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Gibbsian representation for point processes
via hyperedge potentials

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ABSTRACT. We consider marked point processes on the d-dimensional euclidean space, defined in terms of a quasilocal specification based on marked Poisson point processes. We investigate the possibility of constructing uniformly absolutely convergent Hamiltonians in terms of hyperedge potentials in the sense of Georgii [2]. These potentials are natural generalizations of physical multibody potentials which are useful in models of stochastic geometry.

1. INTRODUCTION

In this note we study models for not necessarily translation-invariant Poisson point processes (PPP) in euclidean space $\mathbb{R}^d$ with general marks. Such models are the subject in the infinite-volume statistical mechanics of classical point particles which interact via potentials. They are already very interesting when there are no marks (or internal states of particles), and only the positions of the colorless point particles are relevant. Potentials coming from physics are often pair potentials. Take as an example the famous Lennard-Jones potential. For results on existence of such models in the infinite volume, see [14, 16]. Also more general potentials than pair potentials appear, describing interactions between finite collections of particles. These are quite relevant in physics as well, see for instance the proof of a phase transition for a long (but finite) range potential involving 4-body interactions in [13]. For models from statistical physics with marks, see e.g. the Potts gas in [4]. The famous Widom-Rowlinson model (WRM) is a specific example for this which is proved to have a phase transition in the infinite volume [1, 5, 15].

PPPs also have an interest which is independent from the physical motivation in statistical mechanics in models of stochastic geometry e.g. [7, 8]. In the development of the fundamentals of an infinite-volume theory (existence, uniqueness, variational principle, . . .) also for such systems an important step was made by [2] in the introduction of the more general notion of a hyperedge potential, see [2]. For such potentials one allows the energetic contribution of a finite subset of particles (hyperedges) to depend also on the other points in the cloud, but only up to a finite horizon. This relaxation of the strict locality requirement on the level of potentials incorporates many models from stochastic geometry. In this note we are aiming for uniformly absolutely convergent representations of abstractly given point processes as Gibbs fields in terms of such hyperedge potentials.

To compare, let us recall the simpler situation in statistical mechanics on the lattice $\mathbb{Z}^d$, or more generally countable index sets, where the notion of a quasilocal specification is fundamental for the development of Gibbsian theory in its purest form, see [3]. An uniformly absolutely convergent potential defines a finite-volume Hamiltonian H_Λ in a finite volume $\Lambda \subset \mathbb{Z}^d$, which depends in a quasilocal way on the boundary condition outside of Λ.

On the lattice, going from nice potentials to Gibbsian specifications in lattice statistical mechanics is straightforward, while the opposite is more difficult. However, Kozlov and Sullivan [10, 11, 17] showed how one may construct potentials with various convergence properties. For systems of point particles already going from Hamiltonians to measures is more delicate, for the opposite direction partial results were obtained in [11] where a convergent representation in terms of the (necessarily unique) vacuum potential was obtained, while uniform absolute convergence could not be provided. It is a main aim of our paper to show how uniform absolute convergence can indeed be achieved in the class of Georgii's hyperedge potentials.

The paper is organized as follows. Section 2 contains the setup of Gibbs Point Processes. In Section 3 we discuss the notion of hyperedge potentials in the sense [2], and formulate as our main general result Theorem 3.8, on the uniform absolute convergence of a hyperedge potential. Before doing so, we put in place Theorem 3.6, on which we will build up later, and Corollary 3.7. These concern the convergence of the vacuum potential, and its finite-range property under the assumption of the strict Markov property of the specification. Versions of these statements were obtained for the first time in [11]. In Section 4 we discuss the two-color WRM under independent spin-flip dynamics, see [9]. This model shows quite interesting Gibbs non-Gibbs transitions, depending on activities and time. We explain that in the Gibbsian regimes there is always a hyperedge potential which even has a uniform horizon, and which converges absolutely and uniformly. Finally, the proofs including further comments are provided in Section 5.

2. Gibbs point processes

2.1. **Setup.** We consider the euclidean space $\mathbb{R}^d$ with $d \geq 1$ equipped with its Borel-σ-algebra. Let Ω denote the set of all *locally finite subsets* of $\mathbb{R}^d$, that is, for $\omega \in \Omega$ we have $|\omega_\Lambda| = \#\{\omega \cap \Lambda\} < \infty$ for all bounded sets $\Lambda \subset \mathbb{R}^d$. The polish space E equipped with its Borel-σ-algebra $\mathfrak{E}$ will play the role of a *local state space* or in the language of point processes the *mark space*. We write $\sigma_\omega \in E^\omega$ for the marks of a configuration $\omega \in \Omega$. The *marked configurations* $\boldsymbol{\omega} = (\omega, \sigma_\omega)$ are locally finite subsets of $\mathbb{R}^d \times E$ and we denote $\boldsymbol{\Omega}$ the set of all such marked configurations with $\omega \in \Omega$. Conversely we call $\omega \in \Omega$ the *grey configuration* of $\boldsymbol{\omega} \in \boldsymbol{\Omega}$. We equip $\boldsymbol{\Omega}$ with the σ-algebra $\mathcal{F}$ which is generated by the counting variables $\boldsymbol{\Omega} \ni \boldsymbol{\omega} \mapsto |\boldsymbol{\omega} \cap (\Lambda \times B)|$ for bounded and measurable $\Lambda \Subset \mathbb{R}^d$ and $B \in \mathfrak{E}$, i.e. $\mathcal{F} = \sigma\big(\{\boldsymbol{\omega} : |\omega_\Lambda| = n, \sigma_{\omega_\Lambda} \in B\} : n \in \mathbb{N}, \Lambda \Subset \mathbb{R}^d, B \in \mathfrak{E}^n\big)$. Further we denote by $\boldsymbol{\Omega}_\Lambda$ the set of all marked configurations in the measurable set $\Lambda \subset \mathbb{R}^d$ and equip it with the corresponding trace σ-algebra $\mathcal{F}_\Lambda$ of $\mathcal{F}$ on $\boldsymbol{\Omega}_\Lambda$. We write $f \in \mathcal{F}_\Lambda$ if f is measurable w.r.t. $\mathcal{F}_\Lambda$ and $f \in \mathcal{F}_\Lambda^b$ if f is additionally bounded in the supremum norm $\|\cdot\|$.

2.2. **Gibbs point processes for Poisson modifications.** In this section we setup Gibbsian point processes via Poisson specifications along the lines of λ-specifications for models on fixed geometries as in [3, Chapter 1]. For any $\Lambda \subset \mathbb{R}^d$ and $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ we use the short-hand notation $\boldsymbol{\omega}_\Lambda$ to indicate that points in $\Lambda^c = \mathbb{R}^d \setminus \Lambda$ are eliminated. With $\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_\Delta$ we indicate the configuration which consists of the union of $\boldsymbol{\omega}_\Lambda$ and $\boldsymbol{\omega}_\Delta$ and similar for grey configurations. Let us start by adapting the notion of pre-modifications from [3, Definition 1.31] to the continuum setting.

Definition 2.1 (Pre-modification). *Let $h = (h_\Lambda)_{\Lambda \Subset \mathbb{R}^d}$ be a family of measurable functions $h_\Lambda : \boldsymbol{\Omega}^* \to [0, \infty)$ with common domain $\boldsymbol{\Omega}^* \subset \boldsymbol{\Omega}$. Then h is called a $\boldsymbol{\Omega}^*$-pre-modification if for all $\Lambda \subset \Delta \Subset \mathbb{R}^d$ and $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \boldsymbol{\Omega}^*$,*

$$h_\Delta(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c}) h_\Lambda(\boldsymbol{\omega}'_\Lambda \boldsymbol{\omega}_{\Lambda^c}) = h_\Lambda(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c}) h_\Delta(\boldsymbol{\omega}'_\Lambda \boldsymbol{\omega}_{\Lambda^c}).$$

As the prime example of a pre-modification consider the *Boltzmann weight*

$$h_\Lambda = e^{-H_\Lambda}$$

where the *Hamiltonian* H_Λ is given by

$$(1) \qquad\qquad H_\Lambda(\boldsymbol{\omega}) = \sum_{\eta \Subset \omega : \eta \cap \Lambda \neq \emptyset} \Phi(\eta, \boldsymbol{\omega}).$$

Here the *potentials* $\Phi(\eta, \cdot) : \mathbf{\Omega}^* \to (-\infty, \infty]$ are measurable functions w.r.t. $\mathcal{F}_\eta$, governing the interaction of marked particles at locations η. For example consider the Potts Gas [4] with

(2) $$\Phi(\eta, \boldsymbol{\omega}) = \delta_{\eta = \{x,y\}}[\delta_{\sigma_x \neq \sigma_y} \varphi(x - y) + \psi(x - y)]$$

for some measurable and even functions $\varphi, \psi : \mathbb{R}^d \to] -\infty, \infty]$ which includes also the case of the Widom-Rowlinson model [1, 18].

The introduction of the domain $\mathbf{\Omega}^*$ of *admissible boundary conditions* is necessary since in the continuum setting, due to the possible accumulation of points, even in simple models with infinite-range interactions, Hamiltonians and hence pre-modifications might not be well defined everywhere. More precisely, the sum in (1) is only well-defined for boundary conditions $\boldsymbol{\omega} \in \mathbf{\Omega}_\Lambda^* \subset \mathbf{\Omega}$ such that

$$\sum_{\eta \Subset \omega:\, \eta \cap \Lambda \neq \emptyset} (-\Phi(\eta, \boldsymbol{\omega}) \vee 0) < \infty.$$

Note that $\mathbf{\Omega}_\Delta^* \subset \mathbf{\Omega}_\Lambda^*$ for $\Delta \supset \Lambda$ since

$$\sum_{\eta \Subset \omega:\, \eta \cap \Delta \neq \emptyset} (-\Phi(\eta, \boldsymbol{\omega}) \vee 0) \geq \sum_{\eta \Subset \omega:\, \eta \cap \Lambda \neq \emptyset} (-\Phi(\eta, \boldsymbol{\omega}) \vee 0)$$

and hence it suffices to consider the common domain $\mathbf{\Omega}^* = \lim_{n \uparrow \infty} \mathbf{\Omega}_{\Lambda_n}^*$ where $\Lambda_n = [n/2, n/2]^d$ denotes the centered box of length $n \in \mathbb{N}$. We will give a proper and more general definition of potentials in Section 3.

The notion of a pre-modification can be used to describe a large class of *specifications*.

Definition 2.2 (Specification). *A $\mathbf{\Omega}^*$-specification is a family of proper probability kernels $\gamma = (\gamma_\Lambda)_{\Lambda \Subset \mathbb{R}^d}$ where each $\gamma_\Lambda(\cdot | \boldsymbol{\omega})$ is defined for all $\boldsymbol{\omega} \in \mathbf{\Omega}^*$ with $\gamma_\Lambda(\mathbf{\Omega}^* | \boldsymbol{\omega}) = 1$ and additionally satisfies the following consistency condition. For all measurable $\Lambda \subset \Delta \Subset \mathbb{R}^d$ and $\boldsymbol{\omega} \in \mathbf{\Omega}^*$*

$$\gamma_\Delta(\gamma_\Lambda(d\boldsymbol{\omega}' | \cdot) | \boldsymbol{\omega}) = \gamma_\Delta(d\boldsymbol{\omega}' | \boldsymbol{\omega}).$$

Let us denote by $\boldsymbol{P}$ the (maybe non-stationary) *marked Poisson point process* (PPP) on $(\mathbf{\Omega}, \mathcal{F})$ with *intensity measure* $\mu(dx, du) = \nu(dx)F(du | x)$. Here ν is a σ-finite measure on $\mathbb{R}^d$ which is equivalent to the Lebesgue measure on $\mathbb{R}^d$ and F is a kernel from $\mathbb{R}^d$ to the set of σ-finite measures on $(E, \mathfrak{E})$. By $\boldsymbol{P}_\Lambda$ we denote the restriction of $\boldsymbol{P}$ to $\mathbf{\Omega}_\Lambda$. For $f \in \mathcal{F}^b$ we will often use the short hand notation $\int \boldsymbol{P}(d\boldsymbol{\omega})f(\boldsymbol{\omega}) = \boldsymbol{P}f$. For a given family of density functions $\rho = (\rho_\Lambda)_{\Lambda \Subset \mathbb{R}^d}$, defined on a set of Poisson measure one, probability kernels can be defined via

(3) $$\gamma_\Lambda^\rho(f | \boldsymbol{\omega}_{\Lambda^c}) = \int \boldsymbol{P}_\Lambda(d\boldsymbol{\omega}_\Lambda) f(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c}) \rho_\Lambda(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c}).$$

The following definition labels ρ a *Poisson modification* if the associated γ is a specification, similar to [3, Definition 1.27].

Definition 2.3 (Poisson-modification). *Let $\rho = (\rho_\Lambda)_{\Lambda \Subset \mathbb{R}^d}$ be a family of measurable functions $\rho_\Lambda : \mathbf{\Omega}^* \to [0, \infty)$ with common domain $\mathbf{\Omega}^* \subset \mathbf{\Omega}$. Then, ρ is called a $\mathbf{\Omega}^*$-Poisson modification if the family of probability kernels $\gamma^\rho = (\gamma_\Lambda^\rho)_{\Lambda \Subset \mathbb{R}^d}$ given by (3) is a $\mathbf{\Omega}^*$-specification. A $\mathbf{\Omega}^*$-Poisson modification is called positive if for all $\boldsymbol{\omega} \in \mathbf{\Omega}^*$ and $\Lambda \Subset \mathbb{R}^d$ we have $\rho_\Lambda(\boldsymbol{\omega}) > 0$, it is called vacuum positive if only $\rho_\Lambda(\boldsymbol{\omega}_{\Lambda^c}) > 0$ holds.*

Note that under the PPP, the empty set in finite volumes with positive Lebesgue measure has positive mass. Hence, for all $\boldsymbol{\omega} \in \mathbf{\Omega}^*$ also $\boldsymbol{\omega}_{\Lambda^c} \in \mathbf{\Omega}^*$ for all $\Lambda \Subset \mathbb{R}^d$. As an example note that the Poisson modification of the WRM is not positive but vacuum positive. For a $\mathbf{\Omega}^*$-pre-modification h the normalization $Z_\Lambda(\boldsymbol{\omega}_{\Lambda^c}) = \int \boldsymbol{P}_\Lambda(d\boldsymbol{\omega}_\Lambda) h_\Lambda(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c})$ is referred to as the *partition function*. The conditions on pre-modifications give rise to Poisson modifications. This is the content of the following lemma.

Lemma 2.4. *Let h be a $\mathbf{\Omega}^*$-pre-modification with*

$$0 < Z_\Lambda(\boldsymbol{\omega}_{\Lambda^c}) < \infty$$

for all $\Lambda \Subset \mathbb{R}^d$ and $\boldsymbol{\omega}_{\Lambda^c} \in \mathbf{\Omega}^*$. Then $\rho = (h_\Lambda/Z_\Lambda)_{\Lambda \Subset \mathbb{R}^d}$ is a $\mathbf{\Omega}^*$-Poisson-modification if additionally for all $\Delta \Subset \mathbb{R}^d$ and $\boldsymbol{\omega} \in \mathbf{\Omega}^*$ it holds that $\gamma^\rho_\Delta(\mathbf{\Omega}^*|\boldsymbol{\omega}) = 1$.

Next we give a definition of Gibbs point processes via the DLR equation similar to the one for classical Gibbs measures on deterministic spatial graphs see [3].

Definition 2.5 (Gibbs point processes)**.** *A random field $\mathcal{P}$ is called a Gibbs point process for the $\mathbf{\Omega}^*$-specification γ iff for every $\Lambda \Subset \mathbb{R}^d$ and for any $f \in \boldsymbol{\mathcal{F}}^b$,*

$$(4) \qquad \int \mathcal{P}(d\boldsymbol{\omega})f(\boldsymbol{\omega}) = \int \mathcal{P}(d\boldsymbol{\omega}) \int \gamma_\Lambda(d\boldsymbol{\omega}'_\Lambda|\boldsymbol{\omega})f(\boldsymbol{\omega}'_\Lambda\boldsymbol{\omega}_{\Lambda^c})$$

and $\mathcal{P}(\mathbf{\Omega}^*) = 1$. We denote the set of all such measures $\mathcal{G}(\gamma)$.

Existence of Gibbs point processes and the appearance of phase-transitions of multiple solutions to the so-called DLR equation (4) have been proved in a number of cases, see for example [1, 2, 4]. In the next section we present our main result.

3. Hyperedge potentials and the representation theorem

Let us start by giving a more formal definition of interaction potentials in the continuum. For this let us denote by $\Omega_f = \{\omega \in \Omega : |\omega| < \infty\}$ the set of finite configurations in Ω and $\mathcal{F}_f$ the trace σ-algebra of $\mathcal{F}$ in Ω_f. The product space $\Omega_f \times \mathbf{\Omega}$ carries the product σ-algebra $\mathcal{F}_f \otimes \mathcal{F}$. With $\mathcal{E} \subset \bar{\mathcal{E}} = \{(\eta, \boldsymbol{\omega}) \in \Omega_f \times \mathbf{\Omega} : \eta \subset \omega\}$ we denote a *hypergraph structure* of $\mathbf{\Omega}$ as presented in [2] for models with trivial single-site state-space. For $\boldsymbol{\omega} \in \mathbf{\Omega}$ we write $\mathcal{E}(\boldsymbol{\omega}) = \{\eta \Subset \omega : (\eta, \boldsymbol{\omega}) \in \mathcal{E}\}$. Based on the hypergraph structure we now define *hyperedge potentials*.

Definition 3.1 (Hyperedge Potential)**.** *A hyperedge potential (or simply potential) is a measurable function $\Phi : \mathcal{E} \mapsto (-\infty, \infty]$ with the following properties:*

1. *Finite-horizon: For each $(\eta, \boldsymbol{\omega}) \in \mathcal{E}$ there exists $\Delta(\eta, \boldsymbol{\omega}) \Subset \mathbb{R}^d$ such that if $(\eta, \boldsymbol{\omega}') \in \mathcal{E}$ and $\boldsymbol{\omega}_{\Delta(\eta, \boldsymbol{\omega})} = \boldsymbol{\omega}'_{\Delta(\eta, \boldsymbol{\omega})}$, then $\Phi(\eta, \boldsymbol{\omega}) = \Phi(\eta, \boldsymbol{\omega}')$.*

2. *Well-definedness: For all $\Lambda \Subset \mathbb{R}^d$ the series*

$$H_\Lambda(\boldsymbol{\omega}) = \sum_{\eta \in \mathcal{E}(\boldsymbol{\omega}): \eta \cap \Lambda \neq \emptyset} \Phi(\eta, \boldsymbol{\omega})$$

exists in the sense that $H_\Lambda(\boldsymbol{\omega})$ is the limiting point of the net

$$\left(H_{\Lambda,\Delta}(\boldsymbol{\omega})\right)_{\Delta \Subset \mathbb{R}^d}$$

with

$$H_{\Lambda,\Delta}(\boldsymbol{\omega}) = \sum_{\eta \in \mathcal{E}(\boldsymbol{\omega}_\Delta): \eta \cap \Lambda \neq \emptyset} \Phi(\eta, \boldsymbol{\omega}).$$

For $\Lambda \Subset \mathbb{R}^d$ and $r > 0$ we denote by $B_r(\Lambda) = \{x \in \mathbb{R}^d : |x - y| < r \text{ for some } y \in \Lambda\}$ the r-mollification of Λ. Next we distinguish potentials in view of their finite-horizon properties.

Definition 3.2 (Uniform finite-horizon & vacuum potentials)**.** *We call a potential a*

1. *uniformly finite-horizon potential if for all $(\eta, \boldsymbol{\omega}) \in \mathcal{E}$ the finite-horizon property holds with $\Delta(\eta, \boldsymbol{\omega}) = \Delta(\eta)$.*

2 *r-uniformly finite-horizon potential if for all $(\eta, \boldsymbol{\omega}) \in \mathcal{E}$ the finite-horizon property holds with $\Delta(\eta, \boldsymbol{\omega}) = B_r(\eta)$ with $r > 0$.*

3 *vacuum potential if for all $(\eta, \boldsymbol{\omega}) \in \mathcal{E}$ the finite-horizon property holds with $\Delta(\eta, \boldsymbol{\omega}) = \eta$ and it is vacuum normalized, i.e. for all $\xi \subsetneq \eta$*

$$\Phi(\eta, \boldsymbol{\omega}_\xi) = 0.$$

We can further distinguish different types of potentials w.r.t. their convergence properties. In order to make the connection to the domains $\boldsymbol{\Omega}^*$ of admissible configurations, let us write $\mathcal{E}^*$ for hypergraph structures which are subsets of $\bar{\mathcal{E}}^* = \{(\eta, \boldsymbol{\omega}) \in \Omega_{\mathrm{f}} \times \boldsymbol{\Omega}^* : \eta \subset \omega\}$.

Definition 3.3 (Potential convergence). *We call a potential Φ on $\mathcal{E}^*$*

1 *uniformly convergent if for all $\Lambda \Subset \mathbb{R}^d$ we have*

$$\lim_{\Delta \uparrow \mathbb{R}^d} \sup_{\boldsymbol{\omega} \in \boldsymbol{\Omega}^*} |H_{\Lambda, \Delta}(\boldsymbol{\omega}) - H_\Lambda(\boldsymbol{\omega})| = 0.$$

2 *uniformly absolutely convergent if for all $\Lambda \Subset \mathbb{R}^d$ and $\boldsymbol{\omega} \in \boldsymbol{\Omega}^*$ we have*

$$\sum_{\eta \in \mathcal{E}^*(\boldsymbol{\omega}) : \eta \cap \Lambda \neq \emptyset} |\Phi(\eta, \boldsymbol{\omega})| = \bar{H}_\Lambda(\boldsymbol{\omega}) < \infty \qquad \text{and}$$

$$\lim_{\Delta \uparrow \mathbb{R}^d} \sup_{\boldsymbol{\omega} \in \boldsymbol{\Omega}^*} |\sum_{\eta \in \mathcal{E}^*(\boldsymbol{\omega}) : \emptyset \neq \Lambda \cap \eta \subset \Delta} |\Phi(\eta, \boldsymbol{\omega})| - \bar{H}_\Lambda(\boldsymbol{\omega})| = 0.$$

Clearly, the types of convergence are ordered such that (2) implies (1) and (1) implies well-definedness. Let us note that the definitions above could be extended to also include the case of $\bar{H}(\boldsymbol{\omega}) = \infty$ by requiring the series to be invariant under re-summation, but we omit this here. The next definition describes the fundamental goal behind this work.

Definition 3.4 (Potential representation). *We say that a potential Φ represents the $\boldsymbol{\Omega}^*$-Poisson modification ρ if Φ is defined on a hypergraph structure $\mathcal{E}^*$ and for all $\Lambda \Subset \mathbb{R}^d$ and $\boldsymbol{\omega} \in \boldsymbol{\Omega}^*$*

$$\rho_\Lambda(\boldsymbol{\omega}) = \exp\big(-\sum_{\eta \in \mathcal{E}^*(\boldsymbol{\omega}) : \eta \cap \Lambda \neq \emptyset} \Phi(\eta, \boldsymbol{\omega})\big).$$

Our first result establishes existence of such potentials for given pre-modifications under the condition of vacuum positivity and continuity required to hold only in the direction of the vacuum.

Definition 3.5 (Vacuum quasilocality). *We call a real-valued measurable function f with domain $\boldsymbol{\Omega}^* \subset \boldsymbol{\Omega}$ vacuum quasilocal if for all $\boldsymbol{\omega} \in \boldsymbol{\Omega}^*$ we have that*

$$\lim_{\Lambda \uparrow \mathbb{R}^d} |f(\boldsymbol{\omega}) - f(\boldsymbol{\omega}_\Lambda)| = 0.$$

Moreover, f is called vacuum uniformly log-quasilocal if f is positive and

$$\lim_{\Lambda \uparrow \mathbb{R}^d} \sup_{\boldsymbol{\omega} \in \boldsymbol{\Omega}^*} |\log f(\boldsymbol{\omega}) - \log f(\boldsymbol{\omega}_\Lambda)| = 0.$$

Here and in the sequel, the limits should be understood as limits of nets on $\{\Lambda : \Lambda \Subset \mathbb{R}^d\}$ ordered by inclusion. Clearly, uniform quasilocality w.r.t. the τ-topology, i.e.,

$$\lim_{\Lambda \uparrow \mathbb{R}^d} \sup_{\boldsymbol{\omega}, \boldsymbol{\omega}' \in \boldsymbol{\Omega}} |f(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}'_{\Lambda^c}) - f(\boldsymbol{\omega})| = 0$$

implies vacuum quasilocality but not uniform log-quasilocality even if f is assumed positive. The last implication is true under the additional assumption of uniform positivity which is meaningful for example in lattice systems. But in our continuous setting even for f given as the Poisson modification of the Potts gas we have uniform log-quasilocality but no uniform positivity.

Theorem 3.6. *Suppose ρ is a vacuum positive and vacuum quasilocal $\mathbf{\Omega}^*$-pre-modification such that for all $\Lambda \Subset \mathbb{R}^d$ and $\boldsymbol{\omega}_{\Lambda^c} \in \mathbf{\Omega}^*$ we have*

$$\int \boldsymbol{P}_\Lambda(d\boldsymbol{\omega}_\Lambda)\rho_\Lambda(\boldsymbol{\omega}_\Lambda\boldsymbol{\omega}_{\Lambda^c}) = 1.$$

Then, there exists a unique vacuum potential Φ on $\bar{\mathcal{E}}^*$. Moreover, if ρ is vacuum uniformly log-quasilocal, then Φ is uniformly convergent.

Let us note that for example in the lattice case, potentials can be constructed which are unique w.r.t. certain α-normalizations where α an arbitrary single-site measure. This freedom is not available in the continuum case since the geometry is not fixed. Further we note that the construction of the vacuum potential has been performed multiple times for lattices systems, see for example [6, 10], and even more general point fields in [11, Theorem 1B], but there without the uniform convergence part.

The following statement that finite-range properties of Poisson modifications transfer to their associated vacuum potential is already partially presented in [11, Lemma 2].

Corollary 3.7. *Let ρ be as in Theorem 3.6 and Φ the corresponding vacuum potential. Additionally assume that ρ is of range $r > 0$, i.e., for all $\Lambda \Subset \mathbb{R}^d$, ρ_Λ is $\mathcal{F}_{B_r(\Lambda)}$ measurable. Then, if η is such that there exist $x, y \in \eta$ with $|x - y| > r$, we have $\Phi(\eta, \boldsymbol{\omega}) = 0$.*

The following main result of this paper shows that it is possible to derive a representation for vacuum uniformly log-quasilocal $\mathbf{\Omega}^*$-pre-modifications given by an uniformly absolutely convergent potential. This representation has the uniformly finite-horizon property but is no longer unique.

Theorem 3.8. *Let ρ be vacuum uniformly log-quasilocal. Then, there exists a representation of ρ via an uniformly absolutely convergent potential with uniform finite-horizon property.*

Let us note that the constructed uniformly absolute convergent potential is defined on a much sparser hypergraph structure $\mathcal{E}^*$ described in the remarks following the proof.

4. POTENTIALS FOR A TIME-EVOLVED WIDOM-ROWLINSON MODEL

As an illustration we consider the WRM under independent spin flip as presented in [9]. We start by recalling the model.

4.1. **The WRM under independent spin flip.**
The WRM, as initially proposed in [18], is a hard-core repulsion model with single spin space $E = \{+, -\}$ and $\mathbf{\Omega}$-Poisson-modification given by

$$\chi_\Lambda(\boldsymbol{\omega}) = \mathbb{1}\{\text{for all } x, y \in \omega \text{ with } |x - y| < 2r : \sigma_x = \sigma_y\}.$$

Alternatively, it can be described via the vacuum potential

$$\Phi(\eta, \boldsymbol{\omega}) = \infty \times \mathbb{1}_{|x-y|<2r}\mathbb{1}_{\eta=\{x,y\}}\mathbb{1}_{\sigma_x\neq\sigma_y}.$$

Note that the interaction is of range $2r > 0$. The underlying PPP is given by the superposition of two PPP with spatially homogeneous intensities $\lambda_+ \geq \lambda_- > 0$. It is well known, see for example [1, 5, 15], that the symmetric WRM with $\lambda_+ = \lambda_-$ exhibits a phase-transition in the high-intensity regime. Writing $\mathbf{\Omega}^* = \{\boldsymbol{\omega} \in \mathbf{\Omega} : \boldsymbol{\omega} \text{ has no infinite cluster}\}$, high-intensity here means that for large enough $\lambda_+ + \lambda_-$, the WRM is concentrated on $\mathbf{\Omega} \setminus \mathbf{\Omega}^*$. We speak of low-intensity if the WRM is concentrated on $\mathbf{\Omega}^*$. A cluster $C \subset \omega$ is defined via the property that for all $x, y \in C$ we have $|x - y| < 2r$ and $|x - z| \geq 2r$ for all $z \in \omega \setminus C$.

The dynamics is given by rate-one Poisson flips independently attached to every particle, i.e., the probability, that a site in the plus-state, is still in the plus-state at time $t \geq 0$ is given by

$$p_t(+,+) = \tfrac{1}{2}(1 + e^{-2t})$$

with $p_t(+,-), p_t(-,-)$ and $p_t(-,+)$ defined accordingly. The main findings of [9] are that, depending on asymmetry of the WRM and time, there is a sharp Gibbs-non-Gibbs transition in the sense that the time-evolved WRM can be described as a Gibbs measure for a quasilocal $\boldsymbol{\Omega}$-Poisson modification (respectively $\boldsymbol{\Omega}^*$-Poisson modification) ρ or not. Quasilocality here is defined as continuity w.r.t. the τ-topology. Focussing on the asymmetric case $\lambda_+ > \lambda_-$ with initial WRM being in the plus extremal state, the critical time t_G is given by the unique positive solution of

$$b = \frac{\lambda_- p_t(+,+)}{\lambda_+ p_t(+,-)} = 1.$$

A set of configurations where discontinuities can not appear at the critical time t_G can be defined by $\boldsymbol{\Omega}^+ = \{\boldsymbol{\omega} \in \boldsymbol{\Omega} : \boldsymbol{\omega}$ has no infinite cluster C with $\liminf_{n\uparrow\infty} |C \cap \Lambda_n|^{-1} \sum_{x \in C \cap \Lambda_n} \sigma_x \leq 0\}$. In Table 1 we summarize the results for the asymmetric model. In the table, when we write "no quasilocal Poisson modification" we mean that there exists no $\boldsymbol{\Omega}' \subset \boldsymbol{\Omega}$ such that the time-evolved WRM would be concentrated on $\boldsymbol{\Omega}'$ and there exists a quasilocal $\boldsymbol{\Omega}'$-Poisson modification. In all other cases, the quasilocal Poisson modification can be constructed explicitly and will be introduced in the following subsection.

TABLE 1. Quasilocality (ql) transitions of Poisson-modifications for the time-evolved asymmetric WRM.

time	high intensity	low intensity
$0 < t < t_G$	no ql Poisson modification	ql $\boldsymbol{\Omega}^*$-Poisson modification
$t = t_G$	ql $\boldsymbol{\Omega}^+$-Poisson modification	ql $\boldsymbol{\Omega}^*$-Poisson modification
$t_G < t \leq \infty$	ql $\boldsymbol{\Omega}$-Poisson modification	ql $\boldsymbol{\Omega}$-Poisson modification

4.2. **Uniformly finite-horizon potentials for the time-evolved WRM.** It is one of the nice features of the time-evolved WRM that Poisson modifications can be explicitly constructed. We now use the time-evolved two-color PPP as the a-priori measure, in other words, the underlying point process $\boldsymbol{P}$ is now given by the superposition of two PPP with spatially homogeneous intensity measure

$$F(\{+\}|x) = F(\{+\}) = \lambda_+ p_t(+,+) + \lambda_- p_t(-,+)$$
$$F(\{-\}|x) = F(\{-\}) = \lambda_+ p_t(+,-) + \lambda_- p_t(-,-).$$

The $\boldsymbol{\Omega}$-Poisson modification (respectively $\boldsymbol{\Omega}^*$-Poisson modification, $\boldsymbol{\Omega}^+$-Poisson modification) ρ is given by

$$\rho_\Lambda(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c}) = h_\Lambda(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c})/\boldsymbol{P}_\Lambda(h_\Lambda)(\boldsymbol{\omega}_{\Lambda^c})$$

where

$$h_\Lambda(\boldsymbol{\omega}) = \frac{1}{(1+a)^{|\boldsymbol{\omega}_\Lambda|^+}(1+b)^{|\boldsymbol{\omega}_\Lambda|^-}} \prod_{C \in \mathcal{C}_\Lambda^{\mathrm{f}}(\omega)} (1 + a^{|\boldsymbol{\omega}_C|^+} b^{|\boldsymbol{\omega}_C|^-}).$$

Here we used the following notation: $a = \lambda_- p_t(+,-)/(\lambda_+ p_t(+,+))$; $|\boldsymbol{\omega}_C|^\pm$ denotes the number of plus (respectively minus) spins in $\boldsymbol{\omega}_C$; $\mathcal{C}_\Lambda^{\mathrm{f}}(\omega)$ (respectively $\mathcal{C}_\Lambda^\infty(\omega)$) denotes the set of clusters in ω with nonempty intersection with the volume Λ and which are finite (respectively infinite). Note that in [9] we use notation $a^{|\boldsymbol{\omega}_C|^+} b^{|\boldsymbol{\omega}_C|^-} = \rho(\boldsymbol{\omega}_C)$.

In order to arrive at a potential representation, we write $\mathcal{C}^{\mathrm{f}}(\omega)$ for the set of finite clusters in ω and compute

$$\log h_\Lambda(\boldsymbol{\omega}_\Lambda) = \sum_{C \in \mathcal{C}^{\mathrm{f}}(\omega_\Lambda)} \log(1 + a^{|\boldsymbol{\omega}_C|^+} b^{|\boldsymbol{\omega}_C|^-}) - |\boldsymbol{\omega}_\Lambda|^+ \log(1 + a) - |\boldsymbol{\omega}_\Lambda|^- \log(1 + b).$$

Note that the second and third summand on the r.h.s. form single-site potentials which can be considered as part of the a-priori measure by incorporating them into the mark distribution F. Only the first summand describes interactions. Hence, we can define a potential

$$\Psi(\eta, \boldsymbol{\omega}) = \log(1 + a^{|\boldsymbol{\omega}_\eta|^+} b^{|\boldsymbol{\omega}_\eta|^-}) \mathbb{1}_{\eta \in \mathcal{C}^{\mathrm{f}}(\omega)}$$

which is $2r$-uniformly finite-horizon potential is the sense of Definition 3.2. To see this, note that Ψ assigns an interaction energy to finite clusters of ω. In order to decide wether a subset $\eta \Subset \omega$ is a cluster, it suffices to know $\omega_{B_{2r}(\eta)}$. Note that $a < 1$ by definition. In the low-intensity regime $\boldsymbol{P}(\boldsymbol{\Omega}^*) = 1$ and the number of clusters attached to any finite volume is finite. Hence the $2r$-uniformly finite-horizon Hamiltonian defined via Ψ exists in the domain $\boldsymbol{\Omega}^*$. Existence is also guaranteed at the critical time on the domain $\boldsymbol{\Omega}^+$ since then $b = 1$ and Ψ decays exponentially as the cluster size grows unless the number of plus spins is macroscopically vanishing, but $\boldsymbol{P}(\boldsymbol{\Omega}^+) = 1$. At supercritical times and high-intensities, also $b < 1$ and hence we have exponential decay. Finally, we note that Ψ is defined on the hypergraph structure $\mathcal{E}_{\mathcal{C}} = \{(\eta, \boldsymbol{\omega}) \in \bar{\mathcal{E}} : \eta = \mathcal{C}^{\mathrm{f}}(\omega)\}$ of finite clusters.

4.3. **The vacuum potential for the time-evolved WRM.**

In the following we derive the vacuum potential representation and investigate its decay properties. Using the definition in equation (5) we have

$$\Phi(\eta, \boldsymbol{\omega}) = -\sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \log \frac{\rho_\Lambda(\boldsymbol{\omega}_\xi)}{\rho_\Lambda(\emptyset_\Lambda)} = -\sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \sum_{C \in \mathcal{C}(\xi)} \Psi(C, \boldsymbol{\omega}).$$

Lemma 4.1. *The vacuum potential $\Phi(\eta, \boldsymbol{\omega})$ is non-zero only if η is a cluster.*

In particular Φ is again defined on $\mathcal{E}_{\mathcal{C}}$ and if $\Phi(\eta, \boldsymbol{\omega}) \neq 0$ then

$$\Phi(\eta, \boldsymbol{\omega}) = -\sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \Psi(\xi, \boldsymbol{\omega}).$$

Note that clusters can become infinitely long with positive probability in the high intensity regime, for details see [9]. Further, note that spatial positioning inside clusters do not play any rôle in $\Psi(\eta, \boldsymbol{\omega})$. Hence, for nonzero Φ, we can write

$$\Phi(\eta, \boldsymbol{\omega}) = -\sum_{k=0}^{|\boldsymbol{\omega}_\eta|^+} \binom{|\boldsymbol{\omega}_\eta|^+}{k} (-1)^{|\boldsymbol{\omega}_\eta|^+ - k} \sum_{l=0}^{|\boldsymbol{\omega}_\eta|^-} \binom{|\boldsymbol{\omega}_\eta|^-}{l} (-1)^{|\boldsymbol{\omega}_\eta|^- - l} \kappa(k, l)$$

where $\kappa(k, l) = \log(1 + a^k b^l)$. Expanding the logarithm yields,

$$\Phi(\eta, \boldsymbol{\omega}) = (-1)^{|\eta|+1} \sum_{j=1}^{\infty} (-1)^j \frac{1}{j} (1 - a^j)^{|\boldsymbol{\omega}_\eta|^+} (1 - b^j)^{|\boldsymbol{\omega}_\eta|^-}.$$

The vacuum potential is expected to converge slowly. Let us conclude the discussion by the following (non-optimal) upper bound for the critical-time case where $\Phi(\eta, \boldsymbol{\omega}) = 0$ if $|\boldsymbol{\omega}_\eta|^- > 0$ and hence Φ is given by

$$\varphi(n) = (-1)^{n+1} \sum_{j=1}^{\infty} (-1)^j \frac{1}{j} (1 - \alpha^j)^n \quad \text{with } \alpha = (\lambda_-/\lambda_+)^2 < 1.$$

Lemma 4.2. *We have that $\limsup_{n \uparrow \infty} |\varphi(n)| \log n \leq C$ for some $C > 0$.*

4.4. **Existence of uniformly absolutely convergent potentials.** The time-evolved WRM in the Gibbsian regime $t > t_G$ is uniformly quasilocal in the τ-topology and hence also uniformly vacuum quasilocal, see [9]. The next result shows in particular, that it is also vacuum uniformly log-quasilocal.

Proposition 4.3. *In the WRM under independent spin flip in the regime where the associated $\boldsymbol{\Omega}$-premodification ρ is quasilocal, ρ is even uniformly log-quasilocal.*

As a consequence of Theorem 3.8 the time-evolved WRM in the corresponding regime can thus be written as a Gibbs measures w.r.t. an uniformly absolutely convergent potential. The result of Proposition 4.3 also holds at the critical time t_G, but we do not prove it here.

5. PROOFS

Proof of Lemma 2.4. First note that $\boldsymbol{P}_\Lambda \rho_\Lambda = 1$ and in particular for $f \in \mathcal{F}^b_{\Lambda^c}$ and $\boldsymbol{\omega}_{\Lambda^c} \in \boldsymbol{\Omega}^*$ we have $\gamma^\rho_\Lambda(f|\boldsymbol{\omega}_{\Lambda^c}) = f(\boldsymbol{\omega}_{\Lambda^c})$ and hence γ^ρ is proper. As for the consistency, note that using the concentration on $\boldsymbol{\Omega}^*$ we have that

$$\gamma^\rho_\Delta(\gamma^\rho_\Lambda(f|\cdot)|\boldsymbol{\omega}_{\Delta^c})$$

$$= \int \boldsymbol{P}_\Delta(d\boldsymbol{\omega}'_\Delta)\rho_\Delta(\boldsymbol{\omega}'_\Delta\boldsymbol{\omega}_{\Delta^c}) \int \boldsymbol{P}_\Lambda(d\boldsymbol{\omega}''_\Lambda)\rho_\Lambda(\boldsymbol{\omega}''_\Lambda\boldsymbol{\omega}'_{\Delta\backslash\Lambda}\boldsymbol{\omega}_{\Delta^c})f(\boldsymbol{\omega}''_\Lambda\boldsymbol{\omega}'_{\Delta\backslash\Lambda}\boldsymbol{\omega}_{\Delta^c})$$

$$= \int \boldsymbol{P}_\Delta(d\boldsymbol{\omega}'_\Delta)\frac{h_\Delta(\boldsymbol{\omega}'_\Delta\boldsymbol{\omega}_{\Delta^c})}{\boldsymbol{P}_\Delta h_\Delta(\boldsymbol{\omega}_{\Delta^c})} \int \boldsymbol{P}_\Lambda(d\boldsymbol{\omega}''_\Lambda)\frac{h_\Lambda(\boldsymbol{\omega}''_\Lambda\boldsymbol{\omega}'_{\Delta\backslash\Lambda}\boldsymbol{\omega}_{\Delta^c})}{\boldsymbol{P}_\Lambda h_\Lambda(\boldsymbol{\omega}'_{\Delta\backslash\Lambda}\boldsymbol{\omega}_{\Delta^c})}f(\boldsymbol{\omega}''_\Lambda\boldsymbol{\omega}'_{\Delta\backslash\Lambda}\boldsymbol{\omega}_{\Delta^c})$$

$$= \int \boldsymbol{P}_\Delta(d\boldsymbol{\omega}'_\Delta)\frac{h_\Lambda(\boldsymbol{\omega}'_\Delta\boldsymbol{\omega}_{\Delta^c})}{\boldsymbol{P}_\Delta h_\Delta(\boldsymbol{\omega}_{\Delta^c})} \int \boldsymbol{P}_\Lambda(d\boldsymbol{\omega}''_\Lambda)\frac{h_\Delta(\boldsymbol{\omega}''_\Lambda\boldsymbol{\omega}'_{\Delta\backslash\Lambda}\boldsymbol{\omega}_{\Delta^c})}{\boldsymbol{P}_\Lambda h_\Lambda(\boldsymbol{\omega}'_{\Delta\backslash\Lambda}\boldsymbol{\omega}_{\Delta^c})}f(\boldsymbol{\omega}''_\Lambda\boldsymbol{\omega}'_{\Delta\backslash\Lambda}\boldsymbol{\omega}_{\Delta^c})$$

$$= \int \boldsymbol{P}_{\Delta\backslash\Lambda}(d\boldsymbol{\omega}'_\Delta) \int \boldsymbol{P}_\Lambda(d\boldsymbol{\omega}''_\Lambda)\frac{h_\Delta(\boldsymbol{\omega}''_\Lambda\boldsymbol{\omega}'_{\Delta\backslash\Lambda}\boldsymbol{\omega}_{\Delta^c})}{\boldsymbol{P}_\Delta h_\Delta(\boldsymbol{\omega}_{\Delta^c})}f(\boldsymbol{\omega}''_\Lambda\boldsymbol{\omega}'_{\Delta\backslash\Lambda}\boldsymbol{\omega}_{\Delta^c}) = \gamma^\rho_\Delta(f|\boldsymbol{\omega}_{\Delta^c})$$

where we used the pre-modification property in the forth line. □

Proof of Theorem 3.6. We prove in several steps similar to [3, Theorem 2.30] and [11, Theorem 1]. We claim, that the potential is given by

(5)
$$\Phi(\eta, \boldsymbol{\omega}) = \begin{cases} -\sum_{\xi\subset\eta}(-1)^{|\eta\backslash\xi|}\log\frac{\rho_\Lambda(\boldsymbol{\omega}_\xi)}{\rho_\Lambda(\emptyset_\Lambda)} & \text{if } \rho_\Lambda(\boldsymbol{\omega}_\eta) > 0 \\ +\infty & \text{if } \rho_\Lambda(\boldsymbol{\omega}_\eta) = 0 \end{cases}$$

where the definition is independent of Λ as long as $\eta \subset \Lambda$.

Step 1: Note first, that by vacuum positivity $\rho_\Lambda(\emptyset_\Lambda) > 0$. Further, the pre-modification property for $\xi \subset \eta \subset \Lambda$ with $\eta \backslash \xi \subset \Delta \subset \Lambda$ and $\xi \cap \Delta = \emptyset$ implies that

$$\rho_\Delta(\boldsymbol{\omega}_\eta)\rho_\Lambda(\boldsymbol{\omega}_\xi) = \rho_\Delta(\boldsymbol{\omega}_\xi)\rho_\Lambda(\boldsymbol{\omega}_\eta).$$

By the vacuum positivity assumption $\rho_\Delta(\boldsymbol{\omega}_\xi) > 0$ and thus $\rho_\Lambda(\boldsymbol{\omega}_\eta) > 0$ implies $\rho_\Lambda(\boldsymbol{\omega}_\xi) > 0$ and hence Φ is well-defined.

Step 2: The potential Φ has the following properties.

(1) $\Phi(\eta, \cdot)$ is $\mathcal{F}_\eta$ measurable since the evaluation is only w.r.t. η.

(2) By the inclusion-exclusion principle we have

$$\log\frac{\rho_\Lambda(\boldsymbol{\omega}_\Lambda)}{\rho_\Lambda(\emptyset_\Lambda)} = -\sum_{\eta\subset\boldsymbol{\omega}_\Lambda}\Phi(\eta, \boldsymbol{\omega}).$$

(3) Φ is vacuum normalized, indeed let $\xi \subsetneq \eta$, then

$$-\Phi(\eta, \boldsymbol{\omega}_\xi) = \sum_{\zeta \subset \xi} \sum_{\zeta' \subset \eta \setminus \xi} (-1)^{|\eta \setminus (\zeta \cup \zeta')|} \log \frac{\rho_\Lambda(\boldsymbol{\omega}_\zeta)}{\rho_\Lambda(\emptyset_\Lambda)}$$

$$= \sum_{\zeta \subset \xi} (-1)^{|\xi \setminus \zeta|} \log \frac{\rho_\Lambda(\boldsymbol{\omega}_\zeta)}{\rho_\Lambda(\emptyset_\Lambda)} \sum_{\zeta' \subset \eta \setminus \xi} (-1)^{|(\eta \setminus \xi) \setminus \zeta'|}$$

which is zero since $\sum_{\zeta' \subset \eta \setminus \xi} (-1)^{|(\eta \setminus \xi) \setminus \zeta'|} = 0$.

Step 3: Next we show that the definition of Φ is independent of the volume Λ via the pre-modification property of ρ_Λ. For this, let $\emptyset \neq \eta \subset \Lambda' \subset \Lambda$, then

$$\sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \log \frac{\rho_\Lambda(\boldsymbol{\omega}_\xi)}{\rho_{\Lambda'}(\boldsymbol{\omega}_\xi)} = \log \frac{\rho_\Lambda(\emptyset)}{\rho_{\Lambda'}(\emptyset)} \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} = 0.$$

Step 4: For the existence of the Hamiltonian, note that formely

$$H_{\Lambda,\Delta}(\boldsymbol{\omega}) = \sum_{\emptyset \neq \eta \subset \omega_\Delta} \Phi(\eta, \boldsymbol{\omega}) - \sum_{\emptyset \neq \eta \subset \omega_{\Delta \setminus \Lambda}} \Phi(\eta, \boldsymbol{\omega})$$

$$= \log \frac{\rho_{\Delta \setminus \Lambda}(\boldsymbol{\omega}_{\Delta \setminus \Lambda})}{\rho_{\Delta \setminus \Lambda}(\emptyset_{\Delta \setminus \Lambda})} - \log \frac{\rho_\Delta(\boldsymbol{\omega}_\Delta)}{\rho_\Delta(\emptyset_\Delta)} = \log \frac{\rho_\Delta(\boldsymbol{\omega}_{\Delta \setminus \Lambda})}{\rho_\Delta(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Delta \setminus \Lambda})} = -\log \frac{\rho_\Lambda(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Delta \setminus \Lambda})}{\rho_\Lambda(\boldsymbol{\omega}_{\Delta \setminus \Lambda})}$$

where we used the pre-modification property twice. By vacuum positivity $\rho_\Lambda(\boldsymbol{\omega}_{\Delta \setminus \Lambda}) > 0$ and thus $H_{\Lambda,\Delta}(\boldsymbol{\omega})$ is well defined. Now, by assumption of vacuum quasilocality, as Δ tends to $\mathbb{R}^d$, we have

$$H_\Lambda(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c}) = -\log \frac{\rho_\Lambda(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c})}{\rho_\Lambda(\boldsymbol{\omega}_{\Lambda^c})}.$$

Moreover if ρ_Λ is vacuum uniformly log-quasilocal, it is in particular positive and we have

$$\sup_{\boldsymbol{\omega} \in \boldsymbol{\Omega}^*} |H_{\Lambda,\Delta}(\boldsymbol{\omega}) - \log \frac{\rho_\Lambda(\boldsymbol{\omega}_{\Lambda^c})}{\rho_\Lambda(\boldsymbol{\omega})}| \leq \sup_{\boldsymbol{\omega} \in \boldsymbol{\Omega}^*} (|\log \frac{\rho_\Lambda(\boldsymbol{\omega}_{\Delta \setminus \Lambda})}{\rho_\Lambda(\boldsymbol{\omega}_{\Delta \setminus \Lambda} \boldsymbol{\omega}_{\Delta^c})}| + |\log \frac{\rho_\Lambda(\boldsymbol{\omega}_\Delta)}{\rho_\Lambda(\boldsymbol{\omega}_\Delta \boldsymbol{\omega}_{\Delta^c})}|)$$

which tends to zero as Δ tends to $\mathbb{R}^d$.

Step 5: Note that $h_\Lambda^\Phi(\boldsymbol{\omega}) = \exp(-H_\Lambda(\boldsymbol{\omega})) = \rho_\Lambda(\boldsymbol{\omega})/\rho_\Lambda(\boldsymbol{\omega}_{\Lambda^c})$ and the normalization is given by $\int \boldsymbol{P}_\Lambda(d\boldsymbol{\omega}_\Lambda) h_\Lambda^\Phi(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c}) = 1/\rho_\Lambda(\boldsymbol{\omega}_{\Lambda^c})$. Hence, $\rho^\Phi = \rho$.

Step 6: Finally, for the uniqueness, let Φ' be another vacuum potential with $\rho^\Phi = \rho^{\Phi'}$. Then, $\Phi' - \Phi$ is again a vacuum potential which is equivalent to zero in the sense that

$$H_\Lambda^{\Phi' - \Phi} = \log(h_\Lambda^{\Phi'}/h_\Lambda^\Phi) = \log(Z_\Lambda^{\Phi'}/Z_\Lambda^\Phi)$$

is measurable w.r.t. $\mathcal{F}_{\Lambda^c}$. Then, it suffices to show that $\Psi = \Phi' - \Phi = 0$. But for all $\Lambda \Subset \mathbb{R}^d$ by the inclusion-exclusion principle,

$$\Psi(\eta, \boldsymbol{\omega}) = \sum_{\emptyset \neq \xi \subset \eta} (-1)^{|\eta \setminus \xi|} H_\Lambda^\Psi(\boldsymbol{\omega}_\xi) = H_\Lambda^\Psi(\emptyset) \sum_{\emptyset \neq \xi \subset \eta} (-1)^{|\eta \setminus \xi|} = 0$$

where we used the normalization in the last equation. $\qquad \square$

Proof of Corollary 3.7. Let η be such that there exist $x, y \in \eta$ with $|x - y| = s > r$. Denote $\eta' = \eta \setminus \{x, y\}$ and $B_r(x)$ the open ball with radius r centered at $x \in \mathbb{R}^d$. Then, using the pre-modification property, we have

$$-\Phi(\eta, \boldsymbol{\omega}) = \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \log \frac{\rho_\Lambda(\boldsymbol{\omega}_\xi)}{\rho_\Lambda(\emptyset_\Lambda)}$$

$$= \sum_{\xi_1 \subset \eta'} (-1)^{|\eta' \setminus \xi_1|} [\log \frac{\rho_\Lambda(\boldsymbol{\omega}_{\xi_1} \boldsymbol{x} \boldsymbol{y})}{\rho_\Lambda(\boldsymbol{\omega}_{\xi_1} \boldsymbol{y})} + \log \frac{\rho_\Lambda(\boldsymbol{\omega}_{\xi_1})}{\rho_\Lambda(\boldsymbol{\omega}_{\xi_1} \boldsymbol{x})}]$$

$$= \sum_{\xi_1 \subset \eta'} (-1)^{|\eta' \setminus \xi_1|} [\log \frac{\rho_{B_{s-r}(x)}(\boldsymbol{\omega}_{\xi_1} \boldsymbol{x} \boldsymbol{y})}{\rho_{B_{s-r}(x)}(\boldsymbol{\omega}_{\xi_1} \boldsymbol{y})} - \log \frac{\rho_{B_{s-r}(x)}(\boldsymbol{\omega}_{\xi_1} \boldsymbol{x})}{\rho_{B_{s-r}(x)}(\boldsymbol{\omega}_{\xi_1})}] = 0$$

as required. $\qquad\square$

The key to improve the possibly very poor summability properties of the vacuum potential is to apply a suitable resummation procedure. For the lattice such resummations have been used for the first time in [10] to improve convergence. It is interesting to note that resummations could even be used in certain cases of non-Gibbsian lattice systems, namely for the joint measures of quenched random systems. Here one obtains at least weakly Gibbsian representations. Having a weakly Gibbsian representation means that the Hamiltonians converge absolutely at least on a measure one set of configurations, but possibly not everywhere, see [12]. In the continuum such resummations have not been done so far, to our knowledge. We will explain now, how nice they can be done, and how well indeed it works together with the notion of Georgii's hyperedge potential, see [2], as collected interactions can be naturally indexed with hyperedges when one allows an additional dependence up to a finite horizon.

Proof of Theorem 3.8. Let us start by considering for every $x \in \mathbb{R}^d$ a co-final sequence $(\Delta_{x,m})_{m \geq 1}$ of finite subsets in $\mathbb{R}^d$ to be specified later. Next, let $\geq$ denote a total ordering on $\mathbb{R}^d$ for which every locally finite subset has a least element. For example think of the cycling order where points are ordered first by their euclidean distance to the origin and then by their angles. Let $\Lambda_x = \{y \geq x\}$ and define $A_{x,m} = \Delta_{x,m} \cap \Lambda_x$ with $A_{x,0} = \emptyset$ the part of the sequence such that the x is the left endpoint. For $\eta \Subset \mathbb{R}^d$ we will write $l(\eta)$ and $r(\eta)$ to denote the left and right end points of η in the given ordering. Further we define

$$P_{x,m} = \{\eta \Subset \mathbb{R}^d : l(\eta) = x \text{ and } r(\eta) \in (A_{x,m} \setminus A_{x,m-1})\}$$

the set of finite subsets of $\mathbb{R}^d$ with left end point equal to x and right end point in the m-annulus of the sequence $A_{x,m}$. Note that in particular, $\bigcup_{x,m} P_{x,m} = \{\eta : \eta \Subset \mathbb{R}^d\}$ is a disjoint partition of the set of finite subsets of $\mathbb{R}^d$. This is a certain grading of the set of finite subsets of $\mathbb{R}^d$.

Now we perform the regrouping w.r.t. the unique vacuum potential Φ. For a given $\omega \in \Omega$ and any $x \in \omega$ we denote by $P_{x,m}^\omega = P_{x,m} \cap \{\eta \Subset \omega\}$ the set of subsets of ω in the grading $P_{x,m}$. Such a $P_{x,m}^\omega$ might very well be empty. Note that $P_{x,m}^\omega = P_{x,m}^{\tilde{\omega}}$ if $\omega_{A_{x,m}} = \tilde{\omega}_{A_{x,m}}$. Next, we let $\omega_{x,m} = \omega \cap (\{x\} \cup (A_{x,m} \setminus A_{x,m-1}))$ be the union of finite subsets of ω which have x as their left endpoint and all their other points lying in the m-annulus. In these sets we will accumulate the energy contribution of all $\eta \subset \omega_{x,m}$. In case $P_{x,m}^\omega = \emptyset$ we do not need such a representative as will become clear in the following definition. For $(\eta, \boldsymbol{\omega}) \in \mathcal{E}$ we define $\Psi(\eta, \boldsymbol{\omega}) = 0$ unless $\eta = \omega_{x,m}$ for some pair (x, m) in which case we put

$$\Psi(\omega_{x,m}, \boldsymbol{\omega}) = \sum_{\eta \in P_{x,m}^\omega} \Phi(\eta, \boldsymbol{\omega}).$$

In words, the energy of vacuum interaction potentials within a class is accumulated in one interaction for each class. The sum can contain configurations η with points in $A_{x,m-1}$. Clearly Ψ is not a vacuum

potential. However, note that we have $\Psi(\omega_{x,m}, \boldsymbol{\omega}) = \Psi(\omega_{x,m}, \tilde{\boldsymbol{\omega}})$ if $\omega_{A_{x,m}} = \tilde{\omega}_{A_{x,m}}$ and thus Ψ has the finite-horizon property.
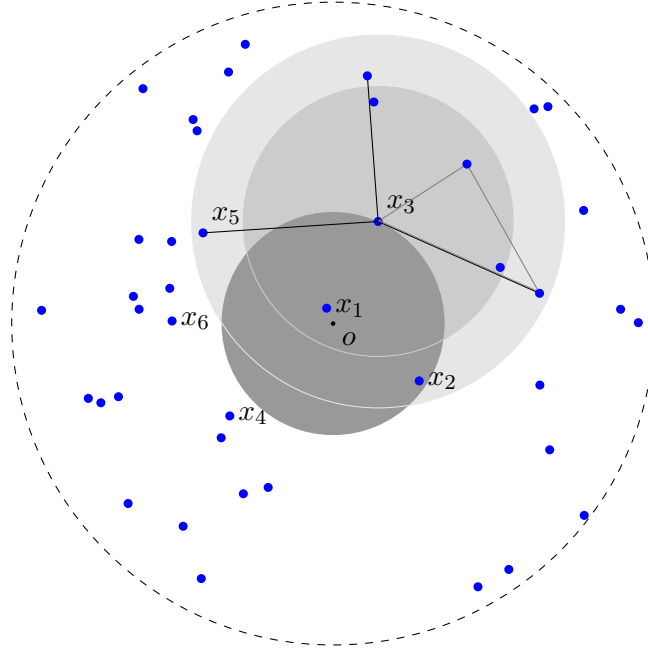

FIGURE 1. Construction of re-summation of vacuum potentials. Indicated numbering of points in configuration ω is according to cyclic ordering; Λ_{x_3} given by the complement of the dark-gray area; two balls $\Delta_{x_3,m}$ and $\Delta_{x_3,m'}$ are given by middle-gray and light and middle-gray area around x_3 including the parts hidden by dark gray area; $\omega_{x,m'}$ is given by the four connected points via black lines; example $\eta \in P_{x,m'}^{\omega}$ is given via points in triangle with gray edges, for this η, $\Delta(\eta) = \Delta_{x_3,m'}$.

What remains to show is that Ψ defines an equivalent Hamiltonian as Φ and that Ψ is indeed uniformly absolutely convergent for a good choice of $\Delta_{x,m}$. W.r.t. the equivalence, note that

$$\sum_{\eta \Subset \omega:\, \eta \cap \Lambda \neq \emptyset} \Psi(\eta, \boldsymbol{\omega}) = \sum_{x \in \omega_\Lambda} \sum_{m=1}^{\infty} \Psi(\omega_{x,m}, \boldsymbol{\omega}) + \sum_{y \in \omega:\, y < l(\omega_\Lambda)} \sum_{m \in \mathbb{N}:\, \omega_{y,m} \cap \Lambda \neq \emptyset} \Psi(\omega_{y,m}, \boldsymbol{\omega})$$

$$= \sum_{x \in \omega_\Lambda} \sum_{m=1}^{\infty} \sum_{\eta \in P_{x,m}^{\omega}} \Phi(\eta, \boldsymbol{\omega}) + \sum_{y \in \omega:\, y < l(\omega_\Lambda)} \sum_{m \in \mathbb{N}:\, \omega_{y,m} \cap \Lambda \neq \emptyset} \sum_{\eta \in P_{y,m}^{\omega}} \Phi(\eta, \boldsymbol{\omega})$$

$$= \sum_{\eta \Subset \omega:\, l(\eta) \in \Lambda} \Phi(\eta, \boldsymbol{\omega}) + \sum_{\eta \Subset \omega:\, l(\eta) < l(\omega_\Lambda),\, \eta \cap \Lambda \neq \emptyset} \Phi(\eta, \boldsymbol{\omega}) + \sum_{\eta \in Q_\Lambda^{\omega}} \Phi(\eta, \boldsymbol{\omega}),$$

where $Q_\Lambda^{\omega} = \{\eta \Subset \omega : l(\eta) < l(\omega_\Lambda),\, \eta \cap \Lambda = \emptyset,$ there exists $m \in \mathbb{N}$ such that $\eta \subset \omega_{y,m} \cap \Lambda \neq \emptyset\}$. Now the last summand only depends on $\boldsymbol{\omega}_{\Lambda^c}$ and hence Ψ and Φ are equivalent.

W.r.t. the uniform absolute convergence, note that for all $\Lambda \Subset \mathbb{R}^d$ and $\boldsymbol{\omega} \in \Omega$,

$$\sum_{\eta \Subset \omega:\, \eta \cap \Lambda \neq \emptyset} |\Psi(\eta, \boldsymbol{\omega})| = \sum_{x \in \omega_\Lambda} \sum_{m=1}^{\infty} |\Psi(\omega_{x,m}, \boldsymbol{\omega})| + \sum_{y \in \omega:\, y < l(\omega_\Lambda)} \sum_{m \in \mathbb{N}:\, \omega_{y,m} \cap \Lambda \neq \emptyset} |\sum_{\eta \in P_{y,m}^{\omega}} \Phi(\eta, \boldsymbol{\omega})|,$$

where the second summand is finite since all the sums involved are in fact finite. Indeed, the first sum is finite due to the definition of the ordering. The second sum is finite by finiteness of Λ and the third sum is finite by the locally finiteness of ω and the assumption that the vacuum Hamiltonian is finite.

In order to prove $\sum_{m=1}^{\infty} |\Psi(\omega_{x,m}, \boldsymbol{\omega})| < \infty$, note that by assumption Φ is uniformly convergent and hence, for every $x \in \mathbb{R}^d$, there exists a co-final sequence $(\Delta_{x,m})_{m \geq 1}$ of balls in $\mathbb{R}^d$ with radius $r_m \in \mathbb{N}$, centered at $x \in \mathbb{R}^d$ such that

$$\sup_{\boldsymbol{\omega} \in \boldsymbol{\Omega}^*} | \sum_{x \in \eta \Subset \omega : \eta \not\subset \Delta_{x,m}} \Phi(\eta, \boldsymbol{\omega})| < m^{-2}$$

and in particular, recalling $\Lambda_x = \{y \geq x\}$, we have

$$\sup_{\boldsymbol{\omega}_{\Lambda_x} \in \boldsymbol{\Omega}^*} | \sum_{x \in \eta \Subset \omega_{\Lambda_x} : \eta \not\subset A_{x,m}} \Phi(\eta, \boldsymbol{\omega})| < m^{-2}.$$

For this choice of $\Delta_{x,m}$ we have

$$\sum_{m=1}^{\infty} |\Psi(\omega_{x,m}, \boldsymbol{\omega})| \leq | \sum_{\eta \Subset \omega : l(\eta)=x, \eta \subset A_{x,1}} \Phi(\eta, \boldsymbol{\omega})| + \sum_{m=2}^{\infty} | \sum_{\eta \Subset \omega : l(\eta)=x, r(\eta) \in A_{x,m} \setminus A_{x,m-1}} \Phi(\eta, \boldsymbol{\omega})|$$

$$\leq | \sum_{\eta \Subset \omega : l(\eta)=x, \eta \subset A_{x,1}} \Phi(\eta, \boldsymbol{\omega})| + 2 \sum_{m=1}^{\infty} | \sum_{\eta \Subset \omega : l(\eta)=x, \eta \not\subset A_{x,m}} \Phi(\eta, \boldsymbol{\omega})|$$

where the first summand consists of only finitely many summands and the second summand is bounded from above by $2 \sum_{m \geq 1} m^{-2} < \infty$ as required. Further, since our choice of the $\Delta_{x,m}$ is independent of $\boldsymbol{\omega}$, also the uniform absolute convergence follows.

Finally note that in order to determine the horizon of $(\eta, \boldsymbol{\omega}) \in \mathcal{E}$, it suffices to consider the case $\eta = \omega_{x,m}$ for some $x \in \mathbb{R}^d$ and $m \geq 1$. But by the definitions, $\Psi(\omega_{x,m}, \boldsymbol{\omega}) = \Psi(\omega_{x,m}, \tilde{\boldsymbol{\omega}})$ if $\boldsymbol{\omega}_{\Delta_{x,m}} = \tilde{\boldsymbol{\omega}}_{\Delta_{x,m}}$ and hence $\Delta(\eta, \boldsymbol{\omega}) = \Delta(\eta)$. $\qquad \square$

Let us make a few more comments on the above proof. (1) The mapping $\eta \mapsto \Delta(\eta)$ is measurable on $\mathbb{B} = \{B_m(x) | x \in \mathbb{R}^d, m \in \mathbb{N}\}$ with σ-algebra $\mathcal{B}(\mathbb{B}) = \sigma(\{B_n(x) \in \mathbb{B} | x \in A, n = m\}, A \in \mathcal{B}(\mathbb{R}^d), m \in \mathbb{N})$. In order to see this, note that the mapping $\eta \mapsto l(\eta)$ is measurable w.r.t. $\mathcal{B}(\mathbb{R}^d)$ since $\{\eta | l(\eta) \in A\} = \{\eta | |\eta \cap A| \geq 1\} \cap \{\eta | |\eta \cap A'| = 0\}$ where $A' = \{x \in \mathbb{R}^d | x < y$ for all $y \in A\}$. Further note that we can decompose $\eta \mapsto (l(\eta), \eta) \mapsto \Delta_{l(\eta)}(\eta) = \Delta(\eta)$ and

$$\{(l(\eta), \eta) | \Delta(\eta) = B_{r_n}(l(\eta))\} = \{(l(\eta), \eta) | |\eta \cap B_{r_n}(l(\eta))^c| = 0\} \cap$$

$$\bigcup_{m \in \mathbb{N}} [\{(l(\eta), \eta) | |\eta| = m\} \cap \{(l(\eta), \eta) | |\eta \cap B_{r_{n-1}}(l(\eta))| \leq m - 1\}]$$

which shows measurability of the second mapping. Finally, since the mapping $n \mapsto r_n, \mathbb{N} \to \mathbb{N}$ is trivially measurable, the result follows.

(2) Instead of balls, the co-final sequence $\Delta_{x,m}$ can also consist of measurable sets. Also in this case measurability of $\eta \mapsto \Delta(\eta)$ follows by measurability of Φ.

(3) Let us also note, that the proof of Theorem 3.8 is not easily adaptable to give absolutely convergent potentials (where we do not require the convergence to hold uniformly in $\boldsymbol{\omega} \in \boldsymbol{\Omega}^*$) in the absence of uniformly convergent vacuum potentials. The reason for this is that in that case the co-final sequence $\Delta_{x,m}$ (which is designed to give a sufficiently quick exhaustion of $\mathbb{R}^d$ such that summability follows) depends on $\boldsymbol{\omega}$. In this case the finite horizon property can not be guaranteed any more.

(4) As can be seen from the proof, the hypergraph structure for Ψ is given by $\mathcal{E}^* = \{(\eta, \boldsymbol{\omega}) \in \bar{\mathcal{E}}^* : \eta = \omega_{l(\eta),m}$ for some $m \in \mathbb{N}\}$.

(5) Finally, note that although the convergence is uniform in general there is no absolute summability with a uniform bound, see for example the Potts gas.

Proof of Lemma 4.1. Assume that η consists of two clusters $\eta_1, \eta_2 \neq \emptyset$, then we have

$$-\Phi(\eta, \boldsymbol{\omega}) = \sum_{\xi_1 \subset \eta_1} (-1)^{|\eta_1 \backslash \xi_1|} \sum_{\xi_2 \subset \eta_2} (-1)^{|\eta_2 \backslash \xi_2|} \sum_{C \in \mathcal{C}(\xi_1 \cup \xi_2)} \Psi(C, \boldsymbol{\omega})$$

$$= \sum_{\xi_1 \subset \eta_1} (-1)^{|\eta_1 \backslash \xi_1|} \sum_{\xi_2 \subset \eta_2} (-1)^{|\eta_2 \backslash \xi_2|} (\Psi(C, \boldsymbol{\omega}_{\xi_1}) + \Psi(C, \boldsymbol{\omega}_{\xi_2}))$$

$$= \sum_{\xi_1 \subset \eta_1} (-1)^{|\eta_1 \backslash \xi_1|} \sum_{\xi_2 \subset \eta_2} (-1)^{|\eta_2 \backslash \xi_2|} \Psi(C, \boldsymbol{\omega}_{\xi_2}) = 0.$$

$\square$

Proof of Lemma 4.2. Let us start by estimating φ using additional cross terms.

$$|\varphi(n)| = |\sum_{j=1,3,\ldots}^{\infty} \left(\frac{1}{j}(1-a^j)^n - \frac{1}{j+1}(1-a^{j+1})^n\right)|$$

$$\leq |\sum_{j=1,3,\ldots}^{\infty} \frac{1}{j}\left((1-a^j)^n - (1-a^{j+1})^n\right)| + |\sum_{j=1,3,\ldots}^{\infty} \left(\frac{1}{j} - \frac{1}{j+1}\right)(1-a^{j+1})^n|$$

$$\leq \sum_{j\geq 1}^{\infty} \frac{1}{j}\left((1-a^{j+1})^n - (1-a^j)^n\right) + \sum_{j\geq 1}^{\infty} \frac{1}{j(j+1)}(1-a^{j+1})^n$$

$$\leq (1-a)^n + 2\sum_{j\geq 1}^{\infty} \frac{1}{j(j+1)}(1-a^{j+1})^n$$

The first term decays exponentially. In order to determine the asymptotic behaviour of the second term, let us split the sum into terms $j \geq J$ and $j < J$, with $J = J(n)$ tending to infinity with n, in a way chosen below. We obtain the upper bound

$$\sum_{j\geq J} \frac{1}{j(j+1)}(1-a^{j+1})^n + \sum_{j<J} \frac{1}{j(j+1)}(1-a^{j+1})^n \leq \frac{1}{J} + (1-a^J)^n$$

where we used twice that $\sum_{j\geq J}(j(j+1))^{-1} = J^{-1}$. As a final step, we optimize over J given n in such a way that the expression

$$(1-a^J)^n = \exp\left(-n(a^J + o(a^J))\right)$$

tends to zero as n tends to infinity. In order to achieve this, take $na^J = n^\varepsilon$ for arbitrary $\varepsilon > 0$. Then $J(n) = ((\varepsilon - 1)/\log a)\log n$ which gives the desired speed of convergence with $C > 2\log(1/a)$. $\square$

Proof of Proposition 4.3. It suffices to consider

$$\sup_{\boldsymbol{\omega} \in \boldsymbol{\Omega}} |\log h_\Lambda(\boldsymbol{\omega}) - \log h_\Lambda(\boldsymbol{\omega}_\Delta)|$$

$$= \sup_{\boldsymbol{\omega} \in \boldsymbol{\Omega}} |\sum_{C \in \mathcal{C}_\Lambda^{\mathrm{f}}(\omega)} \log(1 + a^{|\boldsymbol{\omega}_C|^+} b^{|\boldsymbol{\omega}_C|^-}) - \sum_{C \in \mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Delta)} \log(1 + a^{|\boldsymbol{\omega}_C|^+} b^{|\boldsymbol{\omega}_C|^-})|$$

with $\Delta \supset \Lambda$. Let $d(\Lambda, \Delta^c)$ denotes the set distance between Δ^c and Λ. Note that, in order for a cluster in $\mathcal{C}_\Lambda^{\mathrm{f}}(\omega)$ not to be also contained in $\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Delta)$ it must at least have $d(\Lambda, \Delta^c)/(2r)$ many points since otherwise is would be contained in Δ. Conversely, every cluster in $\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Delta)$ is part of a cluster in $\mathcal{C}_\Lambda^{\mathrm{f}}(\omega) \cup \mathcal{C}_\Lambda^{\infty}(\omega)$. If the cluster would be contained in Δ, then both contributions cancel. Hence, non

canceling clusters in $\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Delta)$ must also have at least $d(\Lambda, \Delta^c)/(2r)$ many points. Moreover, there can only be $K = K(\Lambda, r)$ different clusters attached to Λ. Hence with $c = a \vee b$ we have

$$\sup_{\boldsymbol{\omega} \in \boldsymbol{\Omega}} |\log h_\Lambda(\boldsymbol{\omega}) - \log h_\Lambda(\boldsymbol{\omega}_\Delta)| \leq 2K \log(1 + c^{d(\Lambda, \Delta^c)/(2r)})$$

which tends to zero as Δ tends to $\mathbb{R}^d$ since in the Gibbsian regime $t > t_G$ we have $a, b < 1$. $\square$

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