeed,* (3.10d) **b)** *holds true with the exponents $p = 3$, $p_2 = \frac{3}{2} = \frac{p}{p-1}$, and $p_3 = 2$.*
3. *The density* (3.75c) *satisfies* (3.73) *with the exponents $q = 2m_1$, $q_2 = 2m_2$, and $q_3 = 2m_3$. Moreover,* (3.10d) **a)** *holds true with an exponent $p = \frac{6m_1m_3}{m_1+3m_3} > 3$ if $m_1 \geq 2$ and $m_3 > \frac{m_1}{2m_1-3}$. Otherwise,* (3.10d) **b)** *holds true if $2 \leq m_1, m_3 \in \mathbb{N}$ and $2 \leq m_2 \in \mathbb{N}$ with the exponents $p = \frac{6m_1m_3}{m_1+3m_3} \geq 2$, $p_2 = \frac{6m_2m_3}{2m_2+3m_3} \geq \frac{p}{p-1}$, and $p_3 = 2m_3$.*
4. *The density* (3.75d) *satisfies* (3.73) *with the exponents $q = 2\alpha_1 r_1$, $q_2 = 2\alpha_2 r_2$, and $q_3 = 2\alpha_3 r_3$. Moreover, with $m_i = \alpha_i r_i$ for $i = 1, 2, 3$, the exponents p, p_2, p_3 are determined by statement 3., and* (3.10d) *holds true under the constraints on m_i from 3..*

Proof. We first note that the volumetric energy term h/ι_3^γ is positive. Hence, it can be neglected when deducting the forthcoming coercivity estimates for each of the energy densities (3.75).

Proof of 1.: Firstly, we estimate the determinant term with Young's inequality as follows

$$d(\det\boldsymbol{F} - 1)^2 = d((\det\boldsymbol{F})^2 - 2\det\boldsymbol{F} + 1) \geq d(\tfrac{1}{2}(\det\boldsymbol{F})^2 - 1)\,. \tag{3.76}$$

Using that $U(\boldsymbol{F}^\top\boldsymbol{F}) = |\overline{\boldsymbol{F}}|^2$ we find, again via Young's inequality with the exponent $a = a' = 2$,

$$\begin{aligned}\sum_{i=1}^{2} c_i\big(U(\boldsymbol{F}^\top\boldsymbol{F}) - 3\big)^i &= c_2\big(|\overline{\boldsymbol{F}}|^4 - 6|\overline{\boldsymbol{F}}|^2 + 9\big) + c_1\big(|\overline{\boldsymbol{F}}|^2 - 3\big) = c_2|\overline{\boldsymbol{F}}|^4 + (c_1 - 6c_2)|\overline{\boldsymbol{F}}|^2\\ &\quad + 9c_2 - 3c_1\\ &\geq \tfrac{c_2}{2}|\overline{\boldsymbol{F}}|^4 - \tfrac{(c_1-6c_2)^2}{2c_2} + 9c_2 - 3c_1\end{aligned} \tag{3.77}$$

Similarly as in the proof of Prop. 3.24 we now estimate that

$$\begin{aligned}\tfrac{c_2}{2}|\overline{\boldsymbol{F}}|^4 + \tfrac{d}{4}(\det\boldsymbol{F})^2 + \tfrac{d}{4}(\det\boldsymbol{F})^2 &\geq \big(\tfrac{c_2}{2}\tfrac{6}{6-p}\big)^{(6-p)/6}|\overline{\boldsymbol{F}}|^p\big(\tfrac{d}{4}\tfrac{6}{p}\big)^{p/6}|\det\boldsymbol{F}|^{p/3} + \tfrac{d}{4}(\det\boldsymbol{F})^2\\ &= \big(\tfrac{c_2}{2}\tfrac{6}{6-p}\big)^{(6-p)/6}\big(\tfrac{d}{4}\tfrac{6}{p}\big)^{p/6}|\boldsymbol{F}|^p + \tfrac{d}{4}(\det\boldsymbol{F})^2\,,\end{aligned} \tag{3.78}$$

where we used Young's inequality with the exponents $a = 6/p > 1$ and $a' = \frac{a}{a-1} = \frac{6}{6-p}$. The first choice implies $p < 6$ and for a' we require that $a'p = \frac{6p}{6-p} \overset{!}{=} 4$, which settles the exponent $p = 12/5 \in (2,3)$.

Proof of 2.: We estimate the term with power 3 as follows

$$c_3(|\overline{\boldsymbol{F}}|^2 - 3)^3 = c_3(|\overline{\boldsymbol{F}}|^6 - 9|\overline{\boldsymbol{F}}|^4 + 27|\overline{\boldsymbol{F}}|^2 - 81) \geq \tfrac{c_3}{3}|\overline{\boldsymbol{F}}|^6 - C\,.$$

Here we estimated the quadratic term from below by 0 and for the negative quartic term we used Young's inequality with the exponents $a = 6/4$ and $a' = \frac{a}{a-1} = 3$. We also recall that $f\big(V(\boldsymbol{F}^\top \boldsymbol{F})^{3/2} - 3^{3/2}\big) = f\big(|\overline{\boldsymbol{H}}|^3 - 3^{3/2}\big)$ and combine this and the above estimate with the findings from (3.76) & (3.77), i.e.,

$$
\begin{aligned}
&\tfrac{c_3}{3}|\overline{\boldsymbol{F}}|^6 + \tfrac{c_2}{2}|\overline{\boldsymbol{F}}|^4 + \tfrac{d}{4}(\det \boldsymbol{F})^2 + f|\overline{\boldsymbol{H}}|^3 + \tfrac{d}{8}(\det \boldsymbol{F})^2 + \tfrac{d}{8}(\det \boldsymbol{F})^2 - C \\
&\geq \tfrac{c_3}{3}|\overline{\boldsymbol{F}}|^6 + \tfrac{d}{4}(\det \boldsymbol{F})^2 + f|\overline{\boldsymbol{H}}|^3 + \tfrac{d}{8}(\det \boldsymbol{F})^2 + \tfrac{d}{8}(\det \boldsymbol{F})^2 - C \\
&\geq \big(\tfrac{6}{6-p}\tfrac{c_3}{3}\big)^{(6-p)/6}|\overline{\boldsymbol{F}}|^p\big(\tfrac{6d}{4p}\big)^{p/6}|\det \boldsymbol{F}|^{p/3} + \big(\tfrac{3f}{3-p_2}\big)^{(3-p_2)/3}|\overline{\boldsymbol{H}}|^{p_2}\big(\tfrac{3d}{8p_2}\big)^{p_2/3}|\det \boldsymbol{F}|^{2p_2/3} \\
&\quad + \tfrac{d}{8}(\det \boldsymbol{F})^2 - C \\
&= \big(\tfrac{6}{6-p}\tfrac{c_3}{3}\big)^{(6-p)/6}\big(\tfrac{6d}{4p}\big)^{p/6}|\boldsymbol{F}|^p + \big(\tfrac{3f}{3-p_2}\big)^{(3-p_2)/3}\big(\tfrac{3d}{8p_2}\big)^{p_2/3}|\operatorname{cof} \boldsymbol{F}|^{p_2} + \tfrac{d}{8}(\det \boldsymbol{F})^2 - C\,.
\end{aligned}
$$

To get from the second to third line, we applied Young's inequality with the exponents $a = 6/p > 1$ and $a' = \frac{6}{6-p}$ to the first two summands. Again we find the constraint $p < 6$ and additionally require that $a'p = \frac{6p}{6-p} \stackrel{!}{=} 6$. This yields $p = 3$. Moreover, to the third and the fourth summand we also applied Young's inequality with the exponents $b = 3/p_2$ and $b' = \frac{b}{b-1} = \frac{3}{3-p_2}$ and we have to ensure that $b'p_2 \stackrel{!}{=} 3$. This yields $p_2 = 3/2$. For $p = 3$ we now check that indeed $p_2 = 3/2 \geq \frac{p}{p-1} = 3/2$.

Proof of 3.: Since all the coefficients are positive, gathering all the constants in $C > 0$, we can estimate

$$
\begin{aligned}
&\overline{W}(U(\boldsymbol{F}^\top \boldsymbol{F}), V(\boldsymbol{F}^\top \boldsymbol{F}), \iota_3(\boldsymbol{F}^\top \boldsymbol{F})) > c_{m_1}U(\boldsymbol{F}^\top \boldsymbol{F})^{m_1} + f_{m_2}V(\boldsymbol{F}^\top \boldsymbol{F})^{m_2} + d_{m_3}\iota_3(\boldsymbol{F}^\top \boldsymbol{F})^{m_3} - C \\
&= c_{m_1}|\overline{\boldsymbol{F}}|^{2m_1} + f_{m_2}|\overline{\boldsymbol{H}}|^{2m_2} + d_{m_3}(\det \boldsymbol{F})^{2m_3} - C \\
&\geq c_{m_1}|\overline{\boldsymbol{F}}|^{2m_1} + \tfrac{d_{m_3}}{8}(\det \boldsymbol{F})^{2m_3} + f_{m_2}|\overline{\boldsymbol{H}}|^{2m_2} + \tfrac{d_{m_3}}{8}(\det \boldsymbol{F})^{2m_3} + \tfrac{d_{m_3}}{4}(\det \boldsymbol{F})^{2m_3} - C \\
&\geq \big(\tfrac{2m_1 c_{m_1}}{p}\big)^{p/(2m_1)}|\overline{\boldsymbol{F}}|^p\big(\tfrac{2m_1 d_{m_3}}{8(2m_1-p)}\big)^{(2m_1-p)/(2m_1)}|\det \boldsymbol{F}|^{p/3} \\
&\qquad + \big(\tfrac{2m_2 f_{m_2}}{p_2}\big)^{p_2/(2m_2)}|\overline{\boldsymbol{H}}|^{p_2}\big(\tfrac{2m_2}{2m_2-p_2}\tfrac{d_{m_3}}{8}\big)^{(2m_2-p_2)/(2m_2)}|\det \boldsymbol{F}|^{2p_2/3} + \tfrac{d_{m_3}}{4}(\det \boldsymbol{F})^{2m_3}
\end{aligned}
\tag{3.79}
$$

Here we used the relations (3.58) to arrive at the second line. To get from the third to the fourth line we applied Young's inequality to the first and the second term with the exponents $a = \frac{2m_1}{p}$ and $a' = \frac{a}{a-1} = \frac{2m_1}{2m_1-p}$. We have to ensure that $a'p/3 \stackrel{!}{=} 2m_3$. This yields $p = \frac{6m_1m_3}{m_1+3m_3}$ and we have to make sure that $p \geq 2$. This gives the constraint $m_3 \geq \frac{m_1}{3(m_1-1)}$, which holds true for any $1 < m_1 \in \mathbb{N}$ and $1 < m_3 \in \mathbb{N}$. Moreover, we applied Young's inequality to the third and the fourth summand with the exponents $b = \frac{2m_2}{p_2}$ and $b' = \frac{b}{b-1} = \frac{2m_2}{2m_2-p_2}$. Here we have to make sure that $b'\frac{2p_2}{3} \stackrel{!}{=} 2m_3$, which yields $p_2 = \frac{6m_2m_3}{2m_2+3m_3}$. We now check that $p_2 \geq \frac{p}{p-1}$. Using our findings for p and p_2, this amounts to the constraint $m_2 \geq \frac{m_1m_3}{2m_1m_3-m_1-m_3}$. We note that $2m_1m_3 - m_1 - m_3 > 0$ for any $m_3 > \frac{m_1}{2m_1-1}$, where $\frac{m_1}{2m_1-1} < 1$ for any $1 < m_1 \in \mathbb{N}$. Moreover, we observe that $\frac{m_1m_3}{2m_1m_3-m_1-m_3} \leq 1$, which is equivalent to the constraint $m_3 \geq \frac{m_1}{m_1-1}$, which indeed is satisfied by any $2 \leq m_1, m_3 \in \mathbb{N}$.

Proof of 4.: The density (3.75d) is composed of polynomial terms $P(A) = C(A^\alpha - c^\alpha)^r$ with A a placeholder for ι_3, U, V. Using the polynomial expansion with binomial coefficients b_i we find

$$P(A) = C\sum_{i=0}^{r} b_i (A^\alpha)^{r-i}(-c^\alpha)^i \geq Cb_0 A^{\alpha r} - C\sum_{i=1}^{r} b_i (A^\alpha)^{r-i}(c^\alpha)^i \geq \frac{Cb_r}{2}A^{\alpha r} - B\,, \quad (3.80)$$

where we applied Young's inequality with the exponents $a = r/(r-i) > 1$ for $i > 1$ and $a' = r/i$ to estimate the lower order terms as follows

$$\begin{aligned} b_i(A^\alpha)^{r-i}(c^\alpha)^i &= \Big(\tfrac{b_0 r}{2(r-1)(r-i)}\Big)^{(r-i)/r}(A^\alpha)^{r-i}(c^\alpha)^i b_i\Big(\tfrac{2(r-1)(r-i)}{b_0 r}\Big)^{(r-i)/r} \\ &\leq \Big(\tfrac{b_0}{2(r-1)}\Big)(A^\alpha)^r + \tfrac{i}{r}\Big((c^\alpha)^i b_i\Big(\tfrac{2(r-1)(r-i)}{b_0 r}\Big)^{(r-i)/r}\Big)^{r/i}. \end{aligned}$$

Summation over i thus yields (3.80) by gathering all the constants in B. Now we may invoke the previously proved assertion 3. using that

$$\begin{aligned} &\overline{W}(U(\boldsymbol{F}^\top\boldsymbol{F}), V(\boldsymbol{F}^\top\boldsymbol{F}), \iota_3(\boldsymbol{F}^\top\boldsymbol{F})) > \tfrac{c_1 b_{r_1}}{2}U(\boldsymbol{F}^\top\boldsymbol{F})^{\alpha_1 r_1} + \tfrac{c_2 b_{r_2}}{2}V(\boldsymbol{F}^\top\boldsymbol{F})^{\alpha_2 r_2}\tfrac{c_3 b_{r_3}}{2}\iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{\alpha_3 r_3} - B \\ &= \tfrac{c_1 b_{r_1}}{2}|\overline{\boldsymbol{F}}|^{2\alpha_1 r_1} + \tfrac{c_2 b_{r_2}}{2}|\overline{\boldsymbol{H}}|^{2\alpha_2 r_2} + \tfrac{c_3 b_{r_3}}{2}|(\det \boldsymbol{F})|^{2\alpha_3 r_3} - B \\ &= c_{m_1}|\overline{\boldsymbol{F}}|^{2m_1} + f_{m_2}|\overline{\boldsymbol{H}}|^{2m_2} + d_{m_3}(\det \boldsymbol{F})^{2m_3} - C\,, \end{aligned}$$

i.e., we set $m_i = \alpha_i r_i$ for $i = 1,2,3$ and now continue in (3.79). In this way we find the desired coercivity estimate and the exponents $p = \frac{6m_1 m_3}{m_1+3m_3} = \frac{6\alpha_1 r_1 \alpha_3 r_3}{\alpha_1 r_1 + 3\alpha_3 r_3}$, $p_2 = \frac{6m_2 m_3}{2m_2+3m_3} = \frac{6\alpha_2 r_2 \alpha_3 r_3}{2\alpha_2 r_2 + 3\alpha_3 r_3}$ and $m_3 = \alpha_3 r_3 \geq \frac{m_1}{m_1-1} = \frac{\alpha_1 r_1}{\alpha_1 r_1 - 1}$. □

Stress control (3.10f) **& uniform continuity of the stresses** (3.10g)**:** Relations (3.60)–(3.63) for the derivatives of the invariant functions allow us to give the following statement regarding the assumptions on the stress control (3.10f) and the uniform continuity of the stresses (3.10g):

Proposition 3.27 (Stress control (3.10f) & uniform continuity of the stresses (3.10g))**.** *Let the relations* (3.58)–(3.60) *hold true.*

1 *Assume that there is a constant $K > 0$ such that $\overline{W} : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfies for all $(U, V, \iota_3) \in \mathbb{R}^3_+$:*

$$|\partial_U \overline{W}(U,V,\iota_3)U| + |\partial_V \overline{W}(U,V,\iota_3)V| + |\partial_{\iota_3}\overline{W}(U,V,\iota_3)\iota_3| \leq K(\overline{W}(U,V,\iota_3)+1)\,. \quad (3.81)$$

Then stress control (3.10f) *is true. The assertion remains true if $\iota_3(\boldsymbol{F}^\top\boldsymbol{F})$ is replaced by $\det \boldsymbol{F}$ in* (3.81)*.*

2 *Assume that there is a modulus of continuity $o : [0,\infty] \to [0,\infty]$, $\delta > 0$, so that for all $(\boldsymbol{F}, s) \in \mathbb{R}^{3\times3}\times\mathbb{R}$ and all $C \in \mathrm{GL}_+(3)$ with $|C - \mathbb{1}| \leq \delta$ we have*

$$\begin{aligned} &\Big|\Big(\tfrac{\partial\overline{W}(U,V,\iota_3)}{\partial U}\tfrac{\partial U((C\boldsymbol{F})^\top(C\boldsymbol{F}))}{\partial(C\boldsymbol{F})} + \tfrac{\partial\overline{W}(U,V,\iota_3)}{\partial V}\tfrac{\partial V((C\boldsymbol{F})^\top(C\boldsymbol{F}))}{\partial(C\boldsymbol{F})} + \tfrac{\partial\overline{W}(U,V,\iota_3)}{\partial\iota_3}\tfrac{\partial\iota_3((C\boldsymbol{F})^\top(C\boldsymbol{F}))}{\partial(C\boldsymbol{F})}\Big)(C\boldsymbol{F})^\top \\ &\qquad - \Big(\tfrac{\partial\overline{W}(U,V,\iota_3)}{\partial U}\tfrac{\partial U(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} + \tfrac{\partial\overline{W}(U,V,\iota_3)}{\partial V}\tfrac{\partial V(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}} + \tfrac{\partial\overline{W}(U,V,\iota_3)}{\partial\iota_3}\tfrac{\partial\iota_3(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}}\Big)\boldsymbol{F}^\top\Big| \\ &\leq o(|C-\mathbb{1}|)(\overline{W}(U,V,\iota_3)+\tilde{c}) = o(|C-\mathbb{1}|)(W(\boldsymbol{F})+\tilde{c})\,. \end{aligned} \quad (3.82)$$

Then the uniform continuity of the stresses (3.10g) *is true.*

Proof. **Proof of 1.:** From (3.60) we infer that

$$|\partial_{\boldsymbol{F}} W(\boldsymbol{F})\boldsymbol{F}^\top| \leq \left|\partial_U \overline{W}(U,V,\iota_3)\right| |\partial_{\boldsymbol{F}} U \boldsymbol{F}^\top| + \left|\partial_V \overline{W}(U,V,\iota_3)\right| |\partial_{\boldsymbol{F}} V \boldsymbol{F}^\top| \\ + \left|\partial_{\iota_3} \overline{W}(U,V,\iota_3)\right| |\partial_{\boldsymbol{F}} \iota_3 \boldsymbol{F}^\top| \,,$$

with $U = U(\boldsymbol{F}^\top\boldsymbol{F}), V = V(\boldsymbol{F}^\top\boldsymbol{F})$, and $\iota_3 = \iota_3(\boldsymbol{F}^\top\boldsymbol{F})$. The stress control estimate for the invariant functions (3.62) and assumption (3.81) now allow us to conclude that

$$|\partial_{\boldsymbol{F}} W(\boldsymbol{F})\boldsymbol{F}^\top| \leq c_U|\partial_U \overline{W}(U,V,\iota_3)U| + c_V|\partial_V \overline{W}(U,V,\iota_3)V| + c_{\iota_3}|\partial_{\iota_3} \overline{W}(U,V,\iota_3)\iota_3| \\ \leq \tilde{c}K(\overline{W}(U,V,\iota_3)+1) = \tilde{c}K(W(\boldsymbol{F})+1)\,,$$

which is stress control (3.10f).

Proof of 2.: The condition on continuity of the stresses (3.10g) formulated for the density W, directly follows from (3.82) using the relations $W(\boldsymbol{F}) = \overline{W}(U(\boldsymbol{F}^\top\boldsymbol{F}), V(\boldsymbol{F}^\top\boldsymbol{F}), \iota_3(\boldsymbol{F}^\top\boldsymbol{F}))$ and (3.60) for the first Piola-Kirchhoff stress. ☐

Corollary 3.28 (Energy densities matching with Prop. 3.27)**.** *The densities* (3.75) *introduced in Cor. 3.26 satisfy conditions* (3.81) *&* (3.82)*.*

Proof. Apart from the quadratic term $(\det \boldsymbol{F}-1)^2$, all the other terms contributing to the energy densities $\overline{W}$ from Cor. 3.26 are power laws wrt. the modified invariants. More precisely, these terms take the form $P(A) = C(A^\alpha - c^\alpha)^r - B$, where A is a placeholder for a modified invariant $A(\boldsymbol{F}^\top\boldsymbol{F}) \geq 0$, cf. (3.58c), with $A = \iota_3, U, V$ and with $c = A(\mathbb{1})$ and $B = CA(\mathbb{1})$. Thus, we may restrict the analysis of P to the following scenario

$$A \geq 0\,, \quad c \geq 1\,, \quad B \geq 0\,, \quad C > 0, \quad \alpha \geq 1\,, \quad r > 1\,. \tag{3.83}$$

and it is

$$P'(A) = Cr\alpha A^{\alpha-1}(A^\alpha - c^\alpha)^{r-1}\,. \tag{3.84}$$

To (3.81)**:** In view of (3.84) we have

$$|P'(A)A| = |Cr\alpha A^\alpha (A^\alpha - c^\alpha)^{r-1}| \leq Cr\alpha\big(|(A^\alpha - c^\alpha)^r| + |c^\alpha(A^\alpha - c^\alpha)^{r-1}|\big)\,. \tag{3.85}$$

We now estimate the first term on the right-hand side. For $A^\alpha - c^\alpha \geq 0$ it is $|(A^\alpha - c^\alpha)^r| \leq (A^\alpha - c^\alpha)^r$. For $A^\alpha - c^\alpha < 0$ we use (3.83) to see that $A^\alpha - c^\alpha > -c^\alpha$, so that $|(A^\alpha - c^\alpha)^r| \leq 2c^{\alpha r} + (A^\alpha - c^\alpha)^r$. Hence, by combining the two cases we find

$$|(A^\alpha - c^\alpha)^r| \leq 2c^{\alpha r} + (A^\alpha - c^\alpha)^r \tag{3.86}$$

We now estimate the second term on the right-hand side of (3.85). Here we may use Young's inequality with the exponents $a = r/(r-1)$, $a' = r$, to find $|A^\alpha - c^\alpha|^{r-1} \leq \frac{r-1}{r}|A^\alpha - c^\alpha|^r + \frac{1}{r}$, which can be further processed by (3.86). By combining the estimates for the two terms, then adding and subtracting B, we indeed find a constant $K' > 0$ such that

$$|P'(A)A| \leq K'(P(A)+1)\,. \tag{3.87}$$

Moreover, we may check that the quadratic term $(\det \boldsymbol{F}-1)^2$ and its derivative can be estimated also in the form of (3.87) by applying Young's inequality with the exponent $a = a' = 2$ directly in (3.85). We also note that the above arguments remain true for $P(A) = A^{-\gamma}$, i.e., $-\gamma = \alpha r$ and $c^\alpha = 0,$ which shows the stress control for the volumetric term h/ι_3^γ. This finishes the proof of (3.81).

To (3.82)**:** Again we consider a function of the form $P(A) = C(A^\alpha - c^\alpha)^r - B$ with the properties (3.83) and first derivative P' from (3.84). Revisiting the proof of estimate (3.87) we see that the second derivative $P''(A) = c\alpha r\big((\alpha-1)A^{\alpha-2}(A^\alpha - c^\alpha)^{r-1} + (r-1)(A^\alpha - c^\alpha)^{r-2}A^{\alpha-1}\big)$ satisfies an analogous estimate, more precisely, with a constant $K'' > 0$ we have

$$|P''(A)A| \leq K''(P'(A)+1)\,. \tag{3.88}$$

This will be used to show that

$$|\partial_{\boldsymbol{F}} P(A((C\boldsymbol{F})^\top(C\boldsymbol{F})))(C\boldsymbol{F})^\top - \partial_{\boldsymbol{F}} P(A(\boldsymbol{F}^\top\boldsymbol{F}))\boldsymbol{F}^\top| \leq o(|C-\mathbb{1}|)\big(P(A(\boldsymbol{F}^\top\boldsymbol{F})) + \tilde{c}\big)\,. \tag{3.89}$$

For brevity we now set $\boldsymbol{G} := C\boldsymbol{F}$ to find

$$\begin{aligned}
&|\partial_{\boldsymbol{F}} P(A(\boldsymbol{G}^\top\boldsymbol{G}))\boldsymbol{G}^\top - \partial_{\boldsymbol{F}} P(A(\boldsymbol{F}^\top\boldsymbol{F}))\boldsymbol{F}^\top|\\
&= \big|P'(A(\boldsymbol{G}^\top\boldsymbol{G}))\big(\partial_{\boldsymbol{G}} A(\boldsymbol{G}^\top\boldsymbol{G})\big)\boldsymbol{G}^\top - P'(A(\boldsymbol{F}^\top\boldsymbol{F}))\big(\partial_{\boldsymbol{F}} A(\boldsymbol{F}^\top\boldsymbol{F})\big)\boldsymbol{F}^\top\big|\\
&\leq |P'(A(\boldsymbol{G}^\top\boldsymbol{G})) - P'(A(\boldsymbol{F}^\top\boldsymbol{F}))|\,\big|\partial_{\boldsymbol{F}} A(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^\top\big|\\
&\quad + |P'(A(\boldsymbol{G}^\top\boldsymbol{G}))|\,\big|\partial_{\boldsymbol{G}} A(\boldsymbol{G}^\top\boldsymbol{G})\boldsymbol{G}^\top - \partial_{\boldsymbol{F}} A(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^\top\big|
\end{aligned} \tag{3.90}$$

In order to further process the terms on the right-hand side of (3.90) we recall from [Cia88, p. 11] that $C^{-1} = (\mathbb{1} + (C-\mathbb{1})\mathbb{1})^{-1} = \mathbb{1} - (C-\mathbb{1}) + o(|C-\mathbb{1}|)$ for the inverse of the matrix C with $|C-\mathbb{1}| < \delta$. Hence also $|C^{-1}-\mathbb{1}| < \hat{\delta} < 1$ for $|C-\mathbb{1}| < \delta$ sufficiently small. Thus we may equivalently apply the continuity properties of the invariants (3.63) for the matrices $\boldsymbol{G}$ and $C^{-1}\boldsymbol{F}$.

In this way, also in view of (3.87), we further estimate the second term in (3.90) as follows

$$\begin{aligned}
&|P'(A(\boldsymbol{G}^\top\boldsymbol{G}))|\,\big|\partial_{\boldsymbol{G}} A(\boldsymbol{G}^\top\boldsymbol{G})\boldsymbol{G}^\top - \partial_{\boldsymbol{F}} A(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^\top\big|\\
&\leq |P'(A(\boldsymbol{G}^\top\boldsymbol{G}))|\,\tilde{o}(|C^{-1}-\mathbb{1}|)A(\boldsymbol{G}^\top\boldsymbol{G})\\
&\leq |P'(A(\boldsymbol{G}^\top\boldsymbol{G}))A(\boldsymbol{G}^\top\boldsymbol{G})|\,o(|C-\mathbb{1}|)\\
&\leq K'(P(A(\boldsymbol{G}^\top\boldsymbol{G}))+1)o(|C-\mathbb{1}|)\\
&\leq K'\Big(P(A(\boldsymbol{F}^\top\boldsymbol{F}))+2\Big)o(|C-\mathbb{1}|)
\end{aligned} \tag{3.91}$$

Here we used continuity estimate (3.63b) for $\boldsymbol{G}$ and $C^{-1}\boldsymbol{F}$ to get to the first estimate. The second estimate ist due to the above explained relation between C^{-1} and C close to $\mathbb{1}$. The third estimate follows by (3.87) and the fourth is due to the continuity of P implying that $P(A(\boldsymbol{G}^\top\boldsymbol{G})) \leq P(A(\boldsymbol{F}^\top\boldsymbol{F}))+1,$ for $|A(\boldsymbol{G}^\top\boldsymbol{G}) - A(\boldsymbol{F}^\top\boldsymbol{F})| \leq o(|C-\mathbb{1}|)A(\boldsymbol{F}^\top\boldsymbol{F})$ sufficiently small.

For the first term in (3.90) we apply the stress control for the invariant (3.62) and (3.88) for P', together with the continuity of the invariant (3.63a) and deduce

$$\begin{aligned}
&|P'(A(\boldsymbol{G}^\top\boldsymbol{G}))-P'(A(\boldsymbol{F}^\top\boldsymbol{F}))|\,\big|\partial_{\boldsymbol{F}}A(\boldsymbol{F}^\top\boldsymbol{F})\big|\\
&\le A(\boldsymbol{F}^\top\boldsymbol{F})|P''(A(\boldsymbol{F}^\top\boldsymbol{F})+t(A(\boldsymbol{G}^\top\boldsymbol{G})-A(\boldsymbol{F}^\top\boldsymbol{F})))|\,|A(\boldsymbol{G}^\top\boldsymbol{G})-A(\boldsymbol{F}^\top\boldsymbol{F})|\\
&\le |P''(A(\boldsymbol{F}^\top\boldsymbol{F})+t(A(\boldsymbol{G}^\top\boldsymbol{G})-A(\boldsymbol{F}^\top\boldsymbol{F})))|o(|C-\mathbb{1}|)A(\boldsymbol{F}^\top\boldsymbol{F})\\
&\le |P''(A(\boldsymbol{F}^\top\boldsymbol{F}))+1|\,o(|C-\mathbb{1}|)A(\boldsymbol{F}^\top\boldsymbol{F})^2\\
&\le \big((K''(P'(A(\boldsymbol{F}^\top\boldsymbol{F})+1)A(\boldsymbol{F}^\top\boldsymbol{F})+A(\boldsymbol{F}^\top\boldsymbol{F})^2\big)\,o(|C-\mathbb{1}|)\\
&\le \big(K''\big(K'(P(A(\boldsymbol{F}^\top\boldsymbol{F}))+1)+A(\boldsymbol{F}^\top\boldsymbol{F})\big)+A(\boldsymbol{F}^\top\boldsymbol{F})^2\big)\,o(|C-\mathbb{1}|)\\
&\le \widetilde{K}\big(P(A(\boldsymbol{F}^\top\boldsymbol{F}))+1\big)\,o(|C-\mathbb{1}|)\,,
\end{aligned}\tag{3.92}$$

where we used Young's inequality and the strategy of the above proof of (3.81) to obtain an estimate for the linear term $A(\boldsymbol{F}^\top\boldsymbol{F})$ and the quadratic term $A(\boldsymbol{F}^\top\boldsymbol{F})^2$ in terms of $P(A(\boldsymbol{F}^\top\boldsymbol{F}))$. The combination of (3.91) & (3.92) further estimates (3.90) and (3.82) yields an estimate of the desired form (3.82). Finally we note that the quadratic term $(\det\boldsymbol{F}-1)^2$ can be estimated in a similar manner using the strategy from (3.95c) ahead and the continuity estimate for h/ι_3^γ is due to (3.95d) ahead. This finishes the proof of (3.82). □

Energy densities satisfying assumptions (3.10)**:** Merging the results from Lemma 3.21, Proposition 3.22, and Corollaries 3.26 & 3.28 allows us to conclude:

Corollary 3.29 (Energy densities satisfying assumptions (3.10))**.** *The densities* (3.75a) *&* (3.75c) *introduced in Cor. 3.26 satisfy all the assumptions* (3.10)*.*

Remark 3.30 (Assumptions (3.10) and the anisotropic split (2.16), resp. (2.24))**.** *As explained in Section 2.2 we may apply the anisotropic split* (2.24) *to the modified invariants in order to account for the anisotropy of damage. This neither affects the polyconvexity nor the coercivity properties of the constitutive law. As we have seen in* (2.23)*, the modified invariants ensure the differentiability of the energy density in* $\mathbb{1}$ *with* $\boldsymbol{P}(\mathbb{1},s)=0$*. Thus, also the results on the stress control and the continuity of the stresses remain valid. If the anisotropic split is applied only to energy contributions in* (3.75) *with positive powers (but not to* $\iota_3^{-\gamma}$*), then each of the anisotropically splitted energies* (3.75) *satisfies all the assumptions* (3.10)*.*

3.5 Proof of Lemmata 3.14 & 3.15 on the properties of the invariants

Proof of Lemma 3.14 on the derivatives of the invariants and their stress control: The proof of relations (3.61a)–(3.61c) for the derivatives of ι_1,ι_2,ι_3 can be taken from [Cia88, p. 154].

To find (3.61d) for the derivative of U we calculate with the product rule and the chain rule

$$\frac{\partial U(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}}=\iota_3^{-1/3}(\boldsymbol{F}^\top\boldsymbol{F})\frac{\partial\iota_1(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}}+\iota_1(\boldsymbol{F}^\top\boldsymbol{F})\big(-\tfrac13\iota_3^{-4/3}(\boldsymbol{F}^\top\boldsymbol{F})\frac{\partial\iota_3(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}}\big)\,.$$

In view of (3.61a) and (3.61c) we now conclude (3.61d).

Similarly, the product and chain rule yield for the derivative of V that

$$\frac{\partial V(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}}=\iota_3^{-2/3}(\boldsymbol{F}^\top\boldsymbol{F})\frac{\partial\iota_2(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}}+\iota_2(\boldsymbol{F}^\top\boldsymbol{F})\big(-\tfrac{2}{3}\iota_3^{-5/3}(\boldsymbol{F}^\top\boldsymbol{F})\frac{\partial\iota_3(\boldsymbol{F}^\top\boldsymbol{F})}{\partial\boldsymbol{F}}\big)\,,$$

which, in combination with (3.61b) and (3.61c), results in (3.61e).

To find the stress control (3.62) we use (3.61a)–(3.61e) and deduce

$$|\partial_{\boldsymbol{F}}\iota_1\boldsymbol{F}^\top|=|2\boldsymbol{F}\boldsymbol{F}^\top|\le 2|\boldsymbol{F}|^2=2\iota_1 \tag{3.93a}$$

$$|\partial_{\boldsymbol{F}}\iota_2\boldsymbol{F}^\top|=\big|2\big(\iota_2\boldsymbol{F}^{-\top}-\iota_3\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}\boldsymbol{F}^{-\top}\big)\boldsymbol{F}^\top\big|=2\big|\iota_2\mathbb{1}-\iota_3\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}\big|\le 2(\sqrt{3}+1)\iota_2 \tag{3.93b}$$

$$|\partial_{\boldsymbol{F}}\iota_3\boldsymbol{F}^\top|=|2\iota_3\mathbb{1}|\le 2\sqrt{3}\iota_3\le 4\iota_3\,, \tag{3.93c}$$

$$|\partial_{\boldsymbol{F}}U\boldsymbol{F}^\top|=|2\iota_3^{-1/3}(\boldsymbol{F}\boldsymbol{F}^\top-\tfrac{1}{3}\iota_1\mathbb{1})|\le|2\iota_3^{-1/3}|(|\iota_1|+\tfrac{1}{\sqrt{3}}|\iota_1|)\le 4U\,, \tag{3.93d}$$

$$|\partial_{\boldsymbol{F}}V\boldsymbol{F}^\top|=|2\iota_3^{-2/3}\big(\tfrac{1}{3}\iota_2\mathbb{1}-\iota_3\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}\big)| \tag{3.93e}$$

$$\le 2\iota_3^{-2/3}\big(|\iota_2|+\iota_3|\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}|\big)=4\iota_3^{-2/3}\iota_2=4V\,, \tag{3.93f}$$

where we also used that $|\boldsymbol{F}^\top\boldsymbol{F}|\le|\boldsymbol{F}|^2$ as well as the relations $|\operatorname{cof}\boldsymbol{F}|^2=\operatorname{tr}\operatorname{cof}\boldsymbol{C}=\iota_2(\boldsymbol{C})$ and

$$|\operatorname{cof}\boldsymbol{C}|^2=(\det\boldsymbol{F})^4\big|(\boldsymbol{F}^\top\boldsymbol{F})^{-1}\big|^2\le(\det\boldsymbol{F})^4|\boldsymbol{F}^{-1}|^4=|\operatorname{cof}\boldsymbol{F}|^4=\iota_2(\boldsymbol{C})^2\,. \tag{3.93g}$$

■

Proof of Lemma 3.15 on the continuity properties of the invariants: For shorter notation we set

$$\begin{aligned}F_1(A)&:=|A((C\boldsymbol{F})^\top(C\boldsymbol{F}))-A(\boldsymbol{F}^\top\boldsymbol{F})|\\F_2(A)&:=|\partial_{\boldsymbol{F}}A((C\boldsymbol{F})^\top(C\boldsymbol{F}))(C\boldsymbol{F})^\top-\partial_{\boldsymbol{F}}A(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^\top|\,,\end{aligned}$$

for each invariant function $A\in\{\iota_1,\iota_2,\iota_3,U,V\}$ and we aim to show (3.63), which now reads

$$F_1(A)\le o(|C-\mathbb{1}|)A(\boldsymbol{F}^\top\boldsymbol{F})\,, \tag{3.94a}$$

$$F_2(A)\le o(|C-\mathbb{1}|)A(\boldsymbol{F}^\top\boldsymbol{F})\,, \tag{3.94b}$$

for each invariant function $A\in\{\iota_1,\iota_2,\iota_3,U,V\}$. We start with $F_1(\iota_1)$. In view of (3.58c) we obtain

$$\begin{aligned}F_1(\iota_1)&=|\iota_1((C\boldsymbol{F})^\top(C\boldsymbol{F}))-\iota_1(\boldsymbol{F}^\top\boldsymbol{F})|=\big||C\boldsymbol{F}|^2-|\boldsymbol{F}|^2\big|=\big|(C+\mathbb{1})\boldsymbol{F}:(C-\mathbb{1})\boldsymbol{F}\big|\\&\le(|C|+1)|\boldsymbol{F}|^2\,|C-\mathbb{1}|\le(\sqrt{3}+2)\iota_1(\boldsymbol{F}^\top\boldsymbol{F})|C-\mathbb{1}|=c_{\text{(3.95a)}}\iota_1(\boldsymbol{F}^\top\boldsymbol{F})|C-\mathbb{1}|\,,\end{aligned} \tag{3.95a}$$

where we also used that $|C|\le|\mathbb{1}|+\delta\le\sqrt{3}+1$. For $F_1(\iota_3)$ we have

$$F_1(\iota_3)=|\iota_3((C\boldsymbol{F})^\top(C\boldsymbol{F}))-\iota_3(\boldsymbol{F}^\top\boldsymbol{F})|=\big|\iota_3(C^\top C)-1\big|\iota_3(\boldsymbol{F}^\top\boldsymbol{F})$$

and we shall now determine a modulus of continuity ι_3 using the mean value theorem of differentiability and formula (3.61c). In particular,

$$|\iota_3(C^\top C) - \iota_3(\mathbb{1})| \le \left|2\iota_3(\widetilde{C}^\top\widetilde{C})\widetilde{C}^{-\top}\right| |C - \mathbb{1}| ,$$

where $\widetilde{C} = \mathbb{1} + t(C - \mathbb{1})$ with $t \in [0,1]$ suitably. In order to calculate $\widetilde{C}^{-1}$ we shall invoke [Cia88, p. 11], which states that

$$(\mathbb{1} + BH)^{-1} = \mathbb{1} - BH + o(H) \quad \text{for } |H| < |B|^{-1} . \tag{3.95b}$$

This yields $\widetilde{C}^{-1} = (\mathbb{1} + t(C - \mathbb{1}))^{-1} = \mathbb{1} - t(C - \mathbb{1}) + o(\mathbb{1})$ for $|\mathbb{1}| < |C - \mathbb{1}|^{-1}$. Now we use that the map $\tilde{f} : \mathrm{GL}_+(3) \to \mathbb{R}$, $\tilde{f}(\widetilde{C}) := |2\iota_3(\widetilde{C}^\top\widetilde{C})\widetilde{C}^{-\top}|$ is continuous, since it is the composition of continuous functions. Thus, by continuity, for any $0 < \varepsilon < 1$ there is $0 < \delta \ll 1$ such that for all $C \in \mathrm{GL}_+(3)$ with $t|C - \mathbb{1}| \le \delta$ we have $|\tilde{f}(\widetilde{C}) - \tilde{f}(\mathbb{1})| < \varepsilon < 1$ and hence $\tilde{f}(\widetilde{C}) < 1 + 2\sqrt{3}$. In this way we conclude that

$$|\iota_3(C^\top C) - \iota_3(\mathbb{1})| \le \left|2\iota_3(\widetilde{C}^\top\widetilde{C})\widetilde{C}^{-\top}\right| |C - \mathbb{1}| \le (1 + 2\sqrt{3})|C - \mathbb{1}| = c_{(3.95c)}|C - \mathbb{1}| . \tag{3.95c}$$

In the same fashion we can also show for powers $\iota_3^{-\gamma/3}$ with $\gamma \in \mathbb{N}$ that

$$|\iota_3(C^\top C)^{-\gamma/3} - 1| \le \hat{f}(\widetilde{C})\, |C - \mathbb{1}| \le (1 + 2\gamma/\sqrt{3})|C - \mathbb{1}| = c_{(3.95d)}|C - \mathbb{1}| . \tag{3.95d}$$

More precisely, we deduce that $|\iota_3(C^\top C)^{-\gamma/3} - 1| \le \hat{f}(\widetilde{C})|C - \mathbb{1}|$, where

$$\hat{f}(\widetilde{C}) = \frac{\gamma}{3}|\iota_3(\widetilde{C}^\top\tilde{C})^{-(\gamma+3)/3} 2\iota_3(\widetilde{C}^\top\widetilde{C})\widetilde{C}^{-\top}| \le |\hat{f}(\mathbb{1})| + 1 \quad \text{for } t|C - \mathbb{1}| < \delta ,$$

and $\hat{f}(\mathbb{1}) = 2\gamma/\sqrt{3}$. With the multiplicativity of cof, i.e. $\mathrm{cof}(M_1M_2) = \mathrm{cof}\, M_1 \,\mathrm{cof}\, M_2$, cf. [Cia88, p. 4], we obtain for $F_1(\iota_2)$ that

$$\begin{aligned}
F_1(\iota_2) &= |\iota_2((C\boldsymbol{F})^\top(C\boldsymbol{F})) - \iota_2(\boldsymbol{F}^\top\boldsymbol{F})| = \left||\mathrm{cof}(C\boldsymbol{F})|^2 - |\mathrm{cof}\,\boldsymbol{F}|^2\right| \\
&= \left||\mathrm{cof}\, C\, \mathrm{cof}\,\boldsymbol{F}|^2 - |\mathrm{cof}\,\boldsymbol{F}|^2\right| \\
&= \left|(\mathrm{cof}\, C + \mathbb{1})\,\mathrm{cof}\,\boldsymbol{F} : (\mathrm{cof}\, C - \mathbb{1})\,\mathrm{cof}\,\boldsymbol{F}\right| \\
&\le (|\mathrm{cof}\, C| + 1)|\mathrm{cof}\,\boldsymbol{F}|^2\, |\mathrm{cof}\, C - \mathbb{1}|
\end{aligned}$$

Using (3.95c) we further deduce that

$$\begin{aligned}
|\mathrm{cof}\, C - \mathbb{1}| &= |(\det C)C^{-T} - \mathbb{1}| = |(\det C - 1)\mathbb{1} + \det C(C^{-\top} - \mathbb{1})| \\
&\le |\det C - 1|\, |\mathbb{1}| + |\mathrm{cof}\, C|\, |C - \mathbb{1}| \le (1 + 2\sqrt{3})|C - \mathbb{1}|\sqrt{3} + |\mathrm{cof}\, C|\, |C - \mathbb{1}|
\end{aligned}$$

Furthermore, by continuity we obtain that $|\mathrm{cof}\, C| \le \sqrt{3} + 1$ for $|C - \mathbb{1}|$ small. Hence we conclude

$$\begin{aligned}
F_1(\iota_2) &= |\iota_2((C\boldsymbol{F})^\top(C\boldsymbol{F})) - \iota_2(\boldsymbol{F}^\top\boldsymbol{F})| \\
&\le (\sqrt{3} + 1)|\mathrm{cof}\,\boldsymbol{F}|^2\big((1 + 2\sqrt{3})|C - \mathbb{1}|\sqrt{3} + (\sqrt{3} + 1)|C - \mathbb{1}|\big) \\
&= c_{(3.95e)}\iota_2(\boldsymbol{F}^\top\boldsymbol{F})|C - \mathbb{1}|
\end{aligned} \tag{3.95e}$$

Estimate (3.95d) with $\gamma = 1$ combined with (3.95a) is now used to determine a modulus of continuity for $F_1(U)$:

$$\begin{aligned} F_1(U) &= |U((C\boldsymbol{F})^\top(C\boldsymbol{F})) - U(\boldsymbol{F}^\top\boldsymbol{F})| \\ &= \left|\iota_3(C^\top C)^{-1/3}\iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{-1/3}|C\boldsymbol{F}|^2 - \iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{-1/3}|\boldsymbol{F}|^2\right| \\ &\le \left|\iota_3(C^\top C)^{-1/3} - 1\right| \iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{-1/3}|\boldsymbol{F}|^2 + \iota_3(C^\top C)^{-1/3}\iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{-1/3}\left||C\boldsymbol{F}|^2 - |\boldsymbol{F}|^2\right| \\ &\le \big(c_{\text{(3.95d)}}U(\boldsymbol{F}^\top\boldsymbol{F}) + \big(c_{\text{(3.95d)}}|C-\mathbb{1}|+1\big)\iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{-1/3}c_{\text{(3.95a)}}\iota_1(\boldsymbol{F}^\top\boldsymbol{F})\big)|C-\mathbb{1}| \\ &\le c_{\text{(3.95f)}}U(\boldsymbol{F}^\top\boldsymbol{F})|C-\mathbb{1}|\,. \end{aligned} \tag{3.95f}$$

Similarly, estimate (3.95d) with $\gamma = 2$ in combination with (3.95e) also results in a modulus of continuity for $F_1(V)$:

$$\begin{aligned} F_1(V) &= |V((C\boldsymbol{F})^\top(C\boldsymbol{F})) - V(\boldsymbol{F}^\top\boldsymbol{F})| \\ &= \left|\iota_3(C^\top C)^{-2/3}\iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{-2/3}|\operatorname{cof} C\boldsymbol{F}|^2 - \iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{-2/3}|\operatorname{cof}\boldsymbol{F}|^2\right| \\ &\le \left|\iota_3(C^\top C)^{-2/3} - 1\right| \iota_3(ldsymbolF^\top\boldsymbol{F})^{-2/3}|\operatorname{cof}\boldsymbol{F}|^2 \\ &\quad + \iota_3(C^\top C)^{-2/3}\iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{-2/3}\left||\operatorname{cof}(C\boldsymbol{F})|^2 - |\operatorname{cof}\boldsymbol{F}|^2\right| \\ &\le c_{\text{(3.95d)}}|C-\mathbb{1}|V(\boldsymbol{F}^\top\boldsymbol{F}) + \big(c_{\text{(3.95d)}}|C-\mathbb{1}|+1\big)\iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{-2/3}c_{\text{(3.95e)}}|\operatorname{cof}\boldsymbol{F}|^2\,|C-\mathbb{1}| \\ &\le c_{\text{(3.95g)}}|C-\mathbb{1}|V(\boldsymbol{F}^\top\boldsymbol{F}) \end{aligned} \tag{3.95g}$$

With estimates (3.95) we have obtained moduli of continuity for $F_1(A)$ with $A \in \{\iota_1, \iota_2, \iota_3, U, V\}$ and thus verified (3.94a), resp. (3.63a).

We now aim to prove (3.63b) by deducing moduli of continuity for $F_2(A)$. We start with $F_2(\iota_1)$. In view of (3.61a) we deduce

$$\begin{aligned} F_2(\iota_1) &= |\partial_{\boldsymbol{F}}\iota_1((C\boldsymbol{F})^\top(C\boldsymbol{F}))(C\boldsymbol{F})^\top - \partial_{\boldsymbol{F}}\iota_1(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^\top| \\ &= 2|(C\boldsymbol{F})(C\boldsymbol{F})^\top - \boldsymbol{F}\boldsymbol{F}^\top| \le 2(|C|+1)|C-\mathbb{1}|\,|\boldsymbol{F}\boldsymbol{F}^\top| \\ &\le 2(\sqrt{3}+2)|C-\mathbb{1}|\,|\boldsymbol{F}|^2 = c_{\text{(3.96a)}}\iota_1(\boldsymbol{F}^\top\boldsymbol{F})|C-\mathbb{1}|\,, \end{aligned} \tag{3.96a}$$

where we used that $|C| < |\mathbb{1}| + 1$ for $|C - \mathbb{1}| < \delta < 1$. For $F_2(\iota_3)$ we find via (3.61c) and (3.95c)

$$\begin{aligned} F_2(\iota_3) &= |\partial_{\boldsymbol{F}}\iota_3((C\boldsymbol{F})^\top(C\boldsymbol{F}))(C\boldsymbol{F})^\top - \partial_{\boldsymbol{F}}\iota_3(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^\top| \\ &= 2|\iota_3((C\boldsymbol{F})^\top(C\boldsymbol{F}))\mathbb{1} - \iota_3(\boldsymbol{F}^\top\boldsymbol{F})\mathbb{1}| \\ &\le 2\sqrt{3}\iota_3(\boldsymbol{F}^\top\boldsymbol{F})|\iota_3(C^\top C) - 1| \le 2\sqrt{3}\,c_{\text{(3.95c)}}\iota_3(\boldsymbol{F}^\top\boldsymbol{F})|C-\mathbb{1}| = c_{\text{(3.96b)}}\iota_3(\boldsymbol{F}^\top\boldsymbol{F})|C-\mathbb{1}|\,. \end{aligned} \tag{3.96b}$$

We now turn to the estimates for $F_2(\iota_2)$, $F_2(U)$ and $F_2(V)$. For better readability we will here often use the short-hand $\boldsymbol{G} = C\boldsymbol{F}$. In view of (3.61b) the term $F_2(\iota_2)$ can be estimated as

$$\begin{aligned}
F_2(\iota_2) &= |\partial_{\boldsymbol{G}}\iota_2(\boldsymbol{G}^\top\boldsymbol{G})\boldsymbol{G}^\top - \partial_{\boldsymbol{F}}\iota_2(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^\top| \\
&= 2\big|\big(\iota_2(\boldsymbol{G}^\top\boldsymbol{G})\boldsymbol{G}^{-\top}\boldsymbol{G}^\top - \iota_3(\boldsymbol{G}^\top\boldsymbol{G})\boldsymbol{G}^{-\top}\boldsymbol{G}^{-1}\boldsymbol{G}^{-\top}\boldsymbol{G}^\top\big) \\
&\qquad - \big(\iota_2(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^{-\top}\boldsymbol{F}^\top - \iota_3(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}\boldsymbol{F}^{-\top}\boldsymbol{F}^\top\big)\big| \\
&= 2\big|\big(\iota_2(\boldsymbol{G}^\top\boldsymbol{G})\mathbb{1} - \iota_3(\boldsymbol{G}^\top\boldsymbol{G})\boldsymbol{G}^{-\top}\boldsymbol{G}^{-1}\big) - \big(\iota_2(\boldsymbol{F}^\top\boldsymbol{F})\mathbb{1} - \iota_3(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}\big)\big| \\
&\le 2\sqrt{3}\big|\iota_2(\boldsymbol{G}^\top\boldsymbol{G}) - \iota_2(\boldsymbol{F}^\top\boldsymbol{F})\big| + 2\big|\iota_3(\boldsymbol{G}^\top\boldsymbol{G})\boldsymbol{G}^{-\top}\boldsymbol{G}^{-1} - \iota_3(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}\big| \\
&\le 2\sqrt{3}\big|\iota_2(\boldsymbol{G}^\top\boldsymbol{G}) - \iota_2(\boldsymbol{F}^\top\boldsymbol{F})\big| \\
&\quad + 2\big|\iota_3(\boldsymbol{G}^\top\boldsymbol{G}) - \iota_3(\boldsymbol{F}^\top\boldsymbol{F})\big|\,|\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}| \\
&\quad + 2\iota_3((C\boldsymbol{F})^\top(C\boldsymbol{F}))\big|\boldsymbol{G}^{-\top}\boldsymbol{G}^{-1} - \boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}\big| \\
&\le \Big(2\sqrt{3}c_{\text{(3.95e)}}\iota_2(\boldsymbol{F}^\top\boldsymbol{F}) + (c_{\text{(3.95c)}} + c_{\text{(3.96d)}})\iota_3(\boldsymbol{F}^\top\boldsymbol{F})|\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}|\Big)\,|C-\mathbb{1}| \\
&= c_{\text{(3.96c)}}\iota_2(\boldsymbol{F}^\top\boldsymbol{F})\,|C-\mathbb{1}|
\end{aligned} \tag{3.96c}$$

Here we also used (3.95e), (3.95c), and in analogy to (3.96a) we deduced

$$\begin{aligned}
\big|\boldsymbol{G}^{-\top}\boldsymbol{G}^{-1} - \boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}\big| &\le (|C^{-1}|+1)|\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}|\,|C-\mathbb{1}||C^{-1}| \\
&\le (\sqrt{3}+2)^2|\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}|\,|C-\mathbb{1}| = c_{\text{(3.96d)}}|\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}|\,|C-\mathbb{1}|\,,
\end{aligned} \tag{3.96d}$$

since again by (3.95b) it is $C^{-1} = \mathbb{1} - (C-\mathbb{1}) + o(C-\mathbb{1})$ and hence $|C^{-1}| \le \sqrt{3} + |C-\mathbb{1}| + o(|C-\mathbb{1}|) \le \sqrt{3}+1$ for $|C-\mathbb{1}|$ sufficiently small.

For $F_2(U)$ we apply (3.61d) as well as (3.95d) with $\gamma = 1$, (3.96a), (3.62), and (3.95a) to find

$$\begin{aligned}
|F_2(U)| &= \big|\partial_{\boldsymbol{G}}U(\boldsymbol{G}^\top\boldsymbol{G})\boldsymbol{G}^\top - \partial_{\boldsymbol{F}}U(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^\top\big| \\
&= \big|2\iota_3(\boldsymbol{G}^\top\boldsymbol{G})^{-1/3}\big(\boldsymbol{G} - \tfrac{1}{3}\iota_1(\boldsymbol{G}^\top\boldsymbol{G})\boldsymbol{G}^{-\top}\big)\boldsymbol{G}^\top - 2\iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{-1/3}\big(\boldsymbol{F} - \tfrac{1}{3}\iota_1(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^{-\top}\big)\boldsymbol{F}^\top\big| \\
&= \big|2\iota_3(\boldsymbol{G}^\top\boldsymbol{G})^{-1/3}\big(\boldsymbol{G}\boldsymbol{G}^\top - \tfrac{1}{3}\iota_1(\boldsymbol{G}^\top\boldsymbol{G})\mathbb{1}\big) - 2\iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{-1/3}\big(\boldsymbol{F}\boldsymbol{F}^\top - \tfrac{1}{3}\iota_1(\boldsymbol{F}^\top\boldsymbol{F})\mathbb{1}\big)\big| \\
&= 2\big|\big(\iota_3(C^\top C)^{-1/3} - 1\big)\iota_3(\boldsymbol{F}^\top\boldsymbol{F})^{-1/3}\big(\boldsymbol{F}\boldsymbol{F}^\top - \tfrac{1}{3}\iota_1(\boldsymbol{F}^\top\boldsymbol{F})\mathbb{1}\big) \\
&\qquad + \iota_3(\boldsymbol{G}^\top\boldsymbol{G})^{-1/3}\big(\boldsymbol{G}\boldsymbol{G}^\top - \boldsymbol{F}\boldsymbol{F}^\top - \tfrac{1}{3}(\iota_1(\boldsymbol{G}^\top\boldsymbol{G}) - \iota_1(\boldsymbol{F}^\top\boldsymbol{F}))\mathbb{1}\big)\big| \\
&\le 2|\iota_3(C^\top C)^{-1/3} - 1|\,\big|\partial_{\boldsymbol{F}}U(\boldsymbol{F}^\top\boldsymbol{F})\boldsymbol{F}^\top\big| \\
&\qquad + 2\iota_3(\boldsymbol{G}^\top\boldsymbol{G})^{-1/3}|\boldsymbol{G}\boldsymbol{G}^\top - \boldsymbol{F}\boldsymbol{F}^\top| \\
&\qquad + \tfrac{2}{\sqrt{3}}\iota_3(\boldsymbol{G}^\top\boldsymbol{G})^{-1/3}|\iota_1(\boldsymbol{G}^\top\boldsymbol{G}) - \iota_1(\boldsymbol{F}^\top\boldsymbol{F}))| \\
&\le 2c_{\text{(3.95d)}}4U(\boldsymbol{F}^\top\boldsymbol{F}))|C-\mathbb{1}| \\
&\qquad + (c_{\text{(3.95d)}}|C-\mathbb{1}|+1)\iota_3(\boldsymbol{F}^\top\boldsymbol{F})\,c_{\text{(3.96a)}}\iota_1(\boldsymbol{F}^\top\boldsymbol{F})|C-\mathbb{1}| \\
&\qquad + \tfrac{2}{\sqrt{3}}(c_{\text{(3.95d)}}|C-\mathbb{1}|+1)\iota_3(\boldsymbol{F}^\top\boldsymbol{F})c_{\text{(3.95a)}}\iota_1(\boldsymbol{F}^\top\boldsymbol{F})|C-\mathbb{1}| \\
&\le c_{\text{(3.96e)}}U(\boldsymbol{F}^\top\boldsymbol{F})|C-\mathbb{1}|\,.
\end{aligned} \tag{3.96e}$$

Similarly, we obtain a modulus of continuity for $F_2(V)$ using (3.95g), (3.95d) with $\gamma = 2$, and (3.96d)

$$\begin{aligned}
|F_2(V)| &= \big|\partial_{\boldsymbol{G}} V(\boldsymbol{G}^\top \boldsymbol{G})\boldsymbol{G}^\top - \partial_{\boldsymbol{F}} V(\boldsymbol{F}^\top \boldsymbol{F})\boldsymbol{F}^\top\big| \\
&= \big|2\iota_3(\boldsymbol{G}^\top \boldsymbol{G})^{-2/3}\big(\tfrac{1}{3}\iota_2(\boldsymbol{G}^\top \boldsymbol{G})\boldsymbol{G}^{-\top} - \iota_3(\boldsymbol{G}^\top \boldsymbol{G})\boldsymbol{G}^{-\top}\boldsymbol{G}^{-1}\boldsymbol{G}^{-\top}\big)\boldsymbol{G}^\top \\
&\qquad - 2\iota_3(\boldsymbol{F}^\top \boldsymbol{F})^{-2/3}\big(\tfrac{1}{3}\iota_2(\boldsymbol{F}^\top \boldsymbol{F})\boldsymbol{F}^{-\top} - \iota_3(\boldsymbol{F}^\top \boldsymbol{F})\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}\boldsymbol{F}^{-\top}\big)\boldsymbol{F}^\top\big| \\
&= 2\big|\iota_3(\boldsymbol{G}^\top \boldsymbol{G})^{-2/3}\big(\tfrac{1}{3}\iota_2(\boldsymbol{G}^\top \boldsymbol{G})\mathbb{1} - \iota_3(\boldsymbol{G}^\top \boldsymbol{G})\boldsymbol{G}^{-\top}\boldsymbol{G}^{-1}\big) \\
&\qquad - \iota_3(\boldsymbol{F}^\top \boldsymbol{F})^{-2/3}\big(\tfrac{1}{3}\iota_2(\boldsymbol{F}^\top \boldsymbol{F})\mathbb{1} - \iota_3(\boldsymbol{F}^\top \boldsymbol{F})\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}\big)\big| \\
&\leq \tfrac{2}{\sqrt{3}}\big|\iota_3(\boldsymbol{G}^\top \boldsymbol{G})^{-2/3}\iota_2(\boldsymbol{G}^\top \boldsymbol{G}) - \iota_3(\boldsymbol{F}^\top \boldsymbol{F})^{-2/3}\iota_2(\boldsymbol{F}^\top \boldsymbol{F})\big| \\
&\qquad + 2\big|\iota_3(C^\top C)^{-2/3} - 1\big|\ \iota_3(\boldsymbol{F}^\top \boldsymbol{F})^{-2/3}\ |\boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}| \\
&\qquad + 2\iota_3(C^\top C)^{-2/3}\iota_3(\boldsymbol{F}^\top \boldsymbol{F})^{-2/3}\ |\boldsymbol{G}^{-\top}\boldsymbol{G}^{-1} - \boldsymbol{F}^{-\top}\boldsymbol{F}^{-1}| \\
&\leq \tfrac{2}{\sqrt{3}}c_{\text{(3.95g)}}|C-\mathbb{1}|V(\boldsymbol{F}^\top \boldsymbol{F}) + 2c_{\text{(3.95d)}}|C-\mathbb{1}|V(\boldsymbol{F}^\top \boldsymbol{F}) + 2c_{\text{(3.96d)}}|C-\mathbb{1}|V(\boldsymbol{F}^\top \boldsymbol{F}) \\
&= c_{\text{(3.96f)}}|C-\mathbb{1}|V(\boldsymbol{F}^\top \boldsymbol{F})\,.
\end{aligned} \tag{3.96f}$$

The collection of estimates (3.96) provides moduli of continuity for $F_2(A)$ for $A = \iota_1, \iota_2, \iota_3, U, V$ and thus proves (3.94b), resp. (3.63b). This concludes the proof of (3.63). ■

4 Numerical Examples

In this section we explain shortly the main equations for the discretization within the finite element frameword and we demonstrate the robustness of the proposed model and the analytical results by a series of numerical examples in a next step.

4.1 Discretization

In this subsection the weak forms of the governing equations (cf. Table 1) and the discretization are considered in more detail. The elastic boundary value problem is based on the balance of linear momentum and the crack phase-field evolution, cf. Table 1. For fixed time, these equations are rewritten in the weak form and the coupled problem reads: Find $\boldsymbol{\varphi} : \mathcal{B}_0 \to \mathbb{R}^3$ and $s : \mathcal{B}_0 \to [0,1]$ such that

$$\int_{\mathcal{B}_0} \boldsymbol{P} : \nabla(\delta\boldsymbol{\varphi})\ \mathrm{d}X = \int_{\mathcal{B}_0} \bar{\boldsymbol{b}} \cdot \delta\boldsymbol{\varphi}\ \mathrm{d}X + \int_{\partial\mathcal{B}_0^{\mathrm{N}}} \bar{\boldsymbol{t}} \cdot \delta\boldsymbol{\varphi}\,\mathrm{d}\Gamma \quad \forall \delta\boldsymbol{\varphi} \in \mathbf{U}\,, \tag{4.1}$$

and

$$\int_{\mathcal{B}_0} \dot{s} \cdot \delta s\ \mathrm{d}X + \int_{\mathcal{B}_0} \frac{\partial \Psi}{\partial s} \cdot \delta s\,\mathrm{d}X + 2\mathcal{G}_c l_c \int_{\mathcal{B}_0} \nabla s \nabla(\delta s)\mathrm{d}X \quad - \frac{\mathcal{G}_c}{2l_c}\int_{\mathcal{B}_0} (1-s)\cdot \delta s\,\mathrm{d}X = 0 \quad \forall \delta s \in \mathbf{X}\,. \tag{4.2}$$

The fraction $\frac{\partial \Psi}{\partial s}$ in (4.2) serves as a driving force for the phase-field. Moreover, the spaces of admissible test functions $\mathbf{U}$ and $\mathbf{X}$ are defined as $\mathbf{U} = \{\delta\boldsymbol{\varphi} \in H^1(\mathcal{B}_0;\mathbb{R}^3)\ |\delta\boldsymbol{\varphi} = 0 \text{ on } \partial\mathcal{B}_0^{\mathrm{D}}\}$, where $H^1(\mathcal{B}_0;\mathbb{R}^3)$ denotes the Sobolev functional space of square integrable functions with values in $\mathbb{R}^3$ and with square integrable weak first derivatives. Correspondingly the space of admissible test functions for the phase-field equation can be formulated as $\mathbf{X} = \{\delta s \in H^1(\mathcal{B}_0) \cap L^\infty(\mathcal{B}_0)|\delta s = 1 \text{ on } \partial\mathcal{B}_0^{\mathrm{D}}\}$.

To apply the finite element method in the following the domain $\mathcal{B}_0$ is subdivided into a finite set of non-overlapping elements

$$\mathcal{B}_0 \approx \mathcal{B}_0^h = \bigcup_{e=1}^{n_e} \mathcal{B}_{0e}\,. \tag{4.3}$$

Furthermore, within the discretization we use Lagrangian polynomials for both fields. In particular, the ansatz functions for the mechanical field are denoted by N_i and the shape functions for the phase-field by $\tilde{N}_i$. The values $\hat{\boldsymbol{\varphi}}^{(i)}$ and $\hat{s}^{(i)}$ are the nodal displacements and the nodal values for the phase-field.

$$\boldsymbol{\varphi} \approx \boldsymbol{\varphi}^h = \sum_{i=1}^{n_k} N_i \hat{\boldsymbol{\varphi}}^{(i)}, \qquad \delta\boldsymbol{\varphi}^h = \sum_{i=1}^{n_k} N_i \delta\hat{\boldsymbol{\varphi}}^{(i)}\,, \tag{4.4}$$

$$s \approx s^h = \sum_{i=1}^{n_k} \tilde{N}_i \hat{s}^{(i)}, \qquad \delta s^h = \sum_{i=1}^{n_k} \tilde{N}_i \delta\hat{s}^{(i)}. \tag{4.5}$$

Inserting the proposed approximations (4.4) and (4.5) into the weak formulations (4.1) and (4.2) the final finite element system results after a straightforward calculation. The time integration is based on an implicit Euler-backward scheme regarding the phase-field parameter s, whereby the time interval $[0,T]$ is divided into pairwise disjoint equidistant subintervals with the time step $\triangle t := t_{n+1} - t_n$. At last the system of equations is solved by making use of the Gaussian elimination method. There exist two popular solution strategies for the non-linear system (4.1) and (4.2), the monolithic and the staggered scheme. Making use of the first mentioned ansatz the fully-coupled system is solved in each timestep; using the staggered scheme the solving part is split into the phase-field s and the mechanical field $\boldsymbol{\varphi}$ which means that in each timestep both quantities are solved successively. For the analysis in Sect. 3 we used this latter approach and also for the numerical examples presented in the subsequent exposition we relied on the staggered scheme (3.7). Further information about different solution strategies can be found in [BKKW17].

4.2 Mode-II-sheartest in two dimensions

As a first numerical example we choose a simple mode-II-sheartest in two dimensions and consider a squared plate with a required horizontal notch. The geometry and the related boundary conditions are depicted in Fig. 6. At the lower boundary of the plate the displacements are constrained in horizontal and vertical direction and at the upper side prescribed displacements are applied incrementally in x-direction which are realized by making use of a Dirichlet boundary condition. Furthermore, the mesh presented in Fig. 6 consists of 128×128 elements.

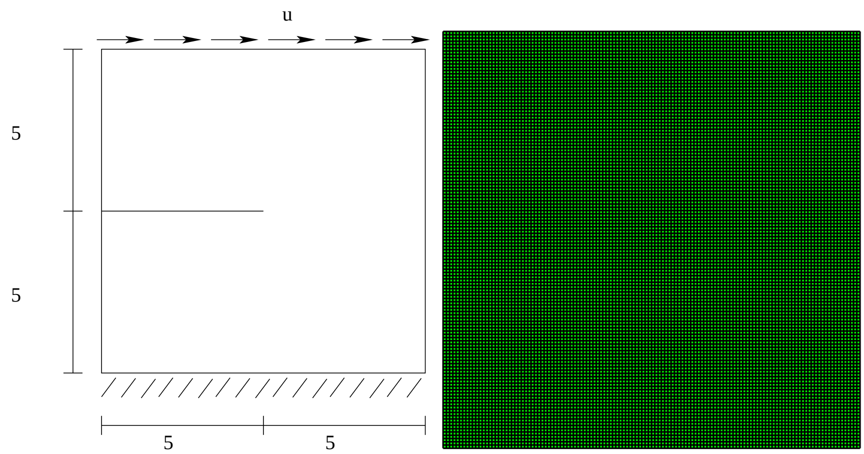

Figure 6: Boundary conditions (left) of a mode-I-tension test and the related mesh based on a hierarchical refinement strategy (right)

The following investigations and simulations are based on the non-linear Yeoh material model (3.75a) and the proposed anisotropic split (2.16) so that the strain energy function can be formulated as

$$W(\boldsymbol{F}, s) = \beta(s) \left(\sum_{i=1}^{2} c_i (m_1^+ (I_1(\boldsymbol{F}^T \boldsymbol{F}))^i + d(m_3^+ (\boldsymbol{F}^T \boldsymbol{F}))^2 \right) + \sum_{i=1}^{2} c_i (m_1^- (I_1(\boldsymbol{F}^T \boldsymbol{F}))^i + d(m_3^- (\boldsymbol{F}^T \boldsymbol{F}))^2, \quad (4.6)$$

with positive coefficients $c_i, d > 0$ for any $i \in \{1, 2\}$ and $\beta(s)$ defined in (2.17). The material parameter are chosen as $c_1 = 2.6923 \times 10^{10}$, $c_2 = 1.3462 \times 10^{10}$ and $d = 2.01923 \times 10^{11}$ which correspond to a Young's modulus of $E = 2.1 \times 10^11 \, \frac{\mathrm{N}}{\mathrm{m}^2}$, a Possion's ratio of $\nu = 0.3$ and the critical energy release rate is adopted as $\mathcal{G}_c = 2.7 \times 10^3 \, \frac{\mathrm{J}}{\mathrm{m}^2}$. The related length-scale parameter l_c depends on the element size h and has to fullfill the inequality $l_c \geq 2h$ in general, cf. [MWH10], which enables the approximation of a diffuse interface zone. In this case using three level refinements the length scale parameter is set to $l_c = 2h_{\mathsf{min}} = 7.8125 \times 10^{-6}$ m. The snapshots of the phase-field and the related crack propagation related to the sheartest are shown in Fig. 7.

In a next step the influence of the timestep size is investigated in more detail. Therefore, the time step size is varied such that a bigger and a smaller time step size is applied. Within this assumption different timestep sizes are examined and the related load-deflection curves are shown in Fig. 8. The results are quite similar regarding the shape of the curve, however, it can be seen that a too big timestep size change the results significantly.

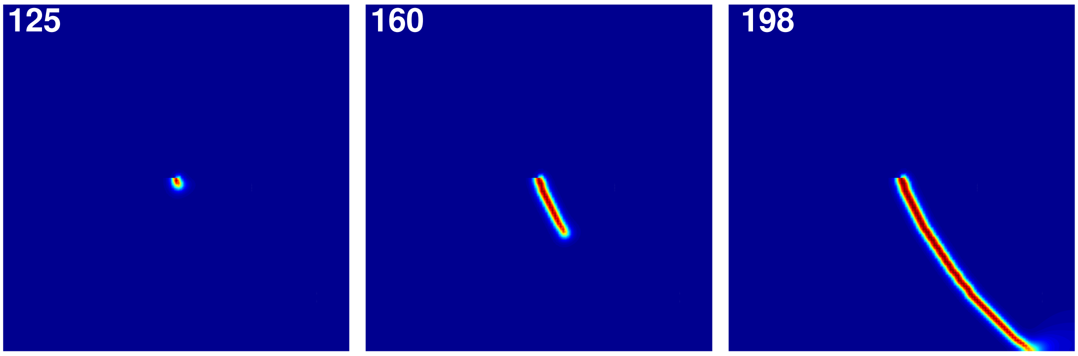

Figure 7: Phase-field snapshots of the mode-II-sheartest at different times steps.

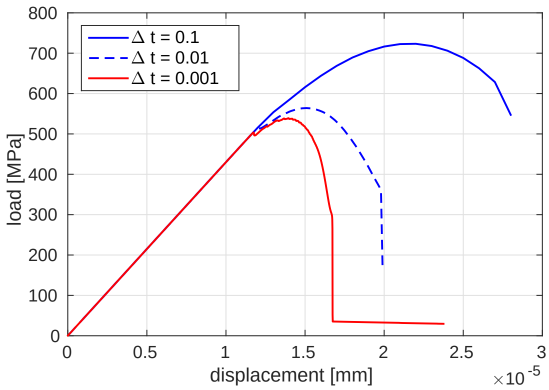

Figure 8: Load-deflection curves for different timestep sizes

4.3 Mode-I-tension test in three dimensions

In a next step we introduce an example in three dimensions. We consider a block with a horizontal notch which consists of $10 \times 4 \times 10$ elements before refinement. The geometry and the related boundary conditions can be found in Fig. 9. The lower surface is constrained in vertical and horizontal direction and the upper surface is constrained in x-direction. Furthermore, at the upper boundary prescribed displacements are applied in y-direction. All the boundary conditions are realized by using Dirichlet boundary conditions. In Fig. 9 the mesh of the block is shown after applying the hierarchical refinement strategy, proposed in [HSD$^+$16].

Also in this example we make use of the proposed Yeoh material model (4.6) with positive coefficients $c_i, d > 0$ for any $i \in \{1, 2\}$ and $\beta(s)$ defined in (2.17). The material parameter are chosen as $c_1 = 2.6923 \times 10^{10}$, $c_2 = 1.3462 \times 10^{10}$ and $d = 2.01923 \times 10^{11}$ which are based on the relations given in [Hol04] and correspond to a Young's modulus of $E = 2.1 \times 10^{1}1 \ \frac{\mathrm{N}}{\mathrm{m}^2}$, a Possion's ratio of $\nu = 0.3$ and the critical energy release rate is adopted as $\mathcal{G}_c = 2.7 \times 10^3 \ \frac{\mathrm{N}}{\mathrm{m}^2}$. The length-scale parameter depends on the mesh size and is chosen as $l_c =$ m. Moreover, the time step size is applied with $\Delta t = 0.01$ sec. In Fig. 10 the crack growth during the simulation can be observed. The crack

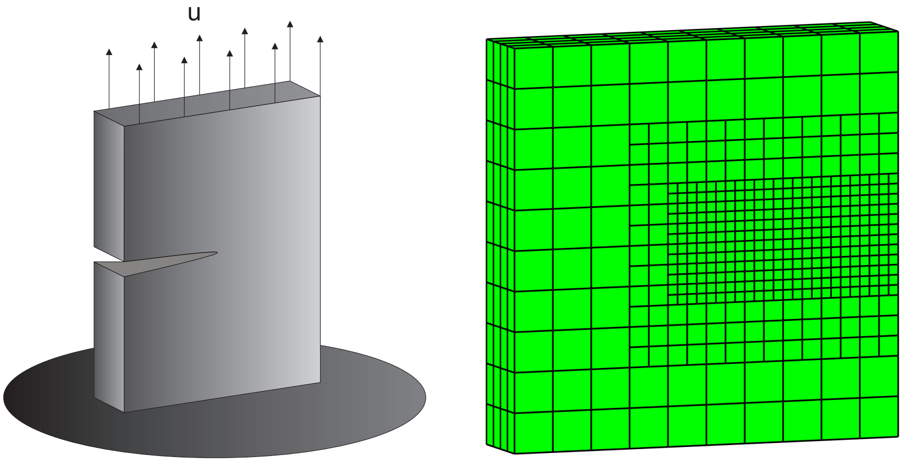

Figure 9: Boundary conditions (left) of a mode-I-tension test and the related mesh based on a hierarchical refinement strategy (right) in three dimensions

propagates within this loading as we expect the crack path.

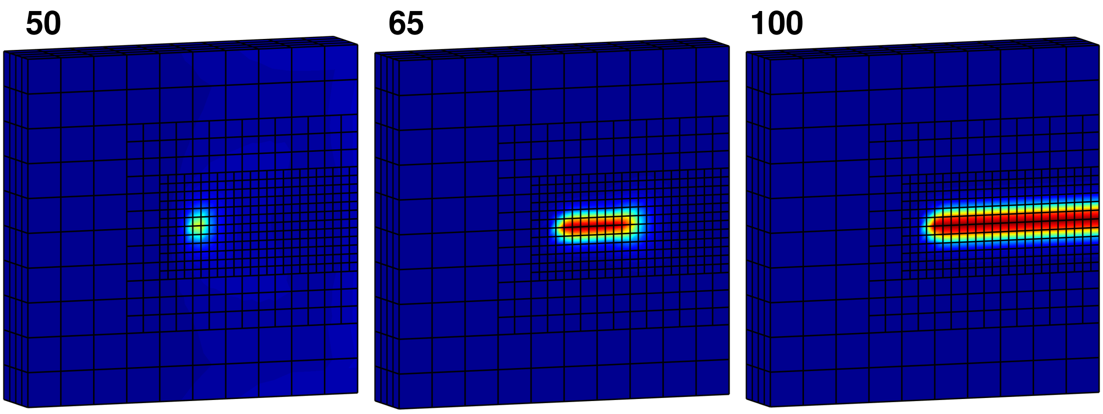

Figure 10: Phase-field snapshots of the mode-I-tension test at different times steps.

Furthermore, the load deflection curves for different timestep sizes are demonstrated in Fig. 11 and the block is cracked at a displacement of $\boldsymbol{u} \approx 0.28 \times 10^{-3}$ m.

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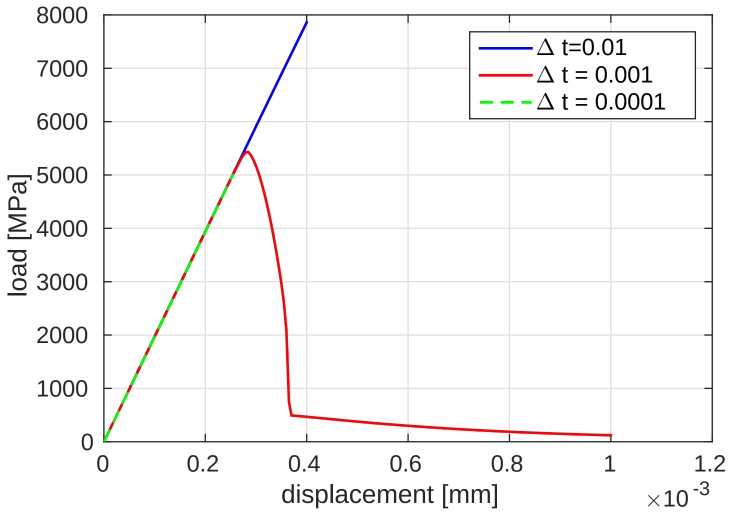

Figure 11: Load-deflection curve

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