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Type II balanced truncation for deterministic bilinear control systems

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Abstract

When solving partial differential equations numerically, usually a high order spatial discretisation is needed. Model order reduction (MOR) techniques are often used to reduce the order of spatially-discretised systems and hence reduce computational complexity. A particular MOR technique to obtain a reduced order model (ROM) is balanced truncation (BT), a method which has been extensively studied for deterministic linear systems. As so-called type I BT it has already been extended to bilinear equations, an important subclass of nonlinear systems. We provide an alternative generalisation of the linear setting to bilinear systems which is called type II BT. The Gramians that we propose in this context contain information about the control. It turns out that the new approach delivers energy bounds which are not just valid in a small neighbourhood of zero. Furthermore, we provide an $\mathcal{H}_\infty$-error bound which so far is not known when applying type I BT to bilinear systems.

1 Introduction

Numerical simulations are one of the conventional methods to study physical phenomena of dynamical systems. However, extracting all the complex system dynamics generally leads to large state-space systems, whose direct simulations are inefficient and involve huge computational cost. Hence, there is a need to consider model order reduction (MOR), aiming at replacing these large-scale systems by systems of much smaller state dimension. MOR for linear systems has been investigated intensively in recent years and is widely used in numerous applications, see e.g. [1, 7, 26]. In this work, we consider MOR for bilinear control systems, which can be considered as a bridge between linear and nonlinear systems. Applications of bilinear systems can be seen in various fields [10, 19, 24].

Several methods for linear systems have been extended to bilinear systems such as balanced truncation (BT) [4] or other balancing related methods [17]. Moreover, interpolation-based MOR has been applied [2, 3, 9, 13]. In this manuscript, we focus on BT for bilinear systems which for linear systems has been studied in e.g. [1, 20]. Later on, the balancing concept for general nonlinear systems has been extended in a series of papers, see e.g. [14, 16, 25].

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BT relies on controllability/reachability and observability Gramians. In [4] Gramians were proposed which we will call type I (bilinear) Gramians in this paper. The drawback of this approach is that only local energy bounds are available [15]. Furthermore, no error bound has been proved so far. The type I bilinear Gramians play a role for stochastic systems as well [4, 8, 23], where they are also used in the context of balancing. Recently, a second way of balancing for stochastic systems was discussed [5, 12, 21, 22]. It is based on another reachability Gramian and called type II ansatz. With this approach an $\mathcal{H}_\infty$-error bound can be achieved which cannot be proved in the ansatz used in [8, 23].

In this paper, we introduce type II bilinear Gramians in Section 2 which are inspired by the type II stochastic Gramians. The difference lies in additional information about the control in the bilinear Gramians. Under the assumption of having bounded controls, we then prove bounds for the reachability and observability energy of the underlying bilinear system using the type II bilinear Gramians. This provides a motivation to balance the bilinear system based on the new Gramians. The procedure will be explained in Section 3. In Section 4, an $\mathcal{H}_\infty$-error bound for type II bilinear BT will be proved, again assuming bounded controls. This error bound is the main result of this paper and has the same structure as in the linear case. Error bounds for BT applied to bilinear system did not exist before.

2 Setting and type II Gramians

We consider the following bilinear deterministic equation

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) + \sum_{i=1}^{m} N_i x(t) u_i(t), \quad x(0) = x_0, \quad t \geq 0, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $N_1, \dots, N_m \in \mathbb{R}^{n\times n}$. Below, $x(t, x_0, u)$, $t \geq 0$, denotes the solution to (1) with initial condition $x_0 \in \mathbb{R}^n$ and control process $u = (u_1, \dots, u_m)^T$. In our framework the state equation (1) is additionally equipped with an output equation

$$y(t, x_0, u) = Cx(t, x_0, u), \quad t \geq 0, \tag{2}$$

where $C \in \mathbb{R}^{p\times n}$. The control u is usually assumed to be an L^2_T function meaning that

$$\|u\|^2_{L^2_T} := \int_0^T u^T(t)u(t)dt = \int_0^T \|u(t)\|_2^2 \, dt < \infty$$

for every $T > 0$. Moreover, a classical assumption is the asymptotic stability of the uncontrolled equation (1), that is

$$\|x(t, x_0, 0)\|_2 \to 0 \tag{3}$$

for $t \to \infty$ and every $x_0 \in \mathbb{R}^n$, being equivalent to the Hurwitz property of A. Later on we will introduce a further condition on u and a stronger assumption on the stability. This is required to define the type II Gramians.

Before alternative Gramians are discussed, the existing theory will be summarised below.

Type I reachability and observability Gramians In [4] and [15] bounds for the controllability/reachability and observability energy of the equations (1) and (2) have been proved using certain Gramians. We call these matrices type I Gramians here. The type I reachability Gramian P_1 is the unique solution to

$$AP_1 + P_1A^T + \sum_{i=1}^{m} N_iP_1N_i^T = -BB^T, \tag{4}$$

whereas the type I observability Gramian is defined as the unique solution of

$$A^TQ_1 + Q_1A + \sum_{i=1}^{m} N_i^TQ_1N_i = -C^TC. \tag{5}$$

Equations (4) and (5) are also considered in the context of model order reduction for stochastic systems [4, 8]. A further discussion about these Gramians can be found in [21]. A unique positive semidefinite solution to the identities (4) and (5) exists if the following stability condition holds:

$$\sigma\left(I_n \otimes A + A \otimes I_n + \sum_{i=1}^{m} N_i \otimes N_i\right) \subset \mathbb{C}_-. \tag{6}$$

Property (6) is also called asymptotic mean square stability because it represents a stability concept for stochastic systems [8, 11, 18, 21]. It is satisfied if A is asymptotically stable and the matrices N_i are relatively small (in some norm) compared to the eigenvalues of A. That is why it is actually enough to assume (3) because equation (1) can be rescaled as

$$\frac{dx(t)}{dt} = Ax(t) + [\frac{1}{\gamma}B][\gamma u(t)] + \sum_{i=1}^{m}[\frac{1}{\gamma}N_i]x(t)[\gamma u_i(t)],$$

where the weighted matrices $\tilde{N}_i = \frac{1}{\gamma}N_i$ can be made arbitrary small with a sufficiently large constant $\gamma > 0$.

No let us introduce the energy functionals. As in [4, 15, 25] the controllability energy is

$$E_c(x_0) := \min_{\substack{u \in L^2(]-\infty,0]) \\ x(-\infty,x_0,u)=0}} \int_{-\infty}^{0} \|u(t)\|_2^2\, dt.$$

In [4] and [15] the observability energy is considered for the case $B = 0$, whereas for general non-linear systems $u \equiv 0$ is assumed [25]. With $B = 0$, the observability energy is

$$E_o(x_0) := \max_{\substack{u \in L^2([0,\infty[) \\ \|u\|_{L^2} < \alpha, B=0}} \int_0^\infty \|y(t, x_0, u)\|_2^2 \, dt, \quad \text{for } \alpha > 0 \text{ small.} \tag{7}$$

The next theorem is a result from [4] and [15].

Theorem 2.1. *Let A be asymptotically stable (and at least one N_i of full rank). If there are positive definite solutions P_1 and Q_1 to (4) and (5), then there exists a neighbourhood $\hat{W}(0)$ of zero such that*

$$x_0^T P_1^{-1} x_0 \leq E_c(x) \quad \textit{and} \quad E_o(x) \leq x_0^T Q_1 x_0 \quad \textit{for } x \in \hat{W}(0).$$

The above inequalities allow us to identify the dominant controllable and observable states (at least locally). Those are contained in the eigenspaces spanned by the eigenvectors of P_1 and Q_1, respectively, that correspond to the small eigenvalues.
More accurate bounds than in Theorem 2.1 were obtained for truncated Gramians [6], which are also computationally cheaper than the type I Gramians.

Type II reachability and observability Gramians We now introduce alternative Gramians which we will see, guarantee an error bound for the bilinear system. The so called type II Gramians are inspired by the stochastic case. A postive definite reachability Gramian P_2 which solves

$$A^T P_2^{-1} + P_2^{-1} A + \sum_{i=1}^m N_i^T P_2^{-1} N_i \leq -P_2^{-1} B B^T P_2^{-1} \tag{8}$$

was initially introduced in [12] in order to show the existence of an $\mathcal{H}_\infty$-error bound for BT applied to stochastic systems. There the balancing was based on P_2 and the type I Gramian Q_1, the solution to (5). The stochastic Gramian P_2 was furthermore analysed in [5] and used in [21, 22] in a more general form.

An inequality is considered in (8) since the existence of a positive definite solution is not ensured when having an equality. A solution to inequality (8) exists if condition (6) is satisfied [12, 21]. As already mentioned above, (6) can be weakened to the assumption of asymptotic stability of A since N_i can be made arbitrary small.

We will not take the matrices Q_1 and P_2 (stochastic type II balancing) to introduce a type II approach for bilinear systems. Moreover, we will modify them further. The idea is to let information about the control enter the new Gramians. This is done by choosing a constant $k > 0$. So, the type II Gramians of the bilinear system (1), (2) are given by a perturbed matrix A:

$$(A+\frac{k^2}{2}I_n)^T P^{-1}+P^{-1}(A+\frac{k^2}{2}I_n)+\sum_{i=1}^m N_i^T P^{-1} N_i \leq -P^{-1}BB^T P^{-1}, \tag{9}$$

$$(A+\frac{k^2}{2}I_n)^T Q+Q(A+\frac{k^2}{2}I_n)+\sum_{i=1}^m N_i^T Q N_i = -C^T C. \tag{10}$$

At the same time we suppose to have a control which is uniformly bounded by the perturbing constant on a finite time interval $[0, T]$

$$\|u(t)\|_2 \leq k, \quad t \in [0, T]. \tag{11}$$

Of course k cannot be arbitrary large because we need the existence of the solutions to (9) and (10). Ideally k is such that

$$\sigma\left(I_n \otimes (A+\frac{k^2}{2}I_n)+(A+\frac{k^2}{2}I_n)\otimes I_n+\sum_{i=1}^m N_i \otimes N_i\right) \subset \mathbb{C}_- \tag{12}$$

holds. We observe that the type II Gramians P, Q are coupled with the control u. We allow large controls (L^∞-sense) if the system is relatively stable (largest real part of the eigenvalues of A is small and N_i are small) and only small controls are admissible if the system is close to be unstable. We can again weaken condition (12) and only assume $\sigma\left(A+\frac{k^2}{2}\right) \subset \mathbb{C}_-$ since we can rescale N_i, i.e., we replace it by $\tilde{N}_i = \frac{1}{\gamma} N_i$. In this situation we pay a price for the rescaling since if we want to bound the rescaled control $\tilde{u} = \gamma u$ as in (11), it is required to have $\|u(t)\|_2 \leq \frac{k}{\gamma}$ which restricts the controls even more. From now on, we assume that (11) and (12) hold at the same time, knowing that we need a less restrictive assumption than (12) when using a smaller bound for the controls.

Let us now investigate how much energy is necessary if we control the system from zero into a certain direction. We desire to bound the reachability energy with the type II Gramian P. Let $(p_k)_{k=1,\dots,n}$ be eigenvectors of P such that they represent an orthonormal basis of $\mathbb{R}^n$. The corresponding eigenvalues are denoted by $(\lambda_k)_{k=1,\dots,n}$. Then, for $t \in [0, T]$

$$\begin{aligned}\langle x(t,0,u), p_k\rangle_2^2 &\leq \lambda_k \sum_{i=1}^n \lambda_i^{-1}\langle x(t,0,u), p_i\rangle_2^2 = \lambda_k \left\|\sum_{i=1}^n \lambda_i^{-\frac{1}{2}}\langle x(t,0,u), p_i\rangle_2\, p_i\right\|_2^2 \\ &= \lambda_k \left\|P^{-\frac{1}{2}} x(t,0,u)\right\|_2^2 = \lambda_k \left[x(t,0,u)^T P^{-1} x(t,0,u)\right].\end{aligned}$$

To shorten the notation we write $x(t)$ instead of $x(t, 0, u)$ from time to time below. So, by the product rule and by inserting (1), we have

$$
\begin{aligned}
x^T(t)P^{-1}x(t) &= \int_0^t x^T(s)P^{-1}dx(s) + \int_0^t dx^T(s)P^{-1}x(s) \qquad (13)\\
&= \int_0^t x^T(s)P^{-1}Ax(s)ds + \int_0^t x^T(s)P^{-1}Bu(s)ds \\
&\quad + \int_0^t x^T(s)A^TP^{-1}x(s)ds + \int_0^t u^T(s)B^TP^{-1}x(s)ds \\
&\quad + \sum_{i=1}^m \left(\int_0^t x^T(s)P^{-1}N_ix(s)u_i(s)ds + \int_0^t x^T(s)u_i(s)N_i^TP^{-1}x(s)ds\right).
\end{aligned}
$$

We analyse the last term above which can be written as

$$
\begin{aligned}
&\sum_{i=1}^m 2\int_0^t x^T(s)P^{-1}N_ix(s)u_i(s)ds = \sum_{i=1}^m 2\int_0^t \left\langle P^{-\frac{1}{2}}x(s)u_i(s), P^{-\frac{1}{2}}N_ix(s)\right\rangle_2 ds \\
&\leq \sum_{i=1}^m \left(\int_0^t \left\|P^{-\frac{1}{2}}x(s)u_i(s)\right\|_2^2 ds + \int_0^t \left\|P^{-\frac{1}{2}}N_ix(s)\right\|_2^2 ds\right) \\
&= \int_0^t x^T(s)P^{-1}x(s)\left\|u(s)\right\|_2^2 ds + \sum_{i=1}^m \int_0^t x^T(s)N_i^TP^{-1}N_ix(s)ds \\
&\leq \int_0^t x^T(s)P^{-1}k^2x(s)ds + \sum_{i=1}^m \int_0^t x^T(s)N_i^TP^{-1}N_ix(s)ds
\end{aligned}
$$

using the bound in (11). Summarising the above steps, we obtain

$$
\begin{aligned}
\langle x(t,0,u), p_k\rangle_2^2 \leq \lambda_k \Bigg[&\int_0^t x^T(s)(A^TP^{-1} + P^{-1}A + \sum_{i=1}^m N_i^TP^{-1}N_i + k^2P^{-1})x(s)\, ds \\
&+2\int_0^t x^T(s)P^{-1}Bu(s)ds\Bigg].
\end{aligned}
$$

Plugging in (9) yields

$$
\begin{aligned}
\langle x(t,0,u), p_k\rangle_2^2 &\leq \lambda_k \int_0^t 2x^T(s)P^{-1}Bu(s) - x^T(s)P^{-1}BB^TP^{-1}x(s)ds \\
&= \lambda_k \int_0^t \left\|u(s)\right\|_2^2 ds - \left\|B^TP^{-1}x(s) - u(s)\right\|_2^2 ds.
\end{aligned}
$$

Consequently, we have

$$
\lambda_k^{-\frac{1}{2}} \sup_{t\in[0,T]} \left|\langle x(t,0,u), p_k\rangle_2\right| \leq \left\|u\right\|_{L^2_T}. \qquad (14)
$$

So, by (14), large controllability energy is needed if λ_k is small especially when a large component in the direction of p_k shall be reached (a component on or outside the unit sphere). This implies that difficult to reach states have a “large” component in the eigenspaces of P belonging to the small eigenvalues.

Remark. *We can replace P by the stochastic type II Gramian P_2 which satisfies (8). This results in the following inequality*

$$x^T(t)P_2^{-1}x(t) \leq \left[\int_0^t x^T(s)(A^TP_2^{-1} + P_2^{-1}A + \sum_{i=1}^m N_i^TP_2^{-1}N_i)x(s)\; ds \right.$$
$$\left. +2\int_0^t x^T(s)P_2^{-1}Bu(s)ds + \int_0^t x^T(s)P_2^{-1}x(s)\left\|u(s)\right\|_2^2 ds\right].$$

Inserting (8), we then see that

$$x^T(t)P_2^{-1}x(t) \leq \int_0^t \left\|u(s)\right\|_2^2 ds + \int_0^t x^T(s)P_2^{-1}x(s)\left\|u(s)\right\|_2^2 ds.$$

By Gronwall's inequality, we obtain

$$x^T(t)P_2^{-1}x(t) \leq \int_0^t \left\|u(s)\right\|_2^2 ds\, \mathrm{e}^{\int_0^t \left\|u(s)\right\|_2^2 ds}$$

which leads to inequality (14) with an additional exponential term but in this case no bound on the control is assumed. Let us suppose that $\left\|u\right\|_{L^2_T} \leq 1$ in case P_2 is used. Then, a small eigenvalue $\lambda_{2,k}$ of P_2 implies that $\langle x(t,0,u), p_{2,k}\rangle_2$ ($p_{2,k}$ is the corresponding eigenvector) is small. This means that no large component in the direction of $p_{2,k}$ can be reached with a small control.

Let us now turn our attention to the type II Gramian Q. We shorten the notation again and write $x_{x_0}(t)$ instead of $x(t, x_0, u)$. The product rule yields

$$x_{x_0}^T(t)Qx_{x_0}(t) - x_0^TQx_0 = \int_0^t x_{x_0}^T(s)Qdx_{x_0}(s) + \int_0^t dx_{x_0}^T(s)Qx_{x_0}(s),$$

where $t \in [0, T]$. Following the steps from (13) onwards, we obtain

$$x_{x_0}^T(t)Qx_{x_0}(t) - x_0^TQx_0 \leq \left[\int_0^t x_{x_0}^T(s)(A^TQ + QA + \sum_{i=1}^m N_i^TQN_i + k^2Q)x_{x_0}(s)\; ds \right.$$
$$\left. +2\int_0^t x_{x_0}^T(s)QBu(s)ds\right].$$

We insert (10) and evaluate the functions at the final time which gives

$$\int_0^T \left\|y(s, x_0, u)\right\|_2^2 ds \leq x_0^TQx_0 + 2\int_0^T x(s, x_0, u)^TQBu(s)ds. \tag{15}$$

As in (7) we assume $B = 0$. This is a natural choice since in the observability problem an unknown initial condition shall be reconstructed from the observations y. Since Bu is a term that does not depend on x_0, it can be assumed to be known and hence be neglected in the considerations. With $B = 0$ and (15) we see that the dominant observable initial states (states which produce little observation energy) are close to the kernel of Q. They are contained in the eigenspaces of Q corresponding to the small eigenvalues.

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Remark. *If we use Q_1 which satisfies (8) instead of Q, we get an extra term in the energy bound but in this case there is no bound on u. So, we have*

$$\int_0^T \|y(s,x_0,u)\|_2^2\,ds \leq x_0^T Q_1 x_0 + 2\int_0^T x(s,x_0,u)^T Q_1 B u(s) ds + \int_0^T x(s,x_0,u)^T Q_1 x(s,x_0,u)\,\|u(s)\|_2^2\,ds.$$

Now, if we say that the control u is small, we can conclude that the observation energy is small if the initial state is close to the kernel of Q_1.

3 Type II balanced truncation

Before considering an $\mathcal{H}_\infty$- error bound for BT based on the type II Gramians P and Q, we summarise the theory of balancing which is similar to the deterministic linear case [1, 20].

States that are difficult to reach can be characterised by P, cf. (14). These states have large components in the span of the eigenvectors corresponding to small eigenvalues of the reachability Gramian P. Similarly, states that are difficult to observe are the ones that have large components in the span of eigenvectors corresponding to small eigenvalues of the observability Gramian Q, see (15). Now, balancing a system relies on the idea to create a system, where dominant reachable and observable states are the same, i.e., reachability and observability Gramians are simultaneously transformed such that they are equal and diagonal. BT for bilinear systems based on the Gramians P_1 and Q_1 (type I ansatz) has already been studied [4] and for related energy functionals compare [15]. For type I BT no error bound could be shown so far. Now, the procedure for the type II approach is explained. This ansatz allows us to show an $\mathcal{H}_\infty$-error bound in Section 4.

We consider a control system consisting of state equation (1) and output equation (2)

$$\begin{aligned}\frac{dx(t)}{dt} &= Ax(t) + Bu(t) + \sum_{i=1}^m N_i x(t) u_i(t), \\ y(t) &= Cx(t), \quad t \geq 0,\end{aligned} \tag{16}$$

Recall that the state equation in (16) is asymptotically stable, i.e., $\sigma\left(A + \frac{k^2}{2}\right) \subset \mathbb{C}_-$ or ideally (12) is satisfied. Introducing a transformation matrix $T \in \mathbb{R}^{n\times n}$ which is assumed to be non-singular, the states are transformed as follows:

$$\hat{x}(t) = Tx(t),$$

such that system (16) becomes

$$\begin{aligned}\frac{d}{dt}\hat{x}(t) &= \hat{A}\hat{x}(t) + \hat{B}u(t) + \sum_{i=1}^m \hat{N}_i x(t) u_i(t), \\ y(t) &= \hat{C}\hat{x}(t), \quad t \geq 0,\end{aligned} \tag{17}$$

where $\hat{A} = TAT^{-1}$, $\hat{B} = TB$, $\hat{C} = CT^{-1}$ and $\hat{N}_i = TN_iT^{-1}$. The input-output map remains the same, only the state and the systems matrices are transformed.

P and Q, the reachability and observability Gramians of system (16), which satisfy (9) and (10) can be transformed into reachability and observability Gramians of the transformed system (17):

$$\hat{P} = TPT^T \quad \text{and} \quad \hat{Q} = T^{-T}QT^{-1}.$$

The above relation is obtained by multiplying (9) and (10) with T^{-T} from the left and T^{-1} from the right. The Hankel singular values (HSVs) $\sigma_1 \geq \ldots \geq \sigma_n$, where $\sigma_i = \sqrt{\lambda_i(PQ)}$ ($i = 1, \ldots, n$), of the original and transformed system are the same. The above transformation is a balancing transformation if the transformed Gramians are equal and diagonal. Such a transformation always exists if $Q > 0$ (observation energy is always non zero for every $x_0 \neq 0$). We also need that $P > 0$ but this is automatically satisfied. A balanced system is obtained by choosing

$$T = \Sigma^{-\frac{1}{2}}U^TL^T \quad \text{and} \quad T^{-1} = KV\Sigma^{-\frac{1}{2}},$$

where $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n) > 0$ is the diagonal matrix of HSVs. Y, Z, L and K are computed as follows. Let $P = KK^T$, $Q = LL^T$ be square root factorisations of P and Q, then an SVD of $K^TL = V\Sigma U^T$ gives the required matrices. With this transformation $\hat{P} = \hat{Q} = \Sigma$. This implies that Σ characterises both the reachability and observability in system (17). The smaller the diagonal entry of Σ, the less important the corresponding state component in the system dynamics of (17).

Below, let T be the balancing transformation as stated above, then we partition the coefficients of the balanced realisation as follows:

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad TB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CT^{-1} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad TN_iT^{-1} = \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{r \times r}$ etc. Furthermore, by setting $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where $x_1(t) \in \mathbb{R}^r$, we obtain the transformed partitioned system

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u(t) + \sum_{i=1}^m \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}u_i(t), \tag{18}$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad t \geq 0. \tag{19}$$

From this system we aim to obtain a approximating system with reduced dimension $r \ll n$. For BT the second row in (18) is truncated and the remaining x_2 components in the first row of (18) and in (19) are set to zero. This leads to a ROM having the same structure as (16), that is,

$$\begin{aligned} \frac{dx_r(t)}{dt} &= A_{11}x_r(t) + B_1u(t) + \sum_{i=1}^m N_{i,11}x_r(t)u_i(t), \qquad (20) \\ y_r(t) &= C_1x_r(t), \quad t \geq 0, \end{aligned}$$

where $A_{11}, N_{i,11} \in \mathbb{R}^{r\times r}$, $B_1 \in \mathbb{R}^{r\times m}$ and $C_1 \in \mathbb{R}^{p\times r}$. In equations (18) and (19), the difficult to reach and observe states are represented by x_2, which correspond to the smallest HSVs $\sigma_{r+1}, \ldots, \sigma_n$, but of course r has to be chosen such that the neglected HSVs are small ($\sigma_{r+1} \ll \sigma_r$).

4 $\mathcal{H}_\infty$-error bound for type II BT

In this section, we show that the $\mathcal{H}_\infty$-error bound, that is known from the linear case [1], is also true for bilinear systems when using the type II approach. Unfortunately, this result could no yet be established when using the type I Gramians.

We recall the original model that we aim to reduce:

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t) + \sum_{i=1}^m N_i x(t) u_i(t), \quad x(0) = 0, \\ y(t) &= Cx(t), \quad t \geq 0, \end{aligned} \tag{21}$$

where the matrices and vectors above are partitioned as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; N_i = \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix}, \; C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

Below, we prove an $\mathcal{H}_\infty$-error bound when the balancing relies on the matrix (in)equalities (9) and (10). We replace equation (10) by an inequality because we do not need the equality in the proof. To simplify the notation, we assume that system (21) is balanced already, i.e., we have applied the balancing transformation from Section 3 already. Hence, the Gramians P and Q are equal and coincide with the diagonal matrix $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2)$, where $\Sigma_1 = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$ is the matrix of large and $\Sigma_2 = \operatorname{diag}(\sigma_{r+1}, \ldots, \sigma_n)$ the matrix of neglected small HSVs.
The following ROM is supposed to be compared with the original model (21):

$$\begin{aligned} \frac{dx_r(t)}{dt} &= A_{11}x_r(t) + B_1 u(t) + \sum_{i=1}^m N_{i,11} x_r(t) u_i(t), \quad x_r(0) = 0, \\ y_r(t) &= C_1 x_r(t), \quad t \geq 0. \end{aligned} \tag{22}$$

The next theorem deals with the L^2_T-error between the full and the reduced order output.

Theorem 4.1. *Let $x(0) = 0$, $x_r(0) = 0$ and $\left\|u(t)\right\|_2 \leq k$, $t \in [0, T]$, $T > 0$, where k is the constant that enters in (9) and (10). Then,*

$$\left\|y - y_r\right\|_{L^2_T} \leq 2(\tilde{\sigma}_1 + \tilde{\sigma}_2 + \ldots + \tilde{\sigma}_\nu) \left\|u\right\|_{L^2_T},$$

where y is the output of the original system (21), y_r is the output of the type II BT approach ROM and $\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_\nu$ are the distinct diagonal entries of $\Sigma_2 = \operatorname{diag}(\sigma_{r+1}, \ldots, \sigma_n) = \operatorname{diag}(\tilde{\sigma}_1 I, \tilde{\sigma}_2 I, \ldots, \tilde{\sigma}_\nu I)$.

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Proof. We sometimes omit the time dependence of the functions in this proof to keep the notation as easy as possible. Inserting for y and y_r yields

$$-\left\|y-y_r\right\|_2^2 = -\left\|C_1(x_1-x_r)+C_2x_2\right\|_2^2 = -\begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix}^T C^TC \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix}.$$

The partitioned matrix (in)equality (10)

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} + \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \sum_{i=1}^m \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix}^T \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix} + k^2 \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \leq -C^TC \tag{23}$$

leads to

$$\begin{aligned}
&-\left\|y-y_r\right\|_2^2 \geq \\
&2(x_1-x_r)^T\Sigma_1 \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix} + \sum_{i=1}^m \left(\begin{bmatrix} N_{i,11} & N_{i,12} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix}\right)^T \Sigma_1 \begin{bmatrix} N_{i,11} & N_{i,12} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix} \\
&+ (x_1-x_r)^T\Sigma_1 k^2 (x_1-x_r) + x_2^T\Sigma_2 k^2 x_2 \\
&+ 2x_2^T\Sigma_2 \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix} + \sum_{i=1}^m \left(\begin{bmatrix} N_{i,21} & N_{i,22} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix}\right)^T \Sigma_2 \begin{bmatrix} N_{i,21} & N_{i,22} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix}.
\end{aligned}$$

Using the above summands, we define

$$\begin{aligned}
\mathcal{T}_1 &:= 2(x_1-x_r)^T\Sigma_1 \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix}, \\
\mathcal{T}_2 &:= \sum_{i=1}^m \left(\begin{bmatrix} N_{i,11} & N_{i,12} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix}\right)^T \Sigma_1 \begin{bmatrix} N_{i,11} & N_{i,12} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix} + (x_1-x_r)^T\Sigma_1 k^2 (x_1-x_r), \\
\mathcal{T}_3 &:= 2x_2^T\Sigma_2 \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix}, \\
\mathcal{T}_4 &:= \sum_{i=1}^m \left(\begin{bmatrix} N_{i,21} & N_{i,22} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix}\right)^T \Sigma_2 \begin{bmatrix} N_{i,21} & N_{i,22} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix} + x_2^T\Sigma_2 k^2 x_2.
\end{aligned}$$

The differential of $x_1 - x_r$ is given by

$$\frac{d(x_1-x_r)}{dt} = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix} + \sum_{i=1}^m \begin{bmatrix} N_{i,11} & N_{i,12} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix} u_i$$

which we insert into the following definition:

$$D_1 := \frac{d\left((x_1(t)-x_r(t))^T\Sigma_1(x_1(t)-x_r(t))\right)}{dt} = 2(x_1(t)-x_r(t))^T\Sigma_1\frac{d(x_1(t)-x_r(t))}{dt}.$$

Hence, we have

$$D_1 = 2(x_1-x_r)^T\Sigma_1\left(\begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix} + \sum_{i=1}^m \begin{bmatrix} N_{i,11} & N_{i,12} \end{bmatrix} \begin{bmatrix} x_1-x_r \\ x_2 \end{bmatrix} u_i\right).$$

We use an elementary estimate for the following inner product

$$\sum_{i=1}^{m} 2(x_1 - x_r)^T \Sigma_1 \left[\begin{smallmatrix} N_{i,11} & N_{i,12} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right] u_i$$
$$= \sum_{i=1}^{m} 2 \left\langle \Sigma_1^{\frac{1}{2}} (x_1 - x_r) u_i, \Sigma_1^{\frac{1}{2}} \left[\begin{smallmatrix} N_{i,11} & N_{i,12} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right] \right\rangle_2$$
$$\leq \sum_{i=1}^{m} \left(\left\| \Sigma_1^{\frac{1}{2}} (x_1 - x_r) u_i \right\|_2^2 + \left\| \Sigma_1^{\frac{1}{2}} \left[\begin{smallmatrix} N_{i,11} & N_{i,12} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right] \right\|_2^2 \right)$$
$$= (x_1 - x_r)^T \Sigma_1 (x_1 - x_r) \left\| u \right\|_2^2 + \sum_{i=1}^{m} \left(\left[\begin{smallmatrix} N_{i,11} & N_{i,12} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right] \right)^T \Sigma_1 \left[\begin{smallmatrix} N_{i,11} & N_{i,12} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right].$$

Using the fact that u is bounded, i.e., $\left\| u \right\|_2 \leq k$ then yields

$$\frac{d}{dt} (x_1(t) - x_r(t))^T \Sigma_1 (x_1(t) - x_r(t)) \leq \mathcal{T}_1 + \mathcal{T}_2.$$

From (21) the variable x_2 satifies

$$\frac{dx_2(t)}{dt} = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + B_2 u(t) + \sum_{i=1}^{m} \begin{bmatrix} N_{i,21} & N_{i,22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} u_i(t).$$

Plugging this into $\frac{dx_2(t)^T \Sigma_2 x_2(t)}{dt} = 2 x_2(t)^T \Sigma_2 \frac{x_2(t)}{dt}$ provides the following relation

$$\frac{dx_2}{dt} = 2 x_2^T \Sigma_2 \left(\begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B_2 u + \sum_{i=1}^{m} \begin{bmatrix} N_{i,21} & N_{i,22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} u_i \right). \tag{24}$$

We find an upper bound for the last term the same way we have done before

$$\sum_{i=1}^{m} 2 x_2^T \Sigma_2 \left[\begin{smallmatrix} N_{i,21} & N_{i,22} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] u_i = \sum_{i=1}^{m} 2 \left\langle \Sigma_2^{\frac{1}{2}} x_2 u_i, \Sigma_2^{\frac{1}{2}} \left[\begin{smallmatrix} N_{i,21} & N_{i,22} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \right\rangle_2$$
$$\leq x_2^T \Sigma_2 x_2 k^2 + \sum_{i=1}^{m} \left(\left[\begin{smallmatrix} N_{i,21} & N_{i,22} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \right)^T \Sigma_2 \left[\begin{smallmatrix} N_{i,21} & N_{i,22} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$$

applying again that the control is bounded by k. This yields

$$\begin{aligned} \frac{d}{dt} \left(x_2(t)^T \Sigma_2 x_2(t) \right) \leq & \left[\mathcal{T}_3 + 2 x_2^T \Sigma_2 (A_{21} x_r + B_2 u) \right] \\ & + \left[\mathcal{T}_4 + 2 \sum_{i=1}^{m} (N_{i,21} x_r)^T \Sigma_2 \begin{bmatrix} N_{i,21} & N_{i,22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right. \\ & \left. - \sum_{i=1}^{m} (N_{i,21} x_r)^T \Sigma_2 N_{i,21} x_r \right]. \end{aligned}$$

Summarising the above computations, we obtain

$$-\|y-y_r\|_2^2 \geq \mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4 \geq \frac{d}{dt}(x_1-x_r)^T\Sigma_1(x_1-x_r)+\frac{d}{dt}x_2^T\Sigma_2 x_2$$
$$-2x_2^T\Sigma_2(A_{21}x_r+B_2u)-2\sum_{i=1}^m (N_{i,21}x_r)^T\Sigma_2\begin{bmatrix} N_{i,21} & N_{i,22}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix}.$$

For the moment we assume that $\Sigma_2=\sigma I$. Since we have zero initial conditions, it holds that

$$\int_0^T \|y(t)-y_r(t)\|_2^2\,dt \leq 2\sigma^2\left(\int_0^T x_2^T\Sigma_2^{-1}(A_{21}x_r+B_2u)dt\right. \tag{25}$$
$$\left.+\int_0^T\sum_{i=1}^m (N_{i,21}x_r)^T\Sigma_2^{-1}\begin{bmatrix} N_{i,21} & N_{i,22}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix}dt\right).$$

Inequality (9) and the Schur complement condition on definiteness imply

$$\begin{bmatrix} A^T\Sigma^{-1}+\Sigma^{-1}A+\sum_{i=1}^m N_i^T\Sigma^{-1}N_i+\Sigma^{-1}k^2 & \Sigma^{-1}B \\ B^T\Sigma^{-1} & -I\end{bmatrix}\leq 0. \tag{26}$$

If we multiply the matrix inequality (26) with $\left[\begin{smallmatrix} x_1+x_r \\ x_2 \\ 2u\end{smallmatrix}\right]^T$ from the left and with $\left[\begin{smallmatrix} x_1+x_r \\ x_2 \\ 2u\end{smallmatrix}\right]$ from the right, we get

$$4\|u\|_2^2 \geq 2(x_1+x_r)^T\Sigma_1^{-1}\left(\left[\begin{smallmatrix} A_{11} & A_{12}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right]+2B_1u\right)+(x_1-x_r)^T\Sigma_1^{-1}k^2(x_1-x_r)$$
$$+\sum_{i=1}^m\left(\left[\begin{smallmatrix} N_{i,11} & N_{i,12}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right]\right)^T\Sigma_1^{-1}\left[\begin{smallmatrix} N_{i,11} & N_{i,12}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right]$$
$$+2x_2^T\Sigma_2^{-1}\left(\left[\begin{smallmatrix} A_{21} & A_{22}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right]+2B_2u\right)+x_2^T\Sigma_2^{-1}k^2x_2$$
$$+\sum_{i=1}^m\left(\left[\begin{smallmatrix} N_{i,21} & N_{i,22}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right]\right)^T\Sigma_2^{-1}\left[\begin{smallmatrix} N_{i,21} & N_{i,22}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right].$$

The above terms are used to define

$$\mathcal{T}_5 := 2(x_1+x_r)^T\Sigma_1^{-1}\left(\left[\begin{smallmatrix} A_{11} & A_{12}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right]+2B_1u\right),$$
$$\mathcal{T}_6 := (x_1+x_r)^T\Sigma_1^{-1}k^2(x_1+x_r)+\sum_{i=1}^m\left(\left[\begin{smallmatrix} N_{i,11} & N_{i,12}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right]\right)^T\Sigma_1^{-1}\left[\begin{smallmatrix} N_{i,11} & N_{i,12}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right],$$
$$\mathcal{T}_7 := 2x_2^T\Sigma_2^{-1}\left(\left[\begin{smallmatrix} A_{21} & A_{22}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right]+2B_2u\right),$$
$$\mathcal{T}_8 := x_2^T\Sigma_2^{-1}k^2x_2+\sum_{i=1}^m\left(\left[\begin{smallmatrix} N_{i,21} & N_{i,22}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right]\right)^T\Sigma_2^{-1}\left[\begin{smallmatrix} N_{i,21} & N_{i,22}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right].$$

Exploiting the following equation

$$\frac{d(x_1(t)+x_r(t))}{dt}=\begin{bmatrix} A_{11} & A_{12}\end{bmatrix}\begin{bmatrix} x_1(t)+x_r(t) \\ x_2(t)\end{bmatrix}+2B_1u(t)$$
$$+\sum_{i=1}^m\begin{bmatrix} N_{i,11} & N_{i,12}\end{bmatrix}\begin{bmatrix} x_1(t)+x_r(t) \\ x_2(t)\end{bmatrix}u_i(t),$$

we derive

$$\frac{d}{dt}\left((x_1+x_r)^T\Sigma_1^{-1}(x_1+x_r)\right) = 2(x_1+x_r)^T\Sigma_1^{-1}\frac{d}{dt}(x_1+x_r)$$
$$= 2(x_1+x_r)^T\Sigma_1^{-1}\left(\left[\begin{smallmatrix} A_{11} & A_{12}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right] + 2B_1u + \sum_{i=1}^m \left[\begin{smallmatrix} N_{i,11} & N_{i,12}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right]u_i\right).$$

Analogously to the above computations the last term can be bounded as follows

$$\sum_{i=1}^m 2(x_1+x_r)^T\Sigma_1^{-1}\left[\begin{smallmatrix} N_{i,11} & N_{i,12}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right]u_i$$
$$= \sum_{i=1}^m 2\left\langle \Sigma_1^{-\frac{1}{2}}(x_1+x_r)u_i, \Sigma_1^{-\frac{1}{2}}\left[\begin{smallmatrix} N_{i,11} & N_{i,12}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right]\right\rangle_2$$
$$\leq (x_1+x_r)^T\Sigma_1^{-1}(x_1+x_r)k^2 + \sum_{i=1}^m \left(\left[\begin{smallmatrix} N_{i,11} & N_{i,12}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right]\right)^T\Sigma_1^{-1}\left[\begin{smallmatrix} N_{i,11} & N_{i,12}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2\end{smallmatrix}\right].$$

This estimate yields

$$\frac{d}{dt}\left((x_1(t)+x_r(t))^T\Sigma_1^{-1}(x_1(t)+x_r(t))\right) \leq \mathcal{T}_5 + \mathcal{T}_6.$$

With equation (24) and the previous steps, we know that

$$\frac{d(x_2^T\Sigma_2^{-1}x_2)}{dt} \leq 2x_2^T\Sigma_2^{-1}\left(\left[\begin{smallmatrix} A_{21} & A_{22}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 \\ x_2\end{smallmatrix}\right] + B_2u\right) + x_2^T\Sigma_2^{-1}x_2k^2$$
$$+ \sum_{i=1}^m \left(\left[\begin{smallmatrix} N_{i,21} & N_{i,22}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 \\ x_2\end{smallmatrix}\right]\right)^T\Sigma_2^{-1}\left[\begin{smallmatrix} N_{i,21} & N_{i,22}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 \\ x_2\end{smallmatrix}\right],$$

such that

$$\frac{d}{dt}\left(x_2(t)^T\Sigma_2^{-1}x_2(t)\right) \leq \left[\mathcal{T}_7 - 2x_2^T\Sigma_2^{-1}(A_{21}x_r + B_2u)\right]$$
$$+ \left[\mathcal{T}_8 - 2\sum_{i=1}^m (N_{i,21}x_r)^T\Sigma_2^{-1}\begin{bmatrix} N_{i,21} & N_{i,22}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} - \sum_{i=1}^m (N_{i,21}x_r)^T\Sigma_2^{-1}N_{i,21}x_r\right].$$

In summary, we obtain

$$4\left\|u(t)\right\|_2^2 \geq \mathcal{T}_5 + \mathcal{T}_6 + \mathcal{T}_7 + \mathcal{T}_8$$
$$\geq \frac{d}{dt}\left((x_1(t)+x_r(t))^T\Sigma_1^{-1}(x_1(t)+x_r(t))\right) + \frac{d}{dt}\left(x_2(t)^T\Sigma_2^{-1}x_2(t)\right)$$
$$+ 2x_2^T\Sigma_2^{-1}(A_{21}x_r + B_2u) + 2\sum_{i=1}^m (N_{i,21}x_r)^T\Sigma_2^{-1}\left[\begin{smallmatrix} N_{i,21} & N_{i,22}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 \\ x_2\end{smallmatrix}\right].$$

Integration of both sides yields

$$4\int_0^T \|u(t)\|_2^2\,dt \geq 2\left[\int_0^T x_2^T\Sigma_2^{-1}(A_{21}x_r + B_2u)dt + \int_0^T \sum_{i=1}^m (N_{i,21}x_r)^T\Sigma_2^{-1}\left[\begin{smallmatrix} N_{i,21} & N_{i,22}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 \\ x_2\end{smallmatrix}\right]dt\right].$$

Combining this inequality with (25) leads to

$$\left(\int_0^T \|y(t) - y_r(t)\|_2^2\,dt\right)^{\frac{1}{2}} \leq 2\sigma\left(\int_0^T \|u(t)\|_2^2\,dt\right)^{\frac{1}{2}}. \tag{27}$$

The proof for general Σ_2 relies on the common idea of removing the Hankel singular values step by step. The error between the outputs y and y_r can be bounded as follows:

$$\|y - y_r\|_{L_T^2} \leq \|y - y_{r_\nu}\|_{L_T^2} + \left\|y_{r_\nu} - y_{r_{\nu-1}}\right\|_{L_T^2} + \ldots + \|y_{r_2} - y_r\|_{L_T^2},$$

where the dimensions r_i of the corresponding states are defined by $r_{i+1} = r_i + m(\tilde{\sigma}_i)$ for $i = 1, 2 \ldots, \nu - 1$. Here, $m(\tilde{\sigma}_i)$ denotes the multiplicity of $\tilde{\sigma}_i$ and $r_1 = r$. In the reduction step from y to y_{r_ν} only the smallest Hankel singular value $\tilde{\sigma}_\nu$ is removed from the system. Hence, by inequality (27), we have

$$\|y - y_{r_\nu}\|_{L_T^2} \leq 2\tilde{\sigma}_\nu \|u\|_{L_T^2}.$$

Inequality (27) can be established as well when comparing the reduced order outputs y_{r_ν} and $y_{r_{\nu-1}}$. Again, only one Hankel singular value, namely $\tilde{\sigma}_{r_{\nu-1}}$, is removed. At the same time, we have the same kind of inequalities in the ROM as before, that are,

$$A_{11}^T\Sigma_1^{-1} + \Sigma_1^{-1}A_{11} + \sum_{i=1}^m N_{i,11}^T\Sigma_1^{-1}N_{i,11} + \Sigma_1^{-1}k^2 \leq -\Sigma_1^{-1}B_1B_1^T\Sigma_1^{-1},$$
$$A_{11}^T\Sigma_1 + \Sigma_1A_{11} + \sum_{i=1}^m N_{i,11}^T\Sigma_1N_{i,11} + \Sigma_1k^2 \leq -C_1^TC_1.$$

So, by repeatedly applying the above arguments, we obtain

$$\left\|y_{r_j} - y_{r_{j-1}}\right\|_{L_T^2} \leq 2\tilde{\sigma}_{r_{j-1}}\|u\|_{L_T^2}$$

for $j = 2, \ldots, \nu$. This provides the claimed result. □

Since the bound in Theorem 4.1 involves only the sum of distinct diagonal entries of Σ_2, the result is also true when using the sum of all diagonal entries instead.

Corollary 4.2. *Let $x(0) = 0$, $x_r(0) = 0$ and $\|u(t)\|_2 \leq k$, $t \in [0, T]$, $T > 0$, where k is the constant that enters in (9) and (10). Then,*

$$\|y - y_r\|_{L^2_T} \leq 2(\sigma_{r+1} + \sigma_{r+2} + \ldots + \sigma_n) \|u\|_{L^2_T},$$

where y is the output of the original system (21), y_r is the output of the type II BT ROM and $\sigma_{r+1}, \ldots, \sigma_n$ are the diagonal entries of Σ_2.

The $\mathcal{H}_\infty$-error of using type II BT depends on the $n-r$ smallest HSVs of the original system. If now only states are neglected that correspond to small values $\sigma_{r+1}, \ldots, \sigma_n$ (dominant reachable and observable states), BT leads to a good approximation of the full order system by Corollary 4.2.

5 Conclusions

We have summerised previous work on balanced truncation for bilinear control systems. We have discussed Gramians that have been studied before and how they can be used to bound controllability and observability energy functionals, however, these bounds only hold in a small neighbourhood of zero. We proposed new Gramians (type II) which contain additional information about the control. With these type II Gramians global energy bounds can be found if the controls are assumed to be bounded in a certain way. These bounds justify the use of the alternative Gramians in the context of balancing. Based on this motivation, we explained the balancing procedure for bilinear systems which is similar to the one in the linear case. Another advantage of using the type II Gramians is the availability of an $\mathcal{H}_\infty$-error bound for balanced truncation of bilinear systems which we proved in this paper. The error bound requires the assumption of having a bounded input to the system. Hence, we have overcome the problem of previous works, where no error bound has been established.

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