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Exit time risk-sensitive stochastic control problems related to systems of cooperative agents

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Abstract

We study sequences, parametrized by the number of agents, of exit time stochastic control problems with risk-sensitive costs structures generate by unbounded costs. We identify a fully characterizing assumption, under which, each of them corresponds to a risk-neutral stochastic control problem with additive cost, and also to a risk-neutral stochastic control problem on the simplex, where the specific information about the state of each agent can be discarded. We finally prove that, under some additional assumptions, the sequence of value functions converges to the value function of a deterministic control problem.

1 Introduction

In this paper, we study many agent exit time stochastic control problems with risk-sensitive cost. The reader with background on physics or chemistry, can think of particles instead of agents. Each agent occupies states that take values in a finite set $\mathcal{X}$, and by controlling the transition rates between states for each individual, we try to keep the system away from a "ruin" set $\mathcal{K}$, for as long as possible and with the least cost. We prove, under suitable assumptions, that for every finite number n of agents the control problem is equivalent to one with an ordinary (additive) cost. Moreover, when $\mathcal{K} \subset \mathcal{X}^n$ can be identified with a subset of the simplex of probability measures $\mathcal{P}(\mathcal{X})$ (in the sense that for every permutation $\sigma \in \mathbb{S}_n$ we have $\sigma\mathcal{K} = \mathcal{K}$), then we can replace the original problem by one on $\mathcal{P}^n(\mathcal{X}) = \mathcal{P}(\mathcal{X}) \cap \frac{1}{n}\mathbb{Z}^d$, getting in this way a control problem whose state is the empirical measure on the states of the individual agents. We also study the behavior as $n \to \infty$ of the sequence of suitable renormalized value functions, and prove uniform convergence to the value function of a deterministic control problem.

We first describe the model without control, which we call the "base" or "nominal" model. Let $\mathcal{X} = \{e_1, \dots, e_d\}$, where e_i is the ith unit vector in $\mathbb{R}^d$. Let also $\boldsymbol{\gamma} = \{\gamma_{xy}\}_{(x,y)\in\mathcal{X}\times\mathcal{X}}$ denote the rates of an ergodic Markov process on $\mathcal{X}$. This process has the generator

$$\mathcal{L}_{l\boldsymbol{\gamma}}[f](x) = \sum_{y\in\mathcal{X}} \gamma_{xy} \left[f(y) - f(x)\right]. \tag{1.1}$$

For $n \in \mathbb{N}$, consider n agents that independently and randomly take different states x_i^n among the elements of $\mathcal{X} = \{e_1, \dots, e_d\}$, and let $\boldsymbol{x}^n = (x_1^n, \dots, x_n^n)$. This process takes values in $\mathcal{X}^n$ and has the generator

$$\mathcal{L}^n_{\boldsymbol{\gamma}}[f](\boldsymbol{x}^n) = \sum_{i=1}^n \sum_{y\in\mathcal{Z}_{x_i^n}} \gamma_{x_i^n y} \left[f(\boldsymbol{x}^n + \boldsymbol{v}^n_{i,x_i^n y}) - f(\boldsymbol{x}^n)\right], \tag{1.2}$$

Here $\mathcal{Z} \doteq \{(x,y) \in \mathcal{X}\times\mathcal{X} : \gamma_{xy} > 0\}$, $\mathcal{Z}_x \doteq \{y \in \mathcal{X} : (x,y) \in \mathcal{Z}\}$ is the set of allowed transitions from x, and $\boldsymbol{v}^n_{i,xy} = (0,\dots,0,\boldsymbol{v}_{xy},0,\dots,0)$ is a $d\times n$ matrix with all columns equal to zero apart from the ith column, which is identically equal to the vector $\boldsymbol{v}_{xy} \doteq y - x$. Since the process is ergodic, $\mathcal{Z}$ generates the hyperplane

$$\mathcal{H} \doteq \left\{ \sum_{(x,y)\in\mathcal{Z}} a_{xy}\boldsymbol{v}_{xy} : a_{xy} > 0,\ (x,y) \in \mathcal{Z} \right\}, \tag{1.3}$$

which coincides with the hyperplane through the origin that is parallel to $\mathcal{P}(\mathcal{X})$.

We claim that the set $\mathcal{H}$ does not change if the a_{xy} are allowed to be arbitrary real numbers. By ergodicity, for any two states $(x,y) \in \mathcal{Z}$ there is a sequence of states $x = x_1, \dots, x_j = x$ with $y = x_2$ and with the property that $(x_i, x_{i+1}) \in \mathcal{Z}$ for $i = 1, \dots, j-1$, and hence $\sum_{i=1}^{j-1} \boldsymbol{v}_{x_i x_{i+1}} = 0$. Repeating this for every possible $(x,y) \in \mathcal{Z}$, there are strictly positive integers b_{xy} such that $\sum_{(x,y)\in\mathcal{Z}} b_{xy}\boldsymbol{v}_{xy} = 0$, which implies the claim.

Next we introduce the empirical measure process. This process is obtained by projecting from $\mathcal{X}^n$ onto $\mathcal{P}^n(\mathcal{X}) = \mathcal{P}(\mathcal{X}) \cap \frac{1}{n}\mathbb{Z}^d \subset \mathcal{P}(\mathcal{X})$, and has the generator

$$\mathcal{M}^n_{\boldsymbol{\gamma}}[f](\boldsymbol{m}) = n \sum_{(x,y)\in\mathcal{Z}} \gamma_{xy} m_x \left[f\left(\boldsymbol{m} + \frac{1}{n}\boldsymbol{v}_{xy}\right) - f(\boldsymbol{m})\right]. \tag{1.4}$$

One can interpret the model introduced above as a collection of independent agents with each evolving according to the transition rates $\boldsymbol{\gamma}$. This is the “preferred” or “nominal” dynamic, and is what would occur if no “outside influence” or other form of control acts on the agents. If a controller should wish to change this behavior, then it must pay a cost to do so. We would like to model the situation where limited information on the system state, and in particular information relating only to the empirical measure of the states of all agents, is used to produce a desired behavior of the group of agents, which again will be characterized in terms of their empirical measure (which is used to characterize how the collective “loads” the system).

Thus we consider for each $n \in \mathbb{N}$ “reward” functions $R^n : \mathcal{P}(\mathcal{X}) \to [0,\infty)$, where we recall

$$\mathcal{P}(\mathcal{X}) \doteq \left\{ \boldsymbol{m} \in \mathbb{R}^{\mathcal{X}} : m_x \geq 0 \text{ for all } x \in \mathcal{X} \text{ and } \sum_{x\in\mathcal{X}} m_x = 1 \right\} \tag{1.5}$$

is the simplex of probability measures on $\mathcal{X}$. It is assumed that the R^n are continuous and that they converge uniformly to some R^∞. We also have *a sequence of unbounded* “cost” functions $\boldsymbol{C}^n = \{C^n_{xy} : [0,\infty) \to [0,\infty]\}_{(x,y)\in\mathcal{Z}}$ that converge on $(0,\infty)$ to some $\boldsymbol{C}^\infty$ in a sense that we are going to define in the sequel. In the controlled setting, the jump rates of each agent can be perturbed from $\boldsymbol{\gamma}$ to $\boldsymbol{u}$. Let $\boldsymbol{\chi}^n$ denote the controlled state occupied by the collection of agents, and for $\boldsymbol{x}^n = \{x^n_i\}_{i\leq n} \in \mathcal{X}^n$ define

$$L(\boldsymbol{x}^n) \doteq \sum_{i=1}^n \delta_{x^n_i}. \tag{1.6}$$

If the problem is of interest over the interval $[0,T]$ then there is a collective risk-sensitive cost (paid by the coordinating controller) equal to

$$\mathbb{E}_{\boldsymbol{x}^n}\left[\exp\left(\int_0^T \left(\sum_{i=1}^n \sum_{y\in\mathcal{Z}_{\chi^n_i(t)}} \gamma_{\chi^n_i(t)y} C^n_{\chi^n_i(t)y}\left(\frac{u_{\chi^n_i(t)y}(t,i)}{\gamma_{\chi^n_i(t)y}}\right) - nR^n(L(\boldsymbol{\chi}^n(t)))\right) dt\right)\right]. \tag{1.7}$$

Here the control process $\boldsymbol{u}$ takes values in a space that will be defined later, and for a collection of $n|\mathcal{Z}|$ independent Poisson random measures (PRM) $\{N^1_{i,xy}\}_{1\leq i\leq n,(x,y)\in\mathcal{Z}}$ with intensity measure equal to Lebesgue measure, the controlled dynamics are given by

$$\chi^n_i(t) = x^n_i + \sum_{(x,y)\in\mathcal{Z}} \boldsymbol{v}_{xy} \int_{(0,t]}\int_{[0,\infty)} 1_{\left[0,1_x(\chi^n_i(s-))u_{xy}(s,i)\right]}(r) N^1_{i,xy}(dsdr). \tag{1.8}$$

Thus χ^n_i changes from state x to y with rate u_{xy}. The formulation of the dynamics in terms of a stochastic differential equation will be convenient in the analysis to follow.

In this paper we present three results. The first is that, under additional assumptions on the cost $\boldsymbol{C}^n$, for each n the risk-sensitive control problem is equivalent to an ordinary control problem the cost function $\boldsymbol{F}^n = \{F^n_{xy}\}_{(x,y)\in\mathcal{Z}}$, where F^n_{xy} is defined by

$$F^n_{xy}(q) \doteq \sup_{u\in(0,\infty)} G^n_{xy}(u,q) \quad \text{and} \quad G^n_{xy}(u,q) \doteq \left[u\ell\left(\frac{q}{u}\right) - \gamma_{xy}C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right)\right], \tag{1.9}$$

$$\ell(q) \doteq q\log q - q + 1 \text{ for } q \geq 0. \tag{1.10}$$

Under the additional conditions we do not end up with a *stochastic game*, as one might expect, but rather a control problem with additive cost. Control problems are often substantially simpler than games, and in particular are often more tractable from a computational perspective.

We also show under appropriate conditions that both the risk sensitive and the ordinary control problems are equivalent to mean field control problems, which consider the projected process on the simplex, and for which costs depend only on the dynamics of the empirical measure. From an analytical point of view, the benefit is that we provide sufficient conditions, in addition to convexity, that the cost function $\boldsymbol{C}^n$ should satisfy in order for the optimal choice of each agent at a specific moment t to be the same for all particles that occupy the same state in $\mathcal{X}$.

The last contribution, again under additional assumptions on the sequence of costs $\{\boldsymbol{C}^n\}_n$, is that the sequence of value functions, suitably renormalized, converges to the value function of a deterministic control problem. This is also helpful in the construction of controls.

1.1 Literature

In ordinary discrete-time (see [1, 19] for an exposition) and continuous-time (see [17] for an exposition) Markov Decision Processes (MDP) problems, one is given the task to control a random processes in order to have an optimal expected result. The most common optimality criteria are

$$J_T(x_0,\pi) = \mathbb{E}_\pi\left[\sum_{n=0}^{T}\beta^n C(X_n,U_n) + \beta^T R(X_T)\right], \text{ or } \quad J(x_0,\pi) = \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}_\pi\left[\sum_{n=0}^{T} C(X_n,U_n)\right] \quad (1.11)$$

for the discrete, and

$$J_T(x_0,\pi) = \mathbb{E}_\pi\left[\int_0^T \beta^t C(X_t,U_t)dt + \beta^T R(X_T)\right], \text{ or } \quad J(x_0,\pi) = \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}_\pi\left[\int_0^T C(X_t,U_t)dt\right] \quad (1.12)$$

for the continuous time case respectively, where C is some cost function that depends on the state $x \in X$ and the control/action $u \in U$, and π is a policy/strategy (way of picking controls). On the LHS of either (1.11) or (1.12), when $\beta = 1, T < \infty$ we have the Total Cost optimality condition, when $\beta < 1, T = \infty$ we have the Discounted Cost optimality condition, and when $\beta = 1$, and T is a stopping time we have an Exit-Time optimality condition. On the RHS of either (1.11) or (1.12), we have the Average Cost optimality condition.

In risk-sensitive MCP problems one deals with optimality conditions of the form

$$\begin{aligned} J_T(x_0,\pi) &= \mathbb{E}_\pi\left[g_\lambda\left(\sum_{n=0}^{T}\beta^t C(X_n,U_n) + \beta^T R(X_T)\right)\right], \text{ or} \\ J(x_0,\pi) &= \limsup_{T\to\infty}\frac{1}{T}g_\lambda^{-1}\left(\mathbb{E}_\pi\left[g_\lambda\left(\sum_{n=0}^{T} C(X_n,U_n)\right)\right]\right) \end{aligned} \quad (1.13)$$

for the discrete, and

$$\begin{aligned} J_T(x_0,\pi) &= \mathbb{E}_\pi\left[g_\lambda\left(\int_0^T \beta^t C(X_t,U_t)dt + \beta^T R(X_T)\right)\right], \text{ or} \\ J(x_0,\pi) &= \limsup_{T\to\infty}\frac{1}{T}g_\lambda^{-1}\left(\mathbb{E}_\pi\left[g_\lambda\left(\int_0^T C(X_t,U_t)dt\right)\right]\right) \end{aligned} \quad (1.14)$$

for the continuous time case respectively, where g_λ is a convex function, that may be dependent on a parameter λ. The idea behind risk-sensitive cost structures, is that if $g_\lambda(y)$ has a Taylor expansion $g_\lambda(x) = \sum_n g_{\lambda,n}x^n$, then we have

$$J_T(x_0,\pi) = \sum_n g_{\lambda,n}\mathbb{E}_\pi\left[\left(\beta^n\sum_{n=0}^{T} C(X_n,U_n) + R(X_T)\right)^n\right],$$

therefore the optimality condition takes into account higher moments. By choosing g and λ, appropriately one can tune how much weight to put in variation or higher moment, therefore seeking or avoiding risk.

One of the most studied cases is when $g_\lambda(x) = e^{-\lambda x}$, (see [23, 22, 20, 18, 15, 11, 10, 9, 7, 6, 2] for the discrete and [12, 13, 16] for the continuous time-case). In our problem λ is integrated in the choice of cost C and R.

In the recent year, following the seminal work of [25], new risk-sensitive criteria were studied [26, 8, 3].

2 Notation and definitions

For a locally compact Polish space $\mathcal{S}$, the space of positive Borel measures on $\mathcal{S}$ is denoted by $\mathcal{M}(\mathcal{S})$. With the subscripts f, c we denote, respectively, the space of finite measures, and the space of measures that are finite on every compact subset. Letting $C_c(S)$ denote the space of continuous functions with compact support, we equip $\mathcal{M}_c(\mathcal{S})$, with the weakest topology such that for every $f \in C_c(S)$, the function $\nu \to \int_S f d\nu$, $\nu \in M_c(S)$, is continuous. $\mathcal{B}(\mathcal{S})$ is the Borel σ-algebra on $\mathcal{S}$ and $\mathcal{P}(\mathcal{S})$ the set of probability measures on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$. Finally, for a second Polish space $\mathcal{S}'$, we let

$$\mathcal{F}(\mathcal{S};\mathcal{S}') = \{f : \mathcal{S} \to \mathcal{S}' : f \text{ measurable}\} \tag{2.1}$$

denote the space of measurable functions from $\mathcal{S}$ to $\mathcal{S}'$.

For the finite set $\mathcal{X}$, let

$$\mathcal{P}_*(\mathcal{X}) = \left\{ \boldsymbol{m} \in \mathbb{R}^{\mathcal{X}} : m_x > 0 \text{ for all } x \in \mathcal{X} \text{ and } \sum_{x\in\mathcal{X}} m_x = 1 \right\}, \tag{2.2}$$

and

$$\mathcal{P}_a(\mathcal{X}) = \left\{ \boldsymbol{m} \in \mathbb{R}^{\mathcal{X}} : m_x \geq a \text{ for all } x \in \mathcal{X} \text{ and } \sum_{x\in\mathcal{X}} m_x = 1 \right\}. \tag{2.3}$$

For a set $K \subset \mathcal{P}(\mathcal{X})$, the closure $\bar{K}$, the complement K^c and the interior K°, will be considered with respect to the restriction of the Euclidean topology on the set $\mathcal{P}(\mathcal{X})$. $\mathcal{D}([0,\infty);\mathcal{S})$ denotes the space of cadlag functions on $\mathcal{S}$, equipped with the Skorohod toplogy (see [5, Section 16]), i.e., the Skorohod space. This space is separable and complete [5, Theorem 16.3], and a set is relatively compact in $\mathcal{D}([0,\infty);\mathcal{S})$, if and only if for every $M < \infty$, its natural projection on $\mathcal{D}([0,M];\mathcal{S})$, is relatively compact [5, Theorem 16.4].

For $\bar{\mathcal{M}} = \mathcal{M}_c([0,\infty)^2)$, let $\mathbb{P}$ be the probability measure on $(\bar{\mathcal{M}}, \mathcal{B}(\bar{\mathcal{M}}))$, under which the canonical map $N(\omega) = \omega$ is a Poisson measure with intensity measure equal to Lebesgue measure on $[0,\infty)^2$... Let

$$\mathcal{G}_t = \sigma\{N((0,s] \times A) : 0 \leq s \leq t, A \in \mathcal{B}([0,\infty))\},$$

and let $\mathcal{F}_t$ be the completion of $\mathcal{G}_t$ under $\mathbb{P}$. Let $\mathcal{P}$ be the corresponding predictable σ-field in $[0,\infty) \times \bar{\mathcal{M}}$. Similary, for natural numbers k, k' we similarly define a measure $\mathbb{P}^{k,k'}$ on $(\bar{\mathcal{M}}^{k'}, \mathcal{B}(\bar{\mathcal{M}}^{k'}))$ under which the maps $N_i(\boldsymbol{\omega}) = \omega_i$, $1 \leq i \leq k'$, are independent Poisson measures with intensity measure equal to k times the Lebesgue measure on $[0,\infty)^2$. $\{\mathcal{G}_t^{k,k'}\}, \{\mathcal{F}_t^{k,k'}\}$, and $\mathcal{P}^{k,k'}$ are defined analogously. Let $\mathcal{A}$ be the class of $\mathcal{P} \setminus \mathcal{B}([0,\infty))$ measurable maps $\phi : [0,\infty) \times \bar{\mathcal{M}} \to [0,\infty)$, and $\mathcal{A}_b$ the subset of these functions that are uniformly bounded from below away from zero and above by a positive constant. Similarly we define $\mathcal{A}^{k,k'}$ to be the set of $\mathcal{P}^{k,k'} \setminus \mathcal{B}([0,\infty)^{k'})$ measurable maps $\boldsymbol{\phi} : [0,\infty) \times \bar{\mathcal{M}}^{k'} \to [0,\infty)^{k'}$, and $\mathcal{A}_b^{k,k'}$ the subset of these functions that all entries are uniformly bounded from below and above by positive constants.

2.1 The many particle control problem

For a subset $\mathcal{K}$ of $\mathcal{X}^n$, we define a risk-sensitive cost $\mathcal{I}^n_{\mathcal{K}} : \mathcal{X}^n \times \mathcal{A}_b^{1,n|\mathcal{Z}|} \to [0,\infty]$ that corresponds to cost/reward up to the first time of hitting of $\mathcal{K}$ by

$$\mathcal{I}^n_{\mathcal{K}}(\boldsymbol{x}^n, \boldsymbol{u}) \doteq \mathbb{E}_{\mathbf{x}^n}\left[e^{\int_0^{T_{\mathcal{K}}} \left(\sum_{i=1}^n \sum_{y\in\mathcal{Z}_{\chi_i^n(t)}} \gamma_{\chi_i^n(t)y} C^n_{\chi_i^n(t)y}\left(\frac{u_{\chi_i^n(t)y}(t,i)}{\gamma_{\chi_i^n(t)y}} \right) - nR^n(L(\boldsymbol{\chi}^n(t))) \right) dt} \right], \tag{2.4}$$

where $\mathbb{E}_{\mathbf{x}^n}$ denotes expected value given $\boldsymbol{\chi}^n(0) = \boldsymbol{x}^n$, the dynamics are given in (1.8), and

$$T_{\mathcal{K}} \doteq \inf\{t \in [0,\infty] : \boldsymbol{\chi}^n(t) \in \mathcal{K}\}. \tag{2.5}$$

We define the value function $\mathcal{W}^n_{\mathcal{K}} : \mathcal{X}^n \to [0,\infty]$ by

$$\mathcal{W}^n_{\mathcal{K}}(\boldsymbol{x}^n) \doteq \inf_{\boldsymbol{u}\in\mathcal{A}_b^{1,n|\mathcal{Z}|}} \mathcal{I}^n_{\mathcal{K}}(\boldsymbol{x}^n, \boldsymbol{u}). \tag{2.6}$$

Similarly, for a set $\mathcal{K} \subset \mathcal{X}^n$ we define the ordinary cost $\mathcal{J}^n_{\mathcal{K}} : \mathcal{X}^n \times \mathcal{A}_b^{1,n|\mathcal{Z}|} \to [0,\infty]$ and corresponding value function $\mathcal{V}^n_{\mathcal{K}} : \mathcal{X}^n \to [0,\infty]$ by

$$\mathcal{J}^n_{\mathcal{K}}(\boldsymbol{x}^n, \boldsymbol{q}) \doteq \mathbb{E}_{\mathbf{x}^n}\left[\int_0^{T_{\mathcal{K}}} \left(\sum_{i=1}^n \sum_{y\in\mathcal{Z}_{\chi_i^n(t)}} F^n_{\chi_i^n(t)y}(q_{\chi_i^n(t)y}(t,i)) + nR^n(L(\boldsymbol{\chi}^n(t))) \right) dt \right], \tag{2.7}$$

where $\boldsymbol{F}^n$ is as in (1.9), and

$$\mathcal{V}^n_{\mathcal{K}}(\boldsymbol{x}^n) \doteq \inf_{\boldsymbol{q}\in\mathcal{A}_b^{1,n|\mathcal{Z}|}} \mathcal{J}^n_{\mathcal{K}}(\boldsymbol{x}^n, \boldsymbol{q}), \tag{2.8}$$

where the dynamics are given by (1.8) with $\boldsymbol{u}$ replaced by $\boldsymbol{q}$, and the stopping time by (2.5). We remark that the reason for two different notations for controls is to aid the reader, by associating one with the risk sensitive problem and one with the regular control problem. Moreover, there are occasions that both variables appear at the same time, as in the definition of $\boldsymbol{F}^n$ or that of the Hamiltonian. Specific conditions on the cost functions will be given in Section 3.1, and properties of $\boldsymbol{F}^n$ will be proved in Lemma 4. Note that for the many particle systems there are $n|\mathcal{Z}|$ PRMs, each with intensity 1.

2.2 The mean-field control problems

Suppose that $\mathcal{K}$ can be identified with a subset of the simplex of probability measures $\mathcal{P}(\mathcal{X})$, in the sense that for every permutation $\sigma \in \mathbb{S}_n$ we have $\sigma\mathcal{K} = \mathcal{K}$. Then we can replace a control problem on $\mathcal{X}^n$ by one on $\mathcal{P}(\mathcal{X})$. In this case $\mathcal{W}^n_{\mathcal{K}}$ and $\mathcal{V}^n_{\mathcal{K}}$ can be considered as functions on $\mathcal{P}^n(\mathcal{X})$, in the sense that we can find $W^n_K, V^n_K : \mathcal{P}^n(\mathcal{X}) \to [0,\infty]$, such that $\mathcal{W}^n_{\mathcal{K}}(\boldsymbol{x}^n) = W^n_K(L(\boldsymbol{x}^n))$ and $\mathcal{V}^n_{\mathcal{K}}(\boldsymbol{x}^n) = V^n_K(L(\boldsymbol{x}^n))$. To see this, pick a starting point $\boldsymbol{x}^n \in \mathcal{X}^n$ and some permutation σ. Then for any admissible control $\boldsymbol{u}$, the total cost generated starting at $\boldsymbol{x}^n$ is the same as starting from $\boldsymbol{x}^n_\sigma$ and picking $\boldsymbol{u}_\sigma$ as control. Therefore, for every $\boldsymbol{x}^n \in \mathcal{X}^n, \sigma \in \mathbb{S}_n$, we have $\mathcal{V}_{\mathcal{K}}(\boldsymbol{x}^n) = \mathcal{V}_{\mathcal{K}}(\boldsymbol{x}^n_\sigma)$.

Define $h^n : \mathcal{D}([0,\infty);\mathcal{P}^n(\mathcal{X})) \times \mathcal{A}_b^{n,|\mathcal{Z}|} \times \mathcal{P}^n(\mathcal{X}) \times \bar{\mathcal{M}}^{n,|\mathcal{Z}|} \to \mathcal{D}([0,\infty);\mathbb{R}^d)$ by

$$h^n\left(\boldsymbol{\mu}, \boldsymbol{u}, \boldsymbol{m}, \frac{1}{n}\boldsymbol{N}^n\right)(t) \doteq \boldsymbol{m} + \sum_{(x,y)\in\mathcal{Z}} \boldsymbol{v}_{xy} \int_{(0,t]}\int_{[0,\infty)} 1_{[0,\mu_x(-s)u_{xy}(s)]}(r)\frac{1}{n}N^n_{xy}(dsdr).$$

Since $\boldsymbol{u} \in \mathcal{A}_b^{n,|\mathcal{Z}|}$ implies the rates $u_{xy}(s)$ are uniformly bounded, one can explicitly construct a unique $\mathcal{D}([0,\infty);\mathcal{P}^n(\mathcal{X}))$-valued process that satisfies

$$\boldsymbol{\mu} = h^n\left(\boldsymbol{\mu}, \boldsymbol{u}, \boldsymbol{m}, \frac{1}{n}\boldsymbol{N}^n\right). \tag{2.9}$$

[14]. Here $\boldsymbol{\mu}$ is the controlled process, $\boldsymbol{u}$ is the control, $\boldsymbol{m}$ is an initial condition, and $\boldsymbol{N}^n/n$ is scaled noise.

Now with $T_K \doteq \inf\{t \in [0,\infty] : \boldsymbol{\mu}(t) \in K\}$, the functions $I^n_K, J^n_V : \mathcal{P}^n(\mathcal{X}) \times \mathcal{A}_b^{n,|\mathcal{Z}|} \to [0,\infty]$ and $W^n_K, V^n_K : \mathcal{P}^n(\mathcal{X}) \to [0,\infty]$ are given by

$$W^n_K(\boldsymbol{m}) \doteq \inf_{\boldsymbol{u}\in\mathcal{A}_b^{n,|\mathcal{Z}|}} I^n_K(\boldsymbol{m}, \boldsymbol{u}) \tag{2.10}$$

$$I^n_K(\boldsymbol{m}, \boldsymbol{u}) \doteq \mathbb{E}_{\boldsymbol{m}}\left[e^{n\int_0^{T_K}\left(\sum_{(x,y)\in\mathcal{Z}} \mu_x(t)\gamma_{xy}C^n_{xy}\left(\frac{u_{xy}(t)}{\gamma_{xy}}\right) - R^n(\boldsymbol{\mu}(t))\right)dt} : \boldsymbol{\mu} = h^n\left(\boldsymbol{\mu}, \boldsymbol{u}, \boldsymbol{m}, \frac{1}{n}\boldsymbol{N}^n\right)\right], \tag{2.11}$$

and

$$V^n_K(\boldsymbol{m}) \doteq \inf_{\boldsymbol{q}\in\mathcal{A}_b^{n,|\mathcal{Z}|}} J^n_K(\boldsymbol{m}, \boldsymbol{q}) \tag{2.12}$$

$$J^n_K(\boldsymbol{m}, \boldsymbol{q}) \doteq \mathbb{E}_{\boldsymbol{m}}\left[\int_0^{T_K}\left(\sum_{(x,y)\in\mathcal{Z}} \mu_x(t)F^n_{xy}(q_{xy}(t)) + R^n(\boldsymbol{\mu}(t))\right)dt : \boldsymbol{\mu} = h^n\left(\boldsymbol{\mu}, \boldsymbol{u}, \boldsymbol{m}, \frac{1}{n}\boldsymbol{N}^n\right)\right]. \tag{2.13}$$

For these control problems, there are $|\mathcal{Z}|$ PRMs, each with intensity n.

3 The equivalence of the control problems

In this section we prove that after a natural renormalization, the value function $\mathcal{W}^n_{\mathcal{K}}$ defined in (2.6) is linked to $\mathcal{V}^n_{\mathcal{K}}$ defined in (2.8) which, as noted before, it is the value function of an ordinary stochastic control problem with a different cost function. Specifically, we show that $-\log(\mathcal{W}^n_K)/n$ equals $\mathcal{V}^n_K$, and that the many particle and the mean field control problem are equivalent:

$$-\frac{1}{n}\log(W^n_K(L(\boldsymbol{x}^n))) = V^n_K(L(\boldsymbol{x}^n)) = \mathcal{V}^n_{\mathcal{K}}(\boldsymbol{x}^n) = -\frac{1}{n}\log(\mathcal{W}^n_{\mathcal{K}}(\boldsymbol{x}^n)). \tag{3.1}$$

3.1 The cost functions

To motivate the main Assumption for the costs C^n, we will first discuss briefly the strategy we are going to use for the proof of (3.1). The proof will use a related Bellman equation. Let $H^n : \mathcal{P}(\mathcal{X}) \times \mathbb{R}^{|\mathcal{Z}|} \to \mathbb{R}$ be given by

$$H^n(\boldsymbol{m}, \boldsymbol{\xi}) \doteq \inf_{\boldsymbol{q}\in(0,\infty)^{|\mathcal{Z}|}} \left\{ \sum_{(x,y)\in\mathcal{Z}} m_x \left(q_{xy}\xi_{xy} + F^n_{xy}(q_{xy}) \right) \right\}, \tag{3.2}$$

where

$$F^n_{xy}(q) \doteq \sup_{u\in(0,\infty)} G^n_{xy}(u,q) \quad \text{and} \quad G^n_{xy}(u,q) \doteq \left[u\ell\left(\frac{q}{u}\right) - \gamma_{xy} C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right) \right]. \tag{3.3}$$

We will show that the equation

$$H^n\left(\boldsymbol{m}, \Delta^n V(\boldsymbol{m})\right) + R^n(\boldsymbol{m}) = 0 \text{ in } \mathcal{P}^n(\mathcal{X}) \setminus K, \tag{3.4}$$

and boundary condition $V(\boldsymbol{m}) = 0$ for $\boldsymbol{m} \in K$ has V^n_K as the unique solution, where by $\Delta^n V(\boldsymbol{m})$ we denote the $|\mathcal{Z}|-$dimensional vector $n\left(V(\boldsymbol{m} + \frac{\boldsymbol{v}_{xy}}{n}) - V(\boldsymbol{m})\right)$, and by $\Delta^n_{xy} V(\boldsymbol{m})$ the component $n\left(V(\boldsymbol{m} + \frac{\boldsymbol{v}_{xy}}{n}) - V(\boldsymbol{m})\right)_{xy}$, $(x,y) \in \mathcal{Z}$.

We are also going to prove that W^n_K is the unique solution to

$$\sup_{\boldsymbol{u}\in(0,\infty)^{|\mathcal{Z}|}} \left\{ \sum_{(x,y)\in\mathcal{Z}} m_x \left(u_{xy} \left(\frac{W(\boldsymbol{m}) - W\left(\boldsymbol{m} + \frac{\boldsymbol{v}_{xy}}{n}\right)}{W(\boldsymbol{m})} \right) - \gamma_{xy} C^n_{xy}\left(\frac{u_{xy}}{\gamma_{xy}}\right) \right) \right\} = -R^n(\boldsymbol{m}), \tag{3.5}$$

with $W(\boldsymbol{m}) = 1$, for $\boldsymbol{m} \in K$. In the sequel, we are going to use the following Lemma:

Lemma 1. *If $\tilde{V} : \mathcal{P}^n(\mathcal{X}) \to [0,\infty)$ is a solution to (3.4) and $\tilde{V}(\boldsymbol{m}) = 0$ for $\boldsymbol{m} \in K$, then $\tilde{W} = e^{-n\tilde{V}} : \mathcal{P}^n(\mathcal{X}) \to (0,\infty)$ is a solution of* (3.5) *and $W(\boldsymbol{m}) = 1$ for $\boldsymbol{m} \in K$.*

Lemma 1 guarantees that for every solution $\tilde{V}$ of (3.4) $e^{-n\tilde{V}}$ is a solution (3.5). Since V^n_K is a solution to (3.4), the lemma implies that $-\frac{1}{n}\log(W^n_K) = V^n_K$. However for Lemma 1 to hold true, we need the following equality to hold true:

$$\begin{aligned} H^n(\mathbf{m}, \boldsymbol{\xi}) &\doteq \inf_{\boldsymbol{q}\in[0,\infty)^{|\mathcal{Z}|}} \left\{ \sum_{(x,y)\in\mathcal{Z}} m_x \left(q_{xy}\xi_{xy} + F^n_{xy}(q_{xy}) \right) \right\} \\ &= \inf_{\boldsymbol{q}\in[0,\infty)^{|\mathcal{Z}|}} \sup_{\boldsymbol{u}\in(0,\infty)^{|\mathcal{Z}|}} \left\{ \sum_{(x,y)\in\mathcal{Z}} m_x \left(q_{xy}\xi_{xy} + G^n_{xy}(u_{xy}, q_{xy}) \right) \right\} \\ &= \sup_{\boldsymbol{u}\in(0,\infty)^{|\mathcal{Z}|}} \inf_{\boldsymbol{q}\in[0,\infty)^{|\mathcal{Z}|}} \left\{ \sum_{(x,y)\in\mathcal{Z}} m_x \left(q_{xy}\xi_{xy} + G^n_{xy}(u_{xy}, q_{xy}) \right) \right\} \\ &= \sup_{\boldsymbol{u}\in(0,\infty)^{|\mathcal{Z}|}} \left\{ \sum_{(x,y)\in\mathcal{Z}} m_x \left(u_{xy}\left(1 - e^{-\xi_{xy}}\right) - \gamma_{xy} C^n_{xy}\left(\frac{u_{xy}}{\gamma_{xy}}\right) \right) \right\} \\ &= \sum_{(x,y)\in\mathcal{Z}} m_x \gamma_{xy} (C^n_{xy})^* \left(1 - e^{-\xi_{xy}}\right), \end{aligned} \tag{3.6}$$

where $(C^n_{xy})^* : (-\infty, 1) \to \mathbb{R}$ is given by $(C^n_{xy})^*(z) = \sup_{u>0}\left[zu - C^n_{xy}(u)\right]$. However for the equality to hold, we need that the Isaac condition is satisfied, i.e., the supremum and infimum are exchangeable. For the proof of the exchange between supremum and infimum, we will apply Sion's Theorem (Corollary 3.3 in [27]), which states that if a continuous $F(u,q)$ is quasi-concave for every u is some convex set U and quasi-convex for every q in some convex set Q, and if one of the two sets is compact, then we can exchange the supremum with the infimum. We would like to apply Sion's Theorem on

$$L^n_{xy}(u,q) = q\xi + u\ell\left(\frac{q}{u}\right) - \gamma_{xy} C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right).$$

Since ℓ is convex, L^n_{xy} is convex with respect to q. It is easy to see that $L^n_{xy}(u,q)$ is not concave with respect to u, however it is possible for L^n_{xy} to be quasi-concave, with respect to u, for every $q \geq 0$. L^n_{xy} to be quasi-concave, with respect to u, for every q, means that it changes monotonicity at most one time. By differentiating with respect to u we get

$$\partial_u L^n_{xy}(u,q) = -\frac{q}{u} + 1 - (C^n_{xy})'\left(\frac{u}{\gamma_{xy}}\right),$$

so what we need for every $q \geq 0$, is the existence of a u_q, such that,

$$\forall u \leq u_q : -\frac{q}{u} + 1 - (C^n_{xy})'\left(\frac{u}{\gamma_{xy}}\right) \geq 0 \quad \text{and} \quad \forall u \geq u_q : -\frac{q}{u} + 1 - (C^n_{xy})'\left(\frac{u}{\gamma_{xy}}\right) \leq 0,$$

or after multiplying with u, and making a renormalization, for every $q \geq 0$, it exists u_q, such that,

$$\forall u \leq u_q : u(C^n_{xy})'(u) - u \leq -q \quad \text{and} \quad \forall u \geq u_q : u(C^n_{xy})'(u) - u \geq -q.$$

If someone considers $-q$, as all the negative level sets, this will translate to the fact that

$$u(C^n_{xy})'(u) - u \text{ is increasing until } \quad (C^n_{xy})'(u) \geq 1, \tag{3.7}$$

and then, $(C^n_{xy})'(u)$ remains bigger than one. By taking another derivative, we have

$$(C^n_{xy})'(u) + u(C^n_{xy})''(u) - 1 \geq 0 \text{ until } \quad (C^n_{xy})'(u) \geq 1,, \tag{3.8}$$

from which we further conclude, that while $(C^n_{xy})'(u) < 1$, the function C^n_{xy} is also convex. In fact, as it is apparent from the last line in (3.6), the values of C^n_{xy}, after $(C^n_{xy})'$ gets bigger than one, are irrelevant, and therefore we can assume that (C^n_{xy}) is convex on the whole $[0,\infty)$. In Remark 25, it is heuristically argued that (3.7), is actually the weakest condition such that in the definition of the Hamiltonian (3.6) one can exchange the supremum with the infimum, and therefore the application of Sion's theorem is optimal (there is no better result that we could have used).

Now we provide the main assumption for C^n_{xy}.

Assumption 2. *For each $n \in \mathbb{N}$, $R^n : \mathcal{P}(\mathcal{X}) \to [0,\infty)$ is a continuous function. Moreover, for every $(x,y) \in \mathcal{Z}$, $C^n_{xy} : [0,\infty) \to [0,\infty]$ is a convex function that satisfies the following:*

1 *there exist $0 \leq u^n_{1,xy} < 1 < u^n_{2,xy} \leq \infty$ such that C^n_{xy} is finite on $(u^n_{1,xy}, u^n_{2,xy})$, continuous as an extended function on $[u^n_{1,xy}, u^n_{2,xy}]$, and $C^n_{xy}(u) = \infty$ outside $[u^n_{1,xy}, u^n_{2,xy}]$. In addition, it exists a $\tilde{u} \in [0,\infty]$, such that*

$$\forall u \leq \tilde{u} : u(C^n_{xy})'(u) - u \text{ is increasing} \quad \text{and} \quad \forall u \geq \tilde{u} : (C^n_{xy})'(u) \geq 1. \tag{3.9}$$

2 $C^n_{xy}(1) = 0.$

Assumption 2.2 is not necessary, but it simplifies the analysis and it is appropriate for the situation being modeled to have zero cost when there is no change from the nominal rates. Now under the Assumption 2, in Lemma 24, we will prove that the Isaac condition is actually satisfied, and therefore the equality (3.6) is true.

Lemma 3. *Under Assumption 2, the cost functions C^n_{xy} satisfy the following on $(u^n_{1,xy}, u^n_{2,xy})$:*

1 *for every $(x,y) \in \mathcal{Z}$ we have $(C^n_{xy})'(u) \geq 1 - \frac{1}{u}$ for $u > 1$, and therefore*

$$\liminf_{u\to\infty} (C^n_{xy})'(u) \geq 1,$$

2 *for every $(x,y) \in \mathcal{Z}$ and $u \in (0,\infty)$ we have $C^n_{xy}(u) \geq -\log u + u - 1$.*

Proof. It follows from the monotonicity that $u(C^n_{xy})'(u) - u \geq -1$ for $u > 1$, which gives the first statement. The second follows by comparing $(C^n_{xy})'(u)$ with the integral $\int_1^u \left[1 - \frac{1}{s}\right] ds$ and using $C^n_{xy}(1) = 0$. □

Example 1. *The family of functions $C^n_{xy}(u) = \frac{1}{pu^p} + \frac{u^q}{q} - \frac{p+q}{pq}$, where $p \geq 1$ and $q \geq 1$, satisfy Assumption 2. Taking the derivative of $C^n_{xy}(u)$ gives $-\frac{1}{u^{p+1}} + u^{q-1}$, and multiplying with u and subtracting u yields $-\frac{1}{u^p} + u^q - u$. Taking the derivative again gives $\frac{p}{u^{p+1}} + qu^{q-1} - 1$, which is always bigger than zero, since $\frac{p}{u^{p+1}}$ and qu^{q-1} are everywhere positive and bigger than one on the intervals $[0,1]$ and $[1,\infty)$, respectively.*

Before proceeding with the proof, we state some properties of F^n_{xy}.

Lemma 4. *For every $n \in \mathbb{N}$ and for every $(x,y) \in \mathcal{Z}$, let F^n_{xy} be as in (1.9), where $\{C^n_{xy}\}$ satisfy Assumption 2. Then the following hold.*

1 $F^n_{xy}(q) \geq \gamma_{xy}\ell\left(\frac{q}{\gamma_{xy}}\right) \geq 0.$

2 $F^n_{xy}(\gamma_{xy}) = 0.$

3 F^n_{xy} *is convex on* $[0,\infty)$.

3.2 Equivalence of the stochastic problems

Theorem 5. *Let $n \in \mathbb{N}$, $\mathcal{K} \subset \mathcal{X}^n$, (resp. $K \subset \mathcal{P}^n(\mathcal{X})$), and $\boldsymbol{C}^n, R^n$ be as in Assumption 2. Further assume that R^n is bounded below in K^c by a positive constant $R^n_{\min}$. Then*

$$V^n_K(\boldsymbol{m}) = -\frac{1}{n}\log(W^n_K(\boldsymbol{m})) \tag{3.10}$$

and

$$\mathcal{V}^n_{\mathcal{K}}(\boldsymbol{x}^n) = -\frac{1}{n}\log(\mathcal{W}^n_{\mathcal{K}}(\boldsymbol{x}^n)). \tag{3.11}$$

If in addition $\mathcal{K} \subset \mathcal{X}^n$ is invariant under permutations, and therefore can be identified with a subset of $\mathcal{P}^n(\mathcal{X})$, then

$$-\frac{1}{n}\log(W^n_K(L(\boldsymbol{x}^n))) = V^n_K(L(\boldsymbol{x}^n)) = \mathcal{V}^n_{\mathcal{K}}(\boldsymbol{x}^n) = -\frac{1}{n}\log(\mathcal{W}^n_{\mathcal{K}}(\boldsymbol{x}^n)). \tag{3.12}$$

The proof of this result appears later in this section. Also, we will only prove the first equality and note that the third follows in a similar manner.

Lemma 6. *For a non-empty set $K \subset \mathcal{P}^n(\mathcal{X})$, the equation (3.4) has at least one solution.*

Proof. For the proof we use the equivalent discrete time stochastic control problem. Thus, with some abuse of notation, we consider a feedback control $\boldsymbol{q} : \mathcal{Z} \times \mathcal{P}^n(\mathcal{X}) \to (0,\infty)$. For such a control the probability of moving from state $\boldsymbol{m}$ to state $\boldsymbol{m} + \frac{1}{n}\boldsymbol{v}_{\tilde{x},\tilde{y}}$ is given by

$$\frac{m_{\tilde{x}} q_{\tilde{x}\tilde{y}}(\boldsymbol{m})}{\sum_{(x,y)\in\mathcal{Z}} m_x q_{xy}(\boldsymbol{m})},$$

and the (conditional) expected cost till the time of transition is given by

$$\frac{\sum_{(x,y)\in\mathcal{Z}} m_x F^n_{xy}(q_{xy}(\boldsymbol{m})) + R^n(\boldsymbol{m})}{n\sum_{(x,y)\in\mathcal{Z}} m_x q_{xy}(\boldsymbol{m})}.$$

Given controlled transition probabilities as above, let $\boldsymbol{\mu}(i)$ be the corresponding controlled process. We define the value function $\bar{V}^n_K(\boldsymbol{m}) : \mathcal{P}(\mathbb{R}^d) \to [0,\infty)$ by

$$\bar{V}^n_K(\boldsymbol{m}) \doteq \inf_{\boldsymbol{q}:\mathcal{Z}\times\mathcal{P}^n(\mathcal{X})\to(0,\infty)} \mathbb{E}_{\boldsymbol{m}}\left[\sum_{i=1}^{T_K} \frac{\sum_{(x,y)\in\mathcal{Z}} \mu_x(i) F^n_{xy}(q_{xy}(\boldsymbol{\mu}(i))) + R^n(\boldsymbol{\mu}(i))}{n\sum_{(x,y)\in\mathcal{Z}} \mu_x(i) q_{xy}(\boldsymbol{\mu}(i))}\right],$$

where $\mathbb{E}_{\boldsymbol{m}}$ denotes expected value given $\boldsymbol{\mu}(0) = \boldsymbol{m}$ and $T_K \doteq \inf\{i \in \mathbb{N} : \boldsymbol{\mu}(i) \in K\}$.

To see that $\bar{V}^n_K(\boldsymbol{m})$ is finite, we just have to use the original rates and note that the total cost is proportional to the expected exit time, which is finite by classical results in Markov chains.

Then by [4, Proposition 1.1 in Chapter 3], we have that this value function satisfies

$$\bar{V}^n_K(\boldsymbol{m}) = \inf_{\boldsymbol{q}\in(0,\infty)^{|\mathcal{Z}|}} \left\{ \frac{\sum_{(x,y)\in\mathcal{Z}} m_x F^n_{xy}(q_{xy}) + R^n(\boldsymbol{m})}{n\sum_{(x,y)\in\mathcal{Z}} m_x q_{xy}} + \sum_{(\tilde{x},\tilde{y})\in\mathcal{Z}} \frac{m_{\tilde{x}} q_{\tilde{x}\tilde{y}}}{\sum_{(x,y)\in\mathcal{Z}} m_x q_{xy}} \bar{V}^n_K\left(\boldsymbol{m} + \frac{1}{n}\boldsymbol{v}_{\tilde{x}\tilde{y}}\right)\right\}.$$

Since $F^n_{xy} \geq 0$ (see Lemma 4) and $R^n_{\min} > 0$, the infimum in the last display can be restricted to $\boldsymbol{q}$ that are bounded away from $\mathbf{0}$. It then follows that $\bar{V}^n_K(\boldsymbol{m})$ satisfies the last display if and only if [with $\Delta^n_{xy}\bar{V}^n_K(\boldsymbol{m}) \doteq n\left(\bar{V}^n_K(\boldsymbol{m}+\frac{\boldsymbol{v}_{xy}}{n}) - \bar{V}^n_K(\boldsymbol{m})\right)$]

$$\inf_{\boldsymbol{q}\in(0,\infty)^{|\mathcal{Z}|}} \left\{ \sum_{(x,y)\in\mathcal{Z}} m_x \left(q_{xy}\Delta^n_{xy}\bar{V}^n_K(\boldsymbol{m}) + F^n_{xy}(q_{xy}) \right) \right\} + R^n(\boldsymbol{m}) = 0.$$

Then using the definition (3.2) this is the same as

$$H^n\left(\boldsymbol{m}, \Delta^n\bar{V}^n_K(\boldsymbol{m})\right) + R^n(\boldsymbol{m}) = 0,$$

and we also have the boundary condition $\bar{V}^n_K(\boldsymbol{m}) = 0$ for all $\boldsymbol{m} \in K$. □

Proof of Lemma 1. Let $\tilde{V}$ be a solution to (3.4). We then have $H^n(\boldsymbol{m}, \Delta^n\tilde{V}(\boldsymbol{m})) + R^n(\boldsymbol{m}) = 0$, or by using the second from the bottom line in (3.6),

$$\sup_{\boldsymbol{u}\in(0,\infty)^{|\mathcal{Z}|}} \left\{ \sum_{(x,y)\in\mathcal{Z}} m_x \left(u_{xy}\left(1 - e^{-n\left(\tilde{V}(\boldsymbol{m}+\frac{\boldsymbol{v}_{xy}}{n}) - \tilde{V}(\boldsymbol{m})\right)}\right) - \gamma_{xy}C^n_{xy}\left(\frac{u_{xy}}{\gamma_{xy}}\right) \right) \right\} + R^n(\boldsymbol{m}) = 0.$$

By making the substitution we have

$$\sup_{\boldsymbol{u}\in(0,\infty)^{|\mathcal{Z}|}} \left\{ \sum_{(x,y)\in\mathcal{Z}} m_x \left(u_{xy}\left(1 - \frac{\tilde{W}(\boldsymbol{m}+\frac{\boldsymbol{v}_{xy}}{n})}{\tilde{W}(\boldsymbol{m})}\right) - \gamma_{xy}C^n_{xy}\left(\frac{u_{xy}}{\gamma_{xy}}\right) \right) \right\} + R^n(\boldsymbol{m}) = 0,$$

which is the same as (3.5). □

Lemma 7. *Let $f : \mathcal{P}^n(\mathcal{X}) \to \mathbb{R}$, $\boldsymbol{m} \in \mathcal{P}^n(\mathcal{X})$, and $\boldsymbol{q} \in \mathcal{A}^{n,|\mathcal{Z}|}_b$ be given, and let $\boldsymbol{\mu}$ solve (2.9). Then*

$$f(\boldsymbol{\mu}(t)) - f(\boldsymbol{m}) - \int_0^t \sum_{(x,y)\in\mathcal{Z}} \mu_x(s)q_{xy}(s)\Delta^n_{xy}f(\boldsymbol{\mu}(s))ds,$$
$$f(\boldsymbol{\mu}(t\wedge T_K)) - f(\boldsymbol{m}) - \int_0^{t\wedge T_K} \sum_{(x,y)\in\mathcal{Z}} \mu_x(s)q_{xy}(s)\Delta^n_{xy}f(\boldsymbol{\mu}(s))ds,$$
$$f(\boldsymbol{\mu}(t\wedge T_K)) - f(\boldsymbol{\mu}(t'\wedge T_K)) - \int_{t'\wedge T_K}^{t\wedge T_K} \sum_{(x,y)\in\mathcal{Z}} \mu_x(s)q_{xy}(s)\Delta^n_{xy}f(\boldsymbol{\mu}(s))ds,$$

are martingales with respect to the filtration $\mathcal{F}_t$.

Proof. By the construction of $\boldsymbol{\mu}$ (also see Ito's formula [21, Chapter 2, Theorem 5.1]), we have

$$f(\boldsymbol{\mu}(t)) - f(\boldsymbol{m}) - \int_{(0,t]} \sum_{(x,y)\in\mathcal{Z}} \int_{[0,\infty)} 1_{[0,\mu_x(s-)q_{xy}(s)]}(r)\Delta^n_{xy}f(\boldsymbol{\mu}(s-))N^n_{xy}(dsdr) = 0.$$

Indeed, the right hand side simply records each jump in $f(\boldsymbol{\mu}(s))$ for $0 < s \leq t$. Also by [21, Chapter 2, Theorem 3.1], for each $(x,y) \in \mathcal{Z}$

$$\int_0^t \mu_x(s-)q_{xy}(s)\Delta^n_{xy}f(\boldsymbol{\mu}(s-))ds - \int_{(0,t]}\int_{[0,\infty)} 1_{[0,\mu_x(s-)q_{xy}(s)]}(r)\Delta^n_{xy}f(\boldsymbol{\mu}(s-))N^n_{xy}(dsdr),$$

is a martingale. By combining the last two displays and using that $s-$ in the ordinary integral can be replaced by s due to left continuity,

$$f(\boldsymbol{\mu}(t)) - f(\boldsymbol{m}) - \int_0^t \sum_{(x,y)\in\mathcal{Z}} \mu_x(s)q_{xy}(s)\Delta^n_{xy}f(\boldsymbol{\mu}(s))ds$$

is a martingale. The second and third formulas then follow from standard properties of martingales. □

Lemma 8. *Let $g : \mathcal{P}^n(\mathcal{X}) \to (0,\infty)$, $\boldsymbol{m} \in \mathcal{P}^n(\mathcal{X})$, and $\boldsymbol{u} \in \mathcal{A}_b^{n,|\mathcal{Z}|}$ be given, and let $\boldsymbol{\mu}$ solve (2.9). Then*

$$\frac{g(\boldsymbol{\mu}(t))}{g(\mathbf{m})} \exp\left\{-\int_0^t \sum_{(x,y)\in\mathcal{Z}} \mu_x(s) u_{xy}(s) \frac{\Delta^n_{xy} g(\boldsymbol{\mu}(s))}{g(\boldsymbol{\mu}(s))} ds\right\},$$
$$\frac{g(\boldsymbol{\mu}(t\wedge T_K))}{g(\boldsymbol{m})} \exp\left\{-\int_0^{t\wedge T_K} \sum_{(x,y)\in\mathcal{Z}} \mu_x(s) u_{xy}(s) \frac{\Delta^n_{xy} g(\boldsymbol{\mu}(s))}{g(\boldsymbol{\mu}(s))} ds\right\}, \tag{3.13}$$
$$\frac{g(\boldsymbol{\mu}(t\wedge T_K))}{g(\boldsymbol{\mu}(t'\wedge T_K))} \exp\left\{-\int_{t'\wedge T_K}^{t\wedge T_K} \sum_{(x,y)\in\mathcal{Z}} \mu_x(s) u_{xy}(s) \frac{\Delta^n_{xy} g(\boldsymbol{\mu}(s))}{g(\boldsymbol{\mu}(s))} ds\right\},$$

are martingales with respect to the filtration $\mathcal{F}_t$.

Proof. The proof is a direct application of the corollary in [24, Page 66]. □

Lemma 9. *Let $\boldsymbol{m} \in \mathcal{P}^n(\mathcal{X})$ and $\boldsymbol{u} \in \mathcal{A}_b^{n,|\mathcal{Z}|}$. There exists a constant $c > 0$, that depends only on the bounds on $\boldsymbol{u}$, the dimension d, the constant $R^n_{\max} = \max\{R^n(\boldsymbol{m}) : \boldsymbol{m} \in \mathcal{P}^n(\mathcal{X})\}$, and the number n of agents, such that for every $t \geq t' \geq 0$,*

$$\mathbb{E}_{\boldsymbol{m}}\left[e^{-nR^n_{\max}(t\wedge T_K - t'\wedge T_K)}\middle|\mathcal{F}_{t'}\right] > c.$$

Furthermore it is true that

$$T_K < \infty \text{ a.s.,} \quad \text{and} \quad \mathbb{E}_{\boldsymbol{m}}\left[e^{-nR^n_{\max}(T_K - t'\wedge T_K)}\middle|\mathcal{F}_{t'}\right] > c.$$

Proof. We claim there exists g such that for all s

$$\sum_{(x,y)\in\mathcal{Z}} \mu_x(s) u_{xy}(s) \frac{\Delta^n_{xy} g(\boldsymbol{\mu}(s))}{g(\boldsymbol{\mu}(s))} \geq nR^n_{\max}. \tag{3.14}$$

To show the existence of such a g we use the following procedure. Since the one particle process with generator given in (1.1) is ergodic, we have that the process on $\mathcal{X}^n$, with generator given in (1.2), as well as the one on $\mathcal{P}^n(\mathcal{X})$, with generator given in (1.4), are also ergodic. We split $\mathcal{P}^n(\mathcal{X})$ into sets $\{K_i\}_{0\leq i\leq i_{\max}}$, where $K_0 = K$, and K_{i+1} is generated inductively as the set of all points in $\mathcal{P}^n(\mathcal{X})$ that do not belong to K_i but such that the process with generator (1.4) can reach K_i in one jump. Since the original process has d states, it is easy to see that $i_{\max} \leq d^n$. Since $\boldsymbol{u} \in \mathcal{A}_b^{n,|\mathcal{Z}|}$ there exist constants $0 < c_1 \leq c_2 < \infty$ such that $c_1 \leq u_{xy}(t) \leq c_2$ for all $t \geq 0$ a.s. Let g be defined by

$$g(\boldsymbol{m}) \doteq \left(\frac{nR^n_{\max} + nd^2c_2 + c_1}{c_1}\right)^{i_{\max}-i}, \quad \text{for} \quad \boldsymbol{m} \in K_i.$$

Let $\boldsymbol{\mu}(\cdot)$ be the process with control $\boldsymbol{u}$... For $0 \leq s \leq t$ suppose that $\boldsymbol{\mu}(s) \in K_i$ for some $i \geq 1$. Then there exists at least one $(\tilde{x},\tilde{y}) \in \mathcal{Z}$ such that $\boldsymbol{\mu}(s) + \frac{v_{\tilde{x}\tilde{y}}}{n} \in K_{i-1}$. Therefore

$$\sum_{(x,y)\in\mathcal{Z}} \mu_x(s) u_{xy}(s) \frac{\Delta^n_{xy} g(\boldsymbol{\mu}(s))}{g(\boldsymbol{\mu}(s))} = \mu_{\tilde{x}}(s) u_{\tilde{x}\tilde{y}}(s) \frac{\Delta^n_{\tilde{x}\tilde{y}} g(\boldsymbol{\mu}(s))}{g(\boldsymbol{\mu}(s))} + n \sum_{(x,y)\in\mathcal{Z},(x,y)\neq(\tilde{x},\tilde{y})} \frac{g(\boldsymbol{\mu}(s) + \frac{\boldsymbol{v}_{xy}}{n})}{g(\boldsymbol{\mu}(s))} \mu_x(s) u_{xy}(s)$$
$$- n \sum_{(x,y)\in\mathcal{Z},(x,y)\neq(\tilde{x},\tilde{y})} \frac{g(\boldsymbol{\mu}(s))}{g(\boldsymbol{\mu}(s))} \mu_x(s) u_{xy}(s) \geq \mu_{\tilde{x}}(s) u_{\tilde{x}\tilde{y}}(s) \frac{\Delta^n_{\tilde{x}\tilde{y}} g(\boldsymbol{\mu}(s))}{g(\boldsymbol{\mu}(s))} - n \sum_{(x,y)\in\mathcal{Z}} \mu_x(s) u_{xy}(s)$$
$$\geq c_1\left(\frac{nR^n_{\max} + nd^2c_2 + c_1}{c_1} - 1\right) - nc_2d^2 \geq nR^n_{\max},$$

where in the next to last inequality we used the fact that $\mu_{\tilde{x}}(s) \geq \frac{1}{n}$ (because otherwise there is no particle at $\tilde{x}$ to move), and that $\Delta^n_{xy} V(\boldsymbol{m}) = n\left(V(\boldsymbol{m} + \frac{\boldsymbol{v}_{xy}}{n}) - V(\boldsymbol{m})\right)$.

Using the last martingale in Lemma 8, we have

$$\mathbb{E}_{\boldsymbol{m}}\left[\frac{g(\boldsymbol{\mu}(t\wedge T_K))}{g(\boldsymbol{\mu}(t'\wedge T_K))} \exp\left\{-\int_{t'\wedge T_K}^{t\wedge T_K} \sum_{(x,y)\in\mathcal{Z}} \mu_x(s) u_{xy}(s) \frac{\Delta^n_{xy} g(\boldsymbol{\mu}(s))}{g(\boldsymbol{\mu}(s))} ds\right\}\middle|\mathcal{F}_{t'}\right] = 1,$$

from which we get

$$\mathbb{E}_{\boldsymbol{m}}\left[\exp\left\{-\int_{t'\wedge T_K}^{t\wedge T_K}\sum_{(x,y)\in\mathcal{Z}}\mu_x(s)u_{xy}(s)\frac{\Delta_{xy}^n g(\boldsymbol{\mu}(s))}{g(\boldsymbol{\mu}(s))}ds\right\}\Big|\mathcal{F}_{t'}\right]\geq c\doteq\frac{\min_{\mathcal{P}^n(\mathcal{X})}g}{\max_{\mathcal{P}^n(\mathcal{X})}g}.$$

By applying equation (3.14)

$$\mathbb{E}_{\boldsymbol{m}}\left[e^{-nR^n_{\max}(t\wedge T_K-t'\wedge T_K)}\Big|\mathcal{F}_{t'}\right]\geq c.$$

Now choose now $\tau>0$ such that $e^{-nR^n_{\max}\tau}\leq\frac{c}{2}$.. We claim that

$$(T_K\leq t'+\tau)\Leftrightarrow(T_K\wedge(t'+2\tau)-t'\wedge T_K)\leq\tau.$$

Indeed if $t'\geq T_K$, then both parts are trivially true. Let assume that $t'\leq T_K$, and $T_K\leq t'+\tau$. Then $T_K\wedge(t'+2\tau)=T_K$, and $t'\wedge T_K=t'$, and therefore $(T_K\wedge(t'+2\tau)-t'\wedge T_K)=T_K-t'\leq\tau$. If on the other hand $t'\leq T_K$ and $(T_K\wedge(t'+2\tau)-t'\wedge T_K)\leq\tau$, we get $(T_K\wedge(t'+2\tau))\leq\tau+t'$, which gives that $T_K\leq(t'+2\tau)$, and therefore $T_K=(T_K\wedge(t'+2\tau))\leq t'+\tau$.

Using the claim

$$\begin{aligned}\mathbb{P}_{\boldsymbol{m}}(T_K\leq t'+\tau|\mathcal{F}_{t'}) &= \mathbb{P}_{\boldsymbol{m}}(T_K\wedge(t'+2\tau)-t'\wedge T_K\leq\tau|\mathcal{F}_{t'})\\ &= \mathbb{P}_{\boldsymbol{m}}\left(e^{-nR^n_{\max}\left(T_K\wedge(t'+2\tau)-t'\wedge T_K\right)}\geq e^{-nR^n_{\max}\tau}|\mathcal{F}_{t'}\right).\end{aligned}$$

Let $E_1\doteq\{e^{-nR^n_{\max}\left(T_K\wedge(t'+2\tau)-t'\wedge T_K\right)}\geq e^{-nR_{\max}\tau}\}$ and $E_2\doteq E_1$. Then since $T_K\wedge(t'+2\tau)-t'\wedge T_K\geq 0$

$$\mathbb{E}_{\boldsymbol{m}}\left[e^{-nR^n_{\max}\left(T_K\wedge(t'+2\tau)-t'\wedge T_K\right)}\Big|\mathcal{F}_{t'}\right]=\mathbb{E}_{\boldsymbol{m}}\left[1_{E_1}e^{-nR^n_{\max}\left(T_K\wedge(t'+2\tau)-t'\wedge T_K\right)}\Big|\mathcal{F}_{t'}\right]$$
$$+\mathbb{E}_{\boldsymbol{m}}\left[1_{E_2}e^{-nR^n_{\max}\left(T_K\wedge(t'+2\tau)-t'\wedge T_K\right)}\Big|\mathcal{F}_{t'}\right]\leq\mathbb{E}_{\boldsymbol{m}}\left[1_{E_1}\Big|\mathcal{F}_{t'}\right]+e^{-R^n_{\max}\tau}.$$

From this we get

$$\begin{aligned}&\mathbb{P}_{\boldsymbol{m}}\left(e^{-nR^n_{\max}\left(T_K\wedge(t'+2\tau)-t'\wedge T_K\right)}\geq e^{-nR^n_{\max}\tau}|\mathcal{F}_{t'}\right)\\ &\geq\mathbb{E}_{\boldsymbol{m}}\left[e^{-nR^n_{\max}\left(T_K\wedge(t'+2\tau)-t'\wedge T_K\right)}\Big|\mathcal{F}_{t'}\right]-e^{-nR^n_{\max}\tau}\geq c-\frac{c}{2}=\frac{c}{2}.\end{aligned}$$

Since the probability depends only on the size of τ, an iteration argument gives that T_K is finite almost surely. The remaining inequality is just an application of the monotone convergence theorem. ☐

Lemma 10. *Given $\boldsymbol{m}\in\mathcal{P}^n(\mathcal{X}),\epsilon>0$ and $\boldsymbol{u}\in\mathcal{A}_b^{n,|\mathcal{Z}|}$ with*

$$\mathbb{E}_{\boldsymbol{m}}\left[e^{n\int_0^{T_K}\left(\sum_{(x,y)\in\mathcal{Z}}\boldsymbol{\mu}_x(t)C^n_{xy}\left(\frac{u_{xy}(t)}{\gamma_{xy}}\right)-R^n(\boldsymbol{\mu}(t))\right)dt}\right]<\infty,$$

there exists $\tilde{\boldsymbol{u}}\in\mathcal{A}_b^{n,|\mathcal{Z}|}$ and $\tau<\infty$, such that

$$\sum_{(x,y)\in\mathcal{Z}}\tilde{\mu}_x(t)\gamma_{xy}C^n_{xy}\left(\frac{\tilde{u}_{xy}(t)}{\gamma_{xy}}\right)-R^n(\tilde{\boldsymbol{\mu}}(t))\leq 0$$

for every $t>\tau$, and

$$I^n_K(\boldsymbol{m},\tilde{\boldsymbol{u}})\leq I^n_K(\boldsymbol{m},\boldsymbol{u})+\epsilon.$$

Proof. Let such $\boldsymbol{m}\in\mathcal{P}^n(\mathcal{X}),\epsilon>0,$ and $\boldsymbol{u}\in\mathcal{A}_b^{n,|\mathcal{Z}|}$ be given, and let $c>0$ from Lemma 9 be such that

$$\mathbb{E}_{\boldsymbol{m}}\left[e^{nR^n_{\max}(T_K-t'\wedge T_K)}\Big|\mathcal{F}_{t'}\right]>c \tag{3.16}$$

for $t'\in[0,\infty)$. Since by Lemma 9 T_K is finite a.s., we can find $\tau<\infty$ such that

$$\mathbb{E}_{\boldsymbol{m}}\left[I_{\{T_K\geq\tau\}}e^{n\int_0^{T_K}\left(\sum_{(x,y)\in\mathcal{Z}}\mu_x(t)\gamma_{xy}C^n_{xy}\left(\frac{u_{xy}(t)}{\gamma_{xy}}\right)-R^n(\boldsymbol{\mu}(t))\right)dt}\right]\leq\epsilon c.$$

Now set $\tilde{\boldsymbol{u}}(t) = \boldsymbol{u}(t)$ for $t \leq \tau$, and $\tilde{\boldsymbol{u}}(t) = \boldsymbol{\gamma}$ so that $C^n_{xy}\left(\tilde{u}_{xy}(t)/\gamma_{xy}\right) = 0$ for $t \geq \tau$. Let $\tilde{\boldsymbol{\mu}}$ and $\tilde{T}_K$ be the corresponding controlled process and stopping time. Then the first claim of the lemma follows. The remaining claim follows from the following display, where the first inequality uses again that $C^n_{xy}(1) = 0$, the following equality uses that $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\mu}}, \tilde{T}_K)$ had the same distribution as the original versions up till time τ, and the second inequality uses (3.16):

$$\begin{aligned}
I^n_K(\boldsymbol{m}, \tilde{\boldsymbol{u}}) &= \mathbb{E}_{\boldsymbol{m}}\left[e^{n\int_0^{T_K}\left(\sum_{(x,y)\in\mathcal{Z}}\tilde{\mu}_x(t)\gamma_{xy}C^n_{xy}\left(\frac{\tilde{u}_{xy}(t)}{\gamma_{xy}}\right) - R^n(\tilde{\boldsymbol{\mu}}(t))\right)dt}\right] \\
&\leq \mathbb{E}_{\boldsymbol{m}}\left[I_{\{T_K\leq\tau\}}e^{n\int_0^{T_K}\left(\sum_{(x,y)\in\mathcal{Z}}\tilde{\mu}_x(t)\gamma_{xy}C^n_{xy}\left(\frac{\tilde{u}_{xy}(t)}{\gamma_{xy}}\right) - R^n(\tilde{\boldsymbol{\mu}}(t))\right)dt}\right] \\
&\quad + \mathbb{E}_{\boldsymbol{m}}\left[I_{\{T_K\geq\tau\}}e^{n\int_0^{T_K\wedge\tau}\left(\sum_{(x,y)\in\mathcal{Z}}\tilde{\mu}_x(t)\gamma_{xy}C^n_{xy}\left(\frac{\tilde{u}_{xy}(t)}{\gamma_{xy}}\right) - R^n(\tilde{\boldsymbol{\mu}}(t))\right)dt}\right] \\
&= \mathbb{E}_{\boldsymbol{m}}\left[I_{\{T_K\leq\tau\}}e^{n\int_0^{T_K}\left(\sum_{(x,y)\in\mathcal{Z}}\mu_x(t)\gamma_{xy}C^n_{xy}\left(\frac{u_{xy}(t)}{\gamma_{xy}}\right) - R^n(\boldsymbol{\mu}(t))\right)dt}\right] \\
&\quad + \mathbb{E}_{\boldsymbol{m}}\Bigg[I_{\{T_K\geq\tau\}}e^{n\int_0^{T_K\wedge\tau}\left(\sum_{(x,y)\in\mathcal{Z}}\mu_x(t)\gamma_{xy}C^n_{xy}\left(\frac{u_{xy}(t)}{\gamma_{xy}}\right) - R^n(\boldsymbol{\mu}(t))\right)dt} \\
&\qquad\qquad \times \frac{\mathbb{E}_{\boldsymbol{m}}\left[e^{n\int_{T_K\wedge\tau}^{T_K}\left(\sum_{(x,y)\in\mathcal{Z}}\mu_x(t)\gamma_{xy}C^n_{xy}\left(\frac{u_{xy}(t)}{\gamma_{xy}}\right) - R^n(\boldsymbol{\mu}(t))\right)dt}\middle|\mathcal{F}_\tau\right]}{\mathbb{E}_{\boldsymbol{m}}\left[e^{n\int_{T_K\wedge\tau}^{T_K}\left(\sum_{(x,y)\in\mathcal{Z}}\mu_x(t)\gamma_{xy}C^n_{xy}\left(\frac{u_{xy}(t)}{\gamma_{xy}}\right) - R^n(\boldsymbol{\mu}(t))\right)dt}\middle|\mathcal{F}_\tau\right]}\Bigg] \\
&\leq \mathbb{E}_{\boldsymbol{m}}\left[e^{n\int_0^{T_K}\left(\sum_{(x,y)\in\mathcal{Z}}\mu_x(t)C^n_{xy}\left(\frac{u_{xy}(t)}{\gamma_{xy}}\right) - R^n(\boldsymbol{\mu}(t))\right)dt}\right] \\
&\quad + \frac{1}{c}\mathbb{E}_{\boldsymbol{m}}\left[I_{\{T_K\geq\tau\}}e^{n\int_0^{T_K}\left(\sum_{(x,y)\in\mathcal{Z}}\mu_x(t)\gamma_{xy}C^n_{xy}\left(\frac{u_{xy}(t)}{\gamma_{xy}}\right) - R^n(\boldsymbol{\mu}(t))\right)dt}\right] \\
&\leq I^n_K(\boldsymbol{m}, \boldsymbol{u}) + \epsilon.
\end{aligned}$$

□

Proof of Theorem 5. We are first going to prove that V^n_K is the unique solution to (3.4). Let $\tilde{V}$ be any solution to (3.4), and let $\boldsymbol{m} \in \mathcal{P}(\mathcal{X})$. Let also $\boldsymbol{q} \in \mathcal{A}_b^{n,|\mathcal{Z}|}$ be given and let $\boldsymbol{\mu}$ solve (2.9). By Lemma 7,

$$\tilde{V}(\boldsymbol{\mu}(t\wedge T_K)) - \tilde{V}(\boldsymbol{m}) - \int_0^{t\wedge T_K}\sum_{(x,y)\in\mathcal{Z}}\mu_x(s)q_{xy}(s)\Delta^n\tilde{V}(\boldsymbol{\mu}(s))ds$$

is a martingale. Taking expectation gives

$$\mathbb{E}_{\boldsymbol{m}}\left[\tilde{V}(\boldsymbol{\mu}(t\wedge T_K))\right] - \mathbb{E}_{\boldsymbol{m}}\left[\int_0^{t\wedge T_K}\sum_{(x,y)\in\mathcal{Z}}\mu_x(s)q_{xy}(s)\Delta^n\tilde{V}(\boldsymbol{\mu}(s))ds\right] = \tilde{V}(\boldsymbol{m})$$

and since $\tilde{V}$ is a solution to (3.4) and by (3.2),

$$\mathbb{E}_{\boldsymbol{m}}\left[\tilde{V}(\boldsymbol{\mu}(t\wedge T_K))\right] + \mathbb{E}_{\boldsymbol{m}}\left[\int_0^{t\wedge T_K}\left(\sum_{(x,y)\in\mathcal{Z}}\mu_x(s)F^n_{xy}(q_{xy}(s)) + R^n(\boldsymbol{\mu}(s))\right)ds\right] \geq \tilde{V}(\boldsymbol{m}).$$

By Lemma 9, $T_K < \infty$ almost surely. Letting $t \to \infty$, Lemma 4 and the monotone convergence theorem imply

$$J^n_K(\boldsymbol{m}, \boldsymbol{q}) = \mathbb{E}_{\boldsymbol{m}}\left[\int_0^{T_K}\sum_{(x,y)\in\mathcal{Z}}\mu_x(t)F^n_{xy}(q_{xy}(s)) + R^n(\boldsymbol{\mu}(s)))ds\right] \geq \tilde{V}(\boldsymbol{m}).$$

Since $\boldsymbol{q} \in \mathcal{A}_b^{n,|\mathcal{Z}|}$ was arbitrary we get $V^n_K(\boldsymbol{m}) \geq \tilde{V}(\boldsymbol{m})$.

For each $\boldsymbol{m} \in \mathcal{P}^n(\mathcal{X})$ let $\bar{\boldsymbol{q}}(\boldsymbol{m})$ satisfy

$$\sum_{(x,y)\in\mathcal{Z}}\left(\bar{q}_{xy}(\boldsymbol{m})n\left(\tilde{V}\left(\boldsymbol{m} + \frac{1}{n}v_{xy}\right) - \tilde{V}(\boldsymbol{m})\right) + m_xF^n_{xy}(\bar{q}_{xy}(\boldsymbol{m}))\right) + R^n(\boldsymbol{m})) \leq R^n_{\min}\epsilon \qquad (3.17)$$

(note that $\bar{q}_{xy}(\boldsymbol{m})$ will be bounded away from zero). We can construct a solution to (2.9) with $\boldsymbol{u}$ replaced by the feedback control $\bar{\boldsymbol{q}}(\boldsymbol{\mu})$, and then obtain $\hat{\boldsymbol{q}} \in \mathcal{A}_b^{|\mathcal{Z}|}$ by setting $\hat{\boldsymbol{q}}(t) = \bar{\boldsymbol{q}}(\boldsymbol{\mu}(t))$. Then

$$\mathbb{E}_{\boldsymbol{m}}\left[\tilde{V}(\boldsymbol{\mu}(t \wedge T_K))\right] - \mathbb{E}_{\boldsymbol{m}}\left[\int_0^{t\wedge T_K} \sum_{(x,y)\in\mathcal{Z}} \mu_x(s)\bar{q}_{xy}(\boldsymbol{\mu}(s))\Delta^n \tilde{V}(\boldsymbol{\mu}(s))ds\right] = \tilde{V}(\boldsymbol{m}),$$

and therefore by (3.17)

$$\mathbb{E}_{\boldsymbol{m}}\left[\tilde{V}(\boldsymbol{\mu}(t \wedge T_K))\right] + \mathbb{E}_{\boldsymbol{m}}\left[\int_0^{t\wedge T_K} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x(t)F^n(\bar{q}_{xy}(\boldsymbol{\mu}(s))) + R^n(\boldsymbol{\mu}(s)) - \epsilon R^n_{\min}\right) ds\right] \leq \tilde{V}(\boldsymbol{m}).$$

Again using Lemma 9 and the monotone convergence theorem gives

$$(1-\epsilon)\mathbb{E}_{\boldsymbol{m}}\left[\int_0^{T_K} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x(t)F^n(\bar{q}_{xy}(\boldsymbol{\mu}(s))) + R^n(\boldsymbol{\mu}(s))\right) ds\right] \leq \tilde{V}(\boldsymbol{m}),$$

and therefore $V_K^n(\boldsymbol{m}) \leq J_K^n(\boldsymbol{m}, \hat{\boldsymbol{q}}) \leq \frac{1}{1-\epsilon}\tilde{V}(\boldsymbol{m})$. Since ϵ is arbitrary we get $V_K^n(\boldsymbol{m}) = \tilde{V}(\boldsymbol{m})$, which implies the uniqueness of $\tilde{V}$.

We now proceed with the proof that W_K^n is the unique solution to

$$\sup_{\boldsymbol{u}\in(0,\infty)^{|\mathcal{Z}|}} \left\{ \sum_{(x,y)\in\mathcal{Z}} \mu_x \left(u_{xy}\left(\frac{W(\boldsymbol{\mu}) - W\left(\boldsymbol{\mu} + \frac{\boldsymbol{v}_{xy}}{n}\right)}{W(\boldsymbol{\mu})}\right) - \gamma_{xy}C^n_{xy}\left(\frac{u_{xy}}{\gamma_{xy}}\right)\right)\right\} = -R^n(\boldsymbol{\mu}). \qquad (3.18)$$

Since V_K^n is a solution to (3.4), by Lemma 1 we get that $\frac{1}{n}\log(V_K^n)$ is a solution to (3.18), and thus uniqueness will imply $\frac{1}{n}\log(V_K^n) = W_K^n$.

Let $\tilde{W}$ be any solution to (3.18), $m \in \mathcal{P}^n(\mathcal{X})$, and $\boldsymbol{u} \in \mathcal{A}_b^{n,|\mathcal{Z}|}$, and let $\boldsymbol{\mu}$ solve (2.9). Further assume that there exists $\tau < \infty$ such that for $t > \tau$

$$\sum_{(x,y)\in\mathcal{Z}} \mu_x(t)\gamma_{xy}C^n_{xy}\left(\frac{u_{xy}(t))}{\gamma_{xy}}\right) - R^n(\boldsymbol{\mu}(t)) \leq 0. \qquad (3.19)$$

To show $J_K^n(\boldsymbol{m}, \boldsymbol{u}) \geq \tilde{W}(\boldsymbol{m})$ we can assume that $J_K^n(\boldsymbol{m}, \boldsymbol{u}) < \infty$, since otherwise there is nothing to prove. By Lemma 8

$$\frac{\tilde{W}(\boldsymbol{\mu}(t\wedge T_K))}{\tilde{W}(\boldsymbol{m})}\exp\left\{-\int_0^{t\wedge T_K} \sum_{(x,y)\in\mathcal{Z}} \mu_x(s)u_{xy}(s)\frac{\Delta^n\tilde{W}(\boldsymbol{\mu}(s))}{\tilde{W}(\boldsymbol{\mu}(s))}ds\right\}$$

is a martingale. Taking expectations gives

$$\mathbb{E}_{\boldsymbol{m}}\left[\tilde{W}(\boldsymbol{\mu}(t\wedge T_K))\exp\left\{-\int_0^{t\wedge T_K} \sum_{(x,y)\in\mathcal{Z}} \mu_x(s)u_{xy}(s)\frac{\Delta^n\tilde{W}(\boldsymbol{\mu}(s))}{\tilde{W}(\boldsymbol{\mu}(s))}ds\right\}\right] = \tilde{W}(\boldsymbol{m}),$$

and by (3.4) and the definition of Δ^n

$$\mathbb{E}_{\boldsymbol{m}}\left[\tilde{W}(\boldsymbol{\mu}(t\wedge T_K))\exp\left\{n\int_0^{t\wedge T_K} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x(s)\gamma_{xy}C^n_{xy}\left(\frac{u_{xy}(s)}{\gamma_{xy}}\right) - R^n(\boldsymbol{\mu}(s))\right)ds\right\}\right] \geq \tilde{W}(\boldsymbol{m}).$$

We claim that

$$\mathbb{E}_{\boldsymbol{m}}\left[\tilde{W}(\boldsymbol{\mu}(t\wedge T_K))\exp\left\{n\int_0^{\tau\wedge T_K} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x(s)\gamma_{xy}C^n_{xy}\left(\frac{u_{xy}(s)}{\gamma_{xy}}\right) - R^n(\boldsymbol{\mu}(s))\right)ds\right\}\right] < \infty. \qquad (3.20)$$

Since $\tilde{W}$ is uniformly bounded this term can be ignored. One can then bound what remains in (3.20) by using

$$\infty > J_K^n(\boldsymbol{m}, \boldsymbol{u}) = \mathbb{E}_{\boldsymbol{m}}\left[\exp\left\{n\int_0^{T_K} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x(s)\gamma_{xy}C^n_{xy}\left(\frac{u_{xy}(s)}{\gamma_{xy}}\right) - R^n(\boldsymbol{\mu}(s))\right)ds\right\}\right],$$

breaking the integral over $[0, T_K]$ into contributions over $[0, \tau \wedge T_K]$ and $[\tau \wedge T_K, T_K]$, and then conditioning on $\mathcal{F}_\tau$ and using the lower bound on the term corresponding to $[\tau \wedge T_K, T_K]$ provided by Lemma 9 (as in the proof of Lemma 10). Since (by Lemma 9) T_K is finite almost surely, and (3.19) holds for $t \geq \tau$, by dominated convergence theorem and (3.20) it follows that

$$J_K^n(\boldsymbol{m}, \boldsymbol{u}) = \mathbb{E}\left[\exp\left\{n \int_0^{T_K} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x(s)\gamma_{xy} C_{xy}^n\left(\frac{u_{xy}(s)}{\gamma_{xy}}\right) - R^n(\boldsymbol{\mu}(s))\right) ds\right\}\right] \geq \tilde{W}(\boldsymbol{m}).$$

By minimizing over all $\boldsymbol{u}$ that satisfy (3.19) and applying Lemma 10, we get $W_K^n(\boldsymbol{m}) \geq \tilde{W}(\boldsymbol{m})$.

Next let $\epsilon \in (0, 1/2)$. For $\boldsymbol{m} \in \mathcal{P}^n(\mathcal{X}), t \geq 0$ we choose $\bar{\boldsymbol{u}}(\boldsymbol{m}, t)$ such that

$$\sum_{(x,y)\in\mathcal{Z}} m_x \left(\bar{u}_{xy}(\boldsymbol{m}, t)\left(\frac{\tilde{W}(\boldsymbol{m}) - \tilde{W}\left(\boldsymbol{m} + \frac{\boldsymbol{v}_{xy}}{n}\right)}{\tilde{W}(\boldsymbol{m})}\right) - \gamma_{xy} C_{xy}^n\left(\frac{\bar{u}_{xy}(\boldsymbol{m}, t)}{\gamma_{xy}}\right)\right) \geq -R^n(\boldsymbol{m}) - \frac{\epsilon}{t^2+1}. \quad (3.21)$$

As before we can solve (2.9) and then generate a corresponding element $\boldsymbol{u}$ of $\mathcal{A}_b^{n,|\mathcal{Z}|}$ by composing $\bar{u}_{xy}(\boldsymbol{m}, t)$ with the solution. It is easy to see that $\boldsymbol{u}$ is an element of $\mathcal{A}_b^{n,|\mathcal{Z}|}$, since very big or very small values of $\bar{u}_{xy}(\boldsymbol{m}, t)$ will make the left hand of (3.21) tend to $-\infty$. Arguing as before, for fixed $t < \infty$

$$\mathbb{E}_{\boldsymbol{m}}\left[\tilde{W}(\boldsymbol{\mu}(t \wedge T_K)) \exp\left\{n \int_0^{T_K \wedge t} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x(s)\gamma_{xy} C_{xy}^n\left(\frac{\bar{u}_{xy}(\boldsymbol{\mu}(s), s)}{\gamma_{xy}}\right) - R^n(\boldsymbol{\mu}(s)) - \frac{\epsilon}{s^2+1}\right) ds\right\}\right] \leq \tilde{W}(\boldsymbol{m}).$$

By sending $t \to \infty$ and using the boundary condition, Fatou's lemma gives

$$\mathbb{E}_{\boldsymbol{m}}\left[\exp\left(\int_0^\infty -\frac{\epsilon}{s^2+1} ds\right) \exp\left\{n \int_0^{T_K} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x(s)\gamma_{xy} C_{xy}^n\left(\frac{\bar{u}_{xy}(\boldsymbol{\mu}(s), s)}{\gamma_{xy}}\right) - R^n(\boldsymbol{\mu}(s))\right) ds\right\}\right] \leq \tilde{W}(\boldsymbol{m}),$$

from which we get $W_K^n(\boldsymbol{m}) \leq \tilde{W}(\boldsymbol{m}) \exp[\epsilon \int_0^\infty 1/(s^2+1) ds]$. Sending ϵ to zero shows $W_K^n(\boldsymbol{m}) \leq \tilde{W}(\boldsymbol{m})$.

The proof that $\mathcal{V}_K^n(\boldsymbol{x}^n) = -\frac{1}{n}\log(\mathcal{W}_K^n(\boldsymbol{x}^n))$ is similar and thus omitted. It remains only to prove $V_K^n(L(\boldsymbol{x}^n)) = \mathcal{V}_K^n(\boldsymbol{x}^n)$.

We have established that V_K^n is the only function that satisfies

$$\inf_{\boldsymbol{q}\in(0,\infty)^{|\mathcal{Z}|}} \left\{\sum_{(x,y)\in\mathcal{Z}} m_x \left(q_{xy}\Delta_{xy}^n V_K^n(\boldsymbol{m}) + F_{xy}^n(q_{xy})\right)\right\} = -R^n(\boldsymbol{m}), \quad (3.22)$$

and that $\mathcal{V}_K^n$ is the only function that satisfies

$$\inf_{\boldsymbol{q}\in(0,\infty)^{n|\mathcal{Z}|}} \left\{\sum_{i=1}^n \sum_{y\in\mathcal{Z}_{x_i^n}} \left(q_{x_i^n y}\Delta_{i,x_i^n y}^n \mathcal{V}_K^n(\boldsymbol{x}^n) + F_{x_i^n y}^n(q_{x_i^n y})\right)\right\} = -nR^n(L(\boldsymbol{x}^n)). \quad (3.23)$$

Since $K \subset \mathcal{X}^n$ is invariant under permutations, and therefore can be identified with a subset of $\mathcal{P}^n(\mathcal{X})$, we have that there exists a function $\bar{V} : \mathcal{P}^n(\mathcal{X}) \to [0, \infty)$ such that $\bar{V}(L(\boldsymbol{x}^n)) = \mathcal{V}_K^n(\boldsymbol{x}^n)$, and therefore (3.23) becomes

$$\inf_{\boldsymbol{q}\in(0,\infty)^{n|\mathcal{Z}|}} \left\{\sum_{i=1}^n \sum_{y\in\mathcal{Z}_{x_i^n}} \left(q_{x_i^n y}\Delta_{i,x_i^n y}^n \bar{V}(L(\boldsymbol{x}^n)) + F_{x_i^n y}^n(q_{x_i^n y})\right)\right\} = -nR^n(L(\boldsymbol{x}^n)).$$

For $\epsilon > 0$, let $\bar{\boldsymbol{q}} \in (0, \infty)^{n|\mathcal{Z}|}$ satisfy

$$\sum_{i=1}^n \sum_{y\in\mathcal{Z}_{x_i^n}} \left[\bar{q}_{x_i^n y}\Delta_{i,x_i^n y}^n \bar{V}(L(\boldsymbol{x}^n)) + F_{x_i^n y}^n(\bar{q}_{x_i^n y})\right] \leq -nR^n(L(\boldsymbol{x}^n)) + \epsilon.$$

Now pick $\tilde{\boldsymbol{q}} \in (0, \infty)^{|\mathcal{Z}|}$ by requiring $nL_x(\boldsymbol{x}^n)\tilde{q}_{xy} = \sum_{i=1}^n I_{x_i^n = x}\bar{q}_{x_i^n y}$, so that

$$\sum_{(x,y)\in\mathcal{Z}} nL_x(\boldsymbol{x}^n)\tilde{q}_{xy}\Delta_{xy}^n \bar{V}(L(\boldsymbol{x}^n)) + \sum_{i=1}^n \sum_{y\in\mathcal{Z}_{x_i^n}} F_{x_i^n y}^n(\bar{q}_{x_i^n y}) \leq -nR^n(L(\boldsymbol{x}^n)) + \epsilon.$$

By using convexity of F^n_{xy} (see Lemma 4) we get

$$\sum_{(x,y)\in\mathcal{Z}} L_x(\boldsymbol{x}^n)\left[\tilde{q}_{xy}\Delta^n_{xy}\bar{V}\left(L(\boldsymbol{x}^n)\right)+F^n_{xy}(\tilde{q}_{xy})\right]\leq -R^n(L(\boldsymbol{x}^n))+\epsilon/n,$$

and sending $\epsilon\downarrow 0$ gives

$$\inf_{\boldsymbol{q}\in(0,\infty)^{|\mathcal{Z}|}}\left\{\sum_{(x,y)\in\mathcal{Z}} L_x(\boldsymbol{x}^n)\left[q_{xy}\Delta^n_{xy}\bar{V}\left(L(\boldsymbol{x}^n)\right)+F^n_{xy}(q_{xy})\right]\right\}\leq -R^n(L(\boldsymbol{x}^n)).$$

The other direction is trivial, and follows if in (3.23) one uses rates that are the same for all particles in the same position. □

4 Discussion regarding convergence

Before we introduce the deterministic control problem, we define the set of admissible controls and controlled trajectories.

Definition 11. *We define the space of paths and controls by*

$$\mathcal{C}\doteq\left\{(\boldsymbol{\mu},\boldsymbol{q})\in\mathcal{D}([0,\infty);\mathcal{P}(\mathcal{X}))\times\mathcal{F}\left([0,\infty);[0,\infty)^{\otimes\mathcal{Z}}\right):\ \mu_x q_{xy}\text{ is locally integrable }\forall(x,y)\in\mathcal{Z}\right\},\tag{4.1}$$

where $\mathcal{F}\left([0,\infty);[0,\infty)^{\otimes\mathcal{Z}}\right)$ *was defined in (2.1). We define* $\Lambda:\mathcal{C}\times\mathcal{P}(\mathcal{X})\to\mathcal{D}([0,\infty);\mathcal{H})$ *by*

$$\Lambda(\boldsymbol{\mu},\boldsymbol{q},\boldsymbol{m})(t)\doteq\boldsymbol{m}+\sum_{(x,y)\in\mathcal{Z}}\boldsymbol{v}_{xy}\int_{[0,t)}\mu_x(s)q_{xy}(s)ds.\tag{4.2}$$

Also we define the set of all deterministic pairs that correspond to a solution of the equation $\boldsymbol{\mu}=\Lambda(\boldsymbol{\mu},\boldsymbol{q},\boldsymbol{m})$*, i.e.,*

$$\mathcal{T}_{\boldsymbol{m}}\doteq\{(\boldsymbol{\mu},\boldsymbol{q})\in\mathcal{C}:\boldsymbol{\mu}=\Lambda(\boldsymbol{\mu},\boldsymbol{q},\boldsymbol{m}),\boldsymbol{\mu}(0)=\boldsymbol{m}\}$$

Finally we introduce the set of controls that generate controlled trajectories

$$\mathcal{U}_{\boldsymbol{m}}\doteq\left\{\boldsymbol{q}\in\mathcal{F}([0;\infty);[0,\infty)^{\otimes\mathcal{Z}}):\exists\boldsymbol{\mu}\in\mathcal{D}([0,\infty);\mathcal{P}(\mathcal{X}))\text{ such that }(\boldsymbol{\mu},\boldsymbol{q})\in\mathcal{T}_{\boldsymbol{m}}\right\}.\tag{4.3}$$

Then the deterministic control problems are given by

$$V_K(\boldsymbol{m})\doteq\inf_{(\boldsymbol{\mu},\boldsymbol{q})\in\mathcal{T}_{\boldsymbol{m}}}J_K(\boldsymbol{m},\boldsymbol{\mu},\boldsymbol{q})\tag{4.4}$$

$$J_K(\boldsymbol{m},\boldsymbol{\mu},\boldsymbol{q})\doteq\left\{\int_0^{T_K}\left(\sum_{(x,y)\in\mathcal{Z}}\mu_x(t)\hat{F}^\infty_{xy}(q_{xy}(t))+R^\infty(\boldsymbol{\mu}(t))\right)dt\right\},\tag{4.5}$$

$$F^\infty(q)\doteq\inf\left\{\liminf_{n\to\infty}F^n(q_n):\{q^n\}\text{ sequence in }[0,\infty)\text{ with }q_n\to q\right\},\tag{4.6}$$

with

$$\hat{F}^\infty(q)\doteq\sup\{F(q):F\text{ convex and }F\leq F^\infty\}.\tag{4.7}$$

and

$$T_K\doteq\inf_{t\in[0,\infty]}\left\{\boldsymbol{\mu}(t)\in K\right\}.$$

In this section we consider sets $K\subset\mathcal{P}(\mathcal{X})$ that satisfy the following assumption.

Assumption 12. $K=\overline{K^\circ}\neq\emptyset$.

For such sets we show that the sequence of values functions V_K^n, where converges uniformly to the function V_K. To simplify the notation we will drop the index that corresponds to the set from the stopping time. We split the study of the convergence in two parts. In the first part, without making any extra assumptions on the cost functions and in great generality, we prove that for any sequence $\{\boldsymbol{m}^n\}$, with $\boldsymbol{m}^n \in \mathcal{P}^n(\mathcal{X})$ converging in $\boldsymbol{m} \in \mathcal{P}(\mathcal{X})$,

$$\liminf_{n\to\infty} V_K^n(\boldsymbol{m}^n) \geq V_K(\boldsymbol{m}).$$

The other direction of the inequality, i.e.,

$$\limsup_{n\to\infty} V_K^n(\boldsymbol{m}^n) \leq V_K(\boldsymbol{m}),$$

is not as straightforward and its analysis can be quite involved. In order to avoid technical issues we will add some assumptions.

Before we present the extra assumptions on $\{\boldsymbol{C}^n\}$ we discuss an almost trivial choice for the cost function that does not depend on n and that will motivate these extra assumptions. As stated in Lemma 3, for every $(x, y) \in \mathcal{Z}$ we have $C_{xy}^n(u) \geq -\log u + u - u$. Actually the function $C_{xy}^n(u) = -\log u + u - 1$ satisfies Assumption 2 and therefore is an eligible cost function.

Setting $C_{xy}^n(u) \equiv \overline{C}(u) = -\log u + u - 1$, we get

$$\begin{aligned} G_{xy}^n(u,q) &= u\ell\left(\frac{q}{u}\right) - \gamma_{xy} C_{xy}^n\left(\frac{u}{\gamma_{xy}}\right) \\ &= q\log\frac{q}{u} - q + u + \gamma_{xy}\log\frac{u}{\gamma_{xy}} - u + \gamma_{xy} \\ &= q\log q + (\gamma_{xy} - q)\log u - q + \gamma_{xy} \end{aligned} \tag{4.8}$$

Examining (4.8) and referring to the definition of F_{xy}^n in (1.9), we observe that if $q_{xy} > \gamma_{xy}$ then the “maximizing player” (the one that picks $\boldsymbol{u}$), can produce an arbitrarily large cost by making u_{xy} as **small** as needed. If $q_{xy} < \gamma_{xy}$, this player can produce an arbitrarily large cost by making u_{xy} as **big** as needed. Hence the minimizing player must keep $q_{xy} = \gamma_{xy}$, and the value function $V(\boldsymbol{m})$ is infinite unless the solution of the equation $\dot{\boldsymbol{\nu}}(t) = \boldsymbol{\nu}(t)\boldsymbol{\gamma}$ passes through K for the specific choice of initial data $\boldsymbol{m}$.

To resolve this difficulty we impose the following assumption on the cost.

Assumption 13. *For every* $(x, y) \in \mathcal{Z}$

1

$$\lim_{u\to 0}\sup_{n\in\mathbb{N}} u(C_{xy}^n)'(u) = -\infty.$$

2

$$\liminf_{u\to\infty}\inf_{n\in\mathbb{N}}\{u(C_{xy}^n)'(u) - u\} \geq 0.$$

Assumption 13 makes $\hat{F}^\infty$ finite on $(0,\infty)$ and allows for some controllability. More specifically, if the first point of Assumption 13 holds true and if $\boldsymbol{m}, \tilde{\boldsymbol{m}} \in \mathcal{P}_a(\mathcal{X})$ for some $a > 0$, then one can observe (see the proof of Lemma 4) that the total cost $V_{\{\tilde{\boldsymbol{m}}\}}(\boldsymbol{m})$ for moving from point $\boldsymbol{m}$ to $\tilde{\boldsymbol{m}}$ is uniformly bounded by $c_a\|\boldsymbol{m} - \tilde{\boldsymbol{m}}\|$, where $c_a > 0$ is an appropriate constant, where the minimizing player picks $\tilde{q}_{xy}(t)$ to be uniformly bounded from above, but big enough to reach the desired point. In particular, the maximizing player cannot impose an abritrarily large cost by taking u_{xy} small. In an analogous fashion, the second point of Assumption 13 implies the minimizer can choose controls so that the total cost $V_{\{\tilde{\boldsymbol{m}}\}}(\boldsymbol{m})$ for moving from point $\boldsymbol{m}$ to $\tilde{\boldsymbol{m}}$ is uniformly bounded by $c_a'\|\boldsymbol{m} - \tilde{\boldsymbol{m}}\|$ by picking $\tilde{q}_{xy}(t)$ bounded from below but small enough.

However, if $\tilde{\boldsymbol{m}}$ is in the natural boundary of the simplex $\mathcal{P}(\mathcal{X})$ an additional complication arises, because to reach the natural boundary it must be true that for at least one $(x, y) \in \mathcal{Z}$ the quantity $\tilde{q}_{xy}(t)$ will scale like $1/\tilde{\mu}_x(t)$. In that case, the first point of Assumption 13 is not enough for a finite cost, since sending $\tilde{q}_{xy}(t)$ to infinity in order to reach the natural boundary may result in an infinite total cost. Taking all these things into account we end up with the following assumption.

Assumption 14. *For every $n \in \mathbb{N}$ let $\boldsymbol{C}^n, R^n$ be as in Assumption 2. Assume that there exist lower semicontinuous functions $\boldsymbol{C}^\infty : (0,\infty)^{\otimes\mathcal{Z}} \to [0,\infty]$, $R^n : \mathcal{P}(\mathcal{X}) \to [0,\infty)$, such that $\{R^n\}_n$ converges uniformly to R^∞, and for all $(x, y) \in \mathcal{Z}$, the following are valid.*

1 *There exist intervals* $(u_{1,xy}, u_{2,xy}) \subset (0,\infty)$ *containing* $1,$ *outside of which* C^n_{xy}, C^∞_{xy} *are infinite, and for which* C^n_{xy}, *converges uniformly on compact subsets to* C^∞_{xy}.

2 *There exists* $p > 0$ *such that*

$$\lim_{u\to 0}\sup_{n\in\mathbb{N}} u^{p+1}(C^n_{xy})'(u) = -\infty. \tag{4.9}$$

3

$$\liminf_{u\to\infty}\inf_{n\in\mathbb{N}}\{u(C^n_{xy})'(u) - u\} \geq 0. \tag{4.10}$$

Now we state the second main theorem of the paper.

Theorem 15. *Let* $\{\boldsymbol{C}^n\}_n$, $\{R^n\}_n$, C^∞, *and* R^∞ *satisfy Assumptions 2 and 14. Let also* K *be a closed subset of* $\mathcal{P}(\mathcal{X})$ *that satisfies Assumption 12. Finally assume that in every compact subset of* K^c, R^∞, *is bounded from below by a positive constant. Then the sequence of functions* V^n_K *defined in (2.12) converges uniformly to* V_K *defined in (4.4).*

Before proceeding with the proof, we state some properties of F^n_{xy}.

Lemma 16. *For every* $n \in \mathbb{N}$ *and for every* $(x,y) \in \mathcal{Z}$, *let* F^n_{xy} *be as in (1.9), where* $\{C^n_{xy}\}$ *satisfy Assumption 14. Then the following hold.*

1 *There exists a constant* $M \in (0,\infty)$ *and a decreasing function* $\bar{M} : (0,\infty) \to (0,\infty)$, *such that for every* $\epsilon > 0$ *and every* $q \geq \epsilon$,

$$F^n_{xy}(q) \leq q \log \frac{q}{\min\left\{\gamma_{xy}\left(\gamma_{xy}/q\right)^{1/p}, M\right\}} + \bar{M}(\epsilon).$$

2 F^n_{xy} *is continuous on the interval* $(0,\infty)$, *and continuous as an extended function on* $[0,\infty)$.

3 F^n_{xy} *converges locally uniformly, on the set* $(0,\infty)$, *to the function*

$$\bar{F}_{xy}(q) = \sup_{u\in(0,\infty)}\left\{u\ell\left(\frac{q}{u}\right) - \gamma_{xy}C^\infty_{xy}\left(\frac{u}{\gamma_{xy}}\right)\right\}.$$

Furthermore, we have $\bar{F}_{xy} = F^\infty_{xy}$, *where* F^∞ *was defined in (4.6). Finally* F^∞_{xy} *is convex on the whole domain* $[0,\infty)$ *and therefore* $\bar{F}_{xy} = F^\infty_{xy} = \hat{F}^\infty_{xy}$.

The proof of the Lemma 16 can be found in Appendix A. It is worth mentioning that it is possible that $F^n_{xy}(0) = \infty$.

In the sequel we will make use of the following remark, which states a property proved in [14, Proposition 4.14]

Remark 17. *There exists* $D \geq 1$ *and* $b_1 > 0, b_2 < \infty$ *such that for every* $\boldsymbol{m} \in \mathcal{P}(X)$, *if* $\boldsymbol{\nu}(\boldsymbol{m},t)$ *is the solution of* $\dot{\boldsymbol{\nu}}(t) = \boldsymbol{\nu}(t)\boldsymbol{\gamma}$ *with initial point* $\boldsymbol{\nu}(0) = \boldsymbol{m}$, *then*

1 $\forall x \in \mathcal{X}, \nu_x(\boldsymbol{m},t) \geq b_1 t^D$

2 $\|\nu(\boldsymbol{m},t) - \boldsymbol{m}\| \leq b_2 t.$

Before proceeding with the proof of Theorem 15, we prove that the function $V(\boldsymbol{m})$ is continuous. We will actually prove something stronger. Recall that $\boldsymbol{\gamma}$ denotes the original unperturbed jump rates and the definitions of $\mathcal{P}_*(\mathcal{X})$ and $\mathcal{P}_a(\mathcal{X})$ in (2.2), (2.3).

Theorem 18. *There is a constant* $\bar{c}$ *that depends only the dimension* d *and the unperturbed rates* $\boldsymbol{\gamma}$, *such that for every* $\boldsymbol{m} \in \mathcal{P}_*(\mathcal{X})$, $\tilde{\boldsymbol{m}} \in \mathcal{P}(\mathcal{X})$ *there exists a control* $\boldsymbol{q} \in \mathcal{U}_{\boldsymbol{m}}$, *that generates a unique* $\boldsymbol{\mu}$ *with* $(\boldsymbol{\mu},\boldsymbol{q}) \in \mathcal{T}_{\boldsymbol{m}}$, *satisfying*

1 $\boldsymbol{\mu}$ *is a constant speed parametrization of the straight line that connects* $\boldsymbol{m}$ *and* $\tilde{\boldsymbol{m}}$,

2 *the exit time* $T_{\{\tilde{\boldsymbol{m}}\}}$ *is equal to* $\|\boldsymbol{m} - \tilde{\boldsymbol{m}}\|$,

3 $\gamma_{xy} \leq q_{xy}(t)$ *and* $\mu_x(t)q_{xy}(t) \leq \bar{c}$.

Furthermore, if $\boldsymbol{m}, \tilde{\boldsymbol{m}} \in \mathcal{P}_a(\mathcal{X})$ *then*

$$\gamma_{xy} \leq q_{xy}(t) \leq \frac{c_1}{a}, \tag{4.11}$$

and we can find a constant $c_a < \infty$ *such that the total cost for applying the control is bounded above by* $c_a \|\boldsymbol{m} - \tilde{\boldsymbol{m}}\|$.

Finally, for every $\epsilon > 0$ *there exists* $\delta > 0$, *such that* $\|\bar{\boldsymbol{m}} - \tilde{\boldsymbol{m}}\| \leq \delta$ *implies* $V_{\{\tilde{\boldsymbol{m}}\}}(\bar{\boldsymbol{m}}), V_{\{\bar{\boldsymbol{m}}\}}(\tilde{\boldsymbol{m}}) \leq \epsilon$, *and therefore as a function of two variables* V *is continuous on* $\mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X})$.

Proof. Let $\boldsymbol{m} \in \mathcal{P}_*(\mathcal{X})$, $\tilde{\boldsymbol{m}} \in \mathcal{P}(\mathcal{X})$. We can find a positive constant $\bar{c}$ that depend only the dimension d and on the rates $\boldsymbol{\gamma}$, and also rates $\boldsymbol{q}$ such that

1 $q_{xy} \geq \gamma_{xy}$,

2 $\sum_{(x,y)\in\mathcal{Z}} m_x q_{xy} v_{xy} = \frac{\tilde{\boldsymbol{m}} - \boldsymbol{m}}{\|\tilde{\boldsymbol{m}} - \boldsymbol{m}\|}$,

3 $\max\{m_x q_{xy}, (x,y) \in \mathcal{Z}\} \leq \bar{c}$.

Indeed, since $\{a_{xy}\boldsymbol{v}_{xy} : a_{xy} \geq 0, (x,y) \in \mathcal{Z}\} = \mathcal{H}$, we can find a constant $c < \infty$ such that for every point $\boldsymbol{m} \in \mathcal{P}_*(\mathcal{X})$, there exist vectors $q_{xy} m_x \boldsymbol{v}_{xy}$ with $q_{xy} m_x \leq c$, and $\sum_{(x,y)\in\mathcal{Z}} m_x q_{xy} v_{xy} = \frac{\tilde{\boldsymbol{m}} - \boldsymbol{m}}{\|\tilde{\boldsymbol{m}} - \boldsymbol{m}\|}$. Now, if for some $(x_1, y_1) \in \mathcal{Z}$ we do not have $q_{x_1 y_1} \geq \gamma_{x_1 y_1}$, then by ergodicity we can pick $x_1, x_2 = y_1, \ldots, x_j$, with $j \leq d$, such that $\sum_{i=1}^{j-1} \boldsymbol{v}_{x_i x_{i+1}} = 0$. If we pick the new $q_{x_i x_{i+1}}$ equal to $\max_{xy}\{\gamma_{xy}\}/m_{x_i}$ plus the original $q_{x_i x_{i+1}}$, then property 2 is still satisfied, but we now also have $q_{x_1 y_1} \geq \gamma_{x_1 y_1}$. We have to repeat the procedure at most $|\mathcal{Z}|$ times to enforce property 1, and can then set $\bar{c} \doteq \max\{m_x q_{xy}, (x,y) \in \mathcal{Z}\}$.

Let $\tilde{\boldsymbol{\mu}}(t) = [(\tilde{\boldsymbol{m}} - \boldsymbol{m})t / \|\tilde{\boldsymbol{m}} - \boldsymbol{m}\| + \boldsymbol{m}]$, and define $\tilde{\boldsymbol{q}} \in \mathcal{U}_{\boldsymbol{m}}$ by

$$\tilde{\mu}_x(t)\tilde{q}_{xy}(t) = m_x q_{xy} \leq \bar{c}. \tag{4.12}$$

Then automatically

$$\sum_{(x,y)\in\mathcal{Z}} \boldsymbol{v}_{xy} \int_{[0,t)} \tilde{\mu}_x(s)\tilde{q}_{xy}(s)ds = t\frac{\tilde{\boldsymbol{m}} - \boldsymbol{m}}{\|\tilde{\boldsymbol{m}} - \boldsymbol{m}\|} = \tilde{\boldsymbol{\mu}}(t) - \boldsymbol{m},$$

and thus $(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{q}}) \in \mathcal{T}_{\boldsymbol{m}}$. This will lead to hitting $\{\tilde{\boldsymbol{m}}\}$ in time $T_{\{\tilde{\boldsymbol{m}}\}} = \|\boldsymbol{m} - \tilde{\boldsymbol{m}}\|$. By the second point in Lemma 4 we get

$$\begin{aligned}
&\inf_{(\boldsymbol{\mu},\boldsymbol{q})\in\mathcal{T}_{\boldsymbol{m}}} J_{\{\tilde{\boldsymbol{m}}\}}(\boldsymbol{m}, \boldsymbol{\mu}, \boldsymbol{q}) \leq J_{\{\tilde{\boldsymbol{m}}\}}(\boldsymbol{m}, \tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{q}}) \leq \sum_{(x,y)\in\mathcal{Z}} \int_0^{T_{\{\tilde{\boldsymbol{m}}\}}} \tilde{\mu}_x(t)\hat{F}^{\infty}_{xy}(\tilde{q}_{xy}(t)) + R_{\max} T_{\{\tilde{\boldsymbol{m}}\}} \\
&\overset{(4.12)}{\leq} \sum_{(x,y)\in\mathcal{Z}} \int_0^{T_{\{\tilde{\boldsymbol{m}}\}}} \left(\tilde{\mu}_x(t)\tilde{q}_{xy}(t) \log \frac{\tilde{q}_{xy}(t)}{\min\left\{\gamma_{xy}\left(\gamma_{xy}/\tilde{q}_{xy}(t)\right)^{1/p}, M\right\}} + \max_{(x,y)\in\mathcal{Z}} \bar{M}(\gamma_{xy}) \right) dt + R_{\max} T_{\{\tilde{\boldsymbol{m}}\}} \\
&\leq \sum_{(x,y)\in\mathcal{Z}} \int_0^{T_{\{\tilde{\boldsymbol{m}}\}}} \left|\tilde{\mu}_x(t)\tilde{q}_{xy}(t)\log \tilde{q}_{xy}(t)\right| dt + \sum_{(x,y)\in\mathcal{Z}} \int_0^{T_{\{\tilde{\boldsymbol{m}}\}}} \left|\tilde{\mu}_x(t)\tilde{q}_{xy}(t)\log\left(\gamma_{xy}/\tilde{q}_{xy}(t)\right)^{1/p}\right| dt + \\
&+ \sum_{(x,y)\in\mathcal{Z}} \int_0^{T_{\{\tilde{\boldsymbol{m}}\}}} \left|\tilde{\mu}_x(t)\tilde{q}_{xy}(t)\log\gamma_{xy}\right| dt + \sum_{(x,y)\in\mathcal{Z}} \int_0^{T_{\{\tilde{\boldsymbol{m}}\}}} \left|\tilde{\mu}_x(t)\tilde{q}_{xy}(t)\log M\right| dt + c' T_{\{\tilde{\boldsymbol{m}}\}} \\
&\overset{(4.12)}{\leq} \bar{c} \sum_{(x,y)\in\mathcal{Z}} \int_0^{T_{\{\tilde{\boldsymbol{m}}\}}} |\log \tilde{q}_{xy}(t)| dt + \bar{c} \sum_{(x,y)\in\mathcal{Z}} \int_0^{T_{\{\tilde{\boldsymbol{m}}\}}} \left|\log\left(\gamma_{xy}/\tilde{q}_{xy}(t)\right)^{1/p}\right| dt + c'' T_{\{\tilde{\boldsymbol{m}}\}} \\
&\overset{(4.12)}{\leq} \bar{c} \sum_{(x,y)\in\mathcal{Z}} \int_0^{T_{\{\tilde{\boldsymbol{m}}\}}} \left|\log \frac{m_x q_{xy}}{\tilde{\mu}_x(t)}\right| dt + \bar{c} \sum_{(x,y)\in\mathcal{Z}} \int_0^{T_{\{\tilde{\boldsymbol{m}}\}}} \left|\log\left(\tilde{\mu}_x(t)\gamma_{xy}/m_x q_{xy}\right)^{1/p}\right| dt + c'' T_{\{\tilde{\boldsymbol{m}}\}} \\
&\overset{(4.12)}{\leq} \bar{c} \sum_{(x,y)\in\mathcal{Z}} \int_0^{T_{\{\tilde{\boldsymbol{m}}\}}} \left|\log \tilde{\mu}_x(t)\right| dt + \bar{c} \sum_{(x,y)\in\mathcal{Z}} \int_0^{T_{\{\tilde{\boldsymbol{m}}\}}} \frac{1}{p} \left|\log \tilde{\mu}_x(t)\right| dt + c''' T_{\{\tilde{\boldsymbol{m}}\}},
\end{aligned} \tag{4.13}$$

where the constants c', c'', c''' depend only on $\boldsymbol{\gamma}, c_1$ and $R_{\max}$.

Now if $\boldsymbol{m}, \tilde{\boldsymbol{m}} \in \mathcal{P}_a(\mathcal{X})$, then all elements are bounded by a constant c_a (that depends on $\gamma, c_1, R_{\max}$, and a) times $T_{\{\tilde{\boldsymbol{m}}\}} = \|\tilde{\boldsymbol{m}} - \boldsymbol{m}\|$, and therefore the first part of the theorem. follows.

Let $1 > \delta > 0$, and $\bar{\boldsymbol{m}}, \tilde{\boldsymbol{m}} \in \mathcal{P}(\mathcal{X})$, with $\|\bar{\boldsymbol{m}} - \tilde{\boldsymbol{m}}\| < \delta$. We take $\boldsymbol{m} = \nu(\bar{\boldsymbol{m}}, \delta)$, where $\boldsymbol{\nu}(\bar{\boldsymbol{m}}, t)$ is the solution of $\dot{\boldsymbol{\nu}}(t) = \boldsymbol{\nu}(t)\boldsymbol{\gamma}$, with initial data $\boldsymbol{\nu}(0) = \bar{\boldsymbol{m}}$. Now by appropriate use of the inequality $\tilde{\mu}_x(t) \geq \min\{m_x, m_x(T_{\{\tilde{\boldsymbol{m}}\}} - t)\}$ and using the last display, we get

$$V_{\{\tilde{\boldsymbol{m}}\}}(\boldsymbol{m}) \leq c'''' \left(\sum_{(x,y)\in\mathcal{Z}} \int_0^{T_{\{\tilde{\boldsymbol{m}}\}}} \big(|\log m_x| + |\log(T_{\{\tilde{\boldsymbol{m}}\}} - t)| \big)\, dt + T_{\{\tilde{\boldsymbol{m}}\}} \right).$$

By a simple change of variable and Remark 17, we have

$$V_{\{\tilde{\boldsymbol{m}}\}}(\boldsymbol{m}) \leq c'''' \left(\sum_{(x,y)\in\mathcal{Z}} \int_0^{b_2\delta} \big(|\log b_1 \delta^D| + |\log t| \big)\, dt + b_2\delta \right). \tag{4.14}$$

Therefore

$$V_{\{\tilde{\boldsymbol{m}}\}}(\bar{\boldsymbol{m}}) \leq V_{\{\boldsymbol{m}\}}(\bar{\boldsymbol{m}}) + V_{\{\tilde{\boldsymbol{m}}\}}(\boldsymbol{m}) \leq \delta R_{\max} + c'''' \left(\sum_{(x,y)\in\mathcal{Z}} \int_0^{b_2\delta} \big(|\log b_1 \delta^D| + |\log t| \big)\, dt + b_2\delta \right),$$

and the right hand side can be made as small as desired by making δ small enough. The estimate for $V_{\{\bar{\boldsymbol{m}}\}}(\tilde{\boldsymbol{m}})$ is proved in a symmetric way. This proves the last statement of the theorem. □

5 Lower bound

For the proof of Theorem 15, we first prove the lower bound: for every sequence $\boldsymbol{m}^n \in \mathcal{P}^n(\mathcal{X})$ and $\boldsymbol{m} \in \mathcal{P}(\mathcal{X})$, with $\boldsymbol{m}^n \to \boldsymbol{m}$, we have

$$\liminf_{n\to\infty} V_K^n(\boldsymbol{m}^n) \geq V_K(\boldsymbol{m}).$$

Without loss of generality we can assume that the liminf is actually a limit, otherwise we can just work with a subsequence. If the limit is ∞ then the conclusion is trivial, therefore we can assume that there is $c \in \mathbb{R}$ such that

$$\sup_{n\in\mathbb{N}} V_K^n(\boldsymbol{m}^n) \leq c. \tag{5.1}$$

Let $\epsilon \in (0,1)$. Recalling (2.12), let $\boldsymbol{q}^n \in \mathcal{A}_b^{n,|\mathcal{Z}|}$ be such that

$$\mathbb{E}_{\boldsymbol{m}^n} \left[\int_0^{T^n} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x^n(t) F_{xy}^n(q_{xy}^n(t)) + R^n(\boldsymbol{\mu}^n(t)) \right) dt \right] < V_K^n(\boldsymbol{m}^n) + \epsilon, \tag{5.2}$$

where $\boldsymbol{\mu}^n = h^n(\boldsymbol{\mu}^n, \boldsymbol{q}^n, \boldsymbol{m}^n, \boldsymbol{N}^n/n)$ and $T^n \doteq \inf\{t \in [0,\infty] : \boldsymbol{\mu}^n(t) \in K\}$. For $\delta > 0$ such that

$$\|\bar{\boldsymbol{m}} - \tilde{\boldsymbol{m}}\| \leq \delta \Rightarrow V_{\bar{\boldsymbol{m}}}(\tilde{\boldsymbol{m}}) \leq \epsilon, \tag{5.3}$$

we define

$$K_\delta \doteq \{m : d(\boldsymbol{m}, K) \leq \delta\} \quad \text{and} \quad T^{n,\delta} \doteq \inf\{t \in [0,\infty] : \boldsymbol{\mu}_n(t) \in K_\delta\}. \tag{5.4}$$

The existence of such a δ is given by Theorem 18. Now for $\boldsymbol{\mu}^n, \boldsymbol{q}^n$ as in (5.2) and $T^{n,\delta}$ as above, we define the sequences $\boldsymbol{\mu}^{n,\delta}(t) = \boldsymbol{\mu}^n(t \wedge T^{n,\delta})$,

$$\boldsymbol{q}^{n,\delta}(t) = \begin{cases} \boldsymbol{q}^n(t) & t \leq T^{n,\delta} \\ \boldsymbol{\gamma} & T > T^{n,\delta} \end{cases} \cdots \tag{5.5}$$

We note that for $t > T^\delta$, $\boldsymbol{q}^{n,\delta}(t)$ does not actually generate $\boldsymbol{\mu}^n$, but we define it this way to simplify some technical issues. We will show that

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{T^n}\left(\sum_{(x,y)\in\mathcal{Z}}\mu_x^n(t)F_{xy}^n(q_{xy}^n(t))+R^n(\boldsymbol{\mu}^n(t))\right)dt\right] &\geq \\ \liminf_{n\to\infty} \mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{T^{n,\delta}}\left(\sum_{(x,y)\in\mathcal{Z}}\mu_x^{n,\delta}(t)F_{xy}^n(q_{xy}^{n,\delta}(t))+R^n(\boldsymbol{\mu}^{n,\delta}(t))\right)dt\right] &\geq V_{K_\delta}(\boldsymbol{m}),\end{aligned} \tag{5.6}$$

and then by an application of Theorem 18 and (5.2) deduce $\lim_{n\to\infty} V_K^n(\boldsymbol{m}^n)+2\epsilon \geq V_K(\boldsymbol{m})$. Since ϵ is arbitrary the lower bound will follow.

Before proceeding we introduce some auxiliary random measures. For $(x,y)\in\mathcal{Z}$, $q_{xy}\in\mathcal{F}([0,\infty);[0,\infty))$, and $t\in[0,\infty)$, define

$$\eta_{xy}(dr;t)\doteq\delta_{q_{x,y}(t)}(dr)\mu_x(t).$$

For each $t\in[0,\infty)$, $(x,y)\in\mathcal{Z}$ we have that $\eta_{xy}(\cdot;t)$ is a subprobability measure on $[0,\infty)$. Also we consider the measures $\theta_{xy}(drdt)=\eta_{xy}(dr;t)dt$ on $[0,\infty)\times[0,\infty)$ as equipped with the topology that generalizes the weak convergence of probability measures to general measures that have at most mass T on $[0,\infty)\times[0,T]$. This can be defined in terms of a distance (a generalization of the Prohorov metric) d_T,and the metric on measures on $[0,\infty)\times[0,\infty)$ is

$$\sum_{T\in\mathbb{N}} 2^{-T}\left[d_T(\boldsymbol{\mu}|_T,\boldsymbol{\nu}|_T)\vee 1\right], \tag{5.7}$$

where $\boldsymbol{\mu}|_T$ denotes the restriction to $[0,T]$ in the last variable.

Let $\boldsymbol{\theta}^{n,\delta}=\{\theta^{n,\delta}\}_{(x,y)\in\mathcal{Z}}$ be the random measures that correspond to $\boldsymbol{\mu}^{n,\delta},\boldsymbol{q}^{n,\delta}$, according to the construction above. We observe that

$$\boldsymbol{\mu}^{n,\delta}(t)=\boldsymbol{m}^n+\sum_{(x,y)\in\mathcal{Z}}\boldsymbol{v}_{xy}\int_0^{t\wedge T^{n,\delta}}\int_0^\infty r\theta_{xy}^{n,\delta}(drds)+\text{ a martingale},$$

where the martingale will converge to zero as $n\to\infty$, and that for every $(x,y)\in\mathcal{Z}$,

$$\mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{T^{n,\delta}}F_{xy}^n(q_{xy}^{n,\delta}(t))\mu_x^{n,\delta}(t)dt\right]=\mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{T^{n,\delta}}\int_0^\infty F_{xy}^n(r)\theta_{xy}^{n,\delta}(drdt)\right]. \tag{5.8}$$

We will split the proof of (5.6) in three parts. First we prove that $(\boldsymbol{\mu}^{n,\delta},\boldsymbol{\theta}^{n,\delta},T^{n,\delta})$ is tight. Then we show that for every limit point $(\boldsymbol{\mu}^\delta,\boldsymbol{\theta}^\delta,T^\delta)$, θ_{xy}^δ has the decomposition $\theta_{xy}^\delta(drdt)=\eta_{xy}^\delta(dr;t)dt$, with $\sum_{y\in\mathcal{X}}\eta_{xy}^\delta([0,\infty);t)=\mu_x^\delta(t)$, and for $\boldsymbol{q}^\delta$ defined by $\mu_x^\delta(t)q_{xy}^\delta(t)=\int_0^\infty r\eta_{xy}^\delta(dr;t)$, that

$$\boldsymbol{\mu}^\delta(t)=\boldsymbol{m}+\sum_{(x,y)\in\mathcal{Z}}\boldsymbol{v}_{xy}\int_0^{t\wedge T^\delta}\int_0^\infty r\theta_{xy}^\delta(drds)=\boldsymbol{m}+\sum_{(x,y)\in\mathcal{Z}}\boldsymbol{v}_{xy}\int_0^{t\wedge T^\delta}\mu_x^\delta(s)q_{xy}^\delta(s)ds.$$

Finally, by an application of Fatou's Lemma, for such a $\boldsymbol{q}^\delta$, we get

$$\begin{aligned}&\liminf_{n\to\infty}\mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{T^{n,\delta}}\int_0^\infty F_{xy}^n(r)\theta_{xy}^{n,\delta}(drdt)\right]\geq\mathbb{E}_{\boldsymbol{m}}\left[\int_0^{T^\delta}\int_0^\infty F_{xy}^\infty(r)\theta_{xy}^\delta(drdt)\right]\\ &\geq\mathbb{E}_{\boldsymbol{m}}\left[\int_0^{T^\delta}\int_0^\infty \hat{F}_{xy}^\infty(r)\theta_{xy}^\delta(drdt)\right]\geq\mathbb{E}_{\boldsymbol{m}}\left[\int_0^{T^\delta}\int_0^\infty \hat{F}_{xy}^\infty(r)\eta_{xy}^\delta(dr;t)dt\right]\\ &\geq\mathbb{E}_{\boldsymbol{m}}\left[\int_0^{T^\delta}\hat{F}_{xy}^\infty\left(\int_0^\infty r\frac{\eta_{xy}^\delta(dr;t)}{\eta_{xy}^\delta([0,\infty);t)}\right)\eta_{xy}^\delta([0,\infty);t)dt\right]=\mathbb{E}_{\boldsymbol{m}}\left[\int_0^{T^\delta}\hat{F}_{xy}^\infty(q_{xy}^\delta(t))\mu_x^\delta(t)dt\right],\end{aligned}$$

where in the second inequality, we used the fact that $\hat{F}_{xy}^\infty\leq F_{xy}^\infty$, and for the fourth, we applied Jensen's inequality. Together with $\boldsymbol{\mu}^{n,\delta}\to\boldsymbol{\mu}^\delta$ and another application of Fatou's Lemma, this gives (5.6).

5.1 Tightness of $(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\theta}^{n,\delta}, T^{n,\delta})$

First, we prove that $(\boldsymbol{\mu}^{n,\delta}(\cdot), T^{n,\delta})$, which takes values in $D([0,\infty);\mathcal{P}(\mathcal{X}))\times[0,\infty)\subset D([0,\infty);\mathbb{R}^d)\times[0,\infty)$, is tight. For that, we introduce some auxiliary random variables $\tilde{\boldsymbol{\mu}}^{n,\delta}$ in $D([0,\infty);\mathbb{R}^d)$, to compare with $\boldsymbol{\mu}^{n,\delta}$, given by

$$\tilde{\boldsymbol{\mu}}^{n,\delta}(t) = \boldsymbol{m}^n + \sum_{(x,y)\in\mathcal{Z}} \boldsymbol{v}_{xy}\int_0^{t\wedge T^{n,\delta}} \mu_x^n(s) q_{xy}^n(s)ds. \tag{5.9}$$

Since $\gamma_{xy}\ell\left(\cdot/\gamma_{xy}\right) \le F_{xy}^n(\cdot)$, recalling (5.1), (5.2) and that for sufficiently large n R^n is bounded away from zero in $K_\delta = \{\boldsymbol{m} : d(\boldsymbol{m},K)\ge\delta\}$ by a constant $R^\delta_{\min}$, we get

$$\mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{T^{n,\delta}}\left(\sum_{(x,y)\in\mathcal{Z}}\mu_x^n(t)\gamma_{xy}\ell\left(\frac{q_{xy}^n(t)}{\gamma_{xy}}\right)\right)dt + R^\delta_{\min}T^{n,\delta}\right] \le c+1, \tag{5.10}$$

which shows tightness of $\{T^{n,\delta}\}$. By setting $\gamma_{\max} = \max\{\gamma_{xy} : (x,y)\in\mathcal{Z}\}$, we get

$$\mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{T^{n,\delta}}\left(\sum_{(x,y)\in\mathcal{Z}}\gamma_{\max}\frac{\mu_x^n(t)\gamma_{xy}}{\gamma_{\max}}\ell\left(\frac{q_{xy}^n(t)}{\gamma_{xy}}\right)\right)dt + R^\delta_{\min}T^{n,\delta}\right] \le c+1.$$

Using the fact that ℓ is convex and $\ell(1)=0$, by Jensen's inequality $a\ell(b)\ge\ell(ab+1-a)$ for $a\in[0,1]$ and $b\ge0$. By setting $a=\frac{\mu_x^n(t)\gamma_{xy}}{\gamma_{\max}}$, the inequality above gives

$$\mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{T^{n,\delta}}\left(\sum_{(x,y)\in\mathcal{Z}}\gamma_{\max}\ell\left(\frac{\mu_x^n(t)}{\gamma_{\max}}q_{xy}^n(t)+1-\frac{(\mu_x^n(t)\gamma_{xy})}{\gamma_{\max}}\right)\right)dt + R^\delta_{\min}T^{n,\delta}\right] \le c+1.$$

By applying Jensen's inequality once more

$$\mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{T^{n,\delta}}|\mathcal{Z}|\gamma_{\max}\ell\left(\frac{1}{|\mathcal{Z}|\gamma_{\max}}\sum_{(x,y)}\mu_x^n(t)q_{xy}^n(t)+\sum_{(x,y)\in\mathcal{Z}}\left[1-\frac{(\mu_x^n(t)\gamma_{xy})}{|\mathcal{Z}|\gamma_{\max}}\right]\right)dt + R^\delta_{\min}T^{n,\delta}\right] \le c+1. \tag{5.11}$$

Now by multiplying with $\frac{1}{|\mathcal{Z}|\gamma_{\max}}$, using (5.9) and the fact that $q\le q'$ implies $\ell(q)\le\ell(q')+1$, we get

$$\mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{T^{n,\delta}}\ell\left(\frac{|\dot{\tilde{\boldsymbol{\mu}}}^{n,\delta}(t)|}{|\mathcal{Z}|\gamma_{\max}}\right)dt + \left(\frac{1}{|\mathcal{Z}|\gamma_{\max}}R^\delta_{\min}-1\right)T^{n,\delta}\right] \le \frac{c+1}{|\mathcal{Z}|\gamma_{\max}}. \tag{5.12}$$

Finally, by using that for every $c>0$ there exists $c_1>0, c_2<\infty$ such that $\ell(cq)\ge c_1\ell(q)-c_2$, we get

$$\mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{T^{n,\delta}}c_1\ell\left(|\dot{\tilde{\boldsymbol{\mu}}}^{n,\delta}(t)|\right)dt + \left(\frac{1}{|\mathcal{Z}|\gamma_{\max}}R^\delta_{\min}-1-c_2\right)T^{n,\delta}\right] \le \frac{c+1}{|\mathcal{Z}|\gamma_{\max}},$$

which implies

$$\mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{T^{n,\delta}}\ell\left(|\dot{\tilde{\boldsymbol{\mu}}}^{n,\delta}(t)|\right)dt + \frac{1}{|\mathcal{Z}|\gamma_{\max}c_1}R^\delta_{\min}T^{n,\delta}\right] \le \frac{c+1}{|\mathcal{Z}|\gamma_{\max}c_1}+\frac{(c_2+1)}{c_1}\mathbb{E}_{\boldsymbol{m}^n}[T^{n,\delta}] \le c',$$

where

$$c' = \frac{c+1}{|\mathcal{Z}|\gamma_{\max}c_1}+\frac{(c+1)(c_2+1)}{c_1}. \tag{5.13}$$

It will follow from the following lemma that $\tilde{\boldsymbol{\mu}}^{n,\delta}$ is a tight sequence in $D([0,\infty);\mathbb{R}^d)$. Let $\mathcal{S}$ be the elements $(\boldsymbol{\mu},T)$ of $C([0,\infty);\mathcal{P}(\mathcal{X}))\times[0,\infty)$ that satisfy $\boldsymbol{\mu}(t)=\boldsymbol{\mu}(T)$ for $t\ge T$.

Lemma 19. *For every positive number a, the function*

$$H(\boldsymbol{\mu},T)=\begin{cases}\int_0^T \ell\left(|\dot{\boldsymbol{\mu}}(t)|\right)dt+aT, & \boldsymbol{\mu}\in AC([0,\infty);\mathbb{R}^d), T\in[0,\infty)\\ \infty, & otherwise,\end{cases} \tag{5.14}$$

is a tightness function on $\mathcal{S}$, where $AC([0,\infty);\mathbb{R}^d)$ is the set of all absolutely continuous functions from $[0,\infty)$ to $\mathbb{R}^d$.

The proof of this lemma is in Appendix 8 (Is better if we use a reference of some sort).

Now we have that

$$|\boldsymbol{\mu}^{n,\delta}(t)-\tilde{\boldsymbol{\mu}}^{n,\delta}(t)|\leq\sum_{(x,y)\in\mathcal{Z}}\left|\int_0^{t\wedge T^{n,\delta}}\mu_x^n(s)q_{xy}^n(s)ds-\int_0^{t\wedge T^{n,\delta}}\int_0^\infty 1_{[0,\mu_x^n(s)q_{xy}^n(s)]}(r)\frac{1}{n}N_{xy}^n(dsdr)\right|,$$

where the summands on the right side, denoted from now on by $Q_{xy,t}^{n,\delta}$, are all martingales with quadratic variation $\mathbb{Q}_{xy,t}^{n,\delta}$ that is bounded above by

$$\frac{1}{n^2}\mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{t\wedge T^{n,\delta}}\int_0^\infty 1_{[0,\mu_x^n(s)q_{xy}^n(s)]}(r)N_{xy}^n(dsdr)\right]=\frac{1}{n}\mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{t\wedge T^{n,\delta}}\mu_x^n(s)q_{xy}^n(s)ds\right]$$
$$\leq\frac{1}{n}\mathbb{E}_{\boldsymbol{m}^n}\left[\int_0^{t\wedge T^{n,\delta}}(\ell(\mu_x^n(s)q_{xy}^n(s))+e)ds\right]\overset{(5.13)}{\leq}\frac{c'+e\mathbb{E}_{\boldsymbol{m}^n}[T^{n,\delta}\wedge t]}{n}\leq\frac{c'+e\mathbb{E}_{\boldsymbol{m}^n}[T^{n,\delta}]}{n}\overset{(5.10)}{\leq}\frac{\left(\frac{(c+1)e}{R_{\min}^\delta}+c'\right)}{n},$$

where in the first inequality of the last line, the estimate $ab\leq e^a+\ell(b)$, with $a=1, b=\mu_x^n(s)q_{xy}^n(s)$ was used. By using the Burkholder-Gundy-Davis inequality, for every $T\in(0,\infty)$

$$\mathbb{E}_{\boldsymbol{m}^n}\left[\sup_{t\in[0,T]}|Q_{xy,t}^{n,\delta}|\right]\leq c_{BGD}\mathbb{E}_{\boldsymbol{m}^n}[\mathbb{Q}_{xy}^{n,\delta}]_T^{1/2}\leq c_{BGD}\sqrt{\frac{\left(\frac{(c+1)e}{R_{\min}^\delta}+c'\right)}{n}}, \tag{5.15}$$

from which we get that $\mathbb{E}_{\boldsymbol{m}^n}[\sup_{t\in[0,T]}|Q_{xy,t}^{n,\delta}|]$ converges to zero as $n\to\infty$. By an application of Lemma 19, we have proved that $\{\tilde{\boldsymbol{\mu}}^{n,\delta}\}$ is tight in $D([0,\infty);\mathbb{R}^d)$ and that $\mathbb{E}_{\boldsymbol{m}^n}\left[d(\boldsymbol{\mu}^{n,\delta},\tilde{\boldsymbol{\mu}}^{n,\delta})\right]\to 0$. From this we conclude that $\{(\boldsymbol{\mu}^{n,\delta},T^{n,\delta})\}$ is tight as well.

To show that the variable $\boldsymbol{\theta}^{n,\delta}$ is tight, we combine (5.8) and (5.1), (5.2) to get

$$\mathbb{E}_{\boldsymbol{m}^n}\left[\sum_{(x,y)\in\mathcal{Z}}\int_0^{T^{n,\delta}}\int_0^\infty F_{xy}^n(r)\theta_{xy}^{n,\delta}(drdt)+\int_0^{T^{n,\delta}}R^n(\boldsymbol{\mu}^{n,\delta}(t))\right]<c+1.$$

Since, by part 1 of Lemma 4, we have $\gamma_{xy}\ell\left(\cdot/\gamma_{xy}\right)\leq F_{xy}^n(\cdot)$, and $q^{n,\delta}=\boldsymbol{\gamma}$ for $t>T^{n,\delta}$, we get

$$\mathbb{E}_{\boldsymbol{m}^n}\left[\sum_{(x,y)\in\mathcal{Z}}\int_0^\infty\int_0^\infty\gamma_{xy}\ell\left(\frac{r}{\gamma_{xy}}\right)\theta_{xy}^{n,\delta}(drdt)\right]=\mathbb{E}_{\boldsymbol{m}^n}\left[\sum_{(x,y)\in\mathcal{Z}}\int_0^{T^{n,\delta}}\int_0^\infty\gamma_{xy}\ell\left(\frac{r}{\gamma_{xy}}\right)\theta_{xy}^{n,\delta}(drdt)\right]<c+1.$$

Now by using the fact that

$$\tilde{H}(\theta)=\int_0^\infty\int_0^T\gamma_{xy}\ell\left(\frac{r}{\gamma_{xy}}\right)\theta(drdt),$$

is a tightness function on the space of measures on $[0,\infty)\times[0,T]$ with mass no greater than T, we conclude that for every $(x,y)\in\mathcal{Z}$, $\theta_{xy}^{n,\delta}$ is tight with the topology introduced in (5.7).

5.2 Distributional limits and and lower bound

From the previous two subsections we have that $(\boldsymbol{\mu}^{n,\delta}, \tilde{\boldsymbol{\mu}}^{n,\delta}, \boldsymbol{\theta}^{n,\delta}, T^{n,\delta})$, is tight. For proving the lower bound, we can assume without loss that the sequence has a distributional limit $(\boldsymbol{\mu}^{\delta}, \tilde{\boldsymbol{\mu}}^{\delta}, \boldsymbol{\theta}^{\delta}, T^{\delta})$. By using the Skorohod representation theorem we can also assume the sequence of variables is on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and that $(\boldsymbol{\mu}^{\delta}, \tilde{\boldsymbol{\mu}}^{\delta}, \boldsymbol{\theta}^{\delta}, T^{\delta})$ is an a.s. pointwise limit.

Consider any $\omega \in \Omega$ for which there is convergence. Since by the definition of $\theta^{n,\delta}$

$$\theta_{xy}^{n,\delta}([0,\infty) \times A) = \int_{A \cap [0,T^{n,\delta}]} \mu_x^{n,\delta}(t)dt, \; \forall A \in \mathcal{B}(\mathbb{R}),$$

for every continuity set A of $\theta_{xy}^{\delta}([0,\infty) \times \cdot)$ we have

$$\begin{aligned}
&\left|\theta_{xy}^{\delta}([0,\infty) \times A) - \int_{A \cap [0,T^{\delta}]} \mu_x^{\delta}(t)dt\right| \\
&\quad \le \left|\theta_{xy}^{\delta}([0,\infty) \times A) - \theta_{xy}^{n,\delta}([0,\infty) \times A)\right| + \left|\int_{A \cap [0,T^{n,\delta}]} \mu_x^{n,\delta}(t)dt - \int_{A \cap [0,T^{\delta}]} \mu_x^{\delta}(t)dt\right| \\
&\quad \le \left|\theta_{xy}^{\delta}([0,\infty) \times A) - \theta_{xy}^{n,\delta}([0,\infty) \times A)\right| + \left|\int_{A \cap [0,T^{\delta}]} \mu_x^{n,\delta}(t)dt - \int_{A \cap [0,T^{\delta}]} \mu_x^{\delta}(t)dt\right| \\
&\qquad + \left|\int_{A \cap [\min\{T^{n,\delta},T^{\delta}\},\max\{T^{\delta},T^{n,\delta}\}]} \boldsymbol{\mu}_x^{n,\delta}(t)dt\right| \\
&\quad \le \left|\theta_{xy}^{\delta}([0,\infty) \times A) - \theta_{xy}^{n,\delta}([0,\infty) \times A)\right| + d(\mu_x^{n,\delta}, \mu_x^{\delta}) + |T^{\delta} - T^{n,\delta}| \to 0.
\end{aligned}$$

Therefore for every continuity set A of $\theta_{xy}^{\delta}([0,\infty) \times \cdot)$

$$\theta_{xy}^{\delta}([0,\infty) \times A) = \int_{A \cap [0,T^{\delta}]} \mu_x^{\delta}(t)dt,$$

from which we conclude that for all $(x,y) \in \mathcal{Z}$, θ_{xy}^{δ} has the decomposition $\theta_{xy}^{\delta}(drdt) = \eta_{xy}^{\delta}(dr;t)dt$, with $\eta_{xy}^{\delta}([0,\infty);t) = \mu_x^{\delta}(t)$. Also, since $\int_0^\infty \int_0^\infty \ell(r)\theta_{xy}^{n,\delta}(drdt)$ is uniformly bounded and ℓ is superlinear, we have convergence of the first moments of the first marginal, i.e.,

$$\int_{\mathbb{R}} f(t) r \theta_{xy}^{n,\delta}(drdt) \to \int_{\mathbb{R}} f(t) r \theta_{xy}^{\delta}(dt), \quad \forall f \in C_b(\mathbb{R}).$$

Hence for $\boldsymbol{q}^{\delta}$ defined by $\mu_x^{\delta}(t) q_{xy}^{\delta}(t) = \int_0^\infty r \eta_{xy}^{\delta}(dr;t)$, we get that for all $(x,y) \in \mathcal{Z}$

$$\int_0^\infty f(t) \mu_x^{n,\delta}(t) q_{xy}^{n,\delta}(t) dt \to \int_0^\infty f(t) \mu_x^{\delta}(t) q_{xy}^{\delta}(t) dt, \quad \forall f \in C_b(\mathbb{R}). \tag{5.16}$$

Using the fact that $d(\boldsymbol{\mu}^{n,\delta}, \tilde{\boldsymbol{\mu}}^{n,\delta}) \to 0$ and (5.9), we get

$$\left|\boldsymbol{\mu}^{n,\delta}(t) - \boldsymbol{m}^n - \sum_{(x,y)\in\mathcal{Z}} \boldsymbol{v}_{xy} \int_0^{T^{n,\delta}\wedge t} \mu_x^{n,\delta}(s) q_{xy}^{n,\delta}(s) ds\right| = \left|\boldsymbol{\mu}^{n,\delta}(t) - \tilde{\boldsymbol{\mu}}^{n,\delta}(t)\right| \to 0, \tag{5.17}$$

for a.e. t. Applying (5.16) for suitable choices of f and using (5.17),

$$\boldsymbol{\mu}^{\delta}(t) = \boldsymbol{m} + \sum_{(x,y)\in\mathcal{Z}} \boldsymbol{v}_{xy} \int_0^{T^{\delta}\wedge t} \mu_x^{\delta}(s) q_{xy}^{\delta}(s) ds$$

for a.e. t, and since the left side is cadlag and the right side is continuous in the last display, equality holds for $t \ge 0$. We conclude that $\boldsymbol{q}^{\delta}$ is the control that generates $\boldsymbol{\mu}^{\delta}$, and we also already noticed that $\mu_x^{\delta}(t) q_{xy}^{\delta}(t) = \int_0^\infty r \eta_{xy}^{\delta}(dr;t)$. Finally, since $\boldsymbol{\mu}^{n,\delta}(T^{n,\delta}) \in K_{\delta}$ and $d(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\mu}^{\delta}) \to 0$, by continuity of $\boldsymbol{\mu}^{\delta}$ we get $\boldsymbol{\mu}^{\delta}(T^{\delta}) \in K_{\delta}$. As discussed below (5.8), this concludes the lower bound proof.

6 Upper bound

Before we proceed with the proof of the upper bound

$$\limsup_{n\to\infty} V_K^n(\boldsymbol{m}^n) \leq V_K(\boldsymbol{m}),$$

we establish some preliminary lemmas. In the following lemmas, we make use of $\mathcal{T}_{\boldsymbol{m}}, \mathcal{U}_{\boldsymbol{m}}$ and $\hat{F}^{\infty}_{xy}$, defined in (4.2), (4.3), and (4.7) respectively. For the properties of $\hat{F}^{\infty}_{xy}$, see Lemma 4.

Lemma 20. *Let $\boldsymbol{m} \in \mathcal{P}_*(\mathcal{X})$, and $\boldsymbol{q} \in \mathcal{U}_{\boldsymbol{m}}$ be such that $(\boldsymbol{\mu}, \boldsymbol{q}) \in \mathcal{T}_{\boldsymbol{m}}$. Given $T < \infty$ and $\epsilon > 0$, we can find $a_1, a_2, a_3 > 0$ and $\tilde{\boldsymbol{q}} \in \mathcal{U}_m$, with $(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{q}}) \in \mathcal{T}_{\boldsymbol{m}}$, such that*

$$a_1 \leq \inf_{(x,y)\in\mathcal{Z}, t\in[0,T]} \tilde{q}_{xy}(t) \leq \sup_{(x,y)\in\mathcal{Z}, t\in[0,T]} \tilde{q}_{xy}(t) \leq a_2, \tag{6.1}$$

$$\inf_{x\in\mathcal{X}, t\in[0,T]} \tilde{\mu}_x(t) > a_3, \quad \sup_{t\in[0,T]} \|\boldsymbol{\mu}(t) - \tilde{\boldsymbol{\mu}}(t)\| < \epsilon, \tag{6.2}$$

and

$$\sum_{(x,y)\in\mathcal{Z}} \int_0^T \tilde{\mu}_x(t) \hat{F}^{\infty}_{xy}(\tilde{q}_{xy}(t)) dt \leq \sum_{(x,y)\in\mathcal{Z}} \int_0^T \mu_x(t) \hat{F}^{\infty}_{xy}(q_{xy}(t)) dt. \tag{6.3}$$

Proof. Recall that $\boldsymbol{m} \in \mathcal{P}_*(\mathcal{X})$ implies $\boldsymbol{m}_x > 0$ for all $x \in \mathcal{X}$. Let $\boldsymbol{\nu}(\boldsymbol{m}, t)$ be the solution to the equation $\dot{\boldsymbol{\nu}}(t) = \boldsymbol{\gamma}\boldsymbol{\nu}(t)$, with initial data m. By Remark 17, we know that there exists $1 \geq a > 0$ such that $\boldsymbol{\nu}(\boldsymbol{m}, t) \in \mathcal{P}_a(\mathcal{X})$, for every $t \in [0, T]$. For $\frac{\epsilon}{2} \geq \delta > 0$, let

$$\boldsymbol{\mu}^{\delta}(\cdot) \doteq \delta\nu(\boldsymbol{m}, \cdot) + (1-\delta)\boldsymbol{\mu}(\cdot), \tag{6.4}$$

and note that $\boldsymbol{\mu}^{\delta}_x(t) > 0$ for every $t \in [0, T]$ and $x \in \mathcal{X}$. Therefore, for δ as above and $(x, y) \in \mathcal{Z}$, we can define

$$q^{\delta}_{xy}(\cdot) = \gamma_{xy} \frac{\delta\nu_x(\boldsymbol{m}, \cdot)}{\mu^{\delta}_x(\cdot)} + q_{xy}(\cdot) \frac{(1-\delta)\mu_x(\cdot)}{\mu^{\delta}_x(\cdot)}. \tag{6.5}$$

Then it is straightforward to check that $(\boldsymbol{\mu}^{\delta}, \boldsymbol{q}^{\delta}) \in \mathcal{T}_{\boldsymbol{m}}$. Moreover, since $\frac{\delta\nu_x(\boldsymbol{m},t)}{\mu^{\delta}_x(t)} + \frac{(1-\delta)\mu_x(t)}{\mu^{\delta}_x(t)} = 1$ for all $t \in [0, T]$, by the convexity of $\hat{F}^{\infty}$ we obtain

$$\begin{aligned}
\sum_{(x,y)\in\mathcal{Z}} \int_0^T \mu^{\delta}_x(t) \hat{F}^{\infty}_{xy}\left(q^{\delta}_{xy}(t)\right) dt &= \sum_{(x,y)\in\mathcal{Z}} \int_0^T \mu^{\delta}_x(t) \hat{F}^{\infty}_{xy} \left(\gamma_{xy} \frac{\delta\nu_x(\boldsymbol{m}, t)}{\mu^{\delta}_x(t)} + q_{xy} \frac{(1-\delta)\mu_x(t)}{\mu^{\delta}_x(t)}\right) dt \\
&\leq \sum_{(x,y)\in\mathcal{Z}} \int_0^T \mu^{\delta}_x(t) \frac{\delta\nu_x(\boldsymbol{m}, t)}{\mu^{\delta}_x(t)} \hat{F}^{\infty}_{xy}\left(\gamma_{xy}\right) dt + \sum_{(x,y)\in\mathcal{Z}} \int_0^T \mu^{\delta}_x(t) \frac{(1-\delta)\mu_x(t)}{\mu^{\delta}_x(t)} \hat{F}^{\infty}_{xy}\left(q_{xy}(t)\right) dt \\
&\leq (1-\delta) \sum_{(x,y)\in\mathcal{Z}} \int_0^T \mu_x(t) \hat{F}^{\infty}_{xy}\left(q_{xy}(t)\right) dt,
\end{aligned} \tag{6.6}$$

where in the second inequality, we used the fact that $\hat{F}^{\infty}(\gamma_{xy}) = 0$ [see Lemma 4]. Therefore, we get a triplet $(\boldsymbol{\mu}^{\delta}, \boldsymbol{q}^{\delta}) \in \mathcal{T}_{\boldsymbol{m}}$ with cost strictly less than the initial one, and with $\boldsymbol{\mu}^{\delta}$ that satisfies

$$\mu^{\delta}_x(t) \geq \delta a \quad \text{and} \quad \frac{(1-\delta)\mu_x(t)}{\mu^{\delta}_x(t)} \leq \frac{(1-\delta)}{\delta a + (1-\delta)} \equiv c < 1, \tag{6.7}$$

for all $t \in [0, T]$.

However, since this triplet does not necessarily satisfy condition (6.1), we modify it even further. Specifically, we pick $M \in (2\gamma_{\max}, \infty)$ big enough such that

$$\sum_{(x,y)\in\mathcal{Z}} \int_0^T \mu^{\delta}_x(t) \left|\min\left\{q^{\delta}_{xy}(t), M\right\} - q^{\delta}_{xy}(t)\right| dt \leq \frac{a\delta(1-\sqrt{c})}{\sqrt{2}}, \tag{6.8}$$

and define

$$\boldsymbol{\mu}^{\delta,M}(t) = \int_0^t \sum_{(x,y)\in\mathcal{Z}} \mu_x^\delta(t) \min\left\{q_{xy}^\delta(t), M\right\} \boldsymbol{v}_{xy} dt. \tag{6.9}$$

Then

$$\begin{aligned}\left|\mu_x^{\delta,M}(t) - \mu_x^\delta(t)\right| \le \left\|\boldsymbol{\mu}^{\delta,M}(t) - \boldsymbol{\mu}^\delta(t)\right\| &\overset{(6.9)}{=} \left\| \sum_{(x,y)\in\mathcal{Z}} \int_0^T \left(\mu_x^\delta(t)\left(q_{xy}^\delta(t) - \min\left\{q_{xy}^\delta(t), M\right\}\right)\right) \boldsymbol{v}_{xy} dt \right\| \\ &\le \sum_{(x,y)\in\mathcal{Z}} \int_0^T \left|\mu_x^\delta(t)\left(q_{xy}^\delta(t) - \min\left\{q_{xy}^\delta(t), M\right\}\right)\right| \|\boldsymbol{v}_{xy}\| dt \\ &\le \sqrt{2} \sum_{(x,y)\in\mathcal{Z}} \int_0^T \left(\mu_x^{\delta,M}(t)\left(q_{xy}^\delta(t) - \min\left\{q_{xy}^\delta(t), M\right\}\right)\right) dt \overset{(6.8)}{\le} a\delta(1-\sqrt{c}),\end{aligned} \tag{6.10}$$

and for $t \in [0,T]$,

$$\mu_x^{\delta,M}(t) \ge \mu_x^\delta(t) - \left|\mu_x^{\delta,M}(t) - \mu_x^\delta(t)\right| \overset{(6.7)}{\ge} a\delta - \left|\mu_x^{\delta,M}(t) - \mu_x^\delta(t)\right| \overset{(6.10)}{\ge} a\delta\sqrt{c}. \tag{6.11}$$

We also get

$$\left|1 - \frac{\mu_x^{\delta,M}(t)}{\mu_x^\delta(t)}\right| \overset{(6.10)}{\le} \frac{a\delta(1-\sqrt{c})}{\min_x \mu_x^\delta} \overset{(6.7)}{\le} (1-\sqrt{c}) \tag{6.12}$$

or

$$\frac{\mu_x^\delta(t)}{\mu_x^{\delta,M}(t)} \ge \frac{1}{2-\sqrt{c}} \qquad \text{and} \qquad \frac{\mu_x^\delta(t)}{\mu_x^{\delta,M}(t)} \le \frac{1}{\sqrt{c}} = \frac{\sqrt{c}}{c}. \tag{6.13}$$

We deduce that $\boldsymbol{\mu}^{\delta,M}(t) \in \mathcal{P}_*(\mathcal{X})$, for all $t \in [0,T]$, and therefore can define

$$q_{xy}^{\delta,M}(t) = \frac{\min\left\{q_{xy}^\delta(t), M\right\} \mu_x^\delta(t)}{\mu_x^{\delta,M}(t)}, \tag{6.14}$$

which will give $(\boldsymbol{\mu}^{\delta,M}, \boldsymbol{q}^{\delta,M}) \in \mathcal{T}_{\boldsymbol{m}}$. We can see that (6.1) is satisfied, since by (6.5) and the LHS of (6.13) for the bound from below and the RHS of (6.13) for the bound from above we have

$$\frac{\gamma_{xy}\delta\nu_x(\boldsymbol{m},\cdot)}{2} \le q_{xy}^{\delta,M}(t) \le M\frac{\sqrt{c}}{c} \tag{6.15}$$

It is worth mentioning at this point that trying to get an estimate for the cost of $(\boldsymbol{\mu}^{\delta,M}, \boldsymbol{q}^{\delta,M})$, with respect to the cost of $(\boldsymbol{\mu}^\delta, \boldsymbol{q}^\delta)$, would require some extra properties of $\hat{F}^\infty$. However, we can obtain an estimate of the cost $(\boldsymbol{\mu}^{\delta,M}, \boldsymbol{q}^{\delta,M})$ with respect to the cost of the initial triplet $(\boldsymbol{\mu}, \boldsymbol{q})$, by utilizing only the convexity of $\hat{F}_{xy}^\infty$, and choosing the right parameters. Using the fact that $\hat{F}_{xy}^\infty$ is increasing on $[\gamma_{xy}, \infty)$ in the first inequality,

$$\begin{aligned}\hat{F}_{xy}^\infty\left(q_{xy}^{\delta,M}(t)\right) &\overset{(6.14)}{=} \hat{F}_{xy}^\infty\left(\frac{\min\left\{q_{xy}^\delta(t), M\right\}\mu_x^\delta(t)}{\mu_x^{\delta,M}(t)}\right) \le \hat{F}_{xy}^\infty\left(\frac{q_{xy}^\delta(t)\mu_x^\delta(t)}{\mu_x^{\delta,M}(t)}\right) \\ &\overset{(6.5)}{=} \hat{F}_{xy}^\infty\left(\frac{\mu_x^\delta(t)}{\mu_x^{\delta,M}(t)}\left(\gamma_{xy}\frac{\delta\nu_x(\boldsymbol{m},t)}{\mu_x^\delta(t)} + q_{xy}(t)\frac{(1-\delta)\mu_x(t)}{\mu_x^\delta(t)}\right)\right) \\ &= \hat{F}_{xy}^\infty\left(\gamma_{xy}\frac{\delta\nu_x(\boldsymbol{m},t)}{\mu_x^{\delta,M}(t)} + q_{xy}(t)\frac{(1-\delta)\mu_x(t)}{\mu_x^{\delta,M}(t)}\right).\end{aligned} \tag{6.16}$$

However, from (6.7) and (6.13), we have

$$\frac{(1-\delta)\mu_x(t)}{\mu_x^{\delta,M}(t)} = \frac{(1-\delta)\mu_x(t)}{\mu_x^\delta(t)}\frac{\mu_x^\delta(t)}{\mu_x^{\delta,M}(t)} \le c\frac{\sqrt{c}}{c} = \sqrt{c} < 1.$$

Therefore using the convexity of $\hat{F}^{\infty}$ we have

$$\begin{aligned}
&\hat{F}^{\infty}_{xy}\left(\gamma_{xy}\frac{\delta\nu_x(\boldsymbol{m},t)}{\mu_x^{\delta,M}(t)}+q_{xy}(t)\frac{(1-\delta)\mu_x(t)}{\mu_x^{\delta,M}(t)}\right)\\
&=\hat{F}^{\infty}_{xy}\left(\frac{\left(1-\frac{(1-\delta)\mu_x(t)}{\mu_x^{\delta,M}(t)}\right)}{\left(1-\frac{(1-\delta)\mu_x(t)}{\mu_x^{\delta,M}(t)}\right)}\gamma_{xy}\frac{\delta\nu_x(\boldsymbol{m},t)}{\mu_x^{\delta,M}(t)}+q_{xy}(t)\frac{(1-\delta)\mu_x(t)}{\mu_x^{\delta,M}(t)}\right) \qquad (6.17)\\
&\leq\left(1-\frac{(1-\delta)\mu_x(t)}{\mu_x^{\delta,M}(t)}\right)\hat{F}^{\infty}_{xy}\left(\gamma_{xy}\frac{\delta\nu_x(\boldsymbol{m},t)}{\mu_x^{\delta,M}(t)-(1-\delta)\mu_x(t)}\right)+\frac{(1-\delta)\mu_x(t)}{\mu_x^{\delta,M}(t)}\hat{F}^{\infty}_{xy}\left(q_{xy}(t)\right).
\end{aligned}$$

Combining (6.16) and (6.17) and then using (6.4), we obtain

$$\begin{aligned}
&\mu_x^{\delta,M}(t)\hat{F}^{\infty}_{xy}\left(q_{xy}^{\delta,M}(t)\right)\\
&\leq\left(\mu_x^{\delta,M}(t)-(1-\delta)\mu_x(t)\right)\hat{F}^{\infty}_{xy}\left(\gamma_{xy}\frac{\delta\nu_x(\boldsymbol{m},t)}{\mu_x^{\delta,M}(t)-(1-\delta)\mu_x(t)}\right)+(1-\delta)\mu_x(t)\hat{F}^{\infty}_{xy}\left(q_{xy}(t)\right)\\
&=\left(\mu_x^{\delta,M}(t)-\mu_x^{\delta}(t)+\delta\nu_x(M,t)\right)\hat{F}^{\infty}_{xy}\left(\gamma_{xy}\frac{\delta\nu_x(\boldsymbol{m},t)}{\mu_x^{\delta,M}(t)-\mu_x^{\delta}(t)+\delta\nu_x(\boldsymbol{m},t)}\right)+(1-\delta)\mu_x(t)\hat{F}^{\infty}_{xy}\left(q_{xy}(t)\right).
\end{aligned} \qquad (6.18)$$

We can make $|\mu_x^{\delta,M}(t)-\mu_x^{\delta}(t)|$ uniformly as close to zero as desired and therefore we can make $\gamma_{xy}\frac{\delta\nu_x(\boldsymbol{m},t)}{\mu_x^{\delta,M}(t)-\mu_x^{\delta}(t)+\delta\nu_x(\boldsymbol{m},t)}$ as close to γ_{xy} as desired by picking M sufficiently large. Since $\hat{F}^{\infty}_{xy}(\gamma_{xy})=0$ and $\hat{F}^{\infty}_{xy}(\cdot)$ is continuous on $(0,\infty)$ by Lemma 4, we can pick $M<\infty$ such that for every $t\in[0,T]$,

$$\hat{F}^{\infty}_{xy}\left(\gamma_{xy}\frac{\delta\nu_x(\boldsymbol{m},t)}{\mu_x^{\delta,M}(t)-\mu_x^{\delta}(t)+\delta\nu_x(\boldsymbol{m},t)}\right)\leq\frac{1}{2T}\int_0^T\mu_x(s)\hat{F}^{\infty}_{xy}(q_{xy}(s))ds. \qquad (6.19)$$

Then from (6.18) and (6.19) and the fact that $\nu_x(m,t)\leq 1$ and (6.10), for $t\in[0,T]$

$$\begin{aligned}
&\sum_{(x,y)\in\mathcal{Z}}\int_0^T\mu_x^{\delta,M}(t)\hat{F}^{\infty}_{xy}\left(q_{xy}^{\delta,M}(t)\right)dt\\
&\leq\sum_{(x,y)\in\mathcal{Z}}\left(\int_0^T(2\delta)\left(\frac{1}{2T}\int_0^T\mu_x(s)\hat{F}^{\infty}_{xy}(q_{xy}(s))ds\right)dt+\int_0^T(1-\delta)\mu_x(t)\hat{F}^{\infty}_{xy}\left(q_{xy}(t)\right)dt\right) \qquad (6.20)\\
&=\sum_{(x,y)\in\mathcal{Z}}\int_0^T\mu_x(t)\hat{F}^{\infty}_{xy}\left(q_{xy}(t)\right)dt.
\end{aligned}$$

□

Next, we are going to prove the following result.

Lemma 21 (Law of large numbers)**.** *Let $T\in(0,\infty)$ be given. There exists a constant $c<\infty$ such that if $(\boldsymbol{\mu}^n,\boldsymbol{\gamma})\in\mathcal{T}^n_{\boldsymbol{m}}$ (see (2.9)), and $(\boldsymbol{\nu},\boldsymbol{\gamma})\in\mathcal{T}_{\boldsymbol{m}}$, then*

$$\mathbb{P}\left(\sup_{t\in[0,T]}\|\boldsymbol{\mu}^n(t)-\boldsymbol{\nu}(\boldsymbol{m},t)\|\geq\epsilon\right)\leq\frac{c}{\epsilon\sqrt{n}}. \qquad (6.21)$$

Proof. We have

$$\begin{aligned}
\|\boldsymbol{\mu}^n(t)-\boldsymbol{\nu}(\boldsymbol{m},t)\|&\leq\sum_{(x,y)}\left|\int_0^t\int_0^\infty 1_{[0,\mu_x^n(s)\gamma_{xy}]}(r)\frac{1}{n}N^n_{xy}(dsdr)-\int_0^t\int_0^\infty 1_{[0,\nu_x(m,s)\gamma_{xy}]}(r)dsdr\right|\\
&\leq\sum_{(x,y)}\left|\int_0^t\int_0^\infty 1_{[0,\mu_x^n(s)\gamma_{xy}]}(r)\frac{1}{n}N^n_{xy}(dsdr)-\int_0^t\int_0^\infty 1_{[0,\mu_x^n(s)\gamma_{xy}]}(r)dsdr\right|\\
&\quad+\sum_{(x,y)}\left|\int_0^t\int_0^\infty 1_{[0,\mu_x^n(s)\gamma_{xy}]}(r)dsdr-\int_0^t\int_0^\infty 1_{[0,\nu_x(m,s)\gamma_{xy}]}(r)ds\right|.
\end{aligned}$$

For a constant K that depends on d and the maximum of γ_{xy},

$$\sum_{(x,y)} \left| \int_0^t \int_0^\infty 1_{[0,\mu_x^n(s)\gamma_{xy}]}(r) ds dr - \int_0^t \int_0^\infty 1_{[0,\nu_x(m,s)\gamma_{xy}]}(r) ds \right| \leq K \sup_{0 \leq s \leq t} \|\mu^n(s) - \nu(m,s)\| .$$

Hence by Gronwall's inequality, for $r \in [0,T]$

$$\|\boldsymbol{\mu}^n(t) - \boldsymbol{\nu}(\boldsymbol{m},t)\| \leq e^{KT} \sup_{0 \leq t \leq r} \sum_{(x,y)} \left| \int_0^t \int_0^\infty 1_{[0,\mu_x^n(s)\gamma_{xy}]}(r) \frac{1}{n} N_{xy}^n(dsdr) - \int_0^t \int_0^\infty 1_{[0,\mu_x^n(s)\gamma_{xy}]}(r) dsdr \right| .$$

Using the Burkholder-Gundy-Davis inequality as was done to obtain (5.15),

$$P\left(\sup_{t \in [0,T]} \left| \int_0^t \int_0^\infty 1_{[0,\mu_x^n(s)\gamma_{xy}]}(r) \frac{1}{n} N_{xy}^n(dsdr) - \int_0^t \int_0^\infty 1_{[0,\mu_x^n(s)\gamma_{xy}]}(r) dsdr \right| \geq \epsilon \right) \leq \frac{\bar{c}}{\epsilon \sqrt{n}},$$

and hence

$$P\left(\sup_{t \in [0,T]} \|\boldsymbol{\mu}^n(t) - \boldsymbol{\nu}(\boldsymbol{m},t)\| \geq \epsilon \right) \leq d^2 \frac{e^{KT}\bar{c}}{\epsilon \sqrt{n}},$$

which is (6.21).

□

We now obtain the following result.

Lemma 22. *The sequence $V^n(\boldsymbol{m})$ bounded, uniformly in n and $\boldsymbol{m} \in \mathcal{P}_a(X)$.*

Proof. Let $\tau = diameter(\mathcal{P}(\mathcal{X}))$. By Remark 17, there exists $a > 0$ such that $\boldsymbol{\nu}(\boldsymbol{m},\tau) \in \mathcal{P}_{2a}(\mathcal{X})$ regardless of the initial data $\boldsymbol{m}$. We can further assume that $\mathcal{P}_a(\mathcal{X}) \cap K^\circ \neq \emptyset$, and in particular that there exists an element $\tilde{\boldsymbol{m}}$ such that $B(\tilde{\boldsymbol{m}}, \frac{a}{2}) \subset \mathcal{P}_a(\mathcal{X}) \cap K^\circ$.

Since $\tilde{\boldsymbol{m}} \in \mathcal{P}_a$, the first part of Theorem 18 implies that for every point $\boldsymbol{m}$ in $\mathcal{P}_a(X)$ we can find a control $\boldsymbol{q_m}$ with the following properties: there is a unique $\boldsymbol{\mu}$ such that $(\boldsymbol{\mu}, \boldsymbol{q_m}) \in \mathcal{T}_{boldsymbolm}$; $\boldsymbol{\mu}$ is a constant speed parametrization of the straight line that connects $\boldsymbol{m}$ to $\tilde{\boldsymbol{m}}$ in time $T_{\{\tilde{\boldsymbol{m}}\}} = \|\boldsymbol{m} - \tilde{\boldsymbol{m}}\|$; and the control $\boldsymbol{q_m}$ satisfies

$$\gamma_{xy} \leq q_{\boldsymbol{m},xy}(t) \leq \frac{c_1}{a}, \tag{6.22}$$

for $t \in [0, T_{\{\tilde{\boldsymbol{m}}\}}]$, $(x,y) \in \mathcal{Z}$, where $c_1 > 0$ is a constant that does not depend on a. For every $\boldsymbol{m}$, we let

$$q_{xy}(\boldsymbol{m},t) = \begin{cases} q_{\boldsymbol{m},xy}(t) & t \leq \|\boldsymbol{m} - \tilde{\boldsymbol{m}}\|, \\ \gamma_{xy} & t > \|\boldsymbol{m} - \tilde{\boldsymbol{m}}\|, \end{cases} \tag{6.23}$$

denote the control that takes $\boldsymbol{m}$ to $\tilde{\boldsymbol{m}}$ in time $\|\boldsymbol{m} - \tilde{\boldsymbol{m}}\|$, in the sense that it was described above, and after that time is equal to the original rates.

For $i \in \mathbb{N}$ we define a control for the interval $i\tau \leq t < (i+1)\tau$ as follows. Let $f(t-)$ denote the limit of $f(s)$ from the left at time t, and recall that $\boldsymbol{\mu}(\boldsymbol{m}, \cdot)$ is the straight line that connects $\boldsymbol{m}$ to $\tilde{\boldsymbol{m}}$ in time $T_{\{\tilde{\boldsymbol{m}}\}}$, where $\tilde{\boldsymbol{m}}$ is fixed and we explicitly indicate the dependence on $\boldsymbol{m}$. Then set

$$q_{xy}^n(t) = \begin{cases} q_{xy}(\boldsymbol{m}, t - i\tau) \frac{\mu_x^n(t-)}{\mu_x(\boldsymbol{m}, t - i\tau)}, & \text{if } \left(\sup_{s \in [i\tau, t]} \|\boldsymbol{\mu}(\boldsymbol{m},t) - \boldsymbol{\mu}^n(t)\| \leq \frac{a}{2} \right) \text{ and } (\boldsymbol{\mu}^n(i\tau) = \boldsymbol{m} \in \mathcal{P}_a(\mathcal{X})) \\ \gamma_{xy}, & \text{otherwise.} \end{cases} \tag{6.24}$$

The idea with these controls is that, within each time interval with length τ, the control considers the starting point $\boldsymbol{m}$, and then attempts to force the process to follow the straight line to $\tilde{\boldsymbol{m}}$. If the process is very close to the boundary of the simplex $\mathcal{P}(\mathcal{X}) \setminus \mathcal{P}_*(\mathcal{X})$, then we just use original rates to push the process inside $\mathcal{P}_a(X)$. Since all controls used are bounded from above and below, the total cost is a multiple of $\mathbb{E}[T^n]$. Thus we need only show this expected exit time is uniformly bounded.

By using (5.15), we can find constant $c < \infty$ such that

$$\mathbb{P}\left(\sup_{t \in [i\tau, (i+1)\tau]} \|\boldsymbol{\mu}^n(t) - \boldsymbol{\mu}(\boldsymbol{m},t)\| \geq \frac{a}{2} \,\middle|\, \boldsymbol{\mu}^n(i\tau) = \boldsymbol{m} \in \mathcal{P}_a(\mathcal{X}) \right) \leq c \frac{2}{\sqrt{n}a}, \tag{6.25}$$

from which we get

$$P(T^n > (i+1)\tau | \boldsymbol{\mu}^n(i\tau) \in \mathcal{P}_a(X)) \le \inf_{\boldsymbol{m}\in\mathcal{P}_a(\mathcal{X})} \mathbb{P}\left(\sup_{t\in[i\tau,(i+1)\tau]} \|\boldsymbol{\mu}^n(t) - \boldsymbol{\mu}(\boldsymbol{m},t)\| \ge \frac{a}{2} \,\middle|\, \boldsymbol{\mu}^n(i\tau) = \boldsymbol{m} \right) \le c\frac{2}{\sqrt{n}a}. \tag{6.26}$$

By Lemma 21, we have that for some $c' < \infty$

$$\mathbb{P}\left(\sup_{t\in[i\tau,(i+1)\tau]} \|\boldsymbol{\mu}^n(t) - \boldsymbol{\nu}(\boldsymbol{m},t)\| \ge \frac{a}{2} \,\middle|\, \boldsymbol{\mu}^n(i\tau) = \boldsymbol{m} \notin \mathcal{P}_a(\mathcal{X}) \right) \le c'\frac{2}{a\sqrt{n}}, \tag{6.27}$$

which implies that

$$\begin{aligned} &\mathbb{P}\left(\boldsymbol{\mu}^n((i+1)\tau) \notin \mathcal{P}_a(\mathcal{X}) \,\middle|\, \boldsymbol{\mu}^n(i\tau) \notin \mathcal{P}_a(\mathcal{X}) \right) \\ &\le \inf_{\boldsymbol{m}\notin\mathcal{P}_a(\mathcal{X})} \mathbb{P}\left(\sup_{t\in[i\tau,(i+1)\tau]} \|\boldsymbol{\mu}^n(t) - \boldsymbol{\nu}(\boldsymbol{m},t)\| \ge \frac{a}{2} \,\middle|\, \boldsymbol{\mu}^n(i\tau) = \boldsymbol{m} \right) \le c'\frac{2}{a\sqrt{n}}. \end{aligned} \tag{6.28}$$

Thus the probability to escape in the next 2τ units of time has a positive lower bound that is independent of n and the starting position. This implies the uniform upper bound on the mean escape time. ☐

Now we proceed with the proof of the upper bound.

Proof of upper bound. We will initially assume that $\boldsymbol{m}$ is in $\mathcal{P}_a(\mathcal{X})$, for some $a > 0$. Let $\epsilon > 0$. By the definition of $V_K(\boldsymbol{m})$, we can find a triplet $(\boldsymbol{\mu}, \boldsymbol{q}) \in \mathcal{T}_{\boldsymbol{m}}$ and a $T \in [0,\infty]$, such that

$$\int_0^T \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x(t)\hat{F}^{\infty}_{xy}\left(q_{xy}(t)\right) + R^{\infty}(\boldsymbol{\mu}(t)) \right) dt \le V_K(\boldsymbol{m}) + \epsilon. \tag{6.29}$$

Since we assumed that R^{∞} is bounded from below by a positive constant for every compact subset of K^c, we can furthermore find a δ such that for finite time $T^{\delta} \in [0,\infty)$ we have

$$\int_0^{T^{\delta}} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x(t)\hat{F}^{\infty}_{xy}\left(q_{xy}(t)\right) + R^{\infty}(\boldsymbol{\mu}(t)) \right) dt \le V_K(\boldsymbol{m}) + \epsilon, \tag{6.30}$$

and $d(\boldsymbol{\mu}(T^{\delta}), K) \le \delta$. By the second part of Theorem 18, we can extend the path so it can reach a point $\tilde{\boldsymbol{m}}$ of K, with extra cost less than ϵ. Since $K = \overline{(K^{\circ})}$, by a second application of Theorem 18, we can assume that $\tilde{\boldsymbol{m}}$ is an internal point of K, by again adding an extra cost less than ϵ.

Let $r > 0$ be such that $B(\tilde{\boldsymbol{m}}, r) \subset K^{\circ}$. From Lemma 20, without any loss of generality, we can assume that there exist $a_1, a_2, a_3 > 0$ such that

$$a_1 \le \inf_{(x,y)\in\mathcal{Z}, t\in[0,T]} q_{xy}(t) \le \sup_{(x,y)\in\mathcal{Z}, t\in[0,T]} q_{xy}(t) \le a_2, \tag{6.31}$$

and

$$\inf_{x\in\mathcal{X}, t\in[0,T]} \mu_x(t) > a_3, \;\; \|\boldsymbol{\mu}(T) - \tilde{\boldsymbol{m}}\| < \frac{r}{2}. \tag{6.32}$$

Finally, by applying the first part of Theorem 18, we can assume the existence of a $r_1 > 0$ such that for every point $\bar{\boldsymbol{m}}$ in the neighborhood $B(\boldsymbol{m}, r_1)$, we can find a path like the one described above, by connecting $\bar{\boldsymbol{m}}$ with a straight line to $\boldsymbol{m}$. Of course this could generate different, though universal, a_1, a_2, a_3 from the initial ones.

Now let $\boldsymbol{m}^n$ be a sequence that converges to $\boldsymbol{m}$... For big enough n, we can assume that $\boldsymbol{m}^n \in B(\boldsymbol{m}, r_1)$. By the continuity of $\hat{F}^\infty$ on compact subsets of $(0,\infty)$, we can find $r_2 > 0$ such that if $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathcal{P}_{\frac{a_3}{2}}(\mathcal{X})$ and $\|\boldsymbol{m}_1 - \boldsymbol{m}_2\| \leq r_2$, then for every $\boldsymbol{q}$ that satisfies (6.31), we have

$$\sum_{(x,y)\in\mathcal{Z}} \left| m_{1,x}\hat{F}^\infty_{xy}(q_{xy}) - m_{2,x}\hat{F}^\infty_{xy}\left(q_{xy}\frac{m_{1,x}}{m_{2,x}}\right)\right| \leq \frac{\epsilon}{T}. \tag{6.33}$$

Now for every $n \in \mathbb{N}$, we define the following control for the time interval $[0,T]$,

$$q^n_{xy}(t) = \begin{cases} q_{xy}(t)\frac{\mu^n_x(t-)}{\mu_x(t)}, & \text{if } \sup_{s\in[0,t]} \|\boldsymbol{\mu}(t) - \boldsymbol{\mu}^n(t)\| \leq r_2 \\ \gamma_{xy}, & \text{otherwise.} \end{cases} \tag{6.34}$$

For every n, we define an auxiliary stopping time $\overline{T}^n = \inf\{t \in [0,T] : \|\boldsymbol{\mu}^n(t) - \boldsymbol{\mu}(t)\| > r_2\}$, and also we define $R_{max} = \sup_{n\in\mathbb{N}, \boldsymbol{m}\in\mathcal{P}(\mathcal{X})} R^n(\boldsymbol{m})$. For sufficiently large n, by uniform convergence of F^n to $\hat{F}^\infty$ on compact subsets of $(0,\infty)$, and the uniform convergence of R^n to R^∞, we can get an estimate of the cost accumulated up to time T, for the triple $(\boldsymbol{\mu}^n, \boldsymbol{q}^n) \in \mathcal{T}^n_{\boldsymbol{m}^n}$. Specifically,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left(\sum_{(x,y)\in\mathcal{Z}} \mu^n_x(t) F^n_{xy}\left(q^n_{xy}(t)\right) + R^n(\boldsymbol{\mu}^n(t))\right) dt\right] \\
&\leq \mathbb{E}\left[\int_0^T \left(\sum_{(x,y)\in\mathcal{Z}} \mu^n_x(t) F^n_{xy}\left(q^n_{xy}(t)\right) + R^n(\boldsymbol{\mu}^n(t))\right) dt \cdot 1_{\left\{|\sup_{t\in[0,T]}\|\boldsymbol{\mu}(t)-\boldsymbol{\mu}^n(t)\|\leq r_2\right\}}\right] \\
&+ \mathbb{P}\left(\sup_{t\in[0,T]} \|\boldsymbol{\mu}^n(t) - \boldsymbol{\mu}(t)\| > r_2\right) \times \\
&\left(\mathbb{E}\left[\int_0^{\overline{T}^n} \left(\sum_{(x,y)\in\mathcal{Z}} \mu^n_x(t) F^n_{xy}\left(q^n_{xy}(t)\right) + R^n(\boldsymbol{\mu}^n(t))\right) dt \middle| \sup_{t\in[0,T]} \|\boldsymbol{\mu}^n(t) - \boldsymbol{\mu}(t)\| > r_2\right] + TR_{max}\right) \\
&\overset{(6.34)}{=} \mathbb{E}\left[\int_0^T \left(\sum_{(x,y)\in\mathcal{Z}} \mu^n_x(t) F^n_{xy}\left(q_{xy}(t)\frac{\mu^n_x(t-)}{\mu_x(t)}\right) + R^n(\boldsymbol{\mu}^n(t))\right) dt \cdot 1_{\left\{|\sup_{t\in[0,T]}\|\boldsymbol{\mu}(t)-\boldsymbol{\mu}^n(t)\|\leq r_2\right\}}\right] \\
&+ \mathbb{P}\left(\sup_{t\in[0,T]} \|\boldsymbol{\mu}^n(t) - \boldsymbol{\mu}(t)\| > r_2\right) \times \\
&\left(\mathbb{E}\left[\int_0^{\overline{T}^n} \left(\sum_{(x,y)\in\mathcal{Z}} \mu^n_x(t) F^n_{xy}\left(q_{xy}(t)\frac{\mu^n_x(t-)}{\mu_x(t)}\right) + R^n(\boldsymbol{\mu}^n(t))\right) dt \middle| \sup_{t\in[0,T]} \|\boldsymbol{\mu}(t) - \boldsymbol{\mu}^n(t)\| > r_2\right] + TR_{max}\right) \\
&\leq \mathbb{E}\left[\int_0^T \left(\sum_{(x,y)\in\mathcal{Z}} \mu^n_x(t) \hat{F}^\infty_{xy}\left(q_{xy}(t)\frac{\mu^n_x(t-)}{\mu_x(t)}\right) + R^\infty(\boldsymbol{\mu}^n(t))\right) dt \cdot 1_{\left\{\sup_{t\in[0,T]}\|\boldsymbol{\mu}(t)-\boldsymbol{\mu}^n(t)\|\leq r_2\right\}}\right] + \epsilon \\
&+ \mathbb{P}\left(\sup_{t\in[0,T]} \|\boldsymbol{\mu}^n(t) - \boldsymbol{\mu}(t)\| > r_2\right) \times \\
&\left(\mathbb{E}\left[\int_0^{\overline{T}^n} \left(\sum_{(x,y)\in\mathcal{Z}} \mu^n_x(t) \hat{F}^\infty_{xy}\left(q_{xy}(t)\frac{\mu^n_x(t-)}{\mu_x(t)}\right) + R^\infty(\boldsymbol{\mu}^n(t))\right) dt \middle| \sup_{t\in[0,T]} \|\boldsymbol{\mu}(t) - \boldsymbol{\mu}^n(t)\| > r_2\right] + \epsilon + TR_{\max}\right).
\end{aligned} \tag{6.35}$$

Then using (6.33) with $m_{1,x} = \mu_x(t), m_{2,x} = \mu^n_x(t-)$, for big enough n we can bound

$$\mathbb{E}\left[\int_0^T \left(\sum_{(x,y)\in\mathcal{Z}} \mu^n_x(t) F^n_{xy}\left(q^n_{xy}(t)\right) + R^n(\boldsymbol{\mu}^n(t))\right) dt\right]$$

above by

$$V_K(\boldsymbol{m}) + 2\epsilon + \mathbb{P}\left(\sup_{t\in[0,T]} \|\boldsymbol{\mu}^n(t) - \boldsymbol{\mu}(t)\| > r_2\right)(V_K(\boldsymbol{m}) + TR_{\max} + 2\epsilon). \tag{6.36}$$

By using (5.15), the probability that there was no exit in the time interval $[0, T]$ is

$$\mathbb{P}(T^n \geq T) \leq \mathbb{P}\left(\sup_{t\in[0,T]} \|\boldsymbol{\mu}^n(t) - \boldsymbol{\mu}(t)\| > r_2\right) \leq c\frac{1}{\sqrt{n}r_2}.$$

Hence the total cost satisfies

$$\begin{aligned}
V_K^n(\boldsymbol{m}^n) &\leq \mathbb{E}\left[\int_0^T \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x^n(t) F_{xy}^n\left(q_{xy}^n(t)\right) + R^n(\boldsymbol{\mu}^n(t))\right) dt + V(\boldsymbol{\mu}^n(T\wedge T^n))\right] \\
&\leq V_K(\boldsymbol{m}) + 2\epsilon + \mathbb{P}\left(\sup_{t\in[0,T]} \|\boldsymbol{\mu}^n(t) - \boldsymbol{\mu}(t)\| > r_2\right)(V_K(\boldsymbol{m}) + TR_{\max} + 2\epsilon) + \mathbb{P}(T^n \geq T)V_{max} \\
&\leq V_K(\boldsymbol{m}) + 2\epsilon + 2(TR_{\max} + V_{\max} + 2\epsilon)\frac{c}{\sqrt{n}r_2}
\end{aligned} \tag{6.37}$$

where $V_{\max}$ is the upper bound identified in Lemma 22. By sending n to infinity we get the upper bound if $\boldsymbol{m} \in \mathcal{P}_a(\mathcal{X})$ for some $a > 0$.

Next let $\boldsymbol{m} \in \mathcal{P}(\mathcal{X}) \setminus \mathcal{P}_*(\mathcal{X})$. Let $t_0 \leq \epsilon$ be such that $V_K(\boldsymbol{\nu}(\boldsymbol{m}, t_0)) \leq V_K(\boldsymbol{m}) + \epsilon$, where $\boldsymbol{\nu}(\boldsymbol{m}, t)$ is the solution to the original equation after time t. We can find a $r > 0$ such that for every $\tilde{\boldsymbol{m}} \in B(\boldsymbol{\nu}(\boldsymbol{m}, t_0), r), V_K(\tilde{\boldsymbol{m}}) \leq V_K(\boldsymbol{m}) + 2\epsilon$. If $q^n(\bar{\boldsymbol{m}}, t)$ is an ϵ optimal control that corresponds to each $\bar{\boldsymbol{m}}$, we define the control

$$q_{xy}^n(t) = \begin{cases} \gamma_{xy}, & t \leq t_0, \\ q_{xy}^n(\boldsymbol{\mu}^n(t_0), t - t_0), & t > t_0, \end{cases}$$

which gives

$$\begin{aligned}
&V_K^n(\boldsymbol{m}^n) \leq \mathbb{E}\left[\int_0^{T^n} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x^n(s) F_{xy}^n\left(q_{xy}^n(s)\right) + R^n(\boldsymbol{\mu}^n(s))\right) dt\right] \\
&\leq \mathbb{E}\left[\int_0^{t_0} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x^n(s) F_{xy}^n\left(q_{xy}^n(s)\right) + R^n(\boldsymbol{\mu}^n(s))\right) dt\right] \\
&+ \mathbb{E}\left[\int_{t_0}^{T^n} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x^n(s) F_{xy}^n\left(q_{xy}^n(s)\right) + R^n(\boldsymbol{\mu}^n(s))\right) dt\right] \\
&\leq \mathbb{E}\left[\int_0^{t_0} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x^n(s) F_{xy}^n\left(\gamma_{xy}^n(s)\right) + R^n(\boldsymbol{\mu}^n(s))\right) dt\right] \\
&+ \mathbb{E}\left[\int_{t_0}^{T^n} \left(\sum_{(x,y)\in\mathcal{Z}} \mu_x^n(s) F_{xy}^n\left(q_{xy}^n(\boldsymbol{\mu}^n(t_0), s - t_0)\right) + R^n(\boldsymbol{\mu}^n(s))\right) dt\right] \leq t_0 R_{\max} + \mathbb{E}\left[V(\boldsymbol{\mu}^n(t_0))\right] \\
&\overset{Lemma\,22}{\leq} \epsilon R_{\max}^n + P\left(\boldsymbol{\mu}^n(t_0) \in B(\boldsymbol{\nu}(\boldsymbol{m}, t_0), r)\right)(V_K(\boldsymbol{m}) + 2\epsilon) + P\left(\boldsymbol{\mu}^n(t_0) \notin B(\boldsymbol{\nu}(\boldsymbol{m}, t_0), r)\right) V_{\max} \\
&\leq V_K(\boldsymbol{m}) + (2 + R_{\max}^n)\epsilon + P\left(\boldsymbol{\mu}^n(t_0) \notin B(\boldsymbol{\nu}(\boldsymbol{m}, t_0), r)\right) V_{\max}.
\end{aligned} \tag{6.38}$$

Now by an application of Lemma 21, we get that the last term goes to zero as n goes to ∞, and since ϵ is arbitrary, we get that

$$\limsup V_K^n(\boldsymbol{m}^n) \leq V_K(\boldsymbol{m}).$$

□

7 Appendix: Properties of F^n_{xy}

Proof of Lemma 4. (1) We have

$$\begin{aligned} F^n_{xy}(q) &= \sup_{u\in(0,\infty)} G^n_{xy}(u,q) = \sup_{u\in(0,\infty)} \left\{ u\ell\left(\frac{q}{u}\right) - \gamma_{xy} C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right)\right\} \\ &\geq \gamma_{xy}\ell\left(\frac{q}{\gamma_{xy}}\right) - \gamma_{xy}C^n_{xy}\left(\frac{\gamma_{xy}}{\gamma_{xy}}\right) \geq \gamma_{xy}\ell\left(\frac{q}{\gamma_{xy}}\right) \geq 0. \end{aligned}$$

(2) We have

$$\begin{aligned} F^n_{xy}(\gamma_{xy}) &= \sup_{u\in(0,\infty)} G^n_{xy}(u,\gamma_{xy}) = \sup_{u\in(0,\infty)} \left\{ u\ell\left(\frac{\gamma_{xy}}{u}\right) - \gamma_{xy} C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right)\right\} \\ &= \sup_{u\in(0,\infty)} \left\{ \gamma_{xy}\log\gamma_{xy} - \gamma_{xy}\log u - \gamma_{xy} + u - \gamma_{xy}C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right)\right\}, \end{aligned}$$

but by applying part 2 of Lemma 3

$$\gamma_{xy}C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right) \geq \gamma_{xy}\log\gamma_{xy} - \gamma_{xy}\log u - \gamma_{xy} + u.$$

Therefore $F^n_{xy}(\gamma_{xy}) \leq 0$. However, by part (1) of this lemma $F^n_{xy}(\gamma_{xy}) \geq 0$, and therefore the equality follows.

(3) By definition $F^n_{xy}(q) = \sup_{u\in(0,\infty)} G^n_{xy}(u,q)$. Let $a \in (0,1)$ and $0 \leq q_1 < q_2 < \infty$, and let $q = aq_1 + (1-a)q_2$. Using the convexity of $G^n_{xy}(u,q)$ for fixed u as a function of q, we have

$$\begin{aligned} F^n_{xy}(aq_1 + (1-a)q_2) &= \sup_{u\in(0,\infty)} G^n_{xy}(u, aq_1 + (1-a)q_2) \\ &\leq \sup_{u\in(0,\infty)} \left\{ aG^n_{xy}(u,q_1) + (1-a)G^n_{xy}(u,q_2)\right\} \\ &\leq a\sup_{u\in(0,\infty)} G^n_{xy}(u,q_1) + (1-a)\sup_{u\in(0,\infty)} G^n_{xy}(u,q_2) \\ &\leq aF^n_{xy}(q_1) + (1-a)F^n_{xy}(q_2). \end{aligned}$$

□

For the proof of Lemma 16, we are going to use the following auxiliary lemma. Recall the definition of G^n_{xy} in (1.9).

Lemma 23. *If $\{C^n\}$ satisfies Assumption 14, then the following hold for every $(x,y) \in \mathcal{Z}$.*

1 *There exists a positive real number M, that does not depend on x, y, such that for the decreasing function $M^1_{xy} : (0,\infty) \to [0,\infty)$, given by*

$$M^1_{xy}(q) \doteq \min\left\{\gamma_{xy}\left(\frac{\gamma_{xy}}{q}\right)^{1/p}, M\right\},$$

we have that $G^n_{xy}(u,q)$ is increasing as a function of u on the interval $(0, M^1_{xy}(q)]$.

2 *There exists a decreasing function $M^2_{xy} : (0,\infty) \to [0,\infty)$, with $M^2_{xy}(q) \geq M^1_{xy}(q)$, such that, $G^n_{xy}(u,q)$ is decreasing as a function of u on the interval $\left[M^2_{xy}(q),\infty\right)$.*

Proof. By taking the derivative with respect to u in the definition (1.9) we get

$$-\frac{q}{u} - (C^n_{xy})'\left(\frac{u}{\gamma_{xy}}\right) + 1. \tag{7.1}$$

(1) By part 2 of Assumption 14 there exists $M \in (0,\infty)$ such that if $u < M$, then

$$-\frac{q}{u} - (C^n_{xy})'\left(\frac{u}{\gamma_{xy}}\right) + 1 \geq -\frac{q}{u} + \left(\frac{\gamma_{xy}}{u}\right)^{p+1} + 1,$$

and by taking $u \leq \gamma_{xy}\left(\gamma_{xy}/q\right)^{1/p}$ we get

$$-\frac{q}{u}+\left(\frac{\gamma_{xy}}{u}\right)^{p+1}+1 \geq -\frac{q}{u}+\frac{q}{u}+1>0.$$

Therefore for

$$M^1_{xy}(q)=\min\left\{\gamma_{xy}\left(\frac{\gamma_{xy}}{q}\right)^{1/p}, M\right\},$$

we have $-\frac{q}{u}-(C^n_{xy})'\left(\frac{u}{\gamma_{xy}}\right)+1 \geq 0$ on the interval $(0, M^1_{xy}(q)]$.

(2) By applying part 3 of Assumption 14, we get that there exists $\tilde{M}^2_{xy}(q)>0$, such that if $u>\tilde{M}^2_{xy}(q)$ then

$$\frac{u}{\gamma_{xy}}(C^n_{xy})'\left(\frac{u}{\gamma_{xy}}\right)-\frac{u}{\gamma_{xy}} \geq -\frac{q}{\gamma_{xy}}. \tag{7.2}$$

Then

$$M^2_{xy}(q) \doteq \max\{M^1_{xy}(q), \tilde{M}^2_{xy}(q)\},$$

is decreasing and bigger than M^1_{xy}, and using (7.2) we get

$$-\frac{q}{u}-(C^n_{xy})'\left(\frac{u}{\gamma_{xy}}\right)+1 \leq -\frac{q}{u}-\frac{\gamma_{xy}}{u}\left(\frac{u}{\gamma_{xy}}(C^n_{xy})'\left(\frac{u}{\gamma_{xy}}\right)-\frac{u}{\gamma_{xy}}\right) \leq 0$$

on the interval $[M^2_{xy}(q), \infty)$. □

Proof of Lemma 16. (1) Let $\epsilon>0$, and $q \geq \epsilon$. By Lemma 23, we have that $G^n_{xy}(u,q)$, as a function of u, is increasing on the interval $(0, M^1_{xy}(q)]$. Therefore for all $u \in (0, M^1_{xy}(q)]$ we have

$$\begin{aligned}
u\ell\left(\frac{q}{u}\right)-\gamma_{xy}C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right) \leq& M^1_{xy}(q)\ell\left(\frac{q}{M^1_{xy}(q)}\right)-\gamma_{xy}C^n_{xy}\left(\frac{M^1_{xy}(q)}{\gamma_{xy}}\right) \leq M^1_{xy}(q)\ell\left(\frac{q}{M^1_{xy}(q)}\right)\\
&\leq q\log\left(\frac{q}{M^1_{xy}(q)}\right)+M^1_{xy}(q) \leq q\log\left(\frac{q}{M^1_{xy}(q)}\right)+M^1_{xy}(\epsilon)\\
&\leq q\log(q)-q\log\left(M^1_{xy}(q)\right)+M^1_{xy}(\epsilon)\\
&\overset{M^1_{xy}(\epsilon)\leq M^2_{xy}(\epsilon)}{\leq} q\log(q)-q\log\left(M^1_{xy}(q)\right)+M^2_{xy}(\epsilon).
\end{aligned}$$

By the second part of Lemma 23, we have that $G^n_{xy}(u,q)$ is decreasing on the interval $(M^2_{xy}(\epsilon), \infty)$. Therefore for all $u \in (M^2_{xy}(\epsilon), \infty)$

$$\begin{aligned}
u\ell\left(\frac{q}{u}\right)-\gamma_{xy}C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right) &\leq M^2_{xy}(\epsilon)\ell\left(\frac{q}{M^2_{xy}(\epsilon)}\right)-\gamma_{xy}C^n_{xy}\left(\frac{M^2_{xy}(\epsilon)}{\gamma_{xy}}\right) \leq M^2_{xy}(\epsilon)\ell\left(\frac{q}{M^2_{xy}(\epsilon)}\right)\\
&\leq q\log\left(\frac{q}{M^2_{xy}(\epsilon)}\right)+M^2_{xy}(\epsilon)\\
&\overset{M^2_{xy}(q)\leq M^2_{xy}(\epsilon)}{\leq} q\log(q)-q\log\left(M^2_{xy}(q)\right)+M^2_{xy}(\epsilon)\\
&\overset{M^1_{xy}(q)\leq M^2_{xy}(q)}{\leq} q\log(q)-q\log\left(M^1_{xy}(q)\right)+M^2_{xy}(\epsilon).
\end{aligned}..$$

Finally for the interval $[M^1_{xy}(q), M^2_{xy}(\epsilon)]$ we have

$$\begin{aligned}
u\ell\left(\frac{q}{u}\right)-\gamma_{xy}C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right) &\leq u\ell\left(\frac{q}{u}\right)=q\log q-q\log u-q+u\\
&\leq q\log q-q\log(M^1_{xy}(q))+M^2_{xy}(\epsilon).
\end{aligned}$$

Now if we recall the definition of M^1_{xy} given in Lemma 23 and set $\bar{M}(q) \doteq \max\{M^2_{xy}(q) : (x,y) \in \mathcal{Z}\}$, then

$$G^n_{xy}(u,q) \leq q \log \frac{q}{\min\left\{\gamma_{xy}\left(\frac{\gamma_{xy}}{q}\right)^{1/p}, M\right\}} + \bar{M}(\epsilon),$$

and by taking supremum over u we end up with

$$F^n_{xy}(q) \leq q \log \frac{q}{\min\left\{\gamma_{xy}\left(\frac{\gamma_{xy}}{q}\right)^{1/p}, M\right\}} + \bar{M}(\epsilon).$$

(2) This is straightforward since F^n_{xy} is finite on the interval $(0,\infty)$, and convex.

(3) Let $1 > \epsilon > 0$. For every $q \in [\epsilon, \frac{1}{\epsilon}]$ and $n \in \mathbb{N}$, we have that $u\ell\left(\frac{q}{u}\right) - \gamma_{xy}C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right)$ is increasing on the interval $(0, M^2_{xy}(\epsilon)]$ and decreasing on $[M^1_{xy}\left(\frac{1}{\epsilon}\right), \infty)$ and since C^n_{xy} converges locally uniformly to C_{xy}, then the same conclusion holds for $u\ell\left(\frac{q}{u}\right) - \gamma_{xy}C_{xy}\left(\frac{u}{\gamma_{xy}}\right)$. It is straight forward to conclude that in all cases, the supremum is achieved on the interval $[M^2_{xy}(\epsilon), M^1_{xy}\left(\frac{1}{\epsilon}\right)]$.

We define the sets

$$A^{a,n}_{\epsilon,xy} = \left\{u : \gamma_{xy}C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right) \leq a \sup_{\epsilon \leq \tilde{u} \leq \frac{1}{\epsilon},\, M^2_{xy}(\epsilon) \leq q \leq M^1_{xy}\left(\frac{1}{\epsilon}\right)} \tilde{u}\ell\left(\frac{q}{\tilde{u}}\right)\right\},$$

and

$$A^{a,\infty}_{\epsilon,xy} = \left\{u : \gamma_{xy}C^\infty_{xy}\left(\frac{u}{\gamma_{xy}}\right) \leq a \sup_{\epsilon \leq \tilde{u} \leq \frac{1}{\epsilon},\, M^2_{xy}(\epsilon) \leq q \leq M^1_{xy}\left(\frac{1}{\epsilon}\right)} \tilde{u}\ell\left(\frac{q}{\tilde{u}}\right)\right\}.$$

By uniform convergence of C^n_{xy} in C^∞_{xy} in compact subsets of $(u_{1,xy}, u_{2,xy})$, and the monotonicity properties of C^n_{xy}, we get that for large enough n, we have $A^{3,\infty}_{\epsilon,xy} \subset A^{2,n}_{\epsilon,xy} \subset A^{1,\infty}_{\epsilon,xy}$.

For every $q \in [\epsilon, \frac{1}{\epsilon}]$, we have

$$\begin{aligned}
F^n_{xy}(q) &= \sup_{u \in \left[M^2_{xy}(\epsilon), M^1_{xy}\left(\frac{1}{\epsilon}\right)\right]} \left\{u\ell\left(\frac{q}{u}\right) - \gamma_{xy}C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right)\right\} \\
&= \sup_{u \in \left[M^2_{xy}(\epsilon), M^1_{xy}\left(\frac{1}{\epsilon}\right)\right] \cap A^{2,n}_{\epsilon,xy}} \left\{u\ell\left(\frac{q}{u}\right) - \gamma_{xy}C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right)\right\} \\
&\leq \sup_{u \in \left[M^2_{xy}(\epsilon), M^1_{xy}\left(\frac{1}{\epsilon}\right)\right] \cap A^{1,\infty}_{\epsilon,xy}} \left\{u\ell\left(\frac{q}{u}\right) - \gamma_{xy}C^\infty_{xy}\left(\frac{u}{\gamma_{xy}}\right)\right\} \\
&\quad + \sup_{u \in \left[M^2_{xy}(\epsilon), M^1_{xy}\left(\frac{1}{\epsilon}\right)\right] \cap A^{1,\infty}_{\epsilon,xy}} \left\{\left|\gamma_{xy}C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right) - \gamma_{xy}C^\infty_{xy}\left(\frac{u}{\gamma_{xy}}\right)\right|\right\} \\
&\leq \sup_{u \in \left[M^2_{xy}(\epsilon), M^1_{xy}\left(\frac{1}{\epsilon}\right)\right]} \left\{u\ell\left(\frac{q}{u}\right) - \gamma_{xy}C^\infty_{xy}\left(\frac{u}{\gamma_{xy}}\right)\right\} \\
&\quad + \sup_{u \in \left[M^2_{xy}(\epsilon), M^1_{xy}\left(\frac{1}{\epsilon}\right)\right] \cap A^{1,\infty}_{\epsilon,xy}} \left\{\left|\gamma_{xy}C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right) - \gamma_{xy}C^\infty_{xy}\left(\frac{u}{\gamma_{xy}}\right)\right|\right\}.
\end{aligned}$$

By doing the same for $\hat{F}^\infty_{xy}$, we get

$$\sup_{q \in \left[\epsilon, \frac{1}{\epsilon}\right]} |F^n_{xy}(q) - \hat{F}^\infty_{xy}(q)| \leq \sup_{u \in \left[M^2_{xy}(\epsilon), M^1_{xy}\left(\frac{1}{\epsilon}\right)\right] \cap A^{1,\infty}_{\epsilon,xy}} \left\{\left|\gamma_{xy}C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right) - \gamma_{xy}C^\infty_{xy}\left(\frac{u}{\gamma_{xy}}\right)\right|\right\},$$

and therefore F^n_{xy} converges locally uniformly to $\hat{F}^\infty_{xy}$ on $A^{1,\infty}_{\epsilon,xy}$. Now by the definition of F^∞_{xy}, i.e.

$$F^\infty_{xy}(q) = \inf\left\{\liminf_{n\to\infty} F^n_{xy}(q^n) : \{q^n\} \text{ sequence in } \mathbb{R} \text{ with } q^n \to q\right\},$$

it is easy to see that since F^n_{xy} converges locally uniformly to $\hat{F}^\infty_{xy}$ on $(0,\infty)$, and so $F^\infty_{xy} = \hat{F}^\infty_{xy}$ on $(0,\infty)$. Also F^∞_{xy} as a pointwise limit of convex functions on $(0,\infty)$, is also a convex function there. It remains to investigate what happens with F^∞_{xy} at zero. Let $q^n \to 0$ with $F^n_{xy}(q^n) \to F^\infty_{xy}(0)$. For $a \in (0,1)$, we have

$$\begin{aligned}
&aF^\infty_{xy}(0) + (1-a)F^\infty_{xy}(q)\\
&= aF^n_{xy}(q^n) + (1-a)F^n_{xy}(q) + a(F^n_{xy}(q^n) - F^\infty_{xy}(0)) + (1-a)(F^\infty_{xy}(q) - F^n_{xy}(q))\\
&\geq F^n_{xy}(aq^n + (1-a)q) + a(F^n_{xy}(q^n) - F^\infty_{xy}(0)) + (1-a)(F^\infty_{xy}(q) - F^n_{xy}(q))
\end{aligned}$$

Now if we take the limit, then by continuity of each F^n_{xy} on $(0,\infty)$ and the uniform convergence on every compact subset of that interval (also on $[(1-a)q, q]$), we have $aF^\infty_{xy}(0) + (1-a)F^\infty_{xy}(q) \geq F^\infty_{xy}((1-a)q)$. □

8 Appendix: Tightness functionals.

Proof of Lemma 19. Let $c_2 > 0$ and $\{(\boldsymbol{\mu}^n, T^n)\}$ be a sequence in S with $\boldsymbol{\mu}^n$ absolutely continuous such that

$$\int_0^{T^{\boldsymbol{n}}} \ell\left(|\dot{\boldsymbol{\mu}}^n(t)|\right) dt + c_1 T^{\boldsymbol{n}} \leq c_2$$

and $|\dot{\boldsymbol{\mu}}^n(t)| = 0$ for $t > T^{\boldsymbol{n}}$. Since all elements are positive, we have that $T^{\boldsymbol{n}} \leq c_2/c_1$... Let $\bar{\boldsymbol{\mu}}^n$ denote the restriction of $\boldsymbol{\mu}^n$ to $[0, c_2/c_1]$. If we prove that $\bar{\boldsymbol{\mu}}^n$ converges along some subsequence then we are done. Using the inequality $ab \leq e^{ca} + \ell(b)/c$, which is valid for $a, b \geq 0$, and $c \geq 1$, we have that

$$|\boldsymbol{\mu}^n(t) - \boldsymbol{\mu}^n(s)| \leq \int_t^s |\dot{\boldsymbol{\mu}}^n(r)| dr \leq (t-s)e^c + \frac{c_2}{c}.$$

This shows that $\{\bar{\boldsymbol{\mu}}^n\}$ are equicontinuous. Since $\bar{\boldsymbol{\mu}}^n(t)$ takes values in the compact set $\mathcal{P}(\mathcal{X})$, by the Arzela-Ascoli theorem there is a convergent subsequence. □

9 Appendix: Properties of Hamiltonians

Lemma 24. *Under Assumption 2 the Isaac condition is satisfied, i.e.,*

$$\begin{aligned}
H^{-,n}(\boldsymbol{m}, \boldsymbol{\xi}) = \quad & \sup_{\boldsymbol{u}\in(0,\infty)^{\otimes\mathcal{Z}}} \inf_{\boldsymbol{q}\in[0,\infty)^{\otimes\mathcal{Z}}} \sum_{(x,y)\in\mathcal{Z}} m_x \left\{q_{xy}\xi_{xy} + G^n_{xy}(u_{xy}, q_{xy})\right\} = \\
& \inf_{\boldsymbol{q}\in[0,\infty)^{\otimes\mathcal{Z}}} \sup_{\boldsymbol{u}\in(0,\infty)^{\otimes\mathcal{Z}}} \sum_{(x,y)\in\mathcal{Z}} m_x \left\{q_{xy}\xi_{xy} + G^n(u_{xy}, q_{xy})\right\} = H^{+,n}(\boldsymbol{m}, \boldsymbol{\xi}).
\end{aligned}$$

Proof. We have

$$\begin{aligned}
H^{-,n}(\boldsymbol{m}, \boldsymbol{\xi}) = \sup_{\boldsymbol{u}\in(0,\infty)^{\otimes\mathcal{Z}}} \inf_{\boldsymbol{q}\in[0,\infty)^{\otimes\mathcal{Z}}} \sum_{(x,y)\in\mathcal{Z}} m_x \left\{q_{xy}\xi_{xy} + G^n_{xy}(u_{xy}, q_{xy})\right\} = \\
\sum_{(x,y)\in\mathcal{Z}} m_x \sup_{u_{xy}\in(0,\infty)} \inf_{q_{xy}\in[0,\infty)} \left\{q_{xy}\xi_{xy} + u_{xy}\ell\left(\frac{q_{xy}}{u_{xy}}\right) - \gamma_{xy} C^n_{xy}\left(\frac{u_{xy}}{\gamma_{xy}}\right)\right\}.
\end{aligned}$$

If we prove the exchange of sup and inf for each $(x,y) \in \mathcal{Z}$, then we are done.

Since ℓ is convex

$$L^n_{xy}(u,q) = q\xi + u\ell\left(\frac{q}{u}\right) - \gamma_{xy} C^n_{xy}\left(\frac{u}{\gamma_{xy}}\right)$$

is convex with respect to q. It is easy to see that $L^n_{xy}(u,q)$ is not concave with respect to u, however under Assumption 2, we can show that L^n_{xy} is quasi-concave with respect to u (i.e., $\{u : L^n_{xy}(u,q) \geq c\}$ is convex for every $q \in [0,\infty)$, and $c \in \mathbb{R}$.)

By differentiating with respect to u we get

$$\partial_u L^n_{xy}(u,q) = -\frac{q}{u} + 1 - (C^n_{xy})' \left(\frac{u}{\gamma_{xy}}\right).$$

If we prove that for each q the set of roots for $\partial_u L^n_{xy}(u,q)$ is an interval or a point we are done, because a real function that changes monotonicity from increasing to decreasing at most once is quasi-concave. However $\partial_u L^n_{xy}(u,q)$ has the same roots as $u(C^n_{xy})'\left(\frac{u}{\gamma_{xy}}\right) - u + q.$. By part 1 of Assumption 1 $u(C^n_{xy})'\left(\frac{u}{\gamma_{xy}}\right) - u + q$ is increasing, which gives what is needed.

To prove the exchange between supremum and infimum, we apply Sion's Theorem (Corollary 3.3 in [27]), which states that if a continuous $F(u,q)$ is quasi-concave for every u is some convex set U and quasi-convex for every q in some convex set Q, and if one of the two sets is compact, then we can exchange the supremum with the infimum. In our case both sets are non-compact, and so this result cannot be applied directly, but it can be applied using the fact that $\lim_{q\to\infty} L^n_{xy}(q,1) = \infty$, as we now explain.

If we prove that

$$\inf_{q\in[0,\infty)} \sup_{u\in(0,\infty)} L^n_{xy}(u,q) = \lim_{r\to\infty} \inf_{q\in[0,\infty)} \sup_{u\in\left[r,\frac{1}{r}\right]} L^n_{xy}(u,q),$$

then we are done, since by Corollary 3.3 in [27]

$$\begin{aligned} \inf_{q\in[0,\infty)} \sup_{u\in(0,\infty)} L^n_{xy}(u,q) &= \lim_{r\to\infty} \inf_{q\in[0,\infty)} \sup_{u\in\left[r,\frac{1}{r}\right]} L^n_{xy}(u,q) = \\ \lim_{r\to\infty} \sup_{u\in\left[r,\frac{1}{r}\right]} \inf_{q\in[0,\infty)} L^n_{xy}(u,q) &= \sup_{u\in(0,\infty)} \inf_{q\in(0,\infty)} L^n_{xy}(u,q). \end{aligned}$$

Let $M \doteq \inf_{q\in[0,\infty)} \sup_{u\in(0,\infty)} L^n_{xy}(u,q)$. We will assume that $M < \infty$, and note that the case $M = \infty$ is treated similarly. Since $\lim_{q\to\infty} L^n_{xy}(q,1) = \infty$, we can find $\tilde{q}$ such that $L^n_{xy}(q,1) > 2M$ for every $q \geq \tilde{q}$. Now we have

$$\inf_{q\in[0,\infty)} \sup_{u\in(0,\infty)} L^n_{xy}(u,q) = \inf_{q\in[0,\tilde{q}]} \sup_{u\in(0,\infty)} L^n_{xy}(u,q),$$

and

$$\inf_{q\in[0,\tilde{q}]} \sup_{u\in\left[r,\frac{1}{r}\right]} L^n_{xy}(u,q) = \inf_{q\in[0,\infty)} \sup_{u\in\left[r,\frac{1}{r}\right]} L^n_{xy}(u,q),$$

which gives

$$\begin{aligned} &\inf_{q\in[0,\infty)} \sup_{u\in(0,\infty)} L^n_{xy}(u,q) = \inf_{q\in[0,\tilde{q}]} \sup_{u\in(0,\infty)} L^n_{xy}(u,q) = \sup_{u\in(0,\infty)} \inf_{q\in[0,\tilde{q}]} L^n_{xy}(u,q) = \\ &\lim_{r\to\infty} \sup_{u\in\left[r,\frac{1}{r}\right]} \inf_{q\in[0,\tilde{q}]} L^n_{xy}(u,q) = \lim_{r\to\infty} \inf_{q\in[0,\tilde{q}]} \sup_{u\in\left[r,\frac{1}{r}\right]} L^n_{xy}(u,q) = \lim_{r\to\infty} \inf_{q\in[0,\infty)} \sup_{u\in\left[r,\frac{1}{r}\right]} L^n_{xy}(u,q). \end{aligned}$$

□

Remark 25. *By observing $H^{+,n}(\boldsymbol{m},\boldsymbol{\xi})$ and $H^{-,n}(\boldsymbol{m},\boldsymbol{\xi})$ we can see that $H^{+,n}(\boldsymbol{m},\boldsymbol{\xi})$ is actually a concave function. If the minmax theorem holds then $H^{-,n}(\boldsymbol{m},\boldsymbol{\xi})$ must be a concave function as well. By using the formula*

$$H^{-,n}(\boldsymbol{m},\boldsymbol{\xi}) = \sum_{(x,y)\in\mathcal{Z}} m_x \gamma_{xy} \left(C^n_{xy}\right)^* \left(-\ell^*\left(-\xi_{xy}\right)\right)$$

we have that $\left(C^n_{xy}\right)^*\left(-\ell^*(\xi)\right) = \left(C^n_{xy}\right)^*\left(1-e^{\xi}\right)$ is concave. By differentiating with respect to ξ we get,

$$e^{2\xi}\left(\left(C^n_{xy}\right)^*\right)''\left(1-e^{\xi}\right) - e^{\xi}\left(\left(C^n_{xy}\right)^*\right)'\left(1-e^{\xi}\right) \leq 0,$$

from which, by using the identity $(f^*)' = (f')^{-1}$, we get

$$e^{2\xi}\left(\left(\left(C^n_{xy}\right)'\right)^{-1}\right)'\left(1-e^{\xi}\right) - e^{\xi}\left(\left(C^n_{xy}\right)'\right)^{-1}\left(1-e^{\xi}\right) \leq 0.$$

By substituting $\tilde{u} = 1 - e^{\xi}$ *we get*

$$\begin{aligned}
(1-\tilde{u})\left(\left(\left(C^n_{xy}\right)'\right)^{-1}\right)'(\tilde{u}) - \left(\left(C^n_{xy}\right)'\right)^{-1}(\tilde{u}) \le 0, &\qquad \textit{with } \tilde{u} \le 1\\
(1-\tilde{u})\,\frac{1}{\left(C^n_{xy}\right)''\left(\left(\left(C^n_{xy}\right)'\right)^{-1}(\tilde{u})\right)} - \left(\left(C^n_{xy}\right)'\right)^{-1}(\tilde{u}) \le 0, &\qquad \textit{with } \tilde{u} \le 1\\
\left(1-\left(C^n_{xy}\right)'(r)\right)\frac{1}{\left(C^n_{xy}\right)''(r)} - r \le 0, &\qquad \textit{with } \left(C^n_{xy}\right)'(r) \le 1\\
1-\left(C^n_{xy}\right)'(r) - r\left(C^n_{xy}\right)''(r) \le 0, &\qquad \textit{with } \left(C^n_{xy}\right)'(r) \le 1\\
r\left(C^n_{xy}\right)''(r) + \left(C^n_{xy}\right)'(r) - 1 \ge 0, &\qquad \textit{with } \left(C^n_{xy}\right)'(r) \le 1.
\end{aligned}$$

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