Oscillations and Dynamical Systems: Normalization Procedures and Averaging*

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Abstract

The present lecture deals with the development of new normalization procedures and averaging algorithms in problems of nonlinear vibrations. Namely, the development of asymptotic methods of perturbation theory is considered, making wide use of group theoretical techniques. Various assumptions about specific group properties are investigated, and are shown to lead to modifications of existing methods, such as the Bogoliubov averaging method and the Poincaré-Birkhoff normal form, as well as to the formulation of new ones. The development of normalization techniques on Lie groups is also treated.

0. Introduction

The idea of introducing coordinate transformations to simplify the analytic expression of a general problem is a powerful one. Symmetry and differential equations have been close partners since the time of the founding masters, namely, Sophus Lie (1842-1899), and his disciples. To this days, symmetry has continued to play a strong role. The ideas of symmetry penetrated deep into various branches of science: mathematical physics, mechanics and so on.

The role of symmetry in perturbation problems of nonlinear mechanics, which was already used by many investigators since 70-th years (G. Hori, A. Kamel, U. Kirchgraber), has been developed considerably in recent time to gain further understanding and development such constructive and powerful methods as averaging and normal form methods.

Normalization techniques in the context with the averaging method was considered in works by A.M. Molchanov [1], A.D. Brjuno [2], S.N. Chow, J. Mallet–Paret [3], Yu.A. Mitropolsky, A.M. Samoilenko [4], J.A. Sanders, F. Verhulst [5].

An approach where Lie series in parameter were used as transformation was considered in works by G. Hori [6], [7], A. Kamel [8], U. Kirchgraber [10], U. Kirchgraber, E. Stiefel [9], V.N. Bogaevsky, A.Ya. Povzner [11], V.F. Zhuravlev, D.N. Klimov [12].

Asymptotic methods of nonlinear mechanics developed by N.M. Krylov, N.N. Bogoliubov and Yu.A. Mitropolsky known as KBM method (see, for example, Bogoliubov N.N. and Mitropolsky Yu.A [18]) is a powerful tool for investigation of nonlinear vibrations.

1. Mathematical Background

Below we are giving the short survey of two methods: Bogoliubov averaging method and normal form method.

1.1. The standard system and Bogoliubov's averaging. The new normalization techniques was developed by Yu.A. Mitropolsky, A.K. Lopatin [13]–[15], A.K. Lopatin [16],[17]. In their works a new method was proposed for investigating systems of differential equations with small parameters. It was a further development of Bogoliubov's averaging method referred to by the authors as "the asymptotic decomposition method". The idea of a new approach originates from Bogoliubov's averaging method (see [18]) but its realization needed to use essentially new apparatus - the theory of continuous transformation groups.

Let us explain the idea of the new approach. As is known, the starting point of investigation by the averaging method is a system in the standard form

$$\frac{dx}{dt} = \varepsilon X(x, t, \varepsilon) \ , \tag{1}$$

where $x = \text{col} \parallel x_1, \ldots, x_n \parallel$, $X(x, t, \varepsilon)$ is an n-dimensional vector.

System (1), upon averaging

$$X_0(\xi, \varepsilon) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(\xi, t, \varepsilon) dt$$

and with a special change of variables, is reduced to the averaged system

$$\frac{d\bar{x}}{dt} = \varepsilon X_0^{(1)}(\bar{x}) + \varepsilon^2 X_0^{(2)}(\bar{x}) + \cdots, \tag{2}$$

which does not explicitly contain the argument t. (To ensure the existence of the average we impose special conditions on the functions $X_j(x, t, \varepsilon)$, $j = \overline{1, n}$. We omit the explicit form of these conditions). Let us rewrite the initial system (1) in the equivalent form

$$\frac{dx}{dt} = \varepsilon X(x, y, \varepsilon), \ \frac{dy}{dt} = 1 \tag{3}$$

and the averaged system (2) correspondingly in the form

$$\frac{d\bar{x}}{dt} = \varepsilon X_0(\bar{x}), \ \frac{d\bar{y}}{dt} = 1, \tag{4}$$

where $X_0(\bar{x}) = X_0^{(1)}(\bar{x}) + \varepsilon X_0^{(2)}(\bar{x}) + \ldots$. Integration of (4) is simpler than that of (3), since the variables are separated: the system for slow variables $\bar{x}$ does not contain a fast variable $\bar{y}$ and is integrated independently.

Everything stated above allows us to interpret the averaging method in the following way: *the averaging method transforms (3) with nonseparated variables into (4) with fast and slow variables separated.*

The described property of the separation of variables with the help of the averaging method has group-theoretical characteristics: *the averaging method transforms (3), which is not invariant with respect to the one-parameter transformation group generated by vector field*

$$\mathbf{W} = \frac{\partial}{\partial y},$$

associated with the system of zero approximation (3), into the averaged system (4), which is invariant with respect to the one-parameter transformation group generated by vector field

$$\mathbf{U} = \frac{\partial}{\partial \bar{y}},$$

associated with the system of zero approximation (4). This statement can be easily proved.

1.2. The normal form method. Consider a system of differential equations with coefficients that are analytical in a neighborhood of zero

$$\begin{gathered} \dot{y}_1 = a_{11}y_1 + \dots + a_{1n}y_n + \sum f^1_{m_1 \dots m_n}\, y_1^{m_1} \dots y_n^{m_n}, \\ \dots\dots\dots \\ \dot{y}_n = a_{n1}y_1 + \dots + a_{nn}y_n + \sum f^n_{m_1 \dots m_n}\, y_1^{m_1} \dots y_n^{m_n}. \end{gathered} \tag{5}$$

Nonlinear terms in the right-hand sides of system (5) are started with terms not lower than second order.

We consider the problem of finding an analytical change of variables

$$y = f(z), \quad z = \|z_1, \, \dots, \, z_n\|, \tag{6}$$

which turns a maximal number of coefficients at nonlinear terms into zero. The limiting case is the linearization of system (5), i.e. transformation of it into

$$\dot{z} = \mathcal{A}z, \quad \mathcal{A} = \|a_{ij}\|, \quad i,j = \overline{1,n} \tag{7}$$

under the action of the variable change (6). Since the procedure pointed out is ultimately reduced to the solvability of linear nonhomogeneous algebraic equations, it turns out that the reduction

$$(5) \overset{y=f(z)}{\longrightarrow} (7) \tag{8}$$

is not always possible. In the general case, a system of nonlinear differential equations is obtained

$$\dot{z} = \mathcal{A}z + F(z).$$

This system is called a normal form.

We refer to the corresponding nonlinear term in above equation as *resonance*. It is clear that only resonance terms remain in the normal form. In particular, linearization by (8) is possible only when there are no resonance terms (see, for example, [12]).

1.3. Generalization of Bogoliubov's averaging method through the symmetry of the standard system. The asymptotic decomposition method is based on the group-theoretical interpretation of the averaging method. Consider the system of ordinary differential equations

$$\frac{dx}{dt} = \omega(x) + \varepsilon\tilde{\omega}(x), \tag{9}$$

where

$$\omega(x) = \operatorname{col}\|\omega_1(x), \dots, \omega_n(x)\|; \; \tilde{\omega}(x) = \operatorname{col}\|\tilde{\omega}_1(x), \dots, \tilde{\omega}_n(x)\|.$$

The differential operator associated with the perturbed system (9) can be represented as

$$\mathbf{U}_0 = \mathbf{U} + \varepsilon\tilde{\mathbf{U}},$$

where

$$\mathbf{U} = \omega_1 \frac{\partial}{\partial x_1} + \cdots + \omega_n \frac{\partial}{\partial x_n}, \ \tilde{\mathbf{U}} = \tilde{\omega}_1 \frac{\partial}{\partial x_1} + \cdots + \tilde{\omega}_n \frac{\partial}{\partial x_n}.$$

By using a certain change of variables in the form of a series in ε

$$x = \varphi(\bar{x}, \varepsilon), \tag{10}$$

system (9) is transformed into a new system

$$\frac{d\bar{x}}{dt} = \omega(\bar{x}) + \sum_{\nu=1}^{\infty} \varepsilon^{\nu} b^{(\nu)}(\bar{x}), \tag{11}$$

which is referred to as *a centralized system*. For this system, $\bar{\mathbf{U}}_0 = \bar{\mathbf{U}} + \varepsilon\tilde{\bar{\mathbf{U}}}$, where

$$\bar{\mathbf{U}} = \omega_1(\bar{x}) \frac{\partial}{\partial \bar{x}_1} + \cdots + \omega_n(\bar{x}) \frac{\partial}{\partial \bar{x}_n},$$

$$\tilde{\bar{\mathbf{U}}} = \sum_{\nu=1}^{\infty} \varepsilon^{\nu} \mathbf{N}_{\nu}, \ \mathbf{N}_{\nu} = b_1^{(\nu)}(\bar{x}) \frac{\partial}{\partial \bar{x}_1} + \cdots + b_n^{(\nu)}(\bar{x}) \frac{\partial}{\partial \bar{x}_n}. \tag{12}$$

We impose a condition on the choice of transformations (10) saying that the centralized system (11) should be invariant with respect to the one-parameter transformation group

$$\bar{x} = e^{s\bar{\mathbf{U}}(\bar{x}_0)} \bar{x}_0, \tag{13}$$

where $\bar{x}_0$ is the vector of new variables. Therefore, after the change of variables (13), system (11) turns into

$$\frac{d\bar{x}_0}{dt} = \omega(\bar{x}_0) + \sum_{\nu=1}^{\infty} \varepsilon^{\nu} b^{(\nu)}(\bar{x}_0),$$

which coincides with the original one up to the notations. This means that we have the identities $[\bar{\mathbf{U}}, \mathbf{N}_{\nu}] \equiv 0$ for $\bar{\mathbf{U}}, \mathbf{N}_{\nu}, \nu = 1, 2, \ldots$.

Presented below is some material which will be needed for understanding the structure of the present lecture as a whole. The essential point in realizing the above-mentioned indicated scheme of the asymptotic decomposition algorithm is that transformations (10) are chosen in the form of a series

$$x = e^{\varepsilon \mathbf{S}} \bar{x}, \tag{14}$$

where

$$\mathbf{S} = \mathbf{S}_1 + \varepsilon \mathbf{S}_2 + \cdots,$$

$$\mathbf{S}_j = \gamma_{j1}(\bar{x}) \frac{\partial}{\partial \bar{x}_1} + \cdots + \gamma_{jn}(\bar{x}) \frac{\partial}{\partial \bar{x}_n}.$$

Coefficients of $\mathbf{S}_j, \gamma_{j1}(\bar{x}), \ldots, \gamma_{jn}(\bar{x})$ are unknown functions. They should be determined by the recurrent sequence of operator equations

$$[\mathbf{U}, \mathbf{S}_{\nu}] = \mathbf{F}_{\nu}. \tag{15}$$

The operator $\mathbf{F}_{\nu}, \nu = 1, 2, \ldots$ is a known function of $\mathbf{U}$ and $\mathbf{S}_1, \ldots, \mathbf{S}_{\nu-1}$, obtained on previous steps.

In the case when $\mathbf{S}$ depends upon ε, Lie series (14) is called a Lie transformation. Thus, the application of a Lie transformation as a change of variables enables us to use the technique of continuous transformation groups.

From the theory of linear operators it is known that the solvability of the nonhomogeneous operator Equation (15) depends on the properties of the solutions of the homogeneous equation

$$[\mathbf{U}, \mathbf{S}_\nu] = 0. \tag{16}$$

Operator (12) $\mathbf{N}_\nu$ is a projection of the right-hand side of the equation onto the kernel of operator (16), which is determined from the condition of solvability in the sense of the nonhomogeneous equation

$$[\mathbf{U}, \mathbf{S}_\nu] = \mathbf{F}_\nu - \mathbf{N}_\nu, \ \nu = 1, 2, \dots . \tag{17}$$

Depending on the way for solving Equations (15) – (17) various modifications of the algorithm of the asymptotic decomposition method are obtained.

The asymptotic decomposition method, being applied to the same objects as the classical asymptotic method, yields identical results. However, the algorithm of the asymptotic decomposition method is, in essence, simpler.

The principal conclusion that can be arrived at after a comparison of the two methods is the following. In the asymptotic decomposition method, the operation of averaging, which is used in Bogoliubov's averaging method, is a certain way of constructing the projection $\operatorname{pr}\mathbf{F}$ of the operator $\mathbf{F}$.

In the asymptotic decomposition method, *the centralized system* is a direct analog *of the averaged* system of Bogoliubov's averaging method.

We refer to the averaging operation used in the asymptotic decomposition method to construct the projection of an operator onto the algebra of the centralizer as the *Bogoliubov projector.*

The last statement means the following. Let us apply the asymptotic decomposition method to Bogoliubov's system in the standard form (3). Let us write out the operator $\mathbf{F}_\nu$ of the right-hand side of (15) as

$$\mathbf{F}_\nu = f_{\nu 1}(x, y)\frac{\partial}{\partial x_1} + \dots + f_{\nu n}(x, y)\frac{\partial}{\partial x_n}.$$

Define the Bogoliubov projection of the operator $\operatorname{pr}\mathbf{F}_\nu$ as

$$\operatorname{pr}\mathbf{F}_\nu = \langle f_{\nu 1}(x, y)\rangle\frac{\partial}{\partial x_1} + \dots + \langle f_{\nu n}(x, y)\rangle\frac{\partial}{\partial x_n},$$

where

$$\langle f_{\nu k}(x, y)\rangle =_{\mathrm{def}} f^0_{\nu k}(x) \tag{18}$$

is an average value for coefficients $f_{\nu k}$. This notion requires exact definition.

In Bogoluibov's averaging method an average value is understand as

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T f_{\nu k}(x, s)ds = f^0_{\nu k}(x) < +\infty, \ k = 1, n.$$

In our further exposition definition (18) will be understand as an average value on the group.

We hope that such a preview will make the main part of the lecture easier to follow. We illustrate further exposition of the material in the next two Subsections using two physically motivated examples: nonlinear oscillators in the plane and the motion of a point on the sphere. There are classical results for the first example and one can compare them with the present approach. The second example is nontrivial as it cannot be considered by existing methods in a similar way.

2. Examples: Models Connected with Nonlinear Oscillator in the Plane

2.1. Algorithm of asymptotic decomposition method in the space of homogeneous polynomials (group GL(2)).

Along with the linear space V over P generated by the elements $x_1, \ldots, x_n$, we consider the linear space $V_{\otimes\nu}$ over P, which is equal to the direct product of the space V taken ν times.

The vector row composed of the basis elements of $V_{\otimes\nu}$ is denoted by $\hat{\hat{x}}_{m_\nu}$. It is evident that $m_1 = n$ and

$$\hat{\hat{x}}_{m_1} = \|x_1, \ldots, x_n\|.$$

Let $\mathcal{Q}$ be a constant matrix of dimension $m_\nu \times n$ with the elements $q_{ij} \in P$, where $i = \overline{1,\ m_\nu}$, $j = \overline{1,\ n}$, and $q_1, \ldots, q_n$ are the rows as elements of this matrix in the equality

$$q = \hat{\hat{x}}_{m_\nu} \mathcal{Q},\ \ q{=}_{\mathrm{def}}\ ||q_1, \ldots, q_n||.$$

For an arbitrary sequence of matrices $\mathcal{Q}$ the totality of differential operators

$$\mathbf{X} = q_1 \frac{\partial}{\partial x_1} + \cdots + q_n \frac{\partial}{\partial x_n},\ \ q_i \in V_{\otimes\nu},$$

yields the linear space over P which is denoted by $\mathcal{B}(V_{\otimes\nu})$. Matrix $\mathcal{Q}$ will be called *as matrix of operator* $\mathbf{X}$.

Consider the system of two equations of the first order

$$\dot{x}_1' = x_2';\ \ \dot{x}_2' = -x_1' + \varepsilon\,(1 - x_1'^2)\,x_2'. \tag{19}$$

The differential operator associated with system (19) is

$$\mathbf{U}_0' = \mathbf{U}' + \varepsilon\,\widetilde{\mathbf{U}}',$$

where

$$\mathbf{U}' = x_2' \frac{\partial}{\partial x_1'} - x_1' \frac{\partial}{\partial x_2'};\ \ \tilde{\mathbf{U}}' = \left(x_2' - x_1'^2\, x_2'\right) \frac{\partial}{\partial x_2'}\,.$$

Write operators these in the form

$$\mathbf{U}' = \hat{\hat{x}}'_{m_1} \mathcal{F} \partial',\ \ \mathcal{F} = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}.$$

Represent the operator $\tilde{\mathbf{U}}$ as the sum

$$\tilde{\mathbf{U}}' = \tilde{\mathbf{U}}'_{\otimes 1} + \tilde{\mathbf{U}}'_{\otimes 3}\quad \tilde{\mathbf{U}}'_{\otimes i} \in \mathcal{B}(V_{\otimes i}),\ \ i = 1,\ 3,$$

where

$$\tilde{\mathbf{U}}'_{\otimes 1} = \hat{\hat{x}}'_{m_1}\, \mathcal{Q}_{m_1,1}\, \partial, \quad \tilde{\mathbf{U}}'_{\otimes 3} = \hat{\hat{x}}'_{m_3}\, \mathcal{Q}_{m_3,1}\, \partial.$$

Calculate two approximations in the transformed operator (12)

$$\mathbf{U}'_0 = \mathbf{U}' + \varepsilon\, \mathbf{N}'_1 + \varepsilon^2\, \mathbf{N}'_2.$$

Calculate the operators $\mathbf{S}_1$ and $\mathbf{S}_2$, which can be obtained from the equations

$$[\mathbf{U},\ \mathbf{S}_1] = \tilde{\mathbf{U}} - \mathrm{pr}\, \tilde{\mathbf{U}};$$

$$[\mathbf{U}_1,\ \mathbf{S}_2] = \left\{ -[\tilde{\mathbf{U}},\ \mathbf{S}_1] - \frac{1}{2}\, [\mathbf{S}_1,\ [\mathbf{U},\, \mathbf{S}_1]] \right\} - \mathrm{pr}\, \{\cdots\} \tag{20}$$

upon the change of variables (14). Solve these equations in two steps. First we find $\mathbf{S}_1$:

$$\mathbf{S}_1 \equiv \mathbf{S}_{\otimes 11} + \mathbf{S}_{\otimes 31},\ \mathbf{S}_{\otimes i1} \in \mathcal{B}\,(V_{\otimes i}),\ i = 1,\ 3,$$

where $\mathbf{S}_{\otimes i1} \equiv \hat{\hat{x}}_{m_i}\, \Gamma_{1i}\, \partial,\ i = 1,\ 3;$ Γ_{1i} are the rectangular matrices of the dimensions $m_i \times n$, which are the solutions of the system of independent algebraic equations

$$\mathcal{F}_i\, \Gamma_{1i} - \Gamma_{1i}\, \mathcal{F} = \mathcal{Q}_{m_i,1} - \mathrm{pr}\, \mathcal{Q}_{m_i,1},\ \mathcal{F} = \mathcal{A}^T,\ i = 1,\ 3. \tag{21}$$

At the second step, we find $\mathbf{S}_2$. We can see that $\mathbf{S}_2 \in \mathcal{B}\,(V_{\otimes 5})$ implies the structure of the right-hand parts of Equation (20). We have to find a solution in the form of the sum

$$\mathbf{S}_2 = \sum_{i=1}^{5}\ \mathbf{S}_{\otimes i2},\ \mathbf{S}_{\otimes i2} = \hat{\hat{x}}_{m_i}\, \Gamma_{2i}\, \partial,\ i = \overline{1,\ 5},$$

where Γ_{2i} are solutions of the system of algebraic equations

$$\mathcal{F}_i\, \Gamma_{2i} - \Gamma_{2i}\, \mathcal{F} = \mathcal{Q}_{m_i,2} - \mathrm{pr}\, \mathcal{Q}_{m_i,2},\ i = \overline{1,\ 5}.$$

Conduct the necessary calculations for the first approximation. Consider Equation (21). The matrices $\mathcal{F}_1,\ \mathcal{F}_2,\ \mathcal{F}_3$ of the representation of the operator $\mathbf{U}$ in the subspaces $V_{\otimes 1},\ V_{\otimes 2},\ V_{\otimes 3}$.

Pass from Equations (21) to the equations in the spaces $\hat{R}^{(m_1,n)},\ \hat{R}^{(m_2,n)}$

$$G^{(i)}_{\mathcal{F}}\, \hat{\Gamma}_{i1} = \hat{\mathcal{Q}}_{m_i,1} - \mathrm{pr}\, \hat{\mathcal{Q}}_{m_i,1},$$

where

$$G^{(i)}_{\mathcal{F}} = \mathcal{F}_i \otimes \mathcal{E}_2 - \mathcal{E}_{m_i} \otimes \mathcal{F}^T,\ i = 1,\ 3;$$

$\hat{\Gamma}_{1i},\ \hat{\mathcal{Q}}_{m_i,1}$ are vector columns composed of rows of the matrices $\Gamma_{1i},\ \mathcal{Q}_{m_i,1}$.

Taking into account that the difference $\hat{\mathcal{Q}}_{m_i,1} - \mathrm{pr}\, \hat{\mathcal{Q}}_{m_i,1}$ belongs to the image $T^{(i)}_{\mathcal{F}}$ of the operator $G^{(i)}_{\mathcal{F}}$ and is orthogonal to the kernel of the operator $G^{(i)T}_{\mathcal{F}}$, we obtain the system of linear algebraic equations for finding $\mathrm{pr}\, \hat{\mathcal{Q}}_{m_i,1}$.

Finally, we have. The operator $\mathbf{U}_0$ in the first approximation:

$$\mathbf{U}_0 = \mathbf{U} + \varepsilon \mathbf{N}_1,$$

where

$$\mathbf{N}_1 = \mathrm{pr}\tilde{\mathbf{U}} = \mathbf{N}_{\otimes 11} + \mathbf{N}_{\otimes 13};$$

$$\mathbf{N}_{\otimes 11} = \hat{\hat{x}}_{m_1}\, \mathcal{Q}_{m_1,1N}\, \partial = \frac{1}{2}\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right);$$

$$\mathbf{N}_{\otimes 31} = \hat{\hat{x}}_{m_3}\, \mathcal{Q}_{m_3,1N}\, \partial = -\,-\frac{1}{8}\left((x_1^2+x_2^2)\, x_1 \frac{\partial}{\partial x_1} + (x_1^2+x_2^2)\, x_2 \frac{\partial}{\partial x_2}\right).$$

After similar calculations we find the centralized system in the second approximation

$$\frac{dx_1}{dt} = x_2 + \varepsilon\left(\frac{1}{2} - \frac{1}{8}\left(x_1^2+x_2^2\right)\right) x_1 +$$

$$+\varepsilon^2\left(-\frac{1}{4} + \frac{3}{8}\left(x_1^2+x_2^2\right) - -\frac{11}{128}\left(x_1^2+x_2^2\right)^2\right) x_2;$$

$$\frac{dx_2}{dt} = -\,-x_1 + \varepsilon\left(\frac{1}{2} - \frac{1}{8}\left(x_1^2+x_2^2\right)\right) x_2 -$$

$$-\varepsilon^2\left(-\frac{1}{4} + \frac{3}{8}\left(x_1^2+x_2^2\right) - -\frac{11}{128}\left(x_1^2+x_2^2\right)^2\right) x_1.$$

We can easily see that upon transformation of the variables by formulae

$$y_1 = \sqrt{x_1^2+x_2^2},\ y_2 = \mathrm{arctg}\,\frac{x_1}{x_2}.$$

the centralized system takes the form

$$\frac{dy_1}{dt} = \varepsilon\left(\frac{1}{2} - \frac{1}{8}\, y_1^2\right) y_1;$$

$$\frac{dy_2}{dt} = 1 - \varepsilon^2\left(\frac{1}{4} - \frac{3}{8}\, y_1^2 + \frac{11}{128}\, y_1^4\right).$$

To pass to the solution of the initial Equations (19) in the second approximation, we have to know the operator $\mathbf{S}_2$. Calculation of $\mathbf{S}_2$ is analogous to that of $\mathbf{S}_1$.

2.2. Procedures of normalization in the representation spaces of the groups GL(2) and SO(2). Consider the nonlinear oscillator (1). All considerations of Subsection 2.1 were based upon the of invariance property of the subspaces $V_{\otimes 1},\ V_{\otimes 2},\ \ldots$ which is associated with the system of zero approximation. The fact of invariance is expressed by the relation

$$\mathbf{U}\hat{\hat{x}}_{m_j} = \hat{\hat{x}}_{m_j}\mathcal{F}_j,\quad j = 1, 2, \ldots$$

where $\mathcal{F}_j$ is the representation matrix of $\mathbf{U}$ in the subspace $V_{\otimes j}$.

A natural question arises: are subspaces $V_{\otimes 1},\ V_{\otimes 2},\ \ldots$ unique invariant subspaces in the linear space of homogeneous polynomials? It turns out that they are not.

Consider the linear space $T_{\otimes}$ that is the direct sum of subspaces $T_{\otimes 1},\ T_{\otimes 2},\ T_{\otimes 3}, \ldots$ with the bases

$$f^{(m_1)} = \|\ x_1,\ x_2\ \|,$$

$$f^{(m_2)} = \left\|\ 2x_1x_2,\ x_2^2 - x_1^2\ \right\|, \tag{22}$$

$$f^{(m_3)} = \left\| \; 3\;(x_1^2 + x_2^2)x_1 - 4x_1^3,\; 4x_2^3 - 3\;(x_1^2 + x_2^2)x_2 \; \right\|,$$
$$\cdots\cdots\cdots$$

It is easy to verify that each subspace $T_{\otimes j}$ is turned into itself by $\mathbf{U}$, i.g. is invariant with respect to it. To do so, it is sufficient to find representation matrices of $\mathbf{U}$ in these subspaces

$$\mathbf{U} f^{(m_j)} = f^{(m_j)} \mathcal{F}_j, \quad \mathcal{F}_j = \begin{Vmatrix} 0 & -j \\ j & 0 \end{Vmatrix}.$$

For a better understanding of the structure of the space $T_{\otimes}$, let us introduce new variables ρ and φ by the formula

$$x_1 = \rho \sin\varphi, \quad x_2 = \rho\cos\varphi.$$

In new variables, bases vectors (22) are written out as follows

$$\hat{\hat{\varphi}}_{m_k} = \left\| \; \rho^k \sin k\varphi,\; \rho^k \cos k\varphi \; \right\|, \quad k = 1, 2, \ldots$$

So, passing to the space $T_{\otimes} \subset T(V)$ means passing from the space of homogeneous polynomials in two variables to the space of trigonometrical functions (Fourier series).

The just-described process of choosing a new representation space for the operator $\mathbf{U}$ has deep group-theoretical background. Let us consider them in detail.

Consider the set of four linearly independent operators

$$\mathbf{V}_{11} = x_1 \frac{\partial}{\partial x_1}, \; \mathbf{V}_{12} = x_1 \frac{\partial}{\partial x_2}, \; \mathbf{V}_{21} = x_2 \frac{\partial}{\partial x_1}, \; \mathbf{V}_{22} = x_2 \frac{\partial}{\partial x_2}, \tag{23}$$

which generate a complete linear finite-dimensional Lie algebra $\mathcal{B}_{GL(2)}$ of order four. From (23), a general linear group $GL(2)$ is restored. To write out the elements of this group in explicit form, let us write down its general element through a Lie series

$$x' = \exp\, \mathbf{V} x, \tag{24}$$

where

$$\mathbf{V} = s_{11}\mathbf{V}_{11} + s_{12}\mathbf{V}_{12} + s_{21}\mathbf{V}_{21} + s_{22}\mathbf{V}_{22},$$

$s_{11}, s_{12}, s_{21}, s_{22}$ are group parameters which range in a neighborhood of zero.

We write down the series (24) in the finite form

$$x' = x e^{\mathcal{F}_1(s)}.$$

The matrix $\mathcal{G}(s) = e^{\mathcal{F}_1(s)}$, where matrix $\mathcal{F}_1$ is the representation matrix of $\mathbf{V}$ in the subspace $V_{\otimes 1}$, determines the general element of $GL(2)$.

In light of the above considerations, we may say that *the linear space of homogeneous polynomials $T(V)$ is the representation space for the general linear group $GL(n)$, $n = 2$.*

The operator $\mathbf{U}$ of the system of zero approximation generates the rotation group $SO(2)$ in the plane. To find the explicit form of the elements of this group, we also make use of a Lie series

$$x' = \exp\,(\varphi\mathbf{U})x.$$

After the corresponding computations we come to the result

$$\begin{Vmatrix} x_1' \\ x_2' \end{Vmatrix} = \begin{Vmatrix} \cos\varphi & \sin\varphi \\ --\sin\varphi & \cos\varphi \end{Vmatrix} \begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix}.$$

Thus, the linear space of the trigonometrical functions $T_{\otimes}$ is the representation space for the rotation group $SO(2)$ in the plane. Let us denote this space by $T_{SO(2)}$.

In the normal form method, the representation space for general linear group $GL(n)$ is chosen as a of representation space. In the asymptotic decomposition method, the representation space for the subgroup of the same $GL(n)$, is chosen as a representation space.

So, the normal form method, which makes use of a universal representation space of the general linear group, does not consider the true algebraic structure of the system of zero approximation.

Contrary to that, the asymptotic decomposition method is based essentially on a deep connection between the representation theory for continuous groups and special functions of mathematical physics. This theory has been intensively developed during the last decades (see Vilenkin N.Ya. [19], Barut A., Roczka R. [20]).

2.3. Asymptotical decomposition algorithm for a perturbed motion on SO(2). Let us consider Van der Pol

$$\dot{x}_1 = x_2; \quad \dot{x}_2 = -x_1 + \varepsilon\,(1 - x_1^2)\,x_2. \tag{25}$$

system as perturbed motion on $SO(2)$. The system of zero approximation (25) yields the group $SO(2)$. Pass to the polar coordinates in (25)

$$x_1 = \rho' \sin\varphi', \;\; x_2 = \rho' \cos\varphi'. \tag{26}$$

Finally, we obtain

$$\frac{d\rho'}{dt} = \varepsilon\frac{\rho'}{2}\left(1 - \frac{\rho'^2}{4} + \cos 2\varphi' + \frac{\rho'^2}{4}\cos 4\varphi'\right),$$

$$\frac{d\varphi'}{dt} = 1 - \varepsilon\frac{1}{2}\left(\sin 2\varphi' - \frac{\rho'^2}{2}\sin 2\varphi' + \frac{\rho'^2}{4}\sin 4\varphi'\right). \tag{27}$$

Write down the operator $\mathbf{U'}_0$ associated with system (27)

$$\mathbf{U'}_0 = \mathbf{U'}_1 + \varepsilon\widetilde{\mathbf{U}}',$$

where

$$\mathbf{U'}_0 = \frac{\partial}{\partial\varphi'}, \quad \widetilde{\mathbf{U'}} = b_1(\rho',\varphi')\frac{\partial}{\partial\rho'} + b_2(\rho',\varphi')\frac{\partial}{\partial\varphi'},$$

$$b_1(\rho',\varphi') = \frac{\rho'}{2}\left(1 - \frac{\rho'^2}{4} + \cos 2\varphi' + \frac{\rho'^2}{4}\cos 4\varphi'\right),$$

$$b_2(\rho',\varphi') = 1 - \varepsilon\frac{1}{2}\left(\sin 2\varphi' - \frac{\rho'^2}{2}\sin 2\varphi' + \frac{\rho'^2}{4}\sin 4\varphi'\right). \tag{28}$$

Fig. 1. Solution in the phase plane for the Van der Pol system: exact solution–contineous lines and approximate one (a centralized system of the first approximation)–dot lines (a) $\varepsilon = 1$, (b) $\varepsilon = 0.5$.

The operator $\mathbf{U}$ has the representation matrix $\mathcal{F}_{m_n}$ in the subspace $T_{\otimes n}$. This matrix can be calculated by

$$\mathbf{U}f^{(n)} = f^{(n)}\mathcal{F}_{m_n}, \ \mathcal{F}_{m_n} = \begin{Vmatrix} 0 & -n \\ n & 0 \end{Vmatrix}.$$

Illustrate application of the asymptotic decomposition method to system (1) in the representation space of $T_{\otimes}$. Calculate only the first approximation. Let one term $\mathbf{S}_1$ be in the transformation (14) and the transformed operator be represented by the sum

$$\mathbf{U}_0 = \mathbf{U} + \varepsilon\mathbf{N}_1.$$

According to the general theory, we should consider the equation

$$[\mathbf{U}, \ \mathbf{S}_1] = \mathbf{F}_1, \ \mathbf{F}_1 =_{\mathrm{def}} \widetilde{\mathbf{U}}. \tag{29}$$

After a change of variables (26), $\partial/\partial x_1$, $\partial/\partial x_2$ turn into $\mathbf{L}_1$, $\mathbf{L}_2$, respectively

$$\mathbf{L}_1 = \sin\varphi\frac{\partial}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}, \ \mathbf{L}_2 = \cos\varphi\frac{\partial}{\partial\rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}.$$

Rewrite $\mathbf{U}, \widetilde{\mathbf{U}}$ in new variables using $\mathbf{L}_1, \mathbf{L}_2$

$$\mathbf{U} \equiv \frac{\partial}{\partial\varphi} \equiv \|\rho\sin\varphi, \ \rho\cos\varphi\| \, \mathcal{F}L,$$

$$\widetilde{\mathbf{U}} = \|\rho\sin\varphi, \ \rho\cos\varphi\| \, \mathcal{Q}_{11}L + \|\rho^3\sin 3\varphi, \ \rho^3\cos 3\varphi\| \, \mathcal{Q}_{31}L.$$

Write the operator of the transformation $\mathbf{S}_1$ following the structure of the right-hand part of Equation (29) in the form

$$\mathbf{S}_1 = \mathbf{S}_{11} + \mathbf{S}_{31},$$

where

$$\mathbf{S}_{11} = \|\rho\sin\varphi, \ \rho\cos\varphi\| \, \Gamma_{11}L, \mathbf{S}_{31} = \|\rho^3\sin 3\varphi, \ \rho^3\cos 3\varphi\| \, \Gamma_{31}L,$$

Γ_{11}, Γ_{31} are the unknown second-order square matrices . In the general case, they depend on the variable ρ.

Substituting $\mathbf{U}, \widetilde{\mathbf{U}}$ and $\mathbf{S}_1$ into Equation (29), we obtain two independent subsystems of linear algebraic equations

$$\mathcal{F}_1\Gamma_{j1} - \Gamma_{j1}\mathcal{F} = \mathcal{Q}_{j1},\ j = 1, 2.$$

All the further calculations are similar to ones done in previous subsection. We are giving the final results.

Thus, the operator $\mathbf{N}_1$ defined by the matrix can be written in the final form

$$\mathbf{N}_1 \equiv \left(\frac{1}{2} - \frac{1}{8}\rho^2\right)\rho\sin\varphi\mathbf{L}_1 + \left(\frac{1}{2} - \frac{1}{8}\rho^2\right)\rho\cos\varphi\mathbf{L}_2 \equiv \rho\left(\frac{1}{2} - \frac{1}{8}\rho^2\right)\frac{\partial}{\partial\rho}.$$

By the operator $\mathbf{U}_0 = \mathbf{U} + \varepsilon\mathbf{N}_1$, we restore the centralized system of the first approximation

$$\dot{\rho} = \frac{\varepsilon}{2}\left(1 - \frac{1}{4}\rho^2\right)\rho,\ \dot{\varphi} = 1.$$

Comparison of the asymptotic decomposition algorithm in representation space $T_\otimes$ of trigonometrical functions described in this Subsection with the analogous algorithm in the space of polynomials $T(V)$ in previous subsection shows a substantial decrease in calculating effort. This fact takes place due to lowering the order of the representation matrices $\mathcal{F}_j$ of the operator $\mathbf{U}$ in the subspaces $T_{\otimes j}$ in comparison with the subspace $V_{\otimes j}$. Really, in the first case, the order of the matrices $\mathcal{F}_j$ is unchangeable and is equal to 2. In the second case, it grows proportionally to the index j.

Finally, let us compare the asymptotic decomposition algorithm with existing methods. If the representation space $T_\otimes$ of the group $SO(2)$ is chosen, we then obtain the results of the Krylov–Bogoliubov asymptotic method. If the representation space $V_\otimes$ of the general linear group $GL(2)$ (the space of homogeneous polynomials) is chosen, then we obtain the results of the normal forms method.

2.4. Group averaging for a perturbed motion on SO(2). Let us apply the asymptotic decomposition algorithm to system (27) with averaging on group $SO(2)$, defined as

$$\langle f(\rho,\varphi)\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f(\rho,\varphi)d\varphi.$$

We restrict ourselves by the first approximation and consider the operator equation

$$[\mathbf{U}, \mathbf{S}_1] = \widetilde{\mathbf{U}} - \mathrm{pr}\,\widetilde{\mathbf{U}}, \tag{30}$$

where $\mathbf{S}_1 = \gamma_1(\rho,\varphi)\partial/\partial\rho + \gamma_2(\rho,\varphi)\partial/\partial\varphi$. Compute the average value of coefficients (28). According to the general theory,

$$\mathrm{pr}\,\widetilde{\mathbf{U}} = \left(\frac{\rho}{2} - \frac{1}{8}\rho^3\right)\frac{\partial}{\partial\rho}.$$

Therefore, the centralized (averaged) system in the first approximation takes the form

$$\dot{\rho} = \varepsilon\left(\frac{\rho}{2} - \frac{1}{8}\rho^3\right),\quad \dot{\varphi} = 1$$

Operator equation (30) is replaced by the system of differential equations

$$\frac{\partial \gamma_j}{\partial \varphi} = b_j(\rho, \varphi) + \langle b_j(\rho, \varphi) \rangle, \ j = 1, 2.$$

Written out, such systems are easily integrated into trigonometric functions.

2.5. Partial group averaging for a perturbed motion on SO(2). Let us return to consideration of the disturbed system (27). We shall change the algorithm of solving of operator equation (30). Let us put in formula (14)

$$\mathbf{S}_j = \gamma_j(\rho, \varphi)\frac{\partial}{\partial \rho}, \ \ j = 1, \cdots.$$

Operator equation (30) can be written in the form

$$[\mathbf{U}, \mathbf{S}_1] = b_1(\rho, \varphi)\frac{\partial}{\partial \rho} - \langle b_1(\rho, \varphi) \rangle \frac{\partial}{\partial \rho},$$

where $\mathbf{S}_1 = \gamma_1(\rho, \varphi)\partial/\partial\rho$. Hence, only variable ρ is transforming. Obviously,

$$\langle b_1(\rho, \varphi) \rangle = \frac{\rho}{2} - \frac{1}{8}\rho^3.$$

Upon calculations the centralized system of the first approximation reduces to

$$\dot{\rho} = \frac{\varepsilon}{2}\left(1 - \frac{1}{4}\rho^2\right)\rho,$$

$$\dot{\varphi} = 1 - \frac{\varepsilon}{2}\left(\sin 2\varphi - \frac{\rho^2}{2}\sin 2\varphi + \frac{\rho^2}{4}\sin 4\varphi\right). \tag{31}$$

For finding $\gamma_1(\rho, \varphi)$ one has differential equation

$$\frac{\partial \gamma_1}{\partial \varphi} = b_1(\rho, \varphi) - \langle b_1(\rho, \varphi) \rangle,$$

which easy to solve

$$\gamma_1(\rho, \varphi) = \frac{\rho}{4}\sin(2\varphi) + \frac{\rho^3}{32}\sin(4\varphi).$$

It is important that analysis of the first equation of system (31) displays existence of stable limit cycle. One can also illustrate this fact graphically. Comparison of the solution of initial disturbed system (27) and the centralized system of the first approximation (31) (previously reduced to initial variables x_1 x_2) is shown in Figure 1.

The advantage of partial group averaging lies in the fact that it enables us to receive approximate equations with much less calculation efforts. Nevertheless, this equations help us to make qualitative analysis of initial nonlinear system.

3. Examples: the Motion of a Point on a Sphere.

3.1. Linear equations. Consider the system of equations of the motion of a point on a sphere.

$$\dot{x}_1 = x_2,$$

Fig. 2. (a) Solution for linear system of the movement of a point on a sphere.
(b) Solution in the phase plane for angle spherical variables, governing the movement of a point on a sphere.

$$\dot{x}_2 = x_3 - x_1, \tag{32}$$
$$\dot{x}_3 = -x_2,$$

To show this it is to be noted that the system has two integrals

$$v_1(x) = x_1 + x_2 = c_1, \tag{33}$$

$$v_2(x) = x_1^2 + x_2^2 + x_3^2 = c_2. \tag{34}$$

Hence, the motion described by system (32) takes place in the circle which is an intersection of sphere (34) with radius $\rho = \sqrt{c_2}$ and plane (33); see Figure 2a. The motion on a sphere is quite complicate. By introducing spherical coordinates in system (32)

$$x_1 = \rho \sin\theta \cos\varphi, x_2 = \rho\cos\theta \sin\varphi, x_3 = \rho\cos\theta, \rho = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

we can clarify this fact. (32) becomes

$$\begin{aligned} \dot{\rho} &= 0, \\ \dot{\theta} &= \sin\varphi, \\ \dot{\varphi} &= -1 + \operatorname{ctg}\theta\cos\varphi, \end{aligned} \tag{35}$$

The trajectories in the phase plane of two last equations of system (35) one can find in Figure 2b.

We are now in position to find the fact which will be important for sequel. Let us show that the motion due to equation (32) may be considered as one-parameter subgroup of rotation sphere group $SO(3)$ in R^3. The operator $\mathbf{U}$ associated with system (32) can be represented as the sum of two operators $\mathbf{U}_1, \mathbf{U}_2$

$$\mathbf{U}_1 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}, \quad \mathbf{U}_2 = x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}.$$

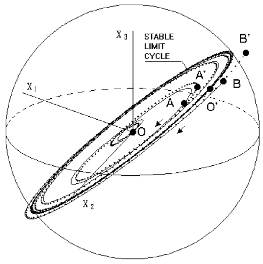

Fig. 3. Solution of nonlinear equations of the movement of a point on a sphere: initial points of trajectories in the plane $x_1 + x_3 = 0$.

Calculate the Poisson bracket of the operators $\mathbf{U}_1, \mathbf{U}_2$ and denote the result by $\mathbf{U}_3$

$$\mathbf{U}_3 =_{\text{def}} [\mathbf{U}_1, \mathbf{U}_2] = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}\,.$$

The operators $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3$ yield the finite-dimensional Lie algebra $\mathcal{B}_3$ as the relations

$$[\mathbf{U}_2, \mathbf{U}_3] = \mathbf{U}_1, \quad [\mathbf{U}_3, \mathbf{U}_1] = \mathbf{U}_2.$$

imply.This algebra is the algebra of the group $SO(3)$ of three-dimensional space rotation. Thus, the solution of system (32) can be written as the Lie series

$$x_j = e^{t(\mathbf{U}_1 + \mathbf{U}_2)} x_j, \quad j = \overline{1,3}. \tag{36}$$

Otherwise speaking, the solution (36) of system (32) is an element of $SO(3)$.

3.2. Nonlinear equations. Now suppose that system (32) is subjected to nonlinear disturbations

$$\begin{aligned} \dot{x}_1 &= x_2 + \varepsilon \frac{x_1}{\rho} F(x), \\ \dot{x}_2 &= x_3 - x_1 + \varepsilon \frac{x_2}{\rho} F(x), \\ \dot{x}_3 &= -x_2 + \varepsilon \frac{x_3}{\rho} F(x), \end{aligned} \tag{37}$$

where

$$\rho = \sqrt{x_1^2 + x_2^2 + x_3^2}, \;\; F(x) = h^2 - \rho^2 + \frac{x_2 - x_1}{\rho} \frac{1}{\sqrt{|\,\rho\,|}},$$

$$\varepsilon -- \text{ positive parameter.}$$

Nonlinear system (37) has limit cycles. Any trajectories with beginning point in plane

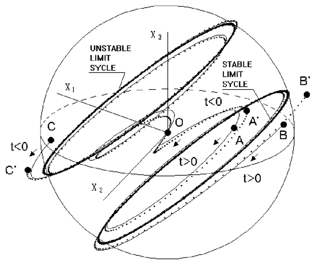

Fig. 4. Solution of nonlinear equations of the movement of a point on a sphere: initial points of trajectories in the plane $x_1 + x_3 = \pm c$.

$$x_1 + x_3 = 0$$

(points A, A', B, B', C, C' in Figure 3) are winding on a limit cycle. Any trajectories with beginning point in plane

$$x_1 + x_3 = \pm c, \quad c = \text{const}$$

or are winding on a limit cycle (points A, A', B, B' in Figure 4), or are unwinding from a limit cycle (points A, A', C, C' in Figure 4).

3.3. Group averaging for a perturbed motion on $SO(3)$. Let us introduce spherical coordinates in system (37)

$$x'_1 = \rho' \sin\theta' \cos\varphi', x'_2 = \rho' \cos\theta' \sin\varphi', x'_3 = \rho' \cos\theta'. \tag{38}$$

(37) becomes

$$\begin{aligned} \dot{\rho}' &= \varepsilon f(\rho', \theta', \varphi'), \\ \dot{\theta}' &= \sin\varphi', \\ \dot{\varphi}' &= -1 + \operatorname{ctg}\theta' \cos\varphi', \end{aligned} \tag{39}$$

where

$$f(\rho', \theta', \varphi') = h^2 - \rho'^2 + (\sin\varphi' - \cos\varphi') \sin\theta' \frac{1}{\sqrt{|\,\rho'\,|}}.$$

For operator associated with system one has

$$\mathbf{U}'_0 = \mathbf{U}' + \varepsilon \tilde{\mathbf{U}}',$$

where

$$\mathbf{U}' = (-1 + \operatorname{ctg}\theta' \cos\varphi') \frac{\partial}{\partial\varphi'} + \sin\varphi' \frac{\partial}{\partial\theta'}$$

Fig. 5. Solution of nonlinear equations of the movement of a point on a sphere: exact solution–contineous lines and approximate one (a centralized system of the first approximation)–dot lines (a) $\varepsilon = 0.5$, (b) $\varepsilon = 0.1$.

$$\tilde{\mathbf{U}}' = f(\rho', \theta', \varphi')\frac{\partial}{\partial \rho'},$$

Now let us apply the asymptotic decomposition algorithm in the first approximation to system (39) using partial group averaging on $SO(3)$. Making use transformations

$$\rho' = e^{\varepsilon \mathbf{S}}\rho, \;\; \theta' = \theta, \;\; \varphi' = \varphi,$$

$$\mathbf{S} = \mathbf{S_1} = \gamma_\mathbf{1}(\rho, \theta, \varphi)\frac{\partial}{\partial \rho}.$$

in accordance with the general theory one has

$$\mathbf{U}_0' = \mathbf{U} + \varepsilon([\mathbf{U}, \mathbf{S}_1] + \mathbf{F}_1) + \varepsilon^2 + \cdots.$$

Let us consider operator equation

$$[\mathbf{U}, \mathbf{S}_1] = \hat{\mathbf{F}}_1 - \mathrm{pr}\hat{\mathbf{F}}_1.$$

where

$$\hat{\mathbf{F}}_1 = f(\rho, \theta, \varphi)\frac{\partial}{\partial \rho}.$$

Let us to remind that $\hat{\mathbf{F}}_1$ is received from $\mathbf{F}_1$ by omitting summands with all derivatives but $\partial/\partial\rho$.

It is worth now to use the fact that coefficients of $\hat{\mathbf{F}}_1$ are functions on group $SO(3)$. For expressions in Fourier series basic spherical functions will be used. After calculation one obtains

$$f(\rho, \theta, \varphi) = f_0 + c_1 Y_1^1 + c_{-1} Y_1^{-1},$$

where

$$f_0 = h^2 - \rho^2,$$

$$Y_1^1 = -\frac{\sqrt{3}}{2\sqrt{2\pi}} e^{i\varphi} \sin\theta, \;\; Y_1^{-1} = \frac{\sqrt{3}}{2\sqrt{2\pi}} e^{-i\varphi} \sin\theta.$$

$$c_1 = \frac{\sqrt{2\pi}}{\sqrt{3}}(1+i), \;\; c_2 = \frac{\sqrt{2\pi}}{\sqrt{3}}(-1+i).$$

Free term f_0 in expressions of f in Fourier series by basic spherical functions is equal to "average on group SO(3)" of this function. It is calculated by formula

$$f_0 = \langle f(\rho,\theta,\varphi)\rangle =_{def} \int_0^{2\pi}\int_0^{\pi} f(\rho,\theta,\varphi)\sin\theta d\theta d\varphi$$

In accordance with the general theory one has

$$\mathrm{pr}\hat{\mathbf{F}}_1 = \mathrm{pr}\left(f(\rho,\theta,\varphi)\frac{\partial}{\partial\rho}\right) = \langle f(\rho,\theta,\varphi)\rangle \frac{\partial}{\partial\rho}.$$

As a result, one obtains an centralized system of the first approximation

$$\begin{aligned} \dot{\rho} &= \varepsilon(h^2-\rho^2), \\ \dot{\theta} &= \sin\varphi, \\ \dot{\varphi} &= -1 + \mathrm{ctg}\,\theta\cos\varphi, \end{aligned} \tag{40}$$

For finding coefficient $\gamma_1(\rho,\theta,\varphi)$ in $\mathbf{S}_1$ one comes to equation

$$\mathbf{U}\gamma_1 = c_1 Y_1^1 + c_{-1} Y_1^{-1},$$

which easy to solve

$$\gamma_1(\rho,\theta,\varphi) = \frac{\sqrt{2\pi}}{2\sqrt{3}}(\cos\varphi + \sin\varphi)\sin\theta \frac{1}{\sqrt{|\,\rho\,|}}.$$

The first equation of system (40) displays existence of two limit cycles $\rho = \pm h$. Trajectories of exact (37) and approximate (40) (reduced to initial variables) systems are shown in Figure 5 under various value of ε.

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