Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Dimension of hyperbolic measures – A proof of the Eckmann–Ruelle conjecture

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submitted: 30th May 1996

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Preprint No. 245
Berlin 1996

1991 Mathematics Subject Classification. 58F11, 28D05.
Key words and phrases. Eckmann–Ruelle conjecture, hyperbolic measures, pointwise dimension.

This paper was written while L.B. was on leave from Instituto Superior Técnico, Department of Mathematics, at Lisbon, Portugal, and J.S. was visiting Penn State. L.B. was supported by Program PRAXIS XXI, Fellowship BD 5236/95, JNICT, Portugal. J.S. was supported by the Leopoldina–Förderpreis. The work of Y.P. was partially supported by the National Science Foundation grant #DMS9403723.

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ABSTRACT. We prove the long-standing Eckmann–Ruelle conjecture in dimension theory of smooth dynamical systems. Namely, we show that the pointwise dimension exists almost everywhere with respect to a Borel probability measure with non-zero Lyapunov exponents invariant under a $C^{1+\alpha}$ diffeomorphism of a smooth Riemannian manifold. This implies in particular that the Hausdorff dimension and box dimension of the measure as well as some other characteristics of dimension type of the measure coincide.

INTRODUCTION

In this paper we obtain an affirmative solution of the long-standing problem in the interface of dimension theory and dynamical systems known as the Eckmann–Ruelle conjecture. This problem was explicitly mentioned as an important open problem by Young at her ICM address [Y3, p.1232] (see also [Y2, p.318]). It can be stated as follows. Let M be a compact smooth Riemannian manifold without boundary, and $f: M \to M$ a $C^{1+\alpha}$ diffeomorphism of M. Let also μ be an f-invariant Borel probability measure on M.

Conjecture. *Assume that μ is hyperbolic, i.e., all the Lyapunov exponents of f do not vanish at μ-almost all points. Then the following limit exists*

$$d(x) \stackrel{\text{def}}{=} \lim_{r \to \infty} \frac{\log \mu(B(x,r))}{\log r} \tag{1}$$

for μ-almost every point $x \in M$ (where $B(x,r)$ denotes the ball of radius r centered at x).

The limit in (1) is called *the pointwise dimension* of μ at x. If this limit does not exist one can consider the lower and upper limits and introduce respectively *the lower and upper pointwise dimension* of μ at x which we denote by $\underline{d}(x)$ and $\overline{d}(x)$. The functions $\underline{d}(x)$ and $\overline{d}(x)$ are measurable and invariant under f.

The existence of the limit in (1) for a Borel probability measure μ on M implies the crucial fact that **all** characteristics of dimension type of the measure coincide (see Proposition 1 below). The common value is a fundamental characteristic of the fractal structure of μ — the *fractal dimension* of μ. It is intimately related to stochastic properties of f with respect to μ.

Since hyperbolic measures play a crucial role in studying physical models with persistent chaotic behavior and fractal structure of invariant sets, the conjecture can be viewed as a mathematical foundation for such study.

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We describe some most important characteristics of dimension type (see for example [F,P]). Let X be a complete separable metric space. For a subset $Z \subset X$ and a number $\alpha \geq 0$, we define the *α-Hausdorff measure of Z* by

$$m_H(Z,\alpha) = \lim_{\varepsilon \to 0} \inf_{\mathcal{G}} \sum_{U \in \mathcal{G}} (\text{diam } U)^\alpha$$

where the infimum is taken over all finite or countable coverings $\mathcal{G}$ of Z by open sets with $\text{diam}\, \mathcal{G} \leq \varepsilon$. We now define the *Hausdorff dimension of Z*, denoted $\dim_H Z$, by

$$\dim_H Z = \inf\{\alpha : m_H(Z,\alpha) = 0\} = \sup\{\alpha : m_H(Z,\alpha) = \infty\}.$$

The Hausdorff dimension was introduced by Hausdorff in 1919. It has since become one of the fundamental notions in dimension theory that is used to characterize sets with complicated "fractal" structure.

We now define the *lower* and *upper box dimensions* of Z (denoted respectively by $\underline{\dim}_B Z$ and $\overline{\dim}_B Z$) by

$$\underline{\dim}_B Z = \inf\{\alpha : \underline{r}_H(Z,\alpha) = 0\} = \sup\{\alpha : \underline{r}_H(Z,\alpha) = \infty\},$$
$$\overline{\dim}_B Z = \inf\{\alpha : \overline{r}_H(Z,\alpha) = 0\} = \sup\{\alpha : \overline{r}_H(Z,\alpha) = \infty\}$$

where

$$\underline{r}_H(Z,\alpha) = \underline{\lim}_{\varepsilon \to 0} \inf_{\mathcal{G}} \sum_{U \in \mathcal{G}} \varepsilon^\alpha \quad \overline{r}_H(Z,\alpha) = \overline{\lim}_{\varepsilon \to 0} \inf_{\mathcal{G}} \sum_{U \in \mathcal{G}} \varepsilon^\alpha$$

and the infimum is taken over all finite or countable coverings $\mathcal{G}$ of Z by open sets of diameter ε. It is easy to see that

$$\dim_H Z \leq \underline{\dim}_B Z \leq \overline{\dim}_B Z.$$

It is now accepted by most experts in dimension theory that the coincidence of the Hausdorff dimension and lower and upper box dimensions of sets is a relatively rare phenomenon and can occur only in some "rigid" situations. One well-known class of sets for which the coincidence usually takes place is the class of Cantor-like sets, i.e., the limit sets for some geometric constructions (see for example [F,PW,B]).

In order to describe the geometric structure of a subset Z invariant under a dynamical system f acting on X, we consider a measure μ supported on Z and introduce the notion of the *Hausdorff dimension of μ* and *lower* and *upper box dimensions of μ* (we denote them by $\dim_H \mu$, $\underline{\dim}_B \mu$, and $\overline{\dim}_B \mu$ respectively). We have

$$\dim_H \mu = \inf\{\dim_H Z \mid \mu(Z) = 1\},$$

$$\underline{\dim}_B \mu = \lim_{\delta \to 0} \inf\{\underline{\dim}_B Z \mid \mu(Z) \geq 1 - \delta\},$$

$$\overline{\dim}_B \mu = \lim_{\delta \to 0} \inf\{\overline{\dim}_B Z \mid \mu(Z) \geq 1 - \delta\}.$$

It follows from the definition that

$$\dim_H \mu \le \underline{\dim}_B \mu \le \overline{\dim}_B \mu.$$

There is another important characteristic of dimension type of μ called the *information dimension* of μ. Given a Borel probability measure μ on X and a partition ξ of X, we define the *entropy of ξ with respect to μ* by

$$H_\mu(\xi) = -\sum_{C_\xi} \mu(C_\xi) \log \mu(C_\xi)$$

where C_ξ is an element of the partition ξ. Given a number $\varepsilon > 0$, we set

$$H_\mu(\varepsilon) = \inf \{H_\mu(\xi) : \operatorname{diam} \xi \le \varepsilon\}$$

where $\operatorname{diam} \xi = \max \operatorname{diam} C_\xi$.

We define the *lower* and *upper information dimensions of μ* by

$$\underline{I}(\mu) = \varliminf_{\varepsilon \to 0} \frac{H_\mu(\varepsilon)}{\log(1/\varepsilon)}, \quad \overline{I}(\mu) = \varlimsup_{\varepsilon \to 0} \frac{H_\mu(\varepsilon)}{\log(1/\varepsilon)}.$$

There is a powerful criterion stated by Young in [Y1] that guarantees the coincidence of the Hausdorff dimension and lower and upper box dimensions of measures as well as their lower and upper information dimensions.

Proposition 1. *Let X be a compact separable metric space of finite topological dimension and μ a Borel probability measure on X. Assume that*

$$\underline{d}(x) = \overline{d}(x) = d \tag{2}$$

for μ-almost every $x \in X$. Then

$$\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = \underline{I}(\mu) = \overline{I}(\mu) = d.$$

A measure μ which satisfies (2) is called *exact dimensional.*

In [P], many other characteristics of dimension type of measures are introduced including *correlation dimension* and *modified Hentschel–Procaccia spectrum for dimensions.* It is shown in [P] that if a measure μ is exact dimensional then all these characteristics coincide and the common value is equal to d.

The Eckmann–Ruelle conjecture provides the most broad class of measures invariant under smooth dynamical systems which are exact dimensional.

In [Y1], Young obtained the affirmative solution to this conjecture for surface diffeomorphisms.

Proposition 2. *Let f be a $C^{1+\alpha}$ diffeomorphism of a smooth compact surface M and μ a hyperbolic ergodic measure with Lyapunov exponents $\lambda_1 > 0 > \lambda_2$. Then*

$$\underline{d} = \overline{d} = h_\mu(f)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right).$$

In [L], Ledrappier proved the conjecture for general SRB-measures (after Sinai, Ruelle, and Bowen). In [PY], Pesin and Yue extended his approach and proved the conjecture for hyperbolic measures satisfying the so-called semi-local product structure (this class includes, for example, Gibbs measures on locally maximal hyperbolic sets).

Let us also point out that neither of the assumptions in the Eckmann–Ruelle conjecture can be omitted. Ledrappier and Misiurewicz [LM] constructed an example of a smooth map of a circle preserving an ergodic measure with zero Lyapunov exponent which is not exact dimensional. In [PW], Pesin and Weiss presented an example of a Hölder homeomorphism with Hölder constant arbitrarily close to 1 whose measure of maximal entropy is not exact dimensional.

Preliminaries

Let M be a smooth Riemannian manifold without boundary, and $f: M \to M$ a $C^{1+\alpha}$ diffeomorphism on M. Let also μ be an f-invariant ergodic Borel probability measure on M.

Given $x \in M$ and $v \in T_xM$ define the *Lyapunov exponent of v at x* by the formula

$$\lambda(x, v) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log \|d_x f^n v\|.$$

If x is fixed then the function $\lambda(x, \cdot)$ can take on only finitely many values $\lambda_1(x) \geq \cdots \geq \lambda_p(x)$ where $p = \dim M$. The functions $\lambda_i(x)$ are measurable and f-invariant. Since μ is ergodic, these functions are constant μ-almost everywhere. We denote these constants by $\lambda_1 \geq \cdots \geq \lambda_p$. The measure μ is said to be *hyperbolic* if $\lambda_i \neq 0$ for every $i = 1, \ldots, p$.

There exists a measurable function $r(x) > 0$ such that for μ-almost every $x \in M$ the sets

$$W^s(x) = \left\{ y \in B(x, r(x)) \mid \overline{\lim_{n\to+\infty}} \frac{1}{n} \log d(f^n x, f^n y) < 0 \right\},$$

$$W^u(x) = \left\{ y \in B(x, r(x)) \mid \underline{\lim_{n\to-\infty}} \frac{1}{n} \log d(f^n x, f^n y) > 0 \right\}$$

are immersed local manifolds called *stable* and *unstable local manifolds* at x. For each $0 < r < r(x)$ we consider the balls $B^s(x, r) \subset W^s(x)$ and $B^u(x, r) \subset W^u(x)$ centered at x with respect to the induced distances on $W^s(x)$ and $W^u(x)$ respectively.

Let ξ be a measurable partition of M. It has a canonical systems of conditional measures: for μ-almost every x there is a probability measure μ_x defined on the element $\xi(x)$ of ξ containing x. The conditional measures μ_x are uniquely characterized by the following property: if $\oplus B_\xi$ is the σ-subalgebra (of the Borel σ-algebra) whose elements are unions of elements of ξ, and $A \subset M$ is a measurable set, then $x \mapsto \mu_x(A \cap \xi(x))$ is $\oplus B_\xi$-measurable and $\mu(A) = \int \mu_x(A \cap \xi(x))\, d\mu(x)$.

In [LY], Ledrappier and Young constructed two measurable partitions ξ^s and ξ^u of M such that for μ-almost every $x \in M$:

(1) $\xi^s(x) \subset W^s(x)$ and $\xi^u(x) \subset W^u(x)$;
(2) $\xi^s(x)$ and $\xi^u(x)$ contain the intersection of an open neighborhood of x with $W^s(x)$ and $W^u(x)$ respectively.

We denote the system of conditional measures of μ with respect to the partitions ξ^s and ξ^u respectively by μ_x^s and μ_x^u, and for any measurable set $A \subset M$ we write $\mu_x^s(A) = \mu_x^s(A \cap \xi^s(x))$ and $\mu_x^u(A) = \mu_x^u(A \cap \xi^u(x))$.

Given $x \in M$, consider the lower and upper pointwise dimensions of μ at x, $\underline{d}(x)$ and $\overline{d}(x)$. Since these functions are measurable and f-invariant they are constant μ-almost everywhere. We denote these constants by $\underline{d}$ and $\overline{d}$ respectively.

In [LY], Ledrappier and Young introduced the quantities

$$d^s(x) \stackrel{\text{def}}{=} \lim_{r\to 0} \frac{\log \mu_x^s(B^s(x,r))}{\log r},$$

$$d^u(x) \stackrel{\text{def}}{=} \lim_{r\to 0} \frac{\log \mu_x^u(B^u(x,r))}{\log r}$$

provided that the corresponding limits exist at $x \in M$. We call them respectively *stable* and *unstable pointwise dimensions* of μ.

Proposition 3 [LY].

(1) *For μ-almost every $x \in M$ the limits $d^s(x)$ and $d^u(x)$ exist and are constant μ-almost everywhere; we denote these constants by d^s and d^u.*
(2) *If μ is a hyperbolic measure then*

$$\overline{d} \le d^s + d^u.$$

When the entropy of f is zero this inequality implies that $\underline{d} = \overline{d} = d^s + d^u = 0$.

In this paper we prove the following statement.

Theorem. *Let f be a $C^{1+\alpha}$ diffeomorphism on a smooth Riemannian manifold without boundary, and μ an f-invariant compactly supported ergodic Borel probability measure. If μ is hyperbolic then it is exact dimensional and its pointwise dimension is equal to the sum of the stable and unstable pointwise dimensions, i.e.,*

$$\underline{d} = \overline{d} = d^s + d^u.$$

Remarks.

(1) The assumption in the theorem that the hyperbolic measure is ergodic is not essential and the theorem holds for any f-invariant compactly supported Borel

probability measure that is hyperbolic. This can be verified in a standard way using the ergodic decomposition of the measure.

(2) It follows immediately from the theorem that the pointwise dimension of an ergodic invariant measure supported on a (uniformly) hyperbolic locally maximal set is exact dimensional. This result has not been known before. We emphasize that in this situation the stable and unstable foliations need not be Lipschitz (in fact they are "generically" not Lipschitz; see [S]), and in general the measure need not have a local product structure despite the fact that the set itself does. This illustrates that the theorem is non-trivial even for measures supported on hyperbolic locally maximal sets.

Description of a special partition

We use the following notations. Let η be a partition. For every integers k, $l \geq 1$, we define the partition $\eta_k^l = \bigvee_{n=-k}^{l} f^{-n}\eta$. We observe that $\eta_k^0(x) \cap \eta_0^l(x) = \eta_k^l(x)$.

¿From now on we assume that μ is hyperbolic. In [LY], Ledrappier and Young constructed a special countable partition $\oplus P$ of M of finite entropy satisfying the following properties. Given $0 < \varepsilon < 1$, there exists a set $\Gamma \subset M$ of measure $\mu(\Gamma) > 1 - \varepsilon/2$, an integer $n_0 \geq 1$, and a number $C > 1$ such that for every $x \in \Gamma$ and any integer $n \geq n_0$, the following statements hold:

(a) for all integers k, $l \geq 1$ we have

$$C^{-1}e^{-(l+k)h-(l+k)\varepsilon} \leq \mu(\oplus P_k^l(x)) \leq Ce^{-(l+k)h+(l+k)\varepsilon}, \tag{3}$$

$$C^{-1}e^{-kh-k\varepsilon} \leq \mu_x^s(\oplus P_k^0(x)) \leq Ce^{-kh+k\varepsilon}, \tag{4}$$

$$C^{-1}e^{-lh-l\varepsilon} \leq \mu_x^u(\oplus P_0^l(x)) \leq Ce^{-lh+l\varepsilon}, \tag{5}$$

where h is the Kolmogorov–Sinai entropy of f with respect to μ;

(b)

$$\xi^s(x) \cap \bigcap_{n\geq 0} \oplus P_0^n(x) \supset B^s(x, e^{-n_0}), \tag{6}$$

$$\xi^u(x) \cap \bigcap_{n\geq 0} \oplus P_n^0(x) \supset B^u(x, e^{-n_0}); \tag{7}$$

(c)

$$e^{-d^s n-n\varepsilon} \leq \mu_x^s(B^s(x, e^{-n})) \leq e^{-d^s n+n\varepsilon}, \tag{8}$$

$$e^{-d^u n-n\varepsilon} \leq \mu_x^u(B^u(x, e^{-n})) \leq e^{-d^u n+n\varepsilon}; \tag{9}$$

(d)

$$\oplus P_{an}^{an}(x) \subset B(x, e^{-n}) \subset \oplus P(x), \tag{10}$$

$$\oplus P_{an}^0(x) \cap \xi^s(x) \subset B^s(x, e^{-n}) \subset \oplus P(x) \cap \xi^s(x), \tag{11}$$

$$\oplus P_0^{an}(x) \cap \xi^u(x) \subset B^u(x, e^{-n}) \subset \oplus P(x) \cap \xi^u(x), \tag{12}$$

where a is the integer part of $2(1+\varepsilon)\max\{1/\lambda_1, -1/\lambda_p, 1\}$;

(e) if $Q_n(x)$ is defined by

$$\{y \in M \mid \oplus P_0^{an}(y) \cap B^u(x, 2e^{-n}) \neq \varnothing \text{ and } \oplus P_{an}^0(y) \cap B^s(x, 2e^{-n}) \neq \varnothing\} \tag{13}$$

then

$$B(x, e^{-n}) \cap \Gamma \subset Q_n(x) \subset B(x, 4e^{-n}); \tag{14}$$

it clearly follows from the definition of $Q_n(x)$ that for each $y \in Q_n(x)$,

$$\oplus P_{an}^{an}(y) \subset Q_n(x).$$

We can also assume that

(f) for every $x \in \Gamma$ and $n \geq n_0$, we have

$$B^s(x, e^{-n}) \cap \Gamma \subset Q_n(x) \cap \xi^s(x) \subset B^s(x, 4e^{-n}), \tag{15}$$

$$B^u(x, e^{-n}) \cap \Gamma \subset Q_n(x) \cap \xi^u(x) \subset B^u(x, 4e^{-n}). \tag{16}$$

The above statements are slightly different versions of statements in [LY]. Property (3) essentially follows from the Shannon–McMillan–Breiman theorem applied to the partition $\oplus P$ while properties (4) and (5) follow from "leaf-wise" versions of this theorem. The inequalities in (8) and (9) are easy consequences of the existence of the stable and unstable pointwise dimensions d^s and d^u (see Proposition 3). Since the Lyapunov exponents at μ-almost every point are constant equal to $\lambda_1, \ldots, \lambda_p$, the properties (10), (11), and (12) follow from (6), (7), and the choice of a indicated above. The inclusions in (14) are based upon the continuous dependence of stable and unstable manifolds in the $C^{1+\alpha}$ topology on the base point (in each Pesin set). We need the following well-known result.

Borel Density Lemma. *Let μ be a finite Borel measure and $A \subset M$ a measurable set. Then for μ-almost every $x \in A$, we have*

$$\lim_{r \to 0} \frac{\mu(B(x,r) \cap A)}{\mu(B(x,r))} = 1.$$

Furthermore, if $\mu(A) > 0$ then, for each $\delta > 0$, there is a set $\Delta \subset A$ with $\mu(\Delta) > \mu(A) - \delta$, and a number $r_0 > 0$ such that for all $x \in \Delta$ and $0 < r < r_0$, we have

$$\mu(B(x,r) \cap A) \geq \frac{1}{2}\mu(B(x,r)).$$

It immediately follows from the Borel Density Lemma that one can choose an integer $n_1 \geq n_0$ and a set $\widehat{\Gamma} \subset \Gamma$ of measure $\mu(\widehat{\Gamma}) > 1 - \varepsilon$ such that for every $n \geq n_1$ and $x \in \widehat{\Gamma}$,

$$\mu(B(x, e^{-n}) \cap \Gamma) \geq \frac{1}{2}\mu(B(x, e^{-n})); \tag{17}$$

$$\mu_x^s(B^s(x, e^{-n}) \cap \Gamma) \geq \frac{1}{2}\mu_x^s(B^s(x, e^{-n})); \tag{18}$$

$$\mu_x^u(B^u(x, e^{-n}) \cap \Gamma) \geq \frac{1}{2}\mu_x^u(B^u(x, e^{-n})). \tag{19}$$

We establish two additional properties of the partitions $\oplus P_0^k$ and $\oplus P_k^0$.

Proposition 4. *There exists a positive constant* $D = D(\widehat{\Gamma}) < 1$ *such that for every* $k \geq 1$ *and* $x \in \Gamma$, *we have:*

$$\mu_x^s(\oplus P_0^k(x) \cap \Gamma) \geq D;$$

$$\mu_x^u(\oplus P_k^0(x) \cap \Gamma) \geq D.$$

Proof. By (6), for every $k \geq 1$ and $x \in \Gamma$, the set $\oplus P_0^k(x) \cap \Gamma$ contains the set $B^s(x, e^{-n_0}) \cap \Gamma$. It follows from (18) and (8) that

$$\mu_x^s(\oplus P_0^k(x) \cap \Gamma) \geq \frac{1}{2}\mu_x^s(B^s(x, e^{-n_0})) \geq \frac{1}{2}e^{-d^s n_0 - n_0\varepsilon} \stackrel{\text{def}}{=} D.$$

The second inequality in the proposition can be proved in a similar fashion using the properties (7), (19), and (9). ☐

The next statement establishes the property of the partition $\oplus P$ which simulates the well-known Markov property.

Proposition 5. *For every* $x \in \Gamma$ *and* $n \geq n_0$, *we have:*

$$\oplus P_{an}^{an}(x) \cap \xi^s(x) = \oplus P_{an}^0(x) \cap \xi^s(x);$$

$$\oplus P_{an}^{an}(x) \cap \xi^u(x) = \oplus P_0^{an}(x) \cap \xi^u(x).$$

Proof. It follows from (11) and (6) that

$$\begin{aligned} \oplus P_{an}^0(x) \cap \xi^s(x) &\subset \oplus P_{an}^0(x) \cap B^s(x, e^{-n}) \subset \oplus P_{an}^0(x) \cap B^s(x, e^{-n_0}) \\ &\subset \oplus P_{an}^0(x) \cap \oplus P_0^{an}(x) \cap \xi^s(x) = \oplus P_{an}^{an}(x) \cap \xi^s(x). \end{aligned}$$

Since $\oplus P_{an}^{an}(x) \subset \oplus P_{an}^0(x)$ this completes the proof of the first identity. The proof of the other identity is similar. ☐

Preparational lemmata

Fix $x \in \widehat{\Gamma}$ and an integer $n \geq n_1$. We consider the following two classes $\oplus R(n)$ and $\oplus F(n)$ of elements of the partition $\oplus P_{an}^{an}$ (we call these elements "rectangles"):

$$\oplus R(n) = \{\oplus P_{an}^{an}(y) \subset \oplus P(x) \mid \oplus P_{an}^{an}(y) \cap \Gamma \neq \varnothing\};$$

$$\oplus F(n) = \{\oplus P_{an}^{an}(y) \subset \oplus P(x) \mid \oplus P_{an}^0(y) \cap \widehat{\Gamma} \neq \varnothing \text{ and } \oplus P_0^{an}(y) \cap \widehat{\Gamma} \neq \varnothing\}.$$

The rectangles in $\oplus R(n)$ carry all the measure of the set $\oplus P(x) \cap \Gamma$, i.e., $\sum_{R \in \oplus R(n)} \mu(R \cap \Gamma) = \mu(\oplus P(x) \cap \Gamma)$. Obviously, the rectangles in $\oplus R(n)$ that intersect $\widehat{\Gamma}$ belong to $\oplus F(n)$. If these were the only ones in $\oplus F(n)$, the measure $\mu|\oplus P(x) \cap \Gamma$ would have the "direct product structure" at the "level" n. One could then use the approach in [L,PY] to estimate the measure of a ball by the product of its stable and unstable measures. In the general case, the rectangles in the class $\oplus F(n)$ are obtained from the rectangles in $\oplus R(n)$ (that intersect $\widehat{\Gamma}$) by "filling in" the gaps in the "product structure" (see Figure 1).

We wish to compare the number of rectangles in $\oplus R(n)$ and $\oplus F(n)$ intersecting a given set. This will allow us to evaluate the deviation of the measure μ from the direct product structure at the level n. Our main observation is that for "typical" points $y \in \widehat{\Gamma}$ the number of rectangles from the class $\oplus R(n)$ intersecting $W^s(y)$ (respectively $W^u(y)$) is "asymptotically" the same up to a factor that grows at most subexponentially with n. However, in general, the distribution of these rectangles along $W^s(y)$ (respectively $W^u(y)$) may be different for different points y. This causes a deviation from the direct product structure. We will use a simple combinatorial argument to show that this deviation grows at most subexponentially with n. One can then say that the measure μ has an "almost direct product structure".

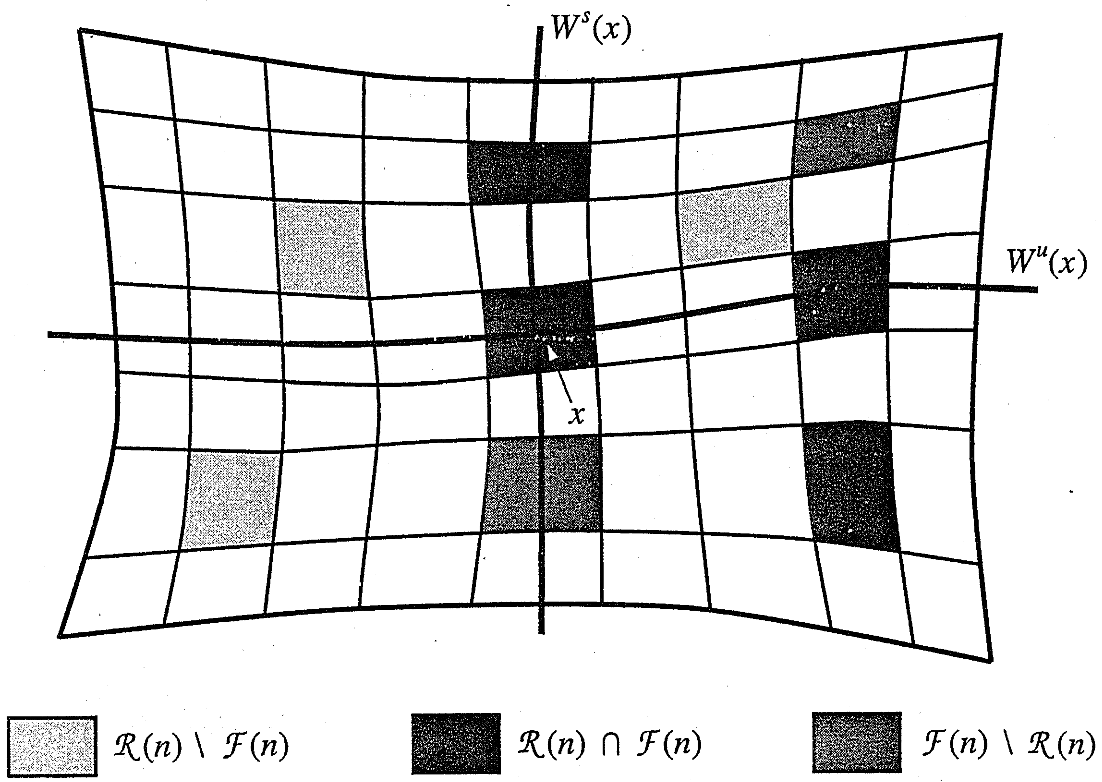

FIGURE 1. The procedure of "filling in" rectangles.

To effect this, for each set $A \subset \oplus P(x)$, we define

$$N(n, A) = \operatorname{card}\{R \in \oplus R(n) \mid R \cap A \neq \varnothing\},$$

$$N^s(n, y, A) = \operatorname{card}\{R \in \oplus R(n) \mid R \cap \xi^s(y) \cap \Gamma \cap A \neq \varnothing\},$$

$$N^u(n, y, A) = \operatorname{card}\{R \in \oplus R(n) \mid R \cap \xi^u(y) \cap \Gamma \cap A \neq \varnothing\},$$

$$\widehat{N}^s(n, y, A) = \operatorname{card}\{R \in \oplus F(n) \mid R \cap \xi^s(y) \cap A \neq \varnothing\},$$

$$\widehat{N}^u(n,y,A) = \operatorname{card}\{R \in \oplus F(n) \mid R \cap \xi^u(y) \cap A \neq \varnothing\}.$$

Note that $N(n, \oplus P(x))$ is the cardinality of the set $\oplus R(n)$, and $N^s(n, y, \oplus P(x))$ (respectively $N^u(n, y, \oplus P(x))$) is the number of rectangles in $\oplus R(n)$ that intersect Γ and the stable (respectively unstable) local manifold at y. The product $\widehat{N}^s(n, y, \oplus P(x)) \times \widehat{N}^u(n, y, \oplus P(x))$ is the cardinality of the set $\oplus F(n)$ for a "typical" point $y \in \oplus P(x)$.

Lemma 1. *For each $y \in \oplus P(x) \cap \Gamma$ and integer $n \geq n_0$, we have:*

$$N^s(n, y, Q_n(y)) \leq \mu_y^s(B^s(y, 4e^{-n})) \cdot Ce^{anh+an\varepsilon};$$

$$N^u(n, y, Q_n(y)) \leq \mu_y^u(B^u(y, 4e^{-n})) \cdot Ce^{anh+an\varepsilon}.$$

where $Q_n(y)$ is defined by (13).

Proof. It follows from (15) that

$$\mu_y^s(B^s(y, 4e^{-n})) \geq \mu_y^s(Q_n(y))$$

$$\geq N^s(n, y, Q_n(y)) \cdot \min\{\mu_y^s(R) \mid R \in \oplus R(n) \text{ and } R \cap \xi^s(y) \cap \Gamma \cap Q_n(y) \neq \varnothing\}.$$

Let $z \in R \cap \xi^s(y) \cap \Gamma \cap Q_n(y)$ for some $R \in \oplus R(n)$. By Proposition 5 we obtain $\mu_y^s(R) = \mu_y^s(\oplus P_{an}^0(z)) = \mu_z^s(\oplus P_{an}^0(z))$. The first inequality in the lemma follows now from (4).

The proof of the second inequality is similar. □

Lemma 2. *For each $y \in \oplus P(x) \cap \widehat{\Gamma}$ and integer $n \geq n_1$, we have:*

$$\mu(B(y, e^{-n})) \leq N(n, Q_n(y)) \cdot 2Ce^{-2anh+2an\varepsilon}.$$

Proof. It follows from (17) and (14) that

$$\frac{1}{2}\mu(B(y, e^{-n})) \leq \mu(B(y, e^{-n}) \cap \Gamma) \leq \mu(Q_n(y) \cap \Gamma)$$

$$\leq N(n, Q_n(y)) \cdot \max\{\mu(R) \mid R \in \oplus R(n) \text{ and } R \cap Q_n(y) \neq \varnothing\}.$$

The desired inequality follows from (3). □

We now estimate the number of rectangles in the classes $\oplus R(n)$ and $\oplus F(n)$.

Lemma 3. *For μ-almost every $y \in \oplus P(x) \cap \widehat{\Gamma}$ there is an integer $n_2(y) \geq n_1$ such that for each $n \geq n_2(y)$, we have:*

$$N(n+2, Q_{n+2}(y)) \leq \widehat{N}^s(n, y, Q_n(y)) \cdot \widehat{N}^u(n, y, Q_n(y)) \cdot 2C^2 e^{4a(h+\varepsilon)} e^{4an\varepsilon}.$$

Proof. By the Borel Density Lemma (with $A = \widehat{\Gamma}$), for μ-almost every $y \in \widehat{\Gamma}$ there is an integer $n_2(y) \geq n_1$ such that for all $n \geq n_2(y)$,

$$2\mu(B(y, e^{-n}) \cap \widehat{\Gamma}) \geq \mu(B(y, e^{-n})).$$

Since $\widehat{\Gamma} \subset \Gamma$, it follows from (14) that for all $n \geq n_2(y)$,

$$\begin{aligned} 2\mu(Q_n(y) \cap \widehat{\Gamma}) &\geq 2\mu(B(y, e^{-n}) \cap \widehat{\Gamma}) \geq \mu(B(y, e^{-n})) \\ &\geq \mu(B(y, 4e^{-n-2})) \geq \mu(Q_{n+2}(y)). \end{aligned} \tag{20}$$

For any $m \geq n_2(y)$, by (3) and property (e), we have

$$\mu(Q_m(y)) = \sum_{\oplus P_{am}^{am}(z) \subset Q_m(y)} \mu(\oplus P_{am}^{am}(z)) \geq N(m, Q_m(y)) \cdot C^{-1} e^{-2amh-2am\varepsilon}.$$

Similarly, for every $n \geq n_2(y)$, we obtain

$$\mu(Q_n(y) \cap \widehat{\Gamma}) = \sum_{\oplus P_{an}^{an}(z) \subset Q_n(y)} \mu(\oplus P_{an}^{an}(z) \cap \widehat{\Gamma}) \leq N_n \cdot C e^{-2anh+2an\varepsilon},$$

where N_n is the number of rectangles $\oplus P_{an}^{an}(z) \in \oplus R(n)$ that have non-empty intersection with $\widehat{\Gamma}$.

Set $m = n + 2$. The last two inequalities together with (20) imply that

$$N(n+2, Q_{n+2}(y)) \leq N_n \cdot 2C^2 e^{4a(h+\varepsilon)+4an\varepsilon}. \tag{21}$$

On the other hand, since $y \in \widehat{\Gamma}$ the intersections $\oplus P_0^{an}(y) \cap \xi^u(y) \cap \widehat{\Gamma}$ and $\oplus P_{an}^0(y) \cap \xi^s(y) \cap \widehat{\Gamma}$ are non-empty.

Consider a rectangle $\oplus P_{an}^{an}(v) \subset Q_n(y)$ that has non-empty intersection with $\widehat{\Gamma}$. Then the rectangles $\oplus P_{an}^0(v) \cap \oplus P_0^{an}(y)$ and $\oplus P_{an}^0(y) \cap \oplus P_0^{an}(v)$ are in $\oplus F(n)$ and intersect respectively the stable and unstable local manifolds at y. Hence, one can associate to any rectangle $\oplus P_{an}^{an}(v) \subset Q_n(y)$ in $\oplus R(n)$ that has non-empty intersection with $\widehat{\Gamma}$, the pair of rectangles $(\oplus P_{an}^0(v) \cap \oplus P_0^{an}(y), \oplus P_{an}^0(y) \cap \oplus P_0^{an}(v))$ in

$$\{R \in \oplus F(n) \mid R \cap \xi^s(y) \cap Q_n(y) \neq \varnothing\} \times \{R \in \oplus F(n) \mid R \cap \xi^u(y) \cap Q_n(y) \neq \varnothing\}.$$

Clearly this correspondence is injective. Therefore,

$$\widehat{N}^s(n, y, Q_n(y)) \cdot \widehat{N}^u(n, y, Q_n(y)) \geq N_n.$$

The desired inequality follows from (21). □

Our next goal is to compare the growth rate in n of the number of rectangles in $\oplus F(n)$ with the number of rectangles in $\oplus R(n)$. We start with an auxiliary result.

Lemma 4. *For each $x \in \widehat{\Gamma}$ and integer $n \geq n_1$, we have:*

$$\widehat{N}^s(n, x, \oplus P(x)) \leq D^{-1} C^2 e^{anh+3an\varepsilon};$$

$$\widehat{N}^u(n, x, \oplus P(x)) \leq D^{-1} C^2 e^{anh+3an\varepsilon}.$$

Proof. Since the partition $\oplus P$ is countable we can find points y_i such that the union of the rectangles $\oplus P_0^{an}(y_i)$ is $\oplus P(x)$, and these rectangles are mutually disjoint. Without loss of generality we can assume that $y_i \in \widehat{\Gamma}$ whenever $\oplus P_0^{an}(y_i) \cap \widehat{\Gamma} \neq \varnothing$.

We have

$$N(n, \oplus P(x)) \geq \sum_i N^s(n, y_i, \oplus P_0^{an}(y_i)) \geq \sum_{i:\oplus P_0^{an}(y_i)\cap\widehat{\Gamma}\neq\varnothing} N^s(n, y_i, \oplus P_0^{an}(y_i)). \quad (22)$$

We now estimate $N^s(n, y_i, \oplus P_0^{an}(y_i))$ for $y_i \in \widehat{\Gamma}$ from below. By Propositions 4 and 5, and (4),

$$\begin{aligned} N^s(n, y_i, \oplus P_0^{an}(y_i)) &\geq \frac{\mu_{y_i}^s(\oplus P_0^{an}(y_i) \cap \Gamma)}{\max\{\mu_z^s(\oplus P_{an}^{an}(z)) \mid z \in \xi^s(y_i) \cap \oplus P(x) \cap \Gamma \neq \varnothing\}} \\ &\geq \frac{D}{\max\{\mu_z^s(\oplus P_{an}^{an}(z)) \mid z \in \xi^s(y_i) \cap \oplus P(x) \cap \Gamma \neq \varnothing\}} \\ &= \frac{D}{\max\{\mu_z^s(\oplus P_{an}^{0}(z)) \mid z \in \xi^s(y_i) \cap \oplus P(x) \cap \Gamma \neq \varnothing\}} \\ &\geq DC^{-1}e^{anh-an\varepsilon}. \end{aligned} \quad (23)$$

Similarly (3) implies that

$$N(n, \oplus P(x)) \leq \frac{\mu(\oplus P(x))}{\min\{\mu(\oplus P_{an}^{an}(z)) \mid z \in \oplus P(x) \cap \Gamma\}} \leq Ce^{2anh+2an\varepsilon}. \quad (24)$$

We now observe that

$$\widehat{N}^u(n, x, \oplus P(x)) = \operatorname{card}\{i \mid \oplus P_0^{an}(y_i) \cap \widehat{\Gamma} \neq \varnothing\}. \quad (25)$$

Putting (22), (23), (24), and (25) together we conclude that

$$\begin{aligned} Ce^{2anh+2an\varepsilon} &\geq N(n, \oplus P(x)) \\ &\geq \sum_{i:\oplus P_0^{an}(y_i)\cap\widehat{\Gamma}\neq\varnothing} N^s(n, y_i, \oplus P_0^{an}(y_i)) \\ &\geq \widehat{N}^u(n, x, \oplus P(x)) \cdot DC^{-1}e^{anh-an\varepsilon}. \end{aligned}$$

This yields $\widehat{N}^u(n, x, \oplus P(x)) \leq D^{-1}C^2e^{anh+3an\varepsilon}$. The other inequality can be proved in a similar way. □

We emphasize that the procedure of "filling in" rectangles to obtain the class $\oplus F(n)$ may substantially increase the number of rectangles in the neighborhood of some points. However, the next lemma shows that this procedure of "filling in" does not add too many rectangles at almost every point.

Lemma 5. *For μ-almost every $y \in \oplus P(x) \cap \widehat{\Gamma}$ we have:*

$$\varlimsup_{n\to+\infty} \frac{\widehat{N}^s(n,y,Q_n(y))}{N^s(n,y,Q_n(y))} e^{-7an\varepsilon} < 1;$$

$$\varlimsup_{n\to+\infty} \frac{\widehat{N}^u(n,y,Q_n(y))}{N^u(n,y,Q_n(y))} e^{-7an\varepsilon} < 1.$$

Proof. By (15) and (18), for each $n \geq n_1$ and $y \in \widehat{\Gamma}$,

$$\mu_y^s(Q_n(y)) \geq \mu_y^s(B^s(y,e^{-n}) \cap \Gamma) \geq \frac{1}{2}\mu_y^s(B^s(y,e^{-n})).$$

Since $\oplus P_{an}^{an}(z) \subset \oplus P_{an}^0(z)$ for every z, by virtue of (4) and (8) we obtain

$$\begin{aligned} N^s(n,y,Q_n(y)) &\geq \frac{\mu_y^s(Q_n(y))}{\max\{\mu_z^s(\oplus P_{an}^{an}(z)) \mid z \in \xi^s(y) \cap \oplus P(x) \cap \Gamma \neq \varnothing\}} \\ &\geq \frac{1}{2}\frac{\mu_y^s(B^s(y,e^{-n}))}{\max\{\mu_z^s(\oplus P_{an}^0(z)) \mid z \in \xi^s(y) \cap \oplus P(x) \cap \Gamma \neq \varnothing\}} \\ &\geq \frac{1}{2C}\frac{e^{-d^s n - n\varepsilon}}{e^{-anh+an\varepsilon}}. \end{aligned}$$

Let us consider the set

$$F = \left\{ y \in \widehat{\Gamma} \mid \varlimsup_{n\to+\infty} \frac{\widehat{N}^s(n,y,Q_n(y))}{N^s(n,y,Q_n(y))} e^{-7an\varepsilon} \geq 1 \right\}.$$

For each $y \in F$ there exists an increasing sequence $\{m_j\}_{j=1}^\infty = \{m_j(y)\}_{j=1}^\infty$ of positive integers such that

$$\widehat{N}^s(m_j,y,Q_{m_j}(y)) \geq \frac{1}{2}N^s(m_j,y,Q_{m_j}(y))e^{7am_j\varepsilon} \geq \frac{1}{4C}e^{-d^s m_j + am_j h + 5am_j\varepsilon} \tag{26}$$

for all j (note that $a > 1$).

We wish to show that $\mu(F) = 0$. Assume on the contrary that $\mu(F) > 0$. Let $F' \subset F$ be the set of points $y \in F$ for which it exists the limit

$$\lim_{r\to 0} \frac{\log \mu_y^s(B^s(y,r))}{\log r} = d^s.$$

Clearly $\mu(F') = \mu(F) > 0$. Then we can find $y \in F$ such that

$$\mu_y^s(F) = \mu_y^s(F') = \mu_y^s(F' \cap \oplus P(y) \cap \xi^s(y)) > 0.$$

It follows from Frostman's lemma that

$$\dim_H(F' \cap \xi^s(y)) = d^s. \tag{27}$$

Let us consider the countable collection of balls

$$\delta B = \{B(z, e^{-m_j(z)}) \mid z \in F' \cap \xi^s(y);\ j = 1, 2, \dots\}.$$

By the Besicovitch covering lemma (see for example [G]) one can find a subcover $\delta C \subset \delta B$ of $F' \cap \xi^s(y)$ of arbitrarily small diameter and finite multiplicity $\rho = \rho(\dim M)$. This means that for any $L > 0$ one can choose a sequence of points $\{z_i \in F' \cap \xi^s(y)\}_{i=1}^{\infty}$ and a sequence of integers $\{t_i\}_{i=1}^{\infty}$, where $t_i \in \{m_j(z_i)\}_{j=1}^{\infty}$ and $t_i > L$ for each i, such that the collection of balls

$$\delta C = \{B(z_i, e^{-t_i}) \mid i = 1, 2, \dots\}$$

comprises a cover of $F' \cap \xi^s(y)$ whose multiplicity does not exceed ρ. We write $Q(i) = Q_{t_i}(z_i)$.

The Hausdorff sum corresponding to this cover is

$$\sum_{B \in \delta C} (\operatorname{diam} B)^{d^s - \varepsilon} = \sum_{i=1}^{\infty} e^{-t_i(d^s - \varepsilon)}.$$

By (26), we obtain

$$\begin{aligned}
\sum_{i=1}^{\infty} e^{-t_i(d^s-\varepsilon)} &\le \sum_{i=1}^{\infty} \widehat{N}^s(t_i, z_i, Q(i)) \cdot 4Ce^{-at_ih - 4at_i\varepsilon} \\
&\le 4C \sum_{q=1}^{\infty} e^{-aqh - 4aq\varepsilon} \sum_{i:t_i=q} \widehat{N}^s(q, z_i, Q(i)).
\end{aligned}$$

Since the multiplicity of the subcover δC is at most ρ, each set $Q(i)$ appears in the sum $\sum_{i:t_i=q} \widehat{N}^s(q, z_i, Q(i))$ at most ρ times. Hence $\sum_{i:t_i=q} \widehat{N}^s(q, z_i, Q(i)) \le \rho \widehat{N}^s(q, y, \oplus P(y))$. It follows from Lemma 4 that

$$\begin{aligned}
\sum_{B \in \delta C} (\operatorname{diam} B)^{d^s-\varepsilon} &\le 4C \sum_{q=1}^{\infty} e^{-aqh-4aq\varepsilon} \rho \widehat{N}^s(q, y, \oplus P(y)) \\
&\le 4D^{-1}C^3\rho \sum_{q=1}^{\infty} e^{-aqh-4aq\varepsilon+aqh+3aq\varepsilon} = 4D^{-1}C^3\rho \sum_{q=1}^{\infty} e^{-aq\varepsilon} < \infty.
\end{aligned}$$

Since L can be chosen arbitrarily large (and so also the numbers t_i), it follows that $\dim_H(F' \cap \xi^s(y)) \le d^s - \varepsilon < d^s$. This contradicts (27). Hence $\mu(F) = 0$ and this yields the first inequality in the lemma. The proof of the second inequality repeats the same arguments. ∎

Proof of the Eckmann–Ruelle conjecture

Proof. By Proposition 3 we only need to prove that $\underline{d} \geq d^s + d^u$.

Given $\varepsilon > 0$, let the set $\widehat{\Gamma}$ be as in the previous sections. By Lemmas 2 and 3, for μ-almost every $y \in \oplus P(x) \cap \widehat{\Gamma}$ and $n \geq n_2(y)$, we obtain

$$\mu(B(y, e^{-n-2})) \leq \widehat{N}^s(n, y, Q_n(y)) \cdot \widehat{N}^u(n, y, Q_n(y)) \cdot 4C^3 e^{4a(h+\varepsilon)} e^{-2anh+6an\varepsilon}.$$

By Lemma 5, for μ-almost every $y \in \oplus P(x) \cap \widehat{\Gamma}$ there exists an integer $n_3(y) \geq n_2(y)$ such that for all $n \geq n_3(y)$ we have

$$\widehat{N}^s(n, y, Q_n(y)) < N^s(n, y, Q_n(y)) e^{7an\varepsilon},$$

$$\widehat{N}^u(n, y, Q_n(y)) < N^u(n, y, Q_n(y)) e^{7an\varepsilon}.$$

This implies that

$$\mu(B(y, e^{-n-2})) \leq N^s(n, y, Q_n(y)) \cdot N^u(n, y, Q_n(y)) \cdot 4C^3 e^{4a(h+\varepsilon)} e^{-2anh+20an\varepsilon}.$$

By Lemma 1 we obtain

$$\mu(B(y, e^{-n-2})) \leq \mu_y^s(B^s(y, 4e^{-n})) \mu_y^u(B^u(y, 4e^{-n})) \cdot 4C^5 e^{4a(h+\varepsilon)} e^{22an\varepsilon}.$$

This implies that

$$\varliminf_{n \to +\infty} \frac{\log \mu(B(y, e^{-n}))}{-n} \geq d^s + d^u - 22a\varepsilon$$

for μ-almost every $y \in \widehat{\Gamma}$. Since $\mu(\widehat{\Gamma}) > 1 - \varepsilon$ and $\varepsilon > 0$ is arbitrarily small, we conclude that

$$\underline{d} = \varliminf_{r \to 0} \frac{\log \mu(B(y, r))}{\log r} = \varliminf_{n \to +\infty} \frac{\log \mu(B(y, e^{-n}))}{-n} \geq d^s + d^u$$

for μ-almost every $y \in M$. This completes the proof. □

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