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Stability index for invariant manifolds of stochastic systems

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Abstract. A lot of works has been devoted to stability analysis of a stationary point for linear and non-linear systems of stochastic differential equations. Here we consider the stability of an invariant compact manifold of a non-linear system. To this end we derive a linearized system for orthogonal displacements of a solution from the manifold. For this system, we introduce notions of Lyapunov exponents, moment Lyapunov exponents, and stability index. The stability index controls the asymptotic behavior of solutions of the input system in a neighborhood of the manifold. Most extensively we study these problems in the case when the invariant manifold is an orbit.

1. Introduction

Consider an autonomous system of stochastic differential equations in the sense of Ito

$$dX = a_0(X)dt + \sum_{r=1}^{q} a_r(X)dw_r(t) \tag{1.1}$$

where X is a d-dimensional vector, $a_r(x)$, $r = 0, 1, ..., q$, are d-dimensional vector functions, and $w_r(t)$, $r = 1, ..., q$, are independent standard Wiener processes on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Let the origin be a stationary point for the system (1.1), i.e.,

$$a_r(0) = 0, \; r = 0, 1, ..., q$$

The linearized system for (1.1) has a form

$$dX = A_0 X dt + \sum_{r=1}^{q} A_r X dw_r(t) \tag{1.2}$$

where $A_r = \{a_r^{ij}\}$ is a $d \times d$-matrix with the elements $a_r^{ij} = \dfrac{\partial a_r^i}{\partial x^j}(0)$, $i, j = 1, ..., d$.

In the deterministic case, the solutions $X_x(t)$, $X_x(0) = x$, of the nonlinear system and the solutions of the linearized one usually have many common features in their asymptotic behavior if x is sufficiently small. The stochastic case is far intricate, and a great many asymptotic characteristics for (1.2) do not reflect the behavior of the solutions of (1.1). For example, such an important characteristic for the system (1.2) as the moment Lyapunov exponent

$$g(p) = \lim_{t\to\infty} \frac{1}{t} \ln E|X_x(t)|^p \tag{1.3}$$

is usually positive for sufficiently large $p > 0$ even for stable systems because of large deviations. At the same time, a situation is possible for the system (1.1) when all

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its trajectories $X_x(t)$ for $|x| \le r$, $r > 0$ is some number, $0 \le t < \infty$, are uniformly bounded. In this case the limit in (1.3) for the system (1.1) is always non-positive.

Recently Arnold and Khasminskii have proved a theorem in which they have indicated a characteristic that precisely relates (1.1) and (1.2) in the sense of asymptotic behavior of solutions. The characteristic is called stability index in [3].

Let $\mathbf{M}$ be an invariant manifold for (1.1), i.e., $x \in \mathbf{M}$ implies $X_x(t) \in \mathbf{M}$, $t \ge 0$. The manifold $\mathbf{M}$ is supposed to be not a stationary point. In the present paper we investigate the asymptotic behavior of the distance $\rho(X_x(t), \mathbf{M})$ for $x \notin \mathbf{M}$. Instead of (1.2), we derive a linearized system for orthogonal displacements of the solution from the manifold (briefly linearized orthogonal system). Then we introduce the notions of Lyapunov exponent, moment Lyapunov exponents, and stability index for the linearized orthogonal system. They are analogous to the known ones for the system (1.2). Finally we prove an analogue of the Arnold-Khasminskii theorem and thereby introduce the concept of stability index for invariant manifolds of the system (1.1).

In Section 2 we review some well-known results (the Khasminskii theorem [13], the Arnold-Oeljeklaus-Pardoux theorem [4], and the Baxendale theorem [6]) for the system (1.2) and the Arnold-Khasminskii theorem for the system (1.1). In Section 3 we give some auxiliary consequences of the Stroock-Varadhan support theorem. Most extensively we study the stability problems in the case when $\mathbf{M}$ is an orbit in $\mathbf{R}^d$. Sections 4–7 are devoted to the orbital stability provided that the orbit is a phase trajectory of a deterministic system and, besides that, the system noise vanishes on the trajectory. The orbital stability with diffusion on the very trajectory is considered in Section 8. And finally, the stability index for general invariant manifolds is studied in Section 9.

2. Preliminary

A large literature has been devoted to studying asymptotic properties of the linear autonomous stochastic system (1.2). Various characteristics of the asymptotic behavior of its solutions such as Lyapunov exponents, moment Lyapunov exponents, stability index, rotation numbers, and some others are derived and studied in [13], [14], [1]-[6] (see also references therein).

The first results related to Lyapunov exponents for the system (1.2) are due to Khasminskii [13], [14]. Adopting some ideas of Furstenberg [9], Khasminskii uses the new coordinates

$$\lambda = \frac{x}{|x|}, \quad \rho = \ln|x|, \quad x \ne 0,$$

shows that the projection of X onto the unit sphere $\mathbf{S}^{d-1}$, i.e., the process

$$\Lambda_\lambda(t) := \frac{X_x(t)}{|X_x(t)|}$$

is also a Markov diffusion process, and introduces a system for Λ. The Khasminskii system has the following form

$$d\Lambda = h_0(\Lambda)dt + \sum_{r=1}^{q} h_r(\Lambda)dw_r(t) \tag{2.1}$$

where the vector fields $h_r(\lambda)$, $r = 0, 1, ..., q$, on $\mathbf{S}^{d-1}$ are equal to

$$h_0(\lambda) = A_0\lambda - (A_0\lambda, \lambda)\lambda - \frac{1}{2}\sum_{r=1}^{q}(A_r\lambda, A_r\lambda)\lambda - \sum_{r=1}^{q}(A_r\lambda, \lambda)A_r\lambda + \frac{3}{2}\sum_{r=1}^{q}(A_r\lambda, \lambda)^2\lambda$$

$$h_r(\lambda) = A_r\lambda - (A_r\lambda, \lambda)\lambda, \ r = 1, ..., q \tag{2.2}$$

In what follows we shall suppose the Lie algebra condition to be fulfilled:

$$\dim LA(h_1, ..., h_q) = d - 1 \ \text{ for all } \lambda \in \mathbf{S}^{d-1} \tag{2.3}$$

where LA denotes the Lie algebra generated by the given vector fields. For many things some weaker conditions would be sufficient but in order to avoid some complications we impose (2.3).

In [13], [14] the following theorem is proved.

Theorem 2.1 (Khasminskii). *Under the condition* (2.3) *the process* Λ *is ergodic, there exists an invariant measure* $\mu(\lambda)$, *and, for any* $x \neq 0$, *there exists the limit (which does not depend on* x*)*

$$P\text{-a.s.} \lim_{t\to\infty} \frac{1}{t} \ln |X_x(t)| = \lim_{t\to\infty} \frac{1}{t} E \ln |X_x(t)| = \int_{\mathbf{S}^{d-1}} Q(\lambda) d\mu(\lambda) := \lambda^* \tag{2.4}$$

where

$$Q(\lambda) = (A_0\lambda, \lambda) + \frac{1}{2} \sum_{r=1}^{q} (A_r\lambda, A_r\lambda) - \sum_{r=1}^{q} (A_r\lambda, \lambda)^2 \tag{2.5}$$

The limit λ^* is called Lyapunov exponent of the system (1.2).

The next essential step in studying asymptotic properties of the solutions of (1.2) was connected with introducing the concept of moment Lyapunov exponents (the idea, used for another object of study, goes back to Molchanov [17]). First results here are due to Arnold [1] for the real noise case and to Arnold, Oeljeklaus, and Pardoux [4] for the white noise case.

Ito's formula gives for every $p \in \mathbf{R}$

$$d|X(t)|^p = (pQ(\Lambda) + \frac{1}{2} p^2 R(\Lambda)) |X(t)|^p dt + p \sum_{r=1}^{q} (A_r\Lambda, \Lambda) |X(t)|^p dw_r(t) \tag{2.6}$$

where $Q(\lambda)$ is from (2.5), and

$$R(\lambda) = \sum_{r=1}^{q} (A_r\lambda, \lambda)^2 \tag{2.7}$$

Let $X(0) = \lambda$, $|\lambda| = 1$. For any $p \in \mathbf{R}$, a strongly continuous semigroup $T_t(p)$ of positive operators on $\mathbf{C}(\mathbf{S}^{d-1})$ can be introduced :

$$T_t(p) f(\lambda) = E f(\Lambda_\lambda(t)) |X_\lambda(t)|^p, \ f \in \mathbf{C}(\mathbf{S}^{d-1}) \tag{2.8}$$

Let $L(p)$ be a generator of the semigroup $T_t(p)$. Under Lie algebra condition (2.3), any operator $T_t(p)$, $t > 0$, $-\infty < p < \infty$, is compact and irreducible (even strongly positive). We recall that a positive operator T in $\mathbf{C}(\mathbf{S}^{d-1})$ is called irreducible if $\{0\}$ and $\mathbf{C}(\mathbf{S}^{d-1})$ are the only T-invariant closed ideals and T is called strongly positive if $Tf(\lambda) > 0$, $\lambda \in \mathbf{S}^{d-1}$, for any $f \geq 0$, $f \neq 0$. The generalized Perron-Frobenius theorem ensures for each $p \in \mathbf{R}$ the existence of a strictly positive eigenfunction for $T_t(p)$ (and, consequently, for $L(p)$) corresponding to the principal eigenvalue. Some properties of the function $g(p)$ and a connection between $g(p)$ and $L(p)$ are given in [4].

Theorem 2.2 (Arnold-Oeljeklaus-Pardoux). *Under Lie algebra condition* (2.3) *the limit* $g(p)$ *in* (1.3) *exists for any* $p \in \mathbf{R}$ *and is independent of* x, $x \neq 0$. *The limit* $g(p)$ *is a convex analytic function of* $p \in \mathbf{R}$, $g(0) = 0$, $g(p)/p$ *is increasing, and*

$$g'(0) = \lim_{p \to 0} \frac{g(p)}{p} = \lambda^* \tag{2.9}$$

Further, the moment Lyapunov exponent $g(p)$ *is an eigenvalue for* $L(p)$ *with a strictly positive eigenfunction* $e_p(\lambda)$:

$$L(p)e_p(\lambda) = g(p)e_p(\lambda), \; e_p(\lambda) > 0, \; \lambda \in \mathbf{S}^{d-1} \tag{2.10}$$

The eigenvalue $g(p)$ *is simple and* $g(p)$ *strictly dominates the real part of any other point of the spectrum of* $L(p)$.

Let us note that if the matrix A_0 in (1.2) is replaced by $A_0 + \alpha I$, where α is a scalar and I is the identity matrix, then $g(p)$ is replaced by $g(p) + \alpha p$, and the new Lyapunov exponent is equal to $\lambda^* + \alpha$. Therefore in many cases we can restrict ourselves to the case $\lambda^* < 0$. If $\lambda^* < 0$ then the trivial solution of the system (1.2) is a.s. asymptotically stable. It is well known (see, for instance, [14]) and follows from (2.9) that in this case $g(p) < 0$ for all sufficiently small p, i.e., the solution $X = 0$ of (1.2) is p-stable for such p. It is shown in [4] that $g(p) \to \infty$ for $p \to \infty$ unless there exists a non-singular matrix G such that GA_rG^{-1}, $r = 1, ..., q$, are skew-symmetric matrices. If $g(p) \to \infty$ for $p \to \infty$ then the equation

$$g(p) = 0 \tag{2.11}$$

has a unique positive root γ^*. It is clear that the solution $X = 0$ of (1.2) is p-stable for $0 < p < \gamma^*$ and p-unstable for $p > \gamma^*$.

The concept of moment Lyapunov exponent was further developed by many other authors and especially by Baxendale. In particular Baxendale shows in [6] that the root γ^* of (2.11) is connected with the asymptotic behavior of the probability $P\{\sup_{t\geq 0} |X_x(t)| > \delta\}$, $|x|/\delta \to 0$, if $\gamma^* < 0$ and of the probability $P\{\inf_{t\geq 0} |X_x(t)| < \delta\}$, $|x|/\delta \to \infty$, if $\gamma^* > 0$.

Theorem 2.3 (Baxendale). *Assume* (2.3). *If* $\lambda^* < 0$ *and the equation* (2.11) *has a positive root* $\gamma^* > 0$ *then there exists* $K \geq 1$ *such that for all* $\delta > 0$ *and for all* x *with* $|x| < \delta$

$$\frac{1}{K}(|x|/\delta)^{\gamma^*} \leq P\{\sup_{t\geq 0} |X_x(t)| > \delta\} \leq K(|x|/\delta)^{\gamma^*} \tag{2.12}$$

If $\lambda^* > 0$ *and the equation* (2.11) *has a negative root* $\gamma^* < 0$ *then there exists* $K \geq 1$ *such that for all* $\delta > 0$ *and for all* x *with* $|x| > \delta$

$$\frac{1}{K}(|x|/\delta)^{\gamma^*} \leq P\{\inf_{t\geq 0} |X_x(t)| < \delta\} \leq K(|x|/\delta)^{\gamma^*} \tag{2.13}$$

Thus, Baxendale has established that the probability with which a solution of the linear system (1.2) exceeds a threshold is controlled by the number γ^*. Arnold and Khasminskii call this number stability index. Their main result in [3] consists in proving that the estimates (2.12)–(2.13) remain true for the nonlinear system as well.

Theorem 2.4. (Arnold-Khasminskii). *Let the system of linear approximation* (1.2) *for the system* (1.1) *be such that the condition* (2.3) *is fulfilled. Assume that the stability index γ^* for* (1.2) *does not vanish, $\gamma^* \neq 0$. Then*

Case $\gamma^* > 0$: There exists a sufficiently small $\rho > 0$ and positive constants a_1, a_2 such that for any $\delta \in (0,\ \rho)$ and all $|x| < \delta$ the solution $X_x(t)$ of (1.1) *satisfies the inequalities*

$$a_1(|x|/\delta)^{\gamma^*} \leq P\{\sup_{t\geq 0} |X_x(t)| > \delta\} \leq a_2(|x|/\delta)^{\gamma^*} \tag{2.14}$$

Case $\gamma^* < 0$: There exists a sufficiently small $\rho > 0$, positive constants a_3, a_4, and a constant $0 < \alpha < 1$ such that for any $\delta \in (0,\ \alpha\rho)$ and all $|x| < \delta$ the solution $X_x(t)$ of (1.1) *satisfies the inequalities*

$$c_3(|x|/\delta)^{\gamma^*} \leq P\{\inf_{0\leq t<\tau} |X_x(t)| < \delta\} \leq c_4(|x|/\delta)^{\gamma^*} \tag{2.15}$$

Here $\tau := \inf\{t : \ |X_x(t)| > \rho\}$.

Remark 2.1. As a matter of fact Arnold and Khasminskii have proved a more general theorem. They consider the situation when a nonlinear system is close to a homogeneous one in a neighborhood of the origin. The point is that the theory of moment Lyapunov exponents can be carried over to stochastic systems with positive homogeneous coefficients of degree one. For such systems, the stability index can also be introduced and the estimates (2.12)–(2.13) can be established (see [3] and references therein).

3. Invariant manifolds of a diffusion process

A set $\mathbf{S} \subset \mathbf{R}^d$ is said to be invariant for (1.1) if $x \in \mathbf{S}$ implies $X_x(t) \in \mathbf{S}$, $t \geq 0$. One can to find out whether the set $\mathbf{S}$ is invariant by the Stroock-Varadhan support theorem (see, for instance, [12]). This theorem has a more simple formulation for equations in the sense of Stratonovich

$$dX = a_0(X)dt + \sum_{r=1}^{q} a_r(X) \circ dw_r(t) \tag{3.1}$$

Suppose $a_0(x)$ to have bounded continuous first order derivatives and $a_r(x)$, $r = 1, ..., q$, to have bounded continuous second order derivatives in $\mathbf{R}^d$. Let $\mathbf{C}$ be a space of d-dimensional continuous functions on $[0, \infty)$ with the topology of the uniform convergence on finite closed intervals. Introduce $\mathbf{S}_{Str}(x) \subset \mathbf{C}$:

$$\mathbf{S}_{Str}(x) = \Big\{X(t),\ 0 \leq t < \infty :\ X(t) = x + \int_0^t a_0(X(s))ds +$$

$$\sum_{r=1}^{q} \int_0^t a_r(X(s))W_r'(s)ds,\ W_r \in \mathbf{W}\Big\}$$

where $W_r(s)$, $r = 1, ..., q$, are arbitrary smooth functions, $\mathbf{W}$ is a set of functions with continuous derivative on $[0, \infty)$.

Theorem 3.1 (Stroock-Varadhan support theorem*). Let $x \in \mathbf{R}^d$, $X_x(t)$ be a solution of the system* (3.1)*, $\mathbf{P}_x$ be its distribution, $\mathbf{S}(\mathbf{P}_x)$ be the support of $\mathbf{P}_x$ (i.e., the smallest closed set of $\mathbf{C}$ with measure equal to* 1*). Then*

$$\mathbf{S}(\mathbf{P}_x) = \bar{\mathbf{S}}_{Str}(x) \tag{3.2}$$

We shall use the following simple consequence of this theorem.

Corollary 3.1. *Let* $\mathbf{S} \subset \mathbf{R}^d$ *be a closed set. If for any* $x \in \mathbf{S}$ *every point of any trajectory from* $\mathbf{S}_{Str}(x)$ *belongs to* $\mathbf{S}$ *(what we shall write shortly:* $\mathbf{S}_{Str}(x;t) \subset \mathbf{S}$*) then* $\mathbf{S}$ *is invariant for* (3.1).

Using the well known connection between stochastic equations in the sense of Ito and of Stratonovich one can formulate the Stroock-Varadhan's theorem for the Ito system (1.1). We restrict ourselves to the following consequence:

Corollary 3.2. *Let* $\mathbf{S} \subset \mathbf{R}^d$ *be a closed set and*

$$\mathbf{S}_{Ito}(x;t) = \Big\{ X(t) : X(t) = x + \int_0^t a_0(X(s))ds - \frac{1}{2}\sum_{r=1}^{q}\int_0^t \frac{\partial a_r}{\partial x}(X(s))a_r(X(s))ds +$$

$$\sum_{r=1}^{q}\int_0^t a_r(X(s))W_r'(s)ds,\ W_r \in \mathbf{W} \Big\}$$

If $\mathbf{S}_{Ito}(x;t) \subset \mathbf{S}$ *for every* $x \in \mathbf{S}$ *then* $\mathbf{S}$ *is invariant for* (1.1).

Remark 3.1. Let $\mathbf{S}_{Str}(x;t) \subset \mathbf{S}$ (respectively $\mathbf{S}_{Ito}(x;t) \subset \mathbf{S}$) be fulfilled. Then Corollary 3.1 (respectively Corollary 3.2) is valid if $a_0(x)$ has bounded continuous first order derivatives and $a_r(x)$, $r = 1,...,q$, have bounded continuous second order derivatives in some neighborhood of $\mathbf{S}$.

Example 3.1. Consider the Khasminskii system (2.1) in $\mathbf{R}^d$. The set $\mathbf{S}_{Ito}(\lambda)$ has a form

$$\mathbf{S}_{Ito}(\lambda) = \{\Lambda(t),\ 0 \le t < \infty :\ \Lambda(t) = \lambda +$$

$$\int_0^t (A_0\Lambda - (A_0\Lambda,\Lambda)\Lambda - \frac{1}{2}\sum_{r=1}^{q} A_r^2\Lambda + \frac{1}{2}\sum_{r=1}^{q}(A_r^2\Lambda,\Lambda)\Lambda)ds +$$

$$\sum_{r=1}^{q}\int_0^t (A_r\Lambda - (A_r\Lambda,\Lambda)\Lambda)W_r' ds \Big\}$$

For $\Lambda(t)$ from $\mathbf{S}_{Ito}(\lambda)$ we have

$$\frac{d(1-(\Lambda,\Lambda))}{dt} = -(1-(\Lambda,\Lambda))(2(A_0\Lambda,\Lambda) - \sum_{r=1}^{q}(A_r^2\Lambda,\Lambda)) -$$

$$2(1-(\Lambda,\Lambda))\sum_{r=1}^{q}(A_r\Lambda,\Lambda)W_r',\ \Lambda(0) = \lambda$$

From here it easily follows that the unit sphere $\mathbf{S}^{d-1} = \{\lambda : (\lambda,\lambda) = 1\}$ is an invariant manifold for (2.1). It is clear also that the sets $\{\lambda : \lambda = 0\}$, $\{\lambda : 0 < (\lambda,\lambda) < 1\}$, $\{\lambda : (\lambda,\lambda) > 1\}$ are invariant ones for (2.1).

Example 3.2. Let $x = \xi(t)$ be a T-periodic solution of the deterministic system

$$\frac{dX}{dt} = a_0(X)$$

and $\mathbf{M}$ be the phase trajectory of this solution $(a_0(\xi(t)) \neq 0,\ 0 \leq t < T)$. Consider the following system in the sense of Stratonovich

$$dX = (\alpha_0(X)a_0(X) + b_0(X))dt + \sum_{r=1}^{q}(\alpha_r(X)a_0(X) + b_r(X)) \circ dw_r(t) \qquad (3.3)$$

where $b_r(x) = 0,\ r = 0, 1, ..., q$, if $x \in \mathbf{M}$, i.e., $b_r(\xi(t)) = 0,\ 0 \leq t < T$, and $\alpha_r(x)$ are scalars, $\alpha_0(x) \neq 0$ if $x \in \mathbf{M}$.

Using Corollary 3.1 it is not difficult to prove that $\mathbf{M}$ is an invariant manifold for (3.3).

Example 3.3. Let a k-dimensional sufficiently smooth manifold $\mathbf{M} \subset \mathbf{R}^d$ defined by equations

$$m_i(x^1, ..., x^d) = 0,\ i = 1, ..., d-k \qquad (3.4)$$

be an invariant manifold for (1.1).

Thus, the system (1.1) defines a diffusion process $P(t)$ on the manifold $\mathbf{M}$. Let us find a formula for a generator of the process $P(t)$.

For certainty, let the system (3.4) be resolvable with respect to $x^{k+1}, ..., x^d$ in some piece $\mathbf{M}_0$ of $\mathbf{M}$:

$$x^i = \varphi^i(x^1, ..., x^k),\ i = k+1, ..., d$$

We have in $\mathbf{M}_0$:

$$dX^i = a_0^i(X^1, ..., X^k, \varphi^{k+1}(X^1, ..., X^k), ..., \varphi^d(X^1, ..., X^k))dt +$$

$$\sum_{r=1}^{q} a_r^i(X^1, ..., X^k, \varphi^{k+1}(X^1, ..., X^k), ..., \varphi^d(X^1, ..., X^k))dw_r(t),$$

$$i = 1, ..., k, k+1, ..., d \qquad (3.5)$$

Due to Ito's formula

$$dX^m = \sum_{i=1}^{k} \frac{\partial \varphi^m}{\partial x^i}(a_0^i dt + \sum_{r=1}^{q} a_r^i dw_r(t)) + \frac{1}{2}\sum_{i,j=1}^{k} \frac{\partial^2 \varphi^m}{\partial x^i \partial x^j} \sum_{r=1}^{q} a_r^i a_r^j dt,\ m = k+1, ..., d \qquad (3.6)$$

where the arguments are the same as in (3.5).

Comparing the last $d-k$ equalities from (3.5) with (3.6) we obtain for the points from $\mathbf{M}_0$

$$\sum_{i=1}^{k} \frac{\partial \varphi^m}{\partial x^i}(x^1, ..., x^k) a_r^i(x^1, ..., x^k, \varphi^{k+1}(x^1, ..., x^k), ..., \varphi^d(x^1, ..., x^k)) =$$

$$a_r^m(x^1, ..., x^k, \varphi^{k+1}(x^1, ..., x^k), ..., \varphi^d(x^1, ..., x^k)),\ m = k+1, ..., d \qquad (3.7)$$

and (with the same arguments as in (3.7))

$$\sum_{i=1}^{k} \frac{\partial \varphi^m}{\partial x^i} a_0^i + \frac{1}{2}\sum_{i,j=1}^{k} \frac{\partial^2 \varphi^m}{\partial x^i \partial x^j} \sum_{r=1}^{q} a_r^i a_r^j = a_0^m,\ m = k+1, ..., d \qquad (3.8)$$

Let $g \in C^2(\mathbf{M})$ and let g have compact support. The function g can be expressed in $\mathbf{M}_0$ in terms of $x^1, ..., x^k$. There exists (and not unique) a function $\breve{g}(x^1, ..., x^d)$ which is C^2-function with compact support and which is an extension of g :

$$g(x^1, ..., x^k) = \breve{g}(x^1, ..., x^k, \varphi^{k+1}(x^1, ..., x^k), ..., \varphi^d(x^1, ..., x^k)) \tag{3.9}$$

Let us denote the generator for the process $P(t)$ by L and for the process $X(t)$ by $\breve{L}$. We have for $P \in \mathbf{M}_0$ (see the first k equalities from (3.5), and then (3.9), (3.7), and (3.8))

$$Lg(P) = Lg(x^1, ..., x^k) = \sum_{i=1}^{k} a_0^i \frac{\partial g}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{k} \sum_{r=1}^{q} a_r^i a_r^j \frac{\partial^2 g}{\partial x^i \partial x^j} =$$

$$\sum_{i=1}^{k} a_0^i \left(\frac{\partial \breve{g}}{\partial x^i} + \sum_{m=k+1}^{d} \frac{\partial \breve{g}}{\partial x^m} \cdot \frac{\partial \varphi^m}{\partial x^i}\right) +$$

$$\frac{1}{2} \sum_{i,j=1}^{k} \sum_{r=1}^{q} a_r^i a_r^j \left(\frac{\partial^2 \breve{g}}{\partial x^i \partial x^j} + \sum_{m=k+1}^{d} \frac{\partial^2 \breve{g}}{\partial x^i \partial x^m} \cdot \frac{\partial \varphi^m}{\partial x^j} + \sum_{m=k+1}^{d} \frac{\partial^2 \breve{g}}{\partial x^m \partial x^j} \cdot \frac{\partial \varphi^m}{\partial x^i}\right) +$$

$$\frac{1}{2} \sum_{i,j=1}^{k} \sum_{r=1}^{q} a_r^i a_r^j \left(\sum_{m=k+1}^{d} \sum_{l=k+1}^{d} \frac{\partial^2 \breve{g}}{\partial x^m \partial x^l} \cdot \frac{\partial \varphi^l}{\partial x^j} \cdot \frac{\partial \varphi^m}{\partial x^i} + \sum_{m=k+1}^{d} \frac{\partial \breve{g}}{\partial x^m} \cdot \frac{\partial^2 \varphi^m}{\partial x^i \partial x^j}\right) =$$

$$\sum_{i=1}^{d} a_0^i \frac{\partial \breve{g}}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{r=1}^{q} a_r^i a_r^j \frac{\partial^2 \breve{g}}{\partial x^i \partial x^j} = \breve{L}\breve{g}(x^1, ..., x^k, \varphi^{k+1}(x^1, ..., x^k), ..., \varphi^d(x^1, ..., x^k))$$

i.e., we prove a formula

$$Lg(P) = \breve{L}\breve{g}(P), \quad P \in \mathbf{M} \tag{3.10}$$

4. The linearized system for orthogonal displacement

Consider the system of deterministic differential equations

$$dX = a_0(X)dt \tag{4.1}$$

We suppose $x = \xi(t)$ to be a T-periodic solution of the system (4.1), $a_0(\xi(t)) \neq 0$ for every $0 \leq t < T$. Let $\mathbf{M}$ be the phase trajectory (orbit) of this solution. Results on orbital stability related to the first Lyapunov method see, for example, in [11], [18] and results related to the second Lyapunov method see in [15]. A method of orbital Lyapunov functions has been proposed in [15] for deterministic systems (4.1) and it has been extended in [16] to stochastic systems of the form

$$dX = a_0(X)dt + \sum_{r=1}^{q} a_r(X)dw_r(t) \tag{4.2}$$

It is assumed that

$$a_r(\xi(t)) = 0, \ 0 \leq t < T, \ r = 1, ..., q \tag{4.3}$$

and consequently $x = \xi(t)$ remains a T-periodic solution for the system (4.2) as well.

Some sufficient conditions for mean square orbital stability have been obtained in [16] provided that there is a sufficiently small neighborhood of orbit $\mathbf{M}$ which is invariant for the system (4.2).

We suppose U to be a tubular neighborhood (a toroidal tube) of the orbit $\mathbf{M}$ such that for any point $x \in U$ one can uniquely find a quantity $\vartheta(x)$, $0 \le \vartheta(x) < T$ for which $\xi(\vartheta(x))$ is the point on the trajectory $\mathbf{M}$ that is nearest to x. It is clear that the vector

$$\delta(x) = x - \xi(\vartheta(x))$$

is a displacement from the orbit normal to the vector $a_0(\xi(\vartheta(x)))$, i.e.,

$$\sum_{j=1}^{d}(x^j - \xi^j(\vartheta(x))) \cdot a_0^j(\xi(\vartheta(x))) = 0 \tag{4.4}$$

We suppose also that all the functions $a_r(x)$, $r = 0, 1, ..., q$, $x \in U$, are sufficiently smooth. Since we are interested in the local behavior of solutions of the system (4.2) close to $\mathbf{M}$ and $\mathbf{M}$ is a compact, without any loss we can consider the coefficients $a_r(x)$ to have uniformly bounded derivatives in $\mathbf{R}^d$. Let $|\delta(x)| \le r$ where r is sufficiently small.

Differentiating (4.4) with respect to x^i and taking into account the equality

$$\xi'(\vartheta(x)) = a_0(\xi(\vartheta(x)))$$

we obtain

$$a_0^i(\xi(\vartheta(x))) - |a_0(\xi(\vartheta(x)))|^2 \cdot \frac{\partial \vartheta}{\partial x^i}(x) +$$

$$\sum_{j=1}^{d}(x^j - \xi^j(\vartheta(x))) \cdot (A_0(\xi(\vartheta(x)))a_0(\xi(\vartheta(x))))^j \cdot \frac{\partial \vartheta}{\partial x^i}(x) = 0$$

where $A_0(x)$ is a matrix with the elements $a_0^{ij}(x) = \dfrac{\partial a_0^i}{\partial x^j}(x)$, $i, j = 1, ..., d$.

From here

$$\frac{\partial \vartheta}{\partial x^i}(x) = \frac{a_0^i(\xi(\vartheta(x)))}{\varphi(x)} \tag{4.5}$$

where

$$\varphi(x) = |a_0(\xi(\vartheta(x)))|^2 - (A_0(\xi(\vartheta(x)))a_0(\xi(\vartheta(x))), x - \xi(\vartheta(x)))$$

Using Ito's formula for $\delta(X)$ we find

$$d\delta^k(X) = a_0^k(X)dt + \sum_{r=1}^{q} a_r^k(X)dw_r(t) -$$

$$\frac{a_0^k(\xi(\vartheta(X)))}{\varphi(X)} \cdot \sum_{i=1}^{d} a_0^i(\xi(\vartheta(X))) \cdot (a_0^i(X)dt + \sum_{r=1}^{q} a_r^i(X)dw_r(t)) -$$

$$\frac{1}{2}(A_0(\xi(\vartheta(X)))a_0(\xi(\vartheta(X))))^k \cdot \frac{1}{\varphi^2(X)} \sum_{i,j=1}^{d} a_0^i(\xi(\vartheta(X)))a_0^j(\xi(\vartheta(X))) \sum_{r=1}^{q} a_r^i(X)a_r^j(X)dt -$$

$$\frac{1}{2}a_0^k(\xi(\vartheta(X))) \sum_{i,j=1}^{d} \frac{\partial^2 \vartheta}{\partial x^i \partial x^j}(X) \sum_{r=1}^{q} a_r^i(X)a_r^j(X)dt \tag{4.6}$$

In view of (4.3)

$$a_r(\xi(\vartheta(X))) = 0,\ X \in U$$

and we have

$$a_r^k(X) = (A_r(\xi(\vartheta(X)))\delta(X))^k + O(|\delta(X)|^2) \tag{4.7}$$

where $A_r(x)$ is the matrix with the elements $a_r^{ij}(x) = \dfrac{\partial a_r^i}{\partial x^j}(x),\ i,j = 1,...,d.$

Here and below all the O are uniform with respect to $0 \leq \vartheta < T$ and $|\delta(x)| \leq r$. Consequently

$$\sum_{r=1}^{q} a_r^i(X) a_r^j(X) = O(|\delta(X)|^2)$$

and (4.6) can be rewritten as

$$d\delta^k(X) = \frac{1}{\varphi(X)}(a_0^k(X)\varphi(X) - a_0^k(\xi(\vartheta(X))) \cdot \sum_{i=1}^{d} a_0^i(\xi(\vartheta(X)))a_0^i(X))dt +$$

$$\sum_{r=1}^{q}(a_r^k(X) - \frac{a_0^k(\xi(\vartheta(X))) \cdot \sum_{i=1}^{d} a_0^i(\xi(\vartheta(X)))a_r^i(X)}{\varphi(X)})dw_r(t) + O(|\delta(X)|^2)dt \tag{4.8}$$

We have (see expression for $\varphi(x)$)

$$a_0^k(X)\varphi(X) - a_0^k(\xi(\vartheta(X))) \cdot \sum_{i=1}^{d} a_0^i(\xi(\vartheta(X)))a_0^i(X) =$$

$$a_0^k(X) \cdot |a_0(\xi(\vartheta(X)))|^2 - a_0^k(\xi(\vartheta(X))) \cdot \sum_{i=1}^{d} a_0^i(\xi(\vartheta(X)))a_0^i(X) -$$

$$a_0^k(X) \cdot (A_0(\xi(\vartheta(X)))a_0(\xi(\vartheta(X))), X - \xi(\vartheta(X))) =$$

$$(a_0^k(X) - a_0^k(\xi(\vartheta(X)))) \cdot |a_0(\xi(\vartheta(X)))|^2 -$$

$$a_0^k(\xi(\vartheta(X))) \cdot \sum_{i=1}^{d} a_0^i(\xi(\vartheta(X))) \cdot (a_0^i(X) - a_0^i(\xi(\vartheta(X)))) -$$

$$a_0^k(X) \cdot (A_0(\xi(\vartheta(X)))a_0(\xi(\vartheta(X))), \delta(X)) \tag{4.9}$$

But

$$a_0^i(X) - a_0^i(\xi(\vartheta(X))) = (A_0(\xi(\vartheta(X)))\delta(X))^i + O(|\delta(X)|^2) \tag{4.10}$$

and $\varphi(X)$ is representable in the form

$$\varphi(X) = |a_0(\xi(\vartheta(X)))|^2 + O(|\delta(X)|) \tag{4.11}$$

From (4.9)–(4.11) we obtain the following expression for the drift coefficient in (4.8):

$$\frac{1}{\varphi(X)}(a_0^k(X)\varphi(X) - a_0^k(\xi(\vartheta(X))) \cdot \sum_{i=1}^{d} a_0^i(\xi(\vartheta(X)))a_0^i(X)) + O(|\delta(X)|^2) =$$

$$(A_0(\xi(\vartheta(X)))\delta(X))^k - \frac{(A_0(\xi(\vartheta(X)))\delta(X), a_0(\xi(\vartheta(X))))}{|a_0(\xi(\vartheta(X)))|^2} a_0^k(\xi(\vartheta(X)))-$$

$$\frac{(A_0(\xi(\vartheta(X)))a_0(\xi(\vartheta(X))), \delta(X))}{|a_0(\xi(\vartheta(X)))|^2} a_0^k(\xi(\vartheta(X))) + O(|\delta(X)|^2) \tag{4.12}$$

The diffusion coefficients can be obtained analogously due to (4.7):

$$a_r^k(X) - \frac{a_0^k(\xi(\vartheta(X))) \cdot \sum_{i=1}^d a_0^i(\xi(\vartheta(X)))a_r^i(X)}{\varphi(X)} = (A_r(\xi(\vartheta(X)))\delta(X))^k -$$

$$\frac{(A_r(\xi(\vartheta(X)))\delta(X), a_0(\xi(\vartheta(X))))}{|a_0(\xi(\vartheta(X)))|^2} a_0^k(\xi(\vartheta(X))) + O(|\delta(X)|^2) \tag{4.13}$$

Now we can write the system for $\delta(X)$ in the following form

$$d\delta(X) = (A_0 - \frac{a_0 a_0^\top (A_0 + A_0^\top)}{|a_0|^2}) \cdot \delta(X)dt +$$

$$\sum_{r=1}^q (A_r - \frac{a_0 a_0^\top A_r}{|a_0|^2})\delta(X)dw_r(t) + O(|\delta(X)|^2)dt + \sum_{r=1}^q O(|\delta(X)|^2)dw_r(t) \tag{4.14}$$

where a_0 and A_k, $k = 0, 1, ..., q$, have the quantity $\xi(\vartheta(X(t)))$ as their argument.

It is not difficult to obtain

$$d\vartheta(X(t)) = dt + O(|\delta(X)|)dt + \sum_{r=1}^q O(|\delta(X)|)dw_r(t) \tag{4.15}$$

as well.

The relations (4.14), (4.15) can be considered as stochastic differential equations for the process $(\vartheta(X), \delta(X))$ in view of a replacement $X = \xi(\vartheta(X)) + \delta(X)$. The process $(\vartheta(X), \delta(X))$ belongs to a d-dimensional manifold since $a_0^\top(\xi(\vartheta(X)))\delta(X) = 0$.

Let us introduce a linear system of stochastic differential equations with periodic coefficients (a linearized orthogonal system for orbit)

$$d\Delta = B_0(t)\Delta dt + \sum_{r=1}^q B_r(t)\Delta dw_r(t) \tag{4.16}$$

where

$$B_0(t) = A_0(\xi(t)) - \frac{a_0(\xi(t))a_0^\top(\xi(t))(A_0(\xi(t)) + A_0^\top(\xi(t)))}{|a_0(\xi(t))|^2}, \tag{4.17}$$

$$B_r(t) = A_r(\xi(t)) - \frac{a_0(\xi(t))a_0^\top(\xi(t))A_r(\xi(t))}{|a_0(\xi(t))|^2}, \ r = 1, ..., q \tag{4.18}$$

Let us note that $\xi(t)$ can be defined for all t as a T-periodic function.

Lemma 4.1. *If $\Delta(t_0)$ is orthogonal to $a_0(\xi(s + t_0))$* for some s, $0 \le s < \infty$, $t_0 \ge 0$, *then $\Delta(t)$ is orthogonal to $a_0(\xi(s + t))$ for all $t \ge t_0$, i.e.,*

$$a_0^\top(\xi(s+t))\Delta(t) = \sum_{i=1}^d a_0^i(\xi(s+t)) \cdot \Delta^i(t) \equiv 0, \ t \ge t_0 \tag{4.19}$$

Proof. The proof consists in simple checking the identity

$$d(\sum_{i=1}^{d} a_0^i(\xi(s+t)) \cdot \Delta^i(t)) \equiv 0,\ t \geq t_0$$

Remark 4.1. The following form

$$a_r(x) = \alpha_r(|\delta(x)|),\ r = 1, ..., q \tag{4.20}$$

is fairly natural for the diffusion coefficients.

In (4.20) the functions $\alpha_r^k(\rho)$ of the scalar argument $\rho \geq 0$ are supposed to be sufficiently smooth, $\alpha_r(0) = 0$. But the derivatives $\partial|\delta(x)|/\partial x^i$ do not exist for x belonging to the phase trajectory $\mathbf{M}$, and the same is true for the derivatives $\partial\alpha_r(|\delta(x)|)/\partial x^i$ if $\alpha_r'(0) \neq 0$. Therefore, A_r, $r = 1, ..., q$, do not exist and one cannot use (4.7).

Instead of (4.13) we can write in the case (4.20)

$$a_r^k(X) - \frac{a_0^k(\xi(\vartheta(X))) \cdot \sum_{i=1}^d a_0^i(\xi(\vartheta(X)))a_r^i(X)}{\varphi(X)} =$$

$$\sigma_r^k \cdot |\delta(X)| - \frac{\sigma_r^\top a_0(\xi(\vartheta(X)))) \cdot |\delta(X)|}{|a_0(\xi(\vartheta(X)))|^2} a_0^k(\xi(\vartheta(X))) + O(|\delta(X)|^2)$$

where the d-dimensional vector σ_r is equal to $\alpha_r'(0)$.

Let us write down an analogue of the linearized orthogonal system in the case (4.20):

$$d\Delta = B_0(t)\Delta dt + |\Delta| \sum_{r=1}^{q} b_r(t) dw_r(t) \tag{4.21}$$

where the matrix $B_0(t)$ is the same as in (4.17) and the vector $b_r(t)$ is equal to

$$b_r(t) = \sigma_r^k - \frac{\sigma_r^\top a_0(\xi(t))}{|a_0(\xi(t))|^2} a_0^k(\xi(t)),\ r = 1, ..., q \tag{4.22}$$

It is not difficult to verify that Lemma 4.1 is true for the system (4.21) as well. The system (4.21) is not linear but it is homogeneous of degree one. In the case of stationary point it is known that the theory of moment Lyapunov exponent can be carried over to such systems (see [3] and references therein). In the case of orbit the same can be done (the concept of moment Lyapunov exponent for the system (4.16) is given in the next section).

Remark 4.2. The behavior of $\delta(X)$ has also been considered in the deterministic theory of orbital stability. For instance, in [18] a new coordinate system is introduced in every hyperplane passing through a point $\xi(s)$, $0 \leq s < T$, orthogonally to the orbit. The point $\xi(s)$ is taken as an origin and $d-1$ mutually orthogonal axis are drawn in the hyperplane through $\xi(s)\ :\ O_{\xi(s)}y^1, ..., O_{\xi(s)}y^{d-1}$. Directions of the vectors $O_{\xi(s)}y^i$, $i = 1, ..., d-1$, are supposed to be some continuously differentiable functions of s. The old coordinates $x^1, ..., x^d$ are expressed in terms of new ones $s, y^1, ..., y^{d-1}$ by formulas:

$$x^i = \sum_{j=1}^{d-1} b_{ij}(s) y^j + \xi^i(s),\ i = 1, ..., d$$

where the T-periodic functions $b_{ij}(s)$ depend on the choice of the axis $O_{\xi(s)}y^1, ...,$ $O_{\xi(s)}y^{d-1}$. After that a system of $d-1$ differential equations for $y^1, ..., y^{d-1}$ can be derived and linearized. The linearized system is a linear system with periodic coefficients. This system is used in studying orbital stability (see [18]). A disadvantage of such an approach consists in the linearized system being dependent on the choice of the coordinate axis what leads to the non-constructiveness of the system. At the same time the system (4.16) has an explicit form. True, its dimension is equal to d and we have to use (4.19). But this does not lead to any serious complications (see the next section).

Remark 4.3. Another system exploited in the deterministic theory of orbital stability is a system of the first approximation in a neighborhood of the orbit $\mathbf{M}$. Such a system for (4.2) has evidently the following form

$$dX = A_0(\xi(t))Xdt + \sum_{r=1}^{q} A_r(\xi(t))Xdw_r(t) \tag{4.23}$$

It should be noted that due to (4.3) $X(t) = a_0(\xi(t))$ is a solution of (4.23). Mention also the following connection (what can be checked by direct evaluations) between solutions of the systems (4.16) and (4.23): if $X(t)$ is any solution of the system (4.23) then

$$\Delta(t) = X(t) - \frac{(X(t), a_0(\xi(t+s)))}{|a_0(\xi(t+s))|^2} a_0(\xi(t+s)) \tag{4.24}$$

is a solution of the system (4.16) for any s, $0 \le s < \infty$, and the relation (4.19) is satisfied. Clearly $\Delta(t)$ from (4.24) is the projection of $X(t)$ on the hyperplane that is orthogonal to the orbit at the point $\xi(t+s)$.

In the author's opinion, it is the linearized orthogonal system (4.16) that to a considerable extent corresponds to stability problems of invariant manifolds (even in the deterministic case). However some questions (for instance, the behavior of a phase of a perturbed motion) require in addition the system of the first approximation.

Remark 4.4. Consider the Stratonovich system (3.1). As before we suppose that $x = \xi(t)$ is a T-periodic solution of the system (4.1) and that (4.3) is fulfilled. The linearized orthogonal system in this case is

$$d\Delta = B_0(t)\Delta dt + \sum_{r=1}^{q} B_r(t)\Delta \circ dw_r(t)$$

with the same matrices $B_0(t)$ and $B_r(t)$ as in (4.17) and in (4.18).

5. Moment Lyapunov exponents and stability index for a linearized orthogonal system

Due to the T-periodicity of $B_k(t)$, $k = 0, 1, ..., q$, the system (4.16) reduces to the following autonomous system

$$d\Delta = B_0(\Theta)\Delta dt + \sum_{r=1}^{q} B_r(\Theta)\Delta dw_r(t) \tag{5.1}$$

$$d\Theta = dt, \ \Theta(0) = \vartheta \tag{5.2}$$

where Θ is considered to be a cyclical variable.

Let $\Delta(0) \neq 0$ and

$$\sum_{i=1}^{d} a_0^i(\xi(\vartheta)) \cdot \Delta^i(0) = 0 \tag{5.3}$$

Introduce

$$\Lambda = \frac{\Delta}{|\Delta|} \tag{5.4}$$

and consider the process (Θ, Λ).

The Khasminskii system has now the following form

$$d\Lambda = b_0(\Theta, \Lambda)dt + \sum_{r=1}^{q} b_r(\Theta, \Lambda)dw_r(t) \tag{5.5}$$

$$d\Theta = dt, \ \Theta(0) = \vartheta \tag{5.6}$$

The vectors $b_0(\vartheta, \lambda)$ and $b_r(\vartheta, \lambda)$ are equal to

$$b_0(\vartheta, \lambda) = B_0\lambda - (B_0\lambda, \lambda)\lambda -$$

$$\frac{1}{2}\sum_{r=1}^{q}(B_r\lambda, B_r\lambda)\lambda - \sum_{r=1}^{q}(B_r\lambda, \lambda)B_r\lambda + \frac{3}{2}\sum_{r=1}^{q}(B_r\lambda, \lambda)^2\lambda \tag{5.7}$$

$$b_r(\vartheta, \lambda) = B_r\lambda - (B_r\lambda, \lambda)\lambda, \ \ r = 1, ..., q \tag{5.8}$$

where $B_k = B_k(\vartheta)$, $k = 0, 1, ..., q$.

Clearly due to (5.3) and (5.4) we have (see Lemma 4.1)

$$a_0^\top(\xi(\vartheta + t))\Lambda(t) = 0, \ \Lambda^\top(t)\Lambda(t) = 1 \tag{5.9}$$

i.e., (Θ, Λ) is a Markov process on the $(d-1)$-dimensional compact manifold $\mathbf{D}$ which is defined in the space of $d+1$ variables ϑ, $\lambda^1, ..., \lambda^d$ by the following equations

$$\mathbf{D} = \{(\vartheta, \lambda) : \ a_0^\top(\xi(\vartheta))\lambda = 0, \ \lambda^\top\lambda = 1\}$$

The manifold $\mathbf{D}$ is invariant for the $(d+1)$-dimensional process defined by the system (5.5)–(5.6).

Ito's formula gives for every $-\infty < p < \infty$

$$d|\Delta(t)|^p = (pQ(\Theta, \Lambda) + \frac{1}{2}p^2R(\Theta, \Lambda))|\Delta(t)|^p dt + p\sum_{r=1}^{q}(B_r\Lambda, \Lambda)|\Delta(t)|^p dw_r(t) \tag{5.10}$$

where

$$Q(\vartheta, \lambda) = (B_0(\vartheta)\lambda, \lambda) + \frac{1}{2}\sum_{r=1}^{q}(B_r(\vartheta)\lambda, B_r\lambda) - \sum_{r=1}^{q}(B_r(\vartheta)\lambda, \lambda)^2 \tag{5.11}$$

$$R(\vartheta, \lambda) = \sum_{r=1}^{q}(B_r(\vartheta)\lambda, \lambda)^2 \tag{5.12}$$

Let $\Delta(0) = \lambda$, $\lambda^\top\lambda = 1$. The following formula defines a strongly continuous semigroup of positive operators on $\mathbf{C}(\mathbf{D})$:

$$T_t(p)f(\vartheta, \lambda) = Ef(\Theta_\vartheta(t), \Lambda_{\vartheta,\lambda}(t))|\Delta_{\vartheta,\lambda}(t)|^p, \ (\vartheta, \lambda) \in \mathbf{D}, \ f \in \mathbf{C}(\mathbf{D}) \tag{5.13}$$

This fact can be proved by direct checking the definition of a strongly continuous semigroup.

Our most urgent goal is to find a generator $A(p)$ for the semigroup $T_t(p)$.

Let $\mathbf{D} \subset U \subset \mathbf{R}^{d+1}$ where U is an open set and $\bar{U}$ is compact ($\bar{U}$ is the closure of U). Let $\check{f}$ be an C^2-extension of $f \in C^2(\mathbf{D})$ and let $\check{f}$ vanish beyond U.

We have

$$f(\Theta_\vartheta(t), \Lambda_{\vartheta,\lambda}(t)) = \check{f}(\Theta_\vartheta(t), \Lambda_{\vartheta,\lambda}(t)), \ t \geq 0, \ (\vartheta, \lambda) \in \mathbf{D} \tag{5.14}$$

Ito's formula gives

$$df(\Theta_\vartheta(t), \Lambda_{\vartheta,\lambda}(t))|\Delta_{\vartheta,\lambda}(t)|^p = d\check{f}(\Theta_\vartheta(t), \Lambda_{\vartheta,\lambda}(t))|\Delta_{\vartheta,\lambda}(t)|^p =$$

$$(\frac{\partial \check{f}}{\partial \vartheta} + (\frac{\partial \check{f}}{\partial \lambda}, b_0) + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{r=1}^{q}\frac{\partial^2 \check{f}}{\partial \lambda^i \partial \lambda^j} b_r^i b_r^j) \cdot |\Delta_\lambda(t)|^p dt +$$

$$p\sum_{r=1}^{q}(\frac{\partial \check{f}}{\partial \lambda}, b_r) \cdot (B_r \Lambda_{\vartheta,\lambda}, \Lambda_{\vartheta,\lambda}) \cdot |\Delta_{\vartheta,\lambda}(t)|^p dt + \check{f} \cdot (pQ + \frac{1}{2}p^2 R) \cdot |\Delta_{\vartheta,\lambda}(t)|^p dt +$$

$$\sum_{r=1}^{q}(\frac{\partial \check{f}}{\partial \lambda}, b_r) \cdot |\Delta_{\vartheta,\lambda}(t)|^p dw_r(t) + \check{f} \cdot p\sum_{r=1}^{q}(B_r \Lambda, \Lambda) \cdot |\Delta_{\vartheta,\lambda}(t)|^p dw_r(t) \tag{5.15}$$

From (5.13), (5.14) and (5.15) it follows

$$T_t(p)f(\vartheta, \lambda) - f(\vartheta, \lambda) = E\check{f}(\Theta_\vartheta(t), \Lambda_{\vartheta,\lambda}(t))|\Delta_{\vartheta,\lambda}(t)|^p - \check{f}(\vartheta, \lambda) =$$

$$E\int_0^t (\frac{\partial \check{f}}{\partial \vartheta} + (\frac{\partial \check{f}}{\partial \lambda}, b_0) + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{r=1}^{q}\frac{\partial^2 \check{f}}{\partial \lambda^i \partial \lambda^j} b_r^i b_r^j) \cdot |\Delta_{\vartheta,\lambda}(s)|^p ds +$$

$$E\int_0^t p\sum_{r=1}^{q}(\frac{\partial \check{f}}{\partial \lambda}, b_r) \cdot (B_r \Lambda_{\vartheta,\lambda}, \Lambda_{\vartheta,\lambda}) \cdot |\Delta_{\vartheta,\lambda}(s)|^p ds + E\int_0^t \check{f} \cdot (pQ + \frac{1}{2}p^2 R) \cdot |\Delta_{\vartheta,\lambda}(s)|^p ds$$

and, consequently,

$$A(p)f(\vartheta, \lambda) = \frac{\partial \check{f}}{\partial \vartheta} + (\frac{\partial \check{f}}{\partial \lambda}, b_0) + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{r=1}^{q}\frac{\partial^2 \check{f}}{\partial \lambda^i \partial \lambda^j} b_r^i b_r^j +$$

$$p\sum_{r=1}^{q}(\frac{\partial \check{f}}{\partial \lambda}, b_r) \cdot (B_r \lambda, \lambda) + \check{f} \cdot (pQ + \frac{1}{2}p^2 R), \ (\vartheta, \lambda) \in \mathbf{D} \tag{5.16}$$

Formula (5.15) can be rewritten in the form

$$df(\Theta_\vartheta(t), \Lambda_{\vartheta,\lambda}(t))|\Delta_{\vartheta,\lambda}(t)|^p = A(p)f(\Theta_\vartheta(t), \Lambda_{\vartheta,\lambda}(t)) \cdot |\Delta_{\vartheta,\lambda}(t)|^p dt +$$

$$\sum_{r=1}^{q}(\frac{\partial \check{f}}{\partial \lambda}, b_r) \cdot |\Delta_{\vartheta,\lambda}(t)|^p dw_r(t) + f \cdot p\sum_{r=1}^{q}(B_r \Lambda_{\vartheta,\lambda}, \Lambda_{\vartheta,\lambda}) \cdot |\Delta_{\vartheta,\lambda}(t)|^p dw_r(t) \tag{5.17}$$

It should be noted that due to the cyclicity of Θ any operator $T_t(p)$, $t > 0$, $-\infty < p < \infty$, is neither compact no irreducible (see, for instance, Section 7 below) in contrast to the operator (2.8). But the whole semigroup (5.13) can be irreducible. We recall that a positive semigroup $T_t(p)$ in $\mathbf{C}(\mathbf{D})$ is called irreducible if $\{0\}$ and $\mathbf{C}(\mathbf{D})$ are the only invariant closed ideals for all $T_t(p)$, $t \geq 0$.

A simple sufficient condition of the irreducibility consists in

$$\dim L(b_1(\vartheta,\lambda),...,b_q(\vartheta,\lambda)) = d-2 \text{ for any } (\vartheta,\lambda)\in \mathbf{D} \tag{5.18}$$

where L denotes the linear hull spanned by the given vector fields.

It follows due to [10], [8] that the spectrum $\sigma(A(p))$ of the generator $A(p)$ of the positive semigroup $T_t(p)$ is not empty and

$$s(A(p)) := \sup\{Re\mu : \mu \in \sigma(A(p))\} =$$

$$\max\{\mu \in \mathbf{R} : \mu \in \sigma(A(p))\},\ -\infty < s(A(p)) < \infty$$

Moreover the resolvent $R(\mu, A(p))$ is strongly positive for $\mu > s(A(p))$ because $T_t(p)$ is irreducible, and

$$R(\mu, A(p))f(\vartheta,\lambda) = \int_0^\infty e^{-\mu t}T_t(p)f(\vartheta,\lambda)dt \tag{5.19}$$

Let us show that under some natural assumption the resolvent $R(\mu, A(p))$ is compact. To this end consider the following system

$$d\Lambda = b_0(\Theta,\Lambda)dt + p\sum_{r=1}^{q}(B_r(\Theta)\Lambda,\Lambda)b_r(\Theta,\Lambda)dt + \sum_{r=1}^{q} b_r(\Theta,\Lambda)dw_r(t),\ \Lambda(0)=\lambda \tag{5.20}$$

$$d\Theta = dt,\ \Theta(0)=\vartheta,\ (\vartheta,\lambda)\in\mathbf{D} \tag{5.21}$$

instead of (5.5)–(5.6). It is not difficult to verify that (5.9) is true for the system (5.20)–(5.21) as well and that the manifold $\mathbf{D}$ is invariant for the process (Θ,Λ) defined by this system. Due to Girsanov's theorem the semigroup (5.13) has the following representation (see the analogous transformation in [4])

$$T_t(p)f(\vartheta,\lambda) = Ef(\Theta_\vartheta(t),\Lambda_{\vartheta,\lambda}(t))\cdot$$

$$\exp\left\{\int_0^t (pQ(\Theta_\vartheta(s),\Lambda_{\vartheta,\lambda}(s)) + \frac{1}{2}p^2R(\Theta_\vartheta(s),\Lambda_{\vartheta,\lambda}(s)))ds\right\} \tag{5.22}$$

where $f\in\mathbf{C}(\mathbf{D})$, $(\vartheta,\lambda)\in\mathbf{D}$, and $\Theta_\vartheta(t),\Lambda_{\vartheta,\lambda}(t)$ is the solution of (5.20)–(5.21).

Let $P(t,(\vartheta,\lambda),(d\tilde\vartheta\times d\tilde\lambda))$ be the transition probability function of the Markov process (Θ,Λ). Here $d\tilde\vartheta$ is an element of the length on the orbit $\mathbf{M}$, and $d\tilde\lambda$ is an element of the area on the sphere $\tilde\lambda^\top\tilde\lambda = 1$. Suppose that

$$P(t,(\vartheta,\lambda),(d\tilde\vartheta\times d\tilde\lambda)) = \delta(t+\vartheta, d\tilde\vartheta)p(t,(\vartheta,\lambda),\tilde\lambda)d\tilde\lambda \tag{5.23}$$

where

$$\delta(t+\vartheta,d\tilde\vartheta) = \begin{cases} 1,\ t+\vartheta\in d\tilde\vartheta \\ 0,\ t+\vartheta\notin d\tilde\vartheta \end{cases}$$

and the density $p(t,(\vartheta,\lambda),\tilde\lambda)$ over $\tilde\lambda$ is continuous with respect to t, ϑ, λ, $\tilde\lambda$ under $t\ge t_0$ for any $t_0>0$.

Then

$$T_t(p)f(\vartheta,\lambda) = \int_{\tilde\lambda^\top\tilde\lambda=1} f(\vartheta+t,\tilde\lambda)\varphi(t,\vartheta,\tilde\lambda)p(t,(\vartheta,\lambda),\tilde\lambda)d\tilde\lambda$$

where

$$\varphi(t,\vartheta,\tilde\lambda) = \exp\left\{\int_0^t (pQ(\vartheta+s,\tilde\lambda) + \frac{1}{2}p^2R(\vartheta+s,\tilde\lambda))ds\right\}$$

and

$$R(\mu, A(p))f(\vartheta, \lambda) =$$

$$\int_{\tilde{\lambda}^\top \tilde{\lambda}=1} \int_{\vartheta}^{\infty} f(t, \tilde{\lambda}) \exp\{\mu(\vartheta - t)\} \varphi(t-\vartheta, \vartheta, \tilde{\lambda}) p(t-\vartheta, (\vartheta, \lambda), \tilde{\lambda}) dt d\tilde{\lambda} \quad (5.24)$$

Now it is not difficult to prove directly that under sufficiently large $\mu > 0$ the representation (5.24) implies the compactness of the operator $R(\mu, A(p))$. Due to Hilbert's resolvent equality the resolvent $R(\mu, A(p))$ is compact for any $\mu \in \rho(A(p))$ where $\rho(A(p))$ is the resolvent set of $A(p)$.

Apparently, the assumption (5.23) is fulfilled not only under the condition (5.18) but also under some weaker one, for instance, under the condition (just as in [4] and [6])

$$\dim LA(b_1(\vartheta, \lambda), ..., b_q(\vartheta, \lambda)) = d - 2 \text{ for any } (\vartheta, \lambda) \in \mathbf{D}$$

which is analogous to (2.3).

Now we formulate a basic hypothesis which is supposed to hold below in a lot of statements.

Hypothesis (H). *For each $p \in \mathbf{R}$ the positive semigroup $T_t(p)$ is irreducible and its resolvent is compact.*

Let us show that the hypothesis (H) ensures the existence of a strictly positive eigenfunction $h_p(\vartheta, \lambda)$ of $A(p)$ corresponding to an eigenvalue $g(p)$:

$$A(p)h_p(\vartheta, \lambda) = g(p)h_p(\vartheta, \lambda) \quad (5.25)$$

The eigenvalue $g(p)$ is real and simple. But in contrast to [4] the real part of any other point of the spectrum of $A(p)$ is not always strictly less than $g(p)$. It can be equal to $g(p)$, i.e., $g(p)$ is more or equal to the real part of any other point of the spectrum of $A(p)$.

Indeed, let $\mu > s(A(p))$. The relation

$$\sigma(R(\mu, A(p))) = (\mu - \sigma(A(p)))^{-1}$$

implies $(\mu - s(A(p)))^{-1} \in \sigma(R(\mu, A(p)))$ because $s(A(p)) \in \sigma(A(p))$. Since $R(\mu, A(p))$ is compact and strongly positive, the number $(\mu - s(A(p)))^{-1}$ is a simple isolated eigenvalue of $R(\mu, A(p))$ which exceeds a module of any other eigenvalue of $R(\mu, A(p))$. Moreover there exists a unique $h_p \in \mathbf{C}(\mathbf{D})$ with $h_p(\vartheta, \lambda) > 0$ for all $(\vartheta, \lambda) \in \mathbf{D}$, $||h_p|| = 1$, and a unique positive measure ν_p over $\mathbf{D}$ with $||\nu_p|| = 1$ such that they are correspondingly an eigenfunction of the operator $R(\mu, A(p))$ and an eigendistribution of the conjugate operator $R^*(\mu, A(p))$. Denoting $s(A(p))$ by $g(p)$ we get (5.25) and the equality

$$A^*(p)\nu_p = g(p)\nu_p \quad (5.26)$$

Further, as $(\mu - s(A(p)))^{-1}$ is a pole of the resolvent of the operator $R(\mu, A(p))$, the number $s(A(p)) = g(p)$ is a pole of $R(\mu, A(p))$ (see [10]). In such a case the generalized Perron-Frobenius theorem [10] (see also [8]) sets besides (5.25) and (5.26) that all the points from $\sigma(A(p))$ with real part $g(p)$ are $g(p) + i\alpha k$, $k = 0, \pm 1, \pm 2, ...$, for some $\alpha \geq 0$, and they are all simple isolated eigenvalues of $A(p)$. Thus, the above-mentioned assertion is justified.

We underline that the noted above distinction from [4] is not any obstacle for carrying over the theory of moment Lyapunov exponents to the system (5.1)–(5.2).

Now we are ready to formulate a number of theorems relating to stability properties of the system (5.1)–(5.2). These theorems are analogous to the corresponding ones from [14], [4], and [6] and their proofs are not adduced here.

The following theorem is an analogue of the Khasminskii theorem (see Theorem 2.1).

Theorem 5.1. *Assume* (H). *Then the process* (Θ, Λ) *on* $\mathbf{D}$ *is ergodic, there exists an invariant measure* $\mu(\vartheta, \lambda)$ *and, for any* $\vartheta, \delta \neq 0$ *with* $a_0^\top(\xi(\vartheta))\delta = 0$, *there exists the limit (which does not depend on* ϑ, δ*)*

$$P\text{-a.s.} \lim_{t\to\infty} \frac{1}{t} \ln |\Delta_{\vartheta,\delta}(t)| = \lim_{t\to\infty} \frac{1}{t} E \ln |\Delta_{\vartheta,\delta}(t)| = \int_{\mathbf{D}} Q(\vartheta, \lambda) d\mu(\vartheta, \lambda) : \quad = \lambda^* \tag{5.27}$$

The limit λ^* is called Lyapunov exponent of the system (5.1)–(5.2).

The following theorem is an analogue of the Arnold-Oeljeklaus-Pardoux theorem (see Theorem 2.2).

Theorem 5.2. *Assume* (H). *Then for all* $\vartheta, \delta \neq 0$ *with* $a_0^\top(\xi(\vartheta))\delta = 0$ *the limit (which is called the* p^{th}*-moment Lyapunov exponent for* (5.1)–(5.2))

$$\lim_{t\to\infty} \frac{1}{t} \ln E|\Delta_{\vartheta,\delta}(t)|^p = g(p) \tag{5.28}$$

exists for any $p \in \mathbf{R}$ *and is independent of* (ϑ, δ). *The limit* $g(p)$ *is a convex analytic function of* $p \in \mathbf{R}$, $g(p)/p$ *is increasing,* $g(0) = 0$ *and* $g'(0) = \lambda^*$.

Further, the moment Lyapunov exponent $g(p)$ *is an eigenvalue for* $A(p)$ *with a strictly positive eigenfunction* $h_p(\vartheta, \lambda)$:

$$A(p)h_p(\vartheta, \lambda) = g(p)h_p(\vartheta, \lambda), \; h_p(\vartheta, \lambda) > 0, \; (\vartheta, \lambda) \in \mathbf{D} \tag{5.29}$$

The eigenvalue $g(p)$ *is simple and* $g(p)$ *is more or equal to the real part of any other point of the spectrum of* $A(p)$.

These results can be applied (as in the case of a stationary point) to study the behavior of $P\{\sup_{t\geq 0} |\Delta_{\vartheta,\delta}(t)| > \rho\}$, $|\delta| \ll \rho$, for asymptotically stable systems ($\lambda^* < 0$), and of $P\{\inf_{t\geq 0} |\Delta_{\vartheta,\delta}(t)| < \rho\}$, $|\delta| \gg \rho$, for unstable systems ($\lambda^* > 0$) (of course it is supposed that $a_0^\top(\xi(\vartheta))\delta = 0$).

The following theorem is an analogue of the Baxendale theorem (see Theorem 2.3).

Theorem 5.3. *Assume* (H). *If* $g'(0) = \lambda^* < 0$ *and the equation*

$$g(p) = 0 \tag{5.30}$$

has a root $\gamma^* > 0$ *then there exists* $K \geq 1$ *such that for all* $\rho > 0$ *and for all* δ *with* $|\delta| < \rho$ and $a_0^\top(\xi(\vartheta))\delta = 0$

$$\frac{1}{K}(|\delta|/\rho)^{\gamma^*} \leq P\{\sup_{t\geq 0} |\Delta_{\vartheta,\delta}(t)| > \rho\} \leq K(|\delta|/\rho)^{\gamma^*} \tag{5.31}$$

If $g'(0) = \lambda^* > 0$ *and the equation* (5.30) *has a root* $\gamma^* < 0$ *then there exists* $K \geq 1$ *such that for all* $\rho > 0$ *and for all* δ *with* $|\delta| > \rho$ and $a_0^\top(\xi(\vartheta))\delta = 0$

$$\frac{1}{K}(|\delta|/\rho)^{\gamma^*} \leq P\{\inf_{t\geq 0} |\Delta_{\vartheta,\delta}(t)| < \rho\} \leq K(|\delta|/\rho)^{\gamma^*} \tag{5.32}$$

The root γ^* is called stability index of the linearized orthogonal system (5.1)–(5.2).

Example 5.1. Clearly from (4.16), the matrix of the second moments

$$M(t) = E\Delta_{\vartheta,\delta}(t)\Delta^{\top}_{\vartheta,\delta}(t)$$

satisfies the following deterministic system

$$\frac{dM}{dt} = B_0(t)M + MB_0^{\top}(t) + \sum_{r=1}^{q} B_r(t)MB_r^{\top}(t)$$

$$M(0) = \delta\delta^{\top},\ a_0^{\top}(\xi(\vartheta))\delta = 0$$

Consequently, (H) implies

$$g(2) = \lim_{t\to\infty} \frac{1}{t}\ln(\mathrm{tr}M(t))$$

If $g(2) < 0$ then $\lambda^* < 0$ and if in addition the equation (5.30) has a root then $\gamma^* > 2$. If $\lambda^* < 0$ and $g(2) > 0$ then $0 < \gamma^* < 2$.

6. The Arnold-Khasminskii theorem and stability index for orbit

The following theorem is an analogue of the Arnold-Khasminskii theorem (see Theorem 2.4).

Theorem 6.1. *Let the linearized orthogonal system* (5.1)–(5.2) *for the system* (4.2) *be such that the hypothesis* (H) *is fulfilled. Assume that the stability index* γ^* *of* (5.1)–(5.2) *does not vanish,* $\gamma^* \neq 0$.
Then
1. Case $\gamma^* > 0$: *There exists a sufficiently small* $\rho > 0$ *and positive constants* c_1, c_2 *such that for all* $x : |\delta(x)| < \rho$ *the solution* $X_x(t)$ *of* (4.2) *satisfies the inequalities*

$$c_1(|\delta(x)|/\rho)^{\gamma^*} \leq P\{\sup_{t\geq 0}|\delta(X_x(t))| > \rho\} \leq c_2(|\delta(x)|/\rho)^{\gamma^*} \tag{6.1}$$

2. Case $\gamma^* < 0$: *There exists a sufficiently small* $\rho > 0$, *positive constants* c_3, c_4 *and a constant* $0 < \alpha < 1$ *such that for any* $\rho_0 \in (0, \alpha\rho)$ *and all* $x : \rho_0 < |\delta(x)| < \alpha\rho$

$$c_3(|\delta(x)|/\rho_0)^{\gamma^*} \leq P\{\inf_{0\leq t<\tau}|\delta(X_x(t))| < \rho_0\} \leq c_4(|\delta(x)|/\rho_0)^{\gamma^*} \tag{6.2}$$

Here $\tau := \inf\{t : |\delta(X_x(t))| > \rho\}$.

Proof. Let $f(\vartheta, \lambda) \in C^2(\mathbf{D})$. Let

$$\vartheta = \vartheta(X_x(t)),\ \delta = \delta(X_x(t)),\ \Gamma = \Gamma(X_x(t)) = \delta(X_x(t))/|\delta(X_x(t))|$$

Clearly $(\vartheta(X_x(t)), \Gamma(X_x(t))) \in \mathbf{D}$. In view of (4.14) it is not difficult to evaluate

$$d\Gamma(X_x(t)) = b_0(\vartheta(X_x(t)), \Gamma(X_x(t)))dt + \sum_{r=1}^{q}(b_r(\vartheta(X_x(t)), \Gamma(X_x(t)))dw_r(t) +$$

$$O(|\delta(X_x(t))|)dt + \sum_{r=1}^{q} O(|\delta(X_x(t))|)dw_r(t)$$

Further, analogously to (5.14), (5.15) and due to (4.15) and (5.16) we get

$$df(\vartheta,\Gamma)|\delta|^p = A(p)f(\vartheta,\Gamma)\cdot|\delta|^p dt+$$

$$\sum_{r=1}^{q}(\frac{\partial \check{f}}{\partial\lambda}(\vartheta,\Gamma),b_r(\vartheta,\Gamma))\cdot|\delta|^p dw_r(t)+f(\vartheta,\Gamma)\cdot p\sum_{r=1}^{q}(B_r(\vartheta)\Gamma,\Gamma)\cdot|\delta|^p dw_r(t)+$$

$$O(|\delta|^{p+1})dt+\sum_{r=1}^{q}O(|\delta|^{p+1})dw_r(t) \tag{6.3}$$

Case 1. Let $\gamma^*>0$ be the stability index for (5.1)–(5.2) and $h_{\gamma^*}(\vartheta,\lambda)$, $h_{\gamma^*+c}(\vartheta,\lambda)$ be strictly positive solutions of the equations

$$A(\gamma^*)h_{\gamma^*}=0,\ A(\gamma^*+c)h_{\gamma^*+c}=g(\gamma^*+c)h_{\gamma^*+c} \tag{6.4}$$

where $0<c<1$ and $g(\gamma^*+c)>0$.

Introduce the following function

$$V_{\mp}(x)=h_{\gamma^*}(\vartheta(x),\delta(x)/|\delta(x)|)\cdot|\delta(x)|^{\gamma^*}\mp h_{\gamma^*+c}(\vartheta(x),\delta(x)/|\delta(x)|)\cdot|\delta(x)|^{\gamma^*+c} \tag{6.5}$$

Due to (6.3) and (6.4)

$$dV_{\mp}(X_x(t))=\mp g(\gamma^*+c)h_{\gamma^*+c}(\vartheta,\Gamma)\cdot|\delta|^{\gamma^*+c}dt+$$

$$\sum_{r=1}^{q}(\frac{\partial \check{h}_{\gamma^*}}{\partial\lambda}(\vartheta,\Gamma),b_r(\vartheta,\Gamma))\cdot|\delta|^{\gamma^*}dw_r(t)+h_{\gamma^*}(\vartheta,\Gamma)\cdot\gamma^*\sum_{r=1}^{q}(B_r(\vartheta)\Gamma,\Gamma)\cdot|\delta|^{\gamma^*}dw_r(t)+$$

$$O(|\delta|^{\gamma^*+1})dt+\sum_{r=1}^{q}O(|\delta|^{\gamma^*+c})dw_r(t) \tag{6.6}$$

Let the eigenfunctions h_{γ^*} and h_{γ^*+c} have already been chosen. It is clear from (6.5) and (6.6) that there exists a sufficiently small $\rho>0$ such that $V_-(x)>0$ for all x with $0<|\delta(x)|<\rho$ and $V_-(X_x(t\wedge\tau_{x,\rho}))$ is a supermartingale where

$$\tau_{x,\rho}:\ =\inf\{t:\ |\delta(X_x(t))|>\rho\}$$

Hence there exist positive constants a_1 and a_2 such that the following inequalities hold:

$$a_1|\delta(x)|^{\gamma^*}\ge V_-(x)\ge EV_-(X_x(t\wedge\tau_{x,\rho}))\ge a_2\rho^{\gamma^*}P\{\sup_{0\le s\le t}|\delta(X_x(s))|>\rho\}$$

and therefore

$$P\{\sup_{t\ge 0}|\delta(X_x(t))|>\rho\}=\lim_{t\to\infty}P\{\sup_{0\le s\le t}|\delta(X_x(s))|>\rho\}\le\frac{a_1}{a_2}(|\delta(x)|/\rho)^{\gamma^*} \tag{6.7}$$

As $V_+(x)>0$ (see (6.5)) and $V_+(X_x(t\wedge\tau_{x,\rho}))$ is a submartingale for sufficiently small ρ (see (6.6)) we have

$$a_3|\delta(x)|^{\gamma^*}\le V_+(x)\le EV_+(X_x(\tau_{x,\varepsilon}\wedge\tau_{x,\rho}))\le a_4\rho^{\gamma^*}P\{\sup_{t>0}|\delta(X_x(t))|>\rho\}+a_5\varepsilon^{\gamma^*} \tag{6.8}$$

where a_3, a_4, a_5 are some positive constants which do not depend on ε, $\varepsilon<|\delta(x)|<\rho$ and

$$\tau_{x,\varepsilon}:\ =\inf\{t:\ |\delta(X_x(t))|<\varepsilon\}$$

Relations (6.7) and (6.8) give (6.1) provided ρ is smallest from (6.7) and (6.8). Case 1 is proved.

Case 2. Let $\gamma^* < 0$. Then there exists a sufficiently small c, $0 < c < 1$, such that $g(\gamma^* + c) < 0$ in (6.4). Now $V_+(X_x(t \wedge \tau_{x,\rho}))$ is a supermartingale for sufficiently small ρ and for x with $0 < |\delta(x)| < \rho$. We have for some positive a_1, a_2 and for x with $\rho_0 < |\delta(x)| < \rho$:

$$a_1|\delta(x)|^{\gamma^*} \geq V_+(x) \geq EV_+(X_x(t \wedge \tau_{x,\rho})) \geq a_2\rho_0^{\gamma^*} P\{\inf_{0\leq t\leq \tau_{x,\rho}} |\delta(X_x(t))| < \rho_0\} \quad (6.9)$$

Relation (6.9) implies the second part of (6.2).

Further $V_-(X_x(t \wedge \tau_{x,\rho}))$ is a submartingale for sufficiently small ρ and there exist positive constants a_3, a_4, a_5 such that for all x with $\rho_0 < |\delta(x)| < \rho$:

$$a_3|\delta(x)|^{\gamma^*} \leq V_-(x) \leq EV_-(X_x(\tau_{x,\rho_0} \wedge \tau_{x,\rho})) \leq a_4\rho_0^{\gamma^*} P\{\inf_{0\leq t\leq \tau_{x,\rho}} |\delta(X_x(t))| < \rho_0\} + a_5\rho^{\gamma^*}$$

where a_3, a_4, a_5 do not depend on ρ_0 and ρ.

If $\rho_0 < |\delta(x)| < \alpha\rho$ then

$$a_4\rho_0^{\gamma^*} P\{\inf_{0\leq t\leq \tau_{x,\rho}} |\delta(X_x(t))| < \rho_0\} \geq a_3|\delta(x)|^{\gamma^*} - a_5\rho^{\gamma^*} \geq$$

$$\frac{1}{2}a_3|\delta(x)|^{\gamma^*} + \frac{1}{2}a_3|\alpha\rho|^{\gamma^*} - a_5\rho^{\gamma^*} \quad (6.10)$$

If $0 < \alpha < 1$ is such that $\frac{1}{2}a_3\alpha^{\gamma^*} - a_5 > 0$ then (6.10) implies the first part of (6.2). Theorem 6.1 is proved.

The root γ^* is called stability index of the orbit $\mathbf{M}$ of the system (4.2).

7. Stability of orbits on the plane

Clearly $\Lambda(t)$ is deterministic in two-dimensional case ($n = 2$) :

$$\Lambda^1(t) = \mp\frac{a_0^2(\xi(t))}{|a_0(\xi(t))|}, \quad \Lambda^2(t) = \pm\frac{a_0^1(\xi(t))}{|a_0(\xi(t))|}$$

It is possible to evaluate directly that (5.10) can be rewritten for $n = 2$ in the following form

$$d|\Delta(t)|^p = (pQ(\Theta) + \frac{1}{2}p^2R(\Theta)) \cdot |\Delta(t)|^p dt +$$

$$p\sum_{r=1}^{q} c_r(\Theta) \cdot |\Delta(t)|^p dw_r(t), \quad |\Delta(0)| = 1 \quad (7.1)$$

where

$$c_r(\vartheta) = \lambda^\top(\vartheta)A_r(\vartheta)\lambda(\vartheta)$$

with

$$\lambda(\vartheta) = \frac{1}{|a_0(\xi(\vartheta))|}\begin{bmatrix} -a_0^2(\xi(\vartheta)) \\ a_0^1(\xi(\vartheta)) \end{bmatrix}$$

$$A_r(\vartheta) = \begin{bmatrix} \frac{\partial a_r^1}{\partial x^1}(\xi(\vartheta)) & \frac{\partial a_r^1}{\partial x^2}(\xi(\vartheta)) \\ \frac{\partial a_r^2}{\partial x^1}(\xi(\vartheta)) & \frac{\partial a_r^2}{\partial x^2}(\xi(\vartheta)) \end{bmatrix}, \ r = 0, 1, ..., q$$

and

$$R(\vartheta) = \sum_{r=1}^{q} c_r^2(\vartheta) \ , \ \ Q(\vartheta) = \lambda^\top(\vartheta)A_0(\vartheta)\lambda(\vartheta) - \frac{1}{2}R(\vartheta)$$

All the functions $\lambda(\vartheta)$, $A_r(\vartheta)$ and so on are T-periodic.

The semigroup $T_t(p)$ is defined on the space of continuous T-periodic functions:

$$T_t(p)f(\vartheta) = f(\vartheta+t)E|\Delta(t)|^p = f(\vartheta+t)\exp\left\{\int_0^t (pQ(s) + \frac{1}{2}p^2R(s))ds\right\} \tag{7.2}$$

and its generator $A(p)$ has a form

$$A(p)f(\vartheta) = \frac{df}{d\vartheta}(\vartheta) + (pQ(\vartheta) + \frac{1}{2}p^2R(\vartheta))f(\vartheta)$$

From the equation

$$A(p)h_p(\vartheta) = g(p)h_p(\vartheta)$$

we obtain an eigenfunction

$$h_p(\vartheta) = \exp\left\{g(p)\vartheta - \int_0^\vartheta (pQ(s) + \frac{1}{2}p^2R(s))ds\right\}$$

where the eigenvalue $g(p)$ is equal to

$$g(p) = \frac{1}{2T}\int_0^T R(s)ds \cdot p^2 + \frac{1}{T}\int_0^T Q(s)ds \cdot p \tag{7.3}$$

It is possible to prove that

$$\int_0^T Q(s)ds = \int_0^T \mathrm{tr}A_0(s)ds$$

Therefore

$$\lambda^* = g'(0) = \frac{1}{T}\int_0^T \mathrm{tr}A_0(s)ds$$

and if $\int_0^T R(s)ds \neq 0$, $\int_0^T \mathrm{tr}A_0(s)ds \neq 0$ then the stability index is equal to

$$\gamma^* = -2 \cdot \frac{\int_0^T Q(s)ds}{\int_0^T R(s)ds} \tag{7.4}$$

So all the characteristics in two-dimensional case can be evaluated in explicit form.

In connection with the contents of Section 5 we can note that as is obvious from the formula (7.2), any operator $T_t(p)$, $0 \leq t < \infty$, $-\infty < p < \infty$, is noncompact and, for instance, for $t_k = kT$, $k = 0, 1, ...$, the operator $T_{t_k}(p)$ is not irreducible. We note also that the spectrum $\sigma(A(p))$ consists of the eigenvalues $g(p) + 2\pi ik/T$, $k = 0, \pm 1, \pm 2, ...$.

Example 7.1. Consider the Van der Pol equation with a multiple noise written in the form of the Ito system

$$dX^1 = X^2dt, \ \ dX^2 = -X^1dt + \varepsilon X^2(1 - {X^1}^2)dt + \sigma(X^1, X^2)dw(t) \tag{7.5}$$

It is known that an asymptotically stable orbit $x = \xi(t)$ for the deterministic Van der Pol equation for small $\varepsilon > 0$ differs little from a circle of radius 2 :

$$\xi^1 = 2\cos t + O(\varepsilon), \quad \xi^2 = 2\sin t + O(\varepsilon), \quad T = 2\pi + O(\varepsilon)$$

Suppose

$$\sigma(x^1, x^2) = \alpha\sqrt{\varepsilon}(x^1 - \xi^1(\psi(x^1, x^2)) + \beta\sqrt{\varepsilon}(x^2 - \xi^2(\psi(x^1, x^2)) \tag{7.6}$$

where α, β are some constants and $\xi(\psi(x^1, x^2) = (\xi^1(\psi(x^1, x^2), \ \xi^2(\psi(x^1, x^2))$ is the nearest to (x^1, x^2) point on the orbit $x = \xi(t)$.

One can evaluate for the system (7.5)

$$\lambda(\vartheta) = \begin{bmatrix} \cos\vartheta + O(\varepsilon) \\ \sin\vartheta + O(\varepsilon) \end{bmatrix},$$

$$A_0(\vartheta) = \begin{bmatrix} 0 & 1 \\ -1 - 4\varepsilon\sin 2\vartheta + O(\varepsilon^2) & \varepsilon(1 - 4\cos^2\vartheta) + O(\varepsilon^2) \end{bmatrix},$$

Further

$$A_1(\vartheta) = \begin{bmatrix} 0 & 0 \\ \dfrac{\partial\sigma}{\partial x^1}(\xi(\vartheta)) & \dfrac{\partial\sigma}{\partial x^2}(\xi(\vartheta)) \end{bmatrix}$$

and since

$$\frac{\partial\psi}{\partial x^1}(\xi(\vartheta)) = -\frac{1}{2}\sin\vartheta + O(\varepsilon), \ \frac{\partial\psi}{\partial x^2}(\xi(\vartheta)) = \frac{1}{2}\cos\vartheta + O(\varepsilon), \ \psi(\xi(\vartheta)) \doteq \vartheta$$

we have

$$A_1(\vartheta) = \sqrt{\varepsilon}(\alpha\cos\vartheta + \beta\sin\vartheta)\begin{bmatrix} 0 & 0 \\ \cos\vartheta + O(\varepsilon^{3/2}) & \sin\vartheta + O(\varepsilon^{3/2}) \end{bmatrix},$$

$$c_1(\vartheta) = \lambda^\top(\vartheta)A_1(\vartheta)\lambda(\vartheta) = \sqrt{\varepsilon}(\alpha\sin\vartheta\cos\vartheta + \beta\sin^2\vartheta) + O(\varepsilon^{3/2}),$$

$$R(\vartheta) = c_1^2(\vartheta) = \varepsilon(\alpha\sin\vartheta\cos\vartheta + \beta\sin^2\vartheta)^2 + O(\varepsilon^2),$$

$$Q(\vartheta) = \lambda^\top(\vartheta)A_0(\vartheta)\lambda(\vartheta) - \frac{1}{2}R(\vartheta) =$$

$$-3\varepsilon\sin^2 2\vartheta + \varepsilon\sin^2\vartheta - \frac{\varepsilon}{2}(\alpha\sin\vartheta\cos\vartheta + \beta\sin^2\vartheta)^2 + O(\varepsilon^2)$$

From here

$$\int_0^T R(s)ds = \frac{\pi}{4}\varepsilon(\alpha^2 + 3\beta^2) + O(\varepsilon^2), \quad \int_0^T Q(s)ds = -2\pi\varepsilon - \frac{\pi}{8}\varepsilon(\alpha^2 + 3\beta^2) + O(\varepsilon^2)$$

and the stability index is equal to

$$\gamma^* = 1 + \frac{16}{\alpha^2 + 3\beta^2} + O(\varepsilon)$$

Now consider the Stratonovich stochastic differential system

$$dX^1 = X^2dt, \quad dX^2 = -X^1dt + \varepsilon X^2(1 - X^{1^2})dt + \sigma(X^1, X^2) \circ dw(t) \tag{7.7}$$

with the same σ as in (7.6). The corresponding Ito system is

$$dX^1 = X^2 dt, \;\; dX^2 = -X^1 dt + \varepsilon X^2(1 - X^{1^2})dt +$$

$$\frac{1}{2}\frac{\partial \sigma}{\partial x^2}(X^1, X^2) \cdot \sigma(X^1, X^2)dt + \sigma(X^1, X^2)dw(t) \tag{7.8}$$

Let us mark all the corresponding values for the system (7.7) by means of bar as opposed to (7.5). We have

$$\bar{\lambda}(\vartheta) = \lambda(\vartheta)$$

$$\bar{A}_0(\vartheta) = A_0(\vartheta) + \frac{1}{2}\begin{bmatrix} 0 & 0 \\ \frac{\partial \sigma}{\partial x^1}(\xi(\vartheta)) \cdot \frac{\partial \sigma}{\partial x^2}(\xi(\vartheta)) & \frac{\partial \sigma}{\partial x^2}(\xi(\vartheta)) \cdot \frac{\partial \sigma}{\partial x^2}(\xi(\vartheta)) \end{bmatrix} =$$

$$A_0(\vartheta) + \frac{\varepsilon}{2}(\alpha \cos\vartheta + \beta \sin\vartheta)^2 \begin{bmatrix} 0 & 0 \\ \cos\vartheta \sin\vartheta + O(\varepsilon^2) & \sin^2\vartheta + +O(\varepsilon^2) \end{bmatrix}$$

$$\bar{A}_1(\vartheta) = A_1(\vartheta), \;\; \bar{c}_1(\vartheta) = c_1(\vartheta), \;\; \bar{R}(\vartheta) = R(\vartheta)$$

$$\bar{Q}(\vartheta) = \lambda^\top(\vartheta)\bar{A}_0(\vartheta)\lambda(\vartheta) - \frac{1}{2}\bar{R}(\vartheta) = -3\varepsilon \sin^2 2\vartheta + \varepsilon \sin^2\vartheta + O(\varepsilon^2)$$

Now

$$\int_0^T \bar{Q}(s)ds = -2\pi\varepsilon + O(\varepsilon^2)$$

and

$$\bar{\gamma}^* = \frac{16}{\alpha^2 + 3\beta^2} + O(\varepsilon)$$

So the Van der Pol equation possesses good stability properties with respect to both the noise in the sense of Ito and the noise in the sense of Stratonovich.

8. Stability of orbits with nonvanishing diffusion

Let an orbit

$$\mathbf{M}: \; x = \xi(\vartheta), \; 0 \le \vartheta < T$$

be an invariant manifold for the system (4.2). Let $\xi'(\vartheta) \neq 0, \; 0 \le \vartheta < T$. In contrast to Section 4 we do not suppose that this orbit is a phase trajectory for (4.1) and we do not suppose (4.3), i.e., it may be a diffusion not only in a neighborhood of the orbit but also on the very orbit.

In a neighborhood of $\mathbf{M}$ we can introduce new variables $\delta = x - \xi(\vartheta(x))$ and $\vartheta = \vartheta(x)$. The dimension of (ϑ, δ) is equal to $d+1$ but due to the restriction

$$(\delta, \xi'(\vartheta)) = 0 \tag{8.1}$$

the number of free variables is equal to d.

We have

$$d(X - \xi(\vartheta(X))) = a_0(X)dt + \sum_{r=1}^{q} a_r(X)dw_r(t) - \xi'(\vartheta(X))\sum_{i=1}^{d}\frac{\partial \vartheta}{\partial x^i}(X)dX^i -$$

$$\frac{1}{2}\sum_{i,j=1}^{d}(\xi^{''}(\vartheta(X))\frac{\partial\vartheta}{\partial x^i}(X)\frac{\partial\vartheta}{\partial x^j}(X)+\xi^{'}(\vartheta(X))\frac{\partial^2\vartheta}{\partial x^i\partial x^j}(X))dX^idX^j =$$

$$b_0(X)dt+\sum_{r=1}^{q}b_r(X)dw_r(t) \tag{8.2}$$

where

$$b_0(x)=a_0(x)-\xi^{'}(\vartheta(x))\sum_{i=1}^{d}\frac{\partial\vartheta}{\partial x^i}(x)a_0^i(x)-$$

$$\frac{1}{2}\sum_{i,j=1}^{d}(\xi^{''}(\vartheta(x))\frac{\partial\vartheta}{\partial x^i}(x)\frac{\partial\vartheta}{\partial x^j}(x)+\xi^{'}(\vartheta(x))\frac{\partial^2\vartheta}{\partial x^i\partial x^j}(x))\sum_{r=1}^{q}a_r^i(x)a_r^j(x) \tag{8.3}$$

$$b_r(x)=a_r(x)-\xi^{'}(\vartheta(x))\sum_{i=1}^{d}\frac{\partial\vartheta}{\partial x^i}(x)a_r^i(x),\ r=1,...,q \tag{8.4}$$

If $X(0)\in\mathbf{M}$ then $X(t)\in\mathbf{M}$ for all $t\geq 0$ as $\mathbf{M}$ is the invariant manifold for (4.2). Therefore $X(t)\equiv\xi(\vartheta(X(t)))$ and in view of (8.2) the following lemma is natural.

Lemma 8.1. *Let the orbit* $\mathbf{M}$ *be an invariant manifold for the system* (4.2). *Then the coefficients* $b_i(x)$, $i=0,1,...,q$, *vanish on the orbit, i.e.,*

$$b_0(\xi(\vartheta))=0,\ 0\leq\vartheta<T \tag{8.5}$$

$$b_r(\xi(\vartheta))=0,\ 0\leq\vartheta<T,\ r=1,...,q \tag{8.6}$$

Proof. Let us make use of the Stroock-Varadhan support theorem. If $X(0)=\xi(\vartheta)$ then $X(t)$ due to Corollary 3.2 also belongs to $\mathbf{M}$ for all $t\geq 0$ and, consequently, $X^{'}(0)$ is collinear to $\xi^{'}(\vartheta)$, i.e., the following vector

$$X^{'}(0)=a_0(\xi(\vartheta))-\frac{1}{2}\sum_{r=1}^{q}\frac{\partial a_r}{\partial x}(\xi(\vartheta))a_r(\xi(\vartheta))+\sum_{r=1}^{q}a_r(\xi(\vartheta))W_r^{'}(0)$$

is collinear to $\xi^{'}(\vartheta)$ under any $W_r^{'}(0)$.
From here it follows (if we put $W_r^{'}(0)=0,\ r=1,...,q$)

$$a_0(\xi(\vartheta))-\frac{1}{2}\sum_{r=1}^{q}\frac{\partial a_r}{\partial x}(\xi(\vartheta))a_r(\xi(\vartheta))=\xi^{'}(\vartheta)\cdot\frac{(a_0(\xi(\vartheta)),\xi^{'}(\vartheta))}{|\xi^{'}(\vartheta)|^2}-$$

$$\frac{1}{2}\frac{\xi^{'}(\vartheta)}{|\xi^{'}(\vartheta)|^2}\cdot\sum_{r=1}^{q}(\frac{\partial a_r}{\partial x}(\xi(\vartheta))a_r(\xi(\vartheta)),\xi^{'}(\vartheta)) \tag{8.7}$$

and

$$a_r(\xi(\vartheta))=\xi^{'}(\vartheta)\cdot\frac{(a_r(\xi(\vartheta)),\xi^{'}(\vartheta))}{|\xi^{'}(\vartheta)|^2},\ r=1,...,q \tag{8.8}$$

From the following identity with respect to x

$$(x-\xi(\vartheta(x)),\xi^{'}(\vartheta(x)))=0$$

we have

$$\frac{\partial\vartheta}{\partial x^i}(x) = \frac{\xi^{i'}(\vartheta(x))}{|\xi'(\vartheta(x))|^2 - (x - \xi(\vartheta(x)), \xi''(\vartheta(x)))}\ ,\ i = 1, ..., d \tag{8.9}$$

and, consequently,

$$\frac{\partial\vartheta}{\partial x^i}(\xi(\vartheta)) = \frac{\xi^{i'}(\vartheta)}{|\xi'(\vartheta)|^2}\ ,\ 0 \le \vartheta < T;\ \ i = 1, ..., d \tag{8.10}$$

Differentiating (8.9) with respect to x^j and setting $x = \xi(\vartheta)$ in obtained expression we find

$$\frac{\partial^2\vartheta}{\partial x^i \partial x^j}(\xi(\vartheta)) =$$

$$\frac{1}{|\xi'(\vartheta)|^4} \cdot (\xi^{i'}(\vartheta)\xi^{j''}(\vartheta) + \xi^{i''}(\vartheta)\xi^{j'}(\vartheta) - \frac{3\xi^{i'}(\vartheta)\xi^{j'}(\vartheta)}{|\xi'(\vartheta)|^2}(\xi'(\vartheta), \xi''(\vartheta))) \tag{8.11}$$

The relations (8.8), (8.10) and (8.4) imply (8.6).

Let us prove (8.5). The equality (8.8) gives

$$(a_r(\xi(\vartheta)), \xi''(\vartheta)) = \frac{1}{|\xi'(\vartheta)|^2} \cdot (a_r(\xi(\vartheta)), \xi'(\vartheta)) \cdot (\xi'(\vartheta), \xi''(\vartheta)) \tag{8.12}$$

Now we obtain from (8.3), (8.10), (8.11), and (8.12)

$$b_0(\xi(\vartheta)) = a_0(\xi(\vartheta)) - \xi'(\vartheta) \cdot \frac{(a_0(\xi(\vartheta)), \xi'(\vartheta))}{|\xi'(\vartheta)|^2} - \psi(\vartheta)$$

where

$$\psi(\vartheta) = \frac{1}{2}\xi''(\vartheta) \cdot \frac{1}{|\xi'(\vartheta)|^4} \cdot \sum_{r=1}^{q}(a_r(\xi(\vartheta)), \xi'(\vartheta))^2 -$$

$$\frac{1}{2}\xi'(\vartheta) \cdot \frac{(\xi'(\vartheta), \xi''(\vartheta))}{|\xi'(\vartheta)|^6} \cdot \sum_{r=1}^{q}(a_r(\xi(\vartheta)), \xi'(\vartheta))^2$$

Using (8.7) we get rid of a_0

$$b_0(\xi(\vartheta)) = \frac{1}{2}\sum_{r=1}^{q}\frac{\partial a_r}{\partial x}(\xi(\vartheta))a_r(\xi(\vartheta)) -$$

$$\frac{1}{2}\frac{\xi'(\vartheta)}{|\xi'(\vartheta)|^2} \cdot \sum_{r=1}^{q}(\frac{\partial a_r}{\partial x}(\xi(\vartheta))a_r(\xi(\vartheta)), \xi'(\vartheta)) - \psi(\vartheta) \tag{8.13}$$

Let us substitute a_r according to (8.8) in the two first terms of (8.13):

$$b_0(\xi(\vartheta)) = \frac{1}{2}\sum_{r=1}^{q}\frac{\partial a_r}{\partial x}(\xi(\vartheta))\xi'(\vartheta) \cdot \frac{(a_r(\xi(\vartheta)), \xi'(\vartheta))}{|\xi'(\vartheta)|^2} -$$

$$\frac{1}{2}\frac{\xi'(\vartheta)}{|\xi'(\vartheta)|^2} \cdot \sum_{r=1}^{q}(\frac{\partial a_r}{\partial x}(\xi(\vartheta))\xi'(\vartheta), \xi'(\vartheta)) \cdot \frac{(a_r(\xi(\vartheta)), \xi'(\vartheta))}{|\xi'(\vartheta)|^2} - \psi(\vartheta) \tag{8.14}$$

After differentiating (8.8) with respect to ϑ we obtain

$$\frac{\partial a_r}{\partial x}(\xi(\vartheta))\xi'(\vartheta) = \xi''(\vartheta) \cdot \frac{(a_r(\xi(\vartheta)), \xi'(\vartheta))}{|\xi'(\vartheta)|^2} + \xi'(\vartheta) \cdot \frac{(a_r(\xi(\vartheta)), \xi''(\vartheta))}{|\xi'(\vartheta)|^2} +$$

$$\frac{\xi'(\vartheta)}{|\xi'(\vartheta)|^2} \cdot (\frac{\partial a_r}{\partial x}(\xi(\vartheta))\xi'(\vartheta), \xi'(\vartheta)) - 2\frac{\xi'(\vartheta)}{|\xi'(\vartheta)|^4} \cdot (a_r(\xi(\vartheta)), \xi'(\vartheta)) \cdot (\xi'(\vartheta), \xi''(\vartheta))$$

Substituting this expression in the first term of (8.14) we find

$$b_0(\xi(\vartheta)) = \frac{1}{2}\frac{\xi'(\vartheta)}{|\xi'(\vartheta)|^4} \cdot (a_r(\xi(\vartheta)), \xi'(\vartheta)) \cdot (a_r(\xi(\vartheta)), \xi''(\vartheta)) -$$

$$\frac{1}{2}\frac{\xi'(\vartheta)}{|\xi'(\vartheta)|^6} \cdot (a_r(\xi(\vartheta)), \xi'(\vartheta))^2 \cdot (\xi'(\vartheta), \xi''(\vartheta))$$

Finally, due to (8.12) we obtain (8.5). Lemma 8.1 is proved.

Introduce matrices

$$B_r(\vartheta) = \{\frac{\partial b_r^i}{\partial x^j}(\xi(\vartheta))\},\ r = 0, 1, ..., q$$

Due to Lemma 8.1 the system (8.2) can be rewritten in the form

$$d\delta(X) = B_0(\vartheta(X))\delta(X)dt + \sum_{r=1}^{q} B_r(\vartheta(X))\delta(X)dw_r(t) +$$

$$O(|\delta(X)|^2)dt + \sum_{r=1}^{q} O(|\delta(X)|^2)dw_r(t) \tag{8.15}$$

It is not difficult to obtain from (8.10), (8.11) and (8.12)

$$d\vartheta(X) = \alpha_0(\vartheta(X))dt + \sum_{r=1}^{q} \alpha_r(\vartheta(X))dw_r(t) +$$

$$O(|\delta(X)|)dt + \sum_{r=1}^{q} O(|\delta(X)|)dw_r(t) \tag{8.16}$$

where

$$\alpha_r(\vartheta) = \frac{(a_r(\xi(\vartheta)), \xi'(\vartheta))}{|\xi'(\vartheta)|^2},\ r = 1, ..., q$$

$$\alpha_0(\vartheta) = \frac{(a_0(\xi(\vartheta)), \xi'(\vartheta))}{|\xi'(\vartheta)|^2} - \frac{1}{2}\frac{(\xi'(\vartheta), \xi''(\vartheta))}{|\xi'(\vartheta)|^2} \cdot \sum_{r=1}^{q} \alpha_r^2(\vartheta)$$

Consider the following system with respect to variables Δ and Θ (Θ is a cyclical variable)

$$d\Delta = B_0(\Theta)\Delta dt + \sum_{r=1}^{q} B_r(\Theta)\Delta dw_r(t) \tag{8.17}$$

$$d\Theta = \alpha_0(\Theta)dt + \sum_{r=1}^{q} \alpha_r(\Theta)dw_r(t) \tag{8.18}$$

We note that all the coefficients of the system (8.17)–(8.18) are T-periodic functions. The following lemma is an analogue of Lemma 4.1.

Lemma 8.2. *Let* $\Delta(t), \Theta(t)$ *be a solution of the system* (8.17)–(8.18) *such that* $(\Delta(0), \xi'(\Theta(0))) = 0$. *Then for all* $t \geq 0$

$$(\Delta(t), \xi'(\Theta(t))) = 0 \tag{8.19}$$

Proof. In view of (8.15) and (8.16) let us write down the following system with respect to $\bar{\delta}$ and $\bar{\vartheta}$:

$$d\bar{\delta} = B_0(\bar{\vartheta})\bar{\delta}dt + \sum_{r=1}^{q} B_r(\bar{\vartheta})\bar{\delta}dw_r(t) + O(|\bar{\delta}|^2)dt + \sum_{r=1}^{q} O(|\bar{\delta}|^2)dw_r(t) \tag{8.20}$$

$$d\bar{\vartheta} = \alpha_0(\bar{\vartheta})dt + \sum_{r=1}^{q} \alpha_r(\bar{\vartheta})dw_r(t) + O(|\bar{\delta}|)dt + \sum_{r=1}^{q} O(|\bar{\delta}|)dw_r(t) \tag{8.21}$$

Let $\bar{\delta}(0)$, $|\bar{\delta}(0)| \leq r$, and $\bar{\vartheta}(0)$ are such that

$$(\bar{\delta}(0), \xi'(\bar{\vartheta}(0))) = 0$$

Then the solution of the system (8.20), (8.21) has the form

$$\bar{\delta}(t) = \delta(X_x(t)),\ \bar{\vartheta}(t) = \vartheta(X_x(t))$$

where x is defined uniquely from

$$\bar{\vartheta} = \vartheta(x),\ \bar{\delta} = x - \xi(\bar{\vartheta})$$

Hence

$$(\bar{\delta}(t), \xi'(\bar{\vartheta}(t))) = 0,\ t \geq 0 \tag{8.22}$$

Due to the Stroock-Varadhan theorem it is not difficult to obtain that (8.22) is fulfilled for

$$\bar{\delta}(t) = \bar{\delta}(0) + \int_0^t B_0(\bar{\vartheta}(s))\bar{\delta}(s)ds - \frac{1}{2}\sum_{r=1}^{q}\int_0^t (B_r^2(\bar{\vartheta}(s))\bar{\delta}(s) + \alpha_r(\bar{\vartheta}(s))B_r'(\bar{\vartheta}(s))\bar{\delta}(s))ds +$$

$$\sum_{r=1}^{q}\int_0^t B_r(\bar{\vartheta}(s))\bar{\delta}(s)W_r'(s)ds + \int_0^t O(|\bar{\delta}(s)|^2)ds + \sum_{r=1}^{q}\int_0^t O(|\bar{\delta}(s)|^2)W_r'(s)ds \tag{8.23}$$

$$\bar{\vartheta}(t) = \bar{\vartheta}(0) + \int_0^t \alpha_0(\bar{\vartheta}(s))ds - \frac{1}{2}\sum_{r=1}^{q}\int_0^t \alpha_r'(\bar{\vartheta}(s))\alpha_r(\bar{\vartheta}(s))ds +$$

$$\sum_{r=1}^{q}\int_0^t \alpha_r(\bar{\vartheta}(s))W_r'(s)ds + \int_0^t O(|\bar{\delta}(s)|)ds + \sum_{r=1}^{q}\int_0^t O(|\bar{\delta}(s)|)W_r'(s)ds \tag{8.24}$$

where $W_r(s)$, $r = 1, ..., q$, are arbitrary smooth functions.

Let us put $\bar{\delta}(0) = \alpha\delta$, $\alpha > 0$, $\bar{\vartheta}(0) = \vartheta$ and find a derivative of $(\bar{\delta}(t), \xi'(\bar{\vartheta}(t)))$ with respect to t at $t = 0$. If we divide this derivative by α and turn α to zero we obtain some expression what is equal to zero under all the mentioned $W_r(s)$. Thereby we can prove the following relations

$$(B_0(\vartheta)\delta - \frac{1}{2}\sum_{r=1}^{q}(B_r^2(\vartheta)\delta + \alpha_r(\vartheta)B_r^{'}(\vartheta)\delta), \xi^{'}(\vartheta)) +$$

$$(\delta, \xi^{''}(\vartheta)) \cdot (\alpha_0(\vartheta) - \frac{1}{2}\sum_{r=1}^{q}\alpha_r^{'}(\vartheta)\alpha_r(\vartheta)) = 0 \tag{8.25}$$

$$(B_r(\vartheta)\delta, \xi^{'}(\vartheta)) + (\delta, \xi^{''}(\vartheta))\alpha_r(\vartheta) = 0, \; r = 1, ..., q \tag{8.26}$$

for any (δ, ϑ) if only $(\delta, \xi^{'}(\vartheta)) = 0$.

In other words, the relations (8.25), (8.26) take place for every $0 \leq \vartheta < T$ and for any δ if only $(\delta, \xi^{'}(\vartheta)) = 0$. This implies the existence of scalars $k_0(\vartheta)$, $k_1(\vartheta), ..., k_r(\vartheta)$ such that the following identities with respect to ϑ, $0 \leq \vartheta < T$, are fulfilled:

$$(B_0^{\top}(\vartheta) - \frac{1}{2}\sum_{r=1}^{q}((B_r^2(\vartheta))^{\top} + \alpha_r(\vartheta)(B_r^{'}(\vartheta))^{\top}))\xi^{'}(\vartheta) +$$

$$(\alpha_0(\vartheta) - \frac{1}{2}\sum_{r=1}^{q}\alpha_r^{'}(\vartheta)\alpha_r(\vartheta))\xi^{''}(\vartheta) = k_0(\vartheta)\xi^{'}(\vartheta) \tag{8.27}$$

$$B_r^{\top}(\vartheta)\xi^{'}(\vartheta) + \alpha_r(\vartheta)\xi^{''}(\vartheta) = k_r(\vartheta)\xi^{'}(\vartheta), \; r = 1, ..., q \tag{8.28}$$

Let us check now according to Corollary 3.2 that the manifold

$$\mathbf{S} = \{(\delta, \vartheta) : (\delta, \xi^{'}(\vartheta)) = 0\,\}$$

is invariant. For the system (8.17), (8.18) we have

$$\mathbf{S}_{Ito}((\delta, \vartheta), t) = \Big\{(\Delta(t), \, \Theta(t)) : \; \Delta(t) = \delta + \int_0^t B_0(\Theta(s))\Delta(s)ds -$$

$$\frac{1}{2}\sum_{r=1}^{q}\int_0^t B_r^2(\Theta(s))\Delta(s)ds - \frac{1}{2}\sum_{r=1}^{q}\int_0^t \alpha_r(\Theta(s))B_r^{'}(\Theta(s))\Delta(s)ds +$$

$$\sum_{r=1}^{q}\int_0^t B_r(\Theta(s))\Delta(s)W_r^{'}(s)ds, \; \Theta(t) = \vartheta + \int_0^t \alpha_0(\Theta(s))ds -$$

$$\frac{1}{2}\sum_{r=1}^{q}\int_0^t \alpha_r^{'}(\Theta(s))\alpha_r(\Theta(s))ds + \sum_{r=1}^{q}\int_0^t \alpha_r(\Theta(s))W_r^{'}(s)ds, \; W_r \in \mathbf{W}\Big\} \tag{8.29}$$

From (8.29)

$$\frac{d}{dt}(\Delta(t), \xi^{'}(\Theta(t))) = (B_0(\Theta(t))\Delta(t) - \frac{1}{2}\sum_{r=1}^{q}B_r^2(\Theta(t))\Delta(t), \; \xi^{'}(\Theta(t))) -$$

$$(\frac{1}{2}\sum_{r=1}^{q}\alpha_r(\Theta(t))B_r^{'}(\Theta(t))\Delta(t) + \sum_{r=1}^{q}B_r(\Theta(t))\Delta(t)W_r^{'}(t), \; \xi^{'}(\Theta(t))) +$$

$$(\Delta(t), \; \xi^{''}(\Theta(t))) \cdot (\alpha_0(\Theta(t)) - \frac{1}{2}\sum_{r=1}^{q}\alpha_r^{'}(\Theta(t))\alpha_r(\Theta(t)) + \sum_{r=1}^{q}\alpha_r(\Theta(t))W_r^{'}(t)) \tag{8.30}$$

From equalities (8.27), (8.28) we have

$$\frac{d}{dt}(\Delta(t),\xi'(\Theta(t))) = k_0(\Theta(t))\cdot(\Delta(t),\xi'(\Theta(t))) + \sum_{r=1}^{q} k_r(\Theta(t))W_r'(t)\cdot(\Delta(t),\ \xi'(\Theta(t)))$$

Since $(\Delta(0),\ \xi'(\Theta(0))) = (\delta,\xi'(\vartheta)) = 0$ we obtain from here

$$(\Delta(t),\xi'(\Theta(t))) \equiv 0,\ t \geq 0$$

and, consequently, $(\Delta(t),\Theta(t)) \in \mathbf{S}$. Lemma 8.2 is proved.

Now it is not difficult to carry over the results of Section 5 and Section 6 to considered case. At first we write the Khasminskii system in accord the formulas (5.5)–(5.8) and an equation for $|\Delta(t)|^p$ in accord the formulas (5.10)–(5.12). Then we introduce a semigroup of operators $T_t(p)$ on $\mathbf{C}(\mathbf{D})$ by (5.13) where

$$\mathbf{D} = \left\{(\vartheta,\lambda) : (\lambda,\ \xi'(\vartheta)) = 0\ ,\ (\lambda,\lambda) = 1\right\}$$

Finally we obtain the formula (5.17) where $A(p)$ has a different form in comparison with (5.16):

$$A(p)f(\vartheta,\lambda) = \frac{\partial\check{f}}{\partial\vartheta}\alpha_0 + \frac{1}{2}\frac{\partial^2\check{f}}{\partial\vartheta^2}\sum_{r=1}^{q}\alpha_r^2 + (\frac{\partial\check{f}}{\partial\lambda},b_0) + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{r=1}^{q}\frac{\partial^2\check{f}}{\partial\lambda^i\partial\lambda^j}b_r^ib_r^j +$$

$$p\sum_{r=1}^{q}(\frac{\partial\check{f}}{\partial\lambda},b_r)\cdot(B_r\lambda,\lambda) + \check{f}\cdot(pQ + \frac{1}{2}p^2R),\ (\vartheta,\lambda)\in\mathbf{D}$$

We remark that b_0, b_r depend on ϑ, λ here (see formulas (5.5)–(5.8)) unlike b_0, b_r in (8.2)–(8.4) which depend on x. But this does not lead to a confusion.

Theorems 5.1, 5.2, 5.3 and 6.1 can be formulated without any essential alterations now. We note that in the case of a non-degenerate noise of the Khasminskii system in the manifold $\mathbf{D}$ (in contrast to Section 5 such a case is possible here) any operator $T_t(p)$, $t > 0$, is irreducible and compact as in [4].

Example 8.1. Consider for simplicity a particular case of the system (3.3) in the sense of Stratonovich

$$dX = a_0(X)dt + \sum_{r=1}^{q}\alpha_r(X)a_0(X)\circ dw_r(t)$$

or, equivalently, in the sense of Ito

$$dX = a_0(X)dt + \frac{1}{2}\sum_{r=1}^{q}\alpha_r(X)(\alpha_r(X)A_0(X) + a_0(X)\alpha_r'^{\top}(X))a_0(X)dt +$$

$$\sum_{r=1}^{q}\alpha_r(X)a_0(X)dw_r(t) \tag{8.31}$$

where $\alpha_r'^{\top}(x)$ is a vector-row with the elements $\dfrac{\partial\alpha_r}{\partial x^i}(x)$, $i = 1,...,d$, and $A_0(x)$ is a matrix with elements $\dfrac{\partial a_0^i}{\partial x^j}(x)$, $i,j = 1,...,d$.

The system of linear approximation Δ for orthogonal displacement $X-\xi(\vartheta(X))$ from the manifold $\mathbf{M}$ (a linearized orthogonal system) has the following form

$$d\Delta=((1+\beta)B_0+2\alpha A_0B_0-\alpha A_0^2)\Delta dt+$$

$$4\alpha(A_0a_0,a_0)((A_0+A_0^{\top})a_0,\Delta)\frac{a_0}{|a_0|^4}dt-2\alpha(A_0a_0,A_0\Delta)\frac{a_0}{|a_0|^2}dt+$$

$$\alpha\sum_{s=1}^{d}\sum_{i=1}^{d}\frac{\partial^2a_0}{\partial x^i\partial x^s}a_0^i\Delta^s dt-\alpha\sum_{s=1}^{d}\sum_{k=1}^{d}\sum_{i=1}^{d}(\frac{\partial^2a_0^i}{\partial x^k\partial x^s}a_0^k+\frac{\partial a_0^i}{\partial x^k}\frac{\partial a_0^k}{\partial x^s})(a_0^i\Delta^s+a_0^s\Delta^i)\frac{a_0}{|a_0|^2}dt+$$

$$\sum_{r=1}^{q}\alpha_rB_0\Delta dw_r(t) \tag{8.32}$$

where $A_0=A_0(\xi(\Theta)),\ B_0=B_0(\Theta)$ (see the formula (4.17)),

$$\alpha=\alpha(\Theta)=\frac{1}{2}\sum_{r=1}^{q}\alpha_r^2,\ \beta=\beta(\Theta)=\frac{1}{2}\sum_{r=1}^{q}\alpha_r(\alpha_r',a_0) \tag{8.33}$$

and $a_0,\ \alpha_r,\ r=1,...,q$, and all their derivatives in (8.32), (8.33) are evaluated at $\xi(\Theta)$. The equation for cyclical variable Θ has a form

$$d\Theta=(1+\beta(\Theta))dt+\sum_{r=1}^{q}\alpha_r(\xi(\Theta))dw_r(t) \tag{8.34}$$

The derivation of the system (8.32), (8.34) involves a lot of calculations. We mention here the most important of them only. We have

$$\frac{\partial\xi(\vartheta(x))}{\partial x^i}=a_0(\xi(\vartheta(x)))\frac{\partial\vartheta}{\partial x^i}(x)$$

$$\frac{\partial^2\xi(\vartheta(x))}{\partial x^i\partial x^j}=A_0(\xi(\vartheta(x)))a_0(\xi(\vartheta(x)))\frac{\partial\vartheta}{\partial x^i}(x)\frac{\partial\vartheta}{\partial x^j}(x)+a_0(\xi(\vartheta(x)))\frac{\partial^2\vartheta}{\partial x^i\partial x^j}(x)$$

Therefore

$$d(X-\xi(\vartheta(X)))=dX-a_0(\xi(\vartheta(X)))\sum_{i=1}^{d}\frac{\partial\vartheta}{\partial x^i}(X)dX^i-$$

$$\frac{1}{2}\sum_{i,i=1}^{d}(A_0(\xi(\vartheta(X)))a_0(\xi(\vartheta(X)))\frac{\partial\vartheta}{\partial x^i}(X)\frac{\partial\vartheta}{\partial x^j}(X)+a_0(\xi(\vartheta(X)))\frac{\partial^2\vartheta}{\partial x^i\partial x^j}(X))dX^idX^j \tag{8.35}$$

Further

$$\frac{\partial\vartheta}{\partial x^i}(x)=\frac{a_0^i(\xi(\vartheta(x)))}{\varphi(x)},\ \frac{\partial^2\vartheta}{\partial x^i\partial x^j}(x)=$$

$$\frac{(A_0(\xi(\vartheta(x)))a_0(\xi(\vartheta(x))))^ia_0^j(\xi(\vartheta(x)))-a_0^i(\xi(\vartheta(x)))\cdot\frac{\partial\varphi}{\partial x^j}(x)}{\varphi^2(x)} \tag{8.36}$$

where

$$\varphi(x)=|a_0(\xi(\vartheta(x)))|^2-(A_0(\xi(\vartheta(x)))a_0(\xi(\vartheta(x))),x-\xi(\vartheta(x)))$$

$$\frac{\partial\varphi}{\partial x^j}(x) = 3(A_0(\xi(\vartheta(x)))a_0(\xi(\vartheta(x))), a_0(\xi(\vartheta(x))))\frac{a_0^j(\xi(\vartheta(x)))}{\varphi(x)} -$$

$$(A_0(\xi(\vartheta(x)))a_0(\xi(\vartheta(x))))^j -$$

$$\frac{1}{\varphi(x)}\sum_{s=1}^{d}\sum_{k=1}^{d}\sum_{m=1}^{d}(\frac{\partial^2 a_0^m}{\partial x^k \partial x^s}(\xi(\vartheta(x)))a_0^k(\xi(\vartheta(x))) + \frac{\partial a_0^m}{\partial x^k}(\xi(\vartheta(x)))\frac{\partial a_0^k}{\partial x^s}(\xi(\vartheta(x))))\cdot$$

$$\cdot a_0^s(\xi(\vartheta(x))(x - \xi(\vartheta(x)))^m a_0^j(\xi(\vartheta(x)) \tag{8.37}$$

Substituting (8.36) and (8.37) in (8.35) and linearizing it with respect to $X - \xi(\vartheta(X))$ we obtain the system (8.32). The equation (8.34) is obtained from the equality

$$d\vartheta(X) = \sum_{i=1}^{d} \frac{\partial \vartheta}{\partial x^i}(X)dX^i + \frac{1}{2}\sum_{i,i=1}^{d} \frac{\partial^2\vartheta}{\partial x^i \partial x^j}(X))dX^i dX^j$$

by throwing small components.

9. Stability index for many-dimensional invariant manifolds

Let $\mathbf{M}$ be a k-dimensional sufficiently smooth compact invariant manifold for the system (1.1), $1 < k < d$. Conceptually, this case slightly differs from the case of orbit considered in the previous sections. Therefore we only outline the main ideas.

Let some piece $\mathbf{M_0}$ of the manifold $\mathbf{M}$ be defined, for instance, by the following equations in the parametric form

$$x^i = \xi^i(\vartheta^1, ..., \vartheta^k)\ ,\ i = 1, ..., d$$

or, briefly,

$$x = \xi(\vartheta)$$

We suppose the system of tangent vectors

$$\frac{\partial\xi}{\partial\vartheta^1}(\vartheta)\ , ...,\ \frac{\partial\xi}{\partial\vartheta^k}(\vartheta)$$

to be linearly independent. Let x belong to sufficiently small neighborhood of $\mathbf{M_0}$. Then the projection $\xi(\vartheta(x))$ of x on $\mathbf{M}$ is uniquely defined. The functions $\vartheta^1(x), ..., \vartheta^k(x)$ can be found from the following relations

$$(x - \xi(\vartheta(x)), \frac{\partial\xi}{\partial\vartheta^m}(\vartheta(x))) = 0,\ m = 1, ..., k \tag{9.1}$$

Differentiating (9.1) with respect to x^i, $i = 1, ..., d$, we obtain a system of k equations for $\frac{\partial\vartheta^m}{\partial x^i}(x)$, $m = 1, ..., k$, whence one can find them owing to the linear independence of the tangent vectors and to the smallness of $x - \xi(\vartheta(x))$. After that we find the derivatives $\frac{\partial^2\vartheta^m}{\partial x^i \partial x^j}(x)$, $i, j = 1, ..., d;\ m = 1, ..., k$. Next it becomes possible to evaluate

$$d\delta(X) = d(X - \xi(\vartheta(X))) = b_0(X)dt + \sum_{r=1}^{q} b_r(X)dw_r(t) \tag{9.2}$$

and

$$d\vartheta(X) = c_0(\vartheta(X))dt + \sum_{r=1}^{q} c_r(\vartheta(X))dw_r(t) +$$

$$O(|\delta(X)|)dt + \sum_{r=1}^{q} O(|\delta(X)|)dw_r(t) \tag{9.3}$$

Lemma 8.1 also holds here:

$$b_i(\xi(\vartheta)) = 0, \ i = 0, 1, ..., q$$

Therefore we are able to linearize the system (9.2) and to obtain from (9.2) and (9.3) the following $(d + k)$-dimensional system

$$d\Delta = B_0(\Theta)\Delta dt + \sum_{r=1}^{q} B_r(\Theta)\Delta dw_r(t) \tag{9.4}$$

$$d\Theta = \alpha_0(\Theta)dt + \sum_{r=1}^{q} \alpha_r(\Theta)dw_r(t) \tag{9.5}$$

An analogue of Lemma 8.2 is valid for this system:

$$(\Delta(t), \frac{\partial \xi}{\partial \vartheta^m}(\Theta(t))) = 0, \ m = 1, ..., k, \ t \geq 0 \tag{9.6}$$

if only

$$(\Delta(0), \frac{\partial \xi}{\partial \vartheta^m}(\Theta(0))) = 0, \ m = 1, ..., k \tag{9.7}$$

Just as above the system (9.4) implies the Khasminskii system

$$d\Lambda = b_0(\Theta, \Lambda)dt + \sum_{r=1}^{q} b_r(\Theta, \Lambda)dw_r(t) \tag{9.8}$$

where the coefficients $b_i(\vartheta, \lambda)$, $i = 0, 1, ..., q$, have the same expression as in the formulas (5.7), (5.8) (of course, the variable ϑ is k-dimensional here). We remind again that b_0, b_r in (9.8) depend on ϑ, λ unlike b_0, b_r in (9.2) which depend on x. But this does not lead to a confusion.

Due to (9.6) the following $(d - 1)$-dimensional compact manifold

$$\mathbf{D} = \{(\vartheta, \lambda) : \ (\lambda, \lambda) = 1, \ (\lambda, \frac{\partial \xi}{\partial \vartheta^m}(\vartheta)) = 0, \ m = 1, ..., k \ \}$$

is invariant for the system (9.5), (9.8). Under each fixed ϑ the manifold $\mathbf{D}$ gives a unit sphere $\mathbf{S}^{d-k-1}$ of the dimension $d - k - 1$ and, consequently, $\mathbf{D}$ is a torus which is equal to the product $\mathbf{M} \times \mathbf{S}^{d-k-1}$.

Then we can write the equation for $|\Delta(t)|^p$, introduce the semigroup $T_t(p)$ on $\mathbf{C}(\mathbf{D})$, define $A(p)$ and so on as in Section 8 up to the form of a majority of the formulas. We should only have in mind that the parameter ϑ is k-dimensional now and, in connection with that, to introduce the corresponding modifications. As a result we can obtain a Khasminskii-type theorem, an Arnold-Oeljeklaus-Pardoux-type theorem and a Baxendale-type theorem for a linearized orthogonal system in the case of a k-dimensional invariant manifold. Finally an Arnold-Khasminskii-type theorem can be obtained and thereby a stability index of a k-dimensional invariant manifold can be introduced.

Consider specifically the case $k = d - 1$. Let an invariant manifold $\mathbf{M}$ of the system (1.1) be defined by the equation

$$F(x) = F(x^1, ..., x^d) = 0 \tag{9.9}$$

Let $p(x)$ be the projection of the point x on $\mathbf{M}$ (of course, x belongs to a sufficiently small neighborhood of $\mathbf{M}$). Clearly

$$F(p(x)) = 0 \tag{9.10}$$

$$\delta(x) = x - p(x) = k(p(x))\frac{\partial F}{\partial x}(p(x)) \tag{9.11}$$

where $k(p(x))$ is a scalar.

The scalar $k(p(x))$ and the coordinates of the vector $p(x)$ can be found from the system (9.10)–(9.11) consisting of $d+1$ equations.

The equation for Δ has the following form

$$d\Delta = B_0(p(X))\Delta dt + \sum_{r=1}^{q} B_r(p(X))\Delta dw_r(t) \tag{9.12}$$

where $X \in \mathbf{M}$ is the solution of the system (1.1) (and, consequently, $p(X) = X$)

We do not write the system (9.8) for Λ because Λ is uniquely defined by $p(X)$:

$$\Lambda = \frac{\Delta}{|\Delta|} = \pm\frac{\frac{\partial F}{\partial x}(p(X))}{|\frac{\partial F}{\partial x}(p(X))|} \tag{9.13}$$

In view of (9.13) the equation for $|\Delta|^p$ can be written with some coefficients depending only on $X \in \mathbf{M}$:

$$d|\Delta|^p = Q_0(p(X))|\Delta|^p dt + \sum_{r=1}^{q} Q_r(p(X))|\Delta|^p dw_r(t)$$

Therefore we can define a semigroup $T_t(p)$ on $\mathbf{C}(\mathbf{M})$ by the following way

$$T_t(p)f(x^1,...,x^d) = Ef(X_x(t))|\Delta|^p, \ x \in \mathbf{M}$$

Example 9.1. Stability index of the unit sphere for the Khasminskii system.

Consider the Khasminskii system (2.1)–(2.2) in $\mathbf{R}^d$. Here $p(\lambda) = \dfrac{\lambda}{|\lambda|}$ and we have

$$d(\Lambda - \frac{\Lambda}{|\Lambda|}) = (1 - \frac{1}{|\Lambda|})A_0\Lambda dt - (1 - \frac{1}{|\Lambda|^3})(A_0\Lambda, \Lambda)\Lambda dt -$$

$$\frac{1}{2}(1 - \frac{1}{|\Lambda|^3})\sum_{r=1}^{q}(A_r\Lambda, A_r\Lambda)\Lambda dt - (1 - \frac{1}{|\Lambda|^3})\sum_{r=1}^{q}(A_r\Lambda, \Lambda)A_r\Lambda dt +$$

$$\frac{3}{2}(1 - \frac{1}{|\Lambda|^5})\sum_{r=1}^{q}(A_r\Lambda, \Lambda)^2\Lambda dt +$$

$$(1 - \frac{1}{|\Lambda|})\sum_{r=1}^{q} A_r\Lambda dw_r(t) - (1 - \frac{1}{|\Lambda|^3})\sum_{r=1}^{q}(A_r\Lambda, \Lambda)\Lambda dw_r(t)$$

Linearizing this system with respect to $\Lambda - \dfrac{\Lambda}{|\Lambda|}$ we obtain

$$d\Delta = A_0\Delta dt - 3(A_0\Lambda, \Lambda)\Delta dt - \frac{3}{2}\sum_{r=1}^{q}(A_r\Lambda, A_r\Lambda)\Delta dt - 3\sum_{r=1}^{q}(A_r\Lambda, \Lambda)A_r\Delta dt +$$

$$\frac{15}{2}\sum_{r=1}^{q}(A_r\Lambda,\Lambda)^2\Delta dt+\sum_{r=1}^{q}(A_r\Delta-3(A_r\Lambda,\Lambda))\Delta dw_r(t)$$

where Λ is a solution of the system (2.1) on the unit sphere, i.e., $|\Lambda(t)|\equiv 1$.

Let us evaluate

$$d|\Delta|^p=-2p((A_0\Lambda,\Lambda)+\frac{1}{2}\sum_{r=1}^{q}(A_r\Lambda,A_r\Lambda)-\sum_{r=1}^{q}(A_r\Lambda,\Lambda)^2)|\Delta|^p dt+$$

$$\frac{1}{2}(2p)^2\sum_{r=1}^{q}(A_r\Lambda,\Lambda)^2|\Delta|^p dt-2p\sum_{r=1}^{q}(A_r\Lambda,\Lambda)|\Delta|^p dw_r(t)=$$

$$(qQ(\Lambda)+\frac{1}{2}q^2R(\Lambda))|\Delta|^p dt+q\sum_{r=1}^{q}(A_r\Lambda,\Lambda)|\Delta|^p dw_r(t) \tag{9.14}$$

where Q, R are correspondingly from (2.5), (2.7) and $q=-2p$.

Comparing (9.14) with (2.6) we obtain the following

Theorem 9.1. *Assume (2.3). Let*

$$g_0(p)=\lim_{t\to\infty}\frac{\ln E|X_x(t)|^p}{t}$$

be the moment function for the equation (1.2) and let

$$g(p)=\lim_{t\to\infty}\frac{\ln E|\Delta(t)|^p}{t}$$

be the moment function for the invariant unit sphere of the Khasminskii system (2.1) connected with the system (1.2). Then

$$g(p)=g_0(-2p)$$

In particular, if γ_0^* is the stability index for the system (1.2) then the stability index γ^* of the unit sphere for the corresponding Khasminskii system is equal to

$$\gamma^*=-\frac{1}{2}\gamma_0^*$$

References

1. L. Arnold. A formula connecting sample and moment stability of linear stochastic systems. SIAM J. Appl. Math. **44**(1984), pp. 793-802.
2. L. Arnold, H. Crauel, and J.-P. Eckmann (eds.). Lyapunov exponents. Proc., Oberwolfach, 1990. Lecture Notes in Math., vol. 1486, Springer, Berlin, 1991.
3. L. Arnold and R.Z. Khasminskii. Stability index for nonlinear stochastic differential equations. Proc. of Symposia in Pure Math. **57**(1995), pp. 543-551.
4. L. Arnold, E. Oeljeklaus, and E. Pardoux. Almost sure and moment stability for linear Ito equations. Lyapunov exponents (L. Arnold and V. Wihstutz, eds.), Lecture Notes in Math., vol. 1186, Springer, Berlin, 1986, pp. 129-159.
5. L. Arnold and V. Wihstutz (eds.). Lyapunov exponents. Proc., Bremen, 1984. Lecture Notes in Math., vol. 1186, Springer, Berlin, 1986.
6. P. Baxendale. Moment stability and large deviations for linear stochastic differential equations. Proc. Taniguchi Symp. on Probab. Meth. in Math. Physics, Katata and Kyoto 1985 (N. Ikeda, ed.), Kinokuniya, Tokyo, 1987, pp. 31-54.
7. P. Baxendale. Invariant measures for nonlinear stochastic differential equations. Lyapunov exponents (L. Arnold, H. Crauel, and J.-P. Eckmann, eds.), Lecture Notes in Math., vol. 1486, Springer, Berlin, 1991, pp. 123-140.

8. Ph. Clément, H.J.A.M. Heijmans, et al. One-Parameter Semigroups. North-Holland, Amsterdam, 1987.
9. H. Furstenberg. Noncommuting random products. Trans. of the Amer. Math. Soc. **108**(1963), No.3, pp. 377-428.
10. G. Greiner. Zur Perron-Frobenius-Theorie stark stetiger Halbgruppen. Math. Z. **177**(1981), pp. 401-423.
11. F. Hartman. Ordinary Differential Equations. Boston, Birkhäuser, 1982.
12. N. Ikeda and S. Watanabe. Stochastic differential equations and diffusion processes. North-Holland Publ. Comp., Amsterdam, 1981.
13. R.Z. Khasminskii. Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems. Theory Probab. Appl. **12**(1967), pp. 144-147.
14. R.Z. Khasminskii. Stochastic stability of differential equations. Nauka, Moscow, 1969; Engl. transl., Sijthoff and Noordhoff, Alphen, 1980.
15. G.N. Milstein. Stability and stabilization of periodic motions of autonomous systems. Prikl. Mat. Mekh. **41**(1977), No. 4, pp. 744-749.
16. G.N. Milstein and L.B. Ryashko. Stability and stabilization of autonomous system orbits under stochastic perturbations. J. Appl. Maths Mechs. **56**(1992), No. 6, pp. 855-862.
17. S.A. Molchanov. The structure of eigenfunctions of one-dimensional unordered structures. Izvestija AN SSSR, ser. matem. **42**(1978), No.1, pp. 70-103.
18. V.V. Nemytskii and V.V. Stepanov. Qualitative Theory of Differential Equations. Princeton University Press, 1960 (Russian edition, 1949).

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