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Pierre-Étienne Druet

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Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: pierre-etienne.druet@wias-berlin.de

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

We show that fully quasilinear parabolic systems are locally well posed in the Hilbert space scala if the coefficients of the differential operator are smooth enough and the spatial domain is sufficiently regular. In the context of diffusion systems driven by entropy, the uniform parabolicity follows from the second law of thermodynamics.

1 Introduction

Let $N \in \mathbb{N}, T > 0$ a time, and $\Omega \subset \mathbb{R}^3$ a bounded domain. We denote $Q_T :=]0,\, T[\times\Omega$ the parabolic cylinder. We consider the following system of partial differential equations for functions $w_1, \ldots, w_N : \overline{Q_T} \to \mathbb{R}$:

$$\partial_t R_i(t, x,\, w) + \operatorname{div} J^i(t, x,\, w,\, w_x) = f_i(t, x,\, w,\, w_x) \text{ in } [0, T] \times \Omega \tag{1}$$

$$\nu\, J^i(t, x,\, w,\, w_x) = f_{\Gamma,i}(t, x,\, w) \quad \text{on } [0, T] \times \partial\Omega \tag{2}$$

$$w = q^0 \quad \text{in } \{0\} \times \Omega\,. \tag{3}$$

For $i = 1, \ldots, N$, the functions R_i are defined in $\overline{Q_T} \times \mathbb{R}^N$, while for $i = 1, \ldots, N$, $k = 1, 2, 3$ the flux functions J^i_k are defined on $\overline{Q_T} \times \mathbb{R}^N \times \mathbb{R}^{N\times 3}$.

The equations (1) can a. o. be consistently interpreted (see [DG17]) as the reduced form of $N+1$ mass conservative diffusion/reaction equations for an isothermal mixture of chemical species $A_1, \ldots, A_{N+1}$ in dynamical equilibrium. The unknown functions w_i then play the role of the so called *relative chemical potentials* or *entropy variables*. A phenomenological, nonconservative diffusion/reaction system for N species also exhibits this structure. In these contexts, the second law of thermodynamics suggests the following restrictions:

(a) The inequality $\sum_{i=1}^N \sum_{k=1}^3 J^i_k(t, x,\, z,\, D)\, D^i_k > 0$ is valid for all $D \in \mathbb{R}^{N\times 3} \setminus \{0\}$ and all $(t, x,\, z) \in \overline{Q} \times \mathbb{R}^N$;

(b) The matrix $R_z(t, x,\, z)$ is symmetric and positive definite for all $(t, x) \in \overline{Q_T}$ and all $z \in \mathbb{R}^N$.

We will in fact assume the following stronger variational structure:

(1) There is a potential Ψ defined on $\overline{Q_T} \times \mathbb{R}^N \times \mathbb{R}^{N\times 3}$ such that $D \mapsto \Psi(t, x, z,\, D)$ is strictly convex and attains a global minimum in $D = 0$. We define $J^i_k := -\partial_{D^i_k}\Psi$;

(2) There is a function β defined on $\overline{Q_T} \times \mathbb{R}^N$ such that $z \mapsto \beta(t, x,\, z)$ is strictly convex, and $R = \beta_z$.

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For the problem (1), (2) and (3), we prove the local–in–time well posedness in the class

$$\mathcal{V}^3(Q_T;\,\mathbb{R}^N) := W^1_2(0,T;\,W^{5,2}(\Omega;\,\mathbb{R}^N),\,W^{3,2}(\Omega;\,\mathbb{R}^N))\,.$$

For $k,\,\ell\,\in\,\mathbb{N}\cup\{0\}$ and for M being some open subset of a finite dimensional linear space, we introduce the simplified notation $C^{k,\ell}([0,T]\times M) := C^k([0,T];\,C^\ell(M))$.

Theorem 1.1. *Assume that the domain Ω is of class $\mathcal{C}^6$. For $i = 1,\ldots,N$ assume that the function R_i is of class $(C^{1,5}\cap C^{0,6})([0,T]\times\Omega\times\mathbb{R}^N)$. For $i = 1,\ldots,N$, $k = 1,2,3$, let J^i_k belong to $(C^{1,4}\cap C^{0,5})([0,T]\times\Omega\times\mathbb{R}^N)$. We assume that $f\in C^{0,5}([0,T]\times\overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{N\times 3})$ and that $f_\Gamma\in C^{0,5}([0,T]\times\overline{\Omega}\times\mathbb{R}^N)$.*

Assume that the thermodynamic conditions (1)*,* (2) *are valid. Moreover, we assume that the initial condition $q^0\in\mathcal{V}^3(Q_T;\,\mathbb{R}^N)$ and satisfies two compatibility conditions: First*

$$\nu_k(x)\,J^i_k(0,x,\,q^0(x),\,q^0_x(x)) = f_{\Gamma,i}(0,x,\,q^0(x)) \text{ for } i = 1,\ldots,N,\ x\in\partial\Omega\,;$$

Second, we assume for $i = 1,\ldots,N$ and $x\in\partial\Omega$ that

$$\begin{aligned}\nu_k(x)\,\big(J^i_{k,t}(0,x,\,q^0(x),\,q^0_x(x)) &+ J^i_{k,z_j}(0,x,\,q^0(x),\,q^0_x(x))\,F^0_j(x)\\ &+ J^i_{k,D^j_\ell}(0,x,\,q^0(x),\,q^0_x(x))\,F^0_{j,x_\ell}(x)\big)\\ &\qquad = f_{\Gamma,i,t}(0,x,\,q^0(x)) + f_{\Gamma,i,z_j}(0,x,\,q^0(x))\,F^0_j(x)\end{aligned}$$

where F^0 is the vector field given by

$$\begin{aligned}F^0(x) :=&[R_z(0,x,\,q^0(x))]^{-1}\times\\ &\times(\mathrm{div}(J(0,x,\,q^0(x),\,q^0_x(x))) - R_t(0,x,\,q^0(x)) - f(0,x,\,q^0(x),\,q^0_x(x)))\,.\end{aligned}$$

Then, there is $T>0$ dependent on Ω, the coefficients $R,\,J,\,f,\,f_\Gamma$ and the initial condition q^0 such that the problem (1)*,* (2) *and* (3) *possesses a unique solution of class $\mathcal{V}^3$.*

To prove the Theorem 1.1, we show that the linearised operator associated with the system (1), (2) and (3) is invertible in the proposed scala of Hilbert spaces. We make use of the fact that the linearisation in smooth points generates a *uniformly parabolic* operator (Definition 7 in [LSU68], Chapter VII, Paragraph 9). In order to obtain estimates in the Hilbert space scala $L^2W^{k,2}$, we employ the basic method of squaring the operator. From this point of view, our study of the linearisation remains far below the complexity of the results on general linear parabolic systems obtained by the Russian school in the Sixties both in the scala of anisotropic Sobolev and Hölder spaces (see [LSU68], Chapter VII, Paragraph 9 and 10 for an overview and references, or the book [EZ98]).

However, our approach is self-contained and it gives sufficient conditions that are *ready to apply* to fully quasilinear systems. The most original contribution of this paper consists in showing that *the larger system satisfied by the unknown and its derivatives* exhibits the structure of a *reduced quasilinear problem* – that is, the case that the flux function J is linear in the gradient. The larger system is moreover subject to 'mixed' boundary conditions: a subgroup of the variables $w_1,\ldots,w_P$ is subject to natural and the remaining $w_{P+1},\ldots,w_N$ are subject to Dirichlet boundary conditions. We propose an original analysis of this system, which directly yields the main result.

For additional context on quasilinear and doubly nonlinear parabolic systems, we mention that the estimates in L^p spaces for the linearised operator are more complex. The maximal regularity theory is presently available only for the reduced quasilinear case, the case that the flux function J is linear in the gradient variable: [HMPW17].

The reader can also consult: [Ama90], [Ama93] for the L^p theory of diffusion–reaction systems, [DiB93] for the case of the p-Laplace system, [DM05] or [Bur14] for almost everywhere $C^{1,\alpha}$ solutions, or [FS15] for the regularity in time. Concerning weak solutions, we refer to [AL83], [Alt12], [FK95], [Kac97], [Ben13], [Dru17], [HRM16]. Concerning weak solution with degeneracy in ellipticity, we suggest the references [J1̈5], [CDJ18] or [J1̈7]. This is a collection of excellent results concerning non-linear parabolic systems that we do not use directly here.

Our plan for the paper is as follows. In the next section 2 we introduce the notation and we collect the main auxiliary tools for the analysis. The Sections 3, 4 are devoted to the analysis of the reduced quasilinear problem – that is, J is linear in D. These sections contain the fundamental estimates respectively for the case of natural and of Dirichlet boundary conditions. In the Section 5, we perform an intermediate step for systems where a subgroup of the variables $w_1, \ldots, w_P$ is subject to natural and the remaining $w_{P+1}, \ldots, w_N$ are subject to Dirichlet boundary conditions. This allows to show in the Section 6 that the full quasilinear case can be solved by the same method as the reduced quasilinear problem.

2 Preliminaries

2.1 Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^{0,1}$ at least. The Sobolev spaces $W^{k,2}(\Omega)$ for $k = 1, 2, \ldots$ are supposed to be well known. For a function $u \in W^{1,2}(\Omega)$, we denote $\nabla u = (u_{x_1},\, u_{x_2},\, u_{x_3})$ the weak spatial gradient. We also make use of the short cut $u_x = \nabla u$. With $W^{k,2}(\Omega;\, \mathbb{R}^\ell)$, $\ell \in \mathbb{N}$ and $\ell > 1$, we mean the space of vector fields $u = (u_1,\, u_2, \ldots, u_\ell)$ for which each component u_i is a function of class $W^{k,2}(\Omega)$. We extend this way of writing to other functional spaces.

For $T > 0$, we denote $Q = Q_T =]0, T[\times\Omega \subset \mathbb{R}^4$.

We denote S_T the surface $]0, T[\times\partial\Omega$.

The parabolic boundary of Q is denoted $\mathcal{P} := S_T \cup (\{0\} \times \Omega) \cup (\{0\} \times \partial\Omega)$.

2.2 Functional spaces

For $T > 0$, and Banach spaces $X \hookrightarrow Y$ (continuous injection), we denote

$$W^1_2(0,T;\, X,\, Y) := \{u \in L^2(0,T;\, X) \,:\, u_t \in L^2(0,T;\, Y)\},$$
$$\|u\|_{W^1_2(0,T;\, X,\, Y)} := \|u\|_{L^2(0,T;\, X)} + \|u_t\|_{L^2(0,T;\, Y)} \,.$$

We will call *the space of strong solutions* for the problem (1) the Banach space

$$W^1_2(0,T;\, X,\, Y) \text{ with } X = W^{2,2}(\Omega;\, \mathbb{R}^N),\; Y = L^2(\Omega;\, \mathbb{R}^N)\,.$$

This space is isomorphic (in fact identical) with the Sobolev space

$$W^{2,1}_2(Q;\, \mathbb{R}^N) := \{u \in L^2(Q;\, \mathbb{R}^N) \,:\, D^r_t D^s_x u \in L^2(Q;\, \mathbb{R}^N) \text{ for all } 2r + s \leq 2\}\,.$$

Note a peculiarity: In general, we will put the time integrability index first and the spatial index second, *except in the case of* the Sobolev spaces $W^{2\ell,\,\ell}_p(Q)$ and the Hölder spaces $H^{2\ell,\ell}(Q)$ where we employ the notation of the book [LSU68].

A fundamental role in the analysis is played by the state space

$$W^1_2(0,T;\, X,\, Y) \text{ with } X = W^{4,2}(\Omega;\, \mathbb{R}^N),\; Y = W^{2,2}(\Omega;\, \mathbb{R}^N)\,.$$

In the standard Hilbert space scala, this is the largest space embedded into $W^{1,0}_\infty(Q;\, \mathbb{R}^N)$ (see details below).

For $\ell = 0, 1, 2, \ldots$, we make use of the abbreviations

$$\begin{aligned} \mathcal{V}^\ell = \mathcal{V}^\ell(Q_T;\, \mathbb{R}^N) &:= W^1_2(0,T;\, X^{\ell+2},\, X^\ell) \text{ with } X^\ell = W^{\ell,2}(\Omega;\, \mathbb{R}^N) \\ \mathcal{V}^\ell_\Omega = \mathcal{V}^\ell_\Omega(Q_T;\, \mathbb{R}^N) &:= \{v \in \mathcal{V}^\ell \,:\, v = 0 \text{ in } \{0\} \times \Omega\}\,. \end{aligned} \tag{4}$$

which are Banach spaces with the norm

$$\|v\|_{\mathcal{V}^\ell} := \|v\|_{W^1_2(0,T;\, X^{\ell+2},\, X^\ell)}\,.$$

Recall that $W^{0,2} := L^2$. Moreover, we need the space

$$\begin{aligned} \mathcal{V}^{-1} = \mathcal{V}^{-1}([0,T] \times \mathbb{R}^3;\, \mathbb{R}^N) &:= W^1_2(0,T;\, W^{1,2}(\mathbb{R}^3),\, [W^{1,2}(\mathbb{R}^3)]^*) \\ \mathcal{V}^{-1}_{\mathbb{R}^3} = \mathcal{V}^{-1}_{\mathbb{R}^3}([0,T] \times \mathbb{R}^3;\, \mathbb{R}^N) &:= \{v \in \mathcal{V}^{-1} \,:\, v = 0 \text{ in } \{0\} \times \mathbb{R}^3\}\,. \end{aligned} \tag{5}$$

We recall the inequality

$$\max_{t \in [0,T]} \|v(t)\|_{L^2(\mathbb{R}^3)} \leq c\, \|v\|_{W^1_2(0,T;\, W^{1,2}(\mathbb{R}^3),\, [W^{1,2}(\mathbb{R}^3)]^*)}$$

and the fact $\mathcal{V}^{-1},\, \mathcal{V}^{-1}_{\mathbb{R}^3} \hookrightarrow C([0,T];\, L^2(\mathbb{R}^3;\, \mathbb{R}^N))$ with continuous injection (see for instance the survey [Nau]).

We also note the following observation concerning the space $\mathcal{V}^{-1}$.

Lemma 2.1. *Consider $v \in W^1_2(0,T;\, W^{2,2}(\Omega),\, L^2(\Omega))$ such that $v(0) = 0$. Then, the functions v_{x_i} are the restriction to $]0,T[\times\Omega$ of functions $E_\Omega(v_{x_i})$ of class $W^1_2(0,T;\, W^{1,2}(\mathbb{R}^3),\, [W^{1,2}(\mathbb{R}^3)]^*)$ such that $E_\Omega(v_{x_i})(0) = 0$. Moreover $\|E_\Omega(v_x)\|_{\mathcal{V}^{-1}_{\mathbb{R}^3}} \leq c\, \|v\|_{\mathcal{V}^0}$, with a constant c that depends only on Ω.*

Proof. We rely on the existence of a linear extension operator E_Ω which is continuous from $W^{\ell,2}(\Omega)$ into $W^{\ell,2}(\mathbb{R}^3)$ for $\ell = 0, 1, 2, \ldots$ (see Section 6.5.1 of [KJF77]).

Let $u \in W^1_2(0,T;\, W^{2,2}(\Omega),\, L^2(\Omega))$. Then $E_\Omega u \in W^1_2(0,T;\, W^{2,2}(\mathbb{R}^3),\, L^2(\mathbb{R}^3))$. For $\phi \in C^\infty_c(\mathbb{R}^3)$ arbitrary, one has

$$\left| \int_{\mathbb{R}^3} E_\Omega(u_t)\, \phi_x \right| \leq c_\Omega\, \|u_t\|_{L^2(Q_T)}\, \|\phi_x\|_{L^2(]0,T[\times\mathbb{R}^3)}\,.$$

Thus, exploiting that $W^{1,2}(\mathbb{R}^3) = W^{1,2}_0(\mathbb{R}^3)$

$$\|\partial_t(E_\Omega(u))_x\|_{L^2(0,T;\, [W^{1,2}(\mathbb{R}^3)]^*)} \leq c_\Omega\, \|u_t\|_{L^2(Q_T)}\,.$$

This shows that $(E_\Omega u)_x \in W^1_2(0,T;\, W^{1,2}(\mathbb{R}^3),\, [W^{1,2}(\mathbb{R}^3)]^*)$. □

Lemma 2.2. *For $\ell = 1, 2, \ldots$, the operation $\frac{d}{dx}$ belongs to $\mathscr{L}(\mathcal{V}^\ell,\, \mathcal{V}^{\ell-1})$. For $\ell = 0$, the operation $E_\Omega \circ \frac{d}{dx} = \frac{d}{dx} \circ E_\Omega$ belongs to $\mathscr{L}(\mathcal{V}^0,\, \mathcal{V}^{-1})$.*

Proof. The claim for $\ell \geq 1$ is obvious, while for $\ell = 0$ it is a direct consequence of the Lemma 2.1. □

2.3 Trace spaces

For the analysis of boundary conditions, we will need trace spaces. Recall that there is a linear trace operator $\gamma \in \mathscr{L}(L^2(0,T;\, W^{1,2}(\Omega)),\, L^2(S_T))$. For a linear subspace $\mathcal{W}$ of $L^2(0,T;\, W^{1,2}(\Omega))$, we denote

$$\mathsf{Tr}_{S_T}\, \mathcal{W} := \{\gamma(w) \,:\, w \in \mathcal{W}\}\,. \tag{6}$$

This is a Banach space with the norm

$$\|f\|_{\mathsf{Tr}_{S_T}\,\mathcal{W}} := \inf_{w\in\mathcal{W},\,\gamma(w)=f} \|w\|_{\mathcal{W}}\,. \tag{7}$$

Lemma 2.3. *For $\ell \in \{0,1,2\}$ assume that $f \in$ Tr$_{S_T}\, \mathcal{V}^\ell$. Then, if $\partial\Omega$ is of class $\mathcal{C}^{\ell+3}$ the components of $\nabla_\Gamma f$ belong to Tr$_{S_T}\, \mathcal{V}^{\ell-1}$.*

Conversely, if $\nabla_\Gamma f$ belongs to Tr$_{S_T}\, \mathcal{V}^{-1}$, then f belongs to Tr$_{S_T}\, \mathcal{V}^0$.

Proof. Assume that $\bar f \in \mathcal{V}^\ell$ is an arbitrary extension for f into Ω (that means, $\gamma(\bar f) = f$). In the case $\ell = 0$, we extend $\bar f$ to $]0,T[\times\mathbb{R}^3$ with the extension operator of Lemma 2.1.

Extending the tangential vectors suitably into Ω (into $\mathbb{R}^3$ for $\ell = 0$), we obtain that $\tau \cdot \nabla \bar f \in \mathcal{V}^{\ell-1}$, and that

$$\|\tau\cdot\nabla\bar f\|_{\mathcal{V}^{\ell-1}} \leq c\,\|\tau\|_{C^{\ell+2}(\overline{\Omega};\,\mathbb{R}^3)}\,\|\bar f_x\|_{\mathcal{V}^{\ell-1}} \leq c\,\|\tau\|_{C^{\ell+2}(\overline{\Omega};\,\mathbb{R}^3)}\,\|\bar f\|_{\mathcal{V}^\ell}\,.$$

Thus, $\|\tau\cdot\nabla f\|_{\mathsf{Tr}_{S_T}\,\mathcal{V}^{\ell-1}} \leq c_\Omega\,\|\bar f\|_{\mathcal{V}^\ell}$. Since the extension $\bar f$ was arbitrary

$$\|\tau\cdot\nabla f\|_{\mathsf{Tr}_{S_T}\,\mathcal{V}^{\ell-1}} \leq c_\Omega \inf_{\bar f\in\mathcal{V}^\ell\,:\,\gamma(\bar f)=f} \|\bar f\|_{\mathcal{V}^\ell} = c\,\|f\|_{\mathsf{Tr}_{S_T}\,\mathcal{V}^\ell}\,.$$

Now, assume that $f \in \mathsf{Tr}_{S_T}\, W^{1,0}_2(Q;\,\mathbb{R}^N)$ is such that $\nabla_\Gamma f$ belongs to $\mathsf{Tr}_{S_T}\,\mathcal{V}^{-1}$. We want to show that $f \in \mathsf{Tr}_{S_T}\,\mathcal{V}^0$ and that

$$\|f\|_{\mathsf{Tr}_{S_T}\,\mathcal{V}^0} \leq c\,(\|f\|_{L^2(S_T)} + \|\nabla_\Gamma f\|_{\mathsf{Tr}_{S_T}\,\mathcal{V}^{-1}}).$$

The results of [LSU68], Ch. 4, Par. 2 allow to show that the space $W^{2,1}_2(Q_T;\,\mathbb{R}^N)$, which is isomorphic to $\mathcal{V}^0$ satisfies

$$\mathsf{Tr}_{S_T}\, W^{2,1}_2(Q_T) = W_2^{\frac{3}{2},\frac{3}{4}}(S_T)\,.$$

Moreover, we know ([DHP07]) that the Neumann trace $\nu\cdot\nabla$ maps $W^{2,1}_2(Q_T)$ onto

$$W_2^{\frac{1}{4}}(0,T;\,L^2(\partial\Omega)) \cap L^2(0,T;\,W^{\frac{1}{2},2}(\partial\Omega))\,.$$

Due to the Lemma 2.8, every function $a \in \mathsf{Tr}_{S_T}\,\mathcal{V}^{-1}$ can be represented as the Neumann trace of a function from $W^{2,1}_2(Q_T)$. Thus

$$\mathsf{Tr}_{S_T}\,\mathcal{V}^{-1} \subseteq W_2^{\frac{1}{4}}(0,T;\,L^2(\partial\Omega)) \cap L^2(0,T;\,W^{\frac{1}{2},2}(\partial\Omega))\,.$$

Thus, if $\nabla_\Gamma f \in \mathsf{Tr}_{S_T}\,\mathcal{V}^{-1}$, then obviously

$$\begin{aligned} f \in& W_2^{\frac{1}{4}}(0,T;\,W^{1,2}(\partial\Omega)) \cap L^2(0,T;\,W^{\frac{3}{2},2}(\partial\Omega)) \\ &= W_2^{\frac{3}{2},\frac{3}{4}}(S_T) = \mathsf{Tr}_{S_T}\,W^{2,1}_2(Q_T)\,. \end{aligned}$$

□

2.4 Multiplication and Nemicki operators

We notice that $\mathcal{V}^1$ is by definition a subset of $C([0,T];\, W^{2,2}(\Omega;\, \mathbb{R}^N))$. By means of the Sobolev embedding theorem. It follows for all $t \in [0,T]$ that

$$\|D^2 v(t)\|_{L^2(\Omega)}^2 + \|\nabla v(t)\|_{L^6(\Omega)}^2 \leq c_\Omega \, \|v\|_{\mathcal{V}^1}^2 \, . \tag{8}$$

We shall make use of auxiliary inequalities for $u,\, v \in \mathcal{V}^1$:

$$\|u_x \, v_t\|_{L^2(Q)} \leq \|u_x\|_{L^{\infty,4}} \, \|v_t\|_{L^{2,4}} \leq c \, \|u\|_{\mathcal{V}^1} \, \|v\|_{\mathcal{V}^1} \tag{9}$$

$$\|u_x \, v_{x,x}\|_{L^2} \leq \|u_x\|_{L^{\infty,4}} \, \|v_{x,x}\|_{L^{2,4}} \leq c \, \|u\|_{\mathcal{V}^1} \, \|v\|_{\mathcal{V}^1} \, . \tag{10}$$

Moreover, we can prove that $\mathcal{V}^1$ embeds into the Hölder space $H^{\frac{1}{2},\frac{1}{4}}(\overline{Q};\, \mathbb{R}^N)$. For instance, we make use of the fact that the spatial derivative of $u \in \mathcal{V}^1$ belongs to $L^{\infty,6}$ while the time derivative belongs to $L^{2,6}$. We apply the anisotropic embedding theorem of [KP11].

2.4.1 Multiplicator spaces

We say that a Banach space $X \hookrightarrow Y$ (continuous injection) is a multiplicator space for Y if and only if there is a constant $c > 0$ such that $\|x\, y\|_Y \leq c\, \|x\|_X \, \|y\|_Y$ for all $x \in X$ and $y \in Y$.

Lemma 2.4. *The space $\mathcal{V}^1$ is a multiplicator space for the following spaces: $L^2(0,T;\, W^{1,2}(\Omega;\, \mathbb{R}^N))$, $L^2(0,T;\, W^{2,2}(\Omega;\, \mathbb{R}^N))$, $\mathcal{V}^0$, and itself.*

The space $\mathcal{V}^2$ is a multiplicator space for itself.

The space $E_\Omega(\mathcal{V}^1)$ is a multiplicator space for $\mathcal{V}^{-1}$.

Moreover, the multiplicator properties extend on the corresponding trace spaces.

Proof. We will prove the claims for scalar functions. For vector fields, we apply the same inequalities component wise.

Let first $a \in L^2(0,T;\, W^{1,2}(\Omega))$ and $b \in \mathcal{V}^1$. Then

$$\begin{aligned}
\|a\, b\|_{L^2} &\leq \|b\|_{L^\infty} \, \|a\|_{L^2} \\
\|a_x\, b\|_{L^2} &\leq \|b\|_{L^\infty} \, \|a_x\|_{L^2} \\
\|a\, b_x\|_{L^2} &\leq \|b_x\|_{L^{\infty,4}} \, \|a\|_{L^{2,4}} \leq c \|b_x\|_{L^{\infty,6}} \, \|a\|_{L^2 W^{1,2}} \, .
\end{aligned}$$

Thus, invoking (8)

$$\|a\, b\|_{L^2(0,T;\, W^{1,2}(\Omega))} \leq c\, \|b\|_{\mathcal{V}^1} \, \|a\|_{L^2(0,T;\, W^{1,2}(\Omega))} \, . \tag{11}$$

This shows that $\mathcal{V}^1$ is a multiplicator space for $L^2(0,T;\, W^{1,2}(\Omega))$.

We note further that

$$\begin{aligned}
\|a_{x,x}\, b\|_{L^2} &\leq \|b\|_{L^\infty} \, \|a_{x,x}\|_{L^2} \\
\|a_x\, b_x\|_{L^2} &\leq \|b_x\|_{L^{\infty,4}} \, \|a_x\|_{L^{2,4}} \leq c\, \|b\|_{\mathcal{V}^1} \, \|a\|_{L^2 W^{2,2}} \\
\|a\, b_{x,x}\|_{L^2} &\leq \|b_{x,x}\|_{L^{\infty,2}} \, \|a\|_{L^{2,\infty}} \leq c\, \|b_{x,x}\|_{L^{\infty,2}} \, \|a\|_{L^2 W^{2,2}} \, .
\end{aligned} \tag{12}$$

Again (8), (11) and (12) yield

$$\|a\,b\|_{L^2(0,T;\,W^{2,2}(\Omega))} \leq c\,\|b\|_{\mathcal{V}^1}\,\|a\|_{L^2(0,T;\,W^{2,2}(\Omega))}\,. \tag{13}$$

This shows that $\mathcal{V}^1$ is a multiplicator space for $L^2(0,T;\,W^{2,2}(\Omega))$. We also easily show that

$$\|a_t\,b\|_{L^2} \leq \|b\|_{L^\infty}\,\|a_t\|_{L^2} \leq c\,\|b\|_{\mathcal{V}^1}\,\|a_t\|_{L^2}$$
$$\|a\,b_t\|_{L^2} \leq \|b_t\|_{L^{2,4}}\,\|a\|_{L^{\infty,4}} \leq c\,\|b_t\|_{L^2W^{1,2}}\,\|a\|_{L^\infty W^{1,2}}\,.$$

Thus

$$\|(a\,b)_t\|_{L^2} \leq c\,\|b\|_{\mathcal{V}^1}\,\|a\|_{\mathcal{V}_0}\,. \tag{14}$$

Combining (13) and (14) yield

$$\|a\,b\|_{W^1_2(0,T;\,W^{2,2}(\Omega),\,L^2(\Omega))} \leq c\,\|b\|_{\mathcal{V}^1}\,\|a\|_{\mathcal{V}_0}\,.$$

Since also

$$\|a_x\,b\|_{L^{\infty,2}} \leq \|b\|_{L^\infty}\,\|a_x\|_{L^{\infty,2}}$$
$$\|a\,b_x\|_{L^{\infty,2}} \leq \|b_x\|_{L^{\infty,4}}\,\|a\|_{L^{\infty,4}} \leq c\,\|b\|_{L^\infty W^{2,2}}\,\|a\|_{L^\infty W^{1,2}}\,,$$

we obtain that

$$\|a\,b\|_{\mathcal{V}^0} \leq c\,\|b_{\mathcal{V}^1}\,\|a\|_{\mathcal{V}^0}\,. \tag{15}$$

Thus, $\mathcal{V}^1$ is a multiplicator space for $\mathcal{V}^0$.

We note that $d \in \mathcal{V}^1$ if and only if $d_x \in \mathcal{V}^0$. On the other hand, the inequality (15) yields

$$\begin{aligned}\|(a\,b)_x\|_{\mathcal{V}^0} &\leq c\,(\|a_x\|_{\mathcal{V}^0}\,\|b\|_{\mathcal{V}^1} + \|a\|_{\mathcal{V}^1}\,\|b_x\|_{\mathcal{V}^0})\\ &\leq c\,\|a\|_{\mathcal{V}^1}\,\|b_{\mathcal{V}^1}\,.\end{aligned} \tag{16}$$

This shows that $\mathcal{V}^1$ is a multiplicator space for itself.

Similarly, $d \in \mathcal{V}^2$ if and only if $d_x \in \mathcal{V}^1$. Thus, (16) directly implies that $\mathcal{V}^2$ is a multiplicator space for itself. Finally we show that $E_\Omega(\mathcal{V}^1)$ is a multiplicator space for $\mathcal{V}^{-1}$.

Let $\phi \in L^2(0,T;\,W^{1,2}(\mathbb{R}^3))$, and a and b be smooth functions defined on $\mathbb{R}^3$.

Then $((a\,b)_t,\,\phi)_{L^2(\mathbb{R}^3)} = (a_t,\,b\phi)_{L^2(\mathbb{R}^3)} + (b_t,\,a\,\phi)_{L^2(\mathbb{R}^3)}$. Moreover

$$\begin{aligned}|(a_t,\,b\phi)_{L^2([0,T]\times\mathbb{R}^3)}| &\leq \|a_t\|_{L^2(W^{1,2})^*}\,\|b\phi\|_{L^2W^{1,2}}\\ &\leq c\,\|a_t\|_{L^2(W^{1,2})^*}\,\|b\|_{\mathcal{V}^1(]0,T[\times\mathbb{R}^3)}\,\|\phi\|_{L^2W^{1,2}}\\ |(b_t,\,a\,\phi)_{L^2([0,T]\times\mathbb{R}^3)}| &\leq \|b_t\|_{L^{2,4}}\,\|a\|_{L^{\infty,2}}\,\phi\|_{L^{2,4}}\\ &\leq c\,\|b_t\|_{L^2W^{1,2}}\,\|a\|_{L^{\infty,2}}\,\|\phi\|_{L^2W^{1,2}}\,.\end{aligned}$$

Thus $\|(a\,b)_t\|_{L^2(W^{1,2})^*} \leq c\,\|b\|_{\mathcal{V}^1(]0,T[\times\mathbb{R}^3)}\,\|a\|_{\mathcal{V}^{-1}}$. Combining the latter with (11), we obtain that

$$\|a\,b\|_{W^1_2(0,T;\,W^{1,2},\,(W^{1,2})^*)} \leq c\,\|b\|_{\mathcal{V}^1(]0,T[\times\mathbb{R}^3)}\,\|a\|_{\mathcal{V}^{-1}}\,.$$

Choosing $b = E_\Omega(\tilde{b})$ with $\tilde{b} \in \mathcal{V}^1(]0,T[\times\Omega)$, we see that $E_\Omega(\mathcal{V}^1)$ is a multiplicator space for $\mathcal{V}^{-1}$.

Finally, if X is a multiplicator space for Y, and if both $X,\,Y \hookrightarrow L^2(0,T;\,W^{1,2}(\Omega))$, then for all $x \in X$ and $y \in Y$

$$\|\gamma(x)\,\gamma(y)\|_{\mathsf{Tr}_{S_T}\,Y} \leq \|x\,y\|_Y \leq \|x\|_X\,\|y\|_Y\,.$$

Thus $\|\gamma(x)\,\gamma(y)\|_{\mathsf{Tr}_{S_T}\,Y} \leq \|x\|_{\mathsf{Tr}_{S_T}\,X}\,\|y\|_{\mathsf{Tr}_{S_T}\,Y}$, and one sees that the multiplicator property extends to the traces. □

The following consequences are direct.

Lemma 2.5. *For* $u,\, v \,:\, Q \to \mathbb{R}$ *smooth enough:*

1 $\|u_t\, v_x\|_{L^2(0,T;\, W^{1,2}(\Omega))} \leq c\, \|u\|_{\mathcal{V}^1}\, \|v\|_{\mathcal{V}^2}$;

2 $\|u\, v_x\|_{L^2(0,T;\, W^{1,2}(\Omega))} \leq c\, \|u\|_{\mathcal{V}^1}\, \|v\|_{\mathcal{V}^0}$;

3 $\|u\, v_x\|_{L^2(0,T;\, W^{2,2}(\Omega))} \leq c\, \|u\|_{\mathcal{V}^1}\, \|v\|_{\mathcal{V}^1}$;

4 $\|E_{\Omega}(u)\, E_{\Omega}(v_x)\|_{\mathcal{V}^{-1}} \leq c\, \|u\|_{\mathcal{V}^1}\, \|v\|_{\mathcal{V}^0}$;

5 $\|v_x\, u\|_{\mathcal{V}^0} \leq c\, \|u\|_{\mathcal{V}^1}\, \|v\|_{\mathcal{V}^1}$.

6 $\|u\, v\|_{Tr\mathcal{V}^0} \leq c\, \|u\|_{Tr\mathcal{V}^1}\, \|v\|_{Tr\mathcal{V}^0}$.

7 $\|\nabla_{\Gamma}(u\, v)\|_{Tr\mathcal{V}^0} \leq c\, \|u\|_{Tr\mathcal{V}^1}\, \|v\|_{Tr\mathcal{V}^1}$.

Proof. We show (2.5) making use of $v_x \in \mathcal{V}^1$ and of the fact that $\mathcal{V}^1$ is a multiplicator space for $L^2W^{1,2}$. Moreover $\|u_t\|_{L^2(0,T;\, W^{1,2}(\Omega))} \leq \|u\|_{\mathcal{V}^1}$.

For (1), we make use of the same argument and that $\|v_x\|_{L^2W^{1,2}} \leq \|v\|_{\mathcal{V}^0_T}$.

In order to prove (2), we make use of the fact that $\mathcal{V}^1$ is a multiplicator space for $L^2W^{2,2}$, and that $\|v_x\|_{L^2W^{2,2}} \leq \|v\|_{\mathcal{V}^1}$.

In order to prove (3), we make use of the fact that $E_{\Omega}(\mathcal{V}^1)$ is a multiplicator space for $\mathcal{V}^{-1}$, and that $\|E_{\Omega}(v_x)\|_{\mathcal{V}^{-1}} \leq \|v\|_{\mathcal{V}^0}$ (see the Lemma 2.1).

Since $\mathcal{V}^1$ is a multiplicator space for $\mathcal{V}^0$, (4) is obvious.

In order to prove (5), note that for arbitrary extensions $\bar{u},\, \bar{v}$ into Ω of u and v, the Lemma 2.4 yields $\|\bar{u}\, \bar{v}\|_{\mathcal{V}^0} \leq c\, \|\bar{u}\|_{\mathcal{V}^1}\, \|\bar{v}\|_{\mathcal{V}^0}$. Thus

$$\|u\, v\|_{\mathrm{Tr}\,\mathcal{V}^0} \leq c\, \|\bar{u}\|_{\mathcal{V}^1}\, \|\bar{v}\|_{\mathcal{V}^0}\, \forall\, \bar{u} \in \mathcal{V}^1,\, \bar{v} \in \mathcal{V}^0 \,:\, \bar{u} = u,\, \bar{v} = v \text{ on } S_T\,.$$

Thus, (5) follows from the definition of the norm for spaces Tr_{S_T} (cf. (7)).

In order to prove (6), we choose an extension $\tau \in C^2(\overline{\Omega};\, \mathbb{R}^3)$ for an arbitrary tangential vector τ field given on $\partial\Omega$. We then regard $\nabla_{\Gamma} = \tau \cdot \nabla$ as a differential operator that we can apply on bulk functions. We Since $\mathcal{V}^1$ is a multiplicator space for $\mathcal{V}^0$, it follows for arbitrary extensions $\bar{u}$ and $\bar{v}$ that

$$\|\bar{u}\, \tau \cdot \nabla\bar{v}\|_{\mathcal{V}^0} \leq c\, \|\tau \cdot \nabla\bar{v}\|_{\mathcal{V}^0}\, \|\bar{u}\|_{\mathcal{V}^1} \leq c\, \|\tau\|_{C^2(\overline{\Omega})}\, \|\bar{v}\|_{\mathcal{V}^1}\, \|\bar{u}\|_{\mathcal{V}^1}\,.$$

Thus $\|u\, \tau \cdot \nabla v\|_{\mathrm{Tr}_{S_T}\mathcal{V}^0} \leq c\, \|v\|_{\mathrm{Tr}_{S_T}\,\mathcal{V}^1}\, \|u\|_{\mathrm{Tr}_{S_T}\,\mathcal{V}^1}$. The claim (6) follows easily. □

2.4.2 Nemicki operators

Nemicki operators are, beside multiplication operators, another important ingredient for the analysis of quasilinear equations. In the following statement, we for simplicity regard the spaces $\mathcal{V}^{\ell}$ as sets of scalar valued functions ($N = 1$).

Proposition 2.6. *Let* $b \,:\, [0,T] \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$. *For* $u \,:\, [0,T] \times \overline{\Omega} \to \mathbb{R}$ *we define* $N_b(u) := b(t,x,\, u)$. *Then, the following is valid:*

(1) If $b \in L^{\infty}([0,T];\, C^3(\overline{\Omega} \times \mathbb{R}))$, then, the Nemicki operator $u \mapsto N_b(u)$ is bounded and continuous from $\mathcal{V}^1$ into $L^2(0,T;\, W^{3,2}(\Omega))$.

(2) If $b \in C^1([0,T];\, C^2(\overline{\Omega} \times \mathbb{R}))$, then $u \mapsto N_b(u)$ is bounded and continuous from $\mathcal{V}^1$ into $W^1_2(0,T;\, W^{1,2}(\Omega))$.

(3) If $b \in L^{\infty}([0,T];\, C^3(\overline{\Omega} \times \mathbb{R})) \cap C^1([0,T];\, C^2(\overline{\Omega} \times \mathbb{R}))$, the operator N_b is bounded and continuous from $\mathcal{V}^1$ into itself. If moreover $b_u \in L^{\infty}([0,T];\, C^3(\overline{\Omega} \times \mathbb{R})) \cap C^1([0,T];\, C^2(\overline{\Omega} \times \mathbb{R}))$, it is even Fréchet differentiable from $\mathcal{V}^1$ into itself.

(4) If $b,\, b_x,\, b_u \in C^1([0,T];\, C^2(\overline{\Omega} \times \mathbb{R})) \cap L^{\infty}([0,T];\, C^3(\overline{\Omega} \times \mathbb{R}))$, the operator N_b is bounded and continuous from $\mathcal{V}^2$ into itself. If moreover $b_{x,u}$ and $b_{u,u} \in C^1([0,T];\, C^2(\overline{\Omega} \times \mathbb{R})) \cap L^{\infty}([0,T];\, C^3(\overline{\Omega} \times \mathbb{R}))$, it is Fréchet differentiable.

Proof. The derivatives of $f := b(t, x,\, u)$ are given by

$$f_{x_i} = b_{x_i} + b_u\, u_{x_i}$$
$$f_{x_i,x_j} = b_{x_i,x_j} + b_{u,x_j}\, u_{x_i} + b_{u,x_i}\, u_{x_j} + b_{u,u}\, u_{x_i}\, u_{x_j} + b_u\, u_{x_i,x_j}\, .$$

The third derivatives are given by an expression

$$f_{x,x,x} = \sum_{\alpha+\beta \leq 3} c_{\alpha,\beta}\, D^{\alpha}_x u\, D^{\beta}_x u\, .$$

Here, the coefficient $c_{\alpha,\beta}(t, x,\, u)$ are given by b and its derivatives after u and x up to order three. Recalling (9) and the embedding $\mathcal{V}^1 \subset L^{\infty}(Q)$, we obtain that

$$\|f\|_{L^2(0,T;\, W^{3,2}(\Omega))} \leq c(\|u\|_{L^{\infty}(Q)})\, (1 + \|u\|_{\mathcal{V}^1}) = C(\|u\|_{\mathcal{V}^1})\, . \tag{17}$$

This shows that the Nemicki operator is bounded from $\mathcal{V}^1$ into $L^2(0,T;\, W^{3,2}(\Omega))$. We easily show the continuity, proving (1).

We compute the derivatives

$$f_{t,x} = b_{t,x} + b_{u,x}\, u_t + b_{u,t}\, u_x + b_{u,u}\, u_x\, u_t + b_u\, u_{x,t}\, .$$

For $u \in \mathcal{V}^1$, it follows from (9) that $\|u_x\, u_t\|_{L^2}$ is bounded by $\|u\|^2_{\mathcal{V}^1}$. Thus

$$\|f\|_{W^1_2(0,T;\, W^{1,2}(\Omega))} \leq c(\|u\|_{L^{\infty}(Q)})\, (1 + \|u\|^2_{\mathcal{V}^1})\, . \tag{18}$$

This shows that the Nemicki operator is also bounded from $\mathcal{V}^1$ into $W^1_2(0,T;\, W^{1,2}(\Omega))$. Combining (17) and (18), we obtain that N_b is bounded from $\mathcal{V}^1$ into itself, that is,

$$\|N_b(u)\|_{\mathcal{V}^1} \leq c_b(\|u\|_{\mathcal{V}^1})\, . \tag{19}$$

We prove easily the continuity.

If $b_u \in C([0,T];\, C^3(\overline{\Omega} \times \mathbb{R}))$, we can show that the mapping $N_b\, :\, u \mapsto b(t, x,\, u)$ is a Gateaux differentiable operator between $\mathcal{V}^1$ and itself. The directional derivative at u^* is the operator

$$N_b'(u^*)\, u = b_u(t, x,\, u^*)\, u\, .$$

Making use of the Lemma 2.4, we see that

$$\|b_u(t,x,\,u^*)\,u\|_{\mathcal{V}^1} \leq c\,\|u\|_{\mathcal{V}^1}\,\|b_u(t,x,\,u^*)\|_{\mathcal{V}^1}\,.$$

Reinterpreting $b_u(t,x,\,u^*)$ as Nemicki operator $N_{b_u}(u^*)$ and making use of the first step (19), it follows that

$$\|b_u(t,x,\,u^*)\,u\|_{\mathcal{V}^1} \leq c\,\|u\|_{\mathcal{V}^1}\,C(\|u^*\|_{\mathcal{V}^1})\,.$$

Thus $\|N_b'(u^*)\|_{\mathscr{L}(\mathcal{V}^1,\mathcal{V}^1)} \leq c_{b,b_u}(\|u^*\|_{\mathcal{V}^1})$. We can show that $u^* \mapsto N_b'(u^*)$ is continuous from $\mathcal{V}^1$ into $\mathscr{L}(\mathcal{V}^1,\,\mathcal{V}^1)$ which is nothing else but the C^1 property. This shows (3).

To show (4), we consider N_b as an operator on $\mathcal{V}^2$. To show that $N_b(u) \in \mathcal{V}^2$, it suffices to show that $(N_b(u))_x \in \mathcal{V}^1$, that is

$$b_x(t,x,\,u) + b_u(t,x,\,u)\,u_x \in \mathcal{V}^1\,.$$

For pairs $v = (u,\,u_x) \in \mathbb{R}^4$, we introduce $\tilde{b}(t,x,\,v) := b_x(t,x,\,u) + b_u(t,x,\,u)\,u_x$. If $b,\,b_x,\,b_u \in C^1([0,T];\,C^2(\overline{\Omega}\times\mathbb{R})) \cap C([0,T];\,C^3(\overline{\Omega}\times\mathbb{R}))$, then $\tilde{b} \in C^1([0,T];\,C^2(\overline{\Omega}\times\mathbb{R}^4))$ and $\tilde{b} \in C([0,T];\,C^3(\overline{\Omega}\times\mathbb{R}^4))$. It then follows from (3) that the Nemicki operator $N_{\tilde{b}}$ is continuous from $\mathcal{V}^1$ into itself. If moreover $b_{x,u},\,b_{u,u} \in C^1([0,T];\,C^2(\overline{\Omega}\times\mathbb{R})) \cap C([0,T];\,C^3(\overline{\Omega}\times\mathbb{R}))$, the operator is Fréchet differentiable. □

Corollary 2.7. *Let* $c \in C^1([0,T];\,C^3(\overline{\Omega}\times\mathbb{R}^4)) \cap C([0,T];\,C^4(\overline{\Omega}\times\mathbb{R}^4))$*. Then, the Nemicki operator* $N_c:\ u \mapsto c(t,x,\,u,\,u_x)$ *is well defined and Fréchet differentiable from* $\mathcal{V}^2$ *into* $\mathcal{V}^1$*.*

Proof. Define $N_c(u) = c(t,x,\,u,\ ,u_x)$ for $u \in \mathcal{V}^2$. Since $u_x \in \mathcal{V}^1$, we have $N_c(u) = N_{\tilde{c}}(u,u_x)$ with $N_{\tilde{c}}(w) = c(t,x,\,(u,u_x))$. Here $\tilde{c} \in C^1([0,T];\,C^3(\overline{\Omega}\times\mathbb{R}^4)) \cap C([0,T];\,C^4(\overline{\Omega}\times\mathbb{R}^4))$. Thus, from the Proposition 2.6, (3) we can conclude that N_c is a C^1 operator between $\mathcal{V}^2$ and $\mathcal{V}^1$. □

2.5 An extension operator for the oblique derivative problem

In order to homogenise boundary conditions, we need an operator proposed in Lemma 7.19 of [Lie96] and slightly modified in the following Lemma.

Lemma 2.8. *Let* $f \in \mathrm{Tr}_{S_T}\,\mathcal{V}^{-1}$*. If* $\partial\Omega$ *is of class* $\mathcal{C}^3$*, then there exists a function* $U \in \mathcal{V}^0$ *such that*

$$U = 0 \text{ on } S_T, \quad \nu(x)\cdot\nabla U = f(x) \text{ on } S_T$$
$$\|U\|_{\mathcal{V}^0} \leq c\,\|f\|_{\mathrm{Tr}_{S_T}\,\mathcal{V}^{-1}}\,.$$

Proof. We first construct the operator for a flat situation. Assume at first that f is a function of class $C^1_c(]-\infty,\,+\infty[\times\mathbb{R}^3)$. Let $\Phi \in C_c^\infty(\mathbb{R}^4)$ be a smooth function with support in $\overline{B_1(0)}$ such that $\int_{\mathbb{R}^4}\Phi(z)\,dz = 1$. The variable $X \in \mathbb{R}^4$ stands here for $X = (t,x)$ where x in $\mathbb{R}^3$ is the space variable. For $X \in \mathbb{R}^4$ we consider

$$(Tf)(X) := -X_4^{-4}\int_{\mathbb{R}^4}\Phi\big(\tfrac{X-Z}{X_4^\beta}\big)\,f(Z)\,dZ\,. \tag{20}$$

Here $\beta := (2,1,1,1)$. The transformation formula yields the equivalent representation

$$(Tf)(X) = X_4\int_{\mathbb{R}^4}\Phi(Y)\,f(X - X_4^\beta\,Y)\,dY\,. \tag{21}$$

First, for $i = 2, 3$, we can show that

$$\begin{aligned}(Tf)_{X_i} &= -X_4^{-5} \int_{\mathbb{R}^4} \Phi_{X_i}(\tfrac{X-Z}{X_4^{\beta}})\, f(Z)\, dZ \\ &= \int_{\mathbb{R}^4} \Phi_{X_i}(Y)\, f(X - X_4^{\beta}\, Y)\, dY\,.\end{aligned}$$

For $i = 4$

$$\begin{aligned}(Tf)_{X_4} =& - X_4^{-5} \int_{\mathbb{R}^4} \Phi_{X_4}(\tfrac{X-Z}{X_4^{\beta}})\, f(Z)\, dZ \\ &+ X_4^{-5} \int_{\mathbb{R}^4} \sum_{i=1}^{4} \beta_i\, \Phi_{X_i}(\tfrac{X-Z}{X_4^{\beta}})\, \tfrac{X_i - Z_i}{X_4^{\beta_i}}\, f(Z)\, dZ \\ &+ 4\, X_4^{-5} \int_{\mathbb{R}^4} \Phi(\tfrac{X-Z}{X_4^{\beta}})\, f(Z)\, dZ\,.\end{aligned}$$

Introducing $\Psi(Y) := \sum_{i=1}^4 \beta_i\, \Phi_{X_i}(Y)\, Y_i$, then it follows that

$$\begin{aligned}(Tf)_{X_4} =& \int_{\mathbb{R}^4} \Phi_{X_4}(Y)\, f(X - X_4^{\beta}\, Y)\, dY - 4 \int_{\mathbb{R}^4} \Phi(Y)\, f(X - X_4^{\beta}\, Y)\, dY \\ &- \int_{\mathbb{R}^4} \Psi(Y)\, f(X - X_4^{\beta}\, Y)\, dY\,.\end{aligned}$$

For $i = 2, 3$, we make use of

$$\begin{aligned}|(Tf)_{X_i}|^2 &\leq \int_{\mathbb{R}^4} |\Phi_{X_i}(Y)|\, dY \int_{\mathbb{R}^4} |\Phi_{X_i}(Y)|\, f^2(X - X_4^{\beta}\, Y)\, dY \\ &\leq C \int_{\mathbb{R}^4} |\Phi_{X_i}(Y)|\, f^2(X - X_4^{\beta}\, Y)\, dY\,.\end{aligned}$$

Thus, integration over $\mathbb{R}^4$ yields

$$\|(Tf)_{X_i}\|_{L^2}^2 \leq C \int_{\mathbb{R}^4} |\Phi_{X_i}(Y)| \int_{\mathbb{R}^4} f^2(X - X_4^{\beta}\, Y)\, dX\, dY$$

Since by the transformation formula

$$\int_{\mathbb{R}^4} f^2(X - X_4^{\beta}\, Y)\, dX = \frac{1}{1 - Y_4}\, \|f\|_{L^2}^2$$

it follows that

$$\|(Tf)_{X_i}\|_{L^2}^2 \leq C \int_{\mathbb{R}^4} \frac{|\Phi_{X_i}(Y)|}{1 - Y_4}\, dY\, \|f\|_{L^2}^2\,.$$

The choice of Φ now guaranties that $\int_{\mathbb{R}^4} \frac{|\Phi_{X_i}(Y)|}{1-Y_4}\, dY \leq c\, \|\Phi\|_{W^{2,2}(\mathbb{R}^4)}$. Similarly, we show that $\|(Tf)_{X_4}\|_{L^2}^2 \leq C\, \|f\|_{L^2}^2$. Thus, the operator T extends to a bounded linear operator from $L^2(\mathbb{R}^4)$ into $L^2(\mathbb{R};\, W^{1,2}(\mathbb{R}^3))$.

Due to (21), we easily verify that the trace of Tf on the plane $X_4 = 0$ is zero.

Now we verify the claims on the derivatives of Tf and the normal trace. For $i = 2, 3$, the representation (20) also implies that

$$(Tf)_{X_i} = -X_4^{-4} \int_{\mathbb{R}^4} \Phi(\tfrac{X-Z}{X_4^\beta})\, f_{X_i}(Z)\, dZ = T(f_{X_i})\,, \tag{22}$$

Thus $\|(Tf)_{X_i}\|_{L^2W^{1,2}} = \|T(f_{X_i})\|_{L^2W^{1,2}} \leq C\, \|f_{X_i}\|_{L^2}$. For $i = 4$, we make use of (21) to see that

$$\begin{aligned}(Tf)_{X_4} = & \int_{\mathbb{R}^4} \Phi(Y)\, f(X - X_4^\beta\, Y)\, dY + X_4 \int_{\mathbb{R}^4} \Phi(Y)\, \big(f_{X_4}(X - X_4^\beta\, Y) \\ & - \sum_{i=1}^4 f_{X_i}(X - X_4^\beta\, Y)\, \beta_i\, Y_i\, X_4^{\beta_i - 1}\big)\, dY \\ = & \int_{\mathbb{R}^4} \Phi(Y)\, f(X - X_4^\beta\, Y)\, dY + X_4 \int_{\mathbb{R}^4} \Phi(Y)\, \big(f_{X_4}(X - X_4^\beta\, Y) \\ & - \sum_{i=2}^4 f_{X_i}(X - X_4^\beta\, Y)\, Y_i\big)\, dY - 2\, X_4^2 \int_{\mathbb{R}^4} \Phi(Y)\, f_{X_1}(X - X_4^\beta\, Y)\, Y_1\, dY\end{aligned}$$

Now, there exists a vector field $F \in L^2(\mathbb{R}^3;\, \mathbb{R}^3)$ such that $f_{X_1} = \operatorname{div}_x F$ and

$$\|f_{X_1}\|_{L^2(\mathbb{R};\, [W^{1,2}(\mathbb{R}^3)]^*)} = \|F\|_{L^2(\mathbb{R}^3;\, \mathbb{R}^3)}\,.$$

For $i = 1, \ldots, 4$ define $\tilde{\Psi}^i(Y) := \Phi(Y)\, Y_i$ and observe that

$$\begin{aligned}-2\, X_4^2 \int_{\mathbb{R}^4} \Phi(Y)\, f_{X_1}(X - X_4^\beta\, Y)\, Y_1\, dY &= 2\, X_4^{-3} \int_{\mathbb{R}^4} \tilde{\Psi}^1(\tfrac{X-Z}{X_4^\beta})\, \operatorname{div}_x F(Z)\, dZ \\ &= -2\, X_4^{-4} \sum_{i=2,3,4} \int_{\mathbb{R}^4} \tilde{\Psi}^1_{X_i}(\tfrac{X-Z}{X_4^\beta})\, F_i(Z)\, dZ\,.\end{aligned}$$

We obtain the representation

$$\begin{aligned}(Tf)_{X_4} = & \int_{\mathbb{R}^4} \Phi(Y)\, f(X - X_4^\beta\, Y)\, dY \\ & + (Tf_{X_4}) - \sum_{i=2,3,4} \{(T_{\tilde{\Psi}^i} f_{X_i}) - 2\, (T_{\tilde{\Psi}^1_{X_i}}\, F_i)\}\,.\end{aligned} \tag{23}$$

Due to the properties of T:

$$\begin{aligned}\|Tf_{X_4}\|_{L^2W^{1,2}} &\leq C\, \|f_{X_4}\|_{L^2} \\ \|T_{\tilde{\Psi}^i} f_{X_i}\|_{L^2W^{1,2}} &\leq C\, \|f_{X_i}\|_{L^2} \\ \|T_{\tilde{\Psi}^1_{X_i}}\, F_i\|_{L^2W^{1,2}} &\leq C\, \|F\|_{L^2} \leq C\, \|f_{X_1}\|_{L^2(\mathbb{R};\, [W^{1,2}(\mathbb{R}^3)]^*)}\,.\end{aligned}$$

With similar arguments, we also prove that the function $g(X) := \int_{\mathbb{R}^4} \Phi(Y)\, f(X - X_4^\beta\, Y)\, dY$ satisfies $\|g\|_{L^2W^{1,2}} \leq C\, (\|f\|_{L^2W^{1,2}} + \|F\|_{L^2})$. Thus, overall

$$\|Tf\|_{L^2(\mathbb{R};\, W^{2,2}(\mathbb{R}^3))} \leq C\, (\|f\|_{L^2W^{1,2}} + \|f_{X_1}\|_{L^2[W^{1,2}]^*})\,.$$

Moreover, (23) shows that $(Tf)_{X_4} = f$ on the plane $X_4 = 0$.

For the time derivative ∂_{X_1}, we can show that

$$(Tf)_{X_1} = X_4^{-4} \int_{\mathbb{R}^4} \Phi(\tfrac{X-Z}{X_4^\beta})\, f_{X_1}(Z)\, dZ\,.$$

Therefore $(Tf)_{X_1} = -X_4^{-5} \int_{\mathbb{R}^4} \sum_{i=2,3,4} \Phi_{X_i}(\frac{X-Z}{X_4^\beta})\, F_i(Z)\, dZ$. It follows that

$$\|(Tf)_{X_1}\|_{L^2} \leq C\, \|f_{X_1}\|_{L^2(\mathbb{R};\, [W^{1,2}(\mathbb{R}^3)]^*)}\,.$$

In order to prove the statement in a curved situation, assume that $]0,T[\times\Gamma =]0,T[\times(\partial\Omega\cap\mathcal{U})$ is a piece of surface such that a $\mathcal{C}^2$ diffeomorphism F exists between $\mathbb{R}\times\{(X_2,X_3,X_4)\,:\, |X_4|<1\}$ and $]0,T[\times\mathcal{U}$. We then choose $U(t,x) = (T(f\circ F))(F^{-1}(t,x))$ and the claim follows. □

2.6 A simple operator theoretic tool

Lemma 2.9. *Let $X,\, Y$ be Banach spaces with X reflexive. Let $\{\mathcal{L}(s)\}_{s\in[0,1]} \subset \mathscr{L}(X,\,Y)$ be a family of bounded linear injections such that $\mathcal{L}(0)$ is invertible and such that $\mathcal{L}(s)\to\mathcal{L}(s_0)$ in $\mathscr{L}(X,\,Y)$ for all $s\to s_0\in[0,1]$. Assume moreover that all solutions to $\mathcal{L}(s)\,x = y$ satisfy a uniform bound $\|x\|_X \leq c_1\,\|y\|_Y$. Then, $\mathcal{L}(1)$ is invertible.*

Proof. Define

$$s^* := \sup\{s\in[0,1]\,:\,\mathcal{L}(s) \text{ is invertible }\}\,.$$

Since $\mathcal{L}(0)$ is invertible, the Banach perturbation argument yields that $\mathcal{L}(s)$ is invertible for all $\|\mathcal{L}(s)-\mathcal{L}(0)\|_{\mathscr{L}(X,Y)} < \|[\mathcal{L}(0)]^{-1}\|_{\mathscr{L}(Y,X)}$. Thus, $s^*>0$. We next show that $\mathcal{L}(s^*)$ is invertible. By definition, we can choose a sequence $s^n \nearrow s^*$ such that $\mathcal{L}(s^n)$ is invertible. In particular, for arbitrary $y\in Y$, we can introduce $x^n\in X$ such that $\mathcal{L}(s^n)\,x^n = y$. We make use of the assumption, $\|x^n\|_X \leq c_1\,\|y\|_Y$. Thus, extracting a weakly convergent subsequence $x^n \rightharpoonup x$, and using that

$$\mathcal{L}(s^*)\,x^n = y + (\mathcal{L}(s^*) - \mathcal{L}(s^n))\,x^n$$

we see that $\mathcal{L}(s^*)\,x = y$. Thus, $\mathcal{L}(s^*)$ is surjective, and therefore invertible.

If now $s^*<1$, the Banach perturbation argument yields that $\mathcal{L}(s)$ is invertible for all $s>s^*$ such that $\|\mathcal{L}(s)-\mathcal{L}(s^*)\|_{\mathscr{L}(X,Y)} < \|[\mathcal{L}(s^*)]^{-1}\|_{\mathscr{L}(Y,X)}$. This would contradict the definition of s^*. □

2.7 Preliminaries associated with the elliptic operator

Localisation. Assume that Ω is a domain of class $\mathcal{C}^m$ ($m\geq 1$). We find a *partition of unity* $\zeta_0,\ldots,\zeta_n$ for the domain Ω (see Theorem 5.3.8 of [KJF77]). For $\mu=0,\ldots,n$, we denote $\Omega_\mu = \Omega\cap\operatorname{supp}\zeta_\mu$. It is possible to assume the following:

- The function ζ_0 has a compact support in Ω;
- For $\mu=0,\ldots,n$, there are vector fields $V^{\mu,1},\, V^{\mu,2},\, V^{\mu,3}\in C^{m-1}(\overline{\Omega_\mu};\,\mathbb{R}^3)$ such that

  $\{V^{\mu,1}(x),\, V^{\mu,2}(x),\, V^{\mu,3}(x)\}$ is an orthonormal system of $\mathbb{R}^3$ for all $x\in\overline{\Omega_\mu}$
  $V^{\mu,3}(x) = \nu(x)$ for all $x\in\partial\Omega\cap\operatorname{supp}\zeta_\mu\,.$

  (For extension of the vector ν, see the Section 14.6 of [GT01]).

In particular, we can decompose the gradient of a function as follows:

$$w_x = \sum_{\mu=0}^{m}\sum_{\ell=1}^{3}(V^{\mu,\ell}\cdot\nabla w\,\zeta_\mu)\,V^{\mu,\ell}\,. \tag{24}$$

There the vector fields $V^{\mu,1},\,V^{\mu,2}$ are tangent on $\partial\Omega$ for $\mu=1,\ldots,m$. With this construction we can avoid tedious changes of coordinates to locally flatten the boundary.

Linear elliptic systems. We consider a general second order linear PDE system in divergence form

$$-\partial_{x_k}(M_{i,j,k,\ell}(x)\,v_{j,x_\ell}) = f_i(x)\,. \tag{25}$$

The system is *uniformly elliptic* if there is $0<\nu_0(M)$ such that

$$\nu_0\,\|A\|^2 \leq \sum_{i,j=1}^{N}\sum_{k,\ell=1}^{3} M_{i,j,k,\ell}(x)\,A_{i,k}\,A_{j,\ell}\text{ for all } A\in\mathbb{R}^{N\times 3}\,. \tag{26}$$

We commence with a technical preliminary about the connection between bulk and boundary operator. The proof is completely elementary and might be left as an exercise.

Lemma 2.10. *Assume for $i,j=1,\ldots,N$ and $k,\ell=1,2,3$ that the functions $M_{i,j,k,\ell}$ are of class $C^{0,1}(\overline{\Omega})$. Consider vector fields $V^1,\,V^2,\,V^3$ that we assume of class $C^{1,1}(\overline{\Omega};\,\mathbb{R}^3)$ and orthonormal in $\Omega'\subseteq\Omega$.*

If $v\in W^{2,2}(\Omega;\,\mathbb{R}^N)$ satisfies the equations (25) *in Ω', then the following identities are valid for $i=1,\ldots,N$:*

$$\begin{aligned} V^3_k\,M_{i,j,k,\ell}\,(V^3\cdot\nabla v_j)_{x_\ell} &= -\sum_{\lambda=1}^{2} V^\lambda_k\,M_{i,j,k,\ell}\,(V^\lambda\cdot\nabla v_j)_{x_\ell} - f_i\\ &+\sum_{\lambda=1}^{3} V^\lambda_k\,(M_{i,j,k,\ell}(x)\,(V^\lambda)_{x_\ell}\cdot\nabla\,v_j - V^\lambda\cdot\nabla M_{i,j,k,\ell}\,v_{j,x_\ell})\,. \end{aligned} \tag{27}$$

and

$$\begin{aligned} M_{i,j,V^3,V^3}\,V^3\cdot\nabla(V^3\cdot\nabla v_j) &= -f_i\\ &-\sum_{\lambda=1}^{2} M_{i,j,k,\ell}\,[V^\lambda_k\,(V^\lambda\cdot\nabla v_j)_{x_\ell} + V^3_k\,V^\lambda_\ell\,V^3\cdot\nabla(V^\lambda\cdot\nabla v_j)]\\ &+\sum_{\lambda=1}^{3} V^\lambda_k\,(M_{i,j,k,\ell}\,(V^\lambda)_{x_\ell}\cdot\nabla\,v_j - V^\lambda\cdot\nabla M_{i,j,k,\ell}\,v_{j,x_\ell})\\ &-\sum_{\lambda=1}^{2} V^3_k\,V^\lambda_\ell\,M_{i,j,k,\ell}\,[\{(V^\lambda\cdot\nabla)V^3-(V^3\cdot\nabla)V^\lambda\}\nabla v_j] \end{aligned} \tag{28}$$

The fundamental Hilbert space estimates for linear equations and systems are very well known: see [DN55], [Nir55], [JN57]..

Lemma 2.11. *Assume for $i, j = 1, \ldots, N$ and $k, \ell = 1, 2, 3$ that the functions $M_{i,j,k,\ell}$ are of class $C^{0,1}(\Omega)$ and satisfy (26). If Ω is of class $\mathcal{C}^{1,1}$ the unique solution $v \in W^{2,2}(\Omega;\, \mathbb{R}^N)$ to the equations (25) in Ω in connection with one of the following two boundary conditions:*

$$-\nu_k\, M_{i,j,k,\ell}(x)\, v_{j,x_\ell} = 0 \text{ on } \partial\Omega$$
$$v = 0 \text{ on } \partial\Omega$$

satisfies the estimate $\|v\|_{W^{2,2}(\Omega;\,\mathbb{R}^N)} \leq c(\|M\|_{C^{0,1}(\Omega)})\, \|f\|_{L^2(\Omega;\,\mathbb{R}^N)}$.

3 Reduced quasilinear parabolic system in divergence form

For functions $w_1, \ldots, w_N : \overline{Q}_T \to \mathbb{R}$ we consider the problem:

$$\partial_t R_i(t, x,\, w) + \operatorname{div} J^i(t, x,\, w,\, w_x) = f_i(t, x,\, w,\, w_x) \text{ in } [0, T] \times \Omega \tag{29}$$
$$B_i(t, x,\, w,\, w_x) = f_{\Gamma,i}(t, x,\, w) \quad \text{on } [0, T] \times \partial\Omega \tag{30}$$
$$w = q^0 \quad \text{in } \{0\} \times \Omega\,. \tag{31}$$

The functions $R_1, \ldots, R_N$ are defined on $\overline{Q} \times \mathbb{R}^N$. We denote $(t, x,\, z)$ a generic element of the latter domain.

The functions J^i_k ($i = 1, \ldots, N$ and $k = 1, 2, 3$) as well as f_i ($i = 1, \ldots, N$) are defined on $\overline{Q} \times \mathbb{R}^N \times \mathbb{R}^{N\times 3}$. We denote $(t, x,\, z,\, D)$ a generic element of the latter domain. The functions $f_{\Gamma,1}, \ldots, f_{\Gamma,N}$ are naturally defined on $\overline{S_T} \times \mathbb{R}^N$, but we will for simplicity directly assume that they are extension functions defined in $\overline{Q}_T \times \mathbb{R}^N$.

In this section, we assume that J generates a *restricted quasilinear* operator in divergence form for a fully coupled system, which means that J is linear in the variable D. Therefore

$$J^i_k(t, x,\, w,\, w_x) := -\sum_{j=1}^N \sum_{\ell=1}^3 M_{i,j,k,\ell}(t, x,\, w)\, w_{j,x_\ell} \text{ for } i = 1, \ldots, N,\ k = 1, 2, 3\,. \tag{32}$$

For $i, j = 1, \ldots, N$ and $k, \ell = 1, 2, 3$, the function $M_{i,j,k,\ell}$ is defined on $\overline{Q} \times \mathbb{R}^N$. With $M_{i,j,k,\ell,t}$, $M_{i,j,k,\ell,x}$, $M_{i,j,k,\ell,z}$, we denote the partial derivatives of M in these these variables.

The 'parabolicity' of the system (29) lays in the following assumptions for R and M: There are positive continuous functions $\lambda_0 \leq \lambda_1 \in C(\mathbb{R}^N)$ and $\nu_0 \in C(\mathbb{R}^N)$ such that

$$\frac{|X|^2}{\lambda_1(z)} \leq \sum_{i,j=1}^N R_{i,z_j}(t, x,\, z)\, X_i\, X_j \leq \frac{|X|^2}{\lambda_0(z)} \text{ for all } X \in \mathbb{R}^N \tag{33}$$

$$\nu_0(z)\, \|D\|^2 \leq \sum_{i,j=1}^N \sum_{k,\ell=1}^3 M_{i,j,k,\ell}(t, x,\, z)\, D^i_k\, D^j_\ell \text{ for all } D \in \mathbb{R}^{N\times 3}\,. \tag{34}$$

In this paper, we only consider the natural boundary operator

$$B_i(t, x,\, w,\, w_x) := \sum_{k=1}^3 \nu_k(x)\, J^i_k(t, x,\, w,\, w_x)$$
$$= -\sum_{k=1}^3 \nu_k(x) \left(\sum_{j=1}^N \sum_{\ell=1}^3 M_{i,j,k,\ell}(t, x,\, w)\, w_{j,x_\ell} \right)\,. \tag{35}$$

Our main result in the section is the following Theorem.

Theorem 3.1. *Assume that Ω is a bounded domain of class $\mathcal{C}^5$. Assume for $i = 1, \ldots, N$ that $R_i \in (C^{0,5} \cap C^{1,4})([0,T] \times \overline{\Omega} \times \mathbb{R}^N)$, and that $M_{i,j,k,\ell} \in (C^{0,4} \cap C^{1,3})([0,T] \times \overline{\Omega} \times \mathbb{R}^N)$ for $i, j = 1, \ldots, N$ and $k, \ell = 1, 2, 3$. Assume that R satisfies* (33) *and M satisfies* (34)*. Assume that $f \in C^{0,4}([0,T] \times \overline{\Omega} \times \mathbb{R}^N)$ and that $f^\Gamma \in C^{0,4}([0,T] \times \overline{\Omega} \times \mathbb{R}^N)$. Let $q_0 \in Tr_{\{0\}\times\Omega}\, \mathcal{V}^2$ satisfy the condition*

$$-\nu_k(x)\, M_{i,j,k,\ell}(0, x,\, q^0(x))\, q^0_{j,x_\ell} = f^\Gamma_i(0,\, x,\, q^0(x)) \text{ for all } x \in \partial\Omega\,. \tag{36}$$

Then, there is $T > 0$ depending only on R, M, q^0 and the domain Ω such that the problem (29) *possesses a unique solution $w \in \mathcal{V}^2(Q_T;\, \mathbb{R}^N)$.*

The remainder of the section is devoted to the proof of this statement.

3.1 The fundamental case

In this subsection we consider (29) with zero lower–order terms ($f = 0$ and $f_\Gamma = 0$). The assumed compatibility condition (36) is then

$$-\sum_{k=1}^{3}\sum_{j=1}^{N}\sum_{\ell=1}^{3} \nu_k(x)\, M_{i,j,k,\ell}(0, x,\, q^0(x))\, q^0_{j,x_\ell}(x) = 0 \text{ for all } x \in \partial\Omega\,. \tag{37}$$

By means of the Lemmas 2.6 and 2.7, it can be shown that

- For $R \in (C^{1,4} \cap C^{0,5})([0,T] \times \overline{\Omega} \times \mathbb{R}^N)$, the non-linear mapping $\mathcal{R}$ defined via
$$\big(\mathcal{R}(w)\big)(t,x) := \partial_t R(t, x,\, w)$$
- For M in $(C^{1,3} \cap C^{0,4})([0,T] \times \overline{\Omega} \times \mathbb{R}^N)$, the non-linear mapping $\mathcal{Q}$ defined via
$$\big(\mathcal{Q}(w)\big)(t,x) := \operatorname{div} J(t, x,\, w,\, w_x)$$

map continuously the space $\mathcal{V}^2$ into $L^2(0,T;\, W^{2,2}(\Omega;\, \mathbb{R}^N))$. Moreover, $\mathcal{R}$ and $\mathcal{Q}$ are Fréchet differentiable as operators between these classes. To see this in the case of $\mathcal{R}$, we reinterpret $\mathcal{R}$ as a Nemicki operator N_R between $\mathcal{V}^2$ and itself. This operator is Fréchet differentiable according to Lemma 2.6, . Thus, the composition operator $\frac{d}{dt} \circ R$ is Fréchet differentiable as an operator from $\mathcal{V}^2$ into $L^2(0,T;\, W^{2,2}(\Omega;\, \mathbb{R}^N))$.

Similarly, if $M \in (C^{1,3} \cap C^{0,4})([0,T] \times \overline{\Omega} \times \mathbb{R}^N)$, we can interpret $\mathcal{Q} : w \mapsto M(t, x,\, w)\, w_x$ as a $\mathcal{C}^1$ operator from $\mathcal{V}^2$ into $\mathcal{V}^1$ (Lemma 2.7). Then, $\operatorname{div} \circ \mathcal{Q}$ is Fréchet differentiable from $\mathcal{V}^2$ into $L^2(0,T;\, W^{2,2}(\Omega;\, \mathbb{R}^N))$.

It can be shown under the assumptions $M \in (C^{1,3} \cap C^{0,4})([0,T] \times \overline{\Omega} \times \mathbb{R}^N)$ and $\partial\Omega$ of class $\mathcal{C}^5$ that the non-linear mapping $\mathcal{B} : w \mapsto B(t, x,\, w,\, w_x)$ maps $\text{Tr}_{S_T}\, \mathcal{V}^2$ into the space $\text{Tr}_{S_T}\, \mathcal{V}^1$. To see this, we again simply reinterpret $\mathcal{B}(w)$ as the Nemicki operator $N_c(w)$ with $c(t, x,\, w,\, w_x) = \nu(x)\, M(t, x,\, w)\, w_x$, and we apply the Lemma 2.7. Note that we need here an extension of class $C^4(\overline{\Omega};\, \mathbb{R}^3)$ of the normal vector into Ω (see Section 2.7).

We denote throughout the section $q^0 \in \mathcal{V}^2$ the initial data. We introduce a non-linear operator $G = \{G_1,\, G_2\}$ acting on $\mathcal{V}^2_\Omega$ via

$$G(v) := \{\mathcal{R}(v + q^0) + \mathcal{Q}(v + q^0),\, \mathcal{B}(v + q^0)\} \tag{38}$$

Due to the preliminary remarks, the image of $\mathcal{V}^2_\Omega$ under G is a subspace of $L^2(0,T;\, W^{2,2}(\Omega;\,\mathbb{R}^N))\times \mathrm{Tr}_{S_T}\,\mathcal{V}^1$. Under the compatibility condition (37), it is even a subspace of

$$Z^2 := L^2(0,T;\, W^{2,2}(\Omega;\,\mathbb{R}^N)) \times \mathrm{Tr}_{S_T}\,\mathcal{V}^1_\Omega\,. \tag{39}$$

Moreover, G is Fréchet differentiable as a map between $\mathcal{V}^2_\Omega$ and Z^2. It is readily seen that the resolvability of the problem (29) in $\mathcal{V}^2$ is equivalent with solving the equation $G(v)=0$ in $\mathcal{V}^2_\Omega$. Our analysis of this equation shall rely on the implicit function theorem.

We now investigate the directional derivative G' at a point $v^*\in\mathcal{V}^2_\Omega$ (introduce $w^*:=v^*+q^0\in\mathcal{V}^2$), which is the linear operator given by

$$G'(v^*)\,\xi = \{\mathcal{R}'(w^*)\,\xi + \mathcal{Q}'(w^*)\,\xi,\, \mathcal{B}'(w^*)\,\xi\} \text{ for } \xi\in\mathcal{V}^2_\Omega\,. \tag{40}$$

We identify $G'(v^*)$ as a linear operator of $\mathscr{L}(\mathcal{V}^2_\Omega,\, Z^2)$. For $i=1,\ldots,N$, we here make use of the abbreviations

$$\begin{aligned}
(\mathcal{R}'(w^*)\,\xi)_i &= \frac{d}{dt}\left(\sum_{j=1}^N R_{i,z_j}(t,x,\,w^*)\,\xi_j\right)\\
(\mathcal{Q}'(w^*)\,\xi)_i &= -\sum_{k=1}^3 \frac{d}{dx_k}\left(\sum_{j=1}^N\sum_{\ell=1}^3 M_{i,j,k,\ell}(t,x,\,w^*)\,\xi_{j,x_\ell}\right.\\
&\qquad\left. + \sum_{m=1}^N M_{i,j,k,\ell,z_m}(t,x,\,w^*)\,\xi_m\, w^*_{j,x_\ell}\right)\\
(\mathcal{B}'(w^*)\,\xi)_i &= -\sum_{k=1}^3 \nu_k\left(\sum_{j=1}^N\sum_{\ell=1}^3 M_{i,j,k,\ell}(t,x,\,w^*)\,\xi_{j,x_\ell}\right.\\
&\qquad\left. + \sum_{m=1}^N M_{i,j,k,\ell,z_m}(t,x,\,w^*)\,\xi_m\, w^*_{j,x_\ell}\right)\,.
\end{aligned}$$

For simplicity, we shall split these operators into a *principal part* and a *lower–order part*. We introduce the following principal parts of these operators:

$$\begin{aligned}
(\mathcal{R}'_o(w^*)\,\xi)_i &:= \sum_{j=1}^N R_{i,z_j}(t,x,\,w^*)\,\partial_t\xi_j\\
(\mathcal{Q}'_o(w^*)\,\xi)_i &:= -\sum_{k=1}^3 \partial_{x_k}(\sum_{j=1}^N\sum_{\ell=1}^3 M_{i,j,k,\ell}(t,x,\,w^*)\,\xi_{j,x_\ell})\\
(\mathcal{B}'_o(w^*)\,\xi)_i &:= -\sum_{k=1}^3 \nu_k\,(\sum_{j=1}^N\sum_{\ell=1}^3 M_{i,j,k,\ell}(t,x,\,w^*)\,\xi_{j,x_\ell})\,.
\end{aligned}$$

Remark 3.2. *The operator $\xi\mapsto\{\mathcal{R}'_o(w^*)+\mathcal{Q}'_o(w^*),\,\mathcal{B}'_o(w^*)\}\,\xi$ extends to a linear operator of $\mathscr{L}(\mathcal{V}^0_\Omega,\,Z^0)$. Here $Z^0:=L^2(\Omega;\,\mathbb{R}^N)\times \mathrm{Tr}_{S_T}\,\mathcal{V}^{-1}_{\mathbb{R}^3}$.*

Proof. Consider the inequalities

$$\begin{aligned}
\|\mathcal{R}'_o(w^*)\,\xi\|_{L^2(Q)} &\le \|R_z(t,x,\,w^*)\|_{L^\infty(Q))}\,\|\partial_t\xi\|_{L^2(Q)}\\
\|\mathcal{Q}'_o(w^*)\,\xi\|_{L^2(Q)} &\le (\|M(t,x,\,w^*)\|_{L^\infty(Q)} + \|\frac{d}{dx}\,M(t,x,\,w^*)\|_{L^\infty(Q)})\,\|\xi_x\|_{W^{1,0}_2(Q)}\,.
\end{aligned}$$

Since $w^* \in \mathcal{V}^2$ implies $w^*,\, w^*_x \in L^\infty(Q)$, it follows that

$$\|(\mathcal{R}'_o(w^*) + \mathcal{Q}'_o(w^*))\, \xi\|_{L^2(Q)} \leq c(\|w^*\|_{\mathcal{V}^2})\, \|\xi\|_{\mathcal{V}^0}\, .$$

Moreover, recalling Lemma 2.5, (3) we obtain that

$$\|\mathcal{B}'_o(w^*)\, \xi\|_{\text{Tr}_{S_T}\, \mathcal{V}^{-1}} \leq c\, \|M(t,x,\, w^*)\|_{\mathcal{V}^1}\, \|\xi\|_{\mathcal{V}^0}\, .$$

. □

Our aim is to show that the Fréchet derivative $G'(v^*)$ is invertible at arbitrary $v^* \in \mathcal{V}^2_\Omega$. We shall need several preliminaries.

Lemma 3.3. *Assume that the hypotheses of Theorem 3.1 are valid. Assume moreover that the following data are given:*

1 $w^* \in \mathcal{V}^2$;

2 $f \in L^2(0,T;\, L^2(\Omega;\mathbb{R}^N))$;

3 $f_\Gamma \in \text{Tr}_{S_T}\, \mathcal{V}^{-1}_{\mathbb{R}^3}$;

4 $w^\Gamma \in \mathcal{V}^0_\Omega$.

Then, the second boundary value problem

$$\mathcal{B}'_o(w^*)\, w = f_\Gamma \text{ on } S_T, \quad w(0) = 0 \tag{41}$$

and the first boundary value problem

$$w = w^\Gamma \text{ on } S_T, \quad w(0) = 0 \tag{42}$$

for the system $\mathcal{R}'_o(w^*)\, w + \mathcal{Q}'_o(w^*)\, w = f$ in Q_T both possess a unique solution in the class $\mathcal{V}^0$ of strong solutions. There is a number $c = c(\|w^*\|_{\mathcal{V}^2})$ such that in the case of (41)

$$\|w\|_{\mathcal{V}^0} \leq c\, (\|f\|_{L^2(0,T;\, L^2(\Omega;\mathbb{R}^N))} + \|f_\Gamma\|_{\text{Tr}_{S_T}\mathcal{V}^{-1}})\, .$$

and in the case of (42)

$$\|w\|_{\mathcal{V}^0} \leq c\, (\|f\|_{L^2(0,T;\, L^2(\Omega;\mathbb{R}^N))} + \|w^\Gamma\|_{\text{Tr}_{S_T}\, \mathcal{V}^0})\, .$$

Proof. Since $w^* \in \mathcal{V}^2$, then the scalar components of w^* belong to $H^{\frac{1}{2},\frac{1}{4}}(\overline{Q}) \hookrightarrow L^\infty(Q)$). The matrix $R_{i,z_j}(t,x,\, w^*)$ is therefore uniformly invertible.

We consider the following equivalent equations in connection with (41), (42):

$$\partial_t w - [R_z(t,x,\, w^*)]^{-1} \circ \mathcal{Q}'_o(w^*)\, w = \tilde{f} := [R_z(t,x\, w^*)]^{-1}\, f\, .$$

We first focus on the problem (41). For $t \in [0,T]$, $x \in \partial\Omega$ and $z \in \mathbb{R}^N$, define

$$(M_{\nu,\nu})_{i,j}(t,x,\, z) := \sum_{k,\ell=1}^{3} M_{i,j,k,\ell}(t,x,\, z)\, \nu_k(x)\, \nu_\ell(x), \quad i,j = 1,\ldots,N\, .$$

Due to the assumption (34), the matrix $M_{\nu,\nu}(t,x,\,z)$ is invertible for all $(t,x,\,z) \in S_T \times \mathbb{R}^N$.

Recall that $f_\Gamma \in \text{Tr}_{S_T}\,\mathcal{V}^{-1}_{\mathbb{R}^3}$. In particular, there is a bulk extension $f_\Gamma \in \mathcal{V}^{-1}_{\mathbb{R}^3}$. Employing the Lemma 2.8, we find extensions $U_1,\ldots,U_N \in \mathcal{V}^0$ that satisfy the following conditions

$$\begin{aligned} U_i &= 0 \text{ on } S_T \cup (\{0\}\times\Omega)\\ \nu\cdot\nabla U_i &= h_{\Gamma,i} := -\sum_{j=1}^N [(M_{\nu,\nu})(t,x,\,w^*)]^{-1}_{i,j}\,f_{\Gamma,j}\\ \|U_i\|_{\mathcal{V}^0} &\leq c\,\|h_\Gamma\|_{\text{Tr}_{S_T}\,\mathcal{V}^{-1}}\,. \end{aligned}$$

Making use of these identities, we can show for $i=1,\ldots,N$ that

$$\nu_k\,M_{i,j,k,\ell}(t,x,\,w^*)\,U_{j,x_\ell} = [M_{\nu,\nu}(t,x,\,w^*)]_{i,j}\,\nu\cdot\nabla U_j = f_{\Gamma,i}\,.$$

Moreover, due to the Lemma 2.4, and to the Lemma 2.6, we obtain that

$$\begin{aligned} \|h_\Gamma\|_{\text{Tr}_{S_T}\,\mathcal{V}^{-1}} &\leq c\,\|[(M_{\nu,\nu})(t,x,\,w^*)]^{-1}\|_{\mathcal{V}^1}\,\|f_\Gamma\|_{\text{Tr}_{S_T}\,\mathcal{V}^{-1}}\\ &\leq c(\|w^*\|_{\mathcal{V}^1})\,\|f_\Gamma\|_{\text{Tr}_{S_T}\,\mathcal{V}^{-1}}\,. \end{aligned}$$

Therefore $\|U_i\|_{\mathcal{V}^0} \leq c(\|w^*\|_{\mathcal{V}^1})\,\|f_\Gamma\|_{\text{Tr}_{S_T}\,\mathcal{V}^{-1}}$. We consider instead of w the function $\tilde{w} = w - U$. For simplicity, note that we in fact loose no generality in assuming $f_\Gamma = 0$, and in considering the problem

$$\begin{aligned} \partial_t w + [R_z(t,x,\,w^*)]^{-1}\circ\mathcal{Q}_o'(w^*)\,w &= \tilde{f} \text{ in }]0,T[\times\Omega\\ \mathcal{B}_o'(w^*)\,w &= 0 \text{ on } S_T,\quad w(0)=0\,. \end{aligned}$$

The fundamental idea to solve this problem is to use the Lemma 2.9. For $s\in[0,1]$, introduce

$$M^s_{i,j,k,\ell} = s\,M_{i,j,k,\ell} + (1-s)\,\delta_{i,j}\,\delta_{k,\ell} \tag{43}$$

We introduce operators

$$\begin{aligned} \mathcal{Q}_o'(w^*;\,s)\,w &= -\partial_{x_k}(M^s_{i,j,k,\ell}(t,x,\,w^*)\,w_{j,x_\ell})\\ \mathcal{B}_o'(w^*;\,s)\,w &= -\nu_k(x)\,(M^s_{i,j,k,\ell}(t,x,\,w^*)\,w_{j,x_\ell})\,. \end{aligned} \tag{44}$$

We now introduce a linear operator acting between $\mathcal{V}^0$ and Z^0 via

$$\mathcal{L}(s) = \{\partial_t w + [R_z(t,x,\,w^*)]^{-1}\circ\mathcal{Q}_o'(w^*;\,s)\,w,\,\mathcal{B}_o'(w^*;\,s)\,w\} \tag{45}$$

The injectivity of $\mathcal{L}(s)$ is obvious.

Assume that w is a solution of class $\mathcal{V}^0$ to $\mathcal{L}(s)\,w = (\tilde{f},\,0)$. Then, we multiply in the bulk with $-\mathcal{Q}_o'(w^*;\,s)\,w$ and we integrate over Ω. This procedure yields

$$\begin{aligned} -\int_\Omega \partial_t w\,(\mathcal{Q}_o'(w^*;\,s)\,w) + \int_\Omega [R_z(t,x,\,w^*)]^{-1}\,[\mathcal{Q}_o'(w^*;\,s)\,w]\,[\mathcal{Q}_o'(w^*;\,s)\,w]&\\ = -\int_\Omega \tilde{f}\,[\mathcal{Q}_o'(w^*;\,s)\,w]\,.& \end{aligned}$$

To proceed, we make use of the fact that Q is a reduced quasilinear operator in divergence form. Therefore, making use also of the boundary conditions, we can derive the identity

$$
\begin{aligned}
& - \int_\Omega \partial_t w \, (\mathcal{Q}_o'(w^*;\, s) w) \\
& = \tfrac{1}{2} \, \frac{d}{dt} \int_\Omega \{ s \, M_{i,j,k,\ell}(t,x,w^*) \, w_{j,x_\ell} \, w_{i,x_k} + (1-s) \, |w_x(t)|^2 \} \\
& \quad - s \int_\Omega M_{i,j,k,\ell,z_m}(t,x,w^*) \, w^*_{m,t} \, w_{j,x_\ell} \, w_{i,x_k} \, .
\end{aligned} \tag{46}
$$

Thus, every strong solution of class $\mathcal{V}^0$ satisfies for all $t \in]0,T[$ the identity

$$
\begin{aligned}
& \tfrac{1}{2} \int_\Omega \{ M^s(t,x,w^*(t)) \, w_x(t) \, w_x(t) \} \\
& + \int_0^t \int_\Omega [R_z(t,x,\, w^*)]^{-1} \, [\mathcal{Q}_o'(w^*;\, s) \, w] \, [\mathcal{Q}_o'(w^*;\, s)] \\
& = - \int_0^t \int_\Omega \tilde{f} \, [\mathcal{Q}_o'(w^*;\, s) \, w] + s \int_0^t \int_\Omega M_z(w^*) \, w^*_t \, w_x \, w_x \, .
\end{aligned}
$$

We make use of the assumption (33). It implies that the symmetric part of $[R_z(t,x,\, w^*)]^{-1}$ is uniformly elliptic. Thus, every strong solution w satisfies the bound

$$
\begin{aligned}
& \tfrac{1}{2} \int_\Omega \{ M^s(w^*(t)) \, w_x(t) \, w_x(t) \} + \lambda_0(\|w^*\|_{L^\infty(Q)}) \int_0^t \int_\Omega [\mathcal{Q}_o'(w^*;\, s) \, w]^2 \\
& \leq \frac{2}{\lambda_0(\|w^*\|_{L^\infty(Q)})} \int_0^t \int_\Omega [R_z(t,x,\, w^*)]^{-1} f \cdot f \\
& \quad + c_M(\|w^*\|_{L^\infty(Q)}) \int_0^t \int_\Omega |w^*_t| \, |w_x|^2 \, .
\end{aligned} \tag{47}
$$

We make use of the assumptions (34) on M to see that

$$
\int_\Omega M^s(w^*(t)) \, w_x(t) \, w_x(t) \geq (s \, \nu_0(\|w^*\|_{L^\infty(Q)}) + 1 - s) \int_\Omega |w_x(t)|^2 \, .
$$

Owing to the fact that $w^* \in \mathcal{V}^2$, we know that the components of w^*_t belong to $L^2(0,T;\, W^{2,2}(\Omega))$ which is a subset of $L^{2,\infty}(Q)$. Therefore

$$
\begin{aligned}
\int_\Omega |w^*_t(t)| \, |w_x(t)|^2 & \leq \|w^*_t(t)\|_{L^\infty(\Omega)} \int_\Omega |w_x(t)|^2 \\
& \leq c_0 \, \|w^*_t(t)\|_{W^{2,2}(\Omega)} \int_\Omega |w_x(t)|^2 \, .
\end{aligned}
$$

Thus, the identity (47) and the Gronwall Lemma yield

$$
\begin{aligned}
\int_\Omega |w_x(t)|^2 & \leq e^{2 \, \frac{c_M}{\nu_0} \, \int_0^T \|w^*_t(t)\|_{W^{2,2}(\Omega)}} \, \frac{4}{\lambda_0 \, \nu_0} \int_0^T \int_\Omega [R_z(t,x,\, w^*)]^{-1} f \cdot f \\
& \leq c(w^*) \, \|f\|_{L^2(Q)} \, .
\end{aligned}
$$

In the latter relation, we have defined

$$
c(w^*) := e^{2 \, \frac{c_M}{\nu_0} \, \|w^*_t\|_{L^1(0,T;\, W^{2,2}(\Omega))}} \, \frac{4 \, \lambda_1}{\lambda_0 \, \nu_0} \, .
$$

Thus, it follows that every strong solution satisfies the bound

$$\text{ess sup}_{t\in[0,T]} \int_\Omega |w_x(t)|^2 \leq c(\|w^*\|_{\mathcal{V}^2})\, \|f\|_{L^2(Q)}\,.$$

Returning into (47), the next is readily deduced

$$\int_0^T \int_\Omega [\mathcal{Q}_o'(w^*;\, s)\, w]^2 \leq c(\|w^*\|_{\mathcal{V}^2})\, \|f\|_{L^2(Q)}\,.$$

We invoke the Lemma 2.11 with $M(x) := M(t, x,\, w^*(t, x))$. For domains of class $\mathcal{C}^{1,1}$, we obtain that

$$\begin{aligned}\|w\|_{L^2(0,T;\, W^{2,2}(\Omega;\, \mathbb{R}^N))} &\leq c(\|w^*\|_{W^{1,0}_\infty(Q_T)})\, \|\mathcal{Q}_o'(w^*;\, s)\, w\|_{L^2(Q)}\\ &\leq c(\|w^*\|_{\mathcal{V}^2})\, \|f\|_{L^2(Q)}\,.\end{aligned}$$

It follows that every strong solution to (41) satisfies an estimate

$$\|w\|_{L^\infty(0,T;\, W^{1,2}(\Omega;\, \mathbb{R}^N))} + \|w\|_{L^2(0,T;\, W^{2,2}(\Omega;\, \mathbb{R}^N))} \leq c(\|w^*\|_{\mathcal{V}^2})\, \|f\|_{L^2(Q)}\,. \tag{48}$$

A bound for the time derivative $\partial_t w$ in $L^2(Q;\, \mathbb{R}^N)$ now follows from the equations $\mathcal{L}(s)\, w = (\tilde{f},\, 0)$. Thus, overall, we have shown that $\mathcal{L}(s)\, w = (\tilde{f},\, 0)$ implies

$$\|w\|_{\mathcal{V}^0} \leq c\, (\|f\|_{L^2(Q;\mathbb{R}^N)} + \|f_\Gamma\|_{\text{Tr}_{S_T}\, \mathcal{V}^{-1}})\,,$$

with c independent on s. The existence can now be deduced from the Lemma 2.9.

The proof for the boundary condition $w = w^\Gamma$ on S_T (cf. (42)) is completely similar. Considering without loss of generality $w^\Gamma = 0$, it suffices to repeat the same argument starting from (46). □

Next we prove some technical preliminaries related to differentiating in the equations.

Lemma 3.4. *Assume that $w \in \mathcal{V}^1_\Omega$ is a solution to*

$$\mathcal{R}_o'(w^*)\, w + \mathcal{Q}_o'(w^*)\, w = f \text{ in } Q_T\,.$$

Assume moreover that

1 $w^* \in \mathcal{V}^2$;

2 $f \in L^2(0,T;\, W^{1,2}(\Omega;\, \mathbb{R}^N))$;

Let $V \in C^3(\overline{\Omega};\, \mathbb{R}^3)$ be a given vector field. Then, the vector $\eta = V \cdot \nabla w$ belongs to $\mathcal{V}^0$ and is a solution to

$$(\mathcal{R}_o'(w^*) + \mathcal{Q}_o'(w^*))\, \eta = V \cdot \nabla f + \mathscr{D}(V,\, w^*)\, w \text{ in } Q_T\,.$$

The operator $\mathscr{D}$ satisfies the estimates

$$\begin{aligned}\|\mathscr{D}(V,\, w^*)\|_{\mathscr{L}(\mathcal{V}^0,\, L^2(Q;\mathbb{R}^N))} &\leq c(\|w^*\|_{\mathcal{V}^2})\, \|V\|_{C^2(\overline{\Omega};\, \mathbb{R}^3)}\\ \|\mathscr{D}(V,\, w^*)\|_{\mathscr{L}(\mathcal{V}^1,\, L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^N)))} &\leq c(\|w^*\|_{\mathcal{V}^2})\, \|V\|_{C^3(\overline{\Omega};\, \mathbb{R}^3)}\,.\end{aligned}$$

Proof. We can show directly that the vector $\eta^n := \partial_{x_n} w$ ($n \in \{1,2,3\}$) satisfies the equations

$$\begin{aligned}
&\mathcal{R}_o'(w^*)\,\eta^n + \mathcal{Q}_o'(w^*)\,\eta^n = f_{x_n} + g\\
&g_i = -\frac{d}{dx_n} R_{i,z_j}(t,x,\,w^*)\,w_{j,t} + \frac{d}{dx_k}\left(\frac{d}{dx_n} M_{i,j,k,\ell}(t,x,\,w^*)\,w_{j,x_\ell}\right)\,.
\end{aligned} \tag{49}$$

We introduce the operation

$$g =: \mathscr{D}(e^n,\, w^*)\,w\,, \tag{50}$$

Next we consider $\zeta \in C^2(\overline{\Omega})$ arbitrary. By means of obvious manipulations, the vector $\tilde{\eta}^n := \eta^n\,\zeta$ satisfies the equations

$$\begin{aligned}
&\mathcal{R}_o'(w^*)\,\tilde{\eta}^n + \mathcal{Q}_o'(w^*)\,\tilde{\eta}^n = \zeta\,(f_{x_n} + g) + \tilde{g}\\
&\tilde{g}_i = -\zeta_{x_k}\,M_{i,j,k,\ell}(t,x,\,w^*)\,\eta^n_{j,x_\ell} - \partial_{x_k}(M_{i,j,k,\ell}(t,x,\,w^*)\,\eta^n_j\,\zeta_{x_\ell})\,.
\end{aligned} \tag{51}$$

We introduce the operation

$$\tilde{g} =: \widetilde{\mathscr{D}}(\zeta,\, w^*)\,w\,.$$

Now we choose $\zeta = V_n$ and sum up in the equations (51) for $n = 1,2,3$. The vector $\eta := V\cdot\nabla w$ then satisfies

$$\begin{aligned}
&\mathcal{R}_o'(w^*)\,\eta + \mathcal{Q}_o'(w^*)\,\eta = V\cdot\nabla f + \mathscr{D}(V,\,w^*)\,w\\
&\mathscr{D}(V,\,w^*)\,w := \sum_{n=1}^{2}(V_n\,\mathscr{D}(e^n,\,w^*)\,w + \widetilde{\mathscr{D}}(V_n,\,w^*)\,w\,.
\end{aligned}$$

It remains to verify the estimates. First, we consider g in (49). Owing to the Lemma 2.6, the operator $w^* \mapsto R_{z_j}(t,x,\,w^*)$ is continuous from $\mathcal{V}^2$ into itself. Thus, $w^* \mapsto \frac{d}{dx_n}R_{z_j}(t,x,\,w^*)$ maps continuously into $\mathcal{V}^1$.

It follows that

$$\|\tfrac{d}{dx_n}R_{i,z_j}(t,x,\,w^*)\,w_t\|_{L^2} \leq \|\tfrac{d}{dx_n}R_{i,z_j}(t,x,\,w^*)\|_{L^\infty}\,\|w_t\|_{L^2} \leq c\,\|w^*\|_{\mathcal{V}^2}\,\|w\|_{\mathcal{V}^0}\,.$$

The Lemma 2.5, case (2.5) with $v = R_{i,z_j}(t,x,\,w^*)$ and $u = w$ yields

$$\|w_t\,\tfrac{d}{dx_n}R_{i,z_j}(t,x,\,w^*)\|_{L^2(0,T;\,W^{1,2}(\Omega))} \leq c\,\|w^*\|_{\mathcal{V}^2}\,\|w\|_{\mathcal{V}^1}\,.$$

Similarly, $w^* \mapsto \frac{d}{dx_n}M_{i,j,k,\ell}(t,x,\,w^*)$ is continuous into $\mathcal{V}^1$. The Lemma 2.5, case (1) with $u = \frac{d}{dx_n}M_{i,j,k,\ell}(t,x,\,w^*)$ and $w = v$ yields

$$\begin{aligned}
\|\frac{d}{dx_n}M_{i,j,k,\ell}(t,x,\,w^*)\,w_x\|_{L^2(0,T;\,W^{1,2})} &\leq c\,\|\frac{d}{dx_n}M_{i,j,k,\ell}(t,x,\,w^*)\|_{\mathcal{V}^1}\,\|w\|_{\mathcal{V}^0}\\
&\leq c\,\|w^*\|_{\mathcal{V}^2}\,\|w\|_{\mathcal{V}^0}\,.
\end{aligned}$$

Analogously, employing Lemma 2.5, case (2)

$$\begin{aligned}
\|\frac{d}{dx_n}M_{i,j,k,\ell}(t,x,\,w^*)\,w_x\|_{L^2(0,T;\,W^{2,2})} &\leq c\,\|\frac{d}{dx_n}M_{i,j,k,\ell}(t,x,\,w^*)\|_{\mathcal{V}^1}\,\|w\|_{\mathcal{V}^1}\\
&\leq c\,\|w^*\|_{\mathcal{V}^2}\,\|w\|_{\mathcal{V}^1}\,.
\end{aligned}$$

Second we consider $\tilde{g}$ in (51) and we note that the structure is completely similar to the one of g. The claim follows. □

Lemma 3.5. *Assume that $w \in Tr_{S_T}\mathcal{V}^1$ is a solution to*

$$\mathcal{B}_o'(w^*)\, w = f_\Gamma \text{ on } S_T, \quad f_\Gamma \in Tr_{S_T}\mathcal{V}^0 \,.$$

Let $V \in C^4(\overline{\Omega};\, \mathbb{R}^3)$ be a given vector field such that $V(x) \cdot \nu(x) = 0$ for all $x \in \partial\Omega$. Then, the vector $\eta = V \cdot \nabla w$ belongs to $Tr_{S_T}\mathcal{V}^0$ and is a solution to

$$\mathcal{B}_o'(w^*)\, \eta = V \cdot \nabla f_\Gamma + \mathscr{D}^{\Gamma,II}(V,\, w^*)\, w \text{ in } Q_T \,.$$

The operator $\mathscr{D}^{\Gamma,II}$ satisfies the estimates

$$\|\mathscr{D}^\Gamma(V,\, w^*)\|_{\mathscr{L}(Tr\mathcal{V}^0_\Omega,\, Tr\mathcal{V}^{-1})} \leq c(\|w^*\|_{\mathcal{V}^2})\, \|V\|_{C^4(\overline{\Omega};\, \mathbb{R}^3)}\, \|\nu\|_{C^4(\overline{\Omega};\, \mathbb{R}^3)}$$
$$\|\mathscr{D}^\Gamma(V,\, w^*)\|_{\mathscr{L}(Tr\mathcal{V}^1_\Omega,\, Tr\mathcal{V}^0_\Omega)} \leq c(\|w^*\|_{\mathcal{V}^2})\, \|V\|_{C^4(\overline{\Omega};\, \mathbb{R}^3)}\, \|\nu\|_{C^4(\overline{\Omega};\, \mathbb{R}^3)} \,.$$

Proof. We assume the validity of $\mathcal{B}_o'(w^*)\, w = f_\Gamma$ as an identity in $\text{Tr}_{S_T}\, \mathcal{V}^0 \subset L^2(0,T,\, W^{1,2}(\partial\Omega))$. Thus, we are allowed to apply the tangential differential operator $V \cdot \nabla$ to this identity, and this yields

$$\mathcal{B}_o'(w^*)\, \eta = V \cdot \nabla f_\Gamma + g_\Gamma \text{ on } S_T$$
$$g_{\Gamma,i} = w_{j,x_\ell}\, (-\nu_k\, M_{i,j,k,\ell'}\, (V_\ell)_{x_{\ell'}} + V \cdot \nabla(\nu_k\, M_{i,j,k,\ell})) \,.$$

We introduce the operation

$$g_\Gamma =: \mathscr{D}^{\Gamma,II}(V,\, w^*)\, w \,, \tag{52}$$

We make use of the estimates of Lemma 2.5, case (3) and (4) to show that

$$\|u\, v_x\|_{\text{Tr}_{S_T}\mathcal{V}^{-1}} \leq c\, \|u\|_{\text{Tr}_{S_T}\mathcal{V}^1}\, \|v\|_{\text{Tr}_{S_T}\mathcal{V}^0}$$
$$\|u\, v_x\|_{\text{Tr}_{S_T}\mathcal{V}^0} \leq c\, \|u\|_{\text{Tr}_{S_T}\mathcal{V}^1}\, \|v\|_{\text{Tr}_{S_T}\mathcal{V}^1} \,.$$

We choose $v = w$ and $u = M(t,x,\, w^*)\, (-\nu\, (V)_x + V\, (\nu)_x) - \nu\, V \cdot \nabla M(t,x,\, w^*)$. Owing to the Lemma 2.6, note that

$$\|u\|_{\mathcal{V}^1} \leq \|\nu\|_{C^4}\, \|V\|_{C^4}\, \|M(t,x,\, w^*)\|_{\mathcal{V}^1} + \|\nu\|_{C^3}\, \|V\|_{C^3}\, \|\frac{d}{dx} M(t,x,\, w^*)\|_{\mathcal{V}^1}$$
$$\leq c\, \|\nu\|_{C^4}\, \|V\|_{C^4}\, c(\|w^*\|_{\mathcal{V}^2}) \,.$$

Thus

$$\|\mathscr{D}^{\Gamma,II}(V,\, w^*)\, w\|_{\text{Tr}_{S_T}\mathcal{V}^{-1}} \leq c\, \|\nu\|_{C^4}\, \|V\|_{C^4}\, c(\|w^*\|_{\mathcal{V}^2})\, \|w\|_{\text{Tr}_{S_T}\mathcal{V}^0}$$
$$\|\mathscr{D}^{\Gamma,II}(V,\, w^*)\, w\|_{\text{Tr}_{S_T}\mathcal{V}^0} \leq c\, \|\nu\|_{C^4}\, \|V\|_{C^4}\, c(\|w^*\|_{\mathcal{V}^2})\, \|w\|_{\text{Tr}_{S_T}\mathcal{V}^1} \,.$$

The claim follows. □

Lemma 3.6. *Assume that $w \in \mathcal{V}^1$ is a solution to*

$$\mathcal{B}_o'(w^*)\, w = f_\Gamma \text{ on } S_T, \quad f_\Gamma \in Tr_{S_T}\mathcal{V}^0 \,.$$

Let $V \in C^3(\overline{\Omega};\, \mathbb{R}^3)$ be a given vector field such that $V(x)$ is parallel to $\nu(x)$ for all $x \in \partial\Omega$. Then, the vector $\eta = V \cdot \nabla w$ belongs to $Tr_{S_T}\mathcal{V}^0_\Omega$ and it satisfies

$$\eta = |V|\, ([M_{\nu,\nu}(t,x,\, w^*)]^{-1}\, f_\Gamma + \mathscr{D}^{\Gamma,I}(w^*)\, w) \,.$$

The operator $\mathscr{D}^{\Gamma,I}$ satisfies the estimates

$$\|\mathscr{D}^{\Gamma,I}(V,\, w^*)\, w\|_{Tr\mathcal{V}^0} \leq c(\|w^*\|_{\mathcal{V}^2})\, \|\nu\|_{C^2(\overline{\Omega};\, \mathbb{R}^3)}\, \|V\|_{C^2(\overline{\Omega};\, \mathbb{R}^3)}\, \|\nabla_\Gamma w\|_{Tr\mathcal{V}^0}$$
$$\|\nabla_\Gamma(\mathscr{D}^{\Gamma,I}(V,\, w^*)\, w)\|_{Tr\mathcal{V}^0} \leq c(\|w^*\|_{\mathcal{V}^2_T})\, \|\nu\|_{C^3(\overline{\Omega};\, \mathbb{R}^3)}\, \|V\|_{C^3(\overline{\Omega};\, \mathbb{R}^3)}\, \|\nabla_\Gamma w\|_{Tr\mathcal{V}^1} \,.$$

Proof. Due to the equation $M_{i,j,k,\ell}\,\nu_k\, w_{j,x_\ell} = f_{\Gamma,i}$, we obtain that

$$(M_{i,j,k,\ell}(t,x,\,w^*)\,\nu_k\,\nu_\ell)\,\nu\cdot\nabla w_j + \sum_{m=1}^{2}(M_{i,j,k,\ell}(w^*)\,\nu_k\,\tau^m_\ell)\,\tau^m\cdot\nabla w_j = f_{\Gamma,i}\,.$$

Here, ν is a unit normal to $\partial\Omega$ and $\{\tau^1,\,\tau^2,\,\nu\}$ is chosen as an orthonormal basis of $\mathbb{R}^3$. It therefore follows that

$$|V|^{-1}\,(M_{i,j,k,\ell}(t,x,\,w^*)\,\nu_k\,\nu_\ell)\,\eta_j = f_{\Gamma,i} - \sum_{m=1}^{2}(M_{i,j,k,\ell}(w^*)\,\nu_k\,\tau^m_\ell)\,\tau^m\cdot\nabla w_j\,.$$

By assumption, the matrices $\{M_{i,j,\nu,\nu}(t,x,\,w^*)\}_{i,j=1,\dots,N}$ are uniformly invertible for $w^*\in L^\infty(Q)$. Consequently, for $i=1,\dots,N$

$$\eta_i := |V|\,\sum_{j'=1}^{N}[M_{\nu,\nu}(t,x,\,w^*)]^{-1}_{i,j'}\,\left(f_{\Gamma_{j'}} - \sum_{m=1}^{2}\,M_{j',j,\nu,\tau^m}(t,x,\,w^*)\,\tau^m\cdot\nabla w_j\right)\,.$$

We denote

$$\mathscr{D}^{\Gamma,I}(V,\,w^*)\,w := -|V(x)|\,[M_{\nu,\nu}(t,x,\,w^*)]^{-1}_{i,j'}\,M_{j',j,\nu,\tau^m}(t,x,\,w^*)\,\tau^m\cdot\nabla w_j\,.$$

The estimates directly follow from Lemma 2.5, (5) and (6). □

Lemma 3.7. *Let $W,\,V\in C^4(\overline{\Omega};\mathbb{R}^3)$ be parallel to ν on $\partial\Omega$. Assume that $w\in\mathcal{V}^2_\Omega$ satisfies $(\mathcal{R}'_o(w^*)+\mathcal{Q}'_o(w^*))\,w = f$ in Q_T. For $i=1,\dots,N$, define $\phi_i := W\cdot\nabla(V\cdot\nabla w_i)$. Then*

$$\|\phi\|_{\mathrm{Tr}_{S_T}\,\mathcal{V}^0} \le c(\|w^*\|_{\mathcal{V}^2})\,(\|w\|_{\mathcal{V}^1} + \|\nabla_\Gamma w\|_{\mathrm{Tr}_{S_T}\,\mathcal{V}^1})\,.$$

Proof. Consider the functions $\phi_i := W\cdot\nabla(V\cdot\nabla w_i)$, where both W and V are parallel to ν on $\partial\Omega$. In order to treat this case, we introduce for $i=1,\dots,N$ the abbreviation $A_i := -R_{i,z}(t,x,\,w^*)\cdot\partial_t w + f_i(t,x)$. By assumption, the identity $\mathcal{Q}'_o(w^*)\,w = A$ is valid, that means,

$$-\operatorname{div}(M_i(t,x,\,w^*)\,w_x) = A_i(t,x)\ \text{ for } i=1,\dots,N\,. \tag{53}$$

Thus, invoking the Lemmas 2.4 and 2.6

$$\begin{aligned}\|A\|_{\mathcal{V}^{-1}} &\le \|M(t,x,\,w^*)\,w_x\|_{\mathcal{V}^0} \le \|M(t,x,\,w^*)\|_{\mathcal{V}^1}\,\|w\|_{\mathcal{V}^1}\\ &\le c_M(\|w^*\|_{\mathcal{V}^1})\,\|w\|_{\mathcal{V}^1}\,.\end{aligned}$$

Moreover, for a vector V tangential to $\partial\Omega$ ($\lambda=1,2$), differentiation in (53) yields

$$\begin{aligned}V\cdot\nabla A_i =& -\operatorname{div}(M_i(t,x,\,w^*)\,(V\cdot\nabla w)_x) + (V)_x\,M_i(t,x,\,w^*)\,w_{x,x}\\ &+\operatorname{div}(M_i(t,x,\,w^*)\,w_x\,V_x) - V\,\operatorname{div}(M_{i,z}(t,x,\,w^*)\,w^*_x\,w_x)\,.\end{aligned}$$

Therefore, with the usual multiplicator arguments

$$\|V\cdot\nabla A\|_{\mathcal{V}^{-1}} \le c(\|w^*\|_{\mathcal{V}^2})\,(\|V\cdot\nabla w\|_{\mathcal{V}^1} + \|w\|_{\mathcal{V}^1})\,.$$

The characterisation of Lemma 2.3 now yields

$$\|A\|_{\mathrm{Tr}_{\mathcal{V}^0}} \le c(\|w^*\|_{\mathcal{V}^2})\,(\|V\cdot\nabla w\|_{\mathcal{V}^1} + \|w\|_{\mathcal{V}^1})\,. \tag{54}$$

Now, we reconsider (53). Locally in Ω_μ for $\mu = 1, \ldots, m$, we apply the Lemma 2.10, (28) that yields

$$\begin{aligned}
& - M_{i,j,V^3,V^3}\, V^3 \cdot \nabla (V^3 \cdot \nabla w_j) = A_i + B_i \\
& \quad + \sum_{\lambda=1}^{2} M_{i,j,k,\ell}\, [V_k^\lambda\, (V^\lambda \cdot \nabla w_j)_{x_\ell} + V_k^3\, V_\ell^\lambda\, V^3 \cdot \nabla (V^\lambda \cdot \nabla w_j)] \\
& B_i := - \sum_{\lambda=1}^{3} V_k^\lambda\, (M_{i,j,k,\ell}\, (V^\lambda)_{x_\ell} \cdot \nabla\, w_j - V^\lambda \cdot \nabla M_{i,j,k,\ell}\, w_{j,x_\ell}) \\
& \quad + \sum_{\lambda=1}^{2} V_k^3\, V_\ell^\lambda\, M_{i,j,k,\ell}\, [\{(V^\lambda \cdot \nabla) V^3 - (V^3 \cdot \nabla) V^\lambda\} \nabla w_j]\, .
\end{aligned}$$

Here, it can be shown using the properties of multiplicators and Nemicki operators that $\|B\|_{\mathcal{V}^0} \leq c(\|w^*\|_{\mathcal{V}^2})\, \|w\|_{\mathcal{V}^1}$. Recalling (54), the vector $\phi = V^{\mu,3} \cdot \nabla (V^{\mu,3} \cdot w_x)$ satisfies

$$\|\phi\|_{\mathsf{Tr}_{S_T}\, \mathcal{V}^0} \leq c(\|w^*\|_{\mathcal{V}^2})\, (\sum_{\lambda=1}^{2} \|V^{\mu,\lambda} \cdot \nabla w\|_{\mathcal{V}^1} + \|A\|_{\mathsf{Tr}_{S_T}\, \mathcal{V}^0} + \|B\|_{\mathsf{Tr}_{S_T}\, \mathcal{V}^0})\, .$$

□

Now we have all ingredient to show the principal technical statement of the paper: The invertibility of the principal part G' with respect to $\mathcal{V}^2$.

Corollary 3.8. *Let $w^* \in \mathcal{V}^2$ and $F = (f,\, f_\Gamma) \in Z^2$. Then the problem*

$$(\mathcal{R}_o'(w^*) + \mathcal{Q}_o'(w^*))\, w = f, \quad \mathcal{B}_o'(w^*)\, w = f_\Gamma, \quad w(0) = 0$$

possesses a unique solution of class $\mathcal{V}_\Omega^2$.

Proof. We consider the family $\{\mathcal{L}(s)\}_{s \in [0,1]} \subset \mathscr{L}(\mathcal{V}_T^2,\, Z^2)$ of linear operators defined via (45). The proof strategy is the following: Consider for $s \in [0, 1]$ and $(f,\, f_\Gamma) \in Z^2$ an arbitrary solution to $w \in \mathcal{V}_\Omega^2$ to $\mathcal{L}(s)\, w = (f,\, f_\Gamma)$, and show that a uniform estimate

$$\|w\|_{\mathcal{V}^2} \leq c_1\, \|(f,\, f_\Gamma)\|_{Z^2}\, .$$

is available. Then, by means of Lemma 2.9, the invertibility of $\mathcal{L}(1)$ follows.

For notational brevity, we prove the estimate for $s = 1$. The same argument applies to $\mathcal{L}(s)$ for $s < 1$ since these operators have exactly the same structure (cp. (43), (44)). Consider thus an arbitrary solution $w \in \mathcal{V}_\Omega^2$ to $\mathcal{L}(1)\, w = (f,\, f_\Gamma)$. We afore mention that $w \in \mathcal{V}_\Omega^2$ implies that $w \in C([0, T];\, W^{3,2}(\Omega;\, \mathbb{R}^N))$ and that $w(0) = 0$. Thus, also $w_x(0) = 0$ and $w_{x,x}(0) = 0$.

At first, we recall the estimate of Lemma 3.3

$$\|w\|_{\mathcal{V}^0} \leq c_1\, \|(f,\, f_\Gamma)\|_{Z^0} \tag{55}$$

Consider now an arbitrary vector field $V \in C^3(\overline{\Omega};\, \mathbb{R}^3)$. Then $\eta_i := V \cdot \nabla w_i \in \mathcal{V}_\Omega^1$ satisfies (see Lemma 3.4)

$$(\mathcal{R}_o'(w^*) + \mathcal{Q}_o'(w^*))\, \eta = V \cdot \nabla f + \mathscr{D}(V,\, w^*)\, w \text{ in } Q_T\, .$$

Now we distinguish three cases. Consider first a multiplier V with a compact support in Ω. Then, $\eta_1, \ldots, \eta_N$ are strong solutions to the system (51) supplemented by the conditions $\eta = 0$ on S_T and $\{0\} \times \Omega$.

The Dirichlet case of Lemma 3.3 guaranties that there is a unique solution of class $\mathcal{V}^0_\Omega$ satisfying a continuity estimate

$$\begin{aligned} \|\eta\|_{\mathcal{V}^0} &\leq c(\|w^*\|_{\mathcal{V}^2}) \, \|V \cdot \nabla f + \mathscr{D}(V,\, w^*)\, w\|_{L^2(Q)} \\ &\leq c(\|w^*\|_{\mathcal{V}^2}) \, \|V\|_{C^2(\overline{\Omega})} \, (\|f_x\|_{L^2(Q)} + \|w\|_{\mathcal{V}^0}) \, . \end{aligned} \tag{56}$$

In the second case, we choose $V \in C^3(\overline{\Omega};\, \mathbb{R}^3)$ tangent to $\partial\Omega$. Owing to Lemma 3.5, the vector η satisfies

$$\mathcal{B}'_o(w^*)\, \eta = V \cdot \nabla f_\Gamma + \mathscr{D}^{\Gamma,II}(V,\, w^*)\, w \text{ on } S_T \, .$$

The Lemma 3.3 then ensures that

$$\begin{aligned} \|\eta\|_{\mathcal{V}^0} &\leq c(\|w^*\|_{\mathcal{V}^2}) \, (\|V \cdot \nabla f_\Gamma + \mathscr{D}^{\Gamma,II}(V,\, w^*)\, w\|_{\mathsf{Tr}\mathcal{V}^{-1}}) \\ &\leq c(\|w^*\|_{\mathcal{V}^2}) \, \|V\|_{C^2(\overline{\Omega})} \, (\|f_\Gamma\|_{\mathsf{Tr}\mathcal{V}^0_\Omega} + \|w\|_{\mathsf{Tr}\mathcal{V}^0_\Omega}) \, . \end{aligned} \tag{57}$$

The third case is that V is parallel to ν on $\partial\Omega$. Due to the Lemma 3.6, we see that

$$\eta = M_{\nu,\nu}(t, x,\, w^*)]^{-1}\, f_\Gamma + \mathscr{D}^{\Gamma,I}(V,\, w^*)\, w$$

Owing to the Lemma 3.6

$$\|\eta\|_{\mathsf{Tr}\mathcal{V}^0} \leq c(\|w^*\|_{\mathcal{V}^2}) \, (\|f_\Gamma\|_{\mathsf{Tr}\mathcal{V}^0} + \|\nabla_\Gamma w\|_{\mathsf{Tr}\mathcal{V}^0}) \, .$$

From the Dirichlet case of the Lemma 3.3, it now follows that

$$\|\eta\|_{\mathcal{V}^0} \leq c(\|w^*\|_{\mathcal{V}^2}) \, (\|f_x\|_{L^2(Q)} + \|\nabla_\Gamma w\|_{\mathsf{Tr}\mathcal{V}^0}) \, . \tag{58}$$

In order to obtain a bound for $\|w\|_{\mathcal{V}^1}$, we invoke the construction of Section 2.7, (24) to represent the gradient, and we see that

$$w_x = \sum_{\mu=0}^{m} \sum_{\ell=1}^{3} \eta^{\mu,\ell}\, V^{\mu,\ell} \, . \tag{59}$$

There the vector fields $V^{0,\ell}$ have a compact support for $\ell = 1, 2, 3$, while the vector fields $V^{\mu,1},\ V^{\mu,2}$ are tangent on $\partial\Omega$ for $\mu = 1, \ldots, m$. Thus, invoking (56) and (57)

$$\|\eta^{\mu,\ell}\|_{\mathcal{V}^0} \leq c(\|w^*\|_{\mathcal{V}^2}) \, (\|f_x\|_{L^2(Q)} + \|f_\Gamma\|_{\mathsf{Tr}\mathcal{V}^0_T}) \text{ for } \mu = 0 \text{ and for } \mu > 0, \ell \leq 2 \, .$$

Then, we make use of (58), and it follows that

$$\begin{aligned} \|\eta^{\mu,3}\|_{\mathcal{V}^0} &\leq c(\|w^*\|_{\mathcal{V}^2}) \, (\|f_x\|_{L^2(Q)} + \sum_{\mu\geq 0, \ell\leq 2} \|\eta^{\mu,\ell}\|_{\mathsf{Tr}\mathcal{V}^0}) \\ &\leq c(\|w^*\|_{\mathcal{V}^2}) \, (\|f_x\|_{L^2(Q)} + \|f_\Gamma\|_{\mathsf{Tr}\mathcal{V}^0}) \, . \end{aligned}$$

Thus, we have obtained the estimate

$$\|w_x\|_{\mathcal{V}^0} \leq c(\|w^*\|_{\mathcal{V}^2}) \, (\|f\|_{L^2(0,T;\, W^{1,2})} + \|f_\Gamma\|_{\mathsf{Tr}\mathcal{V}^0}) \, . \tag{60}$$

Now, we go one order further. For $W \in C^2(\overline{\Omega};\, \mathbb{R}^3)$, we see that $\phi_i := W \cdot (V \cdot \nabla w_i) \in \mathcal{V}^0_\Omega$ satisfies (see Lemma 3.4)

$$(\mathcal{R}'_o(w^*) + \mathcal{Q}'_o(w^*))\, \phi = W \cdot \nabla(V \cdot \nabla f + \mathscr{D}(V,\, w^*)\, w) + \mathscr{D}(W,\, w^*)\, V \cdot \nabla w\, .$$

If $W \in C^2(\overline{\Omega};\, \mathbb{R}^3)$ is moreover tangent to $\partial\Omega$ as well as V, then owing to Lemma 3.5, the vector ϕ satisfies

$$\mathcal{B}'_o(w^*)\, \phi = W \cdot \nabla(V \cdot \nabla f_\Gamma + \mathscr{D}^{\Gamma,II}(V,\, w^*)\, w) + \mathscr{D}^{\Gamma,II}(W,\, w^*)\, V \cdot \nabla w \text{ on } S_T\, .$$

If W is tangent and V is normal, then

$$\phi = W \cdot \nabla(M_{\nu,\nu}(t,x,\, w^*)]^{-1}\, f_\Gamma + \mathscr{D}^{\Gamma,I}(V,\, w^*)\, w)\, . \tag{61}$$

Thanks also to (60), we obtain in these cases an estimate

$$\|\phi\|_{\mathcal{V}^0} \leq c(\|w^*\|_{\mathcal{V}^2})\, (\|F\|_{Z^2} + \|w\|_{\mathcal{V}^1}) \leq c(\|w^*\|_{\mathcal{V}^2})\, \|F\|_{Z^2}\, . \tag{62}$$

Owing to (62) and to (24) that we apply to $w \approx V \cdot \nabla w$, we obtain a bound

$$\|(V \cdot \nabla w)_x\|_{\mathcal{V}^0} \leq c(\|w^*\|_{\mathcal{V}^2})\, \|F\|_{Z^2} \tag{63}$$

for each V of class $C^3(\overline{\Omega};\, \mathbb{R}^3)$ which has a compact support or is tangent on $\partial\Omega$.

Finally, consider the functions $\phi_i := W \cdot \nabla(V \cdot \nabla w_i)$, where both W and V are parallel to ν on $\partial\Omega$. Recalling Lemma 3.7 and (63), the vector $\phi = V^{\mu,3} \cdot \nabla(V^{\mu,3} \cdot w)$ satisfies

$$\|\phi\|_{\mathsf{Tr}_{S_T}\, \mathcal{V}^0} \leq c(\|w^*\|_{\mathcal{V}^2})\, \|F\|_{Z^2}\, .$$

Now, the Dirichlet case of Lemma 3.3 yields

$$\|\phi\|_{\mathcal{V}^0} \leq c(\|w^*\|_{\mathcal{V}^2})\, \|F\|_{Z^2}\, . \tag{64}$$

We combine with the estimate (63) on the second tangential and mixed tangential–normal derivatives to obtain overall

$$\|w_{x,x}\|_{\mathcal{V}^0} \leq c(\|w^*\|_{\mathcal{V}^2})\, \|F\|_{Z^2}\, . \tag{65}$$

This establishes that independently on $s \in [0,1]$, all solutions $w \in \mathcal{V}^2_\Omega$ to $\mathcal{L}(s)\, w = F$ satisfy a bound $\|w\|_{\mathcal{V}^2} \leq c_1\, \|(f, f_\Gamma)\|_{Z^2}$. We apply the Lemma 2.9 and are done. □

We can now prove that the complete Fréchet derivative G' is invertible. First we need the following remark.

Remark 3.9. *Let $\mathcal{L}$ be a linear lower-order operator in the following sense: For $\ell = -1, 0, 1$, the operator $\mathcal{L}$ maps $\mathcal{V}^\ell$ continuously into*

$$Z^{\ell+1} := L^2(0,T;\, W^{1+\ell,2}(\Omega;\, \mathbb{R}^N)) \times \mathit{Tr}_{S_T}\, \mathcal{V}^\ell_\Omega\, .$$

Then, every solution $v \in \mathcal{V}^2_\Omega$ to $(G'_o(v^*) + \mathcal{L})\, v = F$ satisfies $\|v\|_{\mathcal{V}^2} \leq c(\|v^*\|_{\mathcal{V}^2})\, (\|F\|_{Z^2} + \|v\|_{\mathcal{V}^{-1}})$, where c depends also on $\max_{\ell=-1,0,1} \|\mathcal{L}\|_{\mathscr{L}(\mathcal{V}^\ell,\, Z^{\ell+1})}$.

Proof. We first obtain (Lemma 3.3)

$$\begin{aligned}\|v\|_{\mathcal{V}^0} &\leq c(\|v^*\|_{\mathcal{V}^2})\,(\|F\|_{Z^2} + \|\mathcal{L}\,v\|_{Z^0})\\ &\leq c(\|v^*\|_{\mathcal{V}^2})\,(\|F\|_{Z^2} + \|\mathcal{L}\|_{\mathscr{L}(\mathcal{V}^{-1},\,Z^0)}\,\|v\|_{\mathcal{V}^{-1}})\,.\end{aligned}$$

Then, the Lemma 3.8, equation (60) yields

$$\begin{aligned}\|v\|_{\mathcal{V}^1} &\leq c(\|v^*\|_{\mathcal{V}^2})\,(\|F\|_{Z^2} + \|\mathcal{L}\,v\|_{Z^1})\\ &\leq c(\|v^*\|_{\mathcal{V}^2})\,(\|F\|_{Z^2} + \|\mathcal{L}\|_{\mathscr{L}(\mathcal{V}^0,\,Z^1)}\,\|v\|_{\mathcal{V}^0})\,.\end{aligned}$$

Finally, Lemma 3.8, equation (65) yields the claim. □

Lemma 3.10. *For $F = (f, f_\Gamma) \in Z^2$ and for all $v^* \in \mathcal{V}^2_\Omega$, the equations $G'(v^*)\,v = F$ have a unique solution $v \in \mathcal{V}^2_\Omega$. We denote $v := (G'(v^*))^{-1}\,F$. Then, $(G'(v^*))^{-1} \in \mathscr{L}(Z^2,\,\mathcal{V}^2_\Omega)$ and*

$$\|(G'(v^*))^{-1}\|_{\mathscr{L}(Z^2,\mathcal{V}^2_\Omega)} \leq C(\|v^* + q^0\|_{\mathcal{V}^2})\,.$$

Proof. The resolvability of $G'(v^*)\,v = F$ means that

$$\begin{aligned}&(\mathcal{R}'_o(w^*) + \mathcal{Q}'_o(w^*))\,v = f - \mathcal{L}_1(w^*)\,v\\ &(\mathcal{L}_1(w^*)\,v)_i = -\frac{d}{dx_k}\,M_{i,j,k,\ell,z_m}(t,x,\,w^*)\,v_m\,w^*_{j,x_\ell} + \partial_t(R_{i,z_j}(t,x,\,w^*))\,v_j\\ &\mathcal{B}'_o(w^*)\,v = f_\Gamma - \mathcal{L}_2(w^*)\,v\\ &(\mathcal{L}_2(w^*)\,v)_i = -\nu_k\,M_{i,j,k,\ell,z_m}(t,x,\,w^*)\,v_m\,w^*_{j,x_\ell}\,.\end{aligned}$$

with $w^* := v^* + q^0$. The first idea is to show that the operator $\mathcal{L}(w^*)\,v = \{\mathcal{L}_1(w^*)\,v,\,\mathcal{L}_2(w^*)\,v\}$ is lower-order in the sense of Remark 3.9. This can be done easily as an exercise (apply Lemma 2.4 and Lemma 2.6).

Second, if we can show that every solution to $(G'_o(v^*) + s\,\mathcal{L}(w^*))\,v = F$ satisfies a uniform bound in $\mathcal{V}^{-1}$. Then, the Remark 3.9 yields automatically a bound in $\mathcal{V}^2$.

We can rely on the fact that for $s = 0$, the operator is $G'_o(v^*)$ which is invertible. If we can also show that $G'_o(v^*) + s\,\mathcal{L}(w^*)$ is injective, we can apply Lemma 2.9 and are done.

Thus, everything is reduced to proving that strong solutions to $(G'_o(v^*) + s\,\mathcal{L}(w^*))\,v = F$ are unique and that they satisfy a bound in $\mathcal{V}^{-1}$. To show this we multiply with v and integrate by parts to obtain that

$$\begin{aligned}&\int_\Omega R_{i,z_j}(t,x,\,w^*)\,v_{j,t}\,v_i + \int_\Omega M_{i,j,k,\ell}(t,x,\,w^*)\,v_{i,x_k}\,v_{j,x_\ell}\\ &= -\int_{\partial\Omega}\{f_{\Gamma,i} + \nu_k\,M_{i,j,k,\ell,z_m}(t,x,\,w^*)\,v_m\,w^*_{j,x_\ell}\}\,v_i\\ &+ \int_\Omega\{f_i + \partial_{x_k}(\sum_{m=1}^N M_{i,j,k,\ell,z_m}(t,x,\,w^*)\,v_m\,w^*_{j,x_\ell}) - \partial_t(R_{i,z_j}(t,x,\,w^*))\,v_j\}\,v_i\,.\end{aligned}$$

If the matrix R_{i,z_j} is symmetric, which is a component of our requirement (a), (b) of parabolicity, then

$$\int_\Omega R_{i,z_j}(t,x,\,w^*)\,v_{j,t}\,v_i = \tfrac{1}{2}\,\frac{d}{dt}\,\int_\Omega R_{i,z_j}(t,x,\,w^*)\,v_j\,v_i - \tfrac{1}{2}\,\int_\Omega \frac{d}{dt}R_{i,z_j}(t,x,\,w^*)\,.$$

Thus

$$
\begin{aligned}
&\tfrac{1}{2}\,\frac{d}{dt}\int_{\Omega} R_{i,z_j}(t,x,\,w^*)\,v_j\,v_i+\int_{\Omega} M_{i,j,k,\ell}(t,x,\,w^*)\,v_{i,x_k}\,v_{j,x_\ell}\\
&=-\int_{\partial\Omega}\{f_{\Gamma,i}+\nu_k\,M_{i,j,k,\ell,z_m}(t,x,\,w^*)\,v_m\,w^*_{j,x_\ell}\}\,v_i\\
&+\int_{\Omega}\{f_i+\partial_{x_k}(\sum_{m=1}^{N} M_{i,j,k,\ell,z_m}(t,x,\,w^*)\,v_m\,w^*_{j,x_\ell})-\tfrac{1}{2}\,\frac{d}{dt}(R_{i,z_j}(t,x,\,w^*))\,v_j\}\,v_i\,.
\end{aligned}
$$

The right-hand I obeys the estimate

$$
\begin{aligned}
|I|\leq c(\|w^*\|_{L^\infty(Q)}+\|w^*_x\|_{L^\infty(Q)})\,\Big(&\int_{\Gamma}\{|f_\Gamma|+|v|\}\,|v|\\
&+\int_{\Omega}\{|f|+(1+|w^*_{x,x}|+|w^*_t|)\,|v|+|v_x|\}\,|v|\Big)\,.
\end{aligned}
$$

Thus, we can employ well known inequalities to attain the structure

$$
\begin{aligned}
|I|\leq&\tfrac{\nu_0(\|w^*\|_{L^\infty(Q)})}{2}\,\|v_x(t)\|^2_{L^2}+c\,(\|f_\Gamma(t)\|^2_{L^2(\partial\Omega)}+\|f(t)\|^2_{L^2(\Omega)})\\
&+c\,(\|w^*_{x,x}(t)\|_{W^{2,2}}+\|w^*_t(t)\|_{W^{2,2}}+1)\,\int_{\Omega}|v(t)|^2\,.
\end{aligned}
$$

where $c=c(\|w^*\|_{L^\infty(Q)}+\|w^*_x\|_{L^\infty(Q)})$. Employing the assumption (34), it follows that

$$
\begin{aligned}
&\tfrac{1}{2}\,\frac{d}{dt}\int_{\Omega} R_{i,z_j}(t,x,\,w^*)\,v_j\,v_i+\tfrac{\nu_0}{2}\,\int_{\Omega}|v_x|^2\leq c\,(\|f_\Gamma(t)\|^2_{L^2(\partial\Omega)}+\|f(t)\|^2_{L^2(\Omega)})\\
&+c\,(\|w^*_{x,x}(t)\|_{W^{2,2}}+\|w^*_t(t)\|_{W^{2,2}}+1)\,\int_{\Omega}|v(t)|^2\,.
\end{aligned}
$$

Thus, the assumption (33) and the Gronwall Lemma yield

$$
\|v\|_{L^\infty(0,T;\,L^2(\Omega;\mathbb{R}^N))}+\|v_x\|_{L^2(Q;\,\mathbb{R}^N)}\leq c(\|w^*\|_{\mathcal{V}^2})\,(\|f\|_{L^2}+\|f_\Gamma\|_{L^2})\,.
$$

From the equations, we now obtain a natural bound for $\|v_t\|_{L^2(0,T;\,[W^{1,2}(\Omega;\,\mathbb{R}^N)]^*)}$. The uniqueness is obvious. □

In order to prove the existence of local strong solutions and to complete the proof of Theorem 3.1, we next apply a more or less standard fixed-point strategy.

For $v\in\mathcal{V}^2_\Omega$, the mapping

$$
\mathcal{T}\,v:=v-[G'(0)]^{-1}\,G(v)
$$

is well defined. Moreover, we define $b^0:=[G'(0)]^{-1}\,G(0)\in\mathcal{V}^2_\Omega$. Since G is a $\mathcal{C}^1$ mapping, we obtain that

$$
\|b^0\|_{\mathcal{V}^2_\Omega}\leq\|[G'(0)]^{-1}\|_{\mathscr{L}(Z^2,\,\mathcal{V}^2_\Omega)}\,\|G(0)\|_{Z^2}=:f_0(T)\,.
$$

It is readily verified that $\limsup_{T\to 0} f_0(T)=0$. For $\delta>0$, we define $\mathcal{M}_\delta:=\{v\in\mathcal{V}^2_\Omega\,:\,\|v-b^0\|_{\mathcal{V}^2}\leq\delta\}$. Obviously, $v\in\mathcal{M}_\delta$ implies that $\|v\|_{\mathcal{V}^2_\Omega}\leq\delta+f_0(T)$.

Note that

$$T\,v - b_0 = v - [G'(0)]^{-1}\,(G(v) - G(0)) = [G'(0)]^{-1}\,(G(v) - G(0) - G'(0)\,v)\,.$$

Thus, for all $\|v\|_{\mathcal{V}^2_\Omega} \in \mathcal{M}_\delta$

$$\begin{aligned}\|T\,v - b_0\|_{\mathcal{V}^2} &\leq \|(G'(0))^{-1}\|_{\mathscr{L}(Z^2,\mathcal{V}^2_\Omega)}\,\|G(v) - G(0) - G'(0)\,v\|_{\mathcal{V}^2_\Omega}\\ &\leq C_0 \sup_{\|w\|_{\mathcal{V}^2}\leq\delta+f_0(T)} \frac{\|G(w) - G(0) - G'(0)\,w\|_{\mathcal{V}^2}}{\|w\|_{\mathcal{V}^2}}\,(\delta + f_0(T))\,.\end{aligned}$$

Due to the continuous differentiability of G, there is $\theta_0 > 0$ such that

$$\sup_{\|w\|_{\mathcal{V}^2}\leq\theta_0} \frac{\|G(w) - G(0) - G'(0)\,w\|_{\mathcal{V}^2}}{\|w\|_{\mathcal{V}^2}} \leq \frac{1}{2C_0}\,.$$

Thus, if $\delta + f_0(T) \leq \theta_0$, we see that $\mathcal{T}$ maps $\mathcal{M}_\delta$ into itself.

With similar arguments, $\mathcal{T}$ is a contraction. Thus, $\mathcal{T}$ possesses a unique fixed point in $v \in \mathcal{M}_\delta$, and $q := q^0 + v$ is a strong solution.

3.2 The case of non zero right-hand side

If f and f_Γ are non-trivial in the equations (29), we introduce $H = \{H_1,\, H_2\}$ as a mapping acting on $\mathcal{V}^2_\Omega$ via

$$\begin{aligned}(H_1(v))_i &:= (G_1(v))_i - f_i(t,x,\,v+q^0,\,(v+q^0)_x)\\ (H_2(v))_i &:= (G_2(v))_i - f_{\Gamma,i}(t,x,\,v+q^0)\end{aligned}$$

Under the assumptions of the Theorem 3.1 for f and f_Γ, we can show that H is a mapping of class $\mathcal{C}^1$ between $\mathcal{V}^2_\Omega$ and $L^2(0,T;\,W^{2,2}(\Omega;\,\mathbb{R}^N)) \times \text{Tr}_{S_T}\,\mathcal{V}^1$. Under the assumption (36), we can even show that the image of H is a subset of Z^2.

We can further verify that the linearisation of H possesses the structure

$$\begin{aligned}H_1'(v^*)\,\xi &= G_1'(v^*)\,\xi + \mathcal{L}_1(w^*)\,\xi\\ \mathcal{L}_1(w^*)\,\xi &:= -f_z(t,x,\,w^*,\,w^*_x)\cdot\xi - f_D(t,x,\,w^*,\,w^*_x)\,:\,\xi_x\\ H_2'(v^*)\,\xi &= G_2'(v^*)\,\xi + \mathcal{L}_2(w^*)\,\xi\\ \mathcal{L}_2(w^*)\,\xi &:= -f_{\Gamma,z}(t,x,\,w^*,\,w^*_x)\cdot\xi\end{aligned}$$

Our manifold calculations based on the multiplicator and Nemicki Lemmas can be use to show that $\mathcal{L}(w^*)$ is lower–order in the sense of Remark 3.9. Thus, $H'(w^*)$ is invertible (compare: Lemma 3.10), and the claim of Theorem 3.1 follows in full generality.

4 The first boundary value problem

The Dirichlet problem for equation (29) can be handled with exactly the same methods if we only adjust the functional setting. We denote $\mathcal{P} := S_T \cup (\{0\}\times\Omega) \cup (\{0\}\times\partial\Omega)$ the parabolic boundary of the domain Q_T. For $\ell = 0, 1, 2$, we denote

$$\mathcal{V}^\ell_{\mathcal{P}} := \{v \in \mathcal{V}^\ell\,:\, v = 0 \text{ on } \mathcal{P}\}\,.$$

We consider

$$\partial_t R_i(t,x,\,w) - \operatorname{div}(M_i(t,x,\,w)\,w_x) = f_i(t,x,\,w,\,w_x) \text{ in } Q_T\,,$$
$$w = w^{\Gamma} \qquad\qquad \text{on } \mathcal{P}\,. \tag{66}$$

Here, we assume that the vector w^{Γ} possesses an extension w^0 of class $\mathcal{V}^2$ into Q_T.

Theorem 4.1. *Assumptions of the theorem 3.1. Instead of* (37)*, we assume that the vector* $q^0 := w^{\Gamma}(0)$ *satisfies*

$$R_t(0,\,x,\,q^0(x)) + R_z(0,x,\,q^0(x))\,w^{\Gamma}_t(0,x)$$
$$= \operatorname{div}(M(0,x,\,q^0)\,q^0_x) + f(0,x,\,q^0(x),\,q^0_x(x)) \text{ for all } x \in \partial\Omega\,. \tag{67}$$

Then, there is $T > 0$ such that the problem (66) *possesses a unique solution of class* $\mathcal{V}^2(Q_T;\,\mathbb{R}^N)$.

The proof strategy is essentially as above. First we consider the case of a zero right-hand side f.

4.1 The case of no lower–order perturbation

On the space $\mathcal{V}^2_{\mathcal{P}}$ we introduce a nonlinear operator G via

$$G(v) := \mathcal{R}(v + w^0) + \mathcal{Q}(v + w^0)\,. \tag{68}$$

As in Section 3, we can verify that G takes values in $L^2(0,T;\,W^{2,2}(\Omega;\,\mathbb{R}^N))$. The result of Theorem 4.1 follows by the methods of Section 3 if we can prove that the principal part of the linearisation is invertible. Thus, for given $w^* \in \mathcal{V}^2$ and right-hand f, we consider for $w \in \mathcal{V}^2_{\mathcal{P}}$ the following problem

$$(\mathcal{R}'_o(w^*) + \mathcal{Q}'_o(w^*))\,w = f \tag{69}$$

In comparison to Section 3, we must however acknowledge an additional subtle point. Indeed, if $w \in \mathcal{V}^2_{\mathcal{P}}$ solves the problem (69), then applying the trace operator γ to the equation yields $\gamma(f + \operatorname{div}(M_i(t,x,\,w)\,w_x)) = 0$ on S_T. Now, since $w \in \mathcal{V}^2_{\Omega}$ implies that $\operatorname{div}(M_i(t,x,\,w)\,w_x) \in \mathcal{V}^0_{\Omega}$, it follows that the resolvability of (69) requires

$$\gamma(f) \in \mathsf{Tr}_{S_T}\,\mathcal{V}^0_{\Omega}\,.$$

Thus, the image space Z is taken here

$$Z := \{f \in L^2(0,T;\,W^{2,2}(\Omega;\,\mathbb{R}^N))\;:\;\gamma(f) \in \mathsf{Tr}_{S_T}\,\mathcal{V}^0_{\Omega}\}\,.$$

In order to apply the same theory, it is necessary that G takes its values in Z. This is the reason why we require (67). Introduce for $s \in [0,1]$ operators $\mathcal{Q}'_o(w^*;\,s)$ in the fashion of (43), (44). Then, everything is reduced to obtaining a uniform bound in $\mathcal{V}^2$ for solutions w to (69).

The bound in $\mathcal{V}^0$ was already proved in Lemma 3.3, that is

$$\|w\|_{\mathcal{V}^0} \leq c\,\|f\|_Z\,. \tag{70}$$

In order to obtain a bound of next order (in the space $\mathcal{V}^1$), we differentiate the equation in the fashion of Lemma 3.4 If $V \in C^2(\overline{\Omega};\,\mathbb{R}^3)$ has compact support or is tangential on $\partial\Omega$, then $\eta = V\cdot\nabla w$

satisfies again the condition $\eta = 0$ on S_T. Thus, the Lemma 3.3 implies that the tangential derivatives satisfy

$$\|\eta\|_{\mathcal{V}^0} \leq c\,\|f\|_Z\,. \tag{71}$$

Employing the equation (69) and Lemma 2.10, we see that the normal derivative $\eta = \nu \cdot \nabla w$ satisfies

$$\begin{aligned} -\,\nu_k(x)\, M_{i,j,k,\ell}(t,x,\,w^*)\,\eta_{j,x_\ell} = & -\sum_{\lambda=1}^{2} \tau_k^\lambda\, M_{i,j,k,\ell}\,(\tau^\lambda\cdot\nabla w_j)_{x_\ell} - A_i \\ & + \sum_{\lambda=1}^{3} V_k^\lambda\,(M_{i,j,k,\ell}\,(V^\lambda)_{x_\ell}\cdot\nabla w_j - V^\lambda\cdot\nabla M_{i,j,k,\ell}\,w_{j,x_\ell})\,. \end{aligned} \tag{72}$$

For $i = 1,\ldots,N$, A_i is here the function $f_i - R_{i,z}(t,x,\,w^*)\,\partial_t w$. Applying the trace operator γ to this identity, and recalling that $\gamma(w) = 0$ for all $w \in \mathcal{V}^2_{\mathcal{P}}$, it follows that $A = f$ on S_T. We call f^Γ the trace of the right-hand (72) of the latter identity, that is

$$\begin{aligned} f_{\Gamma,i} := & -\sum_{\lambda=1}^{2} \tau_k^\lambda\, M_{i,j,k,\ell}\,(\tau^\lambda\cdot\nabla w_j)_{x_\ell} - f_i \\ & + \sum_{\lambda=1}^{3} V_k^\lambda\,(M_{i,j,k,\ell}\,(V^\lambda)_{x_\ell}\cdot\nabla w_j - V^\lambda\cdot\nabla M_{i,j,k,\ell}\,w_{j,x_\ell}) \end{aligned}$$

Thanks to simple multiplicator arguments and to the estimates (70) and (71), we see that the vector f_Γ satisfies an estimate

$$\|f_\Gamma\|_{\mathcal{V}^{-1}} \leq c(\|w^*\|_{\mathcal{V}^2}\,(\|\tau^\lambda\cdot\nabla w\|_{\mathcal{V}^0} + \|w\|_{\mathcal{V}^0}) \leq c(\|w^*\|_{\mathcal{V}^2})\,\|f\|_Z\,.$$

Recall the definition of the principal part of the natural boundary operator. Obviously, every strong solution to (69) satisfies

$$\mathcal{B}'_o(w^*)\,(\nu\cdot\nabla w) = f^\Gamma\,. \tag{73}$$

For $\eta = V\cdot\nabla w$ with V parallel to ν on $\partial\Omega$, we thus can conclude that

$$\|\mathcal{B}'_o(w^*)\,\eta\|_{\mathcal{V}^{-1}} \leq c\,\|f\|_Z\,.$$

We apply the Lemma 3.3, and it follows that $\|\eta\|_{\mathcal{V}_0} \leq c\,\|f\|_Z$. Since now all components of the gradient vector satisfy an estimate in $\mathcal{V}^0$, it follows that

$$\|w\|_{\mathcal{V}^1} \leq c\,\|f\|_Z\,. \tag{74}$$

Next we go one order further, and we consider the second derivatives $\phi := W\cdot\nabla(V\cdot\nabla w)$. If W and V are both tangential vectors, then $W\cdot\nabla(V\cdot\nabla w) = (W\cdot\nabla)V\cdot\nabla w$. Thus, the vector ϕ satisfies a Dirichlet condition with right-hand in $\mathcal{V}^0$. The Lemma 3.3 yields the desired estimate for ϕ in $\mathcal{V}^0$. If W is tangential, but V normal to the boundary, then we differentiate (73) in the direction of W, and we obtain (notations of Lemma 3.5)

$$\mathcal{B}'_o(w^*)\,W\cdot\nabla(\nu\cdot\nabla w) = W\cdot\nabla f_\Gamma + D^{\Gamma,II}(w^*,\,W)\,\nu\cdot\nabla w\,.$$

We verify that the right-hand satisfies a bound in $\mathcal{V}^{-1}_\Omega$. Here it is important that $\gamma(f) \in \mathrm{Tr}\,\mathcal{V}^0_\Omega$. The Lemma 3.3 thus yields a bound in $\mathcal{V}^0$ for the mixed derivatives.

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Finally, if W and V are both normal on $\partial\Omega$, we make use of the Lemma 2.10 that yields

$$\begin{aligned} M_{i,j,\nu,\nu}\, \nu \cdot \nabla(\nu \cdot \nabla w_j) = & -f_i \\ & - \sum_{\lambda=1}^{2} M_{i,j,k,\ell}\, [\tau^{\lambda}_k\, (\tau^{\lambda} \cdot \nabla w_j)_{x_\ell} + \nu_k\, \tau^{\lambda}_\ell\, \nu \cdot \nabla(\tau^{\lambda} \cdot \nabla w_j)] \\ & + \sum_{\lambda=1}^{3} \tau^{\lambda}_k\, (M_{i,j,k,\ell}\, (\tau^{\lambda})_{x_\ell} \cdot \nabla w_j - \tau^{\lambda} \cdot \nabla M_{i,j,k,\ell}\, w_{j,x_\ell}) \\ & - \sum_{\lambda=1}^{2} \nu_k\, \tau^{\lambda}_\ell\, M_{i,j,k,\ell}\, [\{(\tau^{\lambda} \cdot \nabla)\nu - (\nu \cdot \nabla)\tau^{\lambda}\}\nabla w_j] \end{aligned}$$

This provides a bound for $\nu \cdot \nabla(\nu \cdot \nabla w)$ in Tr $\mathcal{V}^0$. The Lemma 3.3 yields the desired estimate for the twice normal derivatives. Now, all spatial derivatives of order two turn out to be uniformly bounded in $\mathcal{V}^0$. Thus every solution $w \in \mathcal{V}^2_{\mathcal{P}}$ to (69) satisfies a uniform bound in this space. We finish the proof of Theorem 4.1 in the case that there is no lower-order perturbation as in Section 3.

4.2 Adding the lower–order term

In order to finally prove the claim in the case of a right-hand side $f(t, x, w, w_x)$, we introduce $H(v) := G(v) + \tilde{G}(v)$, where $\tilde{G}(v) := -f(t, x, v + q^0, (v + q^0)_x)$. The regularity of f and the compatibility condition (67) guaranty that H is $\mathcal{C}^1$ from $\mathcal{V}^2_{\mathcal{P}}$ into the space Z. This is sufficient to prove the invertibility of the linearisation.

5 An necessary extension to one–sided coupling in the leading order

The theory of this section is needed for the full quasilinear case. Let $1 \leq P \leq N$ be a natural number.

Remark 5.1. *In this section, we will decompose the vectors of $\mathbb{R}^N$ according to $X = (\bar{X}, X')$, where $\bar{X} = (X_1, \ldots, X_P) \in \mathbb{R}^P$ and $X' = (X_{P+1}, \ldots, X_N) \in \mathbb{R}^{N-P}$.*

For a vector $w_1, \ldots, w_N$, we consider for $i = 1, \ldots, N$ the equations

$$\partial_t R_i(t, x, w) - \frac{d}{dx_k} (\sum_{j=1}^{N} M_{i,j,\ell,k}(t, x, w)\, w_{j,x_\ell}) = f_i(t, x, w, w_x) \text{ in } Q_T\,, \tag{75}$$

supplemented on S_T with the boundary conditions

$$-\nu_k \sum_{j=1}^{P} M_{i,j,\ell,k}(t, x, w)\, w_{j,x_\ell} = f_{\Gamma,i}(t, x, w) \text{ for } i = 1, \ldots, P \tag{76}$$

$$w_i = D_i(t, x, \bar{w}) \text{ for } i = P + 1, \ldots, N \tag{77}$$

and the initial condition $w(0) = q^0$. The functions $D_{P+1}, \ldots, D_N$ in (77) are defined on $S_T \times \mathbb{R}^P$. For simplicity, we assume throughout the section that they are extension functions defined in $\overline{Q}_T \times \mathbb{R}^P$.

Theorem 5.2. *Assumptions of the Theorem 3.1. We assume that the functions $D_{P+1}, \ldots, D_N$ are of class $(C^{1,4} \cap C^{0,5})([0,T] \times \overline{\Omega} \times \mathbb{R}^P)$. We moreover assume a special diagonal structure:*

1 *The functions $M_{i,j,k,\ell}$ are defined on $\overline{Q} \times \mathbb{R}^N$ and $M_{i,j,k,\ell} = 0$ for $i \leq p$ and $j > p$;*

2 *The functions $R_1, \ldots, R_N$ are defined on $\overline{Q} \times \mathbb{R}^N$; For $i = 1, \ldots, P$, R_i depends only on the components $\bar{w} = (w_1, \ldots, w_P) \in \mathbb{R}^P$. The matrix $\{R_{i,z_j}(t, x, \bar{w})\}_{i,j=1,\ldots,P}$ is symmetric and positive definite for all $\bar{w} \in \mathbb{R}^P$;*

3 *The block $\{R_{i,z_j}(t, x, w)\}_{i,j=P+1,\ldots,N}$ is symmetric and positive definite for all $w \in \mathbb{R}^N$;*

We assume for $i = 1, \ldots, P$ the compatibility condition

$$-\nu_k \, M_{i,j,\ell,k}(0, x, q^0(x)) \, q^0_{j,x_\ell} - f_{\Gamma,i}(0, x, q^0(x)) = 0 \text{ for all } x \in \partial\Omega \,. \tag{78}$$

We assume for $i = P+1, \ldots N$ and all $x \in \partial\Omega$ the following two compatibility conditions: First

$$D_i(0, x, \bar{q^0}(x)) = (q^0(x))'_i \tag{79}$$

and second

$$\begin{aligned}\sum_{j=1}^{P} A_{i,j}(x) \, (R_{j,t}(0, x, \bar{q^0}) - \operatorname{div}(M_j(0, x, q^0) \, \bar{q^0}_x) - f_j(0, x, q^0(x), q^0_x(x)) \\ = R_{i,t}(0, x, q^0) - \operatorname{div}(M_i(0, x, q^0) \, q^0_x) - f_i(0, x, q^0(x), q^0_x(x)) \,.\end{aligned} \tag{80}$$

Here, $\{A_{i,j}(x)\}_{i=P+1,\ldots,N, \, j=1,\ldots,P}$ is the rectangular matrix

$$\begin{aligned}A_{i,j}(x) &:= \sum_{j'=1}^{P} [R_z(0, x, q^0)]^{-1}_{j,j'} \times \\ &\times [R_{i,z_{j'}}(0, x, q^0(x)) + \sum_{k=P+1}^{N} R_{i,z_k}(0, x, q^0(x)) \, D_{k,z_{j'}}(0, x, q^0(x))] \,.\end{aligned} \tag{81}$$

Then, there is $T > 0$ such that the problem (75)*,* (76)*,* (77) *possesses a unique solution of class $\mathcal{V}^2(Q_T; \mathbb{R}^N)$.*

In this section, we introduce the state–space via

$$\mathcal{V}^2_{\Omega,\mathcal{P}}(Q_T; \mathbb{R}^P \times \mathbb{R}^{N-P}) = \mathcal{V}^2_{\Omega}(Q_T; \mathbb{R}^P) \times \mathcal{V}^2_{\mathcal{P}}(Q_T; \mathbb{R}^{N-P}) \,.$$

The elements $v \in \mathcal{V}^2_{\Omega,\mathcal{P}}$ are denoted $v = (\bar{v}, v')$ according to the notation of Remark 5.1. For $v \in \mathcal{V}^2_{\Omega,\mathcal{P}}$ we define

$$w_i = w_i(v) = \begin{cases} v_i + q^0_i & \text{for } i = 1, \ldots, P \\ v_i + D_i(t, x, \bar{v} + \bar{q}^0) & \text{for } i = P+1, \ldots, N \,. \end{cases} \tag{82}$$

Under the compatibility condition (79), we see that $\bar{w}(0) = \bar{q^0}$, that $w'(0) = (q^0)'$, and that $\gamma(w_i) = D_i(t, x, \bar{w})$ on S_T for $i = P+1, \ldots N$.

5.1 The fundamental case

In this subsection we set $f = 0$ and $f_\Gamma = 0$ in (75), (76).

We introduce an operator $G(v)$ acting on $\mathcal{V}^2_{\Omega,\mathcal{P}}$ via

$$G(v) = \{G_1(v),\, G_2(v)\}$$
$$(G_1(v))_i := \partial_t R_i(t, x,\, w) - \frac{d}{dx_k}\,(M_{i,j,\ell,k}(t, x,\, w)\, w_{j,x_\ell}) \text{ for } i = 1, \ldots, N$$
$$(G_2(v))_i := -\nu_k\, M_{i,j,\ell,k}(t, x,\, w)\, w_{j,x_\ell} \text{ for } i = 1, \ldots, P\,.$$

It is readily seen (cp. Section 3) that this operator maps $\mathcal{V}^2_{\Omega,\mathcal{P}}$ into

$$Z := L^2(0,T;\, W^{2,2}(\Omega;\, \mathbb{R}^N)) \times \text{Tr}_{S_T}\, \mathcal{V}^1_\Omega(Q_T;\, \mathbb{R}^P)\,. \tag{83}$$

Under the regularity condition for the function D in Theorem 5.2, the mapping $v \mapsto D_i(t, x,\, \bar{v}+\bar{q}^0)$ is of class $\mathcal{C}^1$ from $\mathcal{V}^2$ into itself. Therefore, we can compose this mapping with other Nemicki operators enjoying the same property, and see that the $\mathcal{C}^1$ property is conserved. We conclude that the operator G is Fréchet differentiable between $\mathcal{V}^2_{\Omega,\mathcal{P}}$ and $L^2(0,T;\, W^{2,2}(\Omega;\, \mathbb{R}^N)) \times \text{Tr}_{S_T}\, \mathcal{V}^1(Q_T;\, \mathbb{R}^P)$.

The compatibility condition (78) ensures that G maps into Z. We next want to study the linearisation $\{G_1'(v^*)\,\xi,\, G_2'(v^*)\,\xi\}$ described hereafter. In the remainder, we define $w^* = w(v^*)$ according to (82). The P first equations of the linearisation for the variables $\bar{\xi} = \xi_1, \ldots, \xi_P$ are governed by the bulk operator

$$\begin{aligned}
&\partial_t(\sum_{j=1}^P R_{i,z_j}(t, x,\, w^*)\, \bar{\xi}_j) - \text{div}(\sum_{j=1}^P M_{i,j}(t, x,\, w^*)\, \bar{\xi}_{j,x})\\
&- \text{div}(\sum_{m=1}^P \left[M_{i,j,z_m}(t, x,\, w^*) + \sum_{j'=P+1}^N M_{i,j,z_{j'}}(t, x,\, w^*)\, D_{j',z_m}(t, x,\, w^*)\right] \bar{\xi}_m\, w^*_{j,x})\\
&+ \text{div}(\sum_{m=P+1}^N M_{i,j,z_m}(t, x,\, w^*)\, \xi'_m\, w^*_{j,x})
\end{aligned}$$

with the associated boundary operator

$$\begin{aligned}
&- \nu_k \sum_{j=1}^P M_{i,j,\ell,k}(t, x,\, w^*)\, \bar{\xi}_{j,x_\ell}\\
&- \nu_k\, (\sum_{m=1}^P \left[M_{i,j,k,z_m}(t, x,\, w^*) + \sum_{j'=P+1}^N M_{i,j,k,z_{j'}}(t, x,\, w^*)\, D_{j',z_m}(t, x,\, w^*)\right] \bar{\xi}_m\, w^*_{j,x})\\
&+ \nu_k\, (\sum_{m=P+1}^N M_{i,j,k,z_m}(t, x,\, w^*)\, \xi'_m\, w^*_{j,x})\,.
\end{aligned}$$

The equations with index $P+1,\ldots,N$ are governed by the operator

$$\begin{aligned}
&\partial_t(\sum_{j=P+1}^N R_{i,z_j}(t,x,\,w^*)\,\xi'_j) - \operatorname{div}(\sum_{j=P+1}^N M_{i,j}(t,x,\,w^*)\,\xi'_{j,x})\\
&+\partial_t(\sum_{j=1}^P [R_{i,z_j}(t,x,\,w^*) + \sum_{j'=P+1}^N R_{i,z_{j'}}(t,x,\,w^*)\,D_{j',z_j}(t,x,\,w^*)]\,\bar{\xi}_j)\\
&-\operatorname{div}(\sum_{j=1}^P M_{i,j}(t,x,\,w^*)\,\xi'_{j,x} + \sum_{j=P+1}^N M_{i,j}(t,x,\,w^*)\,\bar{\xi}_{j,x}))\\
&-\operatorname{div}(\sum_{m=1}^P [M_{i,j,z_m}(t,x,\,w^*) + \sum_{j'=P+1}^N M_{i,j,z_{j'}}(t,x,\,w^*)\,D_{j',z_m}(t,x,\,w^*)]\,\bar{\xi}_m\,w^*_{j,x})\\
&+\operatorname{div}(\sum_{m=P+1}^N M_{i,j,z_m}(t,x,\,w^*)\,\xi'_m\,w^*_{j,x})\,,
\end{aligned}$$

We choose a principal part of the bulk operator via

$$(G'_{1,o}(v^*)\,\xi)_i = \sum_{j=1}^P R_{i,z_j}(t,x,\,w^*)\,\xi_{j,t} - \operatorname{div}(\sum_{j=1}^P M_{i,j}(t,x,\,w^*)\,\bar{\xi}_{j,x}) \text{ for } i \leq P$$

while for $i = P+1,\ldots,N$

$$\begin{aligned}
(G'_{1,o}(v^*)\,\xi)_i = &\sum_{j=P+1}^N R_{i,z_j}(t,x,\,w^*)\,\xi_{j,t} - \operatorname{div}(\sum_{j=P+1}^N M_{i,j}(t,x,\,w^*)\,\xi_{j,x})\\
&+\sum_{j=1}^P [R_{i,z_j}(t,x,\,w^*) + \sum_{j'=P+1}^N R_{i,z_{j'}}(t,x,\,w^*)\,D_{j',z_j}(t,x,\,w^*)]\,\bar{\xi}_{j,t}\\
&-\operatorname{div}(\sum_{j=P+1}^N M_{i,j}(t,x,\,w^*)\,\bar{\xi}_{j,x})\,.
\end{aligned}$$

The principal part of the boundary operator is

$$(G'_{2,o}(v^*)\,\xi)_i = -\nu_k \sum_{j=1}^P M_{i,j,\ell,k}(t,x,\,w^*)\,\xi_{j,x_\ell} \text{ for } i = 1,\ldots,P$$

For the invertibility of the principal part, there is an additional compatibility condition. Recall the definitions (81) and (83).

Lemma 5.3. *Define A as in* (81)*. The principal part $G'_o(v^*)$ is an invertible operator between $\mathcal{V}^2_{\Omega,\mathcal{P}}$ and the Banach space Z_A defined via*

$$Z_A = \{F = (f,\,f_\Gamma) \in Z \,:\, A(x)\,\bar{f}(0,x) = f'(0,x) \textit{ for all } x \in \partial\Omega\}\,.$$

Proof. We consider the system $G'_o(v^*)\,\xi = F$. One can solve for the P first equations making use of the theory of Section 3 (Lemma 3.8). We obtain for the solution $\bar{\xi} \in \mathcal{V}^2_\Omega(Q_T;\,\mathbb{R}^P)$ the bound

$$\|\bar{\xi}\|_{\mathcal{V}^2_\Omega(Q_T;\mathbb{R}^P)} \leq c(\|w^*\|_{\mathcal{V}^2})\,(\|\bar{f}\|_{L^2(0,T;\,W^{2,2}(\Omega;\mathbb{R}^P))} + \|f_\Gamma\|_{\mathrm{Tr}\,\mathcal{V}^1_\Omega(Q_T;\mathbb{R}^P)})\,.$$

Moreover, recalling (53)

$$\|R_z(t,x,\,w^*)\,\bar{\xi}_t - \bar{f}\|_{\mathrm{Tr}_{S_T}\,\mathcal{V}^0_\Omega(Q_T;\,\mathbb{R}^P)} \\ \leq c(\|w^*\|_{\mathcal{V}^2})\,\big(\|\bar{f}\|_{L^2(0,T;\,W^{2,2}(\Omega;\,\mathbb{R}^P))} + \|f_\Gamma\|_{\mathrm{Tr}\,\mathcal{V}^1_\Omega(Q_T;\,\mathbb{R}^P)}\big)\,. \tag{84}$$

In addition, the limit $t \to 0$ in the equations yields the identity

$$\sum_{j=1}^P R_{i,z_j}(0,x,\,w^*(0))\,\xi_{j,t}(0) = f_i(0,x) \text{ for } x \in \partial\Omega\,. \tag{85}$$

Next, we can make use of the Lemmas 2.4 and 2.6 to show that the vector

$$g^1_i := \mathrm{div}(\sum_{j=1}^P M_{i,j}(t,x,\,w^*)\,\xi_{j,x}) \text{ for } i = P+1,\ldots,N\,,$$

belongs to $\mathcal{V}^0_\Omega(Q_T;\,\mathbb{R}^{N-P})$. It satisfies a bound

$$\|g^1\|_{\mathcal{V}^0_\Omega} \leq c(\|w^*\|_{\mathcal{V}^2})\,\|\bar{\xi}\|_{\mathcal{V}^2_\Omega(Q_T;\,\mathbb{R}^P)} \\ \leq c(\|w^*\|_{\mathcal{V}^2})\,\big(\|\bar{f}\|_{L^2(0,T;\,W^{2,2}(\Omega;\,\mathbb{R}^P))} + \|f_\Gamma\|_{\mathrm{Tr}\,\mathcal{V}^1_\Omega(Q_T;\,\mathbb{R}^P)}\big)\,.$$

Further, we denote

$$g^2_i := \sum_{j=1}^P [R_{i,z_j}(t,x,\,w^*) + \sum_{j'=P+1}^N R_{i,z_{j'}}(t,x,\,w^*)\,D_{j',z_j}(t,x,\,w^*)]\,\bar{\xi}_{j,t}\,.$$

Then, obviously

$$\|g^2\|_{L^2(0,T;\,W^{2,2}(\Omega;\,\mathbb{R}^{N-P}))} \leq c(\|w^*\|_{\mathcal{V}^2})\,\|\bar{\xi}\|_{\mathcal{V}^2_\Omega(Q_T;\,\mathbb{R}^P)}\,.$$

Further, for $j = 1,\ldots,P$, we can represent $\bar{\xi}_{j,t} = \sum_{j'=1}^P [R_z]^{-1}_{j,j'}\,(\sum_{k=1}^P R_{j',z_k}\,\bar{\xi}_{k,t} - f_{j'} + f_{j'})$. This allows to decompose

$$g^2_i = g^{2,1}_i + g^{2,2}_i \\ g^{2,1}_i = \sum_{j=1}^P \Big[R_{i,z_j}(t,x,\,w^*) + \sum_{j'=P+1}^N R_{i,z_{j'}}(t,x,\,w^*)\,D_{j',z_j}(t,x,\,w^*)\Big]\times \\ \times\Big[\sum_{j'=1}^P [R_z]^{-1}_{j,j'}\,(\sum_{k=1}^P R_{j',z_k}\,\bar{\xi}_{k,t} - f_{j'})\Big] \\ g^{2,2}_i = \sum_{j=1}^P \Big[R_{i,z_j}(t,x,\,w^*) + \sum_{j'=P+1}^N R_{i,z_{j'}}(t,x,\,w^*)\,D_{j',z_j}(t,x,\,w^*)\Big]\,\Big[\sum_{j'=1}^P [R_z]^{-1}_{j,j'}\,f_{j'}\Big]\,.$$

Making use of (84) and of the regularity of f, it follows for the trace operator γ

$$\|\gamma(g^{2,1})\|_{\mathrm{Tr}_{S_T}\,\mathcal{V}^0(Q_T;\,\mathbb{R}^p)} \leq c(\|w^*\|_{\mathcal{V}^2})\,(\|f\|_{L^2(0,T;\,W^{2,2}(\Omega;\,\mathbb{R}^P))} + \|f_\Gamma\|_{\mathrm{Tr}\,\mathcal{V}^1_\Omega(Q_T;\,\mathbb{R}^P)}) \\ \|\gamma(g^{2,2})\|_{\mathrm{Tr}_{S_T}\,\mathcal{V}^0(Q_T;\,\mathbb{R}^P)} \leq c(\|w^*\|_{\mathcal{V}^2})\,\|\bar{f}\|_{\mathrm{Tr}_{S_T}\,\mathcal{V}^0_\Omega(Q_T;\,\mathbb{R}^P)}\,.$$

Thus, $\gamma(g^2) \in \text{Tr}_{S_T}\, \mathcal{V}^0(Q_T;\, \mathbb{R}^P)$. Moreover, (84) guaranties that $\gamma(g^{2,1}) \in \text{Tr}_{S_T}\, \mathcal{V}^0_\Omega(Q_T;\, \mathbb{R}^{N-P})$. Thanks to (85), we can next consider the limit for $t \to 0$ of $\gamma(g^{2,2}(t))$, and we obtain for $f \in Z_A$ that

$$\begin{aligned}\gamma(g_i^{2,2}(0)) = & \sum_{j=1}^P \Big[R_{i,z_j}(0,x,\, q^0) + \sum_{j'=P+1}^N R_{i,z_{j'}}(0,x,\, q^0)\, D_{j',z_j}(0,x,\, q^0)\Big] \times \\ & \Big[\sum_{j'=1}^P [R_z(0,\, x,\, q^0)]^{-1}_{j,j'}\, f_{j'}(0,x)\Big] \\ & = A_{j,j'}(x)\, f_{j'}(0,x) = f_i(0,x)\,.\end{aligned}$$

Thus, $\gamma(g^{2,2} - f') = 0$, which means that $g^{2,2} - f' \in \text{Tr}_{S_T}\, \mathcal{V}^0_\Omega(Q_T;\, \mathbb{R}^{N-P})$.

Now, the equations $G'_{1,o}(v^*)\, \xi = f$ that have index $i \in \{P+1, \ldots, N\}$ have the form

$$\sum_{j=P+1}^N R_{i,z_j}(t,x,\, w^*)\, \xi_{j,t} - \text{div}(\sum_{j=P+1}^N M_{i,j}(t,x,\, w^*)\, \xi_{j,x}) = f_i - g_i^1 - g_i^2\,,$$

and we have proven that the right-hand side satisfies a bound in $L^2(0,T;\, W^{2,2}(\Omega;\, \mathbb{R}^{N-P})) \cap \text{Tr}_{S_T}\, \mathcal{V}^0_\Omega(Q_T;\, \mathbb{R}^{N-P})$. Thus, we can apply the theory of the Section 4, and the claim follows. □

Since the principal part is invertible, we can verify that the Fréchet derivative is an invertible operator, between $\mathcal{V}^2_{\Omega,\mathcal{P}}$ and the Banach space Z_A. If we can prove that G maps $\mathcal{V}^2_{\Omega,\mathcal{P}}$ into Z_A, the usual arguments provide the local–in–time resolvability. Straightforward calculations show that the compatibility condition (80) is necessary and sufficient. The claim of Theorem 5.2 follows.

5.2 Extension to lower order terms

If f and f_Γ are not zero in (75), (76), we introduce $H(v) = G(v) + \tilde{G}(v)$, where $\tilde{G}(v)$ is the operator

$$\tilde{G}_1(v) = -f(t,x,\, w(v),\, (w(v))_x), \quad \tilde{G}_2(v) := -f_\Gamma(t,x,\, w(v))\,.$$

The regularity of $f,\, f_\Gamma$ and the conditions (78) and (80) guaranty that H maps into Z_A. This is, here also, sufficient for the treatment of lower-order terms.

6 The full quasilinear case

We consider the equation

$$\partial_t R_i(t,x,\, w) - \sum_{k=1}^3 \frac{d}{dx_k} J^i_k(t,x,\, w,\, w_x) = f_i(t,x,\, w,\, w_x) \text{ for } i = 1, \ldots, N\,. \tag{86}$$

For i, the functions R_i are defined in $\overline{Q_T} \times \mathbb{R}^N$, while the flux functions J^i_k are defined on $\overline{Q} \times \mathbb{R}^N \times \mathbb{R}^{N\times 3}$. We denote $(t,\, x,\, z,\, D)$ a point in the latter domain, and $J^i_{k,t}$, $J^i_{k,x}$, J^i_{k,D^j_ℓ} etc. denote the partial derivatives in these variables.

We consider the natural boundary condition

$$-\sum_{k=1}^{3} \nu_k(x)\, J^i_k(t, x,\, w,\, w_x) = f_{\Gamma,i}(t, x,\, w) \text{ for } i = 1, \ldots, N\,. \tag{87}$$

We require the structural conditions (1), (2).

Under this assumption, we observe that

$$\begin{aligned} J^i_k(t, x,\, z,\, D) &= \partial_{D^i_k} \Psi(t, x,\, z,\, D) - \partial_{D^i_k} \Psi(t, x,\, z,\, 0) \\ &= \sum_{j=1}^{N} \sum_{\ell=1}^{3} \int_0^1 D^2_{D^i_k, D^j_\ell} \Psi(t, x,\, z,\, \theta\, D)\, d\theta\, D^j_\ell\,. \end{aligned}$$

We abbreviate

$$M^0_{i,j,k,\ell}(t, x,\, z,\, D) := \int_0^1 D^2_{D^i_k, D^j_\ell} \Psi(t, x,\, z,\, \theta\, D)\, d\theta\,. \tag{88}$$

Then, (86) has for $i = 1, \ldots, N$ the equivalent expression

$$\partial_t R_i(t, x,\, w) - \frac{d}{dx_k}(M^0_{i,j,k,\ell}(t, x,\, w,\, w_x)\, w_{j,x_\ell}) = f_i(t, x,\, w,\, w_x)\,, \tag{89}$$

while (87) reads

$$-\nu_k(x)\, M^0_{i,j,k,\ell}(t, x,\, w,\, w_x)\, w_{j,x_\ell} = f_{\Gamma,i}(t, x,\, w,\, w_x)\,. \tag{90}$$

The treatment of (86), (87) relies in the end on the analysis of the reduced quasi-linear case in the section 5.

It shall rely on the following two compatibility conditions

$$\nu_k(x)\, J^i_k(0, x,\, q^0(x),\, q^0_x(x)) = f_{\Gamma,i}(0, x,\, q^0(x)) \text{ for } i = 1, \ldots, N,\, x \in \partial\Omega\,. \tag{91}$$

Second, we assume for $i = 1, \ldots, N$ and $x \in \partial\Omega$ that

$$\begin{aligned} &\nu_k(x)\, (J^i_{k,t}(0, x,\, q^0(x),\, q^0_x(x)) + J^i_{k,z_j}(0, x,\, q^0(x),\, q^0_x(x))\, F^0_j(x)) \\ &\quad + \nu_k(x)\, J^i_{k,D^j_\ell}(0, x,\, q^0(x),\, q^0_x(x))\, F^0_{j,x_\ell}(x)) \\ &\qquad = f_{\Gamma,i,t}(0, x,\, q^0(x)) + f_{\Gamma,i,z_j}(0, x,\, q^0(x))\, F^0_j(x) \end{aligned} \tag{92}$$

where F^0 is the vector field given by

$$\begin{aligned} F^0(x) :=& [R_z(0, x,\, q^0(x))]^{-1}\, (\operatorname{div}(J(0, x,\, q^0(x),\, q^0_x(x))) - R_t(0, x,\, q^0(x)) \\ & - f(0, x,\, q^0(x),\, q^0_x(x)))\,. \end{aligned}$$

As a preliminary, we will at first explain our main observations for the proof of Theorem 1.1.

6.1 Transformation of the bulk operator

Assume that $w \in \mathcal{V}^2$ is a strong solution to (86). For $m = 1, 2, 3$, the functions $w_{1,x_m}, \ldots, w_{N,x_m}$ belong to $\mathcal{V}^1$ and they satisfy the equations

$$\partial_t(R_{i,x_m} + R_{i,z_j}\, w_{j,x_m}) \\ - \sum_{k=1}^{3} \frac{d}{dx_k}\Big[J^i_{k,x_m} + \sum_{j=1}^{N} J^i_{k,z_j}\, w_{j,x_m} + \sum_{j=1}^{N}\sum_{\ell=1}^{3} J^i_{k,D^j_\ell}\, w_{j,x_\ell,x_m}\Big] = \frac{d}{dx_m} f\,.$$

We make use of a smooth spatial multiplier $\zeta(x)$, and we obtain for $\eta^m_j := w_{j,x_m}\,\zeta$ ($i = 1,\ldots,N$ and $m = 1, 2, 3$) the identities

$$\partial_t(R_{i,x_m}\,\zeta + R_{i,z_j}\,\eta^m_j) - \sum_{k=1}^{3} \partial_{x_k}(\Psi_{D^i_k,D^j_\ell}(t,x,\,w,\,w_x)\,\eta^m_{j,x_\ell}) \\ = \zeta\,(\frac{d}{dx_m} f + \partial_{x_k}(J^i_{k,x_m} + J^i_{k,z_j}\,w_{j,x_m})) \\ - \zeta_{x_k}\, D^2_{D^i_k,D^j_\ell}\Psi(t,x,\,w,\,w_x)\,w_{j,x_\ell,x_m} + \partial_{x_k}(D^2_{D^i_k,D^j_\ell}\Psi(t,x,\,w,\,w_x)\,w_{j,x_m}\,\zeta_{x_\ell})\,. \tag{93}$$

Introduce for $m = 1, 2, 3$, $\zeta \in C^2(\overline{\Omega})$ and $i = 1,\ldots,N$

$$R^{m,\zeta}_i = \zeta\, R_{i,x_m} + R_{i,z_j}\,\eta^m_j$$

$$B^{m,\zeta}_i = \zeta\,(\partial_{x_k}(J^i_{k,x_m} + \sum_{j=1}^{N} J^i_{k,z_j}\,w_{j,x_m}) + \frac{d}{dx_m} f) \\ - \zeta_{x_k}\,\Psi_{D^i_k,D^j_\ell}(t,x,\,w,\,w_x)\,w_{j,x_\ell,x_m} + \partial_{x_k}(D^2_{D^i_k,D^j_\ell}\Psi(t,x,\,w,\,w_x)\,w_{j,x_m}\,\zeta_{x_\ell})\,.$$

The equations (93) possess the structure

$$\partial_t R^{m,\zeta}_i(t,x,\,w,\,\eta^m) - \frac{d}{dx_k}(\Psi_{D^i_k,D^j_\ell}(t,x,\,w,\,w_x)\,\eta^m_{j,x_\ell}) \\ = B^{m,\zeta}_i(t,x,\,w,\,w_x,\,w_{x,x})\,. \tag{94}$$

We next localise the problem in the fashion of Section 2.7. For $\mu = 0,\ldots,n$, $\lambda = 1, 2, 3$ and $\iota = 1,\ldots,N$, we consider the function

$$\eta^{\lambda,\mu}_\iota = V^{\mu,\lambda}\cdot\nabla w_\iota\,\zeta_\mu\,.$$

We define

$$R^{\lambda,\mu}_i(t,x,\,w,\,\eta^{\lambda,\mu}) := \zeta^\mu \sum_{m=1}^{3} V^{\mu,\lambda}_m\,R_{i,x_m}(t,x,\,w) + \sum_{j=1}^{N} R_{i,z_j}(t,x,\,w)\,\eta^{\lambda,\mu}_j$$

$$B^{\lambda,\mu}_i(t,x,\,w,\,w_x,\,w_{x,x}) := \sum_{m=1}^{3} B^{m,\zeta^\mu V^{\mu,\lambda}_m}_i(t,x,\,w,\,w_x,\,w_{x,x})\,.$$

Employing (94), the functions $\eta^{\lambda,\mu}_1,\ldots,\eta^{\lambda,\mu}_N$ belong to $\mathcal{V}^1$ and they satisfy the equations

$$\partial_t R^{\lambda,\mu}_i(t,x,\,w,\,\eta^{\lambda,\mu}) - \frac{d}{dx_k}(\Psi_{D^i_k,D^j_\ell}(t,x,\,w,\,w_x)\,\eta^{\lambda,\mu}_{j,x_\ell}) \\ = B^{\lambda,\mu}_i(t,x,\,w,\,w_x,\,w_{x,x})\,. \tag{95}$$

We now introduce an auxiliary vector $u\,:\;Q_T\to\mathbb{R}^K$ with $K\,=\,N\,(3\,(n+1)+1)$. The first N entries of u are given by $w_1,\ldots,w_N$. Then, for each $\lambda=1,2$, and $\mu=0,\ldots,n$ we add the entries $\eta_1^{\lambda,\mu},\ldots,\eta_N^{\lambda,\mu}$. The last $N\,(n+1)$ remaining entries of u are given by $\eta_1^{3,0},\ldots,\eta_N^{3,n}$. This procedure in fact constructs a bijection P between on the one hand $\alpha\in\{N+1,\ldots,K\}$ and on the other hand $\lambda\in\{1,2,3\}$, $\mu\in\{0,\ldots,n\}$, and $\iota\in\{1,\ldots,N\}$, such that

$$u_{P(\lambda,\mu,i)}:=\eta_i^{\lambda,\mu}\,. \tag{96}$$

In this manner, we define a function $R:\;Q_T\times\mathbb{R}^K\to\mathbb{R}^N$

$$R_\alpha(t,x,\,u):=\begin{cases}R(t,x,\,w) & \text{for } \alpha=1,\ldots,N\\ R^{\lambda,\mu}_\alpha(t,x,\,w,\,\eta^{\lambda,\mu}) & \text{for } \alpha=N+1,\ldots,K,\;\alpha=P(\lambda,\mu,\iota)\end{cases} \tag{97}$$

Next, we recall (59). It follows that there are for $\alpha\,=\,N+1,\ldots,K$ and $i\,=\,1,\ldots,N$ functions $C_{\alpha,i}(x)$ such that

$$w_{i,x}=\sum_{\alpha=N+1}^{K}C_{\alpha,i}(x)\,u_\alpha \tag{98}$$

We define

$$M_{\alpha,\beta}(t,x,\,u):=\begin{cases}M^0_{\alpha,\beta}(t,x,\,w) & \text{for } \alpha,\beta=1,\ldots,N\\ \Psi_{D^i,D^j}(t,x,\,w,\,C(x)\,u) & \text{for } \alpha=P(\mu,\lambda,i) \text{ and } \beta=P(\mu,\lambda,j)\\ 0 & \text{otherwise}\end{cases}\;. \tag{99}$$

Moreover, note that (98) also implies that

$$w_{i,x,x}=\sum_{\alpha=N+1}^{K}C_{\alpha,i,x}(x)\,u_\alpha+\sum_{\alpha=N+1}^{K}C_{\alpha,i}(x)\,u_{\alpha,x}\,. \tag{100}$$

We define

$$f_\alpha(t,x,\,u,\,u_x):=\begin{cases}f_\alpha(t,x,\,w,\,C(x)\,u) & \text{for } \alpha=1,\ldots,N\\ B^{\lambda,\mu}_i(t,x,\,w,\,C(x)\,u,\,C(x)\,u_x+C_x(x)\,u) & \text{for } \alpha=P(\lambda,\mu,i)\end{cases} \tag{101}$$

In this place, we recall also (89) and (95) to see that the vector u satisfies

$$\begin{aligned}\partial_t R_\alpha(t,x,\,u)-\frac{d}{dx_k}\,(\sum_{\beta=1}^{K}\sum_{\ell=1}^{3}M_{\alpha,\beta,k,\ell}(t,x,\,u)\,u_{\beta,x_\ell})&\\ =f_\alpha(t,x,\,u,\,u_x)\,.&\end{aligned} \tag{102}$$

Let $K'\;=\;N\,(2(n+1)+1)$. Then, it is readily seen from the definitions (99) and (101) that this system has sub-diagonal structure with respect to K' as defined by Theorem 5.2 (for $K\;=\;N$ and $K'=P$).

6.2 Transformation of the boundary operator

We next investigate the boundary operator and assume that $w \in \mathcal{V}^2$ satisfies (87) (with $f_\Gamma = 0$). Applying tangential differentiation ($\mu \in \{1,\ldots,n\}$ and $\lambda \in \{1,2\}$), we obtain that

$$\begin{aligned}-\nu_k\, \Psi_{D^i_k, D^j_\ell}\, \eta^{\lambda,\mu}_{j,x_\ell} =& \zeta^\mu\, (V^{\mu,\lambda} \cdot \nabla \nu_k\, \Psi_{D^i_k} + \nu_k\, V^{\mu,\lambda}_m\, [\Psi_{D^i_k, x_m} + \Psi_{D^i_k, z_j}\, w_{j,x_m}]) \\ & - \nu_k\, \Psi_{D^i_k, D^j_\ell}\, w_{j,x_m}\, (V^{\lambda,\mu}_m\, \zeta^(\mu)_{x_\ell}) + V^{\lambda,\mu} \cdot \nabla f_\Gamma\,.\end{aligned}$$

Thus, we can introduce lower order functions $(B^\Gamma)^{V,\mu}_i$ such that

$$-\nu_k\, \Psi_{D^i_k, D^j_\ell}(t,x,\, w,\, w_x)\, \eta^{\lambda,\mu}_{j,x_\ell} = (B^\Gamma)^{V,\mu}_i(t,x,\, w,\, w_x) \text{ for } i = 1,\ldots,N\,. \tag{103}$$

Reorganising the system in the manner of (99) we obtain that

$$-\nu_k\, M_{\alpha,\beta,k,\ell}(t,x,\, u)\, u_{\beta,x_\ell} = f_{\Gamma,\alpha}(t,x,\, u) \forall\, \alpha \leq N - K' = N\,(2(n+1)+1)\,. \tag{104}$$

Next we show that the variables $u_{K'+1},\ldots,u_K$ satisfy Dirichlet conditions on S_T. We introduce for $i = 1,\ldots,N$ and $z^0,\, z^1,\, z^2,\, X \in \mathbb{R}^N$ the functions

$$\mathcal{F}_i(t,x,\, z^0,\, z^1,\, z^2,\, X) := \nu_k(x)\, \Psi_{D^i_k}\Big(t,x,\, z^0,\, \sum_{\lambda=1,2} z^\lambda\, V^{\mu,\lambda}(x) + X\, \nu(x)\Big)\,. \tag{105}$$

We observe that

$$\begin{aligned}&\mathcal{F}_{i,X_j}(t,x,\, z^0,\, z^1,\, z^2,\, X) \\ &\quad = \sum_{k,\ell=1}^3 \nu_k(x)\, \nu_\ell(x)\, \Psi_{D^i_k, D^j_\ell}\Big(t,x,\, z^0,\, \sum_{\lambda=1,2} z^\lambda\, V^{\mu,\lambda}(x) + X\, \nu(x)\Big)\,.\end{aligned}$$

Thus, the matrix F_X is strictly positive definite, and in fact, recalling that we assume (26)

$$\mathcal{F}_X \eta \cdot \eta \geq \nu_0(z^0, z^1, z^2, X)\, \eta^2 \text{ for all } \eta \in \mathbb{R}^N\,.$$

The implicit function theorem implies the existence of an function $D \in C^1(\overline{S}_T \times (\mathbb{R}^N)^3;\, \mathbb{R}^N)$ such that all solutions to $\mathcal{F}(t,x,\, z^0,\, z^1,\, z^2,\, X) = 0$ are globally described by the equation

$$X = D(t,x,\, z^0,\, z^1,\, z^2)\,. \tag{106}$$

Moreover, the derivatives of the function D are given by

$$\begin{aligned}D_t(t,x,\, z^0,\, z^1,\, z^2) &= -[F_X(t,x,\, z^0,\, z^1,\, z^2,\, D)]^{-1}\, F_t(t,x,\, z^0,\, z^1,\, z^2,\, D) \\ D_x(t,x,\, z^0,\, z^1,\, z^2) &= -[F_X(t,x,\, z^0,\, z^1,\, z^2,\, D)]^{-1}\, F_x(t,x,\, z^0,\, z^1,\, z^2,\, D) \\ D_{z^\lambda} &= -[F_X(t,x,\, z^0,\, z^1,\, z^2,\, D)]^{-1}\, F_{z^\lambda}(t,x,\, z^0,\, z^1,\, z^2,\, D)\,.\end{aligned} \tag{107}$$

Now, we observe that the validity of (87) is equivalent with

$$\mathcal{F}(t,x,\, w(x),\, V^{\mu,1} \cdot \nabla w(x),\, V^{\mu,2} \cdot \nabla w(x),\, \nu \cdot \nabla w(x)) = 0\,.$$

This implies that

$$\nu \cdot \nabla w(x) = D(t,x,\, w(x),\, V^{\mu,1} \cdot \nabla w(x),\, V^{\mu,2} \cdot \nabla w)\,.$$

Thus, for all $\alpha = P(3,\, \mu,\, i)$,

$$u_\alpha = D(t,x,\, \bar{u})\,,$$

where $\bar{u}$ is the vector $(u_1, u_2, \ldots, u_{K'})$.

Thus, the boundary operator associated with the equations (102) has the structure

$$-\nu_k\, M_{\alpha,\beta,k,\ell}(t,x,\, u)\, u_{\beta,x_\ell} = f_{\Gamma,\alpha}(t,x,\, u) \text{ for } \alpha = 1,\ldots,K' \tag{108}$$

$$u_\alpha = D_\alpha(t,x,\, \bar{u}) \text{ for } \alpha = K'+1,\ldots,K\,. \tag{109}$$

6.3 Proof of the main result

The problem (102), (108) possesses the structure of a quasilinear system with sub diagonal coupling of leading order. This problem was treated in the Section 5. We next can verify that the coefficients in these relations satisfy the assumptions of Theorem 5.2.

As to the regularity assumptions, the ellipticity assumptions and the sub diagonal structure, this is a straightforward matter. The validity of (91) guaranties that the assumptions (78) and (79) in the statement of Theorem 5.2. To see this in the case of (78), we make use for the N first equations of the relation (90). For the equations with index $N+1, \ldots, K'$, we differentiate tangentially with respect to $\partial\Omega$ the condition (91). This implies (78)

In the case of (79), the implicit definition (106) of the function D in the equations with index $\alpha = K'+1, \ldots, K$ shows that (91) is in fact equivalent with $\nu \cdot \nabla w = D(t, x, w, V^{\mu,1} \cdot \nabla w, V^{\mu,2} \cdot \nabla w)$ for $(t, x) \in [0, T] \times \partial\Omega_{\mu}$.

Thus, it remains to verify the condition (80). This computation is lengthy, but straightforward if we take into account the formulas (107). It turns out that the condition is precisely (92). Applying the Theorem 5.2, we deduce Theorem 1.1.

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