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## On the solution of the generalized airfoil equation

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# 1 Introduction

In the present paper we consider the singular integral equation

$$\begin{aligned} & \frac{1}{\pi} \int_{-1}^1 \frac{f(y)}{x-y} dy + \frac{m(x)}{\pi} \int_{-1}^1 f(y) \ln|x-y| dy \\ & + \frac{1}{\pi} \int_{-1}^1 k(x,y) f(y) dy = g(x), \quad -1 < x < 1, \end{aligned} \quad (1.1)$$

where  $m, k$  and  $g$  are given functions,  $f$  is an unknown solution, and the first integral has to be interpreted in the Cauchy principal value sense. Equation (1.1) arises from the two-dimensional oscillating airfoil in a wind tunnel with subsonic flow (see, for example, [4]), has applications in diffraction theory and two-dimensional elasticity theory (see, for example, [16], [21]).

The analytical as well as the numerical solutions of Equation (1.1) have been studied by many authors [1]–[3], [6]–[9], [10]–[16], [18]–[21], [23]–[27], [29], [30], [32]. (Some of these papers only deal with the case  $m = 0$ .) Schleiff [29] solved Equation (1.1) for  $k = 0$  and  $m, f \in \mathcal{L}_\varrho^2$ , where  $\mathcal{L}_\varrho^2$  means the space of square integrable functions on the interval  $(-1,1)$  with the Chebyshev weight  $\varrho(x) = (1-x^2)^{1/2}$ . Using those results, he constructed a Fredholm integral equation of the second kind equivalent to Equation (1.1). In the present paper we extend Schleiff's results to the cases of spaces  $\mathcal{L}_w^2$  and of weighted Sobolev-type spaces with weights  $w(x) = (1-x)^\alpha(1+x)^\beta$ , where  $|\alpha| = |\beta| = 1/2$  (Section 3). These solvability results then give rise to establishing a numerical procedure for which stability and error estimates in a scale of Sobolev-type norms as well as in weighted uniform norms will be proved (Section 4).

## 2 Preliminaries

Throughout this paper let  $\lambda$  be the Lebesgue measure in the open interval  $\Omega = (-1, 1)$ . Those functions on  $\Omega$  which coincide outside a Lebesgue null set will be identified as usual.

Define functions  $\varrho$  and  $\sigma$  on  $\Omega$  by

$$\varrho(x) = (1-x^2)^{1/2} \quad \text{and} \quad \sigma(x) = (1-x)^{-1/2}(1+x)^{1/2}, \quad x \in \Omega. \quad (2.1)$$

Let  $w$  always stand for any of the functions  $w = \varrho, 1/\varrho, \sigma$  or  $1/\sigma$ . Let  $\pi^{-1}w\lambda$  denote the indefinite integral of the function  $\pi^{-1}w$  with respect to  $\lambda$ . As in [2], let  $\mathcal{L}_w^2$  denote the space  $\mathcal{L}^2(\pi^{-1}w\lambda)$  of complex-valued square integrable functions with respect to the measure  $\pi^{-1}w\lambda$ . Then  $\mathcal{L}_w^2$  becomes a Hilbert space with inner product

$$(f|g)_w = \pi^{-1} \int_{-1}^1 f \bar{g} w d\lambda, \quad f, g \in \mathcal{L}_w^2.$$

The associated norm on  $\mathcal{L}_w^2$  is denoted by  $\|\cdot\|_w$ . The following relationships are then clear:

- (i)  $\mathcal{L}_{1/\varrho}^2 \subset \mathcal{L}^2(\lambda) \subset \mathcal{L}_\varrho^2$ ;
- (ii)  $\mathcal{L}_{1/\varrho}^2 \subset \mathcal{L}_\sigma^2 \subset \mathcal{L}_\varrho^2$ ; and
- (iii)  $\mathcal{L}_{1/\varrho}^2 \subset \mathcal{L}_{1/\sigma}^2 \subset \mathcal{L}_\varrho^2$ .

Furthermore we have

$$\mathcal{L}_\varrho^2 \subset \bigcap_{1 < r < 4/3} \mathcal{L}^r(\lambda). \quad (2.2)$$

Let  $f \in \mathcal{L}^1(\lambda)$ . Then the Cauchy principal value

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow +0} \left( \int_{-1}^{x-\varepsilon} + \int_{x+\varepsilon}^1 \right) \frac{f(y)}{x-y} dy$$

exists for  $\lambda$ -almost every  $x \in \Omega$  and the resulting function  $Hf$  is  $\lambda$ -measurable (see [5, Theorem 8.1.5], for example). So we have a linear operator  $H$  from the space  $\mathcal{L}^1(\lambda)$  into the space of all  $\lambda$ -measurable functions. The following lemma is a special case of the Khvedelidze theorem, which can be found in [15, Theorem 1.2] or [24, Theorem II.3.1], for example.

**Lemma 2.1** *Let  $w = \varrho, 1/\varrho, \sigma$  or  $1/\sigma$ . Then  $H(\mathcal{L}_w^2) \subset \mathcal{L}_w^2$  and the restriction  $H_w$  of  $H$  to the Hilbert space  $\mathcal{L}_w^2$  is a continuous linear operator from  $\mathcal{L}_w^2$  into itself. Furthermore,  $(1/w)H(wf) \in \mathcal{L}_w^2$  for every  $f \in \mathcal{L}_w^2$ .*

A continuous linear operator  $S$  from a Banach space  $X$  into  $X$  is called a *Noether (Fredholm) operator* if its range  $\mathcal{R}(S) = S(X)$  is closed and if both the dimension of its null space  $\mathcal{N}(S) = S^{-1}(\{0\})$  and the co-dimension of  $\mathcal{R}(S)$  in  $X$  are finite. The index  $\text{ind}(S)$  of such an operator  $S$  is defined as

$$\text{ind}(S) := \dim \mathcal{N}(S) - \text{codim } \mathcal{R}(S).$$

**Lemma 2.2** *The following statements hold.*

- (i) *The operator  $H_\varrho : \mathcal{L}_\varrho^2 \rightarrow \mathcal{L}_\varrho^2$  is a surjection with null space*

$$\mathcal{N}(H_\varrho) = \{c/\varrho : c \in \mathbb{C}\},$$

and

$$H_\varrho^{-1}(\{g\}) = -(1/\varrho)H(\varrho g) + \mathcal{N}(H_\varrho), \quad g \in \mathcal{L}_\varrho^2.$$

In particular,  $\text{ind}(H_\varrho) = 1$ .

- (ii) *The operator  $H_{1/\varrho} : \mathcal{L}_{1/\varrho}^2 \rightarrow \mathcal{L}_{1/\varrho}^2$  is an injection with range*

$$\mathcal{R}(H_{1/\varrho}) = \{g \in \mathcal{L}_{1/\varrho}^2 : (g|1)_{1/\varrho} = 0\},$$

and

$$H_{1/\varrho}^{-1}g = -\varrho H(g/\varrho), \quad g \in \mathcal{R}(H_{1/\varrho}).$$

In particular,  $\text{ind}(H_{1/\varrho}) = -1$ .

- (iii) *Let  $w = \sigma$  or  $1/\sigma$ . Then the operator  $H_w : \mathcal{L}_w^2 \rightarrow \mathcal{L}_w^2$  is a bijective isometry, and*

$$H_w^{-1}g = -(1/w)H(wg), \quad g \in \mathcal{L}_w^2.$$

In particular,  $\text{ind}(H_w) = 0$ .

*Proof.* Statement (i) follows from the fact that the restriction of  $H$  to the Banach space  $\mathcal{L}^r(\lambda)$ ,  $1 < r < 4/3$  (cf. (2.2)) has the same property as  $H_\rho$  (see [17, Theorem 13.9] or [26, Proposition 2.4], for example).

Statement (ii) can be proved as in the case of the restriction of  $H$  to the Banach space  $\mathcal{L}^r(\lambda)$ ,  $2 < r < \infty$ , (see [17, Theorem 13.9] or [26, Proposition 2.6], for example).

Statement (iii) has been shown in [30, p.149] for the case when  $w = \sigma$ . The case  $w = 1/\sigma$  can be proved similarly.  $\square$

Let  $s \geq 0$ . We shall define a linear subspace  $\mathcal{L}_{\rho,s}^2$  of  $\mathcal{L}_\rho^2$  as in [2, §2]. Let

$$u_n(x) = \left( 2^{1/2} \sin[(n+1) \arccos x] \right) / \sin(\arccos x), \quad x \in \Omega,$$

for each  $n = 0, 1, 2, \dots$ . Namely,  $2^{-1/2}u_n$ ,  $n = 0, 1, 2, \dots$ , are the Chebyshev polynomials of the second kind. Then  $\{u_n\}_{n=0}^\infty$  is a complete orthonormal sequence in the Hilbert space  $\mathcal{L}_\rho^2$ . Now let  $\mathcal{L}_{\rho,s}^2$  denote the linear subspace of  $\mathcal{L}_\rho^2$  consisting of those functions  $f$  on  $\Omega$  such that

$$\sum_{n=0}^{\infty} (1+n)^{2s} |(f|u_n)_\rho|^2 < \infty.$$

The vector space  $\mathcal{L}_{\rho,s}^2$  becomes a Hilbert space with the inner product given by

$$(f|g)_{\rho,s} = \sum_{n=0}^{\infty} (1+n)^{2s} (f|u_n)_\rho \overline{(g|u_n)_\rho}, \quad f, g \in \mathcal{L}_{\rho,s}^2.$$

The associated norm on  $\mathcal{L}_{\rho,s}^2$  will be denoted by  $\|\cdot\|_{\rho,s}$ . Clearly the Hilbert space  $\mathcal{L}_{\rho,s}^2$  is continuously embedded into  $\mathcal{L}_\rho^2$ . It is worth noting that the definition of  $\mathcal{L}_{\rho,s}^2$  is dependent on  $\{u_n\}_{n=0}^\infty$  so that another complete orthonormal sequence in  $\mathcal{L}_\rho^2$  may define a different linear subspace of  $\mathcal{L}_\rho^2$ .

Let  $t_0 = 1$  and let  $t_n(x) = 2^{1/2} \cos(n \arccos x)$  for every  $x \in \Omega$  and every  $n = 1, 2, \dots$ . So  $t_0, 2^{-1/2}t_1, 2^{-1/2}t_2, \dots$ , are the Chebyshev polynomials of the first kind. Moreover, let

$$p_n(x) = \frac{\cos[(n+2^{-1}) \arccos x]}{\cos(2^{-1} \arccos x)} \quad \text{and} \quad q_n(x) = \frac{\sin[(n+2^{-1}) \arccos x]}{\sin(2^{-1} \arccos x)}$$

for every  $x \in \Omega$  and every  $n = 0, 1, 2, \dots$ . The so-defined functions  $p_n$  and  $q_n$ ,  $n = 0, 1, 2, \dots$ , are Chebyshev polynomials of the third and fourth kind respectively.

If  $s \geq 0$  and if  $w = 1/\rho, \sigma$  or  $1/\sigma$ , then we define the Hilbert space  $\mathcal{L}_{w,s}^2$  with inner product  $(\cdot|\cdot)_{w,s}$  by using  $\{t_n\}_{n=0}^\infty, \{p_n\}_{n=0}^\infty$  or  $\{q_n\}_{n=0}^\infty$  respectively, as  $\mathcal{L}_{\rho,s}^2$ .

Observe that  $\{u_n\}_{n=0}^\infty, \{t_n\}_{n=0}^\infty, \{p_n\}_{n=0}^\infty$  and  $\{q_n\}_{n=0}^\infty$  are unique complete orthonormal sequences of polynomials, with positive leading coefficients, having the property:

$$\deg u_n = \deg t_n = \deg p_n = \deg q_n = n, \quad n = 0, 1, 2, \dots,$$

in the Hilbert spaces  $\mathcal{L}_\rho^2, \mathcal{L}_{1/\rho}^2, \mathcal{L}_\sigma^2$  and  $\mathcal{L}_{1/\sigma}^2$  respectively.

Given a distribution  $v$  on  $\Omega$ , its derivative in the distribution sense will be denoted by  $Dv$ . According to [2, pp.196-197], the space  $\mathcal{L}_{w,s}^2$  can be expressed as follows.

**Lemma 2.3** Let  $w = \varrho, 1/\varrho, \sigma$  or  $1/\sigma$ . Let  $s$  be a positive integer. Then a function  $f \in \mathcal{L}_w^2$  belongs to  $\mathcal{L}_{w,s}^2$  if and only if  $\varrho^j D^j f$  is again an element of  $\mathcal{L}_w^2$  for every  $j = 1, 2, \dots, s$ . Furthermore, the norm  $\|\cdot\|_{w,s}$  on  $\mathcal{L}_{w,s}^2$  is equivalent to the norm:

$$f \mapsto \left( \sum_{j=0}^s \|\varrho^j D^j f\|_w^2 \right)^{1/2}, \quad f \in \mathcal{L}_{w,s}^2.$$

**Definition 2.1** Let  $w = \varrho, 1/\varrho, \sigma$  or  $1/\sigma$ . Let  $s > 0$ . Define

$$(1/w)\mathcal{L}_{1/w,s}^2 = \left\{ \frac{1}{w}f : f \in \mathcal{L}_{1/w,s}^2 \right\} \subset \mathcal{L}_w^2.$$

Equip the vector space  $(1/w)\mathcal{L}_{1/w,s}^2$  with the norm so that the linear map  $f \mapsto (1/w)f$ ,  $f \in \mathcal{L}_{1/w,s}^2$ , from  $\mathcal{L}_{1/w,s}^2$  onto  $(1/w)\mathcal{L}_{1/w,s}^2$  becomes an isometry; in particular,  $(1/w)\mathcal{L}_{1/w,s}^2$  is then a Banach space because so is  $\mathcal{L}_{1/w,s}^2$ .

The Banach space  $(1/w)\mathcal{L}_{1/w,s}^2$  is continuously embedded into  $\mathcal{L}_w^2$  and

$$H_w((1/w)\mathcal{L}_{1/w,s}^2) \subset \mathcal{L}_{w,s}^2 \quad s > 0. \quad (2.3)$$

This inclusion has been shown in [2, Lemma 4.1]. Its proof is based on the following result which is a special case of [33, (25)].

**Lemma 2.4** The following identities hold.

- (i)  $H(\varrho u_n) = t_{n+1}$ ,  $n = 0, 1, 2, \dots$
- (ii)  $H(t_0/\varrho) = 0$  and  $H(t_n/\varrho) = -u_{n-1}$ ,  $n = 1, 2, \dots$
- (iii)  $H(\sigma p_n) = -q_n$ ,  $n = 0, 1, 2, \dots$
- (iv)  $H(q_n/\sigma) = p_n$ ,  $n = 0, 1, 2, \dots$

If  $w = \varrho, 1/\varrho, \sigma$  or  $1/\sigma$  and  $s > 0$ , then let

$$H_{w,s} : (1/w)\mathcal{L}_{1/w,s}^2 \rightarrow \mathcal{L}_{w,s}^2 \quad (2.4)$$

denote the restriction of  $H_w$  to  $(1/w)\mathcal{L}_{1/w,s}^2$ ; see (2.3). The following lemma has essentially been given in [2, Lemma 4.2 (ii)] and its proof is clear in view of Lemma 2.4.

**Lemma 2.5** Let  $s > 0$ . Let  $w = \varrho, 1/\varrho, \sigma$  or  $1/\sigma$ . Then the linear operator  $H_{w,s}$  given by (2.4) enjoys the same property as  $H_w : \mathcal{L}_w^2 \rightarrow \mathcal{L}_w^2$  in Lemma 2.2.

Let  $AC(\Omega)$  denote the space of complex-valued, continuous functions  $f$  on  $\Omega$  for which there is an absolutely continuous function  $g$  on the closed interval  $[-1, 1]$  such that  $f(x) = g(x)$  for every  $x \in \Omega$ .

Let  $w = \varrho, 1/\varrho, \sigma$  or  $1/\sigma$ . Let  $f \in \mathcal{L}_w^2$ . By (2.2), the function  $Lf$  defined by

$$(Lf)(x) = \pi^{-1} \int_{-1}^1 f(y) \ln|y-x| dy, \quad x \in \Omega,$$

belongs to  $AC(\Omega)$  and  $D(Lf) = Hf$ ; see [17, §13], for example. In particular,  $Lf \in \mathcal{L}_w^2$  because  $AC(\Omega) \subset \mathcal{L}_w^2$ .

**Lemma 2.6** *The following identities hold.*

- (i)  $L(t_0/\varrho) = -(\ln 2)t_0 = -(2^{-1/2}\ln 2)u_0;$   
 $L(t_1/\varrho) = -t_1 = -2^{-1}u_1;$  and  
 $L(t_n/\varrho) = -t_n/n = 2^{-1}(u_{n-2} - u_n)/n, \quad n = 2, 3, \dots$
- (ii)  $L(\varrho u_0) = -2^{-1}[(\sqrt{2}\ln 2)t_0 - t_2/2];$  and  
 $L(\varrho u_n) = -2^{-1}[t_n/n - t_{n+2}/(n+2)], \quad n = 1, 2, \dots$
- (iii)  $L(\sigma p_0) = (2^{-1} - \ln 2)q_0 - 2^{-1}q_1;$  and  
 $L(\sigma p_n) = 2^{-1}[q_{n-1}/n - q_n/n(n+1) - q_{n+1}/(n+1)], \quad n = 1, 2, \dots$
- (iv)  $L(q_0/\sigma) = (2^{-1} - \ln 2)p_0 + 2^{-1}p_1;$  and  
 $L(q_n/\sigma) = -2^{-1}[p_{n-1}/n + p_n/n(n+1) - p_{n+1}/(n+1)], \quad n = 1, 2, \dots$

*Proof.* Statement (i) can be found in [28, Corollary, p.138], for instance. Statement (ii) follows from (i) because  $\varrho u_0 = (\sqrt{2}t_0 - t_2)/(2\varrho)$  and  $\varrho u_n = (t_n - t_{n+2})/(2\varrho)$  for every  $n = 1, 2, \dots$ . Statements (iii) has been given in [1, Corollary 3.3] and (iv) can be proved similarly.  $\square$

Let  $L_w : \mathcal{L}_w^2 \rightarrow \mathcal{L}_w^2$  denote the linear operator which assigns  $Lf$  to each function  $f \in \mathcal{L}_w^2$ , when  $w = \varrho, 1/\varrho, \sigma$  or  $1/\sigma$ . By [29, Satz 2] the operator  $L_\varrho$  is continuous. For the remaining cases:  $w = 1/\varrho, \sigma, 1/\sigma$ , the continuity of  $L_w$  follows from the closed graph theorem because  $L_w \subset \mathcal{L}_\varrho^2$ .

**Proposition 2.1** *Let  $s \geq 0$ . Let  $w = \varrho, 1/\varrho, \sigma$  or  $1/\sigma$ . Then  $L_w$  maps the subspace  $(1/w)\mathcal{L}_{1/w,s}^2$  of  $\mathcal{L}_{w,s}^2$  into  $\mathcal{L}_{w,s+1}^2$  and the linear map  $L_{w,s} : (1/w)\mathcal{L}_{1/w,s}^2 \rightarrow \mathcal{L}_{w,s+1}^2$  which assigns  $L_w f$  to each  $f \in (1/w)\mathcal{L}_{1/w,s}^2$  is continuous.*

*Proof.* In view of Definition 2.1, the statement is a direct consequence of the following inequalities:

- (i)  $\|L(f/\varrho)\|_{\varrho,s+1}^2 \leq (5/2)\|f\|_{1/\varrho,s}^2, \quad f \in \mathcal{L}_{1/\varrho,s}^2;$
- (ii)  $\|L(\varrho f)\|_{1/\varrho,s+1}^2 \leq (3(1+3^{2s})/2)\|f\|_{\varrho,s}^2, \quad f \in \mathcal{L}_{\varrho,s}^2;$
- (iii)  $\|L(f/\sigma)\|_{\sigma,s+1}^2 \leq 2(1+2^{2s+1})\|f\|_{1/\sigma,s}^2, \quad f \in \mathcal{L}_{1/\sigma,s}^2;$  and
- (iv)  $\|L(\sigma f)\|_{1/\sigma,s+1}^2 \leq 2(1+2^{2s+1})\|f\|_{\sigma,s}^2, \quad f \in \mathcal{L}_{\sigma,s}^2.$

A routine calculation based on Lemma 2.6 will derive those inequalities.  $\square$

**Remark 2.1** *In the case when  $w = \sigma$ , the statement of Proposition 2.1 has been given in [2, Lemma 5.1 (iv)], without stating constants as above.*

**Remark 2.2** *Let  $s \geq 0$ . The restriction of  $L_\varrho$  to  $(1/\varrho)\mathcal{L}_{1/\varrho,s}^2$  defines a continuous linear operator with values in  $\mathcal{L}_{1/\varrho,s+1}^2$ . In fact,*

$$\|L(f/\varrho)\|_{1/\varrho,s+1}^2 \leq 4\|f\|_{1/\varrho,s}^2, \quad f \in \mathcal{L}_{1/\varrho,s}^2.$$

### 3 The unperturbed generalized airfoil equation

Let  $\Omega = (-1, 1)$ . Let  $w$  stand for any of the functions  $\varrho, 1/\varrho, \sigma$  and  $1/\sigma$  on  $\Omega$  as in Section 2. The main aim of this section is to solve, in  $\mathcal{L}_w^2$ , the singular integral equation

$$(H_w + mL_w)f = g \tag{3.1}$$

for a given  $g \in \mathcal{L}_w^2$ , when  $m \in \mathcal{L}_w^2$ .

In the case when  $w = \varrho$ , the integral equation (3.1) has already been solved by M. Schleiff [29]. We shall deduce the remaining cases from his result, by using the fact that  $\mathcal{L}_w^2 \subset \mathcal{L}_\varrho^2$ . Let  $m \in \mathcal{L}_\varrho^2$ . The Volterra operator  $V$  on  $\mathcal{L}_\varrho^2$  is defined by

$$(Vf)(x) = \int_{-1}^x f \, d\lambda, \quad x \in \Omega, \quad (3.2)$$

for every  $f \in \mathcal{L}_\varrho^2$ . Then  $V(\mathcal{L}_\varrho^2) \subset AC(\Omega)$ . Define a linear operator  $M_\varrho : \mathcal{L}_\varrho^2 \rightarrow \mathcal{L}_\varrho^2$  by

$$M_\varrho f = f + m \left( Vf + \pi^{-1} \int_{-1}^1 f(y)(-\pi/2 + \arcsin y) \, dy \right)$$

for every  $f \in \mathcal{L}_\varrho^2$ . Furthermore, define functions  $a$  and  $b$  on  $\Omega$  by

$$a(x) = \exp[-(Vm)(x)] \quad \text{and} \quad b(x) = \int_x^1 a/\varrho \, d\lambda$$

for every  $x \in \Omega$  respectively. It is clear that  $M_\varrho$  is continuous. Moreover,  $M_\varrho$  is invertible.

**Lemma 3.1** ([29, pp.83-84]). *The linear operator  $M_\varrho : \mathcal{L}_\varrho^2 \rightarrow \mathcal{L}_\varrho^2$  is a surjective isomorphism and its inverse is of the form*

$$M_\varrho^{-1}g = g - (ma) \left[ V(g/a) - \left( \int_{-1}^1 (gb/a) \, d\lambda \right) \left( \int_{-1}^1 a/\varrho \, d\lambda \right)^{-1} \right]$$

for every  $g \in \mathcal{L}_\varrho^2$ . In particular,

$$M_\varrho^{-1}m = \pi \left( \int_{-1}^1 a/\varrho \, d\lambda \right)^{-1} ma.$$

We are now ready to present Schleiff's result in [29], which shows that the operator  $H_\varrho + mL_\varrho$  behaves like  $H_\varrho$  (see Lemma 2.2).

**Proposition 3.1** *Let  $m \in \mathcal{L}_\varrho^2$ . Then the linear operator  $H_\varrho + mL_\varrho : \mathcal{L}_\varrho^2 \rightarrow \mathcal{L}_\varrho^2$  is a continuous surjection such that its null space  $\mathcal{N}(H_\varrho + mL_\varrho)$  is spanned by the function  $\Phi$  defined by*

$$\Phi = [1 - (\ln 2)H(\varrho M_\varrho^{-1}m)](1/\varrho). \quad (3.3)$$

Moreover,

$$(H_\varrho + mL_\varrho)^{-1}(\{g\}) = -(1/\varrho)H(\varrho M_\varrho^{-1}g) + \mathcal{N}(H_\varrho + mL_\varrho) \quad (3.4)$$

for every  $g \in \mathcal{L}_\varrho^2$ . In particular,

$$\text{ind}(H_\varrho + mL_\varrho) = 1.$$



The proof of the following lemma is clear and its proof will be omitted.

**Lemma 3.2** *Let  $w = 1/\varrho$ ,  $\sigma$  or  $1/\sigma$ . Then  $\mathcal{L}_w^2$  is invariant under  $M_\varrho$  and the restriction  $M_w$  of  $M_\varrho$  to  $\mathcal{L}_w^2$  defines an isomorphism from the Banach space  $\mathcal{L}_w^2$  onto  $\mathcal{L}_w^2$ .*

We now concentrate on the case when  $w = \sigma$ . The equalities

$$H_\sigma^{-1}h = -(1/\sigma)H(\sigma h) = -(1/\varrho)H(\varrho h) + (1/\varrho)\pi^{-1} \int_{-1}^1 \sigma h \, d\lambda, \quad h \in \mathcal{L}_\sigma^2, \quad (3.5)$$

hold in the Banach space  $\mathcal{L}_\sigma^2$ . In fact, the first equality in (3.5) has already been given in Lemma 2.2 (iii). The second equality follows easily from the fact that

$$-\sigma(y)/\sigma(x) + \varrho(y)/\varrho(x) = (x - y)\sigma(y)/\varrho(x) \quad (3.6)$$

for all  $x, y \in \Omega$ . By (3.5), we have

$$\int_{-1}^1 H_\sigma^{-1}h \, d\lambda = - \int_{-1}^1 (1/\sigma)H(\sigma h) \, d\lambda = \int_{-1}^1 \sigma h \, d\lambda, \quad h \in \mathcal{L}_\sigma^2, \quad (3.7)$$

because

$$\int_{-1}^1 (1/\varrho)H(\varrho h) \, d\lambda = 0 \quad (3.8)$$

which is a consequence of the Parseval identity (cf. [24, Theorem II.4.4]) and the fact that  $H(1/\varrho) = 0$  (cf. [34, p.174]). From (3.5) and (3.7) it follows that

$$H_\sigma^{-1}h - (1/\varrho)\pi^{-1} \int_{-1}^1 H_\sigma^{-1}h \, d\lambda = -(1/\varrho)H(\varrho h), \quad h \in \mathcal{L}_\sigma^2. \quad (3.9)$$

**Lemma 3.3** *Let  $m \in \mathcal{L}_\sigma^2$ . The function  $\Phi \in \mathcal{L}_\varrho^2$  defined by (3.3) belongs to the space  $\mathcal{L}_\sigma^2$  if and only if*

$$\pi = (\ln 2) \int_{-1}^1 H_\sigma^{-1}M_\sigma^{-1}m \, d\lambda, \quad (3.10)$$

in which case  $\Phi = (\ln 2)H_\sigma^{-1}M_\sigma^{-1}m$  and  $\mathcal{N}(H_\sigma + mL_\sigma) = \text{span} \{\Phi\}$ .

*Proof.* By (3.9) applied to  $h = M_\sigma^{-1}m$ , we have

$$\Phi = (1/\varrho) \left( 1 - \pi^{-1}(\ln 2) \int_{-1}^1 H_\sigma^{-1}M_\sigma^{-1}m \, d\lambda \right) + (\ln 2)H_\sigma^{-1}M_\sigma^{-1}m.$$

The statement now follows from the facts that

$$1/\varrho \notin \mathcal{L}_\sigma^2 \quad (3.11)$$

and that

$$\mathcal{N}(H_\sigma + mL_\sigma) = \mathcal{L}_\sigma^2 \cap \mathcal{N}(H_\varrho + mL_\varrho).$$

□

The index of the operator  $H_\sigma + mL_\sigma$  is the same as that of  $H_\sigma$  as shown in the following theorem.

**Theorem 3.1** *Let  $m$  be a non-zero function belonging to the space  $\mathcal{L}_\sigma^2$ . Then the following statements on the continuous linear operator  $H_\sigma + mL_\sigma : \mathcal{L}_\sigma^2 \rightarrow \mathcal{L}_\sigma^2$  hold.*

(i) *Suppose that (3.10) holds. Then*

$$\mathcal{N}(H_\sigma + mL_\sigma) = \text{span} \{H_\sigma^{-1}M_\sigma^{-1}m\}.$$

*Furthermore, a function  $g \in \mathcal{L}_\sigma^2$  belongs to the range  $\mathcal{R}(H_\sigma + mL_\sigma)$  if and only if*

$$\int_{-1}^1 H_\sigma^{-1}M_\sigma^{-1}g \, d\lambda = 0, \quad (3.12)$$

*in which case*

$$(H_\sigma + mL_\sigma)^{-1}(\{g\}) = H_\sigma^{-1}M_\sigma^{-1}g + \mathcal{N}(H_\sigma + mL_\sigma).$$

(ii) *Suppose that (3.10) does not hold. Then  $H_\sigma + mL_\sigma$  is a bijection and for a given  $g \in \mathcal{L}_\sigma^2$ ,*

$$(H_\sigma + mL_\sigma)^{-1}g = H_\sigma^{-1}M_\sigma^{-1}[g + (c_g \ln 2)m], \quad (3.13)$$

*where  $c_g$  is the constant defined by*

$$c_g = \left( \int_{-1}^1 \sigma M_\sigma^{-1}g \, d\lambda \right) \left( \pi - (\ln 2) \int_{-1}^1 \sigma M_\sigma^{-1}m \, d\lambda \right)^{-1}. \quad (3.14)$$

*Proof.* Recall that  $\Phi$  is the function given by (3.3) which spans  $\mathcal{N}(H_\varrho + mL_\varrho)$ ; see Proposition 3.1. If  $g \in \mathcal{L}_\sigma^2$  and  $c \in \mathcal{C}$ , then we have

$$\begin{aligned} -\frac{1}{\varrho}H(\varrho M_\varrho^{-1}g) + c\Phi &= -\frac{1}{\varrho}H \left[ \varrho(M_\varrho^{-1}g + (c \ln 2)M_\varrho^{-1}m) \right] + \frac{c}{\varrho} \\ &= H_\sigma^{-1}M_\sigma^{-1}[g + (c \ln 2)m] + \frac{1}{\varrho} \left[ c \left( 1 - \frac{\ln 2}{\pi} \int_{-1}^1 H_\sigma^{-1}M_\sigma^{-1}m \, d\lambda \right) \right. \\ &\quad \left. - \frac{1}{\pi} \int_{-1}^1 H_\sigma^{-1}M_\sigma^{-1}g \, d\lambda \right] \end{aligned} \quad (3.15)$$

by applying (3.9) to  $h = M_\sigma^{-1}(g + c(\ln 2)m)$ . Moreover, observe that

$$(H_\sigma + mL_\sigma)^{-1}(\{g\}) = \mathcal{L}_\sigma^2 \cap (H_\varrho + mL_\varrho)^{-1}(\{g\}), \quad g \in \mathcal{L}_\sigma^2. \quad (3.16)$$

(i) Given  $g \in \mathcal{L}_\sigma^2$  and  $c \in \mathcal{C}$ , it follows from (3.10) and (3.15) that

$$\begin{aligned} -(1/\varrho)H(\varrho M_\varrho^{-1}g) + c\Phi &= H_\sigma^{-1}M_\sigma^{-1}(g + c(\ln 2)m) \\ &\quad - (1/\varrho)\pi^{-1} \int_{-1}^1 H_\sigma^{-1}M_\sigma^{-1}g \, d\lambda \end{aligned} \quad (3.17)$$

as elements of  $\mathcal{L}_\varrho^2$ . Hence (3.11) implies that the left-hand side of (3.17) belongs to  $\mathcal{L}_\sigma^2$  if and only if (3.12) hold. Accordingly, given  $g \in \mathcal{L}_\sigma^2$ , it follows from (3.4), (3.16) and (3.17) that

$$(H_\sigma + mL_\sigma)^{-1}(\{g\}) \neq \phi \quad (3.18)$$

if and only if (3.12) holds. Therefore the second half of statement (i) has been established. The first half of (i) has already been given in Lemma 3.3.

(ii) By Lemma 3.3, the operator  $H_\sigma + mL_\sigma$  is injective. To show its surjectivity, let  $g \in \mathcal{L}_\sigma^2$ . The left-hand side of (3.15) is an element of  $\mathcal{L}_\sigma^2$  if and only if  $c$  equals the constant  $c_g$  given by (3.14); we have used (3.7). It then follows from (3.4) and (3.16) that  $H_\sigma + mL_\sigma$  is surjective and that (3.13) holds.  $\square$

**Remark 3.1** Let  $m$  be a non-zero function belonging to  $\mathcal{L}_{1/\sigma}^2$ . Then statements (i) and (ii) of Theorem 3.1 hold with replacement of the subscript  $\sigma$  by the subscript  $1/\sigma$ . The proof will be almost the same if we replace  $\sigma$  by  $1/\sigma$ . The only exceptional relationships to be modified are (3.5), (3.6) and (3.7). The modified versions are as follows:

$$H_{1/\sigma}^{-1}h = -\sigma H(h/\sigma) = -(1/\varrho)H(\varrho h) - (1/\varrho)\pi^{-1} \int_{-1}^1 h/\sigma \, d\lambda, \quad h \in \mathcal{L}_{1/\sigma}^2; \quad (3.5^*)$$

$$-\sigma(x)/\sigma(y) + \varrho(y)/\varrho(x) = (y-x)/(\varrho(x)\sigma(y)), \quad x, y \in \Omega; \quad (3.6^*)$$

$$-\int_{-1}^1 H_{1/\sigma}^{-1}h \, d\lambda = -\int_{-1}^1 \sigma H(h/\sigma) \, d\lambda = -\int_{-1}^1 h/\sigma \, d\lambda, \quad h \in \mathcal{L}_{1/\sigma}^2. \quad (3.7^*)$$

Now we shall consider the case when  $w = 1/\varrho$ . Of course we need to apply Proposition 3.1. For our proof, (3.5) will be replaced by

$$-\varrho H(h/\varrho) = -(1/\varrho) \left[ H(\varrho h) - \pi^{-1} \int_{-1}^1 \mathbf{x}h/\varrho \, d\lambda - \pi^{-1} \mathbf{x} \int_{-1}^1 h/\varrho \, d\lambda \right], \quad h \in \mathcal{L}_{1/\varrho}^2, \quad (3.5^{**})$$

where  $\mathbf{x}$  denotes the identity function on  $\Omega$ . Define continuous linear functionals  $\alpha$  and  $\beta$  on the Banach space  $\mathcal{L}_{1/\varrho}^2$  by

$$\langle \alpha, h \rangle = \pi^{-1} \int_{-1}^1 (M_{1/\varrho}^{-1}h)/\varrho \, d\lambda \quad \text{and} \quad \langle \beta, h \rangle = \pi^{-1} \int_{-1}^1 (\mathbf{x}M_{1/\varrho}^{-1}h)/\varrho \, d\lambda$$

for every  $h \in \mathcal{L}_{1/\varrho}^2$ , respectively. Then the function  $\Phi$  given by (3.3) has the form

$$\Phi = -(\ln 2)\varrho H[(M_{1/\varrho}^{-1}m)/\varrho] - [1 - (\ln 2)\langle \beta, m \rangle](1/\varrho) - (\ln 2)\langle \alpha, m \rangle(\mathbf{x}/\varrho).$$

Since  $\varrho H[(M_{1/\varrho}^{-1}m)/\varrho] \in \mathcal{L}_{1/\varrho}^2$ , the function  $\Phi$  belongs to  $\mathcal{L}_{1/\varrho}^2$  if and only if

$$1 - (\ln 2)\langle \beta, m \rangle = 0 = \langle \alpha, m \rangle \quad (3.19)$$

because neither  $1/\varrho$  nor  $\mathbf{x}/\varrho$  is an element of  $\mathcal{L}_{1/\varrho}^2$ . Now, if  $g \in \mathcal{L}_{1/\varrho}^2$  and  $c \in \mathcal{C}$ , then

$$\begin{aligned} -(1/\varrho)H(\varrho M_{1/\varrho}^{-1}g) + c\Phi &= -\varrho H[(1/\varrho)M_{1/\varrho}^{-1}(g + c(\ln 2)m)] \\ &+ [c - \langle \beta, g + c(\ln 2)m \rangle](1/\varrho) - \langle \alpha, g + c(\ln 2)m \rangle(\mathbf{x}/\varrho). \end{aligned}$$

With the above observations, the proof of the following theorem is straightforward and we shall leave the details with the reader.

**Theorem 3.2** *Let  $m$  be a non-zero function in the Banach space  $\mathcal{L}_{1/\varrho}^2$ . Then the following statements on the continuous linear operator  $H_{1/\varrho} + mL_{1/\varrho} : \mathcal{L}_{1/\varrho}^2 \rightarrow \mathcal{L}_{1/\varrho}^2$  hold.*

(i) *Suppose that (3.19) holds. Then  $H_{1/\varrho} + mL_{1/\varrho}$  has one-dimensional null space:*

$$\mathcal{N}(H_{1/\varrho} + mL_{1/\varrho}) = \text{span} \{ \varrho H[(M_{1/\varrho}^{-1}m)/\varrho] \}.$$

*A function  $g \in \mathcal{L}_{1/\varrho}^2$  belongs to  $\mathcal{R}(H_{1/\varrho} + mL_{1/\varrho})$  if and only if  $\langle \alpha, g \rangle = 0 = \langle \beta, g \rangle$ , in which case*

$$(H_{1/\varrho} + mL_{1/\varrho})^{-1}(\{g\}) = -\varrho H[(M_{1/\varrho}^{-1}g)/\varrho] + \mathcal{N}(H_{1/\varrho} + mL_{1/\varrho})$$

*so that  $\text{codim } \mathcal{R}(H_{1/\varrho} + mL_{1/\varrho}) = 2$ .*

(ii) *Suppose that (3.19) does not hold. Then  $H_{1/\varrho} + mL_{1/\varrho}$  is injective and its range consists of those functions  $g \in \mathcal{L}_{1/\varrho}^2$  such that*

$$(1 - (\ln 2)\langle \beta, m \rangle)\langle \alpha, g \rangle + (\ln 2)\langle \alpha, m \rangle\langle \beta, g \rangle = 0.$$

*For such a function  $g$ ,*

$$(H_{1/\varrho} + mL_{1/\varrho})^{-1}g = -\varrho H[(1/\varrho)M_{1/\varrho}^{-1}(g + c_g(\ln 2)m)],$$

*where  $c_g$  is the constant determined by the two identities:*

$$c_g(\ln 2)\langle \alpha, m \rangle + \langle \alpha, g \rangle = 0 = c_g(1 - (\ln 2)\langle \beta, m \rangle) - \langle \beta, g \rangle.$$

From the above theorem we can see that

$$\text{ind}(H_{1/\varrho}) = -1 = \text{ind}(H_{1/\varrho} + mL_{1/\varrho}),$$

for all  $m \in \mathcal{L}_{1/\varrho}^2$ .

Finally we shall show that  $H_{\sigma,s} + mL_{\sigma,s}$  has the same properties as  $H_{\sigma} + mL_{\sigma}$  for every  $s > 0$ , when  $m$  is smooth. Our arguments can easily be adapted to the remaining cases:  $w = \varrho, 1/\varrho, 1/\sigma$ ; so we shall not discuss them here.

Let us fix a positive number  $s$  and let  $r$  be the smallest positive integer such that  $r \geq s$ . The Hilbert space  $\mathcal{L}_{\sigma,s}^2$  is an intermediate space between  $\mathcal{L}_{\sigma,r-1}^2$  and  $\mathcal{L}_{\sigma,r}^2$ . In fact, let  $\Lambda$  be the linear operator in the Hilbert space  $\mathcal{L}_{\sigma,r-1}^2$ , with domain  $\mathcal{D}(\Lambda) = \mathcal{L}_{\sigma,r}^2$ , defined by

$$\Lambda f = \sum_{n=0}^{\infty} (1+n)^r (f|p_n)_{\sigma} p_n, \quad f \in \mathcal{D}(\Lambda).$$

Then the operator  $\Lambda$  is self-adjoint, positive and unbounded in  $\mathcal{L}_{\sigma, r-1}^2$ . Moreover, if  $0 < \theta < 1$ , then the intermediate space  $[\mathcal{L}_{\sigma, r-1}^2, \mathcal{L}_{\sigma, r}^2]_{\theta}$  is defined as the domain of the linear operator  $\Lambda^{1-\theta}$  in  $\mathcal{L}_{\sigma, r-1}^2$ . Of course,  $\Lambda^{1-\theta}$  has the form

$$\Lambda^{1-\theta}(f) = \sum_{n=0}^{\infty} (1+n)^{r(1-\theta)} (f|p_n)_{\sigma} p_n, \quad f \in \mathcal{D}(\Lambda),$$

(see [22, §2.1 in Chapter 1]). Hence,  $[\mathcal{L}_{\sigma, r-1}^2, \mathcal{L}_{\sigma, r}^2]_{\theta} = \mathcal{L}_{\sigma, r(1-\theta)}^2$ ,  $0 < \theta < 1$ . In particular,

$$[\mathcal{L}_{\sigma, r-1}^2, \mathcal{L}_{\sigma, r}^2]_{1-s/r} = \mathcal{L}_{\sigma, s}^2. \quad (3.20)$$

By  $C_{\rho}^r([-1, 1])$  we denote the space of all  $r$  times differentiable functions  $u : \Omega \rightarrow \mathcal{C}$  such that  $\rho^k D^k u$  has continuous extension to the closed interval  $[-1, 1]$  for each  $k = 0, 1, \dots, r$ . Furthermore, define a norm on  $C_{\rho}^r([-1, 1])$  by

$$\|u\|_{C_{\rho}^r} := \sum_{k=0}^r \|\rho^k D^k u\|_{\infty}, \quad u \in C_{\rho}^r([-1, 1]),$$

where  $\|\cdot\|_{\infty}$  denotes the uniform norm. Given  $m \in C_{\rho}^r([-1, 1])$ , let

$$m\mathcal{L}_{\sigma, \delta}^2 = \{mf : f \in \mathcal{L}_{\sigma, \delta}^2\},$$

which is a linear subspace of  $\mathcal{L}_{\sigma, r-1}^2$  for every  $\delta \in [r-1, r]$ .

**Lemma 3.4** *Let  $m \in C_{\rho}^r([-1, 1])$ . Then the following statements hold.*

(i) *Let  $\delta = r-1$  or  $r$ . Then  $m\mathcal{L}_{\sigma, \delta}^2 \subset \mathcal{L}_{\sigma, \delta}^2$  and the  $\mathcal{L}_{\sigma, \delta}^2$ -valued linear operator:*

$$f \mapsto mf, \quad f \in \mathcal{L}_{\sigma, \delta}^2,$$

*is continuous. Moreover,*

$$\|mf\|_{\sigma, \delta} \leq \text{const} \cdot \|m\|_{C_{\rho}^r} \|f\|_{\sigma, \delta}.$$

(ii) *It follows that  $m\mathcal{L}_{\sigma, s}^2 \subset \mathcal{L}_{\sigma, s}^2$  and the  $\mathcal{L}_{\sigma, s}^2$ -valued linear operator:*

$$f \mapsto mf, \quad f \in \mathcal{L}_{\sigma, s}^2,$$

*is continuous.*

*Proof.* Statement (i) follows from Lemma 2.3 together with Leibnitz's formula. Statement (ii) is a consequence of the interpolation theorem [22, Theorem 5.1, Chapter 1] because of (i) and (3.19).  $\square$

**Lemma 3.5** *The Hilbert space  $\mathcal{L}_{\sigma, s}^2$  is invariant under the Volterra operator  $V$  (see (3.2)) and the restriction of  $V$  to  $\mathcal{L}_{\sigma, s}^2$  is a continuous linear operator from  $\mathcal{L}_{\sigma, s}^2$  into  $\mathcal{L}_{\sigma, s+1}^2$ .*

*Proof* is analogous to that of Lemma 3.4 (ii) because of (3.20).  $\square$

**Corollary 3.1** *Let  $m \in C_\rho^r([-1, 1])$ . The restriction  $M_{\sigma, s}$  of  $M_\sigma$  to  $\mathcal{L}_{\sigma, s}^2$  is an isomorphism onto  $\mathcal{L}_{\sigma, s}^2$ .*

*Proof* follows from Lemmas 3.4 and 3.5. □

Let  $m \in C_\rho^r([-1, 1])$ . It then follows from Proposition 2.1 and Lemma 3.4 that  $mL_{\sigma, s}$  can be regarded as a continuous linear operator from  $(1/\sigma)\mathcal{L}_{1/\sigma, s}^2 (\subset \mathcal{L}_{\sigma, s}^2)$  into  $\mathcal{L}_{\sigma, s}^2$  because  $\mathcal{L}_{\sigma, s+1}^2$  is continuously embedded into  $\mathcal{L}_{\sigma, s}^2$ . We are now ready to present the main result which follows immediately from Theorem 3.1 in view of Corollary 3.1.

**Theorem 3.3** *Let  $s > 0$ . Let  $r$  be the smallest positive integer such that  $r \geq s$ . Suppose that  $m \in C_\rho^r([-1, 1])$  is a non-zero function. Then the linear operator  $H_{\sigma, s} + mL_{\sigma, s} : (1/\sigma)\mathcal{L}_{1/\sigma, s}^2 \rightarrow \mathcal{L}_{\sigma, s}^2$  is continuous, and statements (i) and (ii) of Theorem 3.1 hold with replacement of the subscript  $\sigma$  by the subscripts  $\sigma, s$ .*

## 4 A numerical procedure

The results of Section 3 allow us to consider a numerical procedure for solving singular integral equations of the form

$$(H_w + mL_w + K)f = g \tag{4.1}$$

where  $g \in \mathcal{L}_w^2$  and  $m \in \mathcal{L}_w^2$  are given functions and  $K$  is a given compact linear integral operator acting in  $\mathcal{L}_w^2$ .

Fix a positive integer  $n$ . Let  $w = \rho, 1/\rho, \sigma$  or  $1/\sigma$  and let  $h_n$  be one of the corresponding polynomials  $u_n, t_n, p_n$  or  $q_n$ , respectively. Let  $y_{n,i}, i = 1, \dots, n$ , be the zeros of  $h_n$ , which are known to be distinct and belong to the open interval  $(-1, 1)$ . Define the Lagrangian fundamental polynomials  $l_{n,i}^w, i = 1, 2, \dots, n$ , by

$$l_{n,i}^w(y) = \frac{h_n(y)}{(y - y_{n,i})h_n'(y_{n,i})} = \prod_{\substack{j=1 \\ j \neq i}}^n \frac{y - y_{n,j}}{y_{n,i} - y_{n,j}}, \quad y \in (-1, 1).$$

For an arbitrary continuous function  $u : (-1, 1) \rightarrow \mathcal{C}$ , the Lagrangian interpolation projector  $\mathbf{L}_n^w$  is defined by

$$\mathbf{L}_n^w : u \mapsto \sum_{i=1}^n u(y_{n,i})l_{n,i}^w.$$

Assume that the operator  $K$  has the form

$$Ku(x) = \frac{1}{\pi} \int_{-1}^1 k(x, y)u(y) dy, \quad x \in (-1, 1), \quad u \in \mathcal{L}_w^2.$$

It is well known that  $K$  is a compact operator in  $\mathcal{L}_w^2$  if the kernel function  $k$  satisfies the condition

$$\int_{-1}^1 \int_{-1}^1 |k(x, y)|^2 w(x)/w(y) dy dx < \infty.$$

In the sequel we make the following assumptions about the smoothness of  $k$ . Assume that  $k(\cdot, y) \in \mathcal{L}_{w,s}^2$  uniformly with respect to  $y \in (-1, 1)$ , and  $k(x, \cdot) \in \mathcal{L}_{1/w,r}^2$  uniformly with respect to  $x \in (-1, 1)$ , with some positive real numbers  $s$  and  $r$  to be specified later; in other words, there are constants  $C_1$  and  $C_2$  (independent of both  $x$  and  $y$ ) such that

$$\|k(\cdot, y)\|_{w,s} \leq C_1 \quad \text{and} \quad \|k(x, \cdot)\|_{1/w,r} \leq C_2 \quad (4.2)$$

for all  $x, y \in (-1, 1)$ . Under the above conditions, the operator  $K : \mathcal{L}_w^2 \rightarrow \mathcal{L}_{w,t}^2$  is continuous for all  $t \leq s$  and compact for all  $t < s$  (see, for example, [2, Lemma 4.2]).

In what follows we shall denote by  $\|\cdot\|_{1/w,s}^*$  the norm on the Banach space  $\tilde{\mathcal{L}}_{1/w,s}^2 := (1/w)\mathcal{L}_{1/w,s}^2$  (see Definition 2.1); that is,

$$\|v\|_{1/w,s}^* = \|wv\|_{1/w,s}, \quad v \in \tilde{\mathcal{L}}_{1/w,s}^2.$$

Suppose that  $s > \frac{1}{2}$  and  $r > \frac{1}{2}$ . Introduce the operator  $K_n$  by

$$K_n u(x) = \frac{1}{\pi} \int_{-1}^1 [\mathbf{L}_{n,y}^{1/w} k(x, y)] u(y) dy, \quad x \in (-1, 1),$$

for all  $u \in \mathcal{L}_w^2$ . The subscript  $y$  of  $\mathbf{L}_{n,y}^{1/w}$  indicates that the interpolation is realized with respect to the variable  $y$ . Given  $x \in (-1, 1)$ , the function  $k(x, \cdot) \in \mathcal{L}_{1/w,r}^2$  is continuous on  $(-1, 1)$  by [2, Theorem 2.5] because  $r > \frac{1}{2}$ , and hence we can define  $K_n u(x)$  for each  $u \in \mathcal{L}_w^2$ . The so-defined function  $K_n u$  on  $(-1, 1)$  is continuous again by [2, Theorem 2.5] applied to  $k(\cdot, y) \in \mathcal{L}_{w,s}^2$  with  $s > \frac{1}{2}$ .

Let  $n = 1, 2, \dots$ . Let  $\Pi_n$  denote the space of all polynomials of degree less than  $n$  with complex coefficients. Let  $g \in \mathcal{L}_{w,s}^2$ . We shall seek an approximate solution  $f_n \in \tilde{\mathcal{L}}_{1/w,s}^2$  of Equation (4.1). In other words,  $f_n$  is a solution to the equation

$$f_n - \frac{1}{w} H(w \mathbf{L}_n^w M_w^{-1} \mathbf{L}_n^w K_n f_n) = -\frac{1}{w} H(w \mathbf{L}_n^w M_w^{-1} g). \quad (4.3)$$

It follows from Lemma 2.4 that  $f_n$  is necessarily of the form  $f_n = v_n/w$  for some  $v_n \in \Pi_n$  and that (4.3) is a fully discretized linear algebraic system relative to the coefficients of the unknown polynomial  $v_n$ .

Let us consider the case when  $w = \varrho$ . Introduce the subspace  $\mathcal{L}_\varrho^{2,0}$  of all functions  $u \in \mathcal{L}_\varrho^2$  satisfying

$$\int_{-1}^1 u d\lambda = 0.$$

The range of the operator  $\tilde{K}_n$  defined by

$$\tilde{K}_n u := \frac{1}{\varrho} H(\varrho \mathbf{L}_n^\varrho M_\varrho^{-1} \mathbf{L}_n^\varrho K_n u), \quad u \in \mathcal{L}_\varrho^2,$$

is contained in  $\mathcal{L}_\varrho^{2,0}$  because of (3.8). Moreover, it follows from Proposition 3.1 and (3.8) that the operator  $H_\varrho + m L_\varrho$  restricted to  $\mathcal{L}_\varrho^{2,0}$  is a bijection from  $\mathcal{L}_\varrho^{2,0}$  onto  $\mathcal{L}_\varrho^2$ . Thus the index of the operator  $H_\varrho + m L_\varrho + K : \mathcal{L}_\varrho^{2,0} \rightarrow \mathcal{L}_\varrho^2$  is 0 because  $K$  is compact. So if  $H_\varrho + m L_\varrho + K$  happens to be injective on  $\mathcal{L}_\varrho^{2,0}$ , then it becomes a bijection on  $\mathcal{L}_\varrho^{2,0}$ . For each  $t \in [0, s)$  let

$$\tilde{\mathcal{L}}_{1/\varrho,t}^{2,0} := \tilde{\mathcal{L}}_{1/\varrho,t}^2 \cap \mathcal{L}_\varrho^{2,0}.$$

**Theorem 4.1** *Let  $w = \varrho$ . Assume that the kernel  $k$  of compact operator  $K$  on  $\mathcal{L}_\varrho^2$  satisfies (4.2) with  $s > \frac{1}{2}$  and  $r \geq s + \frac{1}{2}$ . Let  $d$  denote the smallest positive integer such that  $d \geq s$  and let  $m \in C_\varrho^d([-1, 1])$ . Suppose that the homogeneous equation (4.1) possesses only the trivial solution in  $\mathcal{L}_\varrho^{2,0}$ , that is,  $(H_\varrho + mL_\varrho + K)^{-1}(\{0\}) \cap \mathcal{L}_\varrho^{2,0} = \{0\}$ , and that a function  $g \in \mathcal{L}_{\varrho,s}^2$  is given. Assume that  $0 \leq t < s$ .*

*Then the singular integral equation (4.1) has a unique solution  $f$  in  $\tilde{\mathcal{L}}_{1/\varrho,s}^{2,0}$ . Moreover, for all sufficiently large  $n \in \mathbb{N}$ , the system (4.3) is uniquely solvable in  $\tilde{\mathcal{L}}_{1/\varrho,t}^{2,0}$  and the solution  $f_n \in \tilde{\mathcal{L}}_{1/\varrho,t}^{2,0}$  is of the form  $f_n = v_n/\varrho$  for some  $v_n \in \Pi_n$  and satisfies the error estimate*

$$\|f_n - f\|_{1/\varrho,t}^* \leq \text{const} \cdot n^{t-s} \|g\|_{\varrho,s}. \quad (4.4)$$

*Proof.* In this proof the symbol  $c$  stands for a positive constant (not always the same) which is independent of  $n \in \mathbb{N}$ . The identity operators on the various Hilbert spaces to be considered are denoted by  $I$ .

Step I. Let  $0 \leq \delta \leq s$ . The restriction of  $H_{\varrho,\delta} + mL_{\varrho,\delta}$  to  $\tilde{\mathcal{L}}_{1/\varrho,\delta}^{2,0}$  is denoted also by  $H_{\varrho,\delta} + mL_{\varrho,\delta}$  for simplicity. Since  $m \in C_\varrho^d([-1, 1])$  and  $d \geq s \geq \delta$ , the operator  $H_{\varrho,\delta} + mL_{\varrho,\delta}$  is an isomorphism from  $\tilde{\mathcal{L}}_{1/\varrho,\delta}^{2,0}$  onto  $\mathcal{L}_{\varrho,\delta}^2$ , which can be proved as Theorem 3.3.

Step II. We shall show that (4.1) has a unique solution  $f \in \tilde{\mathcal{L}}_{1/\varrho,s}^{2,0}$ . The natural embedding  $Z : \tilde{\mathcal{L}}_{1/\varrho,s}^2 \rightarrow \mathcal{L}_\varrho^2$  is compact because from [2, Conclusion 2.3] the natural embedding from  $\mathcal{L}_{1/\varrho,s}^2$  into  $\mathcal{L}_{1/\varrho}^2$  is compact. Since  $K : \mathcal{L}_\varrho^2 \rightarrow \mathcal{L}_{\varrho,s}^2$  is continuous by the assumption (4.2), the map  $KZ$  is compact, and hence its restriction to  $\tilde{\mathcal{L}}_{1/\varrho,s}^{2,0}$  is an  $\mathcal{L}_{\varrho,s}^2$ -valued compact operator.

Now,

$$\text{ind}(H_{\varrho,s} + mL_{\varrho,s} + K) = \text{ind}(H_{\varrho,s} + mL_{\varrho,s}) = 0.$$

The operator

$$H_{\varrho,s} + mL_{\varrho,s} + K : \tilde{\mathcal{L}}_{1/\varrho,s}^{2,0} \rightarrow \mathcal{L}_{\varrho,s}^2,$$

which is injective by assumption, is a surjective isomorphism. That is, (4.1) has a unique solution  $f$  in  $\tilde{\mathcal{L}}_{1/\varrho,s}^{2,0}$  and

$$\|f\|_{1/\varrho,s}^* \leq \|(H_{\varrho,s} + mL_{\varrho,s} + K)^{-1}\| \cdot \|g\|_{\varrho,s}. \quad (4.5)$$

Step III. In Steps III and IV, we shall establish that the operator

$$I - \tilde{K}_n : \tilde{\mathcal{L}}_{1/\varrho,t}^{2,0} \rightarrow \tilde{\mathcal{L}}_{1/\varrho,t}^{2,0}$$

is invertible, which will imply that (4.3) has a unique solution in  $\tilde{\mathcal{L}}_{1/\varrho,t}^{2,0}$ .

When  $0 \leq \delta \leq s$ , let  $M_{\varrho,\delta}$  denote the restriction of  $M_\varrho$  to  $\mathcal{L}_{\varrho,\delta}^2$ ; then  $M_{\varrho,\delta} : \mathcal{L}_{\varrho,\delta}^2 \rightarrow \mathcal{L}_{\varrho,\delta}^2$  is a surjective isomorphism, which can be proved as  $M_{\sigma,\delta}$  in Corollary 3.1 by using the assumption:  $m \in C_\varrho^d([-1, 1])$  and  $d \geq s \geq \delta$ . Applying Lemma 2.4, define an operator  $A : \mathcal{L}_{\varrho,t}^2 \rightarrow \mathcal{L}_{\varrho,t}^2$  by

$$Av := (1/\varrho)H(\varrho v), \quad v \in \mathcal{L}_{\varrho,t}^2.$$

Let  $W : \tilde{\mathcal{L}}_{1/\varrho,t}^{2,0} \rightarrow \tilde{\mathcal{L}}_{1/\varrho,t}^{2,0}$  be the operator given by

$$Wu := AM_{\varrho,t}^{-1}Ku, \quad u \in \tilde{\mathcal{L}}_{1/\varrho,t}^{2,0}.$$



We claim that the operator  $I - W$  is a surjective isomorphism on  $\tilde{\mathcal{L}}_{1/e,t}^{2,0}$ . In fact,  $H_{e,t} + mL_{e,t}$  is an isomorphism from  $\tilde{\mathcal{L}}_{1/e,t}^{2,0}$  onto  $\mathcal{L}_{e,t}^2$  by Step I. Since  $K : \mathcal{L}_e^2 \rightarrow \mathcal{L}_{e,t}^2$  is compact by the assumption (4.2), the operator  $H_{e,t} + mL_{e,t} + K$  is an isomorphisms from  $\tilde{\mathcal{L}}_{1/e,t}^{2,0}$  onto  $\mathcal{L}_{e,t}^2$ , which can be proved as in the second half of Step II. Now the identity

$$H_{e,t} + mL_{e,t} + K = (H_{e,t} + mL_{e,t})(I - W)$$

on  $\tilde{\mathcal{L}}_{e,t}^{2,0}$  establishes our claim.

The operator  $I - \tilde{K}_n = (I - W) + (W - \tilde{K}_n)$  becomes invertible for a large  $n \in \mathbb{N}$  once we show that

$$\|(W - \tilde{K}_n)u\|_{1/e,t}^* \leq cn^{t-s} \|u\|_{1/e,t}^* \quad (4.6)$$

for every  $u \in \tilde{\mathcal{L}}_{1/e,t}^{2,0}$  and  $n \in \mathbb{N}$ . We shall then have

$$\|(I - \tilde{K}_n)^{-1}\| \leq (1 - \|(I - W)^{-1}\| \cdot \|W - \tilde{K}_n\|)^{-1} \quad (4.7)$$

provided  $\|(I - W)^{-1}\| \cdot \|W - \tilde{K}_n\| < 1$ . This is a consequence of the usual Neumann series argument.

Step IV. The aim of this step is to prove (4.6). To this end fix a function  $u \in \tilde{\mathcal{L}}_{1/e,t}^{2,0}$  and a positive integer  $n$ . Let  $J : \tilde{\mathcal{L}}_{1/e,t} \rightarrow \mathcal{L}_e^2$  be the natural injection. Define a linear operator  $D_n : \mathcal{L}_e^2 \rightarrow \mathcal{L}_{e,t}^2$  by

$$D_n h = (M_e^{-1}K - \mathbf{L}_n^e M_e^{-1} \mathbf{L}_n^e K_n)h, \quad h \in \mathcal{L}_e^2.$$

Using the operator  $A$  given in Step III we have

$$(W - \tilde{K}_n)u = AD_n Ju.$$

Let  $h = Ju$ . Then

$$D_n h = M_{e,t}^{-1}(K - \mathbf{L}_n^e K_n)h + (I - \mathbf{L}_n^e)(M_{e,s}^{-1} - I)\mathbf{L}_n^e K_n h. \quad (4.8)$$

In view of the assumption:  $r \geq s + \frac{1}{2} > s > \frac{1}{2}$ , apply [2, Lemma 4.4] to obtain that, if  $0 \leq \delta \leq s$ , then

$$\|(K - \mathbf{L}_n^e K_n)h\|_{e,\delta} \leq cn^{\delta-s} \|h\|_e. \quad (4.9)$$

Letting  $\delta = t$  in (4.9) we have

$$\|M_{e,t}^{-1}(K - \mathbf{L}_n^e K_n)h\|_{e,t} \leq cn^{t-s} \|M_{e,t}^{-1}\| \cdot \|h\|_e. \quad (4.10)$$

On the other hand, from [2, Theorem 3.4] which requires the assumption (4.2) and  $s > \frac{1}{2}$ , it follows that

$$\|v - L_n^e v\|_{e,t} \leq cn^{t-s} \|v\|_{e,s}, \quad v \in \mathcal{L}_{e,s}^2. \quad (4.11)$$

From (4.9) with  $\delta = s$ , we derive

$$\|\mathbf{L}_n^e K_n h\|_{e,s} \leq c \|h\|_e.$$

This together with (4.11) gives

$$\|(I - \mathbf{L}_n^e)(M_{\varrho,s}^{-1} - I)\mathbf{L}_n^e K_n h\|_{s,t} \leq cn^{t-s} \|M_{\varrho,s}^{-1} - I\| \cdot \|h\|_{\varrho}. \quad (4.12)$$

It then follows from (4.8), (4.10) and (4.12) that

$$\|D_n h\|_{\varrho,t} \leq cn^{t-s} (\|M_{\varrho,t}^{-1}\| + \|M_{\varrho,s}^{-1} - I\|) \|h\|_{\varrho}. \quad (4.13)$$

Therefore we have

$$\|(W - \tilde{K}_n)u\|_{1/\varrho,t}^* \leq \|A\| \cdot \|D_n J u\|_{\varrho,t} \leq cn^{t-s} \|A\| (\|M_{\varrho,t}^{-1}\| + \|M_{\varrho,s}^{-1} - I\|) \|J\| \cdot \|u\|_{1/\varrho,t}^*,$$

which implies (4.6).

Step V. Since (4.6) holds for every  $n \in \mathbb{N}$ , there is an  $N \in \mathbb{N}$  such that  $J - \tilde{K}_n$  is invertible wherever  $n \geq N$ , as observed in Step III. Let  $b = \sup_{n \geq N} \|(I - \tilde{K}_n)^{-1}\|$  which is finite by (4.6) and (4.7). Fix a positive integer  $n$  satisfying  $n \geq N$ . Let

$$f_n := -(I - \tilde{K}_n)^{-1} \left[ \frac{1}{\varrho} H(\varrho \mathbf{L}_n^e M_{\varrho,s}^{-1} g) \right]$$

which is the unique solution of (4.3). As noted before,  $f_n = v_n/\varrho$  for some  $v_n \in \Pi_n$  by applying Lemma 2.4. It is easy to see that

$$\begin{aligned} f_n - f &= (I - \tilde{K}_n)^{-1} \left( -\frac{1}{\varrho} H(\varrho \mathbf{L}_n^e M_{\varrho,s}^{-1} g) - (I - \tilde{K}_n) f \right) \\ &= (I - \tilde{K}_n)^{-1} A[(I - \mathbf{L}_n^e) M_{\varrho,s}^{-1} g - D_n J f]. \end{aligned}$$

By (4.11) we have

$$\|(I - \mathbf{L}_n^e) M_{\varrho,s}^{-1} g\|_{\varrho,t} \leq cn^{t-s} \|M_{\varrho,s}^{-1} g\|_{\varrho,s} \leq cn^{t-s} \|M_{\varrho,s}^{-1}\| \cdot \|g\|_{\varrho,s}. \quad (4.14)$$

Substituting  $Jf$  for  $h$  in (4.13) gives that  $\|D_n J f\|_{\varrho,t} \leq cn^{t-s} \|Jf\|_{\varrho}$ . It then follows from (4.5) that

$$\|D_n J f\|_{\varrho,t} \leq cn^{t-s} \|J\| \cdot \|(H_{\varrho,s} + m L_{\varrho,s} + K)^{-1}\| \cdot \|g\|_{\varrho,s}. \quad (4.15)$$

From (4.14) and (4.15) we finally obtain

$$\begin{aligned} \|f_n - f\|_{1/\varrho,t}^* &\leq \|(I - \tilde{K}_n)^{-1}\| \cdot \|A\| (\|(I - \mathbf{L}_n^e) M_{\varrho,s}^{-1} g\|_{\varrho,t} + \|D_n J f\|_{\varrho,t}) \\ &\leq b \|A\| (cn^{t-s} \|g\|_{\varrho,s}) \end{aligned}$$

wherever  $n \geq N$ . Thus we have established the error estimate (4.4).  $\square$

**Remark 4.1** ([29, Section 4]). *The homogeneous equation (4.1) has a unique solution  $f \in \mathcal{L}_\varrho^2$  satisfying*

$$\int_{-1}^1 f \, d\lambda = \pi C$$

with given  $C \in \mathbb{R}$  if  $k$  and  $m$  fulfil the estimate

$$B < (1 + \alpha \sqrt{\pi} \|m\|_{\varrho})^{-1} \quad (4.16)$$

where

$$\begin{aligned} \pi^2 B^2 &= \int_{-1}^1 \int_{-1}^1 |k(x, y)|^2 (1 - y^2)^{-1/2} (1 - x^2)^{1/2} dy dx \\ &\quad - \int_{-1}^1 \left[ \int_{-1}^1 k(x, y) (1 - y^2)^{-1/2} dy \right]^2 (1 - x^2)^{1/2} dx \end{aligned}$$

and

$$\alpha = \left[ \sup_{-1 < x < 1} a(x) \right] \cdot \left[ \inf_{-1 < x < 1} a(x) \right]^{-1}.$$

If (4.16) is fulfilled then the general solution  $f \in \mathcal{L}_\rho^2$  of (4.1) satisfies the estimate

$$\begin{aligned} \|f\|_\rho^2 &\leq [1 - (1 + \alpha\sqrt{\pi}\|m\|_\rho)B]^{-2} [(1 + \alpha\sqrt{\pi}\|m\|_\rho)\|g\|_\rho \\ &\quad + |C|(\alpha\|m\|_\rho \ln 2 + (1 + \alpha\sqrt{\pi}\|m\|_\rho)\|h\|_\rho)]^2 + \pi|C|^2 \end{aligned}$$

where

$$h(x) = \frac{1}{\pi} \int_{-1}^1 k(x, y) (1 - y^2)^{-1/2} dy.$$

By taking into account Theorems 3.1 (ii) and 3.3, the following assertion can be proved in a similar way to Theorem 4.1.

**Theorem 4.2** *Let  $w = \sigma$ . Suppose that (3.10) does not hold and the homogeneous equation (4.1) possesses only the trivial solution in  $\mathcal{L}_\sigma^2$ . Further, besides the assumptions (4.2) with  $s > \frac{1}{2}$  and  $r > s + \frac{1}{2}$  suppose that  $d$  is the smallest integer satisfying  $d \geq s$  and that  $m \in C_\rho^d([-1, 1])$ . Assume that a function  $g \in \mathcal{L}_{\sigma, s}^2$  is given and  $0 \leq t < s$ .*

*Then (4.1) has a unique solution  $f$  in  $(1/\sigma)\mathcal{L}_{1/\sigma, s}^2$ . Moreover, for all sufficiently large  $n \in \mathbb{N}$ , the system (4.3) is uniquely solvable in  $(1/\sigma)\mathcal{L}_{1/\sigma, t}^2$  such that the solution  $f_n$  is of the form  $v_n/\sigma$  for some  $v_n \in \Pi_n$  and the error estimate*

$$\|f_n - f\|_{1/\sigma, t}^* \leq \text{const} \cdot n^{t-s} \|g\|_{\sigma, s}$$

holds.

Note that if  $t > \frac{1}{2}$  then the following estimates hold:

$$\sup_{-1 < x < 1} 1/\rho(x) |u(x)| \leq \text{const} \cdot \|u\|_{1/\rho, t} \quad \text{if } u \in \mathcal{L}_{1/\rho, t}^2 \quad (4.17)$$

and

$$\sup_{-1 < x < 1} \sqrt{1-x} |v(x)| \leq \text{const} \cdot \|v\|_{1/\sigma, t} \quad \text{if } v \in \mathcal{L}_{1/\sigma, t}^2, \quad (4.18)$$

(see [25, Theorem 7] and [7, Equation (35)]). Thus the estimates of Theorems 4.1 and 4.2 imply error estimates with respect to weighted uniform norms. More precisely,

$$\sup_{-1 < x < 1} |f_n(x) - f(x)| \leq \text{const} \cdot n^{t-s} \|g\|_{\rho, s} \quad (4.19)$$

for all  $t \in (2^{-1}, s)$  under the assumptions of Theorem 4.1, and

$$\sup_{-1 < x < 1} \sqrt{1-x} \sigma(x) |f_n(x) - f(x)| \leq \text{const} \cdot n^{t-s} \|g\|_{\sigma, s} \quad (4.20)$$

for all  $t \in (2^{-1}, s)$  provided the conditions of Theorem 4.2 are satisfied.

Indeed, assuming  $t \in (2^{-1}, s)$  we obtain from (4.18) that

$$\begin{aligned} \sup_{-1 < x < 1} \sqrt{1-x} \sigma(x) |f_n(x) - f(x)| &\leq \text{const} \cdot \|\sigma(f_n - f)\|_{1/\sigma, t} \\ &= \text{const} \cdot \|f_n - f\|_{1/\sigma, t}^* \end{aligned} \quad (4.21)$$

Thus (4.20) follows from (4.21) and Theorem 4.2. Similarly, (4.19) follows from (4.17) together with (4.4).

Remark that (4.3) can be considered as an alternative numerical scheme to the well known collocation method where an approximate solution  $f_n$  of Equation (4.1) is sought in the form  $f_n = v_n/w$ , where the unknown polynomial  $v_n \in \Pi_n$  is determined by the equation

$$(H_w + \mathbf{L}_n^w m L_w + \mathbf{L}_n^w K_n) f_n = \mathbf{L}_n^w g. \quad (4.22)$$

For the method (4.22) the error estimates of Theorems 4.1 and 4.2 hold, too (see [2] for the case of constant  $m$ ; for  $m \in C_\rho^r([-1, 1])$ , the proof is similar to that of Theorem 4.1).

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## References

- [1] D. Berthold, W. Hoppe and B. Silbermann, The numerical solution of the generalized airfoil equation, *J. Integral Equations Appl.* **4** (1992), 309-336.
- [2] D. Berthold, W. Hoppe and B. Silbermann, A fast algorithm for solving the generalized airfoil equation, *J. Comput. Appl. Math.* **43** (1992), 185-219.
- [3] D. Berthold and P. Junghanns, New error bounds for the quadrature method for the solution of Cauchy singular integral equations, *SIAM J. Numer. Anal.* **30** (1993), 1351-1372.
- [4] S.R. Bland, The two-dimensional oscillating airfoil in a wind tunnel in subsonic flow, *SIAM J. Appl. Math.* **18** (1970), 830-848.
- [5] P.L. Butzer and R.J. Nessel, *Fourier Analysis and Approximation Theory*, Birkhäuser Verlag, Basel 1977.
- [6] M.R. Capobianco, The stability and the convergence of a collocation method for a class of Cauchy singular integral equations, *Math. Nachr.* **162** (1993), 45-58.
- [7] G. Chiocchia, S. Pröbldorf and D. Tordella, The lifting line equation for a curved wing in oscillatory motion, Preprint No. 218, Weierstraß-Institut für Angewandte Analysis und Stochastik, Berlin 1996; *ZAMM* (submitted).

- [8] D. Elliott, The classical collocation method for singular integral equations, SIAM J. Numer. Anal. **19** (1982), 816-832.
- [9] D. Elliott, A comprehensive approach to the approximate solution of singular integral equations over the arc  $(-1, 1)$ , J. Integral Equations Appl. **2** (1989), 59-94.
- [10] F. Erdogan, G.D. Gupta and T.S. Cook, Numerical solution of singular integral equations, Mech. of Fractures **1** (1973), 368-425.
- [11] S. Fenyő and H.W. Stolle, Theorie und Praxis der linearen Integralgleichungen, Deutscher Verlag der Wissenschaften, Berlin 1984.
- [12] J.A. Fromme and M.A. Golberg, Numerical solution of a class of integral equations arising in two-dimensional aerodynamics, J. Optim. Theory Appl. **24** (1978), 169-206.
- [13] J.A. Fromme and M.A. Golberg, On the  $L^2$ -convergence of collocation for the generalized airfoil equation, J. Math. Anal. Appl. **71** (1979), 271-286.
- [14] I. Gohberg and N.Y. Krupnik, Einführung in die Theorie der eindimensionalen singulären Integraloperatoren, Birkhäuser Verlag, Basel, Boston, Stuttgart 1979.
- [15] I. Gohberg and N.Y. Krupnik, One-Dimensional Linear Singular Integral Equations, Vol. 1, Birkhäuser Verlag, Basel 1992.
- [16] A.S. Il'inskij, A numerical method for solving an integral equation with a logarithmic singularity of the kernel (Russian), Vestnik Moskov. Univ. Ser. 15, Numerical Mathematics and Kybernetics **4** (1986).
- [17] K. Jörgens, Linear Integral Operators (English translation), Pitman, Boston 1982.
- [18] P. Junghanns, Product integration for the generalized airfoil equation, Beiträge zur Angewandten Analysis und Informatik (ed. E. Schock), Shaker Verlag, 1994, 171-188.
- [19] P. Junghanns and B. Silbermann, Zur Theorie der Näherungsverfahren für singuläre Integralgleichungen auf Intervallen, Math. Nachr. **103** (1981), 199-244.
- [20] P. Junghanns and B. Silbermann, Numerical analysis of the quadrature method for solving linear and nonlinear singular integral equations, Wiss. Schriftenreihe Tech. Univ. Karl-Marx-Stadt **10** (1988).
- [21] A.I. Kalandiya, Mathematical methods of two-dimensional elasticity (Russian), Mir Publishers, Moscow 1973.
- [22] J.L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications (English translation), Springer-Verlag, 1972.
- [23] G. Mastroianni and S. Prößdorf, A quadrature method for Cauchy integral equations with weakly singular perturbation kernel, J. Integral Equations Appl. **4**, No. 2 (1992), 205-228.
- [24] S.G. Mikhlin and S. Prößdorf, Singular Integral Operators (English translation), Springer-Verlag, Berlin 1986.

- [25] G. Monegato and S. Prößdorf, Uniform convergence estimates for a collocation and a discrete collocation method for the generalized airfoil equation. Contributions in Numerical Mathematics (ed. R.P. Agarwal), Vol. 2, World Scientific Series in Applicable Analysis (1993), 285-299.
- [26] S. Okada and D. Elliott, The finite Hilbert transform in  $L^2$ , Math. Nachr. **153** (1991), 43-56.
- [27] S. Prößdorf and B. Silbermann, Numerical analysis for integral and related operator equations, Akademie-Verlag, Berlin 1991, Birkhäuser Verlag, Basel 1991.
- [28] J.B. Reade, Asymptotic behaviour of eigen-values and certain integral equations, Proc. Edinburgh Math. Soc. **22** (1979), 137-144.
- [29] M. Schleiff, Über eine singuläre Integralgleichung mit logarithmischem Zusatzkern, Math. Nachr. **42** (1969), 79-88.
- [30] M. Schleiff, Singuläre Integraloperatoren in Hilbert-Räumen mit Gewichtsfunktion, Math. Nachr. **42** (1969), 145-155.
- [31] I.H. Sloan and E.P. Stephan, Collocation with Chebyshev polynomials for Symm's integral equation on an interval, J. Austral. Math. Soc. Ser. B **34** (1992), 199-211.
- [32] P.S. Theocaris, On the numerical solution of Cauchy-type singular integral equations, Serdica. Bulgar. Math. Publ. **2** (1976), 252-275.
- [33] F.G. Tricomi, On the finite Hilbert transformation, Quart. J. Math. Oxford (2) **2** (1951), 199-211.
- [34] F.G. Tricomi, Integral Equations, Interscience, New York 1957.

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