Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Optimal velocity control of a convective Cahn–Hilliard system with double obstacles and dynamic boundary conditions: A 'deep quench' approach

Pierluigi Colli[1], Gianni Gilardi[1], Jürgen Sprekels[2]

submitted: September 26, 2017

[1] Dipartimento di Matematica "F. Casorati"
Università di Pavia
Via Ferrata, 5
27100 Pavia, Italy
E-Mail: pierluigi.colli@unipv.it
gianni.gilardi@unipv.it

[2] Department of Mathematics
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin, Germany
and
Weierstrass Institute
Mohrenstr. 39
10117 Berlin, Germany
E-Mail: juergen.sprekels@wias-berlin.de

No. 2428
Berlin 2017

2010 *Mathematics Subject Classification.* 49J20, 49K20, 74M15, 35K86, 76R05, 82C26, 80A22.

Key words and phrases. Cahn-Hilliard system, convection term, dynamic boundary conditions, double obstacle potentials, optimal velocity control, optimality conditions.

PC and GG gratefully acknowledge finantial support from the MIUR-PRIN Grant 2015PA5MP7 "Calculus of Variations", the GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e loro Applicazioni) of INDAM (Istituto Nazionale di Alta Matematica) and the IMATI – C.N.R. Pavia.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Optimal velocity control of a convective Cahn–Hilliard system with double obstacles and dynamic boundary conditions: A ‘deep quench’ approach

Pierluigi Colli, Gianni Gilardi, Jürgen Sprekels

Abstract

In this paper, we investigate a distributed optimal control problem for a convective viscous Cahn–Hilliard system with dynamic boundary conditions. Such systems govern phase separation processes between two phases taking place in an incompressible fluid in a container and, at the same time, on the container boundary. The cost functional is of standard tracking type, while the control is exerted by the velocity of the fluid in the bulk. In this way, the coupling between the state (given by the associated order parameter and chemical potential) and control variables in the governing system of nonlinear partial differential equations is bilinear, which presents a difficulty for the analysis. In contrast to the previous paper *Optimal velocity control of a viscous Cahn–Hilliard system with convection and dynamic boundary conditions* by the same authors, the bulk and surface free energies are of double obstacle type, which renders the state constraint nondifferentiable. It is well known that for such cases standard constraint qualifications are not satisfied so that standard methods do not apply to yield the existence of Lagrange multipliers. In this paper, we overcome this difficulty by taking advantage of results established in the quoted paper for logarithmic nonlinearities, using a so-called ‘deep quench approximation’. We derive results concerning the existence of optimal controls and the first-order necessary optimality conditions in terms of a variational inequality and the associated adjoint system.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ denote some open, bounded and connected set having a smooth boundary Γ and unit outward normal ν. We denote by ∂_ν, ∇_Γ, Δ_Γ the outward normal derivative, the tangential gradient, and the Laplace–Beltrami operator on Γ, in this order. Moreover, we fix some final time $T > 0$ and introduce for every $t \in (0,T]$ the sets $Q_t := \Omega \times (0,t)$ and $\Sigma_t := \Gamma \times (0,t)$, where we put, for the sake of brevity, $Q := Q_T$ and $\Sigma := \Sigma_T$. We then consider the following optimal control problem:

($\mathcal{P}_0$) Minimize the cost functional

$$\mathcal{J}((\rho,\rho_\Gamma),u) := \frac{\beta_1}{2}\int_Q |\rho-\widehat{\rho}_Q|^2 + \frac{\beta_2}{2}\int_\Sigma |\rho-\widehat{\rho}_\Sigma|^2 + \frac{\beta_3}{2}\int_\Omega |\rho(T)-\widehat{\rho}_\Omega|^2 + \frac{\beta_4}{2}\int_\Gamma |\rho_\Gamma(T)-\widehat{\rho}_\Gamma|^2 + \frac{\beta_5}{2}\int_Q |u|^2 , \qquad (1.1)$$

DOI 10.20347/WIAS.PREPRINT.2428 Berlin 2017

subject to the state system

$$\partial_t\rho + \nabla\rho\cdot u - \Delta\mu = 0 \quad \text{in } Q\,, \tag{1.2}$$
$$\tau_\Omega\partial_t\rho - \Delta\rho + \xi + \pi(\rho) = \mu \quad \text{in } Q\,, \tag{1.3}$$
$$\xi \in \partial I_{[-1,1]}(\rho) \quad \text{in } Q\,, \tag{1.4}$$
$$\partial_t\rho_\Gamma + \partial_\nu\mu - \Delta_\Gamma\mu_\Gamma = 0 \quad \text{and} \quad \mu_{|\Sigma} = \mu_\Gamma \quad \text{on } \Sigma\,, \tag{1.5}$$
$$\tau_\Gamma\partial_t\rho_\Gamma + \partial_\nu\rho - \Delta_\Gamma\rho_\Gamma + \xi_\Gamma + \pi_\Gamma(\rho_\Gamma) = \mu_\Gamma \quad \text{and} \quad \rho_{|\Sigma} = \rho_\Gamma \quad \text{on } \Sigma\,, \tag{1.6}$$
$$\xi_\Gamma \in \partial I_{[-1,1]}(\rho_\Gamma) \quad \text{on } \Sigma\,, \tag{1.7}$$
$$\rho(0) = \rho_0 \quad \text{in } \Omega, \quad \rho_\Gamma(0) = \rho_{0|\Gamma} \quad \text{on } \Gamma\,, \tag{1.8}$$

and to the control constraint

$$u \in \mathcal{U}_{\rm ad}\,. \tag{1.9}$$

Here, the constants β_i, $1 \le i \le 5$, are nonnegative but not all zero, and $\widehat\rho_Q$, $\widehat\rho_\Sigma$, $\widehat\rho_\Omega$, $\widehat\rho_\Gamma$, are given target functions. Furthermore, π, π_Γ denote smooth functions, while $I_{[-1,1]}$ is the indicator function of the interval $[-1,1]$. Moreover, $\mathcal{U}_{\rm ad}$ is a suitable bounded, closed and convex subset of the control space

$$\mathcal{X} := L^2(0,T;\widetilde U) \cap (L^\infty(Q))^3 \cap (H^1(0,T;L^3(\Omega)))^3\,, \tag{1.10}$$

where

$$\widetilde U \;:=\; \bigl\{ u \in (L^2(\Omega))^3 :\, \operatorname{div} u = 0 \text{ a.e. in } \Omega \text{ and } u\cdot\nu = 0 \text{ a.e. on } \Gamma \bigr\}\,. \tag{1.11}$$

The regularity condition $u \in (H^1(0,T;L^3(\Omega)))^3$ for the admissible controls seems to be unusual at a first glance. However, in view of the bilinear coupling between control and state, it turns out (cf. [13]) that, among other constraints, this is exactly the kind of regularity that guarantees the existence of a unique solution to the state system having sufficient regularity properties.

We note that the state system (1.2)–(1.8) can be seen as a phase field model for a phase separation process taking place in an incompressible fluid in the container Ω and on the container boundary Γ. In this connection, the variables (μ,μ_Γ) and (ρ,ρ_Γ) stand for the chemical potential and the order parameter (usually the density of one of the involved phases, normalized in such a way as to attain its values in the interval [-1,1]) of the phase separation process in the bulk and on the surface, respectively. It is worth noting that the total mass of the order parameter is conserved during the separation process; indeed, integrating (1.2) for fixed $t \in (0,T]$ over Ω, and using the fact that $u \in \mathcal{X}$, as well as (1.5), we readily find that

$$\partial_t\Bigl(\int_\Omega \rho(t) + \int_\Gamma \rho_\Gamma(t) \Bigr) = 0\,. \tag{1.12}$$

We also note that the densities of the local free bulk energy $f + I_{[-1,1]}$ and the local free surface energy $f_\Gamma + I_{[-1,1]}$ are typically of double obstacle type.

In the mathematical literature numerous contributions are dedicated to the questions of well-posedness and asymptotic behavior for various types of Cahn–Hilliard systems: viscous or nonviscous, local or nonlocal, with zero Neumann boundary conditions or dynamic boundary conditions. We omit to (try to) quote a number of contributions since they are too many and we would surely miss some of the important ones. However, let us point out that there are still a few papers dealing with the related optimal control problems: among them, we refer to [7, 9, 12, 16, 22, 29, 32, 33] for the case of Dirichlet or zero Neumann boundary conditions and to [3, 4, 8, 10, 11, 15, 18] for the case of dynamic boundary conditions.

A recent investigation for convective Cahn–Hilliard systems produced the results rigorously proved in [30] for the one-dimensional and in [31] for the two-dimensional case. The papers [17, 28] are devoted to the distributed optimal control of a two-dimensional Cahn–Hilliard/Navier–Stokes system. Let us also mention the contributions [20, 21, 23, 24], which deal with the optimal control of the Cahn–Hilliard/Navier–Stokes system in 3D, but for some time-discretized version.

A key feature of this paper is the use of the fluid velocity as the control variable in the convective Cahn–Hilliard system. From a practical point of view, this control process can be realized by placing either a mechanical stirring device or an ultrasound emitter into the container. In the case of electrically conducting fluids like molten metals, a remarkable option is the possibility of using magnetic fields (cf. [25] for applications of this kind). To the authors' best knowledge, the only existing mathematical contributions, in which the fluid velocity is used as the control in a convective Cahn–Hilliard system in three dimensions of space, are the recent contributions [26] and [14]. While in [26] a nonlocal convective Cahn–Hilliard system with a possibly degenerating mobility and zero Neumann boundary conditions was studied, we considered in [14] a viscous local Cahn–Hilliard system with constant mobility (normalized to unity) and the more difficult dynamic boundary conditions (see also [13] and [6] for related results). However, in [14] only differentiable nonlinearities were admitted.

In this contribution, we investigate the much more challenging nondifferentiable double obstacle case when ξ, ξ_Γ satisfy the inclusions (1.4), (1.7), and we assume dynamic boundary conditions. Moreover, we consider the spatially three-dimensional case. Our approach is guided by a strategy that was introduced by two of the present authors and M. H. Farshbaf-Shaker in [5]: we aim to derive first-order necessary optimality conditions for the double obstacle case by performing a so-called 'deep quench limit' in a family of optimal control problems with differentiable logarithmic nonlinearities that was treated in [14], and for which the corresponding state systems were analyzed in [13]. The general idea is briefly explained as follows: we replace the inclusions (1.4) and (1.7) by the identities

$$\xi = \varphi(\alpha)\, h'(\rho) \quad \text{in } Q, \quad \xi_\Gamma = \varphi(\alpha)\, h'(\rho_\Gamma) \quad \text{on } \Sigma, \tag{1.13}$$

where h is defined by

$$h(\rho) := \begin{cases} (1-\rho)\,\ln(1-\rho) + (1+\rho)\,\ln(1+\rho) & \text{if } \rho \in (-1,1) \\ 2\,\ln(2) & \text{if } \rho \in \{-1,1\} \end{cases}, \tag{1.14}$$

and where

$$\varphi \in C(0,1] \text{ is positive on } (0,1] \text{ and satisfies } \lim_{\alpha \searrow 0} \varphi(\alpha) = 0. \tag{1.15}$$

We remark that we can simply choose $\varphi(\alpha) = \alpha^p$ for some $p > 0$. Now observe that $h(y) \geq 0$ for all $y \in [-1,1]$, $h'(y) = \ln\left(\frac{1+y}{1-y}\right)$ and $h''(y) = \frac{2}{1-y^2} > 0$ for $y \in (-1,1)$. Hence, in particular, we have

$$\lim_{\alpha \searrow 0} \varphi(\alpha)\, h(y) = 0 \quad \forall\, y \in [-1,1], \quad \lim_{\alpha \searrow 0} \varphi(\alpha)\, h'(y) = 0 \quad \forall\, y \in (-1,1),$$

$$\lim_{\alpha \searrow 0} \Big(\varphi(\alpha) \lim_{y \searrow -1} h'(y)\Big) = -\infty, \quad \lim_{\alpha \searrow 0} \Big(\varphi(\alpha) \lim_{y \nearrow +1} h'(y)\Big) = +\infty\,. \tag{1.16}$$

We thus may regard the graph $\varphi(\alpha)\, h'$ as an approximation to the graph of the subdifferential $\partial I_{[-1,1]}$.

Now, for any $\alpha > 0$, the optimal control problem (later to be denoted by $(\mathcal{P}_\alpha)$), which results if in $(\mathcal{P}_0)$ the relations (1.4), (1.7) are replaced by (1.13), is of the type for which in [14] the existence of optimal controls $u^\alpha \in \mathcal{U}_{\rm ad}$ as well as first-order necessary optimality conditions have been derived. Proving a priori estimates (uniform in $\alpha > 0$), and employing compactness and monotonicity arguments, we will be able to show the following existence and approximation result: whenever $\{u^{\alpha_n}\} \subset \mathcal{U}_{\rm ad}$ is a sequence of optimal controls for $(\mathcal{P}_{\alpha_n})$, where $\alpha_n \searrow 0$ as $n \to \infty$, then there exist a subsequence of $\{\alpha_n\}$, which is again indexed by n, and an optimal control $\bar{u} \in \mathcal{U}_{\rm ad}$ of $(\mathcal{P}_0)$ such that

$$u^{\alpha_n} \to \bar{u} \quad \text{weakly-star in } \mathcal{X} \text{ as } n \to \infty\,. \tag{1.17}$$

In other words, optimal controls for $(\mathcal{P}_\alpha)$ are for small $\alpha > 0$ likely to be 'close' to optimal controls for $(\mathcal{P}_0)$. It is natural to ask if the reverse holds, i. e., whether every optimal control for $(\mathcal{P}_0)$ can be approximated by a sequence $\{u^{\alpha_n}\}$ of optimal controls for $(\mathcal{P}_{\alpha_n})$, for some sequence $\alpha_n \searrow 0$.

Unfortunately, we will not be able to prove such a 'global' result that applies to all optimal controls for $(\mathcal{P}_0)$. However, a 'local' result can be established. To this end, let $\bar{u} \in \mathcal{U}_{\rm ad}$ be any optimal control for $(\mathcal{P}_0)$. We introduce the 'adapted' cost functional

$$\widetilde{\mathcal{J}}((\rho,\rho_\Gamma),u) \,:=\, \mathcal{J}((\rho,\rho_\Gamma),u) \,+\, \frac{1}{2}\,\|u-\bar{u}\|^2_{(L^2(Q))^3} \tag{1.18}$$

and consider for every $\alpha \in (0,1]$ the *adapted control problem* of minimizing $\widetilde{\mathcal{J}}$ subject to $u \in \mathcal{U}_{\rm ad}$ and to the constraint that $((\mu,\mu_\Gamma),(\rho,\rho_\Gamma))$ solves the approximating system (1.2), (1.3), (1.5), (1.6), (1.8), (1.13). It will then turn out that the following is true:

(i) There are some sequence $\alpha_n \searrow 0$ and minimizers $\bar{u}^{\alpha_n} \in \mathcal{U}_{\rm ad}$ of the adapted control problem associated with α_n, $n \in \mathbb{N}$, such that

$$\bar{u}^{\alpha_n} \to \bar{u} \quad \text{strongly in } (L^2(Q))^3 \text{ as } n \to \infty. \tag{1.19}$$

(ii) It is possible to pass to the limit as $\alpha \searrow 0$ in the first-order necessary optimality conditions corresponding to the adapted control problems associated with $\alpha \in (0,1]$ in order to derive first-order necessary optimality conditions for problem $(\mathcal{P}_0)$.

The paper is organized as follows: in Section 2, we give a precise statement of the problem under investigation, and we derive some results concerning the state system (1.2)–(1.8) and its α–approximation which is obtained if in $(\mathcal{P}_0)$ the relations (1.4) and (1.7) are replaced by the relations (1.13). In Section 3, we then prove the existence of optimal controls and the approximation result formulated above in (i). The final Section 4 is devoted to the derivation of the first-order necessary optimality conditions, where the strategy outlined in (ii) is employed.

During the course of this analysis, we will make repeated use of Hölder's inequality, of the elementary Young's inequality

$$a\,b \,\le\, \gamma|a|^2 \,+\, \frac{1}{4\gamma}\,|b|^2 \quad \forall\, a,b \in \mathbb{R}, \quad \forall\, \gamma > 0, \tag{1.20}$$

as well as the continuity of the embeddings $H^1(\Omega) \subset L^p(\Omega)$ for $1 \le p \le 6$ and $H^2(\Omega) \subset C^0(\overline{\Omega})$. Notice that the latter embedding is also compact, while this holds true for the former embeddings only if $p < 6$. We will also use the denotations

$$Q^t := \Omega \times (t,T), \quad \Sigma^t := \Gamma \times (t,T), \quad \text{for } 0 \le t < T. \tag{1.21}$$

Moreover, throughout the paper, for a Banach space X we denote by X^* its dual space. Let $\|\cdot\|_X$ stand for the norm in the space X or in a power of it. The only exemption from this rule is for the norms of the L^p spaces and of their powers, which we often denote by $\| \cdot \|_p$, for $1 \le p \le +\infty$. By $\langle v, w \rangle_X$ we will always denote the dual pairing between elements $v \in X^*$ and $w \in X$. Finally, we recall some well-known estimates from trace theory and from the theory of elliptic equations. Namely, there is some constant $C_\Omega > 0$, which depends only on Ω, such that, for every v and v_Γ for which the right-hand sides are meaningful,

$$\|v\|_{H^{3/2}(\Omega)} \le C_\Omega \left(\|v_{|\Gamma}\|_{H^1(\Gamma)} + \|\Delta v\|_{L^2(\Omega)} \right), \tag{1.22}$$

$$\|\partial_\nu v\|_{L^2(\Gamma)} \le C_\Omega \left(\|v\|_{H^{3/2}(\Omega)} + \|\Delta v\|_{L^2(\Omega)} \right), \tag{1.23}$$

$$\|v_\Gamma\|_{H^2(\Gamma)} \le C_\Omega \left(\|v_\Gamma\|_{H^1(\Gamma)} + \|\Delta_\Gamma v_\Gamma\|_{L^2(\Gamma)} \right), \tag{1.24}$$

$$\|v\|_{H^2(\Omega)} \le C_\Omega \left(\|v_{|\Gamma}\|_{H^{3/2}(\Gamma)} + \|\Delta v\|_{L^2(\Omega)} \right). \tag{1.25}$$

2 General setting and the state system

In this section, we introduce the general setting of our control problem and state some results on the state system (1.2)–(1.8). To begin with, we recall the definition (1.10) of $\mathcal{X}$ and introduce the spaces

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := H^2(\Omega), \tag{2.1}$$

$$H_\Gamma := L^2(\Gamma), \quad V_\Gamma := H^1(\Gamma), \quad W_\Gamma := H^2(\Gamma), \tag{2.2}$$

$$\mathcal{H} := H \times H_\Gamma, \quad \mathcal{V} := \{(v, v_\Gamma) \in V \times V_\Gamma : \ v_\Gamma = v_{|\Gamma}\}, \quad \mathcal{W} := \big(W \times W_\Gamma\big) \cap \mathcal{V}. \tag{2.3}$$

In the following, we will often work in the framework of the Hilbert triplet $(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$. Thus, we have

$$\langle (g, g_\Gamma), (v, v_\Gamma) \rangle_{\mathcal{V}} = \int_\Omega g v + \int_\Gamma g_\Gamma v_\Gamma \quad \text{for every } (g, g_\Gamma) \in \mathcal{H} \text{ and } (v, v_\Gamma) \in \mathcal{V}.$$

Next, denote by $(1,1) \in \mathcal{V}$ the pair whose component functions equal unity in Ω and on Γ, respectively, and by $|\Omega|$ and $|\Gamma|$ the volume of Ω and the area of Γ, respectively. We then define the generalized mean value of a functional $g^* \in \mathcal{V}^*$ by

$$\operatorname{mean} g^* := \frac{\langle g^*, (1,1) \rangle_{\mathcal{V}}}{|\Omega| + |\Gamma|}, \tag{2.4}$$

which, if $g^* = (v, v_\Gamma) \in \mathcal{H}$, becomes

$$\operatorname{mean}(v, v_\Gamma) = \frac{\int_\Omega v + \int_\Gamma v_\Gamma}{|\Omega| + |\Gamma|}. \tag{2.5}$$

Observe that the function

$$\mathcal{V} \ni (v, v_\Gamma) \mapsto \int_\Omega |\nabla v|^2 + \int_\Gamma |\nabla_\Gamma v_\Gamma|^2 + |\operatorname{mean}(v, v_\Gamma)|^2$$

yields the square of a Hilbert norm on $\mathcal{V}$ that is equivalent to the natural one, i.e., we have, for some $C_\Omega > 0$ which depends only on Ω,

$$\|(v, v_\Gamma)\|_{\mathcal{V}}^2 \le C_\Omega \Big(\int_\Omega |\nabla v|^2 + \int_\Gamma |\nabla_\Gamma v_\Gamma|^2 + |\operatorname{mean}(v, v_\Gamma)|^2 \Big) \quad \forall\, (v, v_\Gamma) \in \mathcal{V}. \tag{2.6}$$

Next, we set

$$\mathcal{V}_{*0} := \{g^* \in \mathcal{V}^* : \ \mathrm{mean}\, g^* = 0\}, \quad \mathcal{H}_0 := \mathcal{H} \cap \mathcal{V}_{*0} \quad \text{and} \quad \mathcal{V}_0 := \mathcal{V} \cap \mathcal{V}_{*0}. \tag{2.7}$$

Notice the difference between $\mathcal{V}_{*0}$ and the dual space $\mathcal{V}_0^* = (\mathcal{V}_0)^*$. At this point, it is clear that the function

$$\mathcal{V}_0 \ni (v, v_\Gamma) \mapsto \|(v, v_\Gamma)\|_{\mathcal{V}_0}^2 := \int_\Omega |\nabla v|^2 + \int_\Gamma |\nabla_\Gamma v_\Gamma|^2 \tag{2.8}$$

is the square of a Hilbert norm on $\mathcal{V}_0$ which is equivalent to the usual one. This has the consequence (see [13, Sect. 2]) that, for every $g^* \in \mathcal{V}_{*0}$, there exists a unique pair $(\xi, \xi_\Gamma) \in \mathcal{V}_0$ such that

$$\int_\Omega \nabla \xi \cdot \nabla v + \int_\Gamma \nabla_\Gamma \xi_\Gamma \cdot \nabla_\Gamma v_\Gamma = \langle g^*, (v, v_\Gamma) \rangle_{\mathcal{V}} \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}. \tag{2.9}$$

This allows us to define the operator $\mathcal{N} : \mathcal{V}_{*0} \to \mathcal{V}_0$ as follows:

$$\text{For } g^* \in \mathcal{V}_{*0},\ \mathcal{N} g^* \text{ is the unique pair } (\xi, \xi_\Gamma) \in \mathcal{V}_0 \text{ satisfying (2.9).} \tag{2.10}$$

We notice that $\mathcal{N}$ is linear, symmetric, and bijective. Therefore, if we set

$$\|g^*\|_* := \|\mathcal{N} g^*\|_{\mathcal{V}_0}, \quad \text{for } g^* \in \mathcal{V}_{*0}, \tag{2.11}$$

then we obtain a Hilbert norm on $\mathcal{V}_{*0}$ which turns out to be equivalent to the norm induced by the norm of $\mathcal{V}^*$. For future use, we collect some properties of $\mathcal{N}$. By just applying the definition, we readily see that

$$\langle g^*, \mathcal{N} g^* \rangle_{\mathcal{V}} = \|g^*\|_*^2 \quad \text{if } g^* \in \mathcal{V}_{*0}\,, \tag{2.12}$$

$$\int_\Omega \nabla w \cdot \nabla \xi + \int_\Gamma \nabla_\Gamma w_\Gamma \cdot \nabla_\Gamma \xi_\Gamma = \|(w, w_\Gamma)\|_{\mathcal{H}}^2 \quad \text{if } (w, w_\Gamma) \in \mathcal{V}_0 \text{ and } (\xi, \xi_\Gamma) = \mathcal{N}(w, w_\Gamma)\,. \tag{2.13}$$

Moreover, owing to the symmetry of $\mathcal{N}$ (where, here and in the following, $\mathcal{N}$ is also applied to $\mathcal{V}_{*0}$-valued functions in the obvious way), we have, for a.e. $t \in (0, T)$, that

$$\langle \partial_t g^*(t), \mathcal{N} g^*(t) \rangle_{\mathcal{V}} = \frac{1}{2} \frac{d}{dt} \|g^*(t)\|_*^2\,, \quad \text{if } g^* \in H^1(0, T; \mathcal{V}_{*0}), \tag{2.14}$$

$$\int_\Omega \nabla w(t) \cdot \nabla \xi(t) + \int_\Gamma \nabla_\Gamma w_\Gamma(t) \cdot \nabla_\Gamma \xi_\Gamma(t) = \frac{1}{2} \frac{d}{dt} \|(w(t), w_\Gamma(t))\|_{\mathcal{H}}^2\,,$$
$$\text{if } (w, w_\Gamma) \in L^2(0, T; \mathcal{V}),\ \partial_t (w, w_\Gamma) \in L^2(0, T; \mathcal{V}_{*0}) \text{ and } (\xi, \xi_\Gamma) = \mathcal{N}(\partial_t (w, w_\Gamma))\,. \tag{2.15}$$

We now turn our interest to the state system (1.2)–(1.8), observing that with the above notations its weak form reads as follows: we look for functions $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma))$ such that $\mu_{|\Sigma} = \mu_\Gamma$ and

$\rho_{|\Sigma} = \rho_\Gamma$ as well as

$$\int_\Omega \partial_t \rho \, v + \int_\Gamma \partial_t \rho_\Gamma \, v_\Gamma - \int_\Omega \rho u \cdot \nabla v + \int_\Omega \nabla \mu \cdot \nabla v + \int_\Gamma \nabla_\Gamma \mu_\Gamma \cdot \nabla_\Gamma v_\Gamma = 0$$

a.e. in $(0,T)$ and for every $(v, v_\Gamma) \in \mathcal{V}$, (2.16)

$$\tau_\Omega \int_\Omega \partial_t \rho \, v + \tau_\Gamma \int_\Gamma \partial_t \rho_\Gamma \, v_\Gamma + \int_\Omega \nabla \rho \cdot \nabla v + \int_\Gamma \nabla_\Gamma \rho_\Gamma \cdot \nabla_\Gamma v_\Gamma$$
$$+ \int_\Omega (\xi + \pi(\rho)) v + \int_\Gamma (\xi_\Gamma + \pi_\Gamma(\rho_\Gamma)) v_\Gamma = \int_\Omega \mu v + \int_\Gamma \mu_\Gamma v_\Gamma$$

a.e. in $(0,T)$ and for every $(v, v_\Gamma) \in \mathcal{V}$, (2.17)

$$\xi \in \partial I_{[-1,1]}(\rho) \quad \text{a.e. in } Q, \qquad \xi_\Gamma \in \partial I_{[-1,1]}(\rho_\Gamma) \quad \text{a.e. on } \Sigma, \tag{2.18}$$

$$\rho(0) = \rho_0 \quad \text{a.e. in } \Omega, \qquad \rho_\Gamma(0) = \rho_{0|\Gamma} \quad \text{a.e. on } \Gamma. \tag{2.19}$$

We make the following assumptions on the data of our problem:

(A1) $(\rho_0, \rho_{0|\Gamma}) \in \mathcal{W}$, and we have $-1 < \rho_0(x) < 1$ for all $x \in \overline{\Omega}$.

(A2) $\tau_\Omega > 0$ and $\tau_\Gamma > 0$.

(A3) $\pi, \pi_\Gamma \in C^2[-1,1]$.

(A4) The constants β_i, $1 \le i \le 5$, are all nonnegative but not all equal to zero, and it holds $\widehat{\rho}_Q \in L^2(Q)$, $\widehat{\rho}_\Sigma \in L^2(\Sigma)$, $\widehat{\rho}_\Omega \in L^2(\Omega)$, and $\widehat{\rho}_\Gamma \in L^2(\Gamma)$.

(A5) The function $\overline{U} \in L^\infty(Q)$ and the constant $R_0 > 0$ make the admissible set

$$\mathcal{U}_{\rm ad} := \bigl\{ u \in \mathcal{X} : \ |u| \le \overline{U} \text{ a.e. in } Q, \ \|u\|_{\mathcal{X}} \le R_0 \bigr\} \tag{2.20}$$

nonempty.

Remark 2.1. Notice that the conditions $\operatorname{div} u = 0$ in Ω, $u \cdot \nu = 0$ on Γ, encoded in the definition of $\mathcal{X}$, have to be understood in the generalized sense, i.e., they are equivalent to postulating that

$$\int_\Omega u \cdot \nabla v = 0 \quad \forall \, v \in V. \tag{2.21}$$

We thus may infer that $\mathcal{U}_{\rm ad}$ is a bounded, closed and convex subset of $\mathcal{X}$.

The following result is a special case of [13, Thms. 2.3, 2.6].

Theorem 2.2. *Suppose that the assumptions* (A1)–(A3) *and* (A5) *hold true. Then the state system* (1.2)–(1.8) *has for every* $u \in \mathcal{U}_{\rm ad}$ *at least one solution* $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma), (\xi, \xi_\Gamma))$ *such that*

$$(\mu, \mu_\Gamma) \in L^\infty(0,T;\mathcal{W}), \tag{2.22}$$

$$(\rho, \rho_\Gamma) \in W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{W}), \tag{2.23}$$

$$(\xi, \xi_\Gamma) \in L^\infty(0,T;\mathcal{H}). \tag{2.24}$$

Moreover, the component (ρ,ρ_Γ) *is the same for any such solution. In addition, there is some constant* $K_1^*>0$*, which depends only on the data of the problem, such that for any solution* $((\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\xi,\xi_\Gamma))$ *associated with some* $u\in\mathcal{U}_{\rm ad}$ *it holds that*

$$\begin{aligned}&\|(\mu,\mu_\Gamma)\|_{L^\infty(0,T;\mathcal{W})} + \|(\rho,\rho_\Gamma)\|_{W^{1,\infty}(0,T;\mathcal{H})\cap H^1(0,T;\mathcal{V})\cap L^\infty(0,T;\mathcal{W})}\\ &+ \|(\xi,\xi_\Gamma)\|_{L^\infty(0,T;\mathcal{H})} \le K_1^*\,. \end{aligned} \tag{2.25}$$

It follows from Theorem 2.2 that the mapping $\mathcal{U}_{\rm ad}\ni u\mapsto \mathcal{S}_0^2(u):=(\rho,\rho_\Gamma)$ is well defined. Next, we consider for $\alpha\in(0,1]$ the $\alpha-$approximating system

$$\partial_t\rho^\alpha+\nabla\rho^\alpha\cdot u-\Delta\mu^\alpha=0 \quad \text{a.e. in } Q\,, \tag{2.26}$$

$$\tau_\Omega\,\partial_t\rho^\alpha-\Delta\rho^\alpha+\varphi(\alpha)h'(\rho^\alpha)+\pi(\rho^\alpha)=\mu^\alpha \quad \text{a.e. in } Q\,, \tag{2.27}$$

$$\partial_t\rho_\Gamma^\alpha+\partial_\nu\mu^\alpha-\Delta_\Gamma\mu_\Gamma^\alpha=0 \quad\text{and}\quad \mu^\alpha_{|\Sigma}=\mu_\Gamma^\alpha \quad \text{a.e. on } \Sigma\,, \tag{2.28}$$

$$\begin{aligned}&\tau_\Gamma\,\partial_t\rho_\Gamma^\alpha+\partial_\nu\rho^\alpha-\Delta_\Gamma\rho_\Gamma^\alpha+\varphi(\alpha)h'(\rho_\Gamma^\alpha)+\pi_\Gamma(\rho_\Gamma^\alpha)=\mu_\Gamma^\alpha\\ &\quad\text{and}\quad \rho^\alpha_{|\Sigma}=\rho_\Gamma^\alpha \quad \text{a.e. on } \Sigma\,,\end{aligned} \tag{2.29}$$

$$\rho^\alpha(0)=\rho_0 \quad \text{a.e. in } \Omega, \quad \rho_\Gamma^\alpha(0)=\rho_{0|\Gamma} \quad \text{a.e. on } \Gamma\,, \tag{2.30}$$

where h is given by (1.14) and φ satisfies (1.15). The corresponding weak formulation reads as follows: we look for functions $((\mu^\alpha,\mu_\Gamma^\alpha),(\rho^\alpha,\rho_\Gamma^\alpha))$ such that $\mu^\alpha_{|\Sigma}=\mu_\Gamma^\alpha$ and $\rho^\alpha_{|\Sigma}=\rho_\Gamma^\alpha$ as well as

$$\begin{aligned}&\int_\Omega\partial_t\rho^\alpha\,v+\int_\Gamma\partial_t\rho_\Gamma^\alpha\,v_\Gamma-\int_\Omega\rho^\alpha u\cdot\nabla v+\int_\Omega\nabla\mu^\alpha\cdot\nabla v+\int_\Gamma\nabla_\Gamma\mu_\Gamma^\alpha\cdot\nabla_\Gamma v_\Gamma=0\\ &\quad\text{a.e. in } (0,T) \text{ and for every } (v,v_\Gamma)\in\mathcal{V},\end{aligned} \tag{2.31}$$

$$\begin{aligned}&\tau_\Omega\int_\Omega\partial_t\rho^\alpha\,v+\tau_\Gamma\int_\Gamma\partial_t\rho_\Gamma^\alpha\,v_\Gamma+\int_\Omega\nabla\rho^\alpha\cdot\nabla v+\int_\Gamma\nabla_\Gamma\rho_\Gamma^\alpha\cdot\nabla_\Gamma v_\Gamma\\ &\quad+\int_\Omega(\varphi(\alpha)h'(\rho^\alpha)+\pi(\rho^\alpha))v+\int_\Gamma(\varphi(\alpha)h'(\rho_\Gamma^\alpha)+\pi_\Gamma(\rho_\Gamma^\alpha))v_\Gamma=\int_\Omega\mu^\alpha v+\int_\Gamma\mu_\Gamma^\alpha v_\Gamma\\ &\quad\text{a.e. in } (0,T) \text{ and for every } (v,v_\Gamma)\in\mathcal{V},\end{aligned} \tag{2.32}$$

$$\rho^\alpha(0)=\rho_0 \quad \text{a.e. in } \Omega, \qquad \rho_\Gamma^\alpha(0)=\rho_{0|\Gamma} \quad \text{a.e. on } \Gamma. \tag{2.33}$$

Observe that also this system has the property that the unknown representing the order parameter is a conserved quantity: indeed, insertion of $(v,v_\Gamma)=(1,1)\in\mathcal{V}$ in (2.31) and integration over time yield that

$$\widehat{r}\,:=\,\mathrm{mean}\,(\rho_0,\rho_{0|\Gamma})\,=\,\mathrm{mean}\,(\rho^\alpha(t),\rho_\Gamma^\alpha(t)),\ 0\le t\le T,\quad \forall\,\alpha\in(0,1]. \tag{2.34}$$

We have the following result for the approximating system.

Theorem 2.3. *Suppose that the conditions* (A1)–(A3)*,* (A5)*,* (1.14) *and* (1.15) *are satisfied. Then the system* (2.26)–(2.30) *has for every* $\alpha\in(0,1]$ *and for every* $u\in\mathcal{U}_{\rm ad}$ *a unique solution* $((\mu^\alpha,\mu_\Gamma^\alpha),(\rho^\alpha,\rho_\Gamma^\alpha))$ *satisfying* (2.22) *and* (2.23)*. Moreover, there are constants* $\rho_*(\alpha),\rho^*(\alpha)\in(-1,1)$ *and* $K_2^*>0$*, which depend only on the data of the state system, such that the following holds true:*

whenever $((\mu^\alpha,\mu_\Gamma^\alpha),(\rho^\alpha,\rho_\Gamma^\alpha))$ *is the solution to the system* (2.26)–(2.30) *associated with some* $\alpha\in(0,1]$ *and* $u\in\mathcal{U}_{\rm ad}$*, then we have*

$$\rho_*(\alpha)\,\le\rho^\alpha(x,t)\,\le\rho^*(\alpha)\quad\forall\,(x,t)\in\overline{Q}\,, \tag{2.35}$$

$$\|(\mu^\alpha,\mu_\Gamma^\alpha)\|_{L^\infty(0,T;\mathcal{W})}\,+\,\|(\rho^\alpha,\rho_\Gamma^\alpha)\|_{W^{1,\infty}(0,T;\mathcal{H})\cap H^1(0,T;\mathcal{V})\cap L^\infty(0,T;\mathcal{W})}$$
$$+\,\|(\varphi(a)h'(\rho^\alpha),\varphi(\alpha)h'(\rho_\Gamma^\alpha))\|_{L^\infty(0,T;\mathcal{H})}\,\le\,K_2^*\,. \tag{2.36}$$

Remark 2.4. Notice that the pointwise condition (2.35) is meaningful, since it follows from [27, Sect. 8, Cor. 4] and (2.23) that $\rho\in C^0(\overline{Q})$ (and thus, in particular, that $\rho_\Gamma\in C^0(\overline{\Sigma})$).

Remark 2.5. About (2.35), let us point out that, unfortunately, we are unable to show a uniform in $\alpha\in(0,1]$ separation property. In fact, it may well happen that, for $\alpha\searrow0$, we have $\rho_*(\alpha)\searrow-1$ and/or $\rho^*(\alpha)\nearrow+1$.

PROOF OF THEOREM 2.3: The existence of a unique solution with the regularity (2.22) and (2.23), which satisfies the separation property (2.35), is a direct consequence of [13, Thm. 2.8]. In order to establish the global bound (2.36), we now follow the line of a priori estimates carried out in [13], showing that the bounds derived there are in fact independent of $\alpha\in(0,1]$ in our special situation. In the following, we denote by C positive constants that may depend on the data of the system but neither on $u\in\mathcal{U}_{\rm ad}$ nor on $\alpha\in(0,1]$. For the sake of a simpler notation, we will also suppress the superscript α in the calculations, writing it only at the end of each estimation. We also assume that an arbitrary, but fixed, $u\in\mathcal{U}_{\rm ad}$ is given. Observe that then $\|u\|_{\mathcal{X}}\le R_0$, which will be used repeatedly without further reference.

FIRST ESTIMATE:

Let $t\in(0,T]$ be arbitrary and $0<s\le t$. We insert $(v,v_\Gamma)=(\mu,\mu_\Gamma)(s)$ in (2.31) and $(v,v_\Gamma)=(\partial_t\rho,\partial_t\rho_\Gamma)(s)$ in (2.32), add the resulting equations, and integrate over $[0,t]$. Adding the expression $\int_{Q_t}\rho\,\partial_t\rho\,+\,\int_{\Sigma_t}\rho_\Gamma\,\partial_t\rho_\Gamma$ to both sides, we obtain the identity

$$\int_{Q_t}|\nabla\mu|^2+\int_{\Sigma_t}|\nabla_\Gamma\mu_\Gamma|^2+\tau_\Omega\int_{Q_t}|\partial_t\rho|^2+\tau_\Gamma\int_{\Sigma_t}|\partial_t\rho_\Gamma|^2$$
$$+\,\frac12\,\|(\rho,\rho_\Gamma)(t)\|_{\mathcal{V}}^2+\int_\Omega\varphi(\alpha)h(\rho(t))+\int_\Gamma\varphi(\alpha)h(\rho_\Gamma(t))$$
$$=\,\frac12\,\|(\rho_0,\rho_{0|\Gamma}\|_{\mathcal{V}}^2+\int_\Omega\varphi(\alpha)h(\rho_0)+\int_\Gamma\varphi(\alpha)h(\rho_{0|\Gamma})+\int_{Q_t}\rho\,u\cdot\nabla\mu$$
$$+\,\int_{Q_t}(\rho-\pi(\rho))\,\partial_t\rho+\int_{\Sigma_t}(\rho_\Gamma-\pi_\Gamma(\rho_\Gamma))\partial_t\rho_\Gamma\,, \tag{2.37}$$

where, owing to the general assumptions, all of the terms on the left-hand side are nonnegative and the first three terms on the right-hand side are finite and uniformly bounded. Now, recalling the separation property (2.35) and assumption (A3), we conclude from Young's inequality that the last two integrals on the right-hand side are bounded by an expression of the form $C+\frac{\tau_\Omega}{2}\int_{Q_t}|\partial_t\rho|^2+\frac{\tau_\Gamma}{2}\int_{\Sigma_t}|\partial_t\rho_\Gamma|^2$. Moreover, owing to Young's inequality, we have that

$$\int_{Q_t}\rho\,u\cdot\nabla\mu\le\int_0^t\|\rho(s)\|_2\,\|u(s)\|_\infty\,\|\nabla\mu(s)\|_2\,ds\,\le\,\frac12\int_{Q_t}|\nabla\mu|^2\,+\,C\,. \tag{2.38}$$

We thus can infer from Gronwall's lemma the estimate

$$\|(\rho^\alpha,\rho_\Gamma^\alpha)\|^2_{H^1(0,T;\mathcal{H})\cap L^\infty(0,T;\mathcal{V})} + \int_Q |\nabla\mu^\alpha|^2 + \int_\Sigma |\nabla_\Gamma\mu_\Gamma^\alpha|^2 \\ + \|\varphi(\alpha)h(\rho^\alpha)\|_{L^\infty(0,T;L^1(\Omega))} + \|\varphi(\alpha)h(\rho_\Gamma^\alpha)\|_{L^\infty(0,T;L^1(\Gamma))} \le C \quad \forall\,\alpha\in(0,1]. \tag{2.39}$$

SECOND ESTIMATE:

Let $\widehat{m}(t) := \operatorname{mean}(\mu(t),\mu_\Gamma(t))$ for $t\in[0,T]$. Recalling (2.34), we note that $(v,v_\Gamma) := (\rho(t)-\widehat{r},\rho_\Gamma(t)-\widehat{r}) \in \mathcal{V}_0$ for all $t\in[0,T]$. Inserting this in (2.32), where we temporarily omit the argument t, we obtain the identity

$$\int_\Omega \varphi(\alpha)h'(\rho)(\rho-\widehat{r}) + \int_\Gamma \varphi(\alpha)h'(\rho_\Gamma)(\rho_\Gamma-\widehat{r}) \\ = -\tau_\Omega\int_\Omega \partial_t\rho(\rho-\widehat{r}) - \tau_\Gamma\int_\Gamma \partial_t\rho_\Gamma(\rho_\Gamma-\widehat{r}) - \int_\Omega|\nabla\rho|^2 - \int_\Gamma|\nabla_\Gamma\rho_\Gamma|^2 \\ - \int_\Omega \pi(\rho)(\rho-\widehat{r}) - \int_\Gamma \pi_\Gamma(\rho_\Gamma)(\rho_\Gamma-\widehat{r}) + \int_\Omega(\mu-\widehat{m})(\rho-\widehat{r}) \\ + \int_\Gamma(\mu_\Gamma-\widehat{m})(\rho_\Gamma-\widehat{r}). \tag{2.40}$$

At this point, we recall that $-1<\widehat{r}<1$. We thus may argue as in [19, p. 908] to conclude that there exist constants $\delta_0>0$ and $C_0>0$, which do not depend on $\alpha\in(0,1]$, such that

$$\varphi(\alpha)h'(r)(r-\widehat{r}) \ge \delta_0\,\varphi(\alpha)|h'(r)| - C_0 \quad \forall\,r\in(-1,1) \quad \forall\,\alpha\in(0,1].$$

Due to (2.35), the function $\rho-\widehat{r}$ is bounded on $\overline{Q}$; we thus can infer from (2.40), by just employing the Cauchy-Schwarz inequality, that

$$\delta_0\int_\Omega|\varphi(\alpha)h'(\rho)| + \delta_0\int_\Gamma|\varphi(\alpha)h'(\rho_\Gamma)| \\ \le C\big(1 + \|\partial_t\rho\|_H + \|\partial_t\rho_\Gamma\|_{H_\Gamma} + \|(\rho,\rho_\Gamma)\|^2_{\mathcal{V}} + \|(\mu-\widehat{m},\mu_\Gamma-\widehat{m})\|_{\mathcal{H}}\big) \\ \le C\,(1 + \|\partial_t\rho\|_H + \|\partial_t\rho_\Gamma\|_{H_\Gamma} + \|(\mu-\widehat{m},\mu_\Gamma-\widehat{m})\|_{\mathcal{H}}), \tag{2.41}$$

where the last inequality follows from (2.39). Now, we recall the definition (2.8) and the fact that $\|\cdot\|_{\mathcal{V}_0}$ is equivalent to the standard norm on $\mathcal{V}_0$. Therefore,

$$\|(\mu-\widehat{m},\mu_\Gamma-\widehat{m})\|_{\mathcal{H}} \le C\,\|(\mu-\widehat{m},\mu_\Gamma-\widehat{m})\|_{\mathcal{V}_0} = C\,\|(\nabla\mu,\nabla_\Gamma\mu_\Gamma)\|_{\mathcal{H}}.$$

Hence, combining this estimate with (2.39) and (2.41), we can conclude that

$$\|\varphi(\alpha)h'(\rho^\alpha)\|_{L^2(0,T;L^1(\Omega))} + \|\varphi(\alpha)h'(\rho_\Gamma^\alpha)\|_{L^2(0,T;L^1(\Omega))} \le C \quad \forall\,\alpha\in(0,1]. \tag{2.42}$$

At this point, we can insert $(v,v_\Gamma)=(1,1)$ in (2.32), which then yields that the function $t\mapsto \operatorname{mean}(\mu^\alpha(t),\mu_\Gamma^\alpha(t))$ is bounded in $L^2(0,T)$, uniformly in $\alpha\in(0,1]$. In view of (2.39), we have thus shown that

$$\|(\mu^\alpha,\mu_\Gamma^\alpha)\|_{L^2(0,T;\mathcal{V})} \le C \quad \forall\,\alpha\in(0,1]. \tag{2.43}$$

THIRD ESTIMATE:

Next, we take $(v, v_\Gamma) = (\varphi(\alpha)h'(\rho(s)), \varphi(\alpha)h'(\rho_\Gamma(s))) \in \mathcal{V}$ in (2.32), where $0 \le s \le t$ for some $t \in (0, T]$. Integrating over $[0, t]$, we obtain the identity

$$\begin{aligned}
&\tau_\Omega \int_\Omega \varphi(\alpha)h(\rho(t)) + \tau_\Gamma \int_\Gamma \varphi(\alpha)h(\rho_\Gamma(t)) + \int_{Q_t} \varphi(\alpha)h''(\rho)|\nabla\rho|^2 + \int_{\Sigma_t} \varphi(\alpha)h''(\rho_\Gamma)|\nabla_\Gamma\rho_\Gamma|^2 \\
&+ \int_{Q_t} |\varphi(\alpha)h'(\rho)|^2 + \int_{\Sigma_t} |\varphi(\alpha)h'(\rho_\Gamma)|^2 \\
&= \tau_\Omega \int_\Omega \varphi(\alpha)h(\rho_0) + \tau_\Gamma \int_\Gamma \varphi(\alpha)h(\rho_{0|\Gamma}) + \int_{Q_t} (\mu - \pi(\rho))\varphi(\alpha)h'(\rho) \\
&\quad + \int_{\Sigma_t} (\mu_\Gamma - \pi_\Gamma(\rho_\Gamma))\varphi(\alpha)h'(\rho_\Gamma)\,, \qquad (2.44)
\end{aligned}$$

where (note that $h'' \ge 0$) all of the terms on the left-hand side are nonnegative and the first two summands on the right-hand side are bounded uniformly in $\alpha \in (0, 1]$. Hence, in view of (2.43), a simple application of Young's inequality leads to the conclusion that

$$\|(\varphi(\alpha)h'(\rho^\alpha), \varphi(\alpha)h'(\rho_\Gamma^\alpha))\|_{L^2(0,T;\mathcal{H})} \le C \quad \forall\, \alpha \in (0, 1]. \qquad (2.45)$$

Direct comparison in (2.27) then shows that also

$$\|\Delta\rho^\alpha\|_{L^2(0,T;H)} \le C \quad \forall\, \alpha \in (0, 1]. \qquad (2.46)$$

Let us exploit (2.46). Indeed, invoking (2.39), (1.22) and (1.23), we conclude that

$$\|\rho^\alpha\|_{L^2(0,T;H^{3/2}(\Omega))} + \|\partial_\nu\rho^\alpha\|_{L^2(0,T;H_\Gamma)} \le C \quad \forall\, \alpha \in (0, 1]. \qquad (2.47)$$

Then comparison in (2.29), using (2.39), (2.43) and (2.45), implies that

$$\|\Delta_\Gamma\rho_\Gamma^\alpha\|_{L^2(0,T;H_\Gamma)} \le C \quad \forall\, \alpha \in (0, 1], \qquad (2.48)$$

and we conclude from (2.39), (1.24) and (1.25) that

$$\|(\rho^\alpha, \rho_\Gamma^\alpha)\|_{L^2(0,T;\mathcal{W})} \le C \quad \forall\, \alpha \in (0, 1]. \qquad (2.49)$$

Next, since $u \in \mathcal{U}_{\rm ad}$, we readily infer from (2.26) and (2.39) that

$$\|\Delta\mu^\alpha\|_{L^2(0,T;H)} \le C \quad \forall\, \alpha \in (0, 1], \qquad (2.50)$$

whence, in view of (2.43), (1.22) and (1.23),

$$\|\mu^\alpha\|_{L^2(0,T;H^{3/2}(\Omega))} + \|\partial_\nu\mu^\alpha\|_{L^2(0,T;H_\Gamma)} \le C \quad \forall\alpha \in (0, 1]. \qquad (2.51)$$

Hence, by virtue of (2.28) and (2.39), we have that

$$\|\Delta_\Gamma\mu_\Gamma^\alpha\|_{L^2(0,T;H_\Gamma)} \le C \quad \forall\, \alpha \in (0, 1], \qquad (2.52)$$

and we can argue as above to arrive at the estimate

$$\|(\mu^\alpha, \mu_\Gamma^\alpha)\|_{L^2(0,T;\mathcal{W})} \le C \quad \forall\, \alpha \in (0, 1]. \qquad (2.53)$$

FOURTH ESTIMATE:

We now argue formally, noting that the following arguments can be made rigorous by, e.g., using finite differences in time. At first, we note that $\operatorname{mean} \partial_t(\rho, \rho_\Gamma) = 0$ a.e. in $(0,T)$, by (2.34). Hence, $(\xi, \xi_\Gamma)(t) := \mathcal{N}(\partial_t(\rho,\rho_\Gamma)(t)) \in \mathcal{V}_0$ is well defined for a.a. $t \in (0,T)$. Now, we differentiate both (2.31) and (2.32) (formally) with respect to time, test the resulting identities by (ξ, ξ_Γ) and $\partial_t(\rho,\rho_\Gamma)$, respectively, and add the results. Now observe that, by (2.9) and (2.10),

$$\int_{Q_t} \nabla \partial_t \mu \cdot \nabla \xi + \int_{\Sigma_t} \nabla_\Gamma \partial_t \mu_\Gamma \cdot \nabla_\Gamma \xi_\Gamma = \int_{Q_t} \partial_t \mu \, \partial_t \rho + \int_{\Sigma_t} \partial_t \mu_\Gamma \, \partial_t \rho_\Gamma \, .$$

Hence, recalling (2.14), and integrating the expressions containing u (formally) by parts, we arrive at the identity

$$\begin{aligned} &\frac{1}{2} \|\partial_t(\rho,\rho_\Gamma)(t)\|_*^2 + \frac{\tau_\Omega}{2} \int_\Omega |\partial_t \rho(t)|^2 + \frac{\tau_\Gamma}{2} \int_\Gamma |\partial_t \rho_\Gamma(t)|^2 + \int_{Q_t} |\nabla \partial_t \rho|^2 + \int_{\Sigma_t} |\nabla_\Gamma \partial_t \rho_\Gamma|^2 \\ &+ \int_{Q_t} \varphi(\alpha) h''(\rho) |\partial_t \rho|^2 + \int_{\Sigma_t} \varphi(\alpha) h''(\rho_\Gamma) |\partial_t \rho_\Gamma|^2 \\ &= I_0 + \int_{Q_t} \nabla \partial_t \rho \cdot u \xi + \int_{Q_t} \nabla \rho \cdot \partial_t u \, \xi - \int_{Q_t} \pi'(\rho) |\partial_t \rho|^2 - \int_{\Sigma_t} \pi_\Gamma'(\rho_\Gamma) |\partial_t \rho_\Gamma|^2 \, , \end{aligned} \tag{2.54}$$

where

$$I_0 := \frac{1}{2} \|\partial_t(\rho,\rho_\Gamma)(0)\|_*^2 + \frac{\tau_\Omega}{2} \int_\Omega |\partial_t \rho(0)|^2 + \frac{\tau_\Gamma}{2} \int_\Gamma |\partial_t \rho_\Gamma(0)|^2 \, . \tag{2.55}$$

Noting that $\varphi(\alpha) h'' \geq 0$, we may omit the two nonnegative summands in the second line of (2.54), and thus obtain an inequality which has exactly the same form as the inequality [13, Eq. (7.1)]. We thus may repeat the estimates carried out in [13] in order to conclude that (cf., [13, Eq. (7.3)])

$$\|(\rho^\alpha, \rho_\Gamma^\alpha)\|_{W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{V})} \leq C \quad \forall \, \alpha \in (0,1] \tag{2.56}$$

and actually realize that the estimate is uniform with respect to α.

FIFTH ESTIMATE:

Recalling that $\widehat{m}(t) := \operatorname{mean}(\mu(t), \mu_\Gamma(t))$ for $t \in [0,T]$, we test (2.31) by the $\mathcal{V}_0$–valued function $(\mu, \mu_\Gamma) - \widehat{m}(1,1)$. Using the fact that the norm (2.8) is equivalent to the standard norm on $\mathcal{V}_0$, we obtain, for almost every $t \in (0,T)$,

$$\begin{aligned} &\int_\Omega |\nabla \mu|^2 + \int_\Gamma |\nabla_\Gamma \mu_\Gamma|^2 = - \int_\Omega \partial_t \rho (\mu - \widehat{m}) - \int_\Gamma \partial_t \rho_\Gamma (\mu_\Gamma - \widehat{m}) - \int_\Omega \rho u \cdot \nabla \mu \\ &\leq C \, \|\partial_t(\rho,\rho_\Gamma)\|_{\mathcal{H}} \, \|(\mu,\mu_\Gamma) - \widehat{m}(1,1)\|_{\mathcal{V}_0} + \|u\|_{L^\infty(Q)} \, \|\rho\|_2 \, \|\nabla \mu\|_2 \\ &\leq C \left(\|\nabla \mu\|_2 + \|\nabla_\Gamma \mu_\Gamma\|_2 \right) . \end{aligned} \tag{2.57}$$

Consequently, we deduce that

$$\|\nabla \mu^\alpha\|_{L^\infty(0,T;H)} + \|\nabla_\Gamma \mu_\Gamma^\alpha\|_{L^\infty(0,T;H)} \leq C \quad \forall \, \alpha \in (0,1], \tag{2.58}$$

$$\|(\mu^\alpha - \widehat{m}, \mu_\Gamma^\alpha - \widehat{m})\|_{L^\infty(0,T;\mathcal{V})} \leq C \quad \forall \, \alpha \in (0,1]. \tag{2.59}$$

SIXTH ESTIMATE:

At first, we directly obtain from (2.41), (2.56), and (2.59), that

$$\|\varphi(\alpha)h'(\rho^\alpha)\|_{L^\infty(0,T;L^1(\Omega))} + \|\varphi(\alpha)h'(\rho_\Gamma^\alpha)\|_{L^\infty(0,T;L^1(\Omega))} \le C \quad \forall \alpha \in (0,1]. \tag{2.60}$$

Therefore, if we take $(v, v_\Gamma) = (1,1)/(|\Omega| + |\Gamma|)$ in (2.32), for almost every $t \in (0,T)$ we infer that

$$|\mathrm{mean}\,(\mu,\mu_\Gamma)(t)| \le C\,\|\partial_t(\rho,\rho_\Gamma)\|_{L^\infty(0,T;\mathcal{H})} + C\,\|\varphi(\alpha)h'(\rho) + \pi(\rho)\|_{L^\infty(0,T;L^1(\Omega))} \\ + C\,\|\varphi(\alpha)h'(\rho_\Gamma) + \pi_\Gamma(\rho_\Gamma)\|_{L^\infty(0,T;L^1(\Omega))} \le C\,. \tag{2.61}$$

By virtue of (2.59), this shows that

$$\|(\mu^\alpha, \mu_\Gamma^\alpha)\|_{L^\infty(0,T;\mathcal{V})} \le C \quad \forall\,\alpha \in (0,1]. \tag{2.62}$$

At this point, we observe that (2.26), (2.39), (2.56), and the fact that $u \in \mathcal{U}_{\rm ad}$, imply that

$$\|\Delta\mu^\alpha\|_{L^\infty(0,T;H)} \le \|\partial_t\rho^\alpha\|_{L^\infty(0,T;H)} + \|u\cdot\nabla\rho^\alpha\|_{L^\infty(0,T;H)} \le C \quad \forall\,\alpha \in (0,1]. \tag{2.63}$$

In view of (2.62), we are therefore in the same situation as in the third estimation above after the proof of (2.50) (only that we have L^∞ with respect to time in place of L^2). We thus may argue as in the estimates (2.51)–(2.53) to conclude that

$$\|(\mu^\alpha, \mu_\Gamma^\alpha)\|_{L^\infty(0,T;\mathcal{W})} \le C \quad \forall\,\alpha \in (0,1]. \tag{2.64}$$

SEVENTH ESTIMATE:

Finally, we insert $(v, v_\Gamma) = (\varphi(\alpha)h'(\rho), \varphi(\alpha)h'(\rho_\Gamma))$ in (2.32). Employing the estimates shown previously, we readily obtain that, almost everywhere on $(0,T)$,

$$\int_\Omega |\varphi(\alpha)h'(\rho)|^2 + \int_\Gamma |\varphi(\alpha)h'(\rho_\Gamma)|^2 + \int_\Omega \varphi(\alpha)h''(\rho)|\nabla\rho|^2 + \int_\Gamma \varphi(\alpha)h''(\rho_\Gamma)|\nabla_\Gamma\rho_\Gamma|^2 \\ = \int_\Omega \varphi(\alpha)h'(\rho)\,(-\tau_\Omega\partial_t\rho + \mu - \pi(\rho)) + \int_\Gamma \varphi(\alpha)h'(\rho_\Gamma)\,(-\tau_\Gamma\partial_t\rho_\Gamma + \mu_\Gamma - \pi_\Gamma(\rho_\Gamma)) \\ \le C + \frac{1}{2}\int_\Omega |\varphi(\alpha)h'(\rho)|^2 + \frac{1}{2}\int_\Gamma |\varphi(\alpha)h'(\rho_\Gamma)|^2\,. \tag{2.65}$$

Consequently, we have that

$$\|(\varphi(\alpha)h'(\rho^\alpha), \varphi(\alpha)h'(\rho_\Gamma^\alpha))\|_{L^\infty(0,T;\mathcal{H})} \le C \quad \forall\,\alpha \in (0,1]. \tag{2.66}$$

But then, by virtue of (2.27) and the previous estimates, it is clear that

$$\|\Delta\rho^\alpha\|_{L^\infty(0,T;H)} \le C \quad \forall\,\alpha \in (0,1],$$

and, arguing as in the third estimate, we infer that

$$\|(\rho^\alpha, \rho_\Gamma^\alpha)\|_{L^\infty(0,T;\mathcal{W})} \le C \quad \forall \alpha \in (0,1], \tag{2.67}$$

which concludes the proof of the assertion. □

Remark 2.6. By virtue of the well-posedness result given by Theorem 2.3, the control-to-state operator $\mathcal{S}_\alpha : u \mapsto ((\mu^\alpha, \mu_\Gamma^\alpha), (\rho^\alpha, \rho_\Gamma^\alpha))$ is well defined as a mapping between $\mathcal{U}_{\rm ad} \subset \mathcal{X}$ and the space defined by the regularity stated in (2.22), (2.23). In particular, this also holds true for its second component $\mathcal{S}_\alpha^2 : u \mapsto (\rho^\alpha, \rho_\Gamma^\alpha)$.

3 Existence and approximation of optimal controls

In this section, we aim to approximate optimal pairs of ($\mathcal{P}_0$). To this end, we consider for $\alpha \in (0,1]$ the optimal control problem

($\mathcal{P}_\alpha$) Minimize the cost functional $\mathcal{J}((\rho^\alpha, \rho_\Gamma^\alpha), u)$ for $u \in \mathcal{U}_{\rm ad}$, subject to the state system (2.26)–(2.30).

Assuming generally that (A1)–(A5) are fulfilled, we obtain from [14, Thm. 4.1] that this optimal control problem has an optimal pair $(((\mu^\alpha, \mu_\Gamma^\alpha), (\rho^\alpha, \rho_\Gamma^\alpha)), u^\alpha)$, for every $\alpha \in (0,1]$. Our first aim in this section is to prove the following approximation result:

Theorem 3.1. *Suppose that the assumptions* (A1)–(A5)*,* (1.14) *and* (1.15) *are satisfied, and let sequences* $\{\alpha_n\} \subset (0,1]$ *and* $\{u^{\alpha_n}\} \subset \mathcal{U}_{\rm ad}$ *be given such that* $\alpha_n \searrow 0$ *and* $u^{\alpha_n} \to u$ *weakly-star in* $\mathcal{X}$ *for some* $u \in \mathcal{U}_{\rm ad}$*. Then there is a subsequence* $\{\alpha_{n_k}\}$ *of* $\{\alpha_n\}$ *such that for* $k \to \infty$ *it holds, with* $((\mu^{\alpha_n}, \mu_\Gamma^{\alpha_n}), (\rho^{\alpha_n}, \rho_\Gamma^{\alpha_n})) := \mathcal{S}_{\alpha_n}(u^{\alpha_n})$*,* $n \in \mathbb{N}$*,*

$$(\mu^{\alpha_{n_k}}, \mu_\Gamma^{\alpha_{n_k}}) \to (\mu, \mu_\Gamma) \quad \text{weakly-star in } L^\infty(0,T;\mathcal{W}), \tag{3.1}$$

$$(\rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}) \to (\rho, \rho_\Gamma) \quad \text{weakly-star in } W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{W}), \tag{3.2}$$

$$(\varphi(\alpha_{n_k})\, h'(\rho^{\alpha_{n_k}}), \varphi(\alpha_{n_k})\, h'(\rho_\Gamma^{\alpha_{n_k}})) \to (\xi, \xi_\Gamma) \quad \text{weakly-star in } L^\infty(0,T;\mathcal{H}), \tag{3.3}$$

where $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma), (\xi, \xi_\Gamma))$ *is a solution to the state system* (1.2)–(1.8) *associated with* u*. Moreover,* (3.2) *holds true for the entire sequence* $\{\alpha_n\}$*. Finally, with* $\mathcal{S}_0^2(u) := (\rho, \rho_\Gamma)$ *it holds that*

$$\mathcal{J}(\mathcal{S}_0^2(u), u) \le \liminf_{n\to\infty} \mathcal{J}(\mathcal{S}_{\alpha_n}^2(u^{\alpha_n}), u^{\alpha_n}), \tag{3.4}$$

$$\mathcal{J}(\mathcal{S}_0^2(v), v) = \lim_{n\to\infty} \mathcal{J}(\mathcal{S}_{\alpha_n}^2(v), v) \quad \forall\, v \in \mathcal{U}_{\rm ad}. \tag{3.5}$$

PROOF: Let $\{\alpha_n\} \subset (0,1]$ be any sequence such that $\alpha_n \searrow 0$ as $n \to \infty$, and suppose that $u^{\alpha_n} \to u$ weakly-star in $\mathcal{X}$ for some $u \in \mathcal{U}_{\rm ad}$. By virtue of Theorem 2.3, there are a subsequence of $\{\alpha_n\}$, which is again indexed by n, and three pairs $(\mu, \mu_\Gamma), (\rho, \rho_\Gamma), (\xi, \xi_\Gamma)$ such that the convergence results (3.1)–(3.3) hold true. Moreover, from standard compact embedding results (cf. [27, Sect. 8, Cor. 4]) we can infer that

$$\rho^{\alpha_n} \to \rho \quad \text{strongly in } L^2(0,T;V) \cap C^0(\overline{Q}), \tag{3.6}$$

which also yields that

$$\rho_\Gamma^{\alpha_n} \to \rho_\Gamma \quad \text{strongly in } C^0(\overline{\Sigma}). \tag{3.7}$$

In particular, $(\rho(0), \rho_\Gamma(0)) = (\rho_0, \rho_{0|\Gamma})$ and $\rho_\Gamma = \rho_{|\Sigma}$. In addition, we obviously have that

$$\pi(\rho^{\alpha_n}) \to \pi(\rho) \quad \text{strongly in } C^0(\overline{Q}), \tag{3.8}$$

$$\pi_\Gamma(\rho_\Gamma^{\alpha_n}) \to \pi_\Gamma(\rho_\Gamma) \quad \text{strongly in } C^0(\overline{\Sigma}). \tag{3.9}$$

Moreover, it is easily verified that, at least weakly in $L^1(Q)$,

$$\nabla\rho^{\alpha_n} \cdot u^{\alpha_n} \to \nabla\rho \cdot u\,. \tag{3.10}$$

Combining the above convergence results, we may pass to the limit as $n \to \infty$ in the equations (2.26)–(2.30) (written for $\alpha = \alpha_n$ and $u = u^{\alpha_n}$) to find that $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma), (\xi, \xi_\Gamma))$ and u satisfy the equations (1.2), (1.3), (1.5), (1.6), and (1.8). Thus, in order to show that $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma), (\xi, \xi_\Gamma))$ is in fact a solution to the problem (1.2)–(1.8) corresponding to u, it remains to show that $\xi \in \partial I_{[-1,1]}(\rho)$ a. e. in Q and $\xi_\Gamma \in \partial I_{[-1,1]}(\rho_\Gamma)$ a. e. in Σ. To this end, recall that h is convex and bounded in $[-1, 1]$ and that both h and φ are nonnegative. We thus have, for every $n \in \mathbb{N}$,

$$0 \le \int_Q \varphi(\alpha_n)\, h(\rho^{\alpha_n}) \le \int_Q \varphi(\alpha_n)\, h(z) + \int_Q \varphi(\alpha_n)\, h'(\rho^{\alpha_n})\,(\rho^{\alpha_n} - z)$$

$$\text{for all } z \in \mathcal{K} := \{v \in L^2(Q) : |v| \le 1 \text{ a.e. in } Q\}\,. \tag{3.11}$$

Thanks to (1.15), the first two integrals tend to zero as $n \to \infty$. Hence, invoking (3.3) and (3.6), the passage to the limit as $n \to \infty$ yields

$$\int_Q \xi\,(\rho - z) \ge 0 \quad \forall z \in \mathcal{K}. \tag{3.12}$$

Inequality (3.12) entails that ξ is an element of the subdifferential of the extension $\mathcal{I}$ of $I_{[-1,1]}$ to $L^2(Q)$, which means that $\xi \in \partial\,\mathcal{I}(\rho)$ or, equivalently (cf. [2, Ex. 2.3.3., p. 25]), that $\xi \in \partial I_{[-1,1]}(\rho)$ a. e. in Q. Similarly, we can prove that $\xi_\Gamma \in \partial I_{[-1,1]}(\rho_\Gamma)$ a. e. in Σ.

We have thus shown that, for a suitable subsequence of $\{\alpha_n\}$, we have the convergence properties (3.1)–(3.3), where $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma), (\xi, \xi_\Gamma))$ is a solution to the state system (1.2)–(1.8). But, according to Theorem 2.2, the component (ρ, ρ_Γ) is the same for any such solution. This entails that the convergence properties (3.2), (3.6)–(3.9) are in fact valid for the entire sequence $\{\alpha_n\}$. This finishes the proof of the first claims of the theorem.

It remains to show the validity of (3.4) and (3.5). In view of (3.2), the inequality (3.4) is an immediate consequence of the weak and weak-star sequential semicontinuity properties of the cost functional $\mathcal{J}$. To establish the identity (3.5), let $v \in \mathcal{U}_{\rm ad}$ be arbitrary and put $(\rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}) = \mathcal{S}^2_{\alpha_n}(v)$, for $n \in \mathbb{N}$. Taking Theorem 2.3 into account, and arguing as in the first part of this proof, we can conclude that $\{\mathcal{S}^2_{\alpha_n}(v)\}$ converges to $(\rho, \rho_\Gamma) = \mathcal{S}^2_0(v)$ in the sense of (3.2). In particular, we have (recall (3.6) and (3.7))

$$\mathcal{S}^2_{\alpha_n}(v) \to \mathcal{S}^2_0(v) \quad \text{strongly in } C^0(\overline{Q}) \times C^0(\overline{\Sigma}).$$

As the cost functional $\mathcal{J}$ is obviously continuous in the variables (ρ, ρ_Γ) with respect to the strong topology of $C^0(\overline{Q}) \times C^0(\overline{\Sigma})$, we may thus infer that (3.5) is valid. ☐

Corollary 3.2. *Under the assumptions of Theorem 3.1, the optimal control problem* $(\mathcal{P}_0)$ *has a least one solution.*

PROOF: Pick an arbitrary sequence $\{\alpha_n\}$ such that $\alpha_n \searrow 0$ as $n \to \infty$. Then, by virtue of [14, Thm. 4.1], the optimal control problem $(\mathcal{P}_{\alpha_n})$ has for every $n \in \mathbb{N}$ an optimal pair $(((\rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}), (\mu^{\alpha_n}, \mu_\Gamma^{\alpha_n})), u^{\alpha_n})$, where $u^{\alpha_n} \in \mathcal{U}_{\rm ad}$ and $(\rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}) = \mathcal{S}^2_{\alpha_n}(u^{\alpha_n})$. Since $\mathcal{U}_{\rm ad}$ is a bounded subset of $\mathcal{X}$, we may without loss of generality assume that $u^{\alpha_n} \to u$ weakly-star in $\mathcal{X}$ for some $u \in \mathcal{U}_{\rm ad}$. Then, for some solution $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma), (\xi, \xi_\Gamma))$ to the state system (1.2)–(1.8) associated with u, we conclude from Theorem 3.1 the convergence properties (3.2), (3.6), (3.7), and (3.5). Invoking the optimality of $(((\mu^{\alpha_n}, \mu_\Gamma^{\alpha_n}), (\rho^{\alpha_n}, \rho_\Gamma^{\alpha_n})), u^{\alpha_n})$ for $(\mathcal{P}_{\alpha_n})$, we then find, for every $v \in \mathcal{U}_{\rm ad}$, that

$$\begin{aligned} \mathcal{J}((\rho, \rho_\Gamma), u) &= \mathcal{J}(\mathcal{S}^2_0(u), u) \le \liminf_{n\to\infty} \mathcal{J}(\mathcal{S}^2_{\alpha_n}(u^{\alpha_n}), u^{\alpha_n}) \\ &\le \liminf_{n\to\infty} \mathcal{J}(\mathcal{S}^2_{\alpha_n}(v), v) = \lim_{n\to\infty} \mathcal{J}(\mathcal{S}^2_{\alpha_n}(v), v) = \mathcal{J}(\mathcal{S}^2_0(v), v), \end{aligned} \tag{3.13}$$

which yields that u is an optimal control for ($\mathcal{P}_0$) with the associate state $((\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\xi,\xi_\Gamma))$. The assertion is thus proved. $\square$

Corollary 3.2 does not yield any information on whether every solution to the optimal control problem $(\mathcal{P}_0)$ can be approximated by a sequence of solutions to the problems $(\mathcal{P}_\alpha)$. As already announced in the Introduction, we are not able to prove such a general 'global' result. Instead, we can only give a 'local' answer for every individual optimizer of $(\mathcal{P}_0)$. For this purpose, we employ a trick due to Barbu [1]. To this end, let $\bar u \in \mathcal{U}_{\rm ad}$ be an arbitrary optimal control for $(\mathcal{P}_0)$, and let $((\bar\mu,\bar\mu_\Gamma),(\bar\rho,\bar\rho_\Gamma),(\bar\xi,\bar\xi_\Gamma))$ be any associated solution to the state system (1.2)–(1.8) in the sense of Theorem 2.2. In particular, $(\bar\rho,\bar\rho_\Gamma)=\mathcal{S}_0^2(\bar u)$. We associate with this optimal control the *adapted cost functional*

$$\widetilde{\mathcal{J}}((\rho,\rho_\Gamma),u) := \mathcal{J}((\rho,\rho_\Gamma),u) \,+\, \frac{1}{2}\,\|u-\bar u\|^2_{(L^2(Q))^3} \tag{3.14}$$

and a corresponding *adapted optimal control problem*,

($\widetilde{\mathcal{P}}_\alpha$) Minimize $\widetilde{\mathcal{J}}((\rho,\rho_\Gamma),u)$ for $u\in\mathcal{U}_{\rm ad}$, subject to the condition that (2.26)–(2.30) be satisfied.

With a standard direct argument that needs no repetition here, we can show the following result.

Lemma 3.3. *Suppose that the assumptions* (A1)–(A5), (1.14) *and* (1.15) *are satisfied, and let* $\alpha\in(0,1]$. *Then the optimal control problem* $(\widetilde{\mathcal{P}}_\alpha)$ *admits a solution.*

We are now in the position to give a partial answer to the question raised above. We have the following result.

Theorem 3.4. *Let the assumptions* (A1)–(A5), (1.14) *and* (1.15) *be fulfilled, suppose that* $\bar u\in\mathcal{U}_{\rm ad}$ *is an arbitrary optimal control of* $(\mathcal{P}_0)$ *with associated state* $((\bar\mu,\bar\mu_\Gamma),(\bar\rho,\bar\rho_\Gamma),(\bar\xi,\bar\xi_\Gamma))$, *and let* $\{\alpha_n\}\subset(0,1]$ *be any sequence such that* $\alpha_n\searrow 0$ *as* $n\to\infty$. *Then there exist a subsequence* $\{\alpha_{n_k}\}$ *of* $\{\alpha_n\}$, *and, for every* $k\in\mathbb{N}$, *an optimal control* $u^{\alpha_{n_k}}\in\mathcal{U}_{\rm ad}$ *of the adapted problem* $(\widetilde{\mathcal{P}}_{\alpha_{n_k}})$ *with associated state* $((\mu^{\alpha_{n_k}},\mu_\Gamma^{\alpha_{n_k}}),(\rho^{\alpha_{n_k}},\rho_\Gamma^{\alpha_{n_k}}))$ *such that, as* $k\to\infty$,

$$u^{\alpha_{n_k}}\to\bar u \quad \textit{strongly in } (L^2(Q))^3, \tag{3.15}$$

and such that the properties (3.1)–(3.3) *are satisfied, where* $(\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\xi,\xi_\Gamma)$ *are replaced by* $(\bar\mu,\bar\mu_\Gamma),(\bar\rho,\bar\rho_\Gamma),(\bar\xi,\bar\xi_\Gamma)$. *Moreover, we have*

$$\lim_{k\to\infty}\widetilde{\mathcal{J}}((\rho^{\alpha_{n_k}},\rho_\Gamma^{\alpha_{n_k}}),u^{\alpha_{n_k}}) \,=\, \mathcal{J}((\bar\rho,\bar\rho_\Gamma),\bar u)\,. \tag{3.16}$$

PROOF: Let $\alpha_n\searrow 0$ as $n\to\infty$. For any $n\in\mathbb{N}$, we pick an optimal control $u^{\alpha_n}\in\mathcal{U}_{\rm ad}$ for the adapted problem $(\widetilde{\mathcal{P}}_{\alpha_n})$ and denote by $((\mu^{\alpha_n},\mu_\Gamma^{\alpha_n}),(\rho^{\alpha_n},\rho_\Gamma^{\alpha_n}))$ the associated solution to the problem (2.26)–(2.30) for $\alpha=\alpha_n$ and $u=u^{\alpha_n}$. By the boundedness of $\mathcal{U}_{\rm ad}$ in $\mathcal{X}$, there is some subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that

$$u^{\alpha_{n_k}}\to u \quad \text{weakly-star in } \mathcal{X} \quad \text{as } k\to\infty, \tag{3.17}$$

with some $u\in\mathcal{U}_{\rm ad}$. Thanks to Theorem 3.1, the convergence properties (3.1)–(3.3) hold true, where $((\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\xi,\xi_\Gamma))$ is some solution to the state system (1.2)–(1.8). In particular, $(((\mu,\mu_\Gamma),(\rho,\rho_\Gamma),(\xi,\xi_\Gamma)),u)$ is admissible for ($\mathcal{P}_0$).

We now aim to prove that $u = \bar{u}$. Once this is shown, then the uniqueness result of Theorem 2.2 yields that also $(\rho, \rho_\Gamma) = (\bar{\rho}, \bar{\rho}_\Gamma)$, which implies that the properties (3.1)–(3.3) are satisfied, where $(\mu, \mu_\Gamma), (\rho, \rho_\Gamma), (\xi, \xi_\Gamma)$ are replaced by $(\bar{\mu}, \bar{\mu}_\Gamma), (\bar{\rho}, \bar{\rho}_\Gamma), (\bar{\xi}, \bar{\xi}_\Gamma)$.

Now observe that, owing to the weak sequential lower semicontinuity of $\widetilde{\mathcal{J}}$, and in view of the optimality property of $((\bar{\mu}, \bar{\mu}_\Gamma), (\bar{\rho}, \bar{\rho}_\Gamma), (\bar{\xi}, \bar{\xi}_\Gamma), \bar{u})$ for problem $(\mathcal{P}_0)$,

$$\begin{aligned}&\liminf_{k\to\infty} \widetilde{\mathcal{J}}((\rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}) \geq \mathcal{J}((\rho, \rho_\Gamma), u) + \frac{1}{2}\|u - \bar{u}\|^2_{(L^2(Q))^3}\\ &\geq \mathcal{J}((\bar{\rho}, \bar{\rho}_\Gamma), \bar{u}) + \frac{1}{2}\|u - \bar{u}\|^2_{(L^2(Q))^3} . \end{aligned} \tag{3.18}$$

On the other hand, the optimality property of $(((\mu^{\alpha_{n_k}}, \mu_\Gamma^{\alpha_{n_k}}), (\rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}})), u^{\alpha_{n_k}})$ for problem $(\widetilde{\mathcal{P}}_{\alpha_{n_k}})$ yields that for any $k \in \mathbb{N}$ we have

$$\widetilde{\mathcal{J}}((\rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u^{\alpha_{n_k}}) = \widetilde{\mathcal{J}}(\mathcal{S}^2_{\alpha_{n_k}}(u^{\alpha_{n_k}}), u^{\alpha_{n_k}}) \leq \widetilde{\mathcal{J}}(\mathcal{S}^2_{\alpha_{n_k}}(\bar{u}), \bar{u}), \tag{3.19}$$

whence, taking the limit superior as $k \to \infty$ on both sides and invoking (3.5) in Theorem 3.1,

$$\begin{aligned}&\limsup_{k\to\infty} \widetilde{\mathcal{J}}((\rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u^{\alpha_{n_k}})\\ &\leq \widetilde{\mathcal{J}}(\mathcal{S}^2_0(\bar{u}), \bar{u}) = \widetilde{\mathcal{J}}((\bar{\rho}, \bar{\rho}_\Gamma), \bar{u}) = \mathcal{J}((\bar{\rho}, \bar{\rho}_\Gamma), \bar{u}) . \end{aligned} \tag{3.20}$$

Combining (3.18) with (3.20), we have thus shown that $\frac{1}{2}\|u - \bar{u}\|^2_{(L^2(Q))^3} = 0$, so that $u = \bar{u}$ and thus also $(\rho, \rho_\Gamma) = (\bar{\rho}, \bar{\rho}_\Gamma)$. Moreover, (3.18) and (3.20) also imply that

$$\begin{aligned}&\mathcal{J}((\bar{\rho}, \bar{\rho}_\Gamma), \bar{u}) = \widetilde{\mathcal{J}}((\bar{\rho}, \bar{\rho}_\Gamma), \bar{u}) = \liminf_{k\to\infty} \widetilde{\mathcal{J}}((\rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u^{\alpha_{n_k}})\\ &= \limsup_{k\to\infty} \widetilde{\mathcal{J}}((\rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u^{\alpha_{n_k}}) = \lim_{k\to\infty} \widetilde{\mathcal{J}}((\rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u^{\alpha_{n_k}}), \end{aligned} \tag{3.21}$$

which proves (3.16) and, at the same time, also (3.15). This concludes the proof of the assertion. ☐

4 The optimality system

In this section, we aim to establish first-order necessary optimality conditions for the optimal control problem $(\mathcal{P}_0)$. This will be achieved by a passage to the limit as $\alpha \searrow 0$ in the first-order necessary optimality conditions for the adapted optimal control problems $(\widetilde{\mathcal{P}}_\alpha)$ that can by derived as in [14] with only minor and obvious changes. This procedure will yield certain generalized first-order necessary optimality conditions in the limit. In this entire section, we generally assume that h is given by (1.14) and that (1.15) and the assumptions (A1)–(A5) are satisfied. In addition, we assume that the following condition is fulfilled:

(A6) $\quad \tau_\Omega = \tau_\Gamma =: \tau > 0.$

We also assume that a fixed optimal control $\bar{u} \in \mathcal{U}_{\rm ad}$ for $(\mathcal{P}_0)$ is given, along with a corresponding solution $((\bar{\mu}, \bar{\mu}_\Gamma), (\bar{\rho}, \bar{\rho}_\Gamma), (\bar{\xi}, \bar{\xi}_\Gamma))$ to the state system (1.2)–(1.8) in the sense of Theorem 2.2. That is, we have $(\bar{\rho}, \bar{\rho}_\Gamma) = \mathcal{S}^2_0(\bar{u})$, as well as $\bar{\xi} \in \partial I_{[-1,1]}(\bar{\rho})$ a. e. in Q and $\bar{\xi}_\Gamma \in \partial I_{[-1,1]}(\bar{\rho}_\Gamma)$ a. e. on Σ.

We begin our analysis by formulating the adjoint state system for the adapted control problem $(\widetilde{\mathcal{P}}_\alpha)$ corresponding to $\bar{u}$. To this end, let us assume that, for some $\alpha \in (0,1]$, $u^\alpha \in \mathcal{U}_{\rm ad}$ is an arbitrary optimal control for $(\widetilde{\mathcal{P}}_\alpha)$ and that $((\mu^\alpha, \mu_\Gamma^\alpha), (\rho^\alpha, \rho_\Gamma^\alpha))$ is the (unique) solution to the associated state system (2.26)–(2.30). In particular, $((\mu^\alpha, \mu_\Gamma^\alpha), (\rho^\alpha, \rho_\Gamma^\alpha)) = \mathcal{S}_\alpha(u^\alpha)$, the solution enjoys the regularity properties (2.22) and (2.23), and it satisfies the global bounds (2.36) and the separation property (2.35). The associated adjoint system has the following variational form (cf., [14, Eqs. (4.7)–(4.9)]):

$$
\begin{aligned}
&- \langle \partial_t (p^\alpha + \tau q^\alpha, p_\Gamma^\alpha + \tau q_\Gamma^\alpha), (v, v_\Gamma) \rangle_{\mathcal{V}} + \int_\Omega \nabla q^\alpha \cdot \nabla v + \int_\Gamma \nabla_\Gamma q_\Gamma^\alpha \cdot \nabla_\Gamma v_\Gamma \\
&\quad + \int_\Omega (\varphi(\alpha) h''(\rho^\alpha) + \pi'(\rho^\alpha))\, q^\alpha\, v + \int_\Gamma (\varphi(\alpha) h''(\rho_\Gamma^\alpha) + \pi_\Gamma'(\rho_\Gamma^\alpha))\, q_\Gamma^\alpha\, v_\Gamma - \int_\Omega u^\alpha \cdot \nabla p^\alpha\, v \\
&= \int_\Omega \beta_1 (\rho^\alpha - \widehat{\rho}_Q)\, v + \int_\Gamma \beta_2 (\rho_\Gamma^\alpha - \widehat{\rho}_\Sigma)\, v_\Gamma \quad \text{a.e. in } (0,T), \quad \forall\, (v, v_\Gamma) \in \mathcal{V},
\end{aligned}
\tag{4.1}
$$

$$
\int_\Omega \nabla p^\alpha \cdot \nabla v + \int_\Gamma \nabla_\Gamma p_\Gamma^\alpha \cdot \nabla_\Gamma v_\Gamma = \int_\Omega q^\alpha v + \int_\Gamma q_\Gamma^\alpha v_\Gamma
\quad \text{a.e. in } (0,T), \quad \forall\, (v, v_\Gamma) \in \mathcal{V},
\tag{4.2}
$$

$$
\langle (p^\alpha + \tau q^\alpha, p_\Gamma^\alpha + \tau q_\Gamma^\alpha)(T), (v, v_\Gamma) \rangle_{\mathcal{V}} = \int_\Omega \beta_3 (\rho^\alpha(T) - \widehat{\rho}_\Omega) v + \int_\Gamma \beta_4 (\rho_\Gamma^\alpha(T) - \widehat{\rho}_\Gamma) v_\Gamma
\quad \forall\, (v, v_\Gamma) \in \mathcal{V},
\tag{4.3}
$$

which corresponds to the backward problem

$$
\begin{aligned}
&- \partial_t (p^\alpha + \tau q^\alpha) - \Delta q^\alpha + \varphi(\alpha) h''(\rho^\alpha) q^\alpha + \pi'(\rho^\alpha) q^\alpha - u^\alpha \cdot \nabla p^\alpha = \beta_1 (\rho^\alpha - \widehat{\rho}_Q) \\
&\quad \text{and} \quad - \Delta p^\alpha = q^\alpha \quad \text{in } Q,
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
&- \partial_t (p_\Gamma^\alpha + \tau q_\Gamma^\alpha) + \partial_\nu q^\alpha - \Delta_\Gamma q_\Gamma^\alpha + \varphi(\alpha) h''(\rho_\Gamma^\alpha) q_\Gamma^\alpha + \pi_\Gamma'(\rho_\Gamma^\alpha) q_\Gamma^\alpha = \beta_2 (\rho_\Gamma^\alpha - \widehat{\rho}_\Sigma), \\
&\partial_\nu p_\Gamma^\alpha - \Delta_\Gamma p_\Gamma^\alpha = q_\Gamma^\alpha, \quad p^\alpha_{|\Sigma} = p_\Gamma^\alpha \quad \text{and} \quad q^\alpha_{|\Sigma} = q_\Gamma^\alpha \quad \text{on } \Sigma,
\end{aligned}
\tag{4.5}
$$

$$
(p^\alpha + \tau q^\alpha, p_\Gamma^\alpha + \tau q_\Gamma^\alpha)(T) = (\beta_3 (\rho^\alpha(T) - \widehat{\rho}_\Omega), \beta_4 (\rho_\Gamma^\alpha(T) - \widehat{\rho}_\Gamma)) .
\tag{4.6}
$$

According to [14, Thm. 4.4], the adjoint system (4.1)–(4.3) enjoys for every $\alpha \in (0,1]$ a unique solution $((p^\alpha, p_\Gamma^\alpha), (q^\alpha, q_\Gamma^\alpha))$ such that

$$
(p^\alpha, p_\Gamma^\alpha) \in L^\infty(0,T;\mathcal{V}), \quad (q^\alpha, q_\Gamma^\alpha) \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}),
\tag{4.7}
$$

$$
(p^\alpha + \tau q^\alpha, p_\Gamma^\alpha + \tau q_\Gamma^\alpha) \in H^1(0,T;\mathcal{V}^*).
\tag{4.8}
$$

Observe that, owing to (4.7) and (4.8),

$$
(p^\alpha + \tau q^\alpha, p_\Gamma^\alpha + \tau q_\Gamma^\alpha) \in (H^1(0,T;\mathcal{V}^*) \cap L^2(0,T;\mathcal{V})) \subset C^0([0,T];\mathcal{H}),
$$

by continuous embedding. In particular, the final condition (4.3) is in fact satisfied in the form (4.6). Moreover, arguing as in the derivation of [14, Thm. 4.6], we can infer that for any such solution $((p^\alpha, p_\Gamma^\alpha), (q^\alpha, q_\Gamma^\alpha))$ there holds the variational inequality

$$
\int_Q (\rho^\alpha \nabla p^\alpha + \beta_5 u^\alpha + (u^\alpha - \bar{u})) \cdot (v - u^\alpha) \ \ge\ 0 \quad \forall\, v \in \mathcal{U}_{\rm ad} .
\tag{4.9}
$$

We now try to find bounds that are uniform with respect to $\alpha \in (0,1]$. To this end, we define for $\alpha \in (0,1]$ the quantities

$$\varphi_Q^\alpha := \beta_1(\rho^\alpha - \widehat{\rho}_Q),\ \varphi_\Sigma^\alpha := \beta_2(\rho_\Gamma^\alpha - \widehat{\rho}_\Sigma),\ \varphi_\Omega^\alpha := \beta_3(\rho^\alpha(T) - \widehat{\rho}_\Omega),\ \varphi_\Gamma^\alpha := \beta_4(\rho_\Gamma^\alpha(T) - \widehat{\rho}_\Gamma), \tag{4.10}$$

noting that (A3), (A4) and (2.36) imply that

$$\begin{aligned}&\|\varphi_Q^\alpha\|_{L^2(Q)} + \|\varphi_\Sigma^\alpha\|_{L^2(\Sigma)} + \|\varphi_\Omega^\alpha\|_{L^2(\Omega)} + \|\varphi_\Gamma^\alpha\|_{L^2(\Gamma)}\\ &+ \|\pi'(\rho^\alpha)\|_{L^\infty(Q)} + \|\pi_\Gamma'(\rho_\Gamma^\alpha)\|_{L^\infty(\Sigma)} \le K_3^* \quad \forall\,\alpha \in (0,1],\end{aligned} \tag{4.11}$$

with a constant $K_3^* > 0$ that depends only on the data of the system.

In view of the low regularity of the adjoint state variables, the derivation of uniform bounds makes it necessary to argue by approximation, following an idea introduced in the proof of [14, Thm. 4.4]. Namely, for fixed $\alpha \in (0,1]$, we approximate $(\varphi_\Omega^\alpha, \varphi_\Gamma^\alpha)$ by pairs $(\varphi_\Omega^{\alpha,\varepsilon}, \varphi_\Gamma^{\alpha,\varepsilon})$, $\varepsilon \in (0,1]$, which satisfy

$$(\varphi_\Omega^{\alpha,\varepsilon}/\tau, \varphi_\Gamma^{\alpha,\varepsilon}/\tau) \in \mathcal{V}, \quad (\varphi_\Omega^{\alpha,\varepsilon}, \varphi_\Gamma^{\alpha,\varepsilon}) \to (\varphi_\Omega^\alpha, \varphi_\Gamma^\alpha) \quad \text{in } \mathcal{H} \text{ as } \varepsilon \to 0, \tag{4.12}$$

and consider for every $\varepsilon \in (0,1]$ the approximating system

$$\begin{aligned}&- \int_\Omega \partial_t(p^{\alpha,\varepsilon} + \tau q^{\alpha,\varepsilon})v - \int_\Gamma \partial_t(p_\Gamma^{\alpha,\varepsilon} + \tau q_\Gamma^{\alpha,\varepsilon})v_\Gamma + \int_\Omega \nabla q^{\alpha,\varepsilon}\cdot\nabla v + \int_\Gamma \nabla_\Gamma q_\Gamma^{\alpha,\varepsilon}\cdot\nabla_\Gamma v_\Gamma\\ &+ \int_\Omega (\varphi(\alpha)h''(\rho^\alpha) + \pi'(\rho^\alpha))q^{\alpha,\varepsilon}\, v + \int_\Gamma (\varphi(\alpha)h''(\rho_\Gamma^\alpha) + \pi_\Gamma'(\rho_\Gamma^\alpha))q_\Gamma^{\alpha,\varepsilon}\, v_\Gamma\\ &- \int_\Omega u^\alpha\cdot\nabla p^{\alpha,\varepsilon}\, v = \int_\Omega \varphi_Q^\alpha\, v + \int_\Gamma \varphi_\Sigma^\alpha\, v_\Gamma \qquad \forall\,(v,v_\Gamma) \in \mathcal{V} \text{ and a.e. in } (0,T),\end{aligned} \tag{4.13}$$

$$\begin{aligned}&- \varepsilon\int_\Omega \partial_t p^{\alpha,\varepsilon}\, v - \varepsilon\int_\Gamma \partial_t p_\Gamma^{\alpha,\varepsilon}\, v_\Gamma + \int_\Omega \nabla p^{\alpha,\varepsilon}\cdot\nabla v + \int_\Gamma \nabla_\Gamma p_\Gamma^{\alpha,\varepsilon}\cdot\nabla_\Gamma v_\Gamma\\ &= \int_\Omega q^{\alpha,\varepsilon}\, v + \int_\Gamma q_\Gamma^{\alpha,\varepsilon}\, v_\Gamma \qquad \forall\,(v,v_\Gamma) \in \mathcal{V} \text{ and a.e. in } (0,T),\end{aligned} \tag{4.14}$$

$$(p^{\alpha,\varepsilon}, p_\Gamma^{\alpha,\varepsilon})(T) = (0,0), \quad (q^{\alpha,\varepsilon}, q_\Gamma^{\alpha,\varepsilon})(T) = (\varphi_\Omega^{\alpha,\varepsilon}/\tau, \varphi_\Gamma^{\alpha,\varepsilon}/\tau)\,. \tag{4.15}$$

According to [14, Thm. 4.3], the system (4.13)–(4.15) enjoys for every $\varepsilon \in (0,1]$ a unique solution $((p^{\alpha,\varepsilon}, p_\Gamma^{\alpha,\varepsilon}), (q^{\alpha,\varepsilon}, q_\Gamma^{\alpha,\varepsilon}))$ such that

$$(p^{\alpha,\varepsilon}, p_\Gamma^{\alpha,\varepsilon}),\ (q^{\alpha,\varepsilon}, q_\Gamma^{\alpha,\varepsilon}) \in H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{V}). \tag{4.16}$$

Moreover, it was shown in the proof of [14, Thm. 4.4] that there is some sequence $\varepsilon_n \searrow 0$ such that, as $n \to \infty$,

$$(p^{\alpha,\varepsilon_n}, p_\Gamma^{\alpha,\varepsilon_n}) \to (p^\alpha, p_\Gamma^\alpha) \quad \text{weakly-star in } L^\infty(0,T;\mathcal{V}), \tag{4.17}$$

$$(q^{\alpha,\varepsilon_n}, q_\Gamma^{\alpha,\varepsilon_n}) \to (q^\alpha, q_\Gamma^\alpha) \quad \text{weakly-star in } L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}), \tag{4.18}$$

$$\partial_t(p^{\alpha,\varepsilon_n} + \tau q^{\alpha,\varepsilon_n}, p_\Gamma^{\alpha,\varepsilon_n} + \tau q_\Gamma^{\alpha,\varepsilon_n}) \to \partial_t(p^\alpha + \tau q^\alpha, p_\Gamma^\alpha + \tau q_\Gamma^\alpha) \quad \text{weakly in } L^2(0,T;\mathcal{V}^*), \tag{4.19}$$

$$\varepsilon_n\,\partial_t(p^{\alpha,\varepsilon_n}, p_\Gamma^{\alpha,\varepsilon_n}) \to (0,0) \quad \text{strongly in } L^2(0,T;\mathcal{H}), \tag{4.20}$$

where $((p^\alpha, p^\alpha_\Gamma), (q^\alpha, q^\alpha_\Gamma))$ is the solution to the adjoint system (4.1)–(4.3) having the regularity properties (4.7)–(4.8). Notice that (4.17)–(4.19) imply that also

$$(p^{\alpha,\varepsilon_n} + \tau q^{\alpha,\varepsilon_n}, p^{\alpha,\varepsilon_n}_\Gamma + \tau q^{\alpha,\varepsilon_n}_\Gamma) \to (p^\alpha + \tau q^\alpha, p^\alpha_\Gamma + \tau q^\alpha_\Gamma) \quad \text{strongly in } C^0([0,T];\mathcal{V}^*) \cap L^2(0,T;\mathcal{H}), \tag{4.21}$$

so that the Cauchy condition (4.3) is meaningful. In the following, we will always work with the particular sequence $\{\varepsilon_n\}$.

Next, we establish uniform bounds for the approximating solutions. To simplify the notation, we omit in the following estimate the superscript ${}^{\alpha,\varepsilon}$, writing it only at the end of the respective estimations. We also recall the definition of Q^t and Σ^t, for $t \in [0,T)$, given in (1.21), and we denote by C_i, $i \in \mathbb{N}$, positive constants that may depend on the data, but neither on $\varepsilon \in (0,1]$ nor on $\alpha \in (0,1]$.

We test (4.13) by (q, q_Γ), integrate over (t,T), and account for the Cauchy conditions (4.15), to obtain the identity

$$\begin{aligned}
&-\int_{Q^t} \partial_t p\, q - \int_{\Sigma^t} \partial_t p_\Gamma\, q_\Gamma + \frac{\tau}{2}\int_\Omega |q(t)|^2 + \frac{\tau}{2}\int_\Gamma |q_\Gamma(t)|^2 + \int_{Q^t} |\nabla q|^2 + \int_{\Sigma^t} |\nabla_\Gamma q_\Gamma|^2 \\
&+ \int_{Q^t} \varphi(\alpha) h''(\rho^\alpha)|q|^2 + \int_{\Sigma^t} \varphi(\alpha) h''(\rho^\alpha_\Gamma)|q^\alpha_\Gamma|^2 \\
&= \frac{\tau}{2}\int_\Omega |\varphi^{\alpha,\varepsilon}_\Omega/\tau|^2 + \frac{\tau}{2}\int_\Gamma |\varphi^{\alpha,\varepsilon}_\Gamma/\tau|^2 + \int_{Q^t} u^\alpha \cdot \nabla p\, q - \int_{Q^t} \pi'(\rho^\alpha)|q|^2 - \int_{\Sigma^t} \pi'_\Gamma(\rho^\alpha_\Gamma)|q_\Gamma|^2 \\
&+ \int_{Q^t} \varphi^\alpha_Q\, q + \int_{\Sigma^t} \varphi^\alpha_\Sigma\, q_\Gamma\,.
\end{aligned} \tag{4.22}$$

At the same time, we test (4.14) by $-\partial_t(p, p_\Gamma)$ and integrate over (t,T) to obtain the identity

$$\varepsilon\int_{Q^t} |\partial_t p|^2 + \varepsilon\int_{\Sigma^t} |\partial_t p_\Gamma|^2 + \frac{1}{2}\int_\Omega |\nabla p(t)|^2 + \frac{1}{2}\int_\Gamma |\nabla_\Gamma p_\Gamma(t)|^2 = -\int_{Q^t} q\partial_t p - \int_{\Sigma^t} q_\Gamma \partial_t p_\Gamma\,. \tag{4.23}$$

Now, we add (4.22) and (4.23), observing that four terms cancel out and that the two summands in the second line of (4.22) are nonnegative. Omitting these two summands and the first two summands in the left-hand side of (4.23), we then arrive at the inequality

$$\begin{aligned}
&\frac{\tau}{2}\int_\Omega |q(t)|^2 + \frac{\tau}{2}\int_\Gamma |q_\Gamma(t)|^2 + \int_{Q^t} |\nabla q|^2 + \int_{\Sigma^t} |\nabla_\Gamma q_\Gamma|^2 + \frac{1}{2}\int_\Omega |\nabla p(t)|^2 + \frac{1}{2}\int_\Gamma |\nabla_\Gamma p_\Gamma(t)|^2 \\
&\le C_1 + C_2\Bigl(\int_{Q^t} |q|^2 + \int_{\Sigma^t} |q_\Gamma|^2\Bigr) + \int_{Q^t} u^\alpha \cdot \nabla p\, q\,,
\end{aligned} \tag{4.24}$$

where we have used (4.11), (4.12) and Young's inequality. Now, by Young's inequality, and since $u^\alpha \in \mathcal{U}_{\rm ad}$,

$$\begin{aligned}
\int_{Q^t} u^\alpha \cdot \nabla p\, q &\le \|u^\alpha\|_{L^\infty(Q)} \int_t^T \|\nabla p(s)\|_2\, \|q(s)\|_2\, ds \\
&\le \int_{Q^t} |\nabla p|^2 + C_3 \int_{Q^t} |q|^2\,.
\end{aligned} \tag{4.25}$$

Therefore, invoking Gronwall's lemma, we can infer that

$$\|(q^{\alpha,\varepsilon}, q^{\alpha,\varepsilon}_\Gamma)\|_{L^\infty(0,T;\mathcal{H})\cap L^2(0,T;\mathcal{V})} + \operatorname*{sup\,ess}_{t\in(0,T)} \Bigl(\int_\Omega |\nabla p^{\alpha,\varepsilon}(t)|^2 + \int_\Gamma |\nabla_\Gamma p^{\alpha,\varepsilon}_\Gamma(t)|^2\Bigr) \le C_4 \tag{4.26}$$

for all $\alpha \in (0,1]$ and $\varepsilon \in (0,1]$. We thus can conclude from the weak and weak-star sequential lower semicontinuity of norms, taking the limit as $\varepsilon_n \searrow 0$, that

$$\|(q^\alpha, q^\alpha_\Gamma)\|_{L^\infty(0,T;\mathcal{H})\cap L^2(0,T;\mathcal{V})} + \operatorname*{sup\,ess}_{t\in(0,T)} \Big(\int_\Omega |\nabla p^\alpha(t)|^2 + \int_\Gamma |\nabla_\Gamma p^\alpha_\Gamma(t)|^2 \Big) \le C_4 \qquad (4.27)$$

for all $\alpha \in (0,1]$.

Remark 4.1. In the proof of [14, Thm. 4.4], further estimates for the approximations $((p^{\alpha,\varepsilon}, p^{\alpha,\varepsilon}_\Gamma), (q^{\alpha,\varepsilon}, q^{\alpha,\varepsilon}_\Gamma))$ could be derived. However, a closer look at these estimations reveals that the resulting bounds depend on the special choice of $\alpha \in (0,1]$ and may become infinite as $\alpha \searrow 0$. In particular, while it is clear that

$$\operatorname{mean}(q^\alpha(t), q^\alpha_\Gamma(t)) = 0 \quad \text{for all } t \in [0,T] \text{ and } \alpha \in (0,1], \qquad (4.28)$$

as one immediately sees by inserting $(v, v_\Gamma) = (1,1)$ in (4.2), it seems to be impossible to derive a uniform bound for the mean value of $(p^\alpha, p^\alpha_\Gamma)$, the main reason being that the separation constants $\rho_*(\alpha), \rho^*(\alpha)$ introduced in Theorem 2.3, which were implicitly used in the argument to control the expressions $\varphi(\alpha)h''(\rho^\alpha)q^\alpha$ and $\varphi(\alpha)h''(\rho^\alpha_\Gamma)q^\alpha_\Gamma$, may approach ± 1 as $\alpha \searrow 0$. The difficulty becomes apparent if we observe that insertion of $(v, v_\Gamma) = (1,1)$ in (4.1) and integration of the resulting identity over $[t,T]$, where $t \in [0,T]$, yields the representation formula (by also owing to (4.28))

$$\begin{aligned}\operatorname{mean}(p^\alpha(t), p^\alpha_\Gamma(t)) = \frac{1}{|\Omega| + |\Gamma|} \bigg[&- \int_{Q^t} (\varphi(\alpha)h''(\rho^\alpha(t)) + \pi'(\rho^\alpha(t)))q^\alpha(t) \\ &- \int_{\Sigma^t} (\varphi(\alpha)h''(\rho^\alpha_\Gamma(t)) + \pi'_\Gamma(\rho^\alpha_\Gamma(t)))q^\alpha_\Gamma(t) \\ &+ \int_\Omega \beta_3(\rho^\alpha(T) - \widehat{\rho}_\Omega) + \int_\Gamma \beta_4(\rho^\alpha_\Gamma(T) - \widehat{\rho}_\Gamma) + \int_{Q^t} \beta_1(\rho^\alpha - \widehat{\rho}_Q) + \int_{\Sigma^t} \beta_2(\rho^\alpha_\Gamma - \widehat{\rho}_\Sigma) \bigg]. \end{aligned} \qquad (4.29)$$

In order to be able to derive a meaningful adjoint system for problem $(\mathcal{P}_0)$, we thus have to eliminate the mean value of $(p^\alpha, p^\alpha_\Gamma)$ from the problem, thereby avoiding the difficulty mentioned above. To this end, we follow a strategy introduced in [10] and [3]: by recalling (4.28), it follows from (4.2) and the definition (2.10) of the operator $\mathcal{N}$ the identity

$$\begin{aligned}(p^\alpha(t), p^\alpha_\Gamma(t)) - \operatorname{mean}(p^\alpha(t), p^\alpha_\Gamma(t))(1,1) \;=\; \mathcal{N}(q^\alpha(t), q^\alpha_\Gamma(t)) \\ \text{for all } t \in [0,T] \text{ and } \alpha \in (0,1]. \end{aligned} \qquad (4.30)$$

Since $(q^\alpha, q^\alpha_\Gamma)$ is uniformly bounded in $L^\infty(0,T;\mathcal{H})$, in particular, we can infer from [13, Lem. 3.1] that $(\xi^\alpha, \xi^\alpha_\Gamma) := \mathcal{N}(q^\alpha, q^\alpha_\Gamma)$ belongs to $L^\infty(0,T;\mathcal{W} \cap \mathcal{H}_0)$, solves the boundary value problem

$$-\Delta\xi^\alpha(t) = q^\alpha(t) \quad \text{a.e. in } \Omega, \qquad \partial_\nu \xi^\alpha(t) - \Delta_\Gamma \xi^\alpha_\Gamma(t) = q^\alpha_\Gamma(t) \quad \text{a.e. on } \Gamma,$$

for almost every $t \in (0,T)$, and satisfies the uniform bound

$$\|\mathcal{N}(q^\alpha, q^\alpha_\Gamma)\|_{L^\infty(0,T;\mathcal{W})} \le C_5 \quad \forall\, \alpha \in (0,1]. \qquad (4.31)$$

Now recall that $\mathcal{V} = \mathcal{V}_0 \oplus \operatorname{span}\{(1,1)\}$, where $\mathcal{V}_0$ is defined in (2.7). Notice also that, by virtue of [10, Lem. 5.1 and Cor. 5.3], it holds that $V_\Gamma = \{v_\Gamma : (v, v_\Gamma) \in \mathcal{V}_0\}$ and that $\mathcal{H}_0$ is dense in $\mathcal{V}_0$.

We thus can construct the Hilbert triple $\mathcal{V}_0 \subset \mathcal{H}_0 \subset \mathcal{V}_0^*$ with dense and compact embeddings, that is, we identify $\mathcal{H}_0$ with a subspace of $\mathcal{V}_0^*$ in such a way that

$$\langle (w, w_\Gamma), (v, v_\Gamma) \rangle_{\mathcal{V}_0} = \int_\Omega w\, v + \int_\Gamma w_\Gamma\, v_\Gamma \quad \forall\, (w, w_\Gamma) \in \mathcal{H}_0, \quad \forall\, (v, v_\Gamma) \in \mathcal{V}_0\,. \tag{4.32}$$

Notice that the embedding $(H^1(0,T;\mathcal{V}_0^*) \cap L^2(0,T;\mathcal{V}_0)) \subset C^0([0,T];\mathcal{H}_0)$ is continuous. Observe also that, because of the zero mean value condition, the first components v of the elements $(v, v_\Gamma) \in \mathcal{V}_0$ do not span the whole space $C_0^\infty(\Omega)$, so that variational equalities with test functions from $\mathcal{V}_0$ cannot directly be interpreted as equations in the sense of distributions.

At this point, the additional assumption (A6) comes into play. To this end, recall that $(z^\alpha, z_\Gamma^\alpha) := \partial_t(p^\alpha + \tau q^\alpha, p_\Gamma^\alpha + \tau q_\Gamma^\alpha)$ belongs to $L^2(0,T;\mathcal{V}^*)$ and thus also to $L^2(0,T;\mathcal{V}_0^*)$. We now aim to show a global bound for the family $\{(z^\alpha, z_\Gamma^\alpha)\}_{\alpha\in(0,1]}$ that will prove to be fundamental for the subsequent argumentation. To this end, we introduce the spaces

$$\mathcal{Z} := (H^1(0,T;V^*) \times H^1(0,T;V_\Gamma^*)) \cap L^2(0,T;\mathcal{V}_0)\,, \tag{4.33}$$

$$\mathcal{Z}_0 := \{(v, v_\Gamma) \in \mathcal{Z} :\ (v(0), v_\Gamma(0)) = (0,0)\}\,, \tag{4.34}$$

which are Banach spaces when endowed with the natural norm of $\mathcal{Z}$. Moreover, $\mathcal{Z}$ is continuously embedded in $C^0([0,T];\mathcal{H}_0)$, so that the initial condition encoded in (4.34) is meaningful. In addition, $\mathcal{Z}_0$ is a closed subspace of $Y \times Y_\Gamma$, where

$$Y := H^1(0,T;V^*) \cap L^2(0,T;V) \quad \text{and} \quad Y_\Gamma := H^1(0,T;V_\Gamma^*) \cap L^2(0,T;V_\Gamma) \tag{4.35}$$

are Banach spaces when endowed with their natural norms. It then follows (cf., e.g., [8, Prop. 2.6]) that the elements $F \in \mathcal{Z}_0^*$ are exactly those that are of the form

$$\langle F, (\eta, \eta_\Gamma) \rangle_{\mathcal{Z}_0} \ =\ \langle z, \eta \rangle_Y \ +\ \langle z_\Gamma, \eta_\Gamma \rangle_{Y_\Gamma} \quad \text{for all } (\eta, \eta_\Gamma) \in \mathcal{Z}_0, \tag{4.36}$$

with some $z \in Y^*$ and $z_\Gamma \in Y_\Gamma^*$. Thus, we can write

$$\langle F, (\eta, \eta_\Gamma) \rangle_{\mathcal{Z}_0} = \int_0^T \langle z(t), \eta(t) \rangle_V \, dt + \int_0^T \langle z_\Gamma(t), \eta_\Gamma(t) \rangle_{V_\Gamma} \, dt \quad \text{for every } (\eta, \eta_\Gamma) \in \mathcal{Z}_0.$$

Moreover, even though the pair (z, z_Γ) associated with $F \in \mathcal{Z}_0^*$ is not unique, the above representation formula allows us to give a proper meaning to statements like

$$(z^\alpha, z_\Gamma^\alpha) \to (z, z_\Gamma) \quad \text{weakly in } \mathcal{Z}_0^*.$$

Now let $(v, v_\Gamma) \in \mathcal{Z}_0$ be arbitrary. Then $(v, v_\Gamma)(0) = (0,0)$, $\mathrm{mean}\,(v(t), v_\Gamma(t)) = 0$ for all $t \in [0,T]$, and $\mathrm{mean}\,(\partial_t(v, v_\Gamma)(t)) = 0$ for almost every $t \in (0,T)$. Thus, from one side, we have by (4.30) that

$$\begin{aligned} &\langle \partial_t(v, v_\Gamma), (p^\alpha, p_\Gamma^\alpha) \rangle_{\mathcal{V}} = \langle \partial_t(v, v_\Gamma), (p^\alpha, p_\Gamma^\alpha) - \mathrm{mean}(p^\alpha, p_\Gamma^\alpha)(1,1) \rangle_{\mathcal{V}} \\ &= \langle \partial_t(v, v_\Gamma), \mathcal{N}(q^\alpha, q_\Gamma^\alpha) \rangle_{\mathcal{V}} \quad \text{a.e. in } (0,T). \end{aligned} \tag{4.37}$$

On the other hand, using (4.3) and the fact that both (v, v_Γ) and $(p^\alpha + \tau q^\alpha, p_\Gamma^\alpha + \tau q_\Gamma^\alpha)$ belong to $H^1(0,T;\mathcal{V}^*) \cap L^2(0,T;\mathcal{V})$, we see that

$$\begin{aligned} &\int_0^T \langle -(z^\alpha, z_\Gamma^\alpha)(t), (v, v_\Gamma)(t) \rangle_{\mathcal{V}} \, dt \ =\ -\int_\Omega \beta_3(\rho^\alpha(T) - \widehat{\rho}_\Omega) v(T) - \int_\Gamma \beta_4(\rho_\Gamma^\alpha(T) - \widehat{\rho}_\Gamma) v_\Gamma(T) \\ &+ \int_0^T \langle \partial_t(v, v_\Gamma)(t), (p^\alpha + \tau q^\alpha, p_\Gamma^\alpha + \tau q_\Gamma^\alpha)(t) \rangle_{\mathcal{V}} \, dt \ =:\ A^\alpha\,. \end{aligned} \tag{4.38}$$

Thanks to (4.10)–(4.11), the sum of the first two summands on the right-hand side of (4.38), which we denote by A_1^α, satisfies the estimate

$$|A_1^\alpha| \le \|(\varphi_\Omega^\alpha, \varphi_\Gamma^\alpha)\|_{\mathcal{H}} \, \|(v, v_\Gamma)(T)\|_{\mathcal{H}} \le C_6 \, \|(v, v_\Gamma)\|_{\mathcal{Z}_0} \quad \forall \, \alpha \in (0,1] \,, \tag{4.39}$$

where the continuity of the embedding $(H^1(0,T;\mathcal{V}_0^*) \cap L^2(0,T;\mathcal{V}_0)) \subset C^0([0,T];\mathcal{H}_0)$ has been used. Moreover, the third summand on the right-hand side of (4.38), which we denote by A_2^α, satisfies, in view of (4.37), the identity

$$A_2^\alpha = \int_0^T \langle \partial_t (v, v_\Gamma)(t), (\mathcal{N}(q^\alpha(t), q_\Gamma^\alpha(t)) + \tau(q^\alpha(t), q_\Gamma^\alpha(t)) \rangle_{\mathcal{V}_0} \, dt \,. \tag{4.40}$$

In addition, from (4.27) and (4.31) it follows that

$$|A_2^\alpha| \le \int_0^T \|\partial_t (v, v_\Gamma)(t)\|_{\mathcal{V}_0^*} \, \|\mathcal{N}(q^\alpha(t), q_\Gamma^\alpha(t)) + \tau(q^\alpha(t), q_\Gamma^\alpha(t))\|_{\mathcal{V}_0} \, dt \le C_7 \, \|\partial_t (v, v_\Gamma)\|_{L^2(0,T;\mathcal{V}_0^*)} \le C_7 \, \|(v, v_\Gamma)\|_{\mathcal{Z}_0} \quad \forall \, \alpha \in (0,1]. \tag{4.41}$$

From the above estimates, we can infer that

$$\|\partial_t (p^\alpha + \tau q^\alpha, p_\Gamma^\alpha + \tau q_\Gamma^\alpha)\|_{\mathcal{Z}_0^*} \le C_8 \quad \forall \, \alpha \in (0,1], \tag{4.42}$$

and comparison in (4.1) yields that also

$$\|(\varphi(\alpha) h''(\rho^\alpha) q^\alpha, \varphi(\alpha) h''(\rho_\Gamma^\alpha) q_\Gamma^\alpha)\|_{\mathcal{Z}_0^*} \le C_9 \quad \forall \, \alpha \in (0,1]. \tag{4.43}$$

We are now in a position to state the first-order necessary optimality conditions for problem $(\mathcal{P}_0)$.

Theorem 4.2. *Suppose that* (A1)–(A6)*,* (1.14)*,* (1.15) *are satisfied, and let* $\bar{u} \in \mathcal{U}_{\rm ad}$ *be an optimal control with associated state* $((\bar{\mu}, \bar{\mu}_\Gamma), (\bar{\rho}, \bar{\rho}_\Gamma), (\bar{\xi}, \bar{\xi}_\Gamma))$ *in the sense of Theorem 2.2. Then there exist* (q, q_Γ), η, $(\lambda, \lambda_\Gamma)$ *such that the following statements hold true:*

(i) $(q, q_\Gamma) \in L^\infty(0,T;\mathcal{H}_0) \cap L^2(0,T;\mathcal{V}_0)$, $\mathcal{N}(q, q_\Gamma) \in L^\infty(0,T;\mathcal{W} \cap \mathcal{H}_0)$, $(\lambda, \lambda_\Gamma) \in \mathcal{Z}_0^*$, *and* $\eta \in (L^\infty(0,T;H))^3$.

(ii) *Adjoint system:*

$$\langle (\lambda, \lambda_\Gamma), (v, v_\Gamma) \rangle_{\mathcal{Z}_0} + \int_0^T \langle \partial_t (v, v_\Gamma)(t), \mathcal{N}(q(t), q_\Gamma(t)) + \tau(q(t), q_\Gamma(t)) \rangle_{\mathcal{V}_0} \, dt + \int_Q \pi'(\bar{\rho}) q \, v + \int_\Sigma \pi_\Gamma'(\bar{\rho}_\Gamma) q_\Gamma \, v_\Gamma + \int_Q \bar{u} \cdot \eta \, v = \int_Q \beta_1 (\bar{\rho} - \widehat{\rho}_Q) v + \int_\Sigma \beta_2 (\bar{\rho}_\Gamma - \widehat{\rho}_\Sigma) v_\Gamma + \int_\Omega \beta_3 (\bar{\rho}(T) - \widehat{\rho}_\Omega) v(T) + \int_\Gamma \beta_4 (\bar{\rho}_\Gamma(T) - \bar{\rho}_\Gamma) v_\Gamma(T) \qquad \forall \, (v, v_\Gamma) \in \mathcal{Z}_0 \,. \tag{4.44}$$

(iii) *Necessary optimality condition:*

$$\int_Q (\bar{\rho} \, \eta + \beta_5 \, \bar{u}) \cdot (v - \bar{u}) \ge 0 \quad \forall \, v \in \mathcal{U}_{\rm ad} \,. \tag{4.45}$$

PROOF: We pick a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1]$ such that $\alpha_n \searrow 0$ and (cf. Theorem 3.1, Theorem 3.4, and (3.6)–(3.9))

$$u^{\alpha_n} \to \bar{u} \quad \text{strongly in } (L^2(Q))^3, \tag{4.46}$$

$$(\rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}) \to (\bar{\rho}, \bar{\rho}_\Gamma) \quad \text{weakly-star in } W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{W}) \\ \text{and strongly in } L^2(0,T;\mathcal{V}) \cap (C^0(\overline{Q}) \times C^0(\overline{\Sigma})), \tag{4.47}$$

$$(\pi'(\rho^{\alpha_n}), \pi_\Gamma'(\rho_\Gamma^{\alpha_n})) \to (\pi'(\bar{\rho}), \pi_\Gamma'(\bar{\rho}_\Gamma)) \quad \text{strongly in } C^0(\overline{Q}) \times C^0(\overline{\Sigma})\,. \tag{4.48}$$

Moreover, in view of the estimates (4.27), (4.31), and (4.43), we may assume that there are $(q, q_\Gamma) \in L^\infty(0,T;\mathcal{H}_0) \cap L^2(0,T;\mathcal{V}_0)$, $\eta \in (L^\infty(0,T;H))^3$, and $(\lambda, \lambda_\Gamma) \in \mathcal{Z}_0^*$, such that

$$(q^{\alpha_n}, q_\Gamma^{\alpha_n}) \to (q, q_\Gamma) \quad \text{weakly-star in } L^\infty(0,T;\mathcal{H}_0) \cap L^2(0,T;\mathcal{V}_0), \tag{4.49}$$

$$\nabla p^{\alpha_n} \to \eta \quad \text{weakly-star in } (L^\infty(0,T;H))^3, \tag{4.50}$$

$$(\varphi(\alpha_n) h''(\rho^{\alpha_n}) q^{\alpha_n}, \varphi(\alpha_n) h''(\rho_\Gamma^{\alpha_n}) q_\Gamma^{\alpha_n}) \to (\lambda, \lambda_\Gamma) \quad \text{weakly in } \mathcal{Z}_0^*, \tag{4.51}$$

$$\mathcal{N}(q^{\alpha_n}, q_\Gamma^{\alpha_n}) \to \mathcal{N}(q, q_\Gamma) \quad \text{weakly-star in } L^\infty(0,T;\mathcal{W} \cap \mathcal{H}_0). \tag{4.52}$$

Now we can take advantage of the identities (4.38) and (4.40). Indeed, if we restrict ourselves to test functions $(v, v_\Gamma) \in \mathcal{Z}_0$ and invoke the convergence results (4.47)–(4.52), then we may pass to the limit as $n \to \infty$ in the equations (4.1)–(4.3) (written for $\alpha = \alpha_n$) to arrive at the conclusion that we have the identity

$$\begin{aligned} &\langle (\lambda, \lambda_\Gamma), (v, v_\Gamma) \rangle_{\mathcal{Z}_0} + \int_0^T \langle \partial_t (v, v_\Gamma)(t), \mathcal{N}(q(t), q_\Gamma(t)) + \tau(q(t), q_\Gamma(t)) \rangle_{\mathcal{V}_0} \, dt \\ &+ \int_Q \pi'(\bar{\rho}) q \, v + \int_\Sigma \pi_\Gamma'(\bar{\rho}_\Gamma) q_\Gamma \, v_\Gamma + \lim_{n\to\infty} \int_Q u^{\alpha_n} \cdot \nabla p^{\alpha_n} \, v \\ &= \int_Q \beta_1 (\bar{\rho} - \widehat{\rho}_Q) v + \int_\Sigma \beta_2 (\bar{\rho}_\Gamma - \widehat{\rho}_\Sigma) v_\Gamma + \int_\Omega \beta_3 (\bar{\rho}(T) - \widehat{\rho}_\Omega) v(T) \\ &\quad + \int_\Gamma \beta_4 (\bar{\rho}_\Gamma(T) - \bar{\rho}_\Gamma) v_\Gamma(T) \qquad \forall\, (v, v_\Gamma) \in \mathcal{Z}_0\,. \end{aligned} \tag{4.53}$$

Therefore, in order to prove the validity of (4.44), we need to show that

$$\lim_{n\to\infty} \int_Q u^{\alpha_n} \cdot \nabla p^{\alpha_n} \, v = \int_Q \bar{u} \cdot \eta v \quad \forall\, (v, v_\Gamma) \in \mathcal{Z}_0. \tag{4.54}$$

To this end, it suffices to establish the result for all test functions from the set $\widetilde{\mathcal{Z}}_0 := \{(v, v_\Gamma) \in \mathcal{Z}_0 : v \in L^2(0,T;C^0(\overline{\Omega}))\}$. Indeed, since $\widetilde{\mathcal{Z}}_0$ is a dense subset of $\mathcal{Z}_0$, (4.54) then follows from a simple density argument. Now let $(v, v_\Gamma) \in \widetilde{\mathcal{Z}}_0$. We have that

$$\int_Q (u^{\alpha_n} \cdot \nabla p^{\alpha_n} - \bar{u} \cdot \eta) \, v = \int_Q (u^{\alpha_n} - \bar{u}) \cdot \nabla p^{\alpha_n} \, v + \int_Q \bar{u} \cdot (\nabla p^{\alpha_n} - \eta) \, v\,. \tag{4.55}$$

Since $\bar{u} \in \mathcal{U}_{\rm ad}$, we can infer from (4.50) that the second integral on the right-hand side approaches zero as $n \to \infty$. Moreover, we obtain from (4.46), using (4.27) and Hölder's inequality, that

$$\begin{aligned} &\Big| \int_Q (u^{\alpha_n} - \bar{u}) \cdot \nabla p^{\alpha_n} \, v \Big| \le \int_0^T \| u^{\alpha_n}(t) - \bar{u}(t) \|_2 \, \| \nabla p^{\alpha_n}(t) \|_2 \, \| v(t) \|_\infty \, dt \\ &\le \| u^{\alpha_n} - \bar{u} \|_{(L^2(Q))^3} \, \| \nabla p^{\alpha_n} \|_{(L^\infty(0,T;L^2(\Omega)))^3} \, \| v \|_{L^2(0,T;C^0(\overline{\Omega}))} \to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{4.56}$$

and the validity of (4.44) is shown. Next, we take the limit as $n \to \infty$ in the variational inequality (4.9), written for $\alpha = \alpha_n$. Employing (4.46), (4.47), and (4.50), we readily see that (4.45) is fulfilled. This concludes the proof of the assertion. ☐

Remark 4.3. Unfortunately, we are unable to derive a complementarity slackness condition for the adjoint variables. Indeed, although we have the inequality

$$\begin{aligned} &\langle(\varphi(\alpha_n)h''(\rho^{\alpha_n})q^{\alpha_n}, \varphi(\alpha_n)h''(\rho_\Gamma^{\alpha_n})q_\Gamma^{\alpha_n}), (q^{\alpha_n}, q_\Gamma^{\alpha_n})\rangle_{\mathcal{Z}_0} \\ &= \int_Q \varphi(\alpha_n)h''(\rho^{\alpha_n})\,|q^{\alpha_n}|^2 + \int_\Sigma \varphi(\alpha_n)h''(\rho_\Gamma^{\alpha_n})\,|q_\Gamma^{\alpha_n}|^2 \,\geq\, 0 \quad \forall\, n \in \mathbb{N}, \end{aligned} \tag{4.57}$$

the convergence properties (4.49) and (4.51) are not strong enough to guarantee that $\langle(\lambda, \lambda_\Gamma), (q, q_\Gamma)\rangle_{\mathcal{Z}_0} \,\geq\, 0.$

Remark 4.4. Obviously, the adjoint variables are not uniquely determined. It thus may well happen that for different sequences $\alpha_n \searrow 0$ different limits are approached. However, the weak-star limit η in (4.50) must satisfy the variational inequality (4.45).

References

[1] V. Barbu: *Necessary conditions for nonconvex distributed control problems governed by elliptic variational inequalities*. J. Math. Anal. Appl. **80** (1981), 566-597.

[2] H. Brezis: “Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert”. North-Holland Math. Stud. **5**, North-Holland, Amsterdam, 1973.

[3] P. Colli, M.H. Farshbaf-Shaker, G. Gilardi, J. Sprekels: *Optimal boundary control of a viscous Cahn-Hilliard system with dynamic boundary condition and double obstacle potentials*. SIAM J. Control Optim. **53** (2015), 2696-2721.

[4] P. Colli, M.H. Farshbaf-Shaker, G. Gilardi, J. Sprekels: *Second-order analysis of a boundary control problem for the viscous Cahn-Hilliard equation with dynamic boundary conditions.* Ann. Acad. Rom. Sci. Ser. Math. Appl. **7** (2015), 41-66.

[5] P. Colli, M.H. Farshbaf-Shaker, J. Sprekels: *A deep quench approach to the optimal control of an Allen–Cahn equation with dynamic boundary conditions and double obstacles.* Appl. Math. Optim. **71** (2015), 1-24.

[6] P. Colli, T. Fukao: *Equation and dynamic boundary condition of Cahn–Hilliard type with singular potentials.* Nonlinear Anal. **127** (2015), 413-433.

[7] P. Colli, G. Gilardi, E. Rocca, J. Sprekels: *Optimal distributed control of a diffuse interface model of tumor growth.* Nonlinearity **30** (2017), 2518-2546.

[8] P. Colli, G. Gilardi, J. Sprekels: *A boundary control problem for the pure Cahn–Hilliard equation with dynamic boundary conditions.* Adv. Nonlinear Anal. **4** (2015), 311-325.

[9] P. Colli, G. Gilardi, J. Sprekels: *Distributed optimal control of a nonstandard nonlocal phase field system.* AIMS Mathematics **1** (2016), 246-281.

[10] P. Colli, G. Gilardi, J. Sprekels: *A boundary control problem for the viscous Cahn–Hilliard equation with dynamic boundary conditions.* Appl. Math. Optim. **72** (2016), 195-225.

[11] P. Colli, G. Gilardi, J. Sprekels: *Recent results on the Cahn–Hilliard equation with dynamic boundary conditions.* Vestn. Yuzhno-Ural. Gos. Univ., Ser. Mat. Model. Program. **10** (2017), 5-21.

[12] P. Colli, G. Gilardi, J. Sprekels: *Distributed optimal control of a nonstandard nonlocal phase field system with double obstacle potential.* Evol. Equ. Control Theory **6** (2017), 35-58.

[13] P. Colli, G. Gilardi, J. Sprekels: *On a Cahn–Hilliard system with convection and dynamic boundary conditions*. Preprint arXiv:1704.05337 [math.AP] (2017), pp. 1-34.

[14] P. Colli, G. Gilardi, J. Sprekels: *Optimal velocity control of a viscous Cahn–Hilliard system with convection and dynamic boundary conditions*. Preprint arXiv:1709.02335 [math.AP] (2017), pp. 1-29.

[15] P. Colli, J. Sprekels: Optimal boundary control of a nonstandard Cahn–Hilliard system with dynamic boundary condition and double obstacle inclusions. To appear in "Solvability, Regularity, Optimal Control of Boundary Value Problems for PDEs", P. Colli, A. Favini, E. Rocca, G. Schimperna, J. Sprekels (eds.), Springer INdAM Series (see also the WIAS Preprint No. 2370, Berlin 2017, pp. 1-29).

[16] N. Duan, X. Zhao: *Optimal control for the multi-dimensional viscous Cahn–Hilliard equation.* Electron. J. Differential Equations 2015, No. 165, 13 pp.

[17] S. Frigeri, E. Rocca, J. Sprekels: *Optimal distributed control of a nonlocal Cahn–Hilliard/Navier–Stokes system in two dimensions.* SIAM J. Control. Optim. **54** (2016), 221-250.

[18] T. Fukao, N. Yamazaki: *A boundary control problem for the equation and dynamic boundary condition of Cahn–Hilliard type.* To appear in "Solvability, Regularity, Optimal Control of Boundary Value Problems for PDEs", P. Colli, A. Favini, E. Rocca, G. Schimperna, J. Sprekels (eds.), Springer INdAM Series.

[19] G. Gilardi, A. Miranville, G. Schimperna: *On the Cahn–Hilliard equation with irregular potentials and dynamic boundary conditions.* Commun. Pure Appl. Anal. **8** (2009), 881-912.

[20] M. Hintermüller, M. Hinze, C. Kahle, T. Kiel: *A goal-oriented dual-weighted adaptive finite element approach for the optimal control of a nonsmooth Cahn–Hilliard–Navier–Stokes system.* WIAS Preprint No. 2311, Berlin 2016, pp. 1-27.

[21] M. Hintermüller, T. Kiel, D. Wegner: *Optimal control of a semidiscrete Cahn–Hilliard–Navier–Stokes system with non-matched fluid densities*. SIAM J. Control Optim. **55** (2017), 1954-1989.

[22] M. Hintermüller, D. Wegner: *Distributed optimal control of the Cahn–Hilliard system including the case of a double-obstacle homogeneous free energy density*. SIAM J. Control Optim. **50** (2012), 388-418.

[23] M. Hintermüller, D. Wegner: *Optimal control of a semidiscrete Cahn–Hilliard–Navier–Stokes system*. SIAM J. Control Optim. **52** (2014), 747-772.

[24] M. Hintermüller, D. Wegner: *Distributed and boundary control problems for the semidiscrete Cahn–Hilliard/Navier–Stokes system with nonsmooth Ginzburg–Landau energies*. Isaac Newton Institute Preprint Series No. NI14042-FRB (2014), pp. 1-29.

[25] C. Kudla, A.T. Blumenau, F. Büllesfeld, N. Dropka, C. Frank-Rotsch, F. Kiessling, O. Klein, P. Lange, W. Miller, U. Rehse, U. Sahr, M. Schellhorn, G. Weidemann, M. Ziem, G. Bethin, R. Fornari, M. Müller, J. Sprekels, V. Trautmann, P. Rudolph: *Crystallization of 640 kg mc-silicon ingots under traveling magnetic field by using a heater-magnet module.* J. Crystal Growth **365** (2013), 54-58.

[26] E. Rocca, J. Sprekels: *Optimal distributed control of a nonlocal convective Cahn–Hilliard equation by the velocity in three dimensions*. SIAM J. Control Optim. **53** (2015), 1654-1680.

[27] J. Simon: *Compact sets in the space* $L^p(0,T;B)$. Ann. Mat. Pura Appl. (4) **146** (1987), 65-96.

[28] T. Tachim Medjo: *Optimal control of a Cahn–Hilliard–Navier–Stokes model with state constraints.* J. Convex Anal. **22** (2015), 1135-1172.

[29] Q.-F. Wang, S.-i. Nakagiri: *Weak solutions of Cahn–Hilliard equations having forcing terms and optimal control problems*. Mathematical models in functional equations (Japanese) (Kyoto, 1999), Sūrikaisekikenkyūsho Kōkyūroku No. 1128 (2000), 172–180.

[30] X.P. Zhao, C.C. Liu: *Optimal control of the convective Cahn–Hilliard equation*. Appl. Anal. **92** (2013), 1028-1045.

[31] X.P. Zhao, C.C. Liu: *Optimal control of the convective Cahn–Hilliard equation in 2D case*. Appl. Math. Optim. **70** (2014), 61-82.

[32] J. Zheng: *Time optimal controls of the Cahn–Hilliard equation with internal control.* Optimal Control Appl. Methods **36** (2015), 566-582.

[33] J. Zheng, Y. Wang: *Optimal control problem for Cahn–Hilliard equations with state constraint*. J. Dyn. Control Syst. **21** (2015), 257-272.