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Abstract

In this paper, we investigate a distributed optimal control problem for a convective viscous Cahn–Hilliard system with dynamic boundary conditions. Such systems govern phase separation processes between two phases taking place in an incompressible fluid in a container and, at the same time, on the container boundary. The cost functional is of standard tracking type, while the control is exerted by the velocity of the fluid in the bulk. In this way, the coupling between the state (given by the associated order parameter and chemical potential) and control variables in the governing system of nonlinear partial differential equations is bilinear, which presents an additional difficulty for the analysis. The nonlinearities in the bulk and surface free energies are of logarithmic type, which entails that the thermodynamic forces driving the phase separation process may become singular. We show existence for the optimal control problem under investigation, prove the Fréchet differentiability of the associated control-to-state mapping in suitable Banach spaces, and derive the first-order necessary optimality conditions in terms of a variational inequality and the associated adjoint system. Due to the strong nonlinear couplings between state variables and control, the corresponding proofs require a considerable analytical effort.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ denote some open, bounded and connected set having a smooth boundary Γ and unit outward normal ν. We denote by ∂_ν, ∇_Γ, Δ_Γ the outward normal derivative, the tangential gradient, and the Laplace–Beltrami operator on Γ, in this order. Moreover, we fix some final time $T > 0$ and introduce for every $t \in (0,T]$ the sets $Q_t := \Omega \times (0,t)$ and $\Sigma_t := \Gamma \times (0,t)$, where we put, for the sake of brevity, $Q := Q_T$ and $\Sigma := \Sigma_T$. We then consider the following optimal control problem:

(CP) Minimize the cost functional

$$\begin{aligned}\mathcal{J}(\mu,\mu_\Gamma,\rho,\rho_\Gamma,u) &:= \frac{\beta_1}{2}\int_Q |\mu-\widehat{\mu}_Q|^2 + \frac{\beta_2}{2}\int_\Sigma |\mu_\Gamma-\widehat{\mu}_\Sigma|^2 \\ &+ \frac{\beta_3}{2}\int_Q |\rho-\widehat{\rho}_Q|^2 + \frac{\beta_4}{2}\int_\Sigma |\rho-\widehat{\rho}_\Sigma|^2 \\ &+ \frac{\beta_5}{2}\int_\Omega |\rho(T)-\widehat{\rho}_\Omega|^2 + \frac{\beta_6}{2}\int_\Gamma |\rho_\Gamma(T)-\widehat{\rho}_\Gamma|^2 + \frac{\beta_7}{2}\int_Q |u|^2\,, \end{aligned} \tag{1.1}$$

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subject to the state system

$$\partial_t\rho + \nabla\rho\cdot u - \Delta\mu = 0 \quad \text{in } Q\,, \tag{1.2}$$

$$\tau_\Omega\partial_t\rho - \Delta\rho + f'(\rho) = \mu \quad \text{in } Q\,, \tag{1.3}$$

$$\partial_t\rho_\Gamma + \partial_\nu\mu - \Delta_\Gamma\mu_\Gamma = 0 \quad \text{and} \quad \mu_{|\Sigma} = \mu_\Gamma \quad \text{on } \Sigma\,, \tag{1.4}$$

$$\tau_\Gamma\partial_t\rho_\Gamma + \partial_\nu\rho - \Delta_\Gamma\rho_\Gamma + f_\Gamma'(\rho_\Gamma) = \mu_\Gamma \quad \text{and} \quad \rho_{|\Sigma} = \rho_\Gamma \quad \text{on } \Sigma\,, \tag{1.5}$$

$$\rho(0) = \rho_0 \quad \text{in } \Omega, \quad \rho_\Gamma(0) = \rho_{0|\Gamma} \quad \text{on } \Gamma\,, \tag{1.6}$$

and to the control constraint

$$u \in \mathcal{U}_{ad}\,, \tag{1.7}$$

where $\mathcal{U}_{ad}$ is a suitable closed, convex, and bounded subset of the control space $\mathcal{X}$ defined by

$$\mathcal{X} := L^2(0,T;Z) \cap (L^\infty(Q))^3 \cap (H^1(0,T;L^3(\Omega)))^3, \tag{1.8}$$

where

$$Z := \{w \in (L^2(\Omega))^3 : \ \operatorname{div} w = 0 \text{ in } \Omega \ \text{ and } \ w\cdot\nu = 0 \text{ on } \Gamma\}. \tag{1.9}$$

In (1.1), the constants β_i, $1 \le i \le 7$, are nonnegative but not all zero, and $\widehat{\mu}_Q$, $\widehat{\mu}_\Sigma$, $\widehat{\rho}_Q$, $\widehat{\rho}_\Sigma$, $\widehat{\rho}_\Omega$, and $\widehat{\rho}_\Gamma$, are given target functions. We note that the state system (1.2)–(1.6) can be seen as a phase field model for a phase separation process taking place in an incompressible fluid in the container Ω and on the container boundary Γ. In this connection, the variables (μ,μ_Γ) and (ρ,ρ_Γ) stand for the chemical potential and the order parameter (usually the density of one of the involved phases, normalized in such a way as to attain its values in the interval $[-1,1]$) of the phase separation process in the bulk and on the surface, respectively. It is worth noting that the total mass of the order parameter is conserved during the separation process; indeed, integrating (1.2) for fixed $t \in (0,T]$ over Ω, and using the condition $u(t) \in Z$ and (1.4), we readily find that

$$\partial_t\Bigl(\int_\Omega \rho(t) + \int_\Gamma \rho_\Gamma(t)\Bigr) = 0\,. \tag{1.10}$$

We also assume that the densities of the local free bulk energy f and the local free surface energy f_Γ are of logarithmic type, where the latter dominates the former in a sense to be made precise later. In the simplest case, we have

$$f(r) \simeq f_\Gamma(r) \simeq \widehat{c}_1((1+r)\,\ln(1+r) + (1-r)\ln(1-r)) - \widehat{c}_2 r^2, \quad r \in (-1,1), \tag{1.11}$$

with constants (not necessarily the same) $\widehat{c}_1 > 0$ and $\widehat{c}_2 > 0$ such that both f and f_Γ are nonconvex. Notice that the derivatives f' and f_Γ' are singular at the endpoints $r = \pm 1$.

While there are numerous contributions (which cannot be cited here) in the literature that address the questions of well-posedness and asymptotic behavior for various types (viscous or nonviscous, local or nonlocal, zero Neumann boundary conditions or dynamic boundary conditions) of Cahn–Hilliard systems, there are still but a few papers dealing with the associated optimal control problems. In this connection, we refer to [6, 8, 11, 18, 25, 28] for the case of Dirichlet or zero Neumann boundary conditions and to [1, 2, 7, 9, 10, 15] for the case of dynamic boundary conditions.

Recently, a rigorous analysis for convective Cahn–Hilliard systems has been given in [26] for the one-dimensional and in [27] for the two-dimensional case. In [14], the distributed optimal control of a two-dimensional Cahn–Hilliard/Navier–Stokes system was analyzed. We also mention the papers

[16, 17, 19, 20], which deal with the optimal control of three-dimensional Cahn–Hilliard/Navier–Stokes systems, however in the time-discretized version.

A distinguishing feature of this paper is that we use the fluid velocity as the control variable in the convective Cahn–Hilliard system. In practice, this can be realized by placing either a mechanical stirring device or an ultrasound emitter into the container. Another option is, in the case of electrically conducting fluids like molten metals, to make use of magnetic fields (for such an application, see [21]). To the authors' best knowledge, the only existing mathematical contribution, in which the fluid velocity is used as the control in a convective Cahn–Hilliard system in three dimensions of space, is the recent contribution [23]. In comparison with the situation investigated in [23], the main novelties of our paper are the following: while in [23] a nonlocal convective Cahn–Hilliard system with a possibly degenerating mobility and zero Neumann boundary conditions was studied, we consider here a viscous local Cahn–Hilliard system with constant mobility (normalized to unity) and the more difficult dynamic boundary conditions. In the recent paper [12], rather general and strong well-posedness results for this situtation have been established (we also like to quote the contributions [3, 4] for the nonconvective case).

In our analysis, we will take advantage of the results shown in [12]. It turns out that the bilinear coupling between control and state makes it necessary to allow only controls u which, among other constraints, have to obey the somewhat unusual regularity condition $u \in H^1(0,T;L^3(\Omega)^3)$. But, as a matter of fact, this is exactly the kind of regularity that guarantees the existence of a unique solution to the state system having sufficient regularity properties. Under these premises, we will be able to show the Fréchet differentiability of the control-to-state operator in suitable Banach spaces. Finally, we can prove the existence of an optimal control and, in a slightly less general setting, we also derive proper first-order necessary conditions for optimality.

The paper is organized as follows: in the following Section 2, we state the general assumptions for our problem, and we collect known results for the state system (1.2)–(1.6). Section 3 brings an analysis of the differentiability properties of the control-to-state mapping, while in Section 4 we prove existence and the first-order necessary optimality conditions for the control problem.

Throughout this paper, we will denote for a general Banach space X by $\|\cdot\|_X$ its norm and by X^* its dual space. Moreover, $\langle\cdot,\cdot\rangle_X$ denotes the dual pairing between X^* and X. The only exception from this convention for the norms is given by the spaces L^p constructed on Ω, Γ, Q, Σ and their powers, for $1 \le p \le \infty$, whose norms will be denoted by $\|\cdot\|_p$. We will also repeatedly use Young's inequality

$$a\,b \le \delta\,|a|^2 + \tfrac{1}{4\delta}\,|b|^2 \quad \text{for all } a,b \in \mathbb{R} \quad \text{and} \quad \delta > 0, \tag{1.12}$$

as well as the continuity of the embeddings $H^1(\Omega) \subset L^p(\Omega)$ for $1 \le p \le 6$ and $H^2(\Omega) \subset C^0(\overline{\Omega})$. Notice that the latter embedding is also compact, while this holds true for the former only if $p < 6$.

2 General assumptions and the state system

In this section, we introduce the general setting of our control problem and state some known results on the state system (1.2)–(1.6). To begin with, we introduce the spaces

$$H := L^2(\Omega)\,, \quad V := H^1(\Omega) \quad \text{and} \quad W := H^2(\Omega), \tag{2.1}$$

$$H_\Gamma := L^2(\Gamma)\,, \quad V_\Gamma := H^1(\Gamma) \quad \text{and} \quad W_\Gamma := H^2(\Gamma), \tag{2.2}$$

$$\mathcal{H} := H \times H_\Gamma\,, \quad \mathcal{V} := \{(v,v_\Gamma) \in V \times V_\Gamma :\ v_\Gamma = v_{|\Gamma}\} \quad \text{and} \quad \mathcal{W} := \big(W \times W_\Gamma\big) \cap \mathcal{V}\,. \tag{2.3}$$

Moreover, we recall the definition (1.8) of $\mathcal{X}$.

We make the following assumptions on the data of our problem:

(A1) $(\rho_0, \rho_{0|\Gamma}) \in \mathcal{W}$, and we have $-1 < \rho_0(x) < 1$ for all $x \in \overline{\Omega}$.

(A2) $\tau_\Omega > 0$ and $\tau_\Gamma > 0$.

(A3) $f, f_\Gamma \in C^3(-1,1)$ can be written as $f = f_1 + f_2$ and $f_\Gamma = f_{\Gamma 1} + f_{\Gamma 2}$, where $f_2, f_{\Gamma 2} \in C^3[-1,1]$ and

$$\exists\, \gamma_1 > 0,\, \gamma_2 > 0 : \quad |f_1'(r)| \le \gamma_1 |f_{\Gamma 1}'(r)| + \gamma_2 \quad \forall\, r \in (-1,1), \tag{2.4}$$

$$\lim_{r \searrow -1} f_1'(r) = \lim_{r \searrow -1} f_{\Gamma 1}'(r) = -\infty, \qquad \lim_{r \nearrow +1} f_1'(r) = \lim_{r \nearrow +1} f_{\Gamma 1}'(r) = +\infty. \tag{2.5}$$

(A4) The constants β_i, $1 \le i \le 7$, are all nonnegative but not all equal to zero, and it holds $\widehat{\rho}_Q, \widehat{\mu}_Q \in L^2(Q)$, $\widehat{\rho}_\Sigma, \widehat{\mu}_\Sigma \in L^2(\Sigma)$, $\widehat{\rho}_\Omega \in L^2(\Omega)$, and $\widehat{\rho}_\Gamma \in L^2(\Gamma)$.

(A5) The function $\overline{U} \in L^\infty(Q)$ and the constant $R_0 > 0$ make the admissible set

$$\mathcal{U}_{ad} := \bigl\{ u \in \mathcal{X} :\ |u| \le \overline{U} \text{ a.e. in } Q,\ \|u\|_{\mathcal{X}} \le R_0 \bigr\} \tag{2.6}$$

nonempty.

For the following analysis, it is convenient to fix once and for all some open ball in $\mathcal{X}$ that contains $\mathcal{U}_{ad}$. We therefore assume:

(A6) Let $R > 0$ be fixed such that $\mathcal{U}_{ad} \subset \mathcal{U}_R := \{ u \in \mathcal{X} :\ \|u\|_{\mathcal{X}} < R \}$.

Remark 2.1. The condition (2.4) means, loosely speaking, that the thermodynamic force on the boundary (represented by f_Γ') grows faster than the thermodynamic force in the bulk (represented by f'). Moreover, it is easily seen that (A3) is fulfilled for, e.g., the logarithmic case (1.11).

Remark 2.2. We point out that $\mathcal{U}_{ad}$ actually is a closed, convex, and bounded subset of $\mathcal{X}$. However, it is closed in other spaces as well. For the reader's convenience, we spend some words on this point. For $w \in (L^2(\Omega))^3$ with $\operatorname{div} w \in L^2(\Omega)$, the trace $(w \cdot \nu)_{|\Gamma}$ is a well-defined element of $H^{-1/2}(\Gamma)$ (in particular, the definitions (1.8) and (1.9) of $\mathcal{X}$ and Z are meaningful). Moreover, the usual integration-by-parts formula holds true in a generalized form for $w \in (L^2(\Omega))^3$ with $\operatorname{div} w \in L^2(\Omega)$ and $v \in H^1(\Omega)$. Namely, we have that $\int_\Omega w \cdot \nabla v = -\int_\Omega (\operatorname{div} w) v + \langle (w \cdot \nu)_{|\Gamma}, v_{|\Gamma} \rangle$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. In particular, requiring that an element $w \in (L^2(\Omega))^3$ belongs to Z (i.e., it satisfies the conditions $\operatorname{div} w = 0$ in Ω and $w \cdot \nu = 0$ on Γ) is the same as requiring that $\int_\Omega w \cdot \nabla v = 0$ for every $v \in H^1(\Omega)$. Therefore, the whole space $\mathcal{X}$ can be redefined as the space of $u \in (L^\infty(Q) \cap H^1(0,T;L^2(\Omega)))^3$ such that $\int_Q u \cdot \nabla v = 0$ for every $v \in L^2(0,T;H^1(\Omega))$. It follows that $\mathcal{X}$ is a closed subspace of the Banach space $\widetilde{\mathcal{X}} := (L^\infty(Q) \cap H^1(0,T;L^2(\Omega)))^3$ (as well as of each of the spaces $(L^2(Q))^3$ and $(H^1(0,T;L^2(\Omega)))^3$) and $\mathcal{U}_{ad}$ is a closed subset of $\widetilde{\mathcal{X}}$. We also notice that, by the above integration-by-parts formula and the assumptions on u, we can write the convective term in the next variational formulation as it is presented in (2.7) (i.e., the third integral, to be compared with the second term of (1.2)).

We now quote some results for the state system (1.2)–(1.6) that have recently been proved in [12]. Prior to this, we notice that the variational form of (1.2)–(1.6) reads as follows: find functions $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma))$ such that

$$\int_\Omega \partial_t \rho \, v + \int_\Gamma \partial_t \rho_\Gamma \, v_\Gamma - \int_\Omega \rho u \cdot \nabla v + \int_\Omega \nabla \mu \cdot \nabla v + \int_\Gamma \nabla_\Gamma \mu_\Gamma \cdot \nabla_\Gamma v_\Gamma = 0$$
$$\text{a.e. in } (0,T) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}, \tag{2.7}$$

$$\tau_\Omega \int_\Omega \partial_t \rho \, v + \tau_\Gamma \int_\Gamma \partial_t \rho_\Gamma \, v_\Gamma + \int_\Omega \nabla \rho \cdot \nabla v + \int_\Gamma \nabla_\Gamma \rho_\Gamma \cdot \nabla_\Gamma v_\Gamma$$
$$+ \int_\Omega f'(\rho) v + \int_\Gamma f'_\Gamma(\rho_\Gamma) v_\Gamma = \int_\Omega \mu v + \int_\Gamma \mu_\Gamma v_\Gamma$$
$$\text{a.e. in } (0,T) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}, \tag{2.8}$$

$$\rho(0) = \rho_0 \quad \text{a.e. in } \Omega. \tag{2.9}$$

The following result is a combination of the Theorems 2.6, 2.7 and 2.9 in [12].

Theorem 2.3. *Suppose that the assumptions* (A1)–(A3) *and* (A6) *hold true. Then the state system* (2.7)–(2.9) *has for every* $u \in \mathcal{U}_R$ *a unique solution* $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma))$ *such that*

$$(\mu, \mu_\Gamma) \in L^\infty(0,T; \mathcal{W}), \quad (\rho, \rho_\Gamma) \in W^{1,\infty}(0,T; \mathcal{H}) \cap H^1(0,T; \mathcal{V}) \cap L^\infty(0,T; \mathcal{W}). \tag{2.10}$$

Moreover, there are constants $\rho_*, \rho^* \in (-1,1)$ *and* $K_1 > 0$, $K_2 > 0$, *which depend only on the data of the state system and* R, *such that the following holds true:*

(i) *Whenever* $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma))$ *is the solution to the state system associated with some* $u \in \mathcal{U}_R$, *then we have*

$$\rho_* \le \rho(x,t) \le \rho^* \quad \forall\, (x,t) \in \overline{Q}\,, \tag{2.11}$$

$$\|(\mu, \mu_\Gamma)\|_{L^\infty(0,T;\mathcal{W})} \,+\, \|(\rho, \rho_\Gamma)\|_{W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{W})} \,\le\, K_1\,. \tag{2.12}$$

(ii) *Whenever* $u^1, u^2 \in \mathcal{U}_R$ *are given and* $((\mu^i, \mu^i_\Gamma), (\rho^i, \rho^i_\Gamma))$, $i = 1,2$, *are the solutions to the corresponding state systems, then*

$$\begin{aligned} &\|(\mu^1 - \mu^2, \mu^1_\Gamma - \mu^2_\Gamma)\|_{L^\infty(0,T;\mathcal{W})} \\ &+\, \|(\rho^1 - \rho^2, \rho^1_\Gamma - \rho^2_\Gamma)\|_{W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{W})} \\ &\le\, K_2\, \|u^1 - u^2\|_{H^1(0,T;L^3(\Omega)^3)}\,. \end{aligned} \tag{2.13}$$

Remark 2.4. Notice that the pointwise condition (2.11) is meaningful, since it follows from [24, Sect. 8, Cor. 4] and (2.10) that $\rho \in C^0(\overline{Q})$ (and thus, in particular, that $\rho_\Gamma \in C^0(\overline{\Sigma})$).

We point out that the uniform separation property (2.11) also ensures that the possible singularity encoded in the condition (2.5) never becomes active. This implies, in particular, that we may without loss of generality assume that

$$\max_{1 \le j \le 3} \left(\|f^{(j)}(\rho)\|_{C^0(\overline{Q})} \,+\, \|f^{(j)}_\Gamma(\rho_\Gamma)\|_{C^0(\overline{\Sigma})} \right) \,\le\, K_1\,, \tag{2.14}$$

whenever (ρ, ρ_Γ) is the second component pair of a solution to the state system associated with some $u \in \mathcal{U}_R$.

Remark 2.5. By virtue of the well-posedness result given by Theorem 2.3, the control-to-state operator $\mathcal{S} : u \mapsto ((\mu, \mu_\Gamma), (\rho, \rho_\Gamma))$ is well defined as a mapping between $\mathcal{U}_R \subset \mathcal{X}$ and the space defined by the regularity stated in (2.10). Moreover, it is Lipschitz continuous as a mapping from $\mathcal{U}_R$ into the space

$$L^\infty(0,T;\mathcal{W}) \times (W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{W})).$$

3 Fréchet differentiability of the control-to-state operator

In this section, we aim to show the Fréchet differentiability of the control-to-state operator $\mathcal{S}$ in suitable Banach spaces. Throughout this section, we assume that $\overline{u} \in \mathcal{U}_R$ is fixed and that the general assumptions (A1)–(A3), (A5) and (A6) are satisfied, so that the global estimates (2.12) and (2.14) are valid for the associated solution $(\overline{\mu}, \overline{\mu}_\Gamma, \overline{\rho}, \overline{\rho}_\Gamma) = \mathcal{S}(\overline{u})$ to the state system. We then consider the linearized system, where $h \in \mathcal{X}$,

$$\begin{aligned} &\int_\Omega \partial_t \xi \, v + \int_\Gamma \partial_t \xi_\Gamma \, v_\Gamma + \int_\Omega \nabla \eta \cdot \nabla v + \int_\Gamma \nabla_\Gamma \eta_\Gamma \cdot \nabla_\Gamma v_\Gamma \\ &\quad - \int_\Omega \xi \overline{u} \cdot \nabla v - \int_\Omega \overline{\rho}\, h \cdot \nabla v = 0 \\ &\quad \text{a.e. in } (0,T) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\tau_\Omega \int_\Omega \partial_t \xi \, v + \tau_\Gamma \int_\Gamma \partial_t \xi_\Gamma \, v_\Gamma + \int_\Omega \nabla \xi \cdot \nabla v + \int_\Gamma \nabla_\Gamma \xi_\Gamma \cdot \nabla_\Gamma v_\Gamma \\ &\quad + \int_\Omega f''(\overline{\rho}) \xi v + \int_\Gamma f''_\Gamma(\overline{\rho}_\Gamma) \xi_\Gamma v_\Gamma = \int_\Omega \eta v + \int_\Gamma \eta_\Gamma v_\Gamma \\ &\quad \text{a.e. in } (0,T) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}, \end{aligned} \tag{3.2}$$

$$\xi(0) = 0 \quad \text{a.e. in } \Omega, \quad \xi_\Gamma(0) = 0 \quad \text{a.e. on } \Gamma. \tag{3.3}$$

Since $\overline{u} \in \mathcal{U}_R \subset \mathcal{X}$ and $h \in \mathcal{X}$, then $\operatorname{div} \overline{u} = 0$ and $\overline{u} \cdot \nu = 0$, and the same conditions hold for h, so that (3.1)–(3.3) is the weak form of the linear initial-boundary value problem

$$\partial_t \xi - \Delta \eta = -\nabla \xi \cdot \overline{u} - \nabla \overline{\rho} \cdot h \quad \text{a.e. in } Q, \tag{3.4}$$

$$\partial_t \xi_\Gamma + \partial_\nu \eta - \Delta_\Gamma \eta_\Gamma = 0 \quad \text{and} \quad \eta_{|\Sigma} = \eta_\Gamma, \quad \text{a.e. on } \Sigma, \tag{3.5}$$

$$\tau_\Omega \partial_t \xi - \Delta \xi + f''(\overline{\rho}) \xi = \eta \quad \text{a.e. in } Q, \tag{3.6}$$

$$\tau_\Gamma \partial_t \xi_\Gamma + \partial_\nu \xi - \Delta_\Gamma \xi_\Gamma + f''_\Gamma(\overline{\rho}_\Gamma) \xi_\Gamma = \eta_\Gamma \quad \text{and} \quad \xi_{|\Sigma} = \xi_\Gamma, \quad \text{a.e. on } \Sigma, \tag{3.7}$$

$$\xi(0) = 0 \quad \text{a.e. in } \Omega, \qquad \xi_\Gamma(0) = 0 \quad \text{a.e. on } \Gamma. \tag{3.8}$$

However, we only refer to the problem in the form (3.1)–(3.3).

We expect the following to hold true: if the system (3.1)–(3.3) admits for every $h \in \mathcal{X}$ a unique solution $((\eta, \eta_\Gamma), (\xi, \xi_\Gamma)) =: ((\eta^h, \eta^h_\Gamma), (\xi^h, \xi^h_\Gamma))$ in a suitable Banach space, then the Fréchet derivative $D\mathcal{S}(\overline{u})$ of $\mathcal{S}$ at $\overline{u}$ (if it exists), evaluated at h, should have the form $D\mathcal{S}(\overline{u})(h) = ((\eta^h, \eta^h_\Gamma), (\xi^h, \xi^h_\Gamma))$.

Theorem 3.1. *Suppose that the assumptions* (A1)–(A3), (A5) *and* (A6) *are fulfilled, let* $\overline{u} \in \mathcal{U}_R$ *be given, and let* $((\overline{\mu}, \overline{\mu}_\Gamma), (\overline{\rho}, \overline{\rho}_\Gamma)) = \mathcal{S}(\overline{u})$ *be the associated unique solution to the state system* (1.2)–(1.6) *having the regularity properties stated in* (2.10). *Then the system* (3.1)–(3.3) *has for every* $h \in \mathcal{X}$

a unique solution $((\eta, \eta_\Gamma), (\xi, \xi_\Gamma))$ *such that*

$$(\eta, \eta_\Gamma) \in L^2(0,T;\mathcal{W}), \quad (\xi, \xi_\Gamma) \in H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{V}) \cap L^2(0,T;\mathcal{W}). \tag{3.9}$$

Moreover, the linear mapping $h \mapsto ((\eta, \eta_\Gamma), (\xi, \xi_\Gamma))$ *is continuous as a mapping from* $\mathcal{X}$ *into the space*

$$\mathcal{Y} := L^2(0,T;\mathcal{V}) \times \left(H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{V})\right). \tag{3.10}$$

PROOF: We employ a slightly modified Faedo-Galerkin scheme with a proper choice of the Hilbert basis. To this end, we introduce the operator $\mathcal{A} \in \mathcal{L}(\mathcal{V};\mathcal{V}^*)$ by setting

$$\langle \mathcal{A}(w, w_\Gamma), (v, v_\Gamma) \rangle_\mathcal{V} := \int_\Omega \nabla w \cdot \nabla v + \int_\Gamma \nabla_\Gamma w_\Gamma \cdot \nabla_\Gamma v_\Gamma \quad \text{for } (w, w_\Gamma), (v, v_\Gamma) \in \mathcal{V}, \tag{3.11}$$

and notice that $\mathcal{A}$ is nonnegative and weakly coercive. Indeed, we have that

$$\langle \mathcal{A}(v, v_\Gamma), (v, v_\Gamma) \rangle_\mathcal{V} + \|(v, v_\Gamma)\|_\mathcal{H}^2 = \|(v, v_\Gamma)\|_\mathcal{V}^2 \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}. \tag{3.12}$$

Moreover, as the embedding $\mathcal{V} \subset \mathcal{H}$ is compact, the resolvent of $\mathcal{A}$ is compact as well, and the spectrum of $\mathcal{A}$ reduces to a discrete set of eigenvalues, the eigenvalue problem being

$$(e, e_\Gamma) \in \mathcal{V} \setminus \{(0,0)\} \quad \text{and} \quad \mathcal{A}(e, e_\Gamma) = \lambda (e, e_\Gamma). \tag{3.13}$$

More precisely, we can rearrange the eigenvalues and choose the eigenvectors in such a way that

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \dots \quad \text{and} \quad \lim_{j\to\infty} \lambda_j = +\infty, \tag{3.14}$$

$$\mathcal{A}(e^j, e^j_\Gamma) = \lambda_j (e^j, e^j_\Gamma) \quad \text{and}$$

$$\int_\Omega e^i e^j + \int_\Gamma e^i_\Gamma e^j_\Gamma = \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases} \quad \text{for } i,j = 1,2,\dots, \tag{3.15}$$

and $\{(e^j, e^j_\Gamma)\}$ generates a dense subspace of both $\mathcal{V}$ and $\mathcal{H}$. We notice that

$$\int_\Omega \nabla e^i \cdot \nabla e^j + \int_\Gamma \nabla_\Gamma e^i_\Gamma \cdot \nabla_\Gamma e^j_\Gamma = \lambda_i \Big(\int_\Omega e^i e^j + \int_\Gamma e^i_\Gamma e^j_\Gamma \Big) = \lambda_i \delta_{ij} \quad \text{for } i,j = 1,2,\dots.$$

We also observe that every element $(w, w_\Gamma) \in \mathcal{H}$ can be written as

$$(w, w_\Gamma) = \sum_{j=1}^\infty w_j (e^j, e^j_\Gamma) \quad \text{with} \quad \sum_{j=1}^\infty |w_j|^2 = \|(w, w_\Gamma)\|_\mathcal{H}^2 < +\infty,$$

and that (on account of (3.12))

$$(w, w_\Gamma) \in \mathcal{V} \quad \text{if and only if} \quad \sum_{j=1}^\infty (1 + \lambda_j) |w_j|^2 < +\infty.$$

Namely, the last sum yields the square of a norm on $\mathcal{V}$ that is equivalent to $\|\cdot\|_\mathcal{V}$.

At this point, we set

$$\mathcal{V}_n := \operatorname{span}\{(e^j, e^j_\Gamma) : \ 1 \le j \le n\} \quad \text{and} \quad \mathcal{V}_\infty := \bigcup_{j=1}^\infty \mathcal{V}_n = \operatorname{span}\{(e^j, e^j_\Gamma) : \ j \ge 1\}, \tag{3.16}$$

and, for every $n \geq 1$, we look for a quadruple $(\eta^n, \eta^n_\Gamma, \xi^n, \xi^n_\Gamma)$ satisfying

$$(\eta^n, \eta^n_\Gamma) \in L^2(0,T;\mathcal{V}_n) \quad \text{and} \quad (\xi^n, \xi^n_\Gamma) \in H^1(0,T;\mathcal{V}_n), \tag{3.17}$$

$$\int_\Omega \partial_t \xi^n \, v + \int_\Gamma \partial_t \xi^n_\Gamma \, v_\Gamma - \int_\Omega \xi^n \overline{u} \cdot \nabla v - \int_\Omega \overline{\rho} h \cdot \nabla v + \int_\Omega \nabla \eta^n \cdot \nabla v + \int_\Gamma \nabla \eta^n_\Gamma \cdot \nabla v_\Gamma$$
$$+ \frac{1}{n} \int_\Omega \eta^n v + \frac{1}{n} \int_\Gamma \eta^n_\Gamma v_\Gamma = 0 \quad \text{a.e. in } (0,T) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}_n, \tag{3.18}$$

$$\tau_\Omega \int_\Omega \partial_t \xi^n \, v + \tau_\Gamma \int_\Gamma \partial_t \xi^n_\Gamma \, v_\Gamma + \int_\Omega \nabla \xi^n \cdot \nabla v + \int_\Gamma \nabla_\Gamma \xi^n_\Gamma \cdot \nabla_\Gamma v_\Gamma$$
$$+ \int_\Omega f''(\overline{\rho}) \xi^n v + \int_\Gamma f''_\Gamma(\overline{\rho}_\Gamma) \xi^n_\Gamma v_\Gamma = \int_\Omega \eta^n v + \int_\Gamma \eta^n_\Gamma v_\Gamma$$
$$\text{a.e. in } (0,T) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}_n, \tag{3.19}$$

$$\xi^n(0) = 0 \quad \text{a.e. in } \Omega, \qquad \xi^n_\Gamma(0) = 0 \quad \text{a.e. on } \Gamma. \tag{3.20}$$

Existence for the discrete problem. For every fixed $n \in \mathbb{N}$, we are looking for (η^n, η^n_Γ) and (ξ^n, ξ^n_Γ) in the form

$$(\eta^n, \eta^n_\Gamma)(t) = \sum_{j=1}^n \eta^n_j(t)(e^j, e^j_\Gamma) \quad \text{and} \quad (\xi^n, \xi^n_\Gamma)(t) = \sum_{j=1}^n \xi^n_j(t)(e^j, e^j_\Gamma),$$

for some $\eta^n_j \in L^2(0,T)$ and $\xi^n_j \in H^1(0,T)$, $1 \leq j \leq n$. Let us introduce the n-vector functions $\overline{\eta} := (\eta^n_1, \ldots, \eta^n_n)$ and $\overline{\xi} := (\xi^n_1, \ldots, \xi^n_n)$. Then, making the special choices $(v, v_\Gamma) = (e^i, e^i_\Gamma)$ for $i = 1, \ldots, n$, we can rewrite the system (3.18)–(3.19) in the form

$$\overline{\xi}\,'(t) - U(t)\,\overline{\xi}(t) + D_n\,\overline{\eta}(t) = C(t)$$
$$\text{and} \quad B\,\overline{\xi}\,'(t) + D\,\overline{\xi}(t) + G(t)\,\overline{\xi}(t) = \overline{\eta}(t), \tag{3.21}$$

where $D_n := \operatorname{diag}(\lambda_1 + \frac{1}{n}, \ldots, \lambda_n + \frac{1}{n})$, $D := \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and where the matrices $U = (u_{ij}) \in L^2(0,T;\mathbb{R}^{n\times n})$, $G = (g_{ij}) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, and $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ and the vector $C = (c_i) \in L^2(0,T;\mathbb{R}^n)$ are given by

$$u_{ij}(t) := \int_\Omega e^j \overline{u}(t) \cdot \nabla e^i, \quad g_{ij}(t) := \int_\Omega f''(\overline{\rho})(t) e^j e^i + \int_\Gamma f''_\Gamma(\overline{\rho}_\Gamma)(t) e^j_\Gamma e^i_\Gamma,$$
$$b_{ij} := \tau_\Omega \int_\Omega e^j e^i + \tau_\Gamma \int_\Gamma e^j_\Gamma e^i_\Gamma, \quad c_i(t) := \int_\Omega \overline{\rho}(t)\, h(t) \cdot \nabla e^i, \quad \text{for } i,j = 1, \ldots, n.$$

By adding the second identity in (3.21) to the first one multiplied by D_n^{-1}, we obtain the equivalent system

$$(D_n^{-1} + B)\,\overline{\xi}\,'(t) + V(t)\,\overline{\xi}(t) = D_n^{-1} C(t) \quad \text{and} \quad \overline{\eta}(t) = B\,\overline{\xi}\,'(t) + D\,\overline{\xi}(t) + G(t)\,\overline{\xi}(t),$$

where $V := D + G - D_n^{-1} U$ belongs to $L^2(0,T;\mathbb{R}^{n\times n})$ and $D_n^{-1} + B$ is invertible, as we now verify. To this end, we show that B is positive definite. Indeed, for any vector $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, by setting $(v, v_\Gamma) := \sum_{j=1}^n y_j(e^j, e^j_\Gamma)$, we have that

$$(By)\cdot y = \sum_{i,j=1}^n b_{ij} y_j y_i \;=\; \tau_\Omega \int_\Omega \sum_{i=1}^n y_i e^i \sum_{j=1}^n y_j e^j + \tau_\Gamma \int_\Omega \sum_{i=1}^n y_i e^i_\Gamma \sum_{j=1}^n y_j e^j_\Gamma$$
$$= \tau_\Omega \int_\Omega |v|^2 + \tau_\Gamma \int_\Gamma |v_\Gamma|^2 \;\geq\; \min\{\tau_\Omega, \tau_\Gamma\} \, \|(v, v_\Gamma)\|^2_{\mathcal{H}} \;=\; \min\{\tau_\Omega, \tau_\Gamma\} \, \|y\|^2_{\mathbb{R}^n} .$$

Hence, $D_n^{-1} + B$ is positive definite as well, and thus invertible. Therefore, by virtue of standard results for initial value problems for ordinary differential equations, the discrete problem (3.17)–(3.20) has a unique solution having the asserted regularity.

At this point, our aim is to show that the solutions to the discrete problem converge to a solution to (3.1)–(3.3) as n tends to infinity, at least for a subsequence. To this end, we start estimating and find bounds that do not depend on n. In the following, C_i, $i \in \mathbb{N}$, will denote positive constants that may depend on the data of the system and on R, but not on $n \in \mathbb{N}$.

First a priori estimate. We test (3.18), written at the time s, by $(\eta^n, \eta^n_\Gamma)(s)$ and integrate over $(0,t)$ with respect to s to find that

$$\int_{Q_t} \partial_t \xi^n \, \eta^n + \int_{\Sigma_t} \partial_t \xi^n_\Gamma \, \eta^n_\Gamma + \int_{Q_t} |\nabla \eta^n|^2 + \int_{\Sigma_t} |\nabla_\Gamma \eta^n_\Gamma|^2$$
$$+ \frac{1}{n} \int_{Q_t} |\eta^n|^2 + \frac{1}{n} \int_{\Sigma_t} |\eta^n_\Gamma|^2 = \int_{Q_t} \xi^n \overline{u} \cdot \nabla \eta^n + \int_{Q_t} \overline{\rho} \, h \cdot \nabla \eta^n .$$

Next, we test (3.19) by $\partial_t(\xi^n, \xi^n_\Gamma)(s)$, integrate over $(0,t)$ with respect to s, and add the expression $\int_{Q_t} \xi^n \partial_t \xi^n + \int_{\Sigma_t} \xi^n_\Gamma \partial_t \xi^n_\Gamma$ to both sides, for convenience. We infer that

$$\tau_\Omega \int_{Q_t} |\partial_t \xi^n|^2 + \tau_\Gamma \int_{\Sigma_t} |\partial_t \xi^n_\Gamma|^2 + \frac{1}{2} \, \|(\xi^n, \xi^n_\Gamma)(t)\|^2_{\mathcal{V}}$$
$$= \int_{Q_t} (1 - f''(\overline{\rho})) \, \xi^n \partial_t \xi^n + \int_{\Sigma_t} (1 - f''_\Gamma(\overline{\rho})) \, \xi^n_\Gamma \partial_t \xi^n_\Gamma + \int_{Q_t} \eta^n \partial_t \xi^n + \int_{\Sigma_t} \eta^n_\Gamma \partial_t \xi^n_\Gamma .$$

At this point, we add these equalities and notice that four terms cancel and that the remaining terms on the left-hand side are nonnegative. Moreover, we use the global estimates (2.12), (2.14), and Young's inequality. We obtain that

$$\int_{Q_t} |\nabla \eta^n|^2 + \int_{\Sigma_t} |\nabla_\Gamma \eta^n_\Gamma|^2 + \tau_\Omega \int_{Q_t} |\partial_t \xi^n|^2 + \tau_\Gamma \int_{\Sigma_t} |\partial_t \xi^n_\Gamma|^2 + \frac{1}{2} \, \|(\xi^n, \xi^n_\Gamma)(t)\|^2_{\mathcal{V}}$$
$$\leq \int_{Q_t} |\xi^n| \, |\overline{u}| \, |\nabla \eta^n| + \int_{Q_t} |\overline{\rho}| \, |h| \, |\nabla \eta^n| + \frac{\tau_\Omega}{2} \int_{Q_t} |\partial_t \xi^n|^2 + \frac{\tau_\Gamma}{2} \int_{\Sigma_t} |\partial_t \xi^n_\Gamma|^2$$
$$+ C_1 \int_{Q_t} |\xi^n|^2 + C_1 \int_{\Sigma_t} |\xi^n_\Gamma|^2 .$$

On the other hand, the Hölder, Sobolev and Young inequalities yield that

$$\int_{Q_t} |\xi^n| \, |\overline{u}| \, |\nabla \eta^n| \leq \int_0^t \|\xi^n(s)\|_6 \, \|\overline{u}(s)\|_3 \, \|\nabla \eta^n(s)\|_2 \, ds$$
$$\leq \frac{1}{4} \int_{Q_t} |\nabla \eta^n|^2 + C_2 \, \|\overline{u}\|^2_{L^\infty(0,T;L^3(\Omega))} \int_0^t \|\xi^n(s)\|^2_V \, ds .$$

Moreover, we have that

$$\int_{Q_t} |\overline{\rho}|\,|h|\,|\nabla\eta^n| \;\le\; \int_0^t \|\overline{\rho}(s)\|_\infty \|h(s)\|_2\,\|\nabla\eta^n(s)\|_2\,ds$$
$$\le\; \frac14 \int_{Q_t} |\nabla\eta^n|^2 + C_3 \int_{Q_t} |h|^2\,.$$

Therefore, rearranging and applying Gronwall's lemma, we can infer that for all $t\in(0,T]$ it holds

$$\left(\int_{Q_t} |\nabla\eta^n|^2 + \int_{\Sigma_t} |\nabla_\Gamma\eta^n_\Gamma|^2\right)^{1/2} + \|(\xi^n,\xi^n_\Gamma)\|_{H^1(0,t;\mathcal{H})\cap L^\infty(0,t;\mathcal{V})} \le C_4 \|h\|_{L^2(0,t;H)^3}\,. \tag{3.22}$$

Second a priori estimate. We insert $(v,v_\Gamma)=(\eta^n,\eta^n_\Gamma)$ in (3.19). As it turns out, all of the resulting terms can be handled directly by means of Young's inequality to yield an inequality of the form

$$\int_{Q_t} |\eta^n|^2 + \int_{\Sigma_t} |\eta^n_\Gamma|^2 \;\le\; C_5\Big(\|(\xi^n,\xi^n_\Gamma)\|^2_{H^1(0,T;\mathcal{H})\cap L^2(0,T;\mathcal{V})} + \int_{Q_t} |\nabla\eta^n|^2 + \int_{\Sigma_t} |\nabla_\Gamma\eta^n_\Gamma|^2\Big),$$

and it follows from (3.22) that

$$\|(\eta^n,\eta^n_\Gamma)\|_{L^2(0,t;\mathcal{V})} \;\le\; C_6\|h\|_{L^2(0,t;H)^3}\,. \tag{3.23}$$

Existence of a unique solution to the linearized system. We account for (3.22)–(3.23) and use standard weak and weak star compactness results, as well as [24, Sect. 8, Cor. 4]). It follows that, as n tends to infinity,

$$(\eta^n,\eta^n_\Gamma)\to(\eta,\eta_\Gamma) \qquad \text{weakly in } L^2(0,T;\mathcal{V}), \tag{3.24}$$
$$\tfrac1n(\eta^n,\eta^n_\Gamma)\to(0,0) \qquad \text{strongly in } L^2(0,T;\mathcal{V}), \tag{3.25}$$
$$(\xi^n,\xi^n_\Gamma)\to(\xi,\xi_\Gamma) \qquad \text{weakly star in } H^1(0,T;\mathcal{H})\cap L^\infty(0,T;\mathcal{V}) \text{ and strongly in } C^0([0,T];\mathcal{H}), \tag{3.26}$$

at least for a subsequence, which is again indexed by n. In particular, we have $\xi(0)=0$ and $\xi_\Gamma(0)=0$. We also recall that $\overline{u},f''(\overline{\rho})\in L^\infty(Q)$ and $f''_\Gamma(\overline{\rho}_\Gamma)\in L^\infty(\Sigma)$, which implies that

$$\xi^n\overline{u}\to\xi\overline{u} \quad \text{strongly in } L^2(0,T;H)^3\,,$$
$$f''(\overline{\rho})\xi^n\to f''(\overline{\rho})\xi \quad \text{strongly in } L^2(Q)\,,$$
$$f''_\Gamma(\overline{\rho}_\Gamma)\xi^n_\Gamma\to f''_\Gamma(\overline{\rho}_\Gamma)\xi_\Gamma \quad \text{strongly in } L^2(\Sigma)\,.$$

Now, we recall (3.16) for the definition of $\mathcal{V}_\infty$, and take an arbitrary $\mathcal{V}_\infty$-valued step function (v,v_Γ). Since the range of (v,v_Γ) is finite-dimensional, there exists some $m\in\mathbb{N}$ such that $(v,v_\Gamma)(t)\in\mathcal{V}_m$ for a.a. $t\in(0,T)$. It follows that $(v,v_\Gamma)(t)\in\mathcal{V}_n$ for a.a. $t\in(0,T)$ and every $n\ge m$, so that we can test (3.18) and (3.19), written at the time t, by $(v,v_\Gamma)(t)$ and integrate over $(0,T)$. At this point, it is straightforward to deduce that (η,η_Γ) and (ξ,ξ_Γ) satisfy the integrated version of (3.1)–(3.3) for every such step function, namely, we have that

$$\int_Q \partial_t\xi\,v + \int_\Sigma \partial_t\xi_\Gamma\,v_\Gamma - \int_Q \xi\overline{u}\cdot\nabla v - \int_Q \overline{\rho}h\cdot\nabla v + \int_Q \nabla\eta\cdot\nabla v + \int_\Sigma \nabla\eta_\Gamma\cdot\nabla v_\Gamma = 0\,,$$

$$\tau_\Omega \int_Q \partial_t \xi \, v + \tau_\Gamma \int_\Sigma \partial_t \xi \, v_\Gamma + \int_Q \nabla \xi \cdot \nabla v + \int_\Sigma \nabla_\Gamma \xi_\Gamma \cdot \nabla_\Gamma v_\Gamma$$
$$+ \int_Q f''(\overline{\rho}) \xi v + \int_\Sigma f''_\Gamma(\overline{\rho}_\Gamma) \xi_\Gamma v_\Gamma = \int_Q \eta v + \int_\Sigma \eta_\Gamma v_\Gamma \, .$$

By density, the same equations hold true for every $(v, v_\Gamma) \in L^2(0,T;\mathcal{V})$. This implies that (3.1)–(3.2) hold true a.e. in $(0,T)$ and for every $(v, v_\Gamma) \in \mathcal{V}$, as desired. It is thus shown that $((\eta, \eta_\Gamma), (\xi, \xi_\Gamma))$ is a solution to the linearized system (3.1)–(3.3).

Next, we show that there can be no other such solution. To this end, assume that $((\eta^i, \eta^i_\Gamma), (\xi^i, \xi^i_\Gamma))$, $i = 1, 2$, are two solutions such that

$$(\eta^i, \eta^i_\Gamma) \in L^2(0,T;\mathcal{V}) \quad \text{and} \quad (\xi^i, \xi^i_\Gamma) \in H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{V}) \quad \text{for } i = 1, 2.$$

We put $(\eta, \eta_\Gamma) := (\eta^1, \eta^1_\Gamma) - (\eta^2, \eta^2_\Gamma)$ and $(\xi, \xi_\Gamma) := (\xi^1, \xi^1_\Gamma) - (\xi^2, \xi^2_\Gamma)$. Then $((\eta, \eta_\Gamma), (\xi, \xi_\Gamma))$ solves the system (3.1)–(3.3), where, in this case, the expression $\int_\Omega \overline{\rho} h \cdot \nabla v$ is not present. Now, we repeat the two a priori estimates performed above for the approximating system, but this time we proceed directly on the system (3.1)–(3.3). We then recover the estimates (3.22) and (3.23), but this time with zero right-hand sides. Hence, (η, η_Γ) and (ξ, ξ_Γ) vanish, which proves the uniqueness.

Further regularity. We still need to show that $(\xi, \xi_\Gamma) \in L^2(0,T;\mathcal{W})$. This is an immediate consequence of [12, Lem. 3.1]: indeed, we can write (3.2) in the form

$$\int_\Omega \nabla \xi \cdot \nabla v + \int_\Gamma \nabla_\Gamma \xi_\Gamma \cdot \nabla_\Gamma v_\Gamma = \int_\Omega g v + \int_\Gamma g_\Gamma v_\Gamma \quad \forall (v, v_\Gamma) \in \mathcal{V}, \tag{3.27}$$

for a.e. $t \in (0,T)$, where we define $(g, g_\Gamma) \in L^2(0,T;\mathcal{H})$ by

$$g := \eta - f''(\overline{\rho}) \xi - \tau_\Omega \partial_t \xi \, , \quad g_\Gamma := \eta_\Gamma - f''_\Gamma(\overline{\rho}_\Gamma) \xi_\Gamma - \tau_\Gamma \partial_t \xi_\Gamma \, .$$

Obviously, $(g(t), g_\Gamma(t)) \in \mathcal{H}$ for a.e. $t \in (0,T)$. It then follows from [12, Lem. 3.1] that, for a.e. $t \in (0,T)$, it holds $(\xi(t), \xi_\Gamma(t)) \in \mathcal{W}$, as well as

$$\|(\xi(t), \xi_\Gamma(t))\|_{\mathcal{W}} \le C_\Omega \left(\|(\xi(t), \xi_\Gamma(t))\|_{\mathcal{V}} + \|(g(t), g_\Gamma(t))\|_{\mathcal{H}} \right), \tag{3.28}$$

with a constant $C_\Omega > 0$ that depends only on Ω. Since we have $(\xi, \xi_\Gamma) \in L^\infty(0,T;\mathcal{V})$, we conclude that indeed $(\xi, \xi_\Gamma) \in L^2(0,T;\mathcal{W})$. Arguing as above on the equation (3.1), to be written similarly as in (3.27), and observing that

$$\begin{aligned}
&\|\nabla \xi \cdot \overline{u} - \nabla \overline{\rho} \cdot h\|_{L^2(0,T;H)} \\
&\le C_7 \Big(\|\nabla \xi\|_{L^2(0,T;L^6(\Omega)^3)} \|\overline{u}\|_{L^\infty(0,T;L^3(\Omega)^3)} + \|\nabla \overline{\rho}\|_{L^\infty(0,T;L^6(\Omega)^3)} \|h\|_{L^2(0,T;L^3(\Omega)^3)} \Big) \\
&\le C_8 \Big(\|(\xi, \xi_\Gamma)\|_{L^2(0,T;\mathcal{W})} \|\overline{u}\|_{L^\infty(0,T;L^3(\Omega)^3)} + \|(\overline{\rho}, \overline{\rho}_\Gamma)\|_{L^\infty(0,T;\mathcal{W})} \|h\|_{L^2(0,T;L^3(\Omega)^3)} \Big) ,
\end{aligned}$$

it is not difficult to conclude that $(\eta, \eta_\Gamma) \in L^2(0,T;\mathcal{W})$ (cf. [12, Sect. 3]).

At this point, it remains to show the asserted continuity properties of the mapping $h \mapsto ((\eta, \eta_\Gamma), (\xi, \xi_\Gamma))$. Now, it follows from the weak and weak star sequential semicontinuity of norms and from the estimates (3.22) and (3.23) that, for every $h \in \mathcal{X}$,

$$\|(\eta, \eta_\Gamma)\|_{L^2(0,T;\mathcal{V})} + \|(\xi, \xi_\Gamma)\|_{H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{V})} \le C_9 \|h\|_{\mathcal{X}} \, . \tag{3.29}$$

The assertion is thus completely proved. □

We now turn our interest to the Fréchet differentiability. We recall the definitions (1.8) and (3.10) of the spaces $\mathcal{X}$ and $\mathcal{Y}$ and prove the following result.

Theorem 3.2. *Assume that* (A1)–(A3), (A5) *and* (A6) *are fulfilled. Then the control-to-state operator $\mathcal{S}$ is Fréchet differentiable at every $\overline{u} \in \mathcal{U}_R$ as a mapping from the space $\mathcal{X}$ into the space $\mathcal{Y}$. Moreover, for every $\overline{u} \in \mathcal{U}_R$ and every $h \in \mathcal{X}$ we have that the Fréchet derivative $D\mathcal{S}(\overline{u})$ of $\mathcal{S}$ at $\overline{u}$ satisfies $D\mathcal{S}(\overline{u})(h) = (\eta, \eta_\Gamma, \xi, \xi_\Gamma)$, which is the unique solution to the linearized system* (3.1)–(3.3) *associated with h.*

PROOF: Since $\mathcal{U}_R$ is open, there is some $\Lambda > 0$ such that $\overline{u} + h \in \mathcal{U}_R$ whenever $h \in \mathcal{X}$ and $\|h\|_{\mathcal{X}} \le \Lambda$. In the following, we consider only such perturbations h, for which we define the quantities

$$((\mu^h, \mu^h_\Gamma), (\rho^h, \rho^h_\Gamma)) := \mathcal{S}(\overline{u} + h), \quad y^h := \rho^h - \overline{\rho} - \xi^h, \quad y^h_\Gamma := \rho^h_\Gamma - \overline{\rho}_\Gamma - \xi^h_\Gamma,$$
$$z^h := \mu^h - \overline{\mu} - \eta^h, \quad z^h_\Gamma := \mu^h_\Gamma - \overline{\mu}_\Gamma - \eta^h_\Gamma\,.$$

Obviously, we have $y^h_{|\Sigma} = y^h_\Gamma$ and $z^h_{|\Sigma} = z^h_\Gamma$, as well as

$$(y^h, y^h_\Gamma) \in H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{V})\,, \quad (z^h, z^h_\Gamma) \in L^2(0,T;\mathcal{V})\,. \tag{3.30}$$

Since we know already from the previous theorem that the linear mapping $h \mapsto ((\eta^h, \eta^h_\Gamma), (\xi^h, \xi^h_\Gamma))$ is continuous as a mapping from $\mathcal{X}$ into $\mathcal{Y}$, it suffices to show that there is an increasing mapping $Z : (0,\Lambda) \to (0,+\infty)$ such that $\lim_{\lambda\searrow 0} Z(\lambda)/\lambda^2 = 0$ and

$$\|(z^h, z^h_\Gamma)\|^2_{L^2(0,T;\mathcal{V})} + \|(y^h, y^h_\Gamma)\|^2_{H^1(0,T;\mathcal{H})\cap L^\infty(0,T;\mathcal{V})} \le Z(\|h\|_{\mathcal{X}})\,. \tag{3.31}$$

We also recall that $((\mu^h, \mu^h_\Gamma), (\rho^h, \rho^h_\Gamma))$ satisfy the global estimates stated in (2.12), (2.14), and we observe that it follows from Taylor's theorem that there is some $C_1 > 0$ such that

$$\left|f'(\rho^h) - f'(\overline{\rho}) - f''(\overline{\rho})\xi^h\right| \le C_1 \left|y^h\right| + C_1 \left|\rho^h - \overline{\rho}\right|^2 \quad \text{a.e. in } Q, \tag{3.32}$$
$$\left|f'_\Gamma(\rho^h_\Gamma) - f'_\Gamma(\overline{\rho}_\Gamma) - f''_\Gamma(\overline{\rho}_\Gamma)\xi^h_\Gamma\right| \le C_1 \left|y^h_\Gamma\right| + C_1 \left|\rho^h_\Gamma - \overline{\rho}_\Gamma\right|^2 \quad \text{a.e. on } \Sigma, \tag{3.33}$$

where, here and in the remainder of this proof, C and C_i, $i \in \mathbb{N}$, denote positive constants that may depend on the data of the system and R, but not on the special choice of h with $\|h\|_{\mathcal{X}} \le \Lambda$. Moreover, using the state equations and the linearized system, we readily verify that the following identities are valid:

$$\int_\Omega \partial_t y^h v + \int_\Gamma \partial_t y^h_\Gamma v_\Gamma + \int_\Omega \nabla z^h \cdot \nabla v + \int_\Gamma \nabla_\Gamma z^h_\Gamma \cdot \nabla_\Gamma v_\Gamma = \int_\Omega y^h \overline{u} \cdot \nabla v + \int_\Omega (\rho^h - \overline{\rho}) h \cdot \nabla v$$
$$\text{for all } (v, v_\Gamma) \in \mathcal{V} \quad \text{and a.e. in } (0,T), \tag{3.34}$$

$$\tau_\Omega \int_\Omega \partial_t y^h v + \tau_\Gamma \int_\Gamma \partial_t y^h_\Gamma v_\Gamma + \int_\Omega \nabla y^h \cdot \nabla v + \int_\Gamma \nabla_\Gamma y^h_\Gamma \cdot \nabla_\Gamma v_\Gamma$$
$$+ \int_\Omega \bigl(f'(\rho^h) - f'(\overline{\rho}) - f''(\overline{\rho})\xi^h\bigr)\, v + \int_\Omega \bigl(f'_\Gamma(\rho^h_\Gamma) - f'_\Gamma(\overline{\rho}_\Gamma) - f''_\Gamma(\overline{\rho}_\Gamma)\xi^h_\Gamma\bigr)\, v_\Gamma$$
$$= \int_\Omega z^h v + \int_\Gamma z^h_\Gamma v_\Gamma \qquad \text{for all } (v, v_\Gamma) \in \mathcal{V} \quad \text{and a.e. in } (0,T). \tag{3.35}$$

First estimate. For $s \in (0,T)$, we insert $(v, v_\Gamma) = (z^h(s), z^h_\Gamma(s))$ in (3.34) and $(v, v_\Gamma) = (\partial_t y^h(s), \partial_t y^h_\Gamma(s))$ in (3.35). The last position is formal, but the following computations can be justified rigorously by arguing, e.g., as in [5, Appendix]. We then add the two resulting identities, integrate over $(0,t)$, where $t \in (0,T)$, and add on both sides the quantity $\int_{Q_t} y^h\, \partial_t y^h + \int_{\Sigma_t} y^h_\Gamma\, \partial_t y^h_\Gamma$. Observing that some terms cancel out and using the inequalities (3.32) and (3.33), we arrive at the inequality

$$\int_{Q_t} |\nabla z^h|^2 + \int_{\Sigma_t} |\nabla_\Gamma z^h_\Gamma|^2 + \frac{1}{2}\|(y^h(t), y^h(t))\|^2_{\mathcal{V}} + \tau_\Omega \int_{Q_t} |\partial_t y^h|^2 + \tau_\Gamma \int_{\Sigma_t} |\partial_t y^h_\Gamma|^2$$
$$\leq \int_{Q_t} |y^h||\overline{u}||\nabla z^h| + \int_{Q_t} |\rho^h - \overline{\rho}||h||\nabla z^h| + C\int_{Q_t} |y^h||\partial_t y^h| + C\int_{\Sigma_t} |y^h_\Gamma||\partial_t y^h_\Gamma|$$
$$+ C\int_{Q_t} |\partial_t y^h||\rho^h - \overline{\rho}|^2 + C\int_{\Sigma_t} |\partial_t y^h_\Gamma||\rho^h_\Gamma - \overline{\rho}_\Gamma|^2 =: \sum_{j=1}^{6} I_j, \tag{3.36}$$

with obvious notation. We estimate the six terms on the right-hand side individually, using the Hölder, Young and Sobolev inequalities, and invoking (2.13). We obtain the following estimates:

$$\begin{aligned} I_1 &\leq \int_0^t \|y^h(s)\|_6\, \|\overline{u}(s)\|_3\, \|\nabla z^h(s)\|_2\, ds \\ &\leq \frac{1}{4}\int_{Q_t} |\nabla z^h|^2 + C\|\overline{u}\|^2_{L^\infty(0,t;L^3(\Omega)^3)}\|y^h\|^2_{L^2(0,t;V)} \\ &\leq \frac{1}{4}\int_{Q_t} |\nabla z^h|^2 + C\, \|y^h\|^2_{L^2(0,t;V)}\,, \end{aligned} \tag{3.37}$$

$$\begin{aligned} I_2 &\leq \int_0^t \|\rho^h(s) - \overline{\rho}(s)\|_6\, \|h(s)\|_3\, \|\nabla z^h(s)\|_2\, ds \\ &\leq \frac{1}{4}\int_{Q_t} |\nabla z^h|^2 + C\, \|h\|^2_{L^\infty(0,t;L^3(\Omega)^3)}\|\rho^h - \overline{\rho}\|^2_{L^2(0,t;V)} \\ &\leq \frac{1}{4}\int_{Q_t} |\nabla z^h|^2 + C\, \|h\|^4_{H^1(0,t;L^3(\Omega)^3)}\,, \end{aligned} \tag{3.38}$$

$$I_3 \leq \frac{\tau_\Omega}{4}\int_{Q_t} |\partial_t y^h|^2 + C\int_{Q_t} |y^h|^2\,, \tag{3.39}$$

$$I_4 \leq \frac{\tau_\Gamma}{4}\int_{\Sigma_t} |\partial_t y^h_\Gamma|^2 + C\int_{\Sigma_t} |y^h_\Gamma|^2\,, \tag{3.40}$$

$$\begin{aligned} I_5 &\leq C\int_0^t \|\partial_t y^h(s)\|_2\, \|\rho^h(s) - \overline{\rho}(s)\|^2_4\, ds \\ &\leq \frac{\tau_\Omega}{4}\int_{Q_t} |\partial_t y^h|^2 + C\int_0^t \|(\rho^h - \overline{\rho})(s)\|^4_V\, ds \\ &\leq \frac{\tau_\Omega}{4}\int_{Q_t} |\partial_t y^h|^2 + C\, \|h\|^4_{H^1(0,t;L^3(\Omega)^3)}\,, \end{aligned} \tag{3.41}$$

and, by the same token,

$$I_6 \leq \frac{\tau_\Gamma}{4}\int_{\Sigma_t} |\partial_t y^h_\Gamma|^2 + C\, \|h\|^4_{H^1(0,t;L^3(\Omega)^3)}\,. \tag{3.42}$$

At this point, we can combine the estimates (3.36)–(3.42) and infer from Gronwall's lemma that, for every $t \in (0,T]$,

$$\|(y^h, y^h_\Gamma)\|^2_{H^1(0,t;\mathcal{H})\cap L^\infty(0,t;\mathcal{V})} + \|(\nabla z^h, \nabla_\Gamma z^h_\Gamma)\|^2_{L^2(0,t;\mathcal{H}^3)} \leq C_2\, \|h\|^4_{H^1(0,t;L^3(\Omega)^3)}. \tag{3.43}$$

Second estimate. Next, we insert, for $s \in (0,T)$, $(v, v_\Gamma) = (z^h(s), z_\Gamma^h)(s)$ in (3.35) and integrate the resulting equation over $(0,t)$, where $t \in (0,T]$. Using (3.32) and (3.33) once more, we then arrive at the estimate

$$\begin{aligned}
&\int_{Q_t} |z^h|^2 + \int_{\Sigma_t} |z_\Gamma^h|^2 \,\le\, \tau_\Omega \int_{Q_t} |\partial_t y^h||z^h| + \tau_\Gamma \int_{\Sigma_t} |\partial_t y_\Gamma^h||z_\Gamma^h| + \int_{Q_t} |\nabla y^h||\nabla z^h| \\
&\quad + \int_{\Sigma_t} |\nabla_\Gamma y_\Gamma^h|\,|\nabla_\Gamma z_\Gamma^h| + C \int_{Q_t} |y^h|\,|z^h| + C \int_{\Sigma_t} |y_\Gamma^h||z_\Gamma^h| + C \int_{Q_t} |\rho^h - \overline{\rho}|^2 |z^h| \\
&\quad + C \int_{\Sigma_t} |\rho_\Gamma^h - \overline{\rho}_\Gamma|^2 |z_\Gamma^h| \,. \qquad (3.44)
\end{aligned}$$

The sum of the first six summands on the right-hand side, which we denote by J_1, can be estimated using Young's inequality and (3.43). In this way, we readily obtain that

$$J_1 \,\le\, \frac{1}{4} \int_{Q_t} |z^h|^2 + \frac{1}{4} \int_{\Sigma_t} |z_\Gamma^h|^2 + C \|h\|^2_{H^1(0,t;L^3(\Omega)^3)} \,. \qquad (3.45)$$

The remaining two terms, which we denote by J_2 and J_3, can be handled using the Hölder, Young and Sobolev inequalities as well as (2.13). Indeed, we have that

$$\begin{aligned}
J_2 \,&\le\, C \int_0^t \|z^h(s)\|_2 \,\|\rho^h(s) - \overline{\rho}(s)\|_4^2 \, ds \,\le\, \frac{1}{4} \int_{Q_t} |z^h|^2 + C \int_0^t \|(\rho^h - \overline{\rho})(s)\|_V^4 ds \\
&\le\, \frac{1}{4} \int_{Q_t} |z^h|^2 + C \,\|h\|^4_{H^1(0,t;L^3(\Omega)^3)} \,. \qquad (3.46)
\end{aligned}$$

Similar reasoning yields that

$$J_3 \,\le\, \frac{1}{4} \int_{\Sigma_t} |z_\Gamma^h|^2 + C \,\|h\|^4_{H^1(0,t;L^3(\Omega)^3)} \,. \qquad (3.47)$$

Therefore, combining the estimate (3.43) with (3.44)–(3.47), we can conclude that

$$\|(z^h, z_\Gamma^h)\|^2_{L^2(0,t;\mathcal{V})} \,\le\, C_3 \,\|h\|^4_{H^1(0,t;L^3(\Omega)^3)} \quad \text{for all } \, t \in (0,T] \,. \qquad (3.48)$$

In conclusion, the inequality (3.31) is fulfilled with the choice $Z(\lambda) := (C_2 + C_3)\lambda^4$. The assertion is thus proved. □

With the differentiability shown, the road is paved to derive a first-order necessary optimality condition for the control problem under investigation. Indeed, a standard argument (which we do not repeat here) invoking the chain rule for Fréchet derivatives and the convexity of the admissible set $\mathcal{U}_{\rm ad}$ yields the result stated below, where the following abbreviations are used:

$$\varphi_1 := \beta_1(\overline{\mu} - \widehat{\mu}_Q), \quad \varphi_2 := \beta_2(\overline{\mu}_\Gamma - \widehat{\mu}_\Sigma), \qquad (3.49)$$

$$\varphi_3 := \beta_3(\overline{\rho} - \widehat{\rho}_Q), \quad \varphi_4 := \beta_4(\overline{\rho}_\Gamma - \widehat{\rho}_\Sigma), \qquad (3.50)$$

$$\varphi_5 := \beta_5(\overline{\rho}(T) - \widehat{\rho}_\Omega), \quad \varphi_6 := \beta_6(\overline{\rho}_\Gamma(T) - \widehat{\rho}_\Gamma). \qquad (3.51)$$

Corollary 3.3. *Let the assumptions* (A1)–(A5) *be satisfied, and assume that* $\overline{u} \in \mathcal{U}_{ad}$ *is a solution to the control problem* **(CP)** *with associated state* $((\overline{\mu}, \overline{\mu}_\Gamma), (\overline{\rho}, \overline{\rho}_\Gamma)) = \mathcal{S}(\overline{u})$*. Then, with the notation*

(3.49)–(3.51)*, we have that*

$$\int_Q \varphi_1 \,\eta + \int_\Sigma \varphi_2 \,\eta_\Gamma + \int_Q \varphi_3 \,\xi + \int_\Sigma \varphi_4 \,\xi_\Gamma + \int_\Omega \varphi_5 \,\xi(T) + \int_\Gamma \varphi_6 \,\xi_\Gamma(T) + \beta_7 \int_Q \overline{u} \cdot (v - \overline{u}) \geq 0 \quad \textit{for every } v \in \mathcal{U}_{ad}, \tag{3.52}$$

where, for $v \in \mathcal{U}_{ad}$, $(\xi, \xi_\Gamma, \eta, \eta_\Gamma)$ is the solution to the linearized problem corresponding to $h := v - \overline{u}$.

4 The optimal control problem

In this section, we examine deeply the control problem **(CP)** of minimizing the functional (1.1) under the control constraint $u \in \mathcal{U}_{ad}$ and the state constraint (2.7)–(2.9). First of all, we show the existence of an optimal control. Then, we eliminate the solution to the linearized problem from the necessary condition (3.52) already established (with the notations (3.50)–(3.51)), by making use of the solution to a proper adjoint problem. As for the first aim, we have the following result:

Theorem 4.1. *Suppose that the assumptions* (A1)–(A5) *hold true. Then the optimal control problem* **(CP)** *has at least one solution, that is, there exists some $\overline{u} \in \mathcal{U}_{ad}$ such that*

$$\mathcal{J}(\overline{\mu}, \overline{\mu}_\Gamma, \overline{\rho}, \overline{\rho}_\Gamma, \overline{u}) \leq \mathcal{J}(\mu, \mu_\Gamma, \rho, \rho_\Gamma, u) \quad \textit{for every } u \in \mathcal{U}_{ad}, \tag{4.1}$$

where $((\overline{\mu}, \overline{\mu}_\Gamma), (\overline{\rho}, \overline{\rho}_\Gamma))$ and $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma))$ are the solutions to the state system (2.7)–(2.9) *corresponding to the controls $\overline{u}$ and u, respectively.*

Proof: We use the direct method. Thus, we fix a minimizing sequence, i.e., a sequence $\{u_n\}$ of admissible controls such that

$$\lim_{n\to\infty} \mathcal{J}(\mu_n, \mu_{n_\Gamma}, \rho_n, \rho_{n_\Gamma}, u_n) = \Lambda := \inf \mathcal{J}(\mu, \mu_\Gamma, \rho, \rho_\Gamma, u), \tag{4.2}$$

where the infimum is taken over the set of quintuples $(\mu, \mu_\Gamma, \rho, \rho_\Gamma, u)$ that satisfy $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma)) = \mathcal{S}(u)$ for some $u \in \mathcal{U}_{ad}$. By Theorem 2.3, the estimates (2.11)–(2.12) hold true with constants $\rho_*, \rho^* \in (-1, 1)$ and $K_1 > 0$ that do not depend on n. On the other hand, every u_n belongs to $\mathcal{U}_{ad}$. Therefore, we have for a subsequence (still indexed by n)

$$\begin{aligned}
u_n &\to \overline{u} && \text{weakly star in } (L^\infty(Q) \cap H^1(0,T;L^3(\Omega)))^3,\\
(\mu_n, \mu_{n_\Gamma}) &\to (\overline{\mu}, \overline{\mu}_\Gamma) && \text{weakly star in } L^\infty(0,T;\mathcal{W}),\\
(\rho_n, \rho_{n_\Gamma}) &\to (\overline{\rho}, \overline{\rho}_\Gamma) && \text{weakly star in } W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{W})\\
& && \text{and strongly in } L^2(Q) \times L^2(\Sigma).
\end{aligned}$$

By Remark 2.2, we can infer that $\overline{u} \in \mathcal{U}_{ad}$. Moreover, $(f'(\rho_n), f'_\Gamma(\rho_{n_\Gamma}))$ converges to $(f'(\overline{\rho}), f'_\Gamma(\overline{\rho}))$ strongly in $L^2(Q) \times L^2(\Sigma)$ and $\rho_n u_n$ converges to $\overline{\rho}\,\overline{u}$ weakly in $(L^2(Q))^3$. Hence, it is straightforward to verify that $((\overline{\mu}, \overline{\mu}_\Gamma), (\overline{\rho}, \overline{\rho}_\Gamma))$ solves the integrated version of the state system (2.7)–(2.9) with $u = \overline{u}$ and time-dependent test functions $(v, v_\Gamma) \in L^2(0,T;\mathcal{V})$, that is, the system itself. Finally, we have, by semicontinuity and (4.2),

$$\mathcal{J}(\overline{\mu}, \overline{\mu}_\Gamma, \overline{\rho}, \overline{\rho}_\Gamma, \overline{u}) \leq \lim_{n\to\infty} \mathcal{J}(\mu_n, \mu_{n_\Gamma}, \rho_n, \rho_{n_\Gamma}, u_n) = \Lambda\,.$$

Therefore, $\overline{u}$ is an optimal control. □

The final step consists in eliminating the solution to the linearized problem from the necessary condition (3.52), with the notations (3.50)–(3.51), by using the solution to a proper adjoint problem. However, we cannot deal with the general case, unfortunately. Indeed, we are only able to treat a slightly less general situation, namely when

$$\beta_1 = \beta_2 = 0 \tag{4.3}$$

(cf., e.g., [9] for a similar case). Furthermore, for a given optimal control $\overline{u}$, if we let $((\overline{\mu}, \overline{\mu}_\Gamma), (\overline{\rho}, \overline{\rho}_\Gamma)) = \mathcal{S}(\overline{u})$ be the corresponding optimal state, we still keep the notations (3.49)–(3.51), noticing that $\varphi_1 = 0$ and $\varphi_2 = 0$ due to (4.3), and also introduce for brevity

$$\psi := f''(\overline{\rho}) \quad \text{and} \quad \psi_\Gamma := f''_\Gamma(\overline{\rho}_\Gamma) . \tag{4.4}$$

Then the adjoint problem reads as follows: we look for a quadruplet $(p, p_\Gamma, q, q_\Gamma)$ satisfying the regularity requirements

$$(p, p_\Gamma) \in L^\infty(0,T;\mathcal{V}) , \quad (q, q_\Gamma) \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}) , \tag{4.5}$$

$$(p + \tau_\Omega q, p_\Gamma + \tau_\Gamma q_\Gamma) \in H^1(0,T;\mathcal{V}^*) , \tag{4.6}$$

and solving

$$\begin{aligned} & - \langle \partial_t (p + \tau_\Omega q, p_\Gamma + \tau_\Gamma q_\Gamma), (v, v_\Gamma) \rangle_{\mathcal{V}} + \int_\Omega \nabla q \cdot \nabla v + \int_\Gamma \nabla_\Gamma q_\Gamma \cdot \nabla_\Gamma v_\Gamma \\ & \quad + \int_\Omega \psi q v + \int_\Gamma \psi_\Gamma q_\Gamma v_\Gamma - \int_\Omega \overline{u} \cdot \nabla p \, v = \int_\Omega \varphi_3 \, v + \int_\Gamma \varphi_4 \, v_\Gamma \\ & \quad \text{a.e. in } (0,T) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} & \int_\Omega \nabla p \cdot \nabla v + \int_\Gamma \nabla_\Gamma p_\Gamma \cdot \nabla_\Gamma v_\Gamma = \int_\Omega q v + \int_\Gamma q_\Gamma v_\Gamma \\ & \quad \text{a.e. in } (0,T) \text{ and for every } (v, v_\Gamma) \in \mathcal{V}, \end{aligned} \tag{4.8}$$

$$\begin{aligned} & \langle (p + \tau_\Omega q, p_\Gamma + \tau_\Gamma q_\Gamma)(T), (v, v_\Gamma) \rangle_{\mathcal{V}} = \int_\Omega \varphi_5 \, v + \int_\Gamma \varphi_6 v_\Gamma \\ & \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}. \end{aligned} \tag{4.9}$$

We notice that the system (4.7)–(4.8) is the variational formulation of the following boundary value problem:

$$\begin{aligned} & - \partial_t (p + \tau_\Omega q) - \Delta q + \psi q - \overline{u} \cdot \nabla p = \varphi_3 \quad \text{and} \quad - \Delta p = q \quad \text{in } Q, \\ & - \partial_t (p_\Gamma + \tau_\Gamma q_\Gamma) + \partial_\nu q - \Delta_\Gamma q_\Gamma + \psi_\Gamma q_\Gamma = \varphi_4 , \quad \partial_\nu p_\Gamma - \Delta_\Gamma p_\Gamma = q_\Gamma , \\ & p_{|\Sigma} = p_\Gamma \text{ and } q_{|\Sigma} = q_\Gamma \quad \text{on } \Sigma. \end{aligned}$$

However, we only use the weak formulation (4.7)–(4.9).

We discuss the well-posedness of this problem. We prepare our existence result by solving an approximating problem depending on a small parameter $\varepsilon \in (0,1)$. We recall that $\overline{u}$ belongs to $\mathcal{U}_{ad}$. However, our results are valid under the weaker assumption

$$\overline{u} \in L^\infty(0,T;L^3(\Omega)^3). \tag{4.10}$$

We replace $\overline{u}$ in (4.7) by the bounded function u^ε defined a.e. in Q by the conditions

$$u^\varepsilon = \overline{u} \quad \text{where } |\overline{u}| \le 1/\varepsilon \quad \text{and} \quad u^\varepsilon = \frac{1}{\varepsilon}\,\frac{\overline{u}}{|\overline{u}|} \quad \text{where } |\overline{u}| > 1/\varepsilon\,. \tag{4.11}$$

Moreover, we introduce a viscosity term in (4.8). Finally, we approximate the pair $(\varphi_5,\varphi_6) \in \mathcal{H}$ by pairs $(\varphi_5^\varepsilon,\varphi_6^\varepsilon)$ satisfying

$$(\varphi_5^\varepsilon/\tau_\Omega, \varphi_6^\varepsilon/\tau_\Gamma) \in \mathcal{V} \quad \text{for } \varepsilon \in (0,1) \quad \text{and} \quad (\varphi_5^\varepsilon,\varphi_6^\varepsilon) \to (\varphi_5,\varphi_6) \quad \text{in } \mathcal{H} \text{ as } \varepsilon \searrow 0. \tag{4.12}$$

The problem we consider is the following: we look for a quadruplet $(p^\varepsilon, p_\Gamma^\varepsilon, q^\varepsilon, q_\Gamma^\varepsilon)$ satisfying the regularity requirements

$$(p^\varepsilon, p_\Gamma^\varepsilon),\ (q^\varepsilon, q_\Gamma^\varepsilon) \in H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{V}) \tag{4.13}$$

and solving

$$\begin{aligned}
&- \int_\Omega \partial_t(p^\varepsilon + \tau_\Omega q^\varepsilon) v - \int_\Gamma \partial_t(p_\Gamma^\varepsilon + \tau_\Gamma q_\Gamma^\varepsilon) v_\Gamma + \int_\Omega \nabla q^\varepsilon \cdot \nabla v + \int_\Gamma \nabla_\Gamma q_\Gamma^\varepsilon \cdot \nabla_\Gamma v_\Gamma \\
&\quad + \int_\Omega \psi q^\varepsilon v + \int_\Gamma \psi_\Gamma q_\Gamma^\varepsilon v_\Gamma - \int_\Omega u^\varepsilon \cdot \nabla p^\varepsilon\, v = \int_\Omega \varphi_3\, v + \int_\Gamma \varphi_4\, v_\Gamma\,, && (4.14)\\
&- \varepsilon \int_\Omega \partial_t p^\varepsilon\, v - \varepsilon \int_\Gamma \partial_t p_\Gamma^\varepsilon\, v_\Gamma + \int_\Omega \nabla p^\varepsilon \cdot \nabla v + \int_\Gamma \nabla_\Gamma p_\Gamma^\varepsilon \cdot \nabla_\Gamma v_\Gamma \\
&= \int_\Omega q^\varepsilon v + \int_\Gamma q_\Gamma^\varepsilon v_\Gamma\,, && (4.15)\\
&(p^\varepsilon, p_\Gamma^\varepsilon)(T) = (0,0) \quad \text{and} \quad (q^\varepsilon, q_\Gamma^\varepsilon)(T) = (\varphi_5^\varepsilon/\tau_\Omega\,, \varphi_6^\varepsilon/\tau_\Gamma)\,, && (4.16)
\end{aligned}$$

where the equalities (4.14)–(4.15) have to hold for every $(v, v_\Gamma) \in \mathcal{V}$ and a.e. in $(0,T)$. In order to solve this problem, we need a preparatory lemma.

Lemma 4.2. *Let $(\mathbb{V},\mathbb{H},\mathbb{V}^*)$ be a Hilbert triplet with $\mathbb{V}$ separable, and let the operators $\mathbb{A} \in \mathcal{L}(\mathbb{H},\mathbb{H})$, $\mathbb{B} \in \mathcal{L}(\mathbb{V},\mathbb{V}^*)$ and $\mathbb{C}(t) \in \mathcal{L}(\mathbb{V},\mathbb{H})$ satisfy, for some positive constants α, λ and K,*

$$\begin{aligned}
&(\mathbb{A}w, w)_{\mathbb{H}} \ge \alpha \|w\|_{\mathbb{H}}^2 \quad \textit{for every } w \in \mathbb{H}\,, && (4.17)\\
&\langle \mathbb{B}w, w\rangle_{\mathbb{V}} + \lambda \|w\|_{\mathbb{H}}^2 \ge \alpha \|w\|_{\mathbb{V}}^2 \quad \textit{for every } w \in \mathbb{V}\,, && (4.18)\\
&\|\mathbb{C}(t)w\|_{\mathbb{H}} \le K \|w\|_{\mathbb{V}} \quad \textit{for a.a. } t \in (0,T) \textit{ and every } w \in \mathbb{V}\,, && (4.19)\\
&\textit{for every } w_1 \in \mathbb{V} \textit{ and } w_2 \in \mathbb{H}, \\
&\quad \textit{the function} \quad t \mapsto (\mathbb{C}(t)w_1, w_2)_{\mathbb{H}} \quad \textit{is measurable on } (0,T). && (4.20)
\end{aligned}$$

Moreover, assume that $\mathbb{B}$ is symmetric. Then, for every $F \in L^2(0,T;\mathbb{H})$ and $w_T \in \mathbb{V}$, there exists a unique

$$w \in H^1(0,T;\mathbb{H}) \cap L^\infty(0,T;\mathbb{V}) \tag{4.21}$$

satisfying

$$\begin{aligned}
&- \mathbb{A}w'(t) + \mathbb{B}w(t) + \mathbb{C}(t)w(t) = F(t) \quad \textit{in } \mathbb{V}^* \quad \textit{for a.a. } t \in (0,T)\,, && (4.22)\\
&w(T) = w_T\,. && (4.23)
\end{aligned}$$

PROOF: Even nonlinear generalizations of such a result should be known (see, e.g., [13] for a nonlinear case with $\mathbb{C}=0$). However, we did not find any reference that precisely deals with our assumptions. Therefore, we sketch a short proof. Both existence and uniqueness are based on the estimate obtained by formally testing (4.22), written at the time s, by $-w'(s)$ and integrating over (t,T). By doing this, using the symmetry of $\mathbb{B}$ and adding the same quantity to both sides, we obtain

$$\int_t^T \bigl(\mathbb{A}w'(s), w'(s)\bigr)_{\mathbb{H}}\, ds + \frac{1}{2}\,\langle \mathbb{B}w(t), w(t)\rangle_{\mathbb{V}} + \frac{\lambda}{2}\,\|w(t)\|_{\mathbb{H}}^2$$
$$= \frac{1}{2}\,\langle \mathbb{B}w_T, w_T\rangle_{\mathbb{V}} + \frac{\lambda}{2}\,\|w_T\|_{\mathbb{H}}^2 + \lambda \int_t^T \bigl(w(s), w'(s)\bigr)_{\mathbb{H}}\, ds$$
$$+ \int_t^T \bigl(F(s) - \mathbb{C}(s)w(s), w'(s)\bigr)_{\mathbb{H}}\, ds\,.$$

At this point, we account for (4.17)–(4.19), the Young inequality and the Gronwall lemma. We conclude that

$$\|w\|_{H^1(0,T;\mathbb{H})\cap L^\infty(0,T;\mathbb{V})} \le C\bigl(\|F\|_{L^2(0,T;\mathbb{H})} + \|w_T\|_{\mathbb{V}}\bigr),$$

where C depends only on the structural constants and T. This estimate corresponds to the regularity (4.21) and implies that $w=0$ if the data vanish. However, this is formal, as said at the very beginning. To make the existence proof rigorous, we can owe to the separability of $\mathbb{V}$ and use a Faedo-Galerkin scheme. To obtain uniqueness, we test (4.22) by the function $-w_\delta'$ rather than by $-w'$, where $w_\delta \in H^1(0,T;\mathbb{V})$ is obtained by solving the abstract elliptic problem (here $\mathbb{I}:\mathbb{V}\to\mathbb{V}^*$ is the injection)

$$w_\delta(t) + \delta(\mathbb{B} + \lambda\mathbb{I})w_\delta(t) = w(t) \quad \text{for a.a. } t\in(0,T).$$

Then, we use [5, Appendix: Prop. 6.1-6.3 and Rem. 6.4] in letting δ tend to zero. This yields the desired estimate, thus uniqueness if $F=0$ and $w_T=0$. □

Theorem 4.3. *For every $\varepsilon\in(0,1)$, the approximating problem* (4.14)–(4.16) *has a unique solution $(p^\varepsilon, p_\Gamma^\varepsilon, q^\varepsilon, q_\Gamma^\varepsilon)$ satisfying* (4.13).

PROOF: We present the problem in a different form. We term (z,z_Γ) rather than (v,v_Γ) the arbitrary element of $\mathcal{V}$ that appears in (4.14) and add this equation to (4.15) divided by $\varepsilon\tau$, where

$$\tau := \min\{\tau_\Omega\,, \tau_\Gamma\}. \tag{4.24}$$

This yields the identity

$$- \frac{1}{\tau}\int_\Omega \partial_t p^\varepsilon\, v - \frac{1}{\tau}\int_\Gamma \partial_t p_\Gamma^\varepsilon\, v_\Gamma + \frac{1}{\varepsilon\tau}\int_\Omega \nabla p^\varepsilon\cdot\nabla v + \frac{1}{\varepsilon\tau}\int_\Gamma \nabla_\Gamma p_\Gamma^\varepsilon\cdot\nabla_\Gamma v_\Gamma$$
$$- \int_\Omega \partial_t(p^\varepsilon + \tau_\Omega q^\varepsilon) z - \int_\Gamma \partial_t(p_\Gamma^\varepsilon + \tau_\Gamma q_\Gamma^\varepsilon) z_\Gamma + \int_\Omega \nabla q^\varepsilon\cdot\nabla z + \int_\Gamma \nabla_\Gamma q_\Gamma^\varepsilon\cdot\nabla_\Gamma z_\Gamma$$
$$+ \int_\Omega \psi q^\varepsilon z + \int_\Gamma \psi_\Gamma q_\Gamma^\varepsilon z_\Gamma - \int_\Omega u^\varepsilon\cdot\nabla p^\varepsilon\, z$$
$$= \frac{1}{\varepsilon\tau}\int_\Omega q^\varepsilon v + \frac{1}{\varepsilon\tau}\int_\Gamma q_\Gamma^\varepsilon v_\Gamma + \int_\Omega \varphi_3\, z + \int_\Gamma \varphi_4\, z_\Gamma \quad \text{for every } (v,v_\Gamma),(z,z_\Gamma)\in\mathcal{V}. \tag{4.25}$$

As the pairs (v,v_Γ) and (z,z_Γ) are independent from each other, this equation is equivalent to (4.14)–(4.15), and we are going to transform it into an abstract equation like (4.22) in the framework of the Hilbert triplet

$$\mathbb{V}\subset\mathbb{H}\subset\mathbb{V}^* \quad \text{where} \quad \mathbb{V} := \mathcal{V}\times\mathcal{V} \quad \text{and} \quad \mathbb{H} := \mathcal{H}\times\mathcal{H}\,, \quad \text{whence} \quad \mathbb{V}^* = \mathcal{V}^*\times\mathcal{V}^*$$

with a non-standard embedding $\mathbb{H} \subset \mathbb{V}^*$, due to a particular choice of the inner product in $\mathbb{H}$. In order to simplify the notation, we write the elements $((v,v_\Gamma),(z,z_\Gamma))$ of $\mathbb{H}$ as quadruplets (v,v_Γ,z,z_Γ). We set

$$\big((p,p_\Gamma,q,q_\Gamma),(v,v_\Gamma,z,z_\Gamma)\big)_{\mathbb{H}} := \int_\Omega \Big(\frac{1}{\varepsilon\tau_\Omega}\,pv + qz\Big) + \int_\Gamma \Big(\frac{1}{\varepsilon\tau_\Gamma}\,p_\Gamma v_\Gamma + q_\Gamma z_\Gamma\Big)$$
$$\text{for every } (p,p_\Gamma,q,q_\Gamma),\ (v,v_\Gamma,z,z_\Gamma) \in \mathbb{H} \tag{4.26}$$

and notice that $(\,\cdot\,,\,\cdot\,)_{\mathbb{H}}$ actually is an inner product and that the corresponding norm is equivalent to the standard one. Moreover, we define the operators $\mathbb{A}^\varepsilon \in \mathcal{L}(\mathbb{H},\mathbb{H})$, $\mathbb{B} \in \mathcal{L}(\mathbb{V},\mathbb{V}^*)$ and $\mathbb{C}^\varepsilon(t) \in \mathcal{L}(\mathbb{V},\mathbb{H})$ by the formulas

$$\mathbb{A}^\varepsilon(p,p_\Gamma,q,q_\Gamma) := \Big(\frac{\varepsilon\tau_\Omega}{\tau}\,p,\frac{\varepsilon\tau_\Gamma}{\tau}\,p_\Gamma, p+\tau_\Omega q, p_\Gamma + \tau_\Gamma q_\Gamma\Big)$$
$$\text{for every } (p,p_\Gamma,q,q_\Gamma) \in \mathbb{H}\,, \tag{4.27}$$

$$\langle \mathbb{B}^\varepsilon(p,p_\Gamma,q,q_\Gamma),(v,v_\Gamma,z,z_\Gamma)\rangle_{\mathbb{V}}$$
$$:= \frac{1}{\varepsilon\tau}\int_\Omega \nabla p\cdot\nabla v + \frac{1}{\varepsilon\tau}\int_\Gamma \nabla_\Gamma p_\Gamma\cdot\nabla_\Gamma v_\Gamma + \int_\Omega \nabla q\cdot\nabla z + \int_\Gamma \nabla_\Gamma q_\Gamma\cdot\nabla_\Gamma z_\Gamma$$
$$\text{for every } (p,p_\Gamma,q,q_\Gamma),\ (v,v_\Gamma,z,z_\Gamma) \in \mathbb{V}\,, \tag{4.28}$$

$$\mathbb{C}^\varepsilon(t)(p,p_\Gamma,q,q_\Gamma)$$
$$:= \Big(-\frac{\tau_\Omega}{\tau}\,q\,,\,-\frac{\tau_\Gamma}{\tau}\,q_\Gamma\,,\,-u^\varepsilon(t)\cdot\nabla p + \psi(t)\,q\,,\,\psi_\Gamma(t)\,q_\Gamma\Big)$$
$$\text{for a.a. } t\in(0,T) \text{ and every } (p,p_\Gamma,q,q_\Gamma) \in \mathbb{V}\,. \tag{4.29}$$

A simple computation shows that

$$\big(\mathbb{A}^\varepsilon(p,p_\Gamma,q,q_\Gamma),(v,v_\Gamma,z,z_\Gamma)\big)_{\mathbb{H}}$$
$$= \int_\Omega \Big(\frac{1}{\tau}\,pv + (p+\tau_\Omega q)z\Big) + \int_\Gamma \Big(\frac{1}{\tau}\,p_\Gamma v_\Gamma + (p_\Gamma+\tau_\Gamma q_\Gamma)z_\Gamma\Big)$$
$$\text{for every } (p,p_\Gamma,q,q_\Gamma),\ (v,v_\Gamma,z,z_\Gamma) \in \mathbb{H}\,,$$
$$\big(\mathbb{C}^\varepsilon(t)(p,p_\Gamma,q,q_\Gamma),(v,v_\Gamma,z,z_\Gamma)\big)_{\mathbb{H}}$$
$$= -\frac{1}{\varepsilon\tau}\int_\Omega qv - \frac{1}{\varepsilon\tau}\int_\Gamma q_\Gamma v_\Gamma + \int_\Omega \psi(t)qz + \int_\Gamma \psi_\Gamma(t)q_\Gamma z_\Gamma - \int_\Omega u^\varepsilon(t)\cdot\nabla p\, z$$
$$\text{for a.a. } t\in(0,T) \text{ and every } (p,p_\Gamma,q,q_\Gamma) \in \mathbb{V} \text{ and } (v,v_\Gamma,z,z_\Gamma)\in\mathbb{H}.$$

Therefore, the variational equation (4.25) takes the form

$$-\big(\mathbb{A}^\varepsilon\partial_t(p^\varepsilon,p^\varepsilon_\Gamma,q^\varepsilon,q^\varepsilon_\Gamma)(t),(v,v_\Gamma,z,z_\Gamma)\big)_{\mathbb{H}} + \langle \mathbb{B}^\varepsilon(p^\varepsilon,p^\varepsilon_\Gamma,q^\varepsilon,q^\varepsilon_\Gamma)(t),(v,v_\Gamma,z,z_\Gamma)\rangle_{\mathbb{V}}$$
$$+ \big(\mathbb{C}^\varepsilon(t)(p^\varepsilon,p^\varepsilon_\Gamma,q^\varepsilon,q^\varepsilon_\Gamma)(t),(v,v_\Gamma,z,z_\Gamma)\big)_{\mathbb{H}} = \big(F(t),(v,v_\Gamma,z,z_\Gamma)\big)_{\mathbb{H}}$$
$$\text{for a.a. } t\in(0,T) \text{ and every } (v,v_\Gamma,z,z_\Gamma)\in\mathbb{V}\,,$$

with an obvious definition of $F \in L^2(0,T;\mathbb{H})$. Thus, it is a particular case of (4.22). On the other hand, (4.16) is equivalent to

$$\big((p^\varepsilon,p^\varepsilon_\Gamma,q^\varepsilon,q^\varepsilon_\Gamma)(T),(v,v_\Gamma,z,z_\Gamma)\big)_{\mathbb{H}} = \big((0,0,\varphi^\varepsilon_5/\tau_\Omega,\varphi^\varepsilon_6/\tau_\Gamma),(v,v_\Gamma,z,z_\Gamma)\big)_{\mathbb{H}}$$
$$\text{for every } (v,v_\Gamma,z,z_\Gamma)\in\mathbb{H}\,,$$

and $(0,0,\varphi^\varepsilon_5/\tau_\Omega,\varphi^\varepsilon_6/\tau_\Gamma)$ belongs to $\mathbb{V}$ by the first condition in (4.12). Therefore, in order to conclude, it is sufficient to check the properties (4.17)–(4.20) of Lemma 4.2. The second and fourth ones are clear, and $\mathbb{B}^\varepsilon$ is obviously symmetric. Moreover, (4.19) easily follows from the boundedness of u^ε, ψ and ψ_Γ. As for (4.17), we have, for some constant $\alpha > 0$ and every $(v, v_\Gamma, z, z_\Gamma) \in \mathbb{H}$,

$$\begin{aligned}
&\big(\mathbb{A}^\varepsilon(v, v_\Gamma, z, z_\Gamma), (v, v_\Gamma, z, z_\Gamma)\big)_{\mathbb{H}} \\
&= \int_\Omega \Big(\frac{1}{\tau}\, |v|^2 + (v + \tau_\Omega z) z \Big) + \int_\Gamma \Big(\frac{1}{\tau}\, |v_\Gamma|^2 + (v_\Gamma + \tau_\Gamma z_\Gamma) z_\Gamma \Big) \\
&\geq \int_\Omega \Big(\frac{1}{\tau}\, |v|^2 + \tau |z|^2 + vz \Big) + \int_\Gamma \Big(\frac{1}{\tau}\, |v_\Gamma|^2 + \tau |z_\Gamma|^2 + v_\Gamma z_\Gamma \Big) \\
&\geq \frac{1}{2} \int_\Omega \Big(\frac{1}{\tau}\, |v|^2 + \tau |z|^2 \Big) + \frac{1}{2} \int_\Gamma \Big(\frac{1}{\tau}\, |v_\Gamma|^2 + \tau |z_\Gamma|^2 \Big) \geq \alpha \| (v, v_\Gamma, z, z_\Gamma) \|_{\mathbb{H}}^2 \,,
\end{aligned}$$

the last inequalities being due to the Young inequality and to the equivalence between the norm $\| \cdot \|_{\mathbb{H}}$ induced by the inner product (4.26) and the natural norm of $\mathbb{H}$. Therefore, Lemma 4.2 can be applied and the proof is complete. $\square$

Theorem 4.4. *Let the assumptions* (A1)–(A5) *and* (4.3) *be satisfied. Moreover, assume that* $\overline{u} \in \mathcal{U}_{ad}$ *is a solution to the control problem* **(CP)** *and that* $((\overline{\mu}, \overline{\mu}_\Gamma), (\overline{\rho}, \overline{\rho}_\Gamma)) = \mathcal{S}(\overline{u})$ *is the associated state. Then, with the notations* (3.50)–(3.51) *and* (4.4)*, the adjoint problem* (4.7)–(4.9) *has a unique solution* $(p, p_\Gamma, q, q_\Gamma)$ *satisfying* (4.5)–(4.6)*.*

PROOF: In order to show existence, we perform a number of a priori estimates on the solution $(p^\varepsilon, p^\varepsilon_\Gamma, q^\varepsilon, q^\varepsilon_\Gamma)$ to the approximating problem. However, we explicitly write the superscript ε only at the end of each estimate. Moreover, we make use of the same symbol c to denote different constants that do not depend on ε. The symbol c_δ stands for (possibly different) constants that can also depend on the parameter δ. By denoting by $|\Omega|$ and $|\Gamma|$ the volume of Ω and the area of Γ, respectively, we define the mean value function by

$$\mathrm{mean}(v, v_\Gamma) := \frac{\int_\Omega v + \int_\Gamma v_\Gamma}{|\Omega| + |\Gamma|} \quad \text{for every } (v, v_\Gamma) \in \mathcal{H} \tag{4.30}$$

and observe that the Poincaré type inequality

$$\| (v, v_\Gamma) \|_{\mathcal{V}} \leq c \big(\| (\nabla v, \nabla_\Gamma v_\Gamma) \|_{\mathcal{H}^3} + |\, \mathrm{mean}(v, v_\Gamma) | \big) \quad \text{for every } (v, v_\Gamma) \in \mathcal{V} \tag{4.31}$$

holds true with a constant c that depends only on Ω. Finally, we set, for brevity,

$$Q^t := \Omega \times (t, T) \quad \text{and} \quad \Sigma^t := \Gamma \times (t, T) \quad \text{for } t \in (0, T).$$

First a priori estimate. We test (4.15) by $(1, 1)$ and obtain

$$|\, \mathrm{mean}(q, q_\Gamma) | \, (|\Omega| + |\Gamma|) \leq \varepsilon \int_\Omega |\partial_t p| + \varepsilon \int_\Gamma |\partial_t p_\Gamma| \quad \text{a.e. in } (0, T). \tag{4.32}$$

As $\varepsilon^2 \leq \varepsilon$, since $\varepsilon \in (0, 1)$, we infer that

$$\int_t^T |\, \mathrm{mean}(q, q_\Gamma)(s)|^2 \, ds \leq c\, \varepsilon \int_{Q^t} |\partial_t p|^2 + c\, \varepsilon \int_{\Sigma^t} |\partial_t p_\Gamma|^2 \quad \text{for every } t \in [0, T]$$

where c depends only on Ω. On the other hand, owing also to (4.31) and to the Sobolev inequality, we deduce that

$$\int_t^T \|q(s)\|_6^2\,ds \le c\Big(\int_{Q^t} |q|^2 + \int_{Q^t} |\nabla q|^2\Big)$$
$$\le c\Big(\int_{Q^t} |\nabla q|^2 + \int_{\Sigma^t} |\nabla_\Gamma q_\Gamma|^2 + \int_t^T |\operatorname{mean}(q,q_\Gamma)(s)|^2\,ds\Big).$$

Therefore, by combining these inequalities, we conclude that

$$\int_t^T \|q(s)\|_6^2\,ds \le C_\Omega\Big(\int_{Q^t} |\nabla q|^2 + \int_{\Sigma^t} |\nabla_\Gamma q_\Gamma|^2 + \varepsilon \int_{Q^t} |\partial_t p|^2 + \varepsilon \int_{\Sigma^t} |\partial_t p_\Gamma|^2\Big) \qquad (4.33)$$

for every $t \in [0,T]$, with a constant C_Ω that depends only on Ω.

Second a priori estimate. We test (4.14) by (q, q_Γ), integrate over (t,T), account for the Cauchy conditions (4.16), and have

$$-\int_{Q^t} \partial_t p\, q - \int_{\Sigma^t} \partial_t p_\Gamma\, q_\Gamma + \frac{\tau_\Omega}{2}\int_\Omega |q(t)|^2 + \frac{\tau_\Gamma}{2}\int_\Gamma |q_\Gamma(t)|^2 + \int_{Q^t} |\nabla q|^2 + \int_{\Sigma^t} |\nabla_\Gamma q_\Gamma|^2$$
$$= \frac{\tau_\Omega}{2}\int_\Omega |\varphi_5^\varepsilon/\tau_\Omega|^2 + \frac{\tau_\Gamma}{2}\int_\Gamma |\varphi_6^\varepsilon/\tau_\Gamma|^2 - \int_{Q^t} \psi |q|^2 - \int_{\Sigma^t} \psi_\Gamma |q_\Gamma|^2$$
$$+ \int_{Q^t} u^\varepsilon \cdot \nabla p\, q + \int_{Q^t} \varphi_3\, q + \int_{\Sigma^t} \varphi_4\, q_\Gamma\,.$$

At the same time, we test (4.15) by $-\partial_t(p, p_\Gamma)$ and integrate over (t,T) to obtain

$$\varepsilon \int_{Q^t} |\partial_t p|^2 + \varepsilon \int_{\Sigma^t} |\partial_t p_\Gamma|^2 + \frac{1}{2}\int_\Omega |\nabla p(t)|^2 + \frac{1}{2}\int_\Gamma |\nabla_\Gamma p_\Gamma(t)|^2 = -\int_{Q^t} q\partial_t p - \int_{\Sigma^t} q_\Gamma \partial_t p_\Gamma\,.$$

Now, we add this equality to the previous one and observe that four terms cancel each other. Moreover, accounting for (4.11) and (4.33), we treat the transport term as follows:

$$\int_{Q^t} u^\varepsilon \cdot \nabla p\, q \le \|\overline{u}\|_{L^\infty(0,T;L^3(\Omega)^3)} \int_t^T \|\nabla p(s)\|_2\, \|q(s)\|_6\,ds$$
$$\le \delta \int_t^T \|q(s)\|_6^2\,ds + c_\delta \int_{Q^t} |\nabla p|^2$$
$$\le \delta\, C_\Omega\Big(\int_{Q^t} |\nabla q|^2 + \int_{\Sigma^t} |\nabla_\Gamma q_\Gamma|^2 + \varepsilon \int_{Q^t} |\partial_t p|^2 + \varepsilon \int_{\Sigma^t} |\partial_t p_\Gamma|^2\Big) + c_\delta \int_{Q^t} |\nabla p|^2\,,$$

where $\delta > 0$ is arbitrary. Therefore, since ψ and ψ_Γ are bounded, φ_3 and φ_4 are L^2-functions, and (4.12) implies that $(\varphi_5^\varepsilon, \varphi_6^\varepsilon)$ is bounded in $\mathcal{H}$ uniformly with respect to ε, by choosing δ such that $\delta C_\Omega \le 1/2$ and using the Gronwall lemma, we conclude that

$$\|(q^\varepsilon, q_\Gamma^\varepsilon)\|_{L^\infty(0,T;\mathcal{H})\cap L^2(0,T;\mathcal{V})} + \|(\nabla p^\varepsilon, \nabla_\Gamma p_\Gamma^\varepsilon)\|_{L^\infty(0,T;\mathcal{H})} + \varepsilon^{1/2}\|\partial_t(p^\varepsilon, p_\Gamma^\varepsilon)\|_{L^2(0,T;\mathcal{H})} \le c\,. \qquad (4.34)$$

Third a priori estimate. By testing (4.14) by an arbitrary pair $(v, v_\Gamma) \in L^2(0,T;\mathcal{V})$ and accounting for (4.34), we easily deduce that

$$\|\partial_t(p^\varepsilon + \tau_\Omega q^\varepsilon, p_\Gamma^\varepsilon + \tau_\Gamma q_\Gamma^\varepsilon)\|_{L^2(0,T;\mathcal{V}^*)} \le c\,. \qquad (4.35)$$

Fourth a priori estimate. Clearly, (4.35) implies that

$$\|\tfrac{d}{dt}\,\mathrm{mean}(p+\tau_\Omega q, p_\Gamma+\tau_\Gamma q_\Gamma)\|_{L^2(0,T)} \le c\|\partial_t(p+\tau_\Omega q, p_\Gamma+\tau_\Gamma q_\Gamma)\|_{L^2(0,T;\mathcal{V}^*)}\,\|(1,1)\|_{\mathcal{V}} \le c\,,$$

and, in view of (4.16), we infer that

$$\|\mathrm{mean}(p+\tau_\Omega q, p_\Gamma+\tau_\Gamma q_\Gamma)\|_{L^\infty(0,T)} \le c\,. \tag{4.36}$$

On the other hand, even $(\tau_\Omega q, \tau_\Gamma q_\Gamma)$ is bounded in $L^\infty(0,T;\mathcal{H})$ by (4.34), and consequently $\mathrm{mean}(\tau_\Omega q, \tau_\Gamma q_\Gamma)$ is bounded in $L^\infty(0,T)$. Therefore, (4.36) ensures that the same holds for $\mathrm{mean}(p, p_\Gamma)$. By accounting for (4.34) and the Poincaré type inequality (4.31), we conclude that

$$\|(p^\varepsilon, p^\varepsilon_\Gamma)\|_{L^\infty(0,T;\mathcal{V})} \le c\,. \tag{4.37}$$

Existence. We are ready to take the limit as $\varepsilon \searrow 0$. We have, at least for a subsequence,

$$(p^\varepsilon, p^\varepsilon_\Gamma) \to (p, p_\Gamma) \quad \text{weakly star in } L^\infty(0,T;\mathcal{V})\,, \tag{4.38}$$

$$(q^\varepsilon, q^\varepsilon_\Gamma) \to (q, q_\Gamma) \quad \text{weakly star in } L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V})\,, \tag{4.39}$$

$$\partial_t(p^\varepsilon+\tau_\Omega q^\varepsilon, p^\varepsilon_\Gamma+\tau_\Gamma q^\varepsilon_\Gamma) \to \partial_t(p+\tau_\Omega q, p_\Gamma+\tau_\Gamma q_\Gamma) \quad \text{weakly in } L^2(0,T;\mathcal{V}^*)\,, \tag{4.40}$$

$$\varepsilon\,\partial_t(p^\varepsilon, p^\varepsilon_\Gamma) \to 0 \quad \text{strongly in } L^2(0,T;\mathcal{H}). \tag{4.41}$$

As (4.38)–(4.40) imply that (see, e.g., [24, Sect. 8, Cor. 4])

$$(p^\varepsilon+\tau_\Omega q^\varepsilon, p^\varepsilon_\Gamma+\tau_\Gamma q^\varepsilon_\Gamma) \to (p+\tau_\Omega q, p_\Gamma+\tau_\Gamma q_\Gamma)$$
$$\text{strongly in } C^0([0,T];\mathcal{V}^*) \cap L^2(0,T;\mathcal{H}),$$

and since the approximating final data satisfy the convergence property in (4.12), from (4.16) it follows that the Cauchy condition (4.9) is fulfilled by the limiting quadruplet $(p, p_\Gamma, q, q_\Gamma)$. Moreover, by recalling (4.11), we see that u^ε converges to $\overline{u}$ a.e. in Q. By combining this with (4.10) and the inequality $|u^\varepsilon| \le |\overline{u}|$ a.e. in Q, we deduce that u^ε converges to $\overline{u}$ strongly (e.g.) in $(L^{8/3}(Q))^3$. Thus, by also accounting for (4.38), we infer that $u^\varepsilon \cdot \nabla p^\varepsilon$ converges to $\overline{u} \cdot \nabla p$ weakly in $L^{8/7}(Q)$. Therefore, we can take the limit in the integrated version of (4.14)–(4.15) with time-dependent test functions $(v, v_\Gamma) \in L^2(0,T;\mathcal{V})$ with $v \in L^8(Q)$ and conclude that $(p, p_\Gamma, q, q_\Gamma)$ solves the integrated version of (4.7)–(4.8) with the same test functions. By density, since $\overline{u} \cdot \nabla p$ belongs to $L^2(0,T;L^{6/5}(\Omega))$ by (4.10) and (4.38), and $L^2(0,T;V) \subset L^2(0,T;L^6(\Omega))$ by the Sobolev inequality, one can take any element of $L^2(0,T;\mathcal{V})$ as a test function. Hence, this quadruplet also solves the equations (4.7) and (4.8) as they are.

Uniqueness. Since the problem (4.7)–(4.9) is linear, it is sufficient to prove that the unique solution with $(\varphi_3, \varphi_4, \varphi_5, \varphi_6) = (0,0,0,0)$ is $(p, p_\Gamma, q, q_\Gamma) = (0,0,0,0)$. In the next lines, C_i, $i = 1, 2, \ldots$, denote positive constants that depend only on the structural assumptions and the L^∞-norms of $\overline{u}$, ψ and ψ_Γ.

First, we introduce the primitive functions

$$Q(t) := -\int_t^T q(s)\,ds, \quad Q_\Gamma(t) := -\int_t^T q_\Gamma(s)\,ds, \quad t \in [0,T],$$

and integrate (4.7) from t to T in order to obtain

$$\langle (p + \tau_\Omega q, p_\Gamma + \tau_\Gamma q_\Gamma)(t), (v, v_\Gamma)\rangle_{\mathcal{V}} - \int_\Omega \nabla Q(t) \cdot \nabla v - \int_\Gamma \nabla_\Gamma Q_\Gamma(t) \cdot \nabla_\Gamma v_\Gamma$$
$$+ \int_\Omega \int_t^T (\psi q)(s)\, ds\, v + \int_\Gamma \int_t^T (\psi_\Gamma q_\Gamma)(s)\, ds\, v_\Gamma - \int_\Omega \int_t^T (\overline{u} \cdot \nabla p)(s)\, ds\, v$$
$$= 0 \quad \text{for every } t \in [0, T] \text{ and } (v, v_\Gamma) \in \mathcal{V}. \tag{4.42}$$

Next, we test (4.8), written at the time t, by $(p, p_\Gamma)(t)$; at the same time, we take $(v, v_\Gamma) = (q, q_\Gamma)(t)$ in (4.42) and sum the two equalities we obtain by observing that there is a cancellation of four terms. Integrating once more with respect to t, we deduce that

$$\tau_\Omega \int_{Q^t} |q|^2 + \tau_\Gamma \int_{\Sigma^t} |q_\Gamma|^2 + \int_\Omega |\nabla Q(t)|^2$$
$$+ \int_\Gamma |\nabla_\Gamma Q_\Gamma(t)|^2 + \int_{Q^t} |\nabla p|^2 + \int_{\Sigma^t} |\nabla_\Gamma p_\Gamma|^2$$
$$= - \int_{Q^t} \int_s^T (\psi q)(\sigma)\, d\sigma\, q(s)\, ds - \int_{\Sigma^t} \int_s^T (\psi_\Gamma q_\Gamma)(\sigma)\, d\sigma\, q_\Gamma(s)\, ds$$
$$+ \int_{Q^t} \int_s^T (\overline{u} \cdot \nabla p)(\sigma)\, d\sigma\, q(s)\, ds\, . \tag{4.43}$$

We now estimate the terms on the right-hand side of (4.43). Thanks to the Young and Hölder inequalities, and using the L^∞-bound for ψ, we infer that

$$- \int_{Q^t} \int_s^T (\psi q)(\sigma)\, d\sigma\, q(s)\, ds \le \frac{\tau_\Omega}{4} \int_{Q^t} |q|^2 + \frac{1}{\tau_\Omega} \|\psi\|_\infty^2\, T \int_{Q^t} \int_s^T |q(\sigma)|^2\, d\sigma\, ds$$
$$\le \frac{\tau_\Omega}{4} \int_{Q^t} |q|^2 + C_1 \int_t^T \left(\int_{Q^s} |q|^2 \right) ds. \tag{4.44}$$

Arguing similarly for the boundary integral, we have that

$$- \int_{\Sigma^t} \int_s^T (\psi_\Gamma q_\Gamma)(\sigma)\, d\sigma\, q_\Gamma(s)\, ds \le \frac{\tau_\Gamma}{4} \int_{\Sigma^t} |q_\Gamma|^2 + C_2 \int_t^T \left(\int_{\Sigma^s} |q_\Gamma|^2 \right) ds. \tag{4.45}$$

Also the last term of (4.43) can be treated by the same token. Indeed, we see that

$$\int_{Q^t} \int_s^T (\overline{u} \cdot \nabla p)(\sigma)\, d\sigma\, q(s)\, ds \le \frac{\tau_\Omega}{4} \int_{Q^t} |q|^2 + \frac{1}{\tau_\Omega} \|\overline{u}\|_\infty^2\, T \int_{Q^t} \int_s^T |\nabla p(\sigma)|^2\, d\sigma\, ds$$
$$\le \frac{\tau_\Omega}{4} \int_{Q^t} |q|^2 + C_3 \int_t^T \left(\int_{Q^s} |\nabla p|^2 \right) ds. \tag{4.46}$$

At this point, if we combine the inequality (4.43) with the estimates (4.44)–(4.46) and then apply the Gronwall lemma, we deduce that $(q, q_\Gamma) = (0, 0)$ as well as that the vectors ∇p and $\nabla_\Gamma p_\Gamma$ vanish. Hence, by a comparison in (4.42), we finally conclude that $(p, p_\Gamma) = (0, 0)$, and the proof is complete. □

Once the solvability of problem (4.7)–(4.9) is established, we actually can eliminate the solution to the linearized problem in (3.52), as stated in our final result. For its proof, we need a Leibniz rule which is well known under slightly different assumptions.

Lemma 4.5. *Assume that*

$$y \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}) \quad \textit{and} \quad z \in H^1(0,T;\mathcal{V}^*) \cap L^2(0,T;\mathcal{H}) . \tag{4.47}$$

Then the function $t \mapsto (y(t), z(t))_{\mathcal{H}}$ *is absolutely continuous on* $[0,T]$*, and its derivative is given by*

$$\frac{d}{dt}\,(y,z)_{\mathcal{H}} = (y',z)_{\mathcal{H}} + \langle z', y\rangle_{\mathcal{V}} \quad \textit{a.e. in } (0,T). \tag{4.48}$$

PROOF: By the trace method with $p=2$ of the interpolation theory (see, e.g., [22]), the continuous embeddings

$$H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}) \subset C^0([0,T];(\mathcal{V},\mathcal{H})_{1/2}) ,$$
$$H^1(0,T;\mathcal{V}^*) \cap L^2(0,T;\mathcal{H}) \subset C^0([0,T];(\mathcal{H},\mathcal{V}^*)_{1/2}) ,$$

hold true, as well as the duality formula $(\mathcal{H},\mathcal{V}^*)_{1/2} = (\mathcal{V},\mathcal{H})^*_{1/2}$. Therefore, the map

$$t \mapsto \bigl(y(t), z(t)\bigr)_{\mathcal{H}} = \langle z(t), y(t)\rangle_{(\mathcal{V},\mathcal{H})_{1/2}}$$

is continuous on $[0,T]$. Thus, to conclude, it suffices to prove that

$$\langle z(t_2), y(t_2)\rangle_{(\mathcal{V},\mathcal{H})_{1/2}} - \langle z(t_1), y(t_1)\rangle_{(\mathcal{V},\mathcal{H})_{1/2}}$$
$$= \int_{t_1}^{t_2} \bigl((y'(s), z(s))_{\mathcal{H}} + \langle z'(s), y(s)\rangle_{\mathcal{V}}\bigr)\,ds \quad \text{for every } t_1,t_2 \in [0,T]. \tag{4.49}$$

To this end, we approximate z by functions $z_n \in H^1(0,T;H)$ satisfying

$$z_n \to z \quad \text{in } H^1(0,T;\mathcal{V}^*) \cap L^2(0,T;\mathcal{H}).$$

Then, (4.49) holds for y and z_n, as is well known. At this point, one lets n tend to infinity and obtains (4.49) by observing that z_n converges to z also in $C^0([0,T];(\mathcal{V},\mathcal{H})^*_{1/2})$. $\square$

Theorem 4.6. *Let the assumptions* (A1)–(A5) *and* (4.3) *be satisfied. Moreover, assume that* $\overline{u} \in \mathcal{U}_{ad}$ *is a solution to the control problem* **(CP)** *with associated state* $((\overline{\mu},\overline{\mu}_\Gamma),(\overline{\rho},\overline{\rho}_\Gamma)) = \mathcal{S}(\overline{u})$*. Furthermore, with the notations* (3.50)–(3.51) *and* (4.4)*, let* (p,p_Γ,q,q_Γ) *be the solution to the adjoint problem* (4.7)–(4.9) *satisfying the regularity requirements* (4.5)–(4.6)*. Then, we have*

$$\int_Q \bigl(\overline{\rho}\,\nabla p + \beta_7\overline{u}\bigr)\cdot(v-\overline{u}) \geq 0 \quad \textit{for every } v \in \mathcal{U}_{ad}\,. \tag{4.50}$$

PROOF: We fix any $v \in \mathcal{U}_{ad}$ and introduce the linearized problem corresponding to the choice $h = v - \overline{u}$ as in Corollary 3.3. Then, we test (3.1) and (3.2) by (p,p_Γ) and (q,q_Γ), respectively, integrate over $(0,T)$ and sum up. We obtain

$$\int_Q \partial_t\xi\,p + \int_\Sigma \partial_t\xi_\Gamma\,p_\Gamma + \int_Q \nabla\eta\cdot\nabla p + \int_\Sigma \nabla_\Gamma\eta_\Gamma\cdot\nabla_\Gamma p_\Gamma - \int_Q \xi\overline{u}\cdot\nabla p - \int_\Sigma \overline{\rho}\,h\cdot\nabla p$$
$$+ \tau_\Omega\int_Q \partial_t\xi\,q + \tau_\Gamma\int_\Sigma \partial_t\xi_\Gamma\,q_\Gamma + \int_Q \nabla\xi\cdot\nabla q + \int_\Sigma \nabla_\Gamma\xi_\Gamma\cdot\nabla_\Gamma q_\Gamma$$
$$+ \int_Q \psi\xi q + \int_\Sigma \psi_\Gamma\xi_\Gamma q_\Gamma = \int_Q \eta q + \int_\Sigma \eta_\Gamma q_\Gamma\,, \tag{4.51}$$

and we observe that the sum of the terms involving time derivatives can be written as

$$\begin{aligned}&\int_Q \partial_t\xi\, p + \int_\Sigma \partial_t\xi_\Gamma\, p_\Gamma + \tau_\Omega \int_Q \partial_t\xi\, q + \tau_\Gamma \int_\Sigma \partial_t\xi_\Gamma\, q_\Gamma \\ &= \int_0^T \big(\partial_t(\xi,\xi_\Gamma)(s), (p+\tau_\Omega q, p_\Gamma + \tau_\Gamma q_\Gamma(s)\big)_{\mathcal{H}}\, ds.\end{aligned}$$

Now, we test (4.7) and (4.8) by $-(\xi,\xi_\Gamma)$ and $-(\eta,\eta_\Gamma)$, respectively, integrate over $(0,T)$, and sum up. We obtain the identity

$$\begin{aligned}&\int_0^T \langle \partial_t(p+\tau_\Omega q, p_\Gamma + \tau_\Gamma q_\Gamma)(s), (\xi,\xi_\Gamma)(s)\rangle_{\mathcal{V}}\, ds - \int_Q \nabla q\cdot\nabla\xi - \int_\Sigma \nabla_\Gamma q_\Gamma\cdot\nabla_\Gamma\xi_\Gamma \\ &\quad - \int_Q \psi q\xi - \int_\Sigma \psi_\Gamma q_\Gamma \xi_\Gamma + \int_Q \overline{u}\cdot\nabla p\,\xi \\ &- \int_Q \nabla p\cdot\nabla\eta - \int_Q \nabla_\Gamma p_\Gamma\cdot\nabla_\Gamma\eta_\Gamma \\ &= -\int_Q \varphi_3\,\xi - \int_\Sigma \varphi_4\,\xi_\Gamma - \int_Q q\eta - \int_\Sigma q_\Gamma\eta_\Gamma\,. \end{aligned} \tag{4.52}$$

At this point, we add the equalities (4.51) and (4.52) to each other. Then, the most part of the terms cancels out, and the sum of the integrals involving time derivatives can be treated by invoking Lemma 4.5. Hence, we obtain

$$\begin{aligned}&\int_0^T \frac{d}{dt}\big((p+\tau_\Omega q, p_\Gamma+\tau_\Gamma q_\Gamma)(s), (\xi,\xi_\Gamma)(s)\big)_{\mathcal{H}}\, ds - \int_\Sigma \overline{\rho}\, h\cdot\nabla p \\ &= -\int_Q \varphi_3\,\xi - \int_\Sigma \varphi_4\,\xi_\Gamma\,.\end{aligned}$$

On the other hand, thanks to Lemma 4.5, (3.3) and (4.9) with $(v,v_\Gamma) = (\xi,\xi_\Gamma)(T)$, we also have

$$\begin{aligned}&\int_0^T \frac{d}{dt}\big((p+\tau_\Omega q, p_\Gamma+\tau_\Gamma q_\Gamma)(s), (\xi,\xi_\Gamma)(s)\big)_{\mathcal{H}}\, ds \\ &= \big((p+\tau_\Omega q, p_\Gamma+\tau_\Gamma q_\Gamma)(T), (\xi,\xi_\Gamma)(T)\big)_{\mathcal{H}} - \big((p+\tau_\Omega q, p_\Gamma+\tau_\Gamma q_\Gamma)(0), (\xi,\xi_\Gamma)(0)\big)_{\mathcal{H}} \\ &= \big((\varphi_5,\varphi_6), (\xi,\xi_\Gamma)(T)\big)_{\mathcal{H}}\,.\end{aligned}$$

Therefore, (3.52) becomes (4.50).

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