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Abstract

In order to solve partial differential equations numerically and accurately, a high order spatial discretization is usually needed. Model order reduction (MOR) techniques are often used to reduce the order of spatially-discretized systems and hence reduce computational complexity. A particular class of MOR techniques are $\mathcal{H}_2$-optimal methods such as the iterative rational Krylov subspace algorithm (IRKA) and related schemes. However, these methods are used to obtain good approximations on a infinite time-horizon. Thus, in this work, our main goal is to discuss MOR schemes for time-limited linear systems. For this, we propose an alternative time-limited $\mathcal{H}_2$-norm and show its connection with the time-limited Gramians. We then provide first-order optimality conditions for an optimal reduced order model (ROM) with respect to the time-limited $\mathcal{H}_2$-norm. Based on these optimality conditions, we propose an iterative scheme which upon convergences aims at satisfying these conditions. Then, we analyze how far away the obtained ROM is from satisfying the optimality conditions. We test the efficiency of the proposed iterative scheme using various numerical examples and illustrate that the newly proposed iterative method can lead to a better reduced-order compared to unrestricted IRKA in the time interval of interest.

1 Introduction

We consider a continuous linear time-invariant (LTI) system as follows:

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \\ y(t) = Cx(t), \quad t \geq 0, \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$. Generally, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ denote the state, control input and the quantity of interest (output vector), respectively, and in the most cases, the dimension of the state vector is much larger than the numbers of control input and output vectors, i.e., $n \gg m, p$. We also assume that the matrix A is Hurwitz, meaning $\Lambda(A) \subset \mathbb{C}_-$, where $\Lambda(\cdot)$ denotes the spectrum of a matrix. Due to the large dimension of system (1), it is numerically very expensive to simulate the system for various control inputs and perform engineering studies such as optimal control and optimization. One approach to overcome such an issue is *model order reduction* (MOR),

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where we aim at constructing a reduced-order system as follows:

$$\hat{\Sigma}: \begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), & \hat{x}(0) = 0 \\ \hat{y}(t) = \hat{C}\hat{x}(t), & t \geq 0, \end{cases} \tag{2}$$

where $\hat{A} \in \mathbb{R}^{r\times r}$, $\hat{B} \in \mathbb{R}^{r\times m}$, and $\hat{C} \in \mathbb{R}^{p\times r}$ and $r \ll n$ such that $y \approx \hat{y}$ in an appropriate norm for all admissible control inputs u. In the literature, there is a huge collection of methods available which allow us to construct such reduced-order systems, e.g., see [1, 2, 3].

Most of the methods for linear systems such as balanced truncation, e.g., see [1, 4] and the iterative rational Krylov subspace algorithm [5] aim at constructing a reduced-order system which is good for an infinite time horizon. In other words, the output of system (1) is very well approximated by the output of (2) on the time interval $[0, \infty)$. However, from a practical point of view for instance, we are interested in approximating the output y on a finite time interval, e.g., $[0, \bar{T}]$, meaning that

$$y \approx \hat{y} \quad \text{on} \quad [0, \bar{T}]. \tag{3}$$

Due to Equation (3), we expect a better reduced-order system in the time interval $[0, \bar{T}]$ as compared to unconstrained MOR approaches for the same reduced order. Such a problem in a view of balanced truncation was first considered in [6] and its further studied was carried out in [7, 8]. However, in this work, we consider the similar time-limited model reduction problem but in a view of extending the Wilson conditions [9] and the first-order optimality conditions [5, 9, 10]. Generalized optimality conditions for bilinear systems have been studied, e.g., in [11, 12] but their setting (infinite time horizon) clearly differs from the one in this paper.

In Section 2, we first discuss the time-limited $\mathcal{H}_2$-norm for linear systems and provide different representations of the metric induced by this norm which are based on time-limited Gramians. Then, we define the problem setting for time-limited MOR as an optimization problem. Subsequently, in Section 3, we extend the Wilson conditions to time-limited linear systems and derive first order optimality conditions which minimize the time-limited $\mathcal{H}_2$-norm of the error system. Based on these conditions, we propose an iterative scheme, which we aim at constructing a reduced-order system, satisfying approximately the optimality conditions. Later on, we derive expressions, revealing how far away the obtained reduced systems via the proposed iterative scheme are from being optimal. In Section 4, we illustrate the efficiency of the proposed iterative scheme by three benchmark numerical examples for linear systems. Finally, we conclude the paper with a small summary and future work.

2 Time-Limited $\mathcal{H}_2$-Norm and Problem Setting

In this section, we first define the time-limited $\mathcal{H}_2$-norm for linear systems, show its relation to the output error and provide different representations for the time-limited $\mathcal{H}_2$-norm using time-limited Gramians. Subsequently, we define time-limited $\mathcal{H}_2$-model reduction for linear systems. Before we proceed further, we note important relations between the Kronecker product, the vectorization and the trace of a matrix. These are:

$$\operatorname{vec}(XYZ) = (Z^T \otimes X)\operatorname{vec}(Y), \tag{4a}$$
$$\operatorname{tr}(XYZ) = \operatorname{vec}^T(X^T)(I \otimes Y)\operatorname{vec}(Z), \tag{4b}$$

where X, Y and Z are matrices of suitable dimension; $\operatorname{vec}(\cdot)$ and $\operatorname{tr}(\cdot)$ denote the vectorization and the trace of a matrix, and $\otimes$ represents the Kronecker product of two matrices.

We investigate the large scale system (1) and are seeking for a reduced order system (2) having the same structure. Since our goal is to construct a good approximation of the system (1) on a finite time interval $[0, \bar{T}]$ below, where $\bar{T} > 0$ is the terminal time, we first investigate the worst case error between the output of the system (2) and the output of (1) on $[0, \bar{T}]$. In order to find a bound for the error between the output y of the original model and the output $\hat{y}$ of the reduced system, arguments from [13, 14, 15] are used. There an $\mathcal{H}_2$-error bound for stochastic systems applying balanced truncation is derived.

We make use of the explicit representations for the outputs

$$y(t) = C\int_0^t \mathrm{e}^{A(t-s)}\, Bu(s)ds, \quad \hat{y}(t) = \hat{C}\int_0^t \mathrm{e}^{\hat{A}(t-s)}\, \hat{B}u(s)ds,$$

and obtain that

$$\begin{aligned}\|y(t)-\hat{y}(t)\|_2 &= \left\| C\int_0^t \mathrm{e}^{A(t-s)}\, Bu(s)ds - \hat{C}\int_0^t \mathrm{e}^{\hat{A}(t-s)}\, \hat{B}u(s)ds \right\|_2 \\ &\le \int_0^t \left\| \left(C\,\mathrm{e}^{A(t-s)}\, B - \hat{C}\,\mathrm{e}^{\hat{A}(t-s)}\, \hat{B}\right) u(s) \right\|_2 ds \\ &\le \int_0^t \left\| C\,\mathrm{e}^{A(t-s)}\, B - \hat{C}\,\mathrm{e}^{\hat{A}(t-s)}\, \hat{B} \right\|_F \|u(s)\|_2\, ds.\end{aligned}$$

By the inequality of Cauchy-Schwarz and substitution, we have

$$\begin{aligned}\|y(t)-\hat{y}(t)\|_2 &\le \left(\int_0^t \left\| C\,\mathrm{e}^{A(t-s)}\, B - \hat{C}\,\mathrm{e}^{\hat{A}(t-s)}\, \hat{B} \right\|_F^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|u(s)\|_2^2\, ds \right)^{\frac{1}{2}} \\ &\le \left(\int_0^t \left\| C\,\mathrm{e}^{As}\, B - \hat{C}\,\mathrm{e}^{\hat{A}s}\, \hat{B} \right\|_F^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|u(s)\|_2^2\, ds \right)^{\frac{1}{2}} \\ &\le \left(\int_0^{\bar{T}} \left\| C\,\mathrm{e}^{As}\, B - \hat{C}\,\mathrm{e}^{\hat{A}s}\, \hat{B} \right\|_F^2 ds \right)^{\frac{1}{2}} \|u\|_{L^2_{\bar{T}}}.\end{aligned}$$

for $t \in [0, \bar{T}]$. Hence,

$$\max_{t\in[0,\bar{T}]} \|y(t)-\hat{y}(t)\|_2 \le \left\| \Sigma - \hat{\Sigma} \right\|_{\mathcal{H}_{2,\bar{T}}} \|u\|_{L^2_{\bar{T}}}, \tag{5}$$

where $\left\|\Sigma-\hat{\Sigma}\right\|_{\mathcal{H}_{2,\bar{T}}} := \left(\int_0^{\bar{T}} \left\|C\,\mathrm{e}^{As}B-\hat{C}\,\mathrm{e}^{\hat{A}s}\hat{B}\right\|_F^2 ds\right)^{\frac{1}{2}}$. We call $\|\cdot\|_{\mathcal{H}_{2,\bar{T}}}$ the time-limited $\mathcal{H}_2$-norm since $\left\|\Sigma-\hat{\Sigma}\right\|_{\mathcal{H}_{2,\bar{T}}}$ provides the time-domain representation of the metric induced by the $\mathcal{H}_2$-norm if $\bar{T}\to\infty$.

The time-limited $\mathcal{H}_2$-error can also be expressed with the help of the time-limited reachability and observability Gramians [6]. In order to show this, we first provide the following lemma.

Lemma 2.1. *Let $A_1 \in \mathbb{R}^{d_1\times d_1}$, $A_2 \in \mathbb{R}^{d_2\times d_2}$ with $\Lambda(A_1)\cap -\Lambda(A_2)=\emptyset$ and $K_1 \in \mathbb{R}^{d_1\times d_3}$, $K_2 \in \mathbb{R}^{d_2\times d_3}$. Then,*

$$X=\int_0^{\bar{T}} \mathrm{e}^{A_1 s} K_1 K_2^T \mathrm{e}^{A_2^T s} ds$$

uniquely solves the Sylvester equation

$$A_1X+XA_2^T=-K_1K_2^T+\mathrm{e}^{A_1\bar{T}} K_1K_2^T \mathrm{e}^{A_2^T\bar{T}}. \tag{6}$$

Proof. This result is a consequence of the product rule. Setting $g_1(t) := \mathrm{e}^{A_1 t} K_1$ and $g_2(t) := K_2^T \mathrm{e}^{A_2^T t}$, it holds that

$$\begin{aligned} g_1(\bar{T})g_2(\bar{T})-g_1(0)g_2(0) &= \int_0^{\bar{T}} g_1(s)dg_2(s)+\int_0^{\bar{T}} dg_1(s)g_2(s)\\ &= \int_0^{\bar{T}} g_1(s)g_2(s)ds\; A_2^T + A_1\int_0^{\bar{T}} g_1(s)g_2(s)ds, \end{aligned}$$

since $dg_2(s)=g_2(s)A_2^T ds$ and $dg_1(s)=A_1g_1(s)ds$. Now, the solution is unique since (6) can be written equivalently as

$$\underbrace{(I_{d_2}\otimes A_1+A_2\otimes I_{d_1})}_{=:\mathcal{A}_\otimes}\operatorname{vec}(X)=\operatorname{vec}(R_{12}) \tag{7}$$

using (4a). Here, R_{12} is the right-hand side in (6) and I_q denotes the identity matrix of size $q\times q$. Now, the eigenvalues of $\mathcal{A}_\otimes$ are given by $\mu_1^{(i)}+\mu_2^{(j)}$, where $\mu_1^{(i)}$ is the ith eigenvalue of A_1 and $\mu_2^{(j)}$ the jth eigenvalue of A_2. Due the assumption on the spectra of A_1 and A_2, the matrix $\mathcal{A}_\otimes$ is invertible which gives a unique solution to (7). □

The next proposition shows that the time-limited error can be expressed with the help of time-limited Gramians. This result is used later on in order to derive first-order necessary conditions for a minimal error in the time-limited $\mathcal{H}_2$-norm.

Proposition 2.2. *Let Σ and $\hat{\Sigma}$ be the original and reduced-order systems as defined in* (1) *and* (2)*. Then, the time-limited $\mathcal{H}_2$-norm of its difference is given by*

$$\left\|\Sigma-\hat{\Sigma}\right\|^2_{\mathcal{H}_{2,\bar{T}}} = \operatorname{tr}(CP_{\bar{T}}C^T)+\operatorname{tr}(\hat{C}\hat{P}_{\bar{T}}\hat{C}^T)-2\operatorname{tr}(CP_{2,\bar{T}}\hat{C}^T), \tag{8}$$

where $P_{\bar{T}}, P_{2,\bar{T}}$ and $\hat{P}_{\bar{T}}$, respectively, satisfy

$$AP_{\bar{T}} + P_{\bar{T}}A^T = -BB^T + \mathrm{e}^{A\bar{T}}\, BB^T\, \mathrm{e}^{A^T\bar{T}}, \tag{9}$$

$$AP_{2,\bar{T}} + P_{2,\bar{T}}\hat{A}^T = -B\hat{B}^T + \mathrm{e}^{A\bar{T}}\, B\hat{B}^T\, \mathrm{e}^{\hat{A}^T\bar{T}}, \tag{10}$$

$$\hat{A}\hat{P}_{\bar{T}} + \hat{P}_{\bar{T}}\hat{A}^T = -\hat{B}\hat{B}^T + \mathrm{e}^{\hat{A}\bar{T}}\, \hat{B}\hat{B}^T\, \mathrm{e}^{\hat{A}^T\bar{T}}. \tag{11}$$

Proof. The definition of the Frobenius norm and the linearity of the integral yield

$$\begin{aligned}
&\left\|\Sigma - \hat{\Sigma}\right\|^2_{\mathcal{H}_{2,\bar{T}}} = \int_0^{\bar{T}} \left\|C\,\mathrm{e}^{As}\,B - \hat{C}\,\mathrm{e}^{\hat{A}s}\,\hat{B}\right\|_F^2 ds \\
&= \int_0^{\bar{T}} \mathrm{tr}\left(C\,\mathrm{e}^{As}\,BB^T\,\mathrm{e}^{A^Ts}\,C^T\right) ds + \int_0^{\bar{T}} \mathrm{tr}\left(\hat{C}\,\mathrm{e}^{\hat{A}s}\,\hat{B}\hat{B}^T\,\mathrm{e}^{\hat{A}^Ts}\,\hat{C}^T\right) ds \\
&\quad - 2\int_0^{\bar{T}} \mathrm{tr}\left(C\,\mathrm{e}^{As}\,B\hat{B}^T\,\mathrm{e}^{\hat{A}^Ts}\,\hat{C}^T\right) ds \\
&= \mathrm{tr}\left(CP_{\bar{T}}C^T\right) + \mathrm{tr}\left(\hat{C}\hat{P}_{\bar{T}}\hat{C}^T\right) - 2\ \mathrm{tr}\left(CP_{2,\bar{T}}\hat{C}^T\right),
\end{aligned}$$

with $P_{\bar{T}} := \int_0^{\bar{T}} \mathrm{e}^{As}\,BB^T\,\mathrm{e}^{A^Ts}\,ds$, $P_{2,\bar{T}} := \int_0^{\bar{T}} \mathrm{e}^{As}\,B\hat{B}^T\,\mathrm{e}^{\hat{A}^Ts}\,ds$, $\hat{P}_{\bar{T}} := \int_0^{\bar{T}} \mathrm{e}^{\hat{A}s}\,\hat{B}\hat{B}^T\,\mathrm{e}^{\hat{A}^Ts}\,ds$. Due to Lemma 2.1 $P_{\bar{T}}, P_{2,\bar{T}}$ and $\hat{P}_{\bar{T}}$ are the solutions to (9), (10) and (11), respectively. □

The result of Proposition 2.2 has the same structure as the error bound in [8], where the case of time-limited balanced truncation has been investigated. Moreover, the $\mathcal{H}_2$-error bound when applying balanced truncation to stochastic systems has the similar structure as (8) but it is clearly different since other types of generalized matrix equations play a role in the stochastic setting. The next proposition shows that the time-limited $\mathcal{H}_2$-norm of the error system as in Proposition 2.2 can be rewritten using the time-limited observability Gramians.

Proposition 2.3. *Let Σ and $\hat{\Sigma}$ be the original and reduced-order systems as defined in* (1) *and* (2)*. Moreover, let $P_{\bar{T}}, P_{2,\bar{T}}$ and $\hat{P}_{\bar{T}}$ be the solutions to* (9), (10) *and* (11)*, respectively. Then, the following holds:*

$$\begin{aligned}
\mathrm{tr}(CP_{\bar{T}}C^T) &= \mathrm{tr}(B^TQ_{\bar{T}}B), \\
\mathrm{tr}(\hat{C}\hat{P}_{\bar{T}}\hat{C}^T) &= \mathrm{tr}(\hat{B}^T\hat{Q}_{\bar{T}}\hat{B}), \\
\mathrm{tr}(CP_{2,\bar{T}}\hat{C}^T) &= \mathrm{tr}(\hat{B}^TQ_{2,\bar{T}}B),
\end{aligned}$$

where the matrices $Q_{\bar{T}}, Q_{2,\bar{T}}$ and $\hat{Q}_{\bar{T}}$ satisfy

$$A^TQ_{\bar{T}} + Q_{\bar{T}}A = -C^TC + \mathrm{e}^{A^T\bar{T}}\, C^TC\, \mathrm{e}^{A\bar{T}}, \tag{12}$$

$$\hat{A}^TQ_{2,\bar{T}} + Q_{2,\bar{T}}A = -\hat{C}^TC + \mathrm{e}^{\hat{A}^T\bar{T}}\, \hat{C}^TC\, \mathrm{e}^{A\bar{T}}, \tag{13}$$

$$\hat{A}^T\hat{Q}_{\bar{T}} + \hat{Q}_{\bar{T}}\hat{A} = -\hat{C}^T\hat{C} + \mathrm{e}^{\hat{A}^T\bar{T}}\, \hat{C}^T\hat{C}\, \mathrm{e}^{\hat{A}\bar{T}}. \tag{14}$$

Proof. We insert the integral representations of $P_{\bar{T}}, P_{2,\bar{T}}$ and $\hat{P}_{\bar{T}}$ and use basic properties of the trace operator. Thus,

$$\begin{aligned}
\operatorname{tr}(CP_{\bar{T}}C^T) &= \int_0^{\bar{T}} \operatorname{tr}(C\,\mathrm{e}^{As}\,BB^T\,\mathrm{e}^{A^Ts}\,C^T)ds = \int_0^{\bar{T}} \operatorname{tr}(B^T\,\mathrm{e}^{A^Ts}\,C^TC\,\mathrm{e}^{As}\,B)ds,\\
\operatorname{tr}(\hat{C}\hat{P}_{\bar{T}}\hat{C}^T) &= \int_0^{\bar{T}} \operatorname{tr}(\hat{C}\,\mathrm{e}^{\hat{A}s}\,\hat{B}\hat{B}^T\,\mathrm{e}^{\hat{A}^Ts}\,\hat{C}^T)ds = \int_0^{\bar{T}} \operatorname{tr}(\hat{B}^T\,\mathrm{e}^{\hat{A}^Ts}\,\hat{C}^T\hat{C}\,\mathrm{e}^{\hat{A}s}\,\hat{B})ds,\\
\operatorname{tr}(CP_{2,\bar{T}}\hat{C}^T) &= \int_0^{\bar{T}} \operatorname{tr}(C\,\mathrm{e}^{As}\,B\hat{B}^T\,\mathrm{e}^{\hat{A}^Ts}\,\hat{C}^T)ds = \int_0^{\bar{T}} \operatorname{tr}(\hat{B}^T\,\mathrm{e}^{\hat{A}^Ts}\,\hat{C}^TC\,\mathrm{e}^{As}\,B)ds.
\end{aligned}$$

Let us define $Q_{\bar{T}} := \int_0^{\bar{T}} \mathrm{e}^{A^Ts}\,C^TC\,\mathrm{e}^{As}\,ds$, $Q_{2,\bar{T}} := \int_0^{\bar{T}} \mathrm{e}^{\hat{A}^Ts}\,\hat{C}^TC\,\mathrm{e}^{As}\,ds$ and $\hat{Q}_{\bar{T}} := \int_0^{\bar{T}} \mathrm{e}^{\hat{A}^Ts}\,\hat{C}^T\hat{C}\,\mathrm{e}^{\hat{A}s}\,ds$. Then, applying Lemma 2.1 yields the claim. □

From inequality (5), it can be seen that it makes sense to minimize $\left\|\Sigma - \hat{\Sigma}\right\|_{\mathcal{H}_{2,\bar{T}}}^2$ with respect to the reduced order matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$ since this also minimizes the output error. Due to the fact that $\left\|\Sigma - \hat{\Sigma}\right\|_{\mathcal{H}_{2,\bar{T}}}$ is increasing in $\bar{T}$, the time-limited error is less or equal the error in the full $\mathcal{H}_2$-norm $\|\cdot\|_{\mathcal{H}_{2,\infty}}$. Thus, $\|\cdot\|_{\mathcal{H}_{2,\infty}}$ also bounds the output error in (5) but since this bound is larger than the time-limited one, a more accurate reduce order model is expected on $[0, \bar{T}]$ when using the time-limited $\mathcal{H}_2$-norm instead.

3 First-Order Necessary Conditions for Optimality and Model-Order Reduction

In this section, we begin with deriving first-order necessary conditions for time-limited $\mathcal{H}_2$-optimal reduced order systems. In other words, our aim is to construct a reduced-order system $\hat{\Sigma}$ of order r as in (2), such that it minimizes $\|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_{2,\bar{T}}}^2 =: \mathcal{E}$, where Σ is the original systems as in (1). An expression for $\mathcal{E}$ is given in (8). Since the term $\operatorname{tr}(CP_{\bar{T}}C^T)$ in (8) does not depend on the reduced order matrices, we focus on minimizing the expression

$$\mathcal{E}_r := \operatorname{tr}(\hat{C}\hat{P}_{\bar{T}}\hat{C}^T) - 2\operatorname{tr}(CP_{2,\bar{T}}\hat{C}^T). \tag{15}$$

Before proceeding further, we assume that the matrix $\hat{A}$ in (2) is diagonalizable, i.e., there exists an invertible matrix S such that $\hat{A} = S^{-1}DS$, where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_r)$. Using the matrix S as a state-space transformation of (2), the term (15) can be rewritten as

$$\begin{aligned}
\mathcal{E}_r &= \operatorname{tr}(\hat{C}S^{-1}S\hat{P}_{\bar{T}}S^TS^{-T}\hat{C}^T) - 2\operatorname{tr}(CP_{2,\bar{T}}S^TS^{-T}\hat{C}^T)\\
&= \operatorname{tr}(\tilde{C}\tilde{P}_{\bar{T}}\tilde{C}^T) - 2\operatorname{tr}(C\tilde{P}_{2,\bar{T}}\tilde{C}^T),
\end{aligned} \tag{16}$$

where $\tilde{C} = \hat{C}S^{-1}$, $\tilde{P}_{\bar{T}} = S\hat{P}_{\bar{T}}S^T$ and $\tilde{P}_{2,\bar{T}} = P_{2,\bar{T}}S^T$. The matrices $\tilde{P}_{\bar{T}}, \tilde{P}_{2,\bar{T}}$ are the solutions to

$$A\tilde{P}_{2,\bar{T}} + \tilde{P}_{2,\bar{T}}D = -B\tilde{B}^T + \mathrm{e}^{A\bar{T}}\, B\tilde{B}^T\, \mathrm{e}^{D\bar{T}}, \tag{17}$$

$$D\tilde{P}_{\bar{T}} + \tilde{P}_{\bar{T}}D = -\tilde{B}\tilde{B}^T + \mathrm{e}^{D\bar{T}}\, \tilde{B}\tilde{B}^T\, \mathrm{e}^{D\bar{T}}, \tag{18}$$

where $\tilde{B} = S\hat{B}$. Equation (17) is obtained by multiplying (10) with S^T from the right-side, and Equation (18) is derived from multiplying (11) with S and S^T from the left and right sides, respectively, and using that $\mathrm{e}^{\hat{A}\bar{T}} = S^{-1}\,\mathrm{e}^{D\bar{T}}\, S$.

In order to find necessary conditions for a locally minimal transformed error expression (16), we compute the partial derivatives of the form $\partial_x \operatorname{tr}(\tilde{C}\tilde{P}_{\bar{T}}\tilde{C}^T)$ and $\partial_x \operatorname{tr}(C\tilde{P}_{2,\bar{T}}\tilde{C}^T)$ and then set

$$\partial_x \operatorname{tr}(\tilde{C}\tilde{P}_{\bar{T}}\tilde{C}^T) = 2\partial_x \operatorname{tr}(C\tilde{P}_{2,\bar{T}}\tilde{C}^T),$$

where $x = \lambda_i, \tilde{c}_{ki}, \tilde{b}_{ij}, i \in \{1,\dots,r\}, j \in \{1,\dots,m\}, k \in \{1,\dots,p\}$ and $\tilde{c}_{ki}, \tilde{b}_{ij}$ being kj-th and ij-th elements of the matrices $\tilde{C}$ and $\tilde{B}$, respectively.

Let us start with the optimality conditions with respect to $\tilde{c}_{ki}$. With e_i, we denote the i-th column of the identity matrix of suitable dimension that is clear from the context. We then obtain that

$$\begin{aligned}
\partial_{\tilde{c}_{ki}} \operatorname{tr}(\tilde{C}\tilde{P}_{\bar{T}}\tilde{C}^T) &= \partial_{\tilde{c}_{ki}} \operatorname{tr}(\tilde{C}^T\tilde{C}\tilde{P}_{\bar{T}}) \\
&= \operatorname{tr}((\partial_{\tilde{c}_{ki}}\tilde{C}^T)\tilde{C}\tilde{P}_{\bar{T}} + \tilde{C}^T(\partial_{\tilde{c}_{ki}}\tilde{C})\tilde{P}_{\bar{T}}) = \operatorname{tr}(e_i e_k^T\tilde{C}\tilde{P}_{\bar{T}} + \tilde{C}^T e_k e_i^T \tilde{P}_{\bar{T}}) \\
&= 2e_k^T\tilde{C}\tilde{P}_{\bar{T}}e_i,
\end{aligned}$$

where we have used the linearity of the trace, the product rule and the fact that $\tilde{P}_{\bar{T}}$ does not depend on $\tilde{C}$. Since

$$\partial_{\tilde{c}_{ki}} \operatorname{tr}(C\tilde{P}_{2,\bar{T}}\tilde{C}^T) = \operatorname{tr}(C\tilde{P}_{2,\bar{T}}e_i e_k^T) = e_k^T C\tilde{P}_{2,\bar{T}}e_i,$$

the optimality condition with respect to $\tilde{c}_{ki}$ is $e_k^T\tilde{C}\tilde{P}_{\bar{T}}e_i = e_k^T C\tilde{P}_{2,\bar{T}}e_i$ for all $i \in \{1,\dots,r\}$, $k \in \{1,\dots,p\}$. Hence, we obtain

$$\tilde{C}\tilde{P}_{\bar{T}} = C\tilde{P}_{2,\bar{T}}. \tag{19}$$

We now derive the partial derivatives with respect to $\tilde{b}_{ij}$. We rewrite (16) to simplify this procedure by applying Proposition 2.3:

$$\begin{aligned}
\mathcal{E}_r = \operatorname{tr}(\tilde{C}\tilde{P}_{\bar{T}}\tilde{C}^T) - 2\operatorname{tr}(C\tilde{P}_{2,\bar{T}}\tilde{C}^T) &= \operatorname{tr}(\hat{B}^T\hat{Q}_{\bar{T}}\hat{B}) - 2\operatorname{tr}(\hat{B}^T Q_{2,\bar{T}}B) \\
&= \operatorname{tr}(\tilde{B}^T\tilde{Q}_{\bar{T}}\tilde{B}) - 2\operatorname{tr}(\tilde{B}^T\tilde{Q}_{2,\bar{T}}B),
\end{aligned}$$

where $\tilde{Q}_{\bar{T}} = S^{-T}\hat{Q}_{\bar{T}}S^{-1}$ and $\tilde{Q}_{2,\bar{T}} = S^{-T}\hat{Q}_{2,\bar{T}}$. The matrices $\tilde{Q}_{\bar{T}}$ and $\tilde{Q}_{2,\bar{T}}$ satisfy

$$D\tilde{Q}_{2,\bar{T}} + \tilde{Q}_{2,\bar{T}}A = -\tilde{C}^T C + \mathrm{e}^{D\bar{T}}\, \tilde{C}^T C\, \mathrm{e}^{D\bar{T}}, \tag{20}$$

$$D\tilde{Q}_{\bar{T}} + \tilde{Q}_{\bar{T}}D = -\tilde{C}^T\tilde{C} + \mathrm{e}^{D\bar{T}}\, \tilde{C}^T\tilde{C}\, \mathrm{e}^{D\bar{T}}. \tag{21}$$

Equation (20) is obtained by multiplying (13) with S^{-T} from the left side, and we find (21) by multiplying (14) with S^{-T} from the left side and with S^{-1} from the right side. Thus, we have

$$\begin{aligned}\partial_{\tilde{b}_{ij}} \operatorname{tr}(\tilde{B}\tilde{B}^T\tilde{Q}_{\bar{T}}) &= \operatorname{tr}((\partial_{\tilde{b}_{ij}}\tilde{B})\tilde{B}^T\tilde{Q}_{\bar{T}} + \tilde{B}(\partial_{\tilde{b}_{ij}}\tilde{B}^T)\tilde{Q}_{\bar{T}}) = \operatorname{tr}(e_ie_j^T\tilde{B}^T\tilde{Q}_{\bar{T}} + \tilde{B}e_je_i^T\tilde{Q}_{\bar{T}})\\ &= 2e_i^T\tilde{Q}_{\bar{T}}\tilde{B}e_j\end{aligned}$$

using that $\tilde{Q}_{\bar{T}}$ does not depend on $\tilde{B}$ or $\tilde{b}_{ij}$. Since

$$\partial_{\tilde{b}_{ij}} \operatorname{tr}(\tilde{B}^T\tilde{Q}_{2,\bar{T}}B) = \operatorname{tr}(e_je_i^T\tilde{Q}_{2,\bar{T}}B) = e_i^T\tilde{Q}_{2,\bar{T}}Be_j,$$

it is necessary that $e_i^T\tilde{Q}_{\bar{T}}\tilde{B}e_j = e_i^T\tilde{Q}_{2,\bar{T}}Be_j$ for $i \in \{1,\ldots,r\}$, $j \in \{1,\ldots,m\}$, which can equivalently be written as

$$\tilde{Q}_{\bar{T}}\tilde{B} = \tilde{Q}_{2,\bar{T}}B. \tag{22}$$

Next, we first introduce the following lemma in order to derive an optimality condition with respect to the eigenvalues λ_i of $\hat{A}$.

Lemma 3.1. *The partial derivatives $X^{(i)} := \partial_{\lambda_i}\tilde{P}_{\bar{T}}$ and $X_2^{(i)} := \partial_{\lambda_i}\tilde{P}_{2,\bar{T}}$ solve*

$$DX^{(i)} + X^{(i)}D = -e_ie_i^T\tilde{P}_{\bar{T}} - \tilde{P}_{\bar{T}}e_ie_i^T + \bar{T}e_ie_i^T\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^T\,\mathrm{e}^{D\bar{T}} + \bar{T}\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^T\,\mathrm{e}^{D\bar{T}}\,e_ie_i^T, \tag{23}$$

$$AX_2^{(i)} + X_2^{(i)}D = -\tilde{P}_{2,\bar{T}}e_ie_i^T + \bar{T}\,\mathrm{e}^{A\bar{T}}\,B\tilde{B}^T\,\mathrm{e}^{D\bar{T}}\,e_ie_i^T. \tag{24}$$

Proof. The derivative of the left side of equation (17) is

$$AX_2^{(i)} + X_2^{(i)}D + \tilde{P}_{2,\bar{T}}e_ie_i^T$$

applying the product rule. The derivative of corresponding right side is

$$\mathrm{e}^{A\bar{T}}\,B\tilde{B}^T\partial_{\lambda_i}\,\mathrm{e}^{D\bar{T}} = \mathrm{e}^{A\bar{T}}\,B\tilde{B}^T\,\mathrm{e}^{D\bar{T}}\,e_ie_i^T\bar{T},$$

because $\partial_{\lambda_i}\,\mathrm{e}^{D\bar{T}} = \partial_{\lambda_i}\operatorname{diag}(\mathrm{e}^{\lambda_1\bar{T}},\ldots,\mathrm{e}^{\lambda_i\bar{T}},\ldots,\mathrm{e}^{\lambda_r\bar{T}}) = \operatorname{diag}(0,\ldots,\bar{T}\,\mathrm{e}^{\lambda_i\bar{T}},\ldots,0)$. This yields (23). Applying ∂_{λ_i} to the left of equation (18) provides

$$e_ie_i^T\tilde{P}_{\bar{T}} + DX^{(i)} + X^{(i)}D + \tilde{P}_{\bar{T}}e_ie_i^T$$

again using the product rule. Doing the same with the corresponding right side, we have

$$\begin{aligned}\partial_{\lambda_i}(\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^T\,\mathrm{e}^{D\bar{T}}) &= (\partial_{\lambda_i}\,\mathrm{e}^{D\bar{T}})\tilde{B}\tilde{B}^T\,\mathrm{e}^{D\bar{T}} + \mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^T(\partial_{\lambda_i}\,\mathrm{e}^{D\bar{T}})\\ &= \bar{T}e_ie_i^T\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^T\,\mathrm{e}^{D\bar{T}} + \mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^T\,\mathrm{e}^{D\bar{T}}\,e_ie_i^T\bar{T}.\end{aligned}$$

This provides equation (24). □

Let us introduce the infinite Gramian $\tilde{Q}_\infty$ which we get from (21) for $\bar{T} \to \infty$ if the reduced system is asymptotically stable (exponential term on the right side vanishes due to the asymptotic stability of the system). If the asymptotic stability is not given, we can still define $\tilde{Q}_\infty$ as the solution to

$$D\tilde{Q}_\infty + \tilde{Q}_\infty D = -\tilde{C}^T\tilde{C}$$

if D and $-D$ have no common eigenvalues. We insert this matrix equation to

$$\partial_{\lambda_i} \operatorname{tr}(\tilde{C}\tilde{P}_{\bar{T}}\tilde{C}^T) = \operatorname{tr}(\tilde{C}^T\tilde{C}X^{(i)}) = -\operatorname{tr}([D\tilde{Q}_\infty + \tilde{Q}_\infty D]X^{(i)}) = -\operatorname{tr}(\tilde{Q}_\infty[X^{(i)}D + DX^{(i)}]).$$

With Lemma 3.1, we get

$$\begin{aligned}\partial_{\lambda_i} \operatorname{tr}(\tilde{C}\tilde{P}_{\bar{T}}\tilde{C}^T) &= \operatorname{tr}(\tilde{Q}_\infty[e_ie_i^T\tilde{P}_{\bar{T}} + \tilde{P}_{\bar{T}}e_ie_i^T - \bar{T}e_ie_i^T \operatorname{e}^{D\bar{T}} \tilde{B}\tilde{B}^T \operatorname{e}^{D\bar{T}} - \bar{T}\operatorname{e}^{D\bar{T}} \tilde{B}\tilde{B}^T \operatorname{e}^{D\bar{T}} e_ie_i^T]) \\ &= 2e_i^T\tilde{Q}_\infty[\tilde{P}_{\bar{T}} - \bar{T}\operatorname{e}^{D\bar{T}} \tilde{B}\tilde{B}^T \operatorname{e}^{D\bar{T}}]e_i.\end{aligned}$$

Assuming that D and $-A$ have no common eigenvalues, we define the infinite cross Gramian $\tilde{Q}_{2,\infty}$ which satisfies

$$D\tilde{Q}_{2,\infty} + \tilde{Q}_{2,\infty}A^T = -\tilde{C}^TC.$$

Hence, it holds that

$$\begin{aligned}\partial_{\lambda_i} \operatorname{tr}(C\tilde{P}_{2\bar{T}}\tilde{C}^T) &= \operatorname{tr}(\tilde{C}^TCX_2^{(i)}) = -\operatorname{tr}([D\tilde{Q}_{2,\infty} + \tilde{Q}_{2,\infty}A]X_2^{(i)}) \\ &= -\operatorname{tr}(\tilde{Q}_{2,\infty}[X_2^{(i)}D + AX_2^{(i)}]) = \operatorname{tr}(\tilde{Q}_{2,\infty}[\tilde{P}_{2,\bar{T}} - \bar{T}\operatorname{e}^{A\bar{T}} B\tilde{B}^T \operatorname{e}^{D\bar{T}}]e_ie_i^T) \\ &= e_i^T\tilde{Q}_{2,\infty}[\tilde{P}_{2,\bar{T}} - \bar{T}\operatorname{e}^{A\bar{T}} B\tilde{B}^T \operatorname{e}^{D\bar{T}}]e_i\end{aligned}$$

applying Lemma 3.1 again. This leads to the third optimality condition which is

$$e_i^T\tilde{Q}_{2,\infty}[\tilde{P}_{2,\bar{T}} - \bar{T}\operatorname{e}^{A\bar{T}} B\tilde{B}^T \operatorname{e}^{D\bar{T}}]e_i = e_i^T\tilde{Q}_\infty[\tilde{P}_{\bar{T}} - \bar{T}\operatorname{e}^{D\bar{T}} \tilde{B}\tilde{B}^T \operatorname{e}^{D\bar{T}}]e_i \tag{25}$$

for all $i \in \{1, \ldots, r\}$.

Below, the generalized optimality conditions are summarized that have been derived above. Additionally, an equivalent Kronecker formulation is provided that is useful for the error analysis in the optimality conditions. A different type of extended Wilson conditions for bilinear systems has been shown in [12]. Its equivalent Kronecker formulation is presented in [11]. Since the bilinear setting is very different from the the time-limited case, the optimality conditions have a different structure which can be seen in the next theorem.

Theorem 3.2. *Let the reduced-order system (2) be a locally optimal approximation to the original system (1) with respect to $\|\cdot\|_{\mathcal{H}_{2,\bar{T}}}$. Then, conditions (19), (22) and (25) hold or equivalently, we have*

$$\begin{aligned}&(I \otimes \hat{C})\left[(I \otimes \hat{A}) + (D \otimes I)\right]^{-1} (\operatorname{e}^{D\bar{T}} \tilde{B} \otimes \operatorname{e}^{\hat{A}\bar{T}} \hat{B} - \tilde{B} \otimes \hat{B}) \operatorname{vec}(I) \\ &= (I \otimes C)\left[(I \otimes A) + (D \otimes I)\right]^{-1} (\operatorname{e}^{D\bar{T}} \tilde{B} \otimes \operatorname{e}^{A\bar{T}} B - \tilde{B} \otimes B) \operatorname{vec}(I),\end{aligned} \tag{26}$$

$$(\hat{B}^T \otimes I) \Big[(I \otimes D) + (\hat{A}^T \otimes I)\Big]^{-1} (\mathrm{e}^{\hat{A}^T \bar{T}} \hat{C}^T \otimes \mathrm{e}^{D\bar{T}} \tilde{C}^T - \hat{C}^T \otimes \tilde{C}^T) \operatorname{vec}(I)$$
$$= (B^T \otimes I) \left[(I \otimes D) + (A^T \otimes I)\right]^{-1} (\mathrm{e}^{A^T \bar{T}} C^T \otimes \mathrm{e}^{D\bar{T}} \tilde{C}^T - C^T \otimes \tilde{C}^T) \operatorname{vec}(I) \tag{27}$$

and for all $i = 1, \ldots, r$

$$\operatorname{vec}^T(I)(\hat{C} \otimes \tilde{C}) \Big[(I \otimes D) + (\hat{A} \otimes I)\Big]^{-1} (I \otimes e_i e_i^T)$$
$$\times \left(\Big[(I \otimes D) + (\hat{A} \otimes I)\Big]^{-1} (\mathrm{e}^{\hat{A}\bar{T}} \hat{B} \otimes \mathrm{e}^{D\bar{T}} \tilde{B} - \hat{B} \otimes \tilde{B}) - (\bar{T} \mathrm{e}^{\hat{A}\bar{T}} \hat{B} \otimes \mathrm{e}^{D\bar{T}} \tilde{B})\right) \operatorname{vec}(I)$$
$$= \operatorname{vec}^T(I)(C \otimes \tilde{C}) \left[(I \otimes D) + (A \otimes I)\right]^{-1} (I \otimes e_i e_i^T)$$
$$\times \left(\left[(I \otimes D) + (A \otimes I)\right]^{-1} (\mathrm{e}^{A\bar{T}} B \otimes \mathrm{e}^{D\bar{T}} \tilde{B} - B \otimes \tilde{B}) - (\bar{T} \mathrm{e}^{A\bar{T}} B \otimes \mathrm{e}^{D\bar{T}} \tilde{B})\right) \operatorname{vec}(I). \tag{28}$$

Proof. Applying the vec operator to (19) leads to the following equivalent formulation:

$$\operatorname{vec}(\tilde{C}\tilde{P}_{\bar{T}}) = \operatorname{vec}(C\tilde{P}_{2,\bar{T}}).$$

Now, using the vectorization of equation (18), we obtain by (4a) that

$$\operatorname{vec}(\tilde{C}\tilde{P}_{\bar{T}}) = (I \otimes \tilde{C}) \operatorname{vec}(\tilde{P}_{\bar{T}}) = (I \otimes \tilde{C}) \left[(I \otimes D) + (D \otimes I)\right]^{-1} \operatorname{vec}(\mathrm{e}^{D\bar{T}} \tilde{B}\tilde{B}^T \mathrm{e}^{D\bar{T}} - \tilde{B}\tilde{B}^T)$$
$$= (I \otimes \tilde{C}) \left[(I \otimes D) + (D \otimes I)\right]^{-1} (\mathrm{e}^{D\bar{T}} \tilde{B} \otimes \mathrm{e}^{D\bar{T}} \tilde{B} - \tilde{B} \otimes \tilde{B}) \operatorname{vec}(I).$$

Since $(I \otimes \tilde{C}) = (I \otimes \hat{C})(I \otimes S)^{-1}$ and $(\mathrm{e}^{D\bar{T}} \tilde{B} \otimes \mathrm{e}^{D\bar{T}} \tilde{B} - \tilde{B} \otimes \tilde{B}) = (I \otimes S^{-1})^{-1}(\mathrm{e}^{D\bar{T}} \tilde{B} \otimes \mathrm{e}^{\hat{A}\bar{T}} \hat{B} - \tilde{B} \otimes \hat{B})$, we get

$$\operatorname{vec}(\tilde{C}\tilde{P}_{\bar{T}}) = (I \otimes \hat{C}) \Big[(I \otimes \hat{A}) + (D \otimes I)\Big]^{-1} (\mathrm{e}^{D\bar{T}} \tilde{B} \otimes \mathrm{e}^{\hat{A}\bar{T}} \hat{B} - \tilde{B} \otimes \hat{B}) \operatorname{vec}(I).$$

With equation (17), the vectorization of $C\tilde{P}_{2,\bar{T}}$ is given by

$$\operatorname{vec}(C\tilde{P}_{2,\bar{T}}) = (I \otimes C) \operatorname{vec}(\tilde{P}_{2,\bar{T}}) = (I \otimes C) \left[(I \otimes A) + (D \otimes I)\right]^{-1} \operatorname{vec}(\mathrm{e}^{A\bar{T}} B\tilde{B}^T \mathrm{e}^{D\bar{T}} - B\tilde{B}^T)$$
$$= (I \otimes C) \left[(I \otimes A) + (D \otimes I)\right]^{-1} (\mathrm{e}^{D\bar{T}} \tilde{B} \otimes \mathrm{e}^{A\bar{T}} B - \tilde{B} \otimes B) \operatorname{vec}(I)$$

applying (4a) again such that (26) follows. Condition (22) is equivalent to

$$\operatorname{vec}(\tilde{Q}_{\bar{T}}\tilde{B}) = \operatorname{vec}(\tilde{Q}_{2,\bar{T}}B),$$

where with property (4a) it holds that

$$\operatorname{vec}(\tilde{Q}_{\bar{T}}\tilde{B}) = (\tilde{B}^T \otimes I) \operatorname{vec}(\tilde{Q}_{\bar{T}})$$
$$= (\tilde{B}^T \otimes I) \left[(I \otimes D) + (D \otimes I)\right]^{-1} (\mathrm{e}^{D\bar{T}} \tilde{C}^T \otimes \mathrm{e}^{D\bar{T}} \tilde{C}^T - \tilde{C}^T \otimes \tilde{C}^T) \operatorname{vec}(I)$$

inserting the vectorized representation of (21). Using the identities $(\tilde{B}^T \otimes I) = (\hat{B}^T \otimes I)(S^{-T} \otimes I)^{-1}$ and $(\mathrm{e}^{D\bar{T}} \tilde{C}^T \otimes \mathrm{e}^{D\bar{T}} \tilde{C}^T - \tilde{C}^T \otimes \tilde{C}^T) = (S^T \otimes I)^{-1}(\mathrm{e}^{\hat{A}^T\bar{T}} \hat{C}^T \otimes \mathrm{e}^{D\bar{T}} \tilde{C}^T - \hat{C}^T \otimes \tilde{C}^T)$ yields

$$\operatorname{vec}(\tilde{Q}_{\bar{T}}\tilde{B}) = (\hat{B}^T \otimes I) \Big[(I \otimes D) + (\hat{A}^T \otimes I)\Big]^{-1} (\mathrm{e}^{\hat{A}^T\bar{T}} \hat{C}^T \otimes \mathrm{e}^{D\bar{T}} \tilde{C}^T - \hat{C}^T \otimes \tilde{C}^T) \operatorname{vec}(I).$$

Vectorizing (20) leads to

$$\operatorname{vec}(\tilde{Q}_{2,\bar{T}}\tilde{B}) = (B^T \otimes I)\left[(I \otimes D) + (A^T \otimes I)\right]^{-1} (\mathrm{e}^{A^T\bar{T}}\, C^T \otimes \mathrm{e}^{D\bar{T}}\, \tilde{C}^T - C^T \otimes \tilde{C}^T)\operatorname{vec}(I)$$

which gives us equation (27). Condition (25) is equivalent to

$$\operatorname{tr}([\tilde{P}_{2,\bar{T}} - \bar{T}\,\mathrm{e}^{A\bar{T}}\, B\tilde{B}^T\,\mathrm{e}^{D\bar{T}}]e_i e_i^T \tilde{Q}_{2,\infty}) = \operatorname{tr}([\tilde{P}_{\bar{T}} - \bar{T}\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^T\,\mathrm{e}^{D\bar{T}}]e_i e_i^T \tilde{Q}_{\infty})$$

for every $i = 1, \ldots, r$. Taking (4b) into account, we can express the trace using the vec operator:

$$\operatorname{tr}([\tilde{P}_{\bar{T}} - \bar{T}\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^T\,\mathrm{e}^{D\bar{T}}]e_i e_i^T \tilde{Q}_{\infty}) = \operatorname{vec}^T(\tilde{P}_{\bar{T}} - \bar{T}\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^T\,\mathrm{e}^{D\bar{T}})(I \otimes e_i e_i^T)\operatorname{vec}(\tilde{Q}_{\infty}).$$

With the above arguments, we see that

$$\operatorname{vec}(\tilde{Q}_{\infty}) = -(S^{-T} \otimes I)\left[(I \otimes D) + (\hat{A}^T \otimes I)\right]^{-1} (\hat{C}^T \otimes \tilde{C}^T)\operatorname{vec}(I).$$

Combining this with

$$\begin{aligned}(S^{-1} \otimes I)\operatorname{vec}(\bar{T}\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^T\,\mathrm{e}^{D\bar{T}}) &= (\bar{T}\,\mathrm{e}^{\hat{A}\bar{T}}\,\hat{B} \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B})\operatorname{vec}(I),\\ (S^{-1} \otimes I)\operatorname{vec}(\tilde{P}_{\bar{T}}) &= \left[(I \otimes D) + (\hat{A} \otimes I)\right]^{-1} (\mathrm{e}^{\hat{A}\bar{T}}\,\hat{B} \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B} - \hat{B} \otimes \tilde{B})\operatorname{vec}(I)\end{aligned}$$

leads to the following:

$$\begin{aligned}&\operatorname{tr}([\tilde{P}_{\bar{T}} - \bar{T}\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^T\,\mathrm{e}^{D\bar{T}}]e_i e_i^T \tilde{Q}_{\infty})\\ &= \operatorname{vec}^T(I)\left[(\hat{B}^T\,\mathrm{e}^{\hat{A}^T\bar{T}} \otimes \tilde{B}^T\,\mathrm{e}^{D\bar{T}} - \hat{B}^T \otimes \tilde{B}^T)\left[(I \otimes D) + (\hat{A}^T \otimes I)\right]^{-1} - (\bar{T}\hat{B}^T\,\mathrm{e}^{\hat{A}^T\bar{T}} \otimes \tilde{B}^T\,\mathrm{e}^{D\bar{T}})\right]\\ &\quad\times (I \otimes e_i e_i^T)\left[-(I \otimes D) - (\hat{A}^T \otimes I)\right]^{-1} (\hat{C}^T \otimes \tilde{C}^T)\operatorname{vec}(I)\end{aligned}$$

Using (4b) and evaluating the expression

$$\operatorname{tr}([\tilde{P}_{2,\bar{T}} - \bar{T}\,\mathrm{e}^{A\bar{T}}\, B\tilde{B}^T\,\mathrm{e}^{D\bar{T}}]e_i e_i^T \tilde{Q}_{2,\infty}) = \operatorname{vec}^T(\tilde{P}_{2,\bar{T}}^T - \bar{T}\,\mathrm{e}^{D\bar{T}}\,\tilde{B}B^T\,\mathrm{e}^{A^T\bar{T}})(I \otimes e_i e_i^T)\operatorname{vec}(\tilde{Q}_{2,\infty})$$

further by inserting the vectorized form of the matrices yields (28). □

Inspired by the first-order optimality conditions as presented in Theorem 3.2 and IRKA for linear systems in [5], we propose an iterative algorithm, see Algorithm 1, which we refer to as *time-limited IRKA-type algorithm*. The scheme is characterized by an additional term in the right side of the Sylvester equations in comparison to the classical IRKA. These Sylvester equations provide the projection matrices V and W that are used to determine the reduced system (2). However, we would like to point out that the proposed algorithm in general does not construct reduced-order systems which satisfy the first-order necessary conditions for optimality. Thus, our next goal is to derive expressions, which allow us to estimate how far away the obtained reduced-order systems corresponding to Algorithm 1 are from satisfying the optimality conditions exactly.

Theorem 3.3. *Let $\hat{A}$, $\hat{B}$ and $\hat{C}$ be the reduced order matrices computed by Algorithm 1. Then, the difference between the left and the right side in (26) is*

$$E_c = (I \otimes \hat{C})\left[(I \otimes \hat{A}) + (D \otimes I)\right]^{-1} (\mathrm{e}^{D\bar{T}}\,\tilde{B} \otimes (W^TV)^{-1}W^T(\mathrm{e}^{A\,\mathrm{Pr}\,\bar{T}} - \mathrm{e}^{A\bar{T}})B)\operatorname{vec}(I)$$

Algorithm 1 Time-limited IRKA-type Algorithm

Input: The system matrices: A, B, C.
Output: The reduced matrices: $\hat{A}, \hat{B}, \hat{C}$.

1: Make an initial (random) guess of the reduced matrices $\hat{A}, \hat{B}, \hat{C}$.
2: **while** not converged **do**
3: Perform the spectral decomposition of $\hat{A}$ and define:
$\Lambda = S\hat{A}S^{-1},\ \tilde{B} = S\hat{B},\ \tilde{C} = \hat{C}S^{-1}$.
4: Solve for V and W:
$-VD - AV = B\tilde{B}^T - e^{A\bar{T}}B\tilde{B}^Te^{D\bar{T}}$,
$-WD - A^TW = C^T\tilde{C} - e^{A^T\bar{T}}C^T\tilde{C}e^{D\bar{T}}$,
5: Determine the reduced matrices:
$\hat{A} = (W^TV)^{-1}W^TAV, \qquad \hat{B} = (W^TV)^{-1}W^TB, \qquad \hat{C} = CV.$
6: **end while**

and equation (27) is satisfied up to the error term

$$E_b = (\hat{B}^T \otimes I)\left[(I \otimes D) + (\hat{A}^T \otimes I)\right]^{-1}(V^T(\mathrm{e}^{A^T \mathrm{Pr}^T \bar{T}} - \mathrm{e}^{A^T\bar{T}})C^T \otimes \mathrm{e}^{D\bar{T}}\tilde{C}^T)\,\mathrm{vec}(I),$$

where $\mathrm{Pr} := V(W^TV)^{-1}W^T$. *For all* $i = 1, \ldots, r$ *the deviation in (28) is* $E_\lambda^i = E_{\lambda,1}^i + E_{\lambda,2}^i$, *where*

$$\begin{aligned} E_{\lambda,1}^i = &\mathrm{vec}^T(I)(\hat{C} \otimes \tilde{C})\left[(I \otimes D) + (\hat{A} \otimes I)\right]^{-1}(I \otimes e_ie_i^T) \\ &\times \Bigg(\left[(I \otimes D) + (\hat{A} \otimes I)\right]^{-1}((W^TV)^{-1}W^T(\mathrm{e}^{A\,\mathrm{Pr}\,\bar{T}} - \mathrm{e}^{A\bar{T}})B \otimes \mathrm{e}^{D\bar{T}}\tilde{B}) \\ &\quad -(\bar{T}(W^TV)^{-1}W^T(\mathrm{e}^{A\,\mathrm{Pr}\,\bar{T}} - \mathrm{e}^{A\bar{T}})B \otimes \mathrm{e}^{D\bar{T}}\tilde{B})\Bigg)\mathrm{vec}(I) \end{aligned}$$

and the second term is given by

$$\begin{aligned} E_{\lambda,2}^i = &\ \mathrm{vec}^T(I)(C\,\mathrm{e}^{A\bar{T}} \otimes \tilde{C}\,\mathrm{e}^{D\bar{T}}) \\ &\times \left[(V \otimes I)\left[(I \otimes D) + (\hat{A} \otimes I)\right]^{-1}((W^TV)^{-1}W^T \otimes I) - [(I \otimes D) + (A \otimes I)]^{-1}\right] \\ &\times (I \otimes e_ie_i^T)\left[[(I \otimes D) + (A \otimes I)]^{-1}(\mathrm{e}^{A\bar{T}}B \otimes \mathrm{e}^{D\bar{T}}\tilde{B} - B \otimes \tilde{B}) - (\bar{T}\,\mathrm{e}^{A\bar{T}}B \otimes \mathrm{e}^{D\bar{T}}\tilde{B})\right] \\ &\times \mathrm{vec}(I). \end{aligned}$$

Proof. The left side of (26) can be expressed as

$$(I \otimes \hat{C})\left[(I \otimes \hat{A}) + (D \otimes I)\right]^{-1}(\mathrm{e}^{D\bar{T}}\tilde{B} \otimes (W^TV)^{-1}W^T\,\mathrm{e}^{A\bar{T}}B - \tilde{B} \otimes \hat{B})\,\mathrm{vec}(I) + E_c,$$

where we apply that $\mathrm{e}^{\hat{A}\bar{T}}\hat{B} = (W^TV)^{-1}W^T\,\mathrm{e}^{A\,\mathrm{Pr}\,\bar{T}}B$. We set $\hat{K} := (I \otimes \hat{A}) + (D \otimes I)$ and

$K := (I \otimes A) + (D \otimes I)$ and obtain

$$
\begin{aligned}
&(I \otimes \hat{C})\hat{K}^{-1}(\mathrm{e}^{D\bar{T}}\,\tilde{B} \otimes (W^TV)^{-1}W^T\,\mathrm{e}^{A\bar{T}}\,B - \tilde{B} \otimes \hat{B})\,\mathrm{vec}(I)\\
&= (I \otimes \hat{C})\hat{K}^{-1}(I \otimes (W^TV)^{-1}W^T)(\mathrm{e}^{D\bar{T}}\,\tilde{B} \otimes \mathrm{e}^{A\bar{T}}\,B - \tilde{B} \otimes B)\,\mathrm{vec}(I)\\
&= (I \otimes \hat{C})\hat{K}^{-1}(I \otimes (W^TV)^{-1}W^T)K\,\mathrm{vec}(V)\\
&= (I \otimes \hat{C})\hat{K}^{-1}(I \otimes (W^TV)^{-1}W^T)K\,\mathrm{vec}(V(W^TV)^{-1}W^TV)\\
&= (I \otimes \hat{C})\hat{K}^{-1}(I \otimes (W^TV)^{-1}W^T)K(I \otimes V(W^TV)^{-1}W^T)\,\mathrm{vec}(V)\\
&= (I \otimes \hat{C})\hat{K}^{-1}\hat{K}(I \otimes (W^TV)^{-1}W^T)\,\mathrm{vec}(V)\\
&= (I \otimes C)(I \otimes V)(I \otimes (W^TV)^{-1}W^T)\,\mathrm{vec}(V) = (I \otimes C)\,\mathrm{vec}(V)\\
&= (I \otimes C)K^{-1}(\mathrm{e}^{D\bar{T}}\,\tilde{B} \otimes \mathrm{e}^{A\bar{T}}\,B - \tilde{B} \otimes B)\,\mathrm{vec}(I),
\end{aligned}
$$

where the last term above is the right side of (26). The left side of (27) is given by

$$
(\hat{B}^T \otimes I)\left[(I \otimes D) + (\hat{A}^T \otimes I)\right]^{-1}(V^T\,\mathrm{e}^{A^T\bar{T}}\,C^T \otimes \mathrm{e}^{D\bar{T}}\,\tilde{C}^T - \hat{C}^T \otimes \tilde{C}^T)\,\mathrm{vec}(I) + E_b,
$$

taking the identity $\mathrm{e}^{\hat{A}^T\bar{T}}\,\hat{C}^T = V^T\,\mathrm{e}^{A^T\,\mathrm{Pr}^T\,\bar{T}}\,C^T$ into account. So, by setting $\hat{K}_2 := (I \otimes D) + (\hat{A} \otimes I)$ and $K_2 := (I \otimes D) + (A \otimes I)$, we have

$$
\begin{aligned}
&(\hat{B}^T \otimes I)\hat{K}_2^{-T}(V^T\,\mathrm{e}^{A^T\bar{T}}\,C^T \otimes \mathrm{e}^{D\bar{T}}\,\tilde{C}^T - \hat{C}^T \otimes \tilde{C}^T)\,\mathrm{vec}(I)\\
&= (\hat{B}^T \otimes I)\hat{K}_2^{-T}(V^T \otimes I)(\mathrm{e}^{A^T\bar{T}}\,C^T \otimes \mathrm{e}^{D\bar{T}}\,\tilde{C}^T - C^T \otimes \tilde{C}^T)\,\mathrm{vec}(I)\\
&= (\hat{B}^T \otimes I)\hat{K}_2^{-T}(V^T \otimes I)K_2^T\,\mathrm{vec}(W^T)\\
&= (\hat{B}^T \otimes I)\hat{K}_2^{-T}(V^T \otimes I)K_2^T\,\mathrm{vec}(W^TV(W^TV)^{-1}W^T)\\
&= (\hat{B}^T \otimes I)\hat{K}_2^{-T}(V^T \otimes I)K_2^T(W(W^TV)^{-T}V^T \otimes I)\,\mathrm{vec}(W^T)\\
&= (\hat{B}^T \otimes I)\hat{K}_2^{-T}\hat{K}_2^T(V^T \otimes I)\,\mathrm{vec}(W^T)\\
&= (B^T \otimes I)(W(W^TV)^{-T} \otimes I)(V^T \otimes I)\,\mathrm{vec}(W^T) = (B^T \otimes I)\,\mathrm{vec}(W^T)\\
&= (B^T \otimes I)K_2^{-T}(\mathrm{e}^{A^T\bar{T}}\,C^T \otimes \mathrm{e}^{D\bar{T}}\,\tilde{C}^T - C^T \otimes \tilde{C}^T)\,\mathrm{vec}(I)
\end{aligned}
\tag{29}
$$

which is the right side of (27). The left side of (28) is given by

$$
\begin{aligned}
E_{\lambda,1}^i + \mathrm{vec}^T(I)(\hat{C} \otimes \tilde{C})\hat{K}_2^{-1}(I \otimes e_ie_i^T)\Big(&\hat{K}_2^{-1}((W^TV)^{-1}W^T\,\mathrm{e}^{A\bar{T}}\,B \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B} - \hat{B} \otimes \tilde{B})\\
&-(\bar{T}(W^TV)^{-1}W^T\,\mathrm{e}^{A\bar{T}}\,\hat{B} \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B})\Big)\,\mathrm{vec}(I).
\end{aligned}
$$

For the term right of $(I \otimes e_i e_i^T)$ it holds that

$$\begin{aligned}
&\Big[\hat{K}_2^{-1}((W^TV)^{-1}W^T \mathrm{e}^{A\bar{T}} B \otimes \mathrm{e}^{D\bar{T}} \tilde{B} - \hat{B} \otimes \tilde{B})\\
&\quad -(\bar{T}(W^TV)^{-1}W^T \mathrm{e}^{A\bar{T}} \hat{B} \otimes \mathrm{e}^{D\bar{T}} \tilde{B})\Big] \operatorname{vec}(I)\\
&= \hat{K}_2^{-1}((W^TV)^{-1}W^T \otimes I)(\mathrm{e}^{A\bar{T}} B \otimes \mathrm{e}^{D\bar{T}} \tilde{B} - B \otimes \tilde{B}) \operatorname{vec}(I)\\
&\quad - (\bar{T}(W^TV)^{-1}W^T \mathrm{e}^{A\bar{T}} B \otimes \mathrm{e}^{D\bar{T}} \tilde{B}) \operatorname{vec}(I)\\
&= \hat{K}_2^{-1}((W^TV)^{-1}W^T \otimes I)K_2 \operatorname{vec}(V^T) - (\bar{T}(W^TV)^{-1}W^T \mathrm{e}^{A\bar{T}} B \otimes \mathrm{e}^{D\bar{T}} \tilde{B}) \operatorname{vec}(I)\\
&= \hat{K}_2^{-1}((W^TV)^{-1}W^T \otimes I)K_2 \operatorname{vec}(V^TW(W^TV)^{-T}V^T)\\
&\quad - (\bar{T}(W^TV)^{-1}W^T \mathrm{e}^{A\bar{T}} B \otimes \mathrm{e}^{D\bar{T}} \tilde{B}) \operatorname{vec}(I)\\
&= \hat{K}_2^{-1}((W^TV)^{-1}W^T \otimes I)K_2(V(W^TV)^{-1}W^T \otimes I) \operatorname{vec}(V^T)\\
&\quad - (\bar{T}(W^TV)^{-1}W^T \mathrm{e}^{A\bar{T}} B \otimes \mathrm{e}^{D\bar{T}} \tilde{B}) \operatorname{vec}(I)\\
&= ((W^TV)^{-1}W^T \otimes I) \operatorname{vec}(V^T) - (\bar{T}(W^TV)^{-1}W^T \mathrm{e}^{A\bar{T}} B \otimes \mathrm{e}^{D\bar{T}} \tilde{B}) \operatorname{vec}(I)\\
&= (W^TV)^{-1}W^T \otimes I) \Big[K_2^{-1}(\mathrm{e}^{A\bar{T}} B \otimes \mathrm{e}^{D\bar{T}} \tilde{B} - B \otimes \tilde{B}) - (\bar{T} \mathrm{e}^{A\bar{T}} B \otimes \mathrm{e}^{D\bar{T}} \tilde{B})\Big] \operatorname{vec}(I)
\end{aligned}$$

Since $((W^TV)^{-1}W^T \otimes I)$ and $(I \otimes e_i e_i^T)$ commute, it remains to analyze the following term

$$\operatorname{vec}^T(I)(\hat{C} \otimes \tilde{C})\hat{K}_2^{-1}((W^TV)^{-1}W^T \otimes I) = \Big[(W(W^TV)^{-T} \otimes I)\hat{K}_2^{-T}(\hat{C}^T \otimes \tilde{C}^T) \operatorname{vec}(I)\Big]^T.$$

We add a zero such that

$$\begin{aligned}
&(W(W^TV)^{-T} \otimes I)\hat{K}_2^{-T}(\hat{C}^T \otimes \tilde{C}^T) \operatorname{vec}(I)\\
&= (W(W^TV)^{-T} \otimes I)\hat{K}_2^{-T}(V^T \otimes I))[(C^T \otimes \tilde{C}^T) - (\mathrm{e}^{A^T\bar{T}} C^T \otimes \mathrm{e}^{D\bar{T}} \tilde{C}^T)] \operatorname{vec}(I)\\
&\quad + (W(W^TV)^{-T} \otimes I)\hat{K}_2^{-T}(V^T \otimes I))(\mathrm{e}^{A^T\bar{T}} C^T \otimes \mathrm{e}^{D\bar{T}} \tilde{C}^T) \operatorname{vec}(I)
\end{aligned}$$

Using the same steps as in (29), we find

$$\begin{aligned}
&(W(W^TV)^{-T} \otimes I)\hat{K}_2^{-T}(V^T \otimes I))[(C^T \otimes \tilde{C}^T) - (\mathrm{e}^{A^T\bar{T}} C^T \otimes \mathrm{e}^{D\bar{T}} \tilde{C}^T)] \operatorname{vec}(I)\\
&= K_2^{-T}[(C^T \otimes \tilde{C}^T) - (\mathrm{e}^{A^T\bar{T}} C^T \otimes \mathrm{e}^{D\bar{T}} \tilde{C}^T)] \operatorname{vec}(I).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\operatorname{vec}^T(I)(\hat{C} \otimes \tilde{C})\hat{K}_2^{-1}((W^TV)^{-1}W^T \otimes I) = \operatorname{vec}^T(I)(C \otimes \tilde{C})K_2^{-1}\\
&+ \operatorname{vec}^T(I)(C\, \mathrm{e}^{A\bar{T}} \otimes \tilde{C}\, \mathrm{e}^{D\bar{T}}) \Big[(V \otimes I)\hat{K}_2^{-1}((W^TV)^{-1}W^T \otimes I) - K_2^{-1}\Big]. \qquad (30)
\end{aligned}$$

The term in (30) provides $E^i_{\lambda,2}$ which concludes the proof. □

Theorem 3.3 allows us to point out the cases in which Algorithm 1 works well. The method is expected to perform well whenever the error expressions E_b, E_c and E^i_λ are small. By Theorem 3.3, the error in the optimality condition (26) is bounded as follows:

$$\|E_c\|_2 \leq \sqrt{m}k_c \left\|\mathrm{e}^{D\bar{T}} \tilde{B}\right\|_2 \left\|(W^TV)^{-1}W^T(\mathrm{e}^{A\,\mathrm{Pr}\,\bar{T}} - \mathrm{e}^{A\bar{T}})B\right\|_2,$$

where $k_c > 0$ is a suitable constant. Thus, $\|E_c\|_2$ is small if $\left\|(W^TV)^{-1}W^T(\mathrm{e}^{A\,\mathrm{Pr}\,\bar{T}} - \mathrm{e}^{A\bar{T}})B\right\|_2$ is small. At the same time

$$\left\|\mathrm{e}^{D\bar{T}}\,\tilde{B}\right\|_2 \leq \mathrm{e}^{\lambda_{\max}\bar{T}}\left\|\tilde{B}\right\|_2$$

should not be too large which is given if the largest eigenvalue $\lambda_{\max}$ of $\hat{A}$ is small enough or ideally negative (asymptotic stability of the reduced system). Similar conclusions can be made when looking at E_b. It is bounded by

$$\|E_b\|_2 \leq \sqrt{p}k_b\left\|\tilde{C}\,\mathrm{e}^{D\bar{T}}\right\|_2\left\|C(\mathrm{e}^{\mathrm{Pr}\,A\bar{T}} - \mathrm{e}^{A\bar{T}})V\right\|_2$$

with a sufficiently large constant $k_b > 0$. Hence, if $\left\|C(\mathrm{e}^{\mathrm{Pr}\,A\bar{T}} - \mathrm{e}^{A\bar{T}})V\right\|_2$ is small, then condition (27) is approximately satisfied. Now, $\left|E^i_{\lambda,1}\right|$ can be bounded in a similar way as $\|E_c\|_2$ such that it is also small if $\left\|(W^TV)^{-1}W^T(\mathrm{e}^{A\,\mathrm{Pr}\,\bar{T}} - \mathrm{e}^{A\bar{T}})B\right\|_2$ is neglectable, whereas for $\left|E^i_{\lambda,2}\right|$ it is required to have the product

$$\begin{aligned}&\left\|C\,\mathrm{e}^{A\bar{T}}\right\|_2\left\|\tilde{C}\,\mathrm{e}^{D\bar{T}}\right\|_2\\ &\quad\times\left\|(V\otimes I)\left[(I\otimes D) + (\hat{A}\otimes I)\right]^{-1}((W^TV)^{-1}W^T\otimes I) - [(I\otimes D) + (A\otimes I)]^{-1}\right\|_2\end{aligned}$$

small. The asymptotically stable matrix A is also helpful in this context.

4 Numerical Experiments

In this section, we investigate the efficiency of the time-limited IRKA inspired algorithm, see Algorithm 1, and compare it with conventional IRKA (unbounded time), see [5]. All the experiments are done in MATLAB® 8.0.0.783 (R2012b) on a machine Intel®Xeon®CPU X5650 @ 2.67GHz with 48 GB RAM. We run both iterative algorithms until the relative change in the eigenvalues of $\hat{A}$ becomes less a tolerance of 10^{-8}. We initialize conventional IRKA randomly, and we use the reduced-order system obtained by conventional IRKA as an initial guess for Algorithm 1. In Table 1, we list the examples used in order to compare the algorithms. For all examples, we compare the impulse responses of the systems, which is simulated using the `impulse` command from MATLAB. To quantify the quality of reduced-order systems, we determine either the absolute or the relative error, depending on weather the impulse response crosses zero or not. We define the absolute $\mathcal{E}^{(a)}(t)$ and relative errors $\mathcal{E}^{(r)}(t)$, respectively, as follows:

$$\mathcal{E}^{(a)}(t) := \|y^{(\delta)}(t) - y_r^{(\delta)}(t)\| \quad \text{and} \quad \mathcal{E}^{(r)}(t) := \frac{\|y^{(\delta)}(t) - y_r^{(\delta)}(t)\|}{\|y(t)\|}, \tag{31}$$

where $y^{(\delta)}$ and $y_r^{(\delta)}$ are the impulses responses of original and reduced-order systems. In addition to this, we numerically examine how far away the reduced-order systems due to IRKA and Algorithm 1

Example	n	m	p
Heat equation	200	1	1
Clamped beam model	348	1	1
Component 1r of the International Space Station	270	3	3

Table 1: A list of examples with their dimensions (n), the numbers of inputs (m) and outputs (p). These examples are taken from `http://slicot.org/20-site/126-benchmark-examples-for-model-reduction`.

Method	$\mathcal{E}_c$	$\mathcal{E}_b$	$\mathcal{E}_\lambda$
IRKA	2.7×10^{-3}	2.7×10^{-3}	9.10×10^{-3}
TL-IRKA	1.39×10^{-4}	1.39×10^{-4}	1.58×10^{-1}

Table 2: Heat example: relative errors in satisfying the optimality conditions.

are from satisfying the optimality conditions (26) – (28). To measure this, we first define the following quantities:

$$\mathcal{E}_c = \|\mathcal{R}_l^{(c)} - R_r^{(c)}\|/\|\mathcal{R}_l^{(c)}\|, \tag{32a}$$

$$\mathcal{E}_b = \|\mathcal{R}_l^{(b)} - R_r^{(b)}\|/\|\mathcal{R}_l^{(b)}\|, \tag{32b}$$

$$\mathcal{E}_\lambda = \max_i (\mathcal{R}_{\lambda_i}), \qquad \mathcal{R}_{\lambda_i} = \left|\mathcal{R}_l^{(\lambda_i)} - \mathcal{R}_r^{(\lambda_i)}\right| / \left|\mathcal{R}_l^{(\lambda_i)}\right|, \tag{32c}$$

where $\mathcal{R}_l^{(c)}$ and $\mathcal{R}_r^{(c)}$ are the left and right sides of (26); $\mathcal{R}_l^{(b)}$ and $\mathcal{R}_r^{(b)}$ are the left and right sides of (27); $\mathcal{R}_l^{(\lambda_i)}$ and $\mathcal{R}_r^{(\lambda_i)}$ are the left and right sides of (28); $\max(\cdot)$ denotes the maximum.

In the following, we discuss each of these examples in detail. Beginning with the heat example, we compute the reduced-order systems by employing conventional IRKA and Algorithm 1 of order $r = 5$. We consider the terminal time $\bar{T} = 1$. In Figure 1, we compare the impulse response which shows that Algorithm 1 yields a reduced-order system, replicating the systems dynamics better in the time interval $[0, \bar{T}]$. Furthermore, as it has been noted in Section 3, Algorithm 1 does not yield a reduced-order system, satisfying the optimality conditions. Thus, in Table 2 we measure the error of the reduced-order systems obtained via IRKA and Algorithm 1 in the optimality conditions as described in (32). The table shows that for the heat example, Algorithm 1 does a better job in satisfying the two optimality conditions, and in contrast the third condition is satisfied better by the reduced-order system due to conventional IRKA.

As a second example, we have taken a beam model which is reduced to the order $r = 10$ using the IRKA and Algorithm 1. For this, we set the terminal time to $\bar{T} = 2$. Next, we compare the impulse responses of the original and reduced-order systems in Figure 2. Clearly, we observe that Algorithm 1 produces a better reduced-order system as compared to IRKA at least within the time interval of interest. Furthermore, in Table 3, we measure the error of the obtained reduced-order systems in the optimality conditions, where we make a similar observation as in the heat example.

Lastly, we present the results for the model of a space station. We first set the terminal time to $\bar{T} = 1$. For this example, we construct reduced systems of order $r = 20$ via IRKA and Algorithm 1 and compare the quality of them using the impulse response. Since the example has 3 inputs and

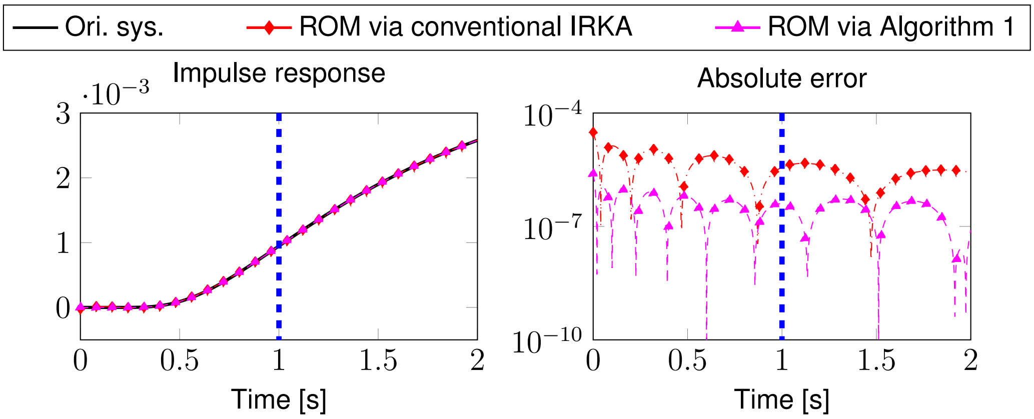

Figure 1: Heat example: a comparison of the impulse response of the original system and reduced-order system obtained via IRKA and Algorithm 1.

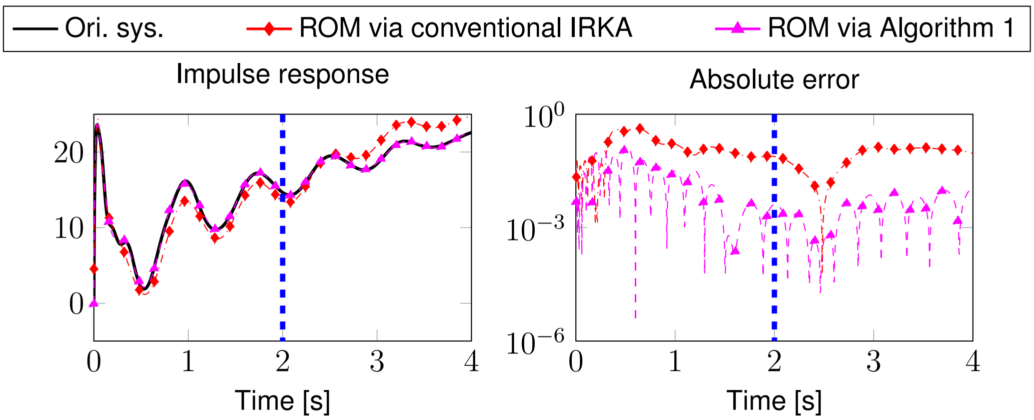

Figure 2: Beam example: a comparison of the impulse response of the original system and reduced-order system obtained via IRKA and Algorithm 1.

Method	$\mathcal{E}_c$	$\mathcal{E}_b$	$\mathcal{E}_\lambda$
IRKA	5.96×10^{-2}	5.96×10^{-2}	9.47×10^{-2}
TL-IRKA	3.94×10^{-4}	3.94×10^{-4}	1.26×10^{-1}

Table 3: Beam example: maximum relative error in satisfying the optimality conditions.

3 outputs, for brevity we refrain to plot the impulse response, but we rather plot the norm absolute error which is shown in Figure 3. We observe that Algorithm 1 constructs a reduced-order system which replicates the dynamics better within the time interval of interest. For this example, we again compute how far away the reduced-order systems are from satisfying the optimality conditions exactly in Table 4. For this example as well, Algorithm 1 does a better job than IRKA in satisfying the first two conditions, but fails to perform better for the third conditions. However, importantly, Algorithm 1 yields a better reduced-order system.

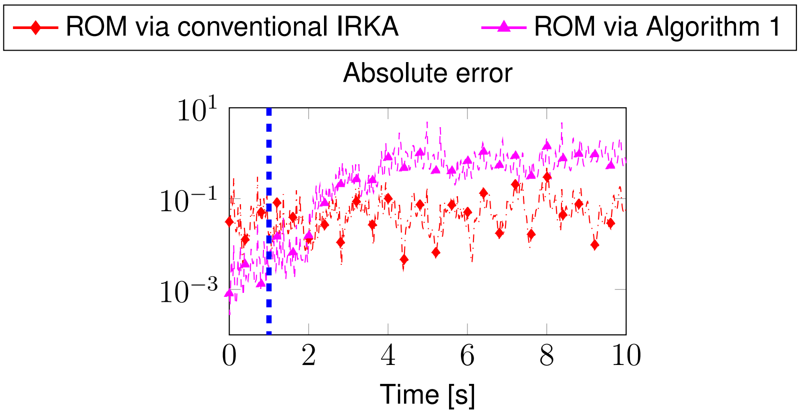

Figure 3: ISS example: a comparison of the impulse response of the original system and reduced-order system obtained via IRKA and Algorithm 1.

Method	$\mathcal{E}_c$	$\mathcal{E}_b$	$\mathcal{E}_\lambda$
IRKA	2.61×10^{-1}	1.62×10^{-1}	1.08×10^{-1}
TL-IRKA	6.00×10^{-2}	5.43×10^{-3}	4.46×10^{-1}

Table 4: ISS example: relative error in satisfying the optimality conditions.

5 Conclusions

In this work, we have studied large scale linear time-invariant systems which we aimed to reduce. We showed that the error between the original and the reduced system on a finite time interval can be bounded using the so-called time-limited $\mathcal{H}_2$-norm. In order to find a reduced order model with a small output error, we minimized the $\mathcal{H}_2$-norm with respect to the reduced order system matrices. This resulted in necessary conditions for optimality using representation of the time-limited $\mathcal{H}_2$-norm based on the time-limited Gramians. Reduced systems satisfying theses condition are expected to perform well on the finite time interval of interest. Based on these optimality conditions, we propose an iterative scheme which is inspired by the iterative rational Krylov algorithm [5]. Moreover, the error of the proposed iterative algorithm in the derived optimality conditions has been analyzed to point out the cases in which the proposed method works particularly well. We concluded this paper by comparing conventional IRKA, an algorithm leading to a good reduced system on an infinite time horizon, with the proposed iterative scheme in several numerical experiments. The simulations showed that time-limited IRKA can outperform IRKA on the finite time interval of interest.

As we have seen, the proposed iterative-type algorithm for the time-limited problem does not satisfy the optimality conditions exactly. Therefore, it would be worthwhile to come up with an improved algorithm, allowing us to construct a reduced-order system which satisfy the derived optimality conditions exactly.

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