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Towards pressure-robust mixed methods for the incompressible Navier–Stokes equations

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Abstract

In this contribution, classical mixed methods for the incompressible Navier–Stokes equations that relax the divergence constraint and are discretely inf-sup stable, are reviewed. Though the relaxation of the divergence constraint was claimed to be harmless since the beginning of the 1970ies, *Poisson locking* is just replaced by another more subtle kind of locking phenomenon, which is sometimes called *poor mass conservation*. Indeed, divergence-free mixed methods and classical mixed methods behave qualitatively in a different way: divergence-free mixed methods are *pressure-robust*, which means that, e.g., their velocity error is independent of the continuous pressure. The lack of pressure-robustness in classical mixed methods can be traced back to a consistency error of an appropriately defined *discrete Helmholtz projector*. Numerical analysis and numerical examples reveal that *really locking-free* mixed methods must be discretely inf-sup stable and pressure-robust, simultaneously. Further, a recent discovery shows that locking-free, pressure-robust mixed methods do not have to be divergence-free. Indeed, relaxing the divergence constraint in the *velocity trial functions* is harmless, if the relaxation of the divergence constraint in *some velocity test functions* is repaired, accordingly.

1 Introduction

This paper studies the treatment of the *divergence constraint* in the discretization theory for the incompressible Navier–Stokes equations

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \qquad \mathbf{x}\in\mathcal{D}, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0, \qquad \mathbf{x}\in\mathcal{D}, \tag{2}$$

$$\mathbf{u} = \mathbf{0}, \qquad \mathbf{x}\in\partial\mathcal{D}, \tag{3}$$

for a bounded Lipschitz domain $\mathcal{D}\subset\mathbb{R}^d$ with $d\in\{2,3\}$. The equations (1) and (2) model *momentum balance* and *mass conservation* in a fluid with a constant density ρ.

The main focus of this paper is concerned with pressure-robustness that is related to the following observation: *arbitrarily large* forces in the momentum balance do not influence the *velocity field* $\mathbf{u}$, if they are irrotational. Indeed, they are balanced by the *pressure gradient*, because the pressure p is a *Lagrangian multiplier* for the *divergence constraint* (2). In the language of functional analysis, this observation can be expressed in terms of the *Helmholtz projector* [Soh12]. According to the Helmholtz decomposition, arbitrary $\mathbf{L}^2$ vector fields can be decomposed in a divergence-free and an irrotational part. The Helmholtz projector is defined by the divergence-free part of a vector field. Defined appropriately, the Helmholtz projector of every gradient field is zero, i.e., for all $\phi\in H^1(\mathcal{D})$, it holds $\mathbb{P}(\nabla\phi)\equiv\mathbf{0}$ [Soh12]. In classical, discretely inf-sup stable discretizations of the incompressible Navier–Stokes equations, a *discrete Helmholtz projector* $\mathbb{P}_h$ can be identified[LM16a], where $\mathbb{P}_h(\nabla\phi)=\mathbf{0}$ holds *only approximately*. This leads to numerical issues, that have accompanied mixed

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methods since their early beginnings [GLCL80, FF85, TS87, TTK88, PFC89, DGT94, GLBB97, Sch97, Fro98, Cod99, GJ05, OR04, GMT07, Lin09, GLRW12, Lin14b, LM16c, JLM+16] and that are sometimes called *poor mass conservation* [PFC89, GJ05, GMT07, Lin09, GLRW12, Lin14b], because they have their origin in the *relaxation of the divergence constraint*. Please note, that the relaxation of the divergence constrained was introduced, in order to achieve *discrete inf-sup stability* more easily.

The significance of *discrete inf-sup stability* [BF91, GR86] was recognized in the early 1970ies and nowadays the theory of discretely inf-sup stable mixed methods is a cornerstone of the discretization theory for PDEs. Discretely inf-sup stable mixed methods introduce a *space of discretely divergence-free* vector fields $\mathbf{V}_h^0$, the discrete counterpart of $\mathbf{V}^0 = \{\mathbf{v} \in \mathbf{H}_0^1(\mathcal{D}) : \nabla \cdot \mathbf{v} = 0\}$, in which the velocity solution of the discrete Stokes problem is searched for. In general functions from $\mathbf{V}_h^0$ are only *approximately* divergence-free. If all discretely-divergence-free vector fields are divergence-free in $\mathbf{L}^2$ sense, then the inf-sup stable mixed method is called *divergence-free*. However, the construction of *efficient* divergence-free inf-sup stable mixed methods was regarded as nearly impossible in the 1970ies, which is the reason, why classical inf-sup stable mixed methods like the Crouzeix–Raviart [CR73] and the Taylor–Hood finite element methods that *relax the divergence constraint* were proposed. Indeed, the first divergence-free inf-sup stable mixed method on unstructured, shape-regular triangulations in 3D was only presented in 2005 in a seminal paper by S. Zhang [Zha05]. However, nowadays many divergence-free mixed finite element methods are known [Qin94, Zha05, QZ07, Zha09, Zha11a, Zha11b, GN14b, GN14a, Nei15, LS16].

Discrete inf-sup stability guarantees that any $\mathbf{v} \in \mathbf{V}$ is approximated by the discrete Stokes projector

$$\mathbb{S}_h(\mathbf{v}) := \underset{\mathbf{v}_h \in \mathbf{V}_h^0}{\arg\min} \|\nabla(\mathbf{v} - \mathbf{v}_h)\|_{L^2} \tag{4}$$

with the same asymptotic approximation order, as if $\mathbf{v}$ would be approximated by constrained and unconstrained discrete vector fields, together. The so-called (Poisson) *locking phenomenon* [BS92a, BS92b], which was found in inappropriate discretizations of the Navier–Stokes equations in the 1960ies and 1970ies, means just the lack of such an optimal approximation property for the space $\mathbf{V}_h^0$. With the help of the *discrete Stokes* projector (4) and the *discrete Helmholtz* projector

$$\mathbb{P}_h(\mathbf{v}) := \underset{\mathbf{v}_h \in \mathbf{V}_h^0}{\arg\min} \|\mathbf{v} - \mathbf{v}_h\|_{L^2},$$

the discrete velocity solution $\mathbf{u}_h$ of classical inf-sup stable mixed methods for discretizations of the Stokes problem can be represented as

$$\mathbf{u}_h = \mathbb{S}_h(\mathbf{u}) - \frac{1}{\nu} \left(\mathbb{P}_h^T \circ \Delta_h \circ \mathbb{P}_h\right)^{-1} \left(\mathbb{P}_h(\nabla p)\right), \tag{5}$$

where $\mathbb{P}_h^T \circ \Delta_h \circ \mathbb{P}_h$ denotes a discrete vector Laplacian in $\mathbf{V}_h^0$. Exploiting the consequences of inf-sup stability and the consistency of the discrete Helmholtz projector, one obtains that $\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2} = \mathcal{O}(h^{\min(k,l+1)})$ for $\mathbf{u} \in \mathbf{V}^0 \cap \mathbf{H}^{k+1}(\mathcal{D}), p \in H^{l+1}(\mathcal{D})$. Note that typically, e.g. in the case of the Crouzeix–Raviart and the Taylor–Hood finite element methods, the discrete pressure space in inf-sup stable mixed methods is approximated with one order less than the discrete velocity space, i.e., $l = k - 1$, leading to an (asymptotically optimal) overall convergence order k.

The rather recently started investigation of pressure-robustness [LM16a, JLM+16] is motivated by the disturbing finding that classical *inf-sup stable mixed methods* for the incompressible Stokes equations that relax the divergence constraint are not really *locking-free* [ALM17]. In discretization theory, *locking-free* means in general that a discretization scheme behaves in a robust way, if a critical parameter in the PDE attains certain limit values [BS92b, BS92a] . Looking at (5), is is obvious that the expression $\nu^{-1}(\mathbb{P}_h(\nabla p))$ can be made arbitrarily large on a fixed triangulation by varying p and/or ν in the interval $0 < \nu \ll 1$. This becomes immediately clear in hydrostatic situations with $\mathbf{f} = \nabla\phi$ and $(\mathbf{u}, p) = (\mathbf{0}, \phi)$, where ϕ is assumed not to be an element of the discrete pressure space. Then, the *relative error* of the discrete velocity field $\mathbf{u}_h$ from (5) is infinitely large, since it holds $\mathbf{u} = \mathbf{0} = \mathbb{S}_h(\mathbf{u})$. Avoiding this *locking phenomenon* is possible by — but not only by as shown below — *divergence-free* discretizations, for which the space of *discretely divergence-free vector fields* only contains vector fields, which belong to $\mathbf{L}^2$ and are divergence-free in a L^2 sense, i.e., where the so-called *distributional divergence* of a vector field is in L^2. For such schemes it holds exactly $\mathbb{P}_h(\nabla p) = \mathbf{0}$, hence their discrete velocity solution is just characterized by

$$\mathbf{u}_h = \mathbb{S}_h(\mathbf{u}). \tag{6}$$

Here, no PDE parameter is involved and no locking behavior is observed. Obviously, this seems to be the ultimate goal of discretization theory. But the reader may be reminded again, that only in 2005 the first *divergence-free* and *simultaneously inf-sup stable* discretization was constructed for unstructured 3D triangulations [Zha05].

Historically, researchers focused on constructing better and better inf-sup stable discretizations that *relax the divergence constraint*, and tried to apply and analyze them for more and more difficult problems like the transient incompressible Navier–Stokes equations. However, simple benchmarks [LM16c, JLM$^+$16, LM16a, ALM17] demonstrate that classical mixed methods may suffer from — possibly extremely large — *space-discretization errors*, whenever the exterior force $\mathbf{f}$, the time-derivative $\mathbf{u}_t$ or the nonlinear convection term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ are gradient fields, i.e., whenever $\mathbb{P}(\mathbf{f}) = \mathbf{0}$, $\mathbb{P}(\mathbf{u}_t) = \mathbf{0}$ or $\mathbb{P}((\mathbf{u}\cdot\nabla)\mathbf{u}) = \mathbf{0}$ hold.

A recent key observation that goes back to [Lin14a] is that *pressure-robust discretizations* of the incompressible Navier–Stokes equations need not to be *divergence-free*. For example, in the incompressible Stokes problem, pressure-robustness emanates exclusively from the $\mathbf{L}^2$-orthogonality between ∇p and the velocity test function $\mathbf{v}_h$ needed in exactly one term of the Stokes problem: the discretization of the exterior force by $(\mathbf{f}, \mathbf{v}_h)$. By replacing the *velocity test function* $\mathbf{v}_h$ by a slightly modified one, say $\Pi(\mathbf{v}_h)$, one can restore *pressure-robustness*, if discretely divergence-free $\mathbf{v}_h \in \mathbf{V}_h^0$ are mapped by Π to divergence-free $\mathbf{L}^2$ vector fields. This simple idea introduces a new class of *pressure-robust* discretizations for the incompressible Stokes and Navier–Stokes equations into the literature. In case of the transient Navier–Stokes equations also the discretization of $\mathbf{u}_t$ and $(\mathbf{u}\cdot\nabla)\mathbf{u}$ has to be changed to lead to a really locking-free discretization [LM16a, ALM17]. The consistency error that is introduced by the application of the operator Π does not compromise the overall convergence rate of the method and, which is the key feature, does not depend on the pressure or on ν. Moreover, also discrete pressure estimates that are pressure-robust can be established.

The rest of this paper is structured as follows. Section 2.1 introduces the weak formulation of the Stokes model problem and some fundamental mathematical preliminaries like the Helmholtz decomposition and the concept of inf-sup stability. Section 3 studies the a priori velocity and pressure error of the steady Stokes equations and explains how the discrete Helmholtz projector enters the estimates. Section 4 elaborates on and demonstrates the locking behavior of classical finite element methods that relax the divergence constraint. Section 5 extends the theory to the transient Stokes equations and Section 6 explains what pressure-robustness means in case of the steady Navier-Stokes equations. Finally, Section 7 explains a new approach to restore the pressure-robustness property of classical schemes by some simple modification in the space discretization of certain terms.

2 Mathematical preliminaries

This section collects some mathematical preliminaries connected to the analysis of the Navier–Stokes equations.

2.1 Weak formulation of the Stokes equations and inf-sup stability

Here and throughout, $\mathbf{V} := \mathbf{H}^1_0(\mathcal{D})$ denotes the usual Sobolev space of weakly differentiable vector-valued $\mathbf{L}^2$ functions with zero boundary data along $\partial\mathcal{D}$ and $Q := L^2_0(\mathcal{D})$ denotes the space of L^2 functions with zero integral mean. These spaces are equipped with the norm $\|\nabla \mathbf{v}\|^2_{L_2} := (\nabla \mathbf{v}, \nabla \mathbf{v})$ for any $\mathbf{v} \in \mathbf{V}$ and $\|q\|^2_{L^2} := (q, q)$ for any $q \in Q$, respectively, where $(\bullet, \bullet)$ is the appropriate L^2 scalar product.

Moreover, the subspace of weakly differentiable and divergence-free vector fields is denoted by

$$\mathbf{V}^0 := \{\mathbf{v} \in \mathbf{V} : \nabla \cdot \mathbf{v} = 0\}$$

and its orthogonal complement with respect to $(\mathbf{V}, \|\nabla \bullet\|_{L_2})$ reads

$$\mathbf{V}^{\perp} := \{\mathbf{w} \in \mathbf{V} : (\nabla \mathbf{w}, \nabla \mathbf{v}) = 0 \text{ for all } \mathbf{v} \in \mathbf{V}^0\}.$$

The weak formulation of the Stokes problem seeks $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that, for all $(\mathbf{v}, q) \in \mathbf{V} \times Q$,

$$\begin{aligned} \nu a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \\ b(\mathbf{u}, q) &= 0, \end{aligned} \tag{7}$$

with the bilinear forms

$$\begin{aligned} a &: \mathbf{H}^1(\mathcal{D}) \times \mathbf{H}^1(\mathcal{D}) \to \mathbb{R}, \qquad a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v}), \\ b &: \mathbf{H}^1(\mathcal{D}) \times L^2(\mathcal{D}) \to \mathbb{R}, \qquad b(\mathbf{v}, q) = (\nabla \cdot \mathbf{v}, q). \end{aligned}$$

The data is assumed to be in $\mathbf{f} \in \mathbf{L}^2(\mathcal{D})$ and, by classical PDE theory [GR86], the bilinear form $a(\bullet, \bullet)$ is continuous and coercive.

Furthermore, functional analysis reveals that the divergence operator $\nabla \cdot : \mathbf{V} \to Q$ is continuous and surjective, which is equivalent to the famous continuous inf-sup condition

$$\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V} \setminus \{\mathbf{0}\}} \frac{(q, \nabla \cdot \mathbf{v})}{\|\nabla \mathbf{v}\|_{L_2} \|q\|_{L^2}} =: \beta > 0, \tag{8}$$

where β denotes the (positive) inf-sup constant for the Stokes problem with homogeneous Dirichlet boundary conditions [GR86]. An immediate consequence of the inf-sup condition is:

Lemma 2.1. *For every $q \in Q$, there is a $\mathbf{v} \in \mathbf{V}$ such that*

$$\nabla \cdot \mathbf{v} = q \qquad \textit{holds with} \qquad \|\nabla \mathbf{v}\|_{L_2} \leq \frac{1}{\beta} \|q\|_{L^2}.$$

Hence, the divergence operator $\nabla \cdot : \mathbf{V}^{\perp} \to Q$ is linear, continuous and bijective and also its inverse mapping is continuous. This allows to prove existence of a unique solution $(\mathbf{u}, p) \in \mathbf{V} \times Q$ for the continuous incompressible Stokes equations (7). Similarly, an appropriate *discrete inf-sup stability* was found to be fundamental for the (optimal) convergence of the discrete Stokes problem [GR86].

2.2 The Helmholtz projector and a fundamental orthogonality

We introduce the famous space of divergence-free resp. solenoidal $\mathbf{L}^2_\sigma$ vector fields [Soh12] in a bounded, polyhedral Lipschitz domain $\mathcal{D} \subset \mathbb{R}^d$ with $d \in \{2, 3\}$ by

$$\mathbf{L}^2_\sigma := \{\mathbf{w} \in \mathbf{L}^2(\mathcal{D}) : \text{ for all } \psi \in C^\infty(\mathcal{D}) \text{ holds } \quad -(\mathbf{w}, \nabla\psi) = 0\}, \tag{9}$$

Restricting ψ to $\psi \in C^\infty_0(\mathcal{D})$ the mapping $\psi \to -(\mathbf{w}, \nabla\psi)$ denotes the distributional divergence of $\mathbf{w}$ [JLM+16]. Therefore, all vector fields in $\mathbf{L}^2_\sigma(\mathcal{D})$ are weakly divergence-free. Further it holds $\mathbf{w} \cdot \mathbf{n} = 0$ everywhere on the boundary of $\mathcal{D}$ with outer normal $\mathbf{n}$. A density argument shows the following orthogonality.

Lemma 2.2 ($\mathbf{L}^2$ orthogonality of divergence-free vector fields and gradient fields)**.** *For all $\mathbf{w} \in \mathbf{L}^2_\sigma(\mathcal{D})$ and for all $\psi \in H^1(\mathcal{D})$ it holds*

$$(\mathbf{w}, \nabla\psi) = 0.$$

Here, $H^1(\mathcal{D})$ denotes the Sobolev space of weakly differentiable scalar fields. One of the most important concepts in the functional analysis for the incompressible Navier–Stokes equations is stated in the following lemma.

Lemma 2.3 (Helmholtz decomposition and Helmholtz projector)**.** *Every vector field $\mathbf{f} \in \mathbf{L}^2(\mathcal{D})$ is uniquely decomposable as*

$$\mathbf{f} = \nabla\phi + \mathbb{P}(\mathbf{f}),$$

into some $\phi \in H^1(\mathcal{D})/\mathbb{R}$ and the Helmholtz projector $\mathbb{P}(\mathbf{f}) \in \mathbf{L}^2_\sigma(\mathcal{D})$.

Proof. The weak problem "search for $\phi \in H^1(\mathcal{D})/\mathbb{R}$ such that

$$(\nabla\phi, \nabla\psi) = (\mathbf{f}, \nabla\psi) \quad \text{for all } \psi \in H^1(\mathcal{D})/\mathbb{R}$$

is uniquely solvable. Hence, $\mathbf{w} := \mathbf{f} - \nabla\phi \in \mathbf{L}^2(\mathcal{D})$, satisfies $(\mathbf{w}, \nabla\psi) = 0$ for all $\psi \in H^1(\mathcal{D})/\mathbb{R}$ which shows $\mathbf{w} \in \mathbf{L}^2_\sigma(\mathcal{D})$.

In order to show that the above Helmholtz decomposition is unique, assume two different decompositions

$$\mathbf{f} = \nabla\phi_1 + \mathbf{w}_1 = \nabla\phi_2 + \mathbf{w}_2$$

with ϕ_1, $\phi_2 \in H^1(\mathcal{D})/\mathbb{R}$ and $\mathbf{w}_1$, $\mathbf{w}_2 \in L^2_\sigma(\mathcal{D})$. This is equivalent to

$$\underbrace{\mathbf{w}_1 - \mathbf{w}_2}_{\in \mathbf{L}^2_\sigma(\mathcal{D})} = \nabla(\underbrace{\phi_2 - \phi_1}_{\in H^1(\mathcal{D})/\mathbb{R}}).$$

Lemma 2.2 shows $\mathbf{w}_1 = \mathbf{w}_2$ and $\phi_1 = \phi_2$. This concludes the proof. □

A direct consequence of Lemma 2.2 are the following properties of the Helmholtz projector.

Lemma 2.4. *For every vector field $\mathbf{f} \in \mathbf{L}^2(\mathcal{D})$, its Helmholtz projector satisfies*

$$\mathbb{P}(\mathbf{f}) = \underset{\mathbf{w} \in \mathbf{L}^2_\sigma(\mathcal{D})}{\arg\min} \|\mathbf{f} - \mathbf{w}\|_{L_2}.$$

In particular, for all $\psi \in H^1(\mathcal{D})$, it holds $\mathbb{P}(\nabla\psi) = \mathbf{0}$.

Another direct consequence of the Helmholtz decomposition and the Helmholtz projector is the following *fundamental invariance property*:

Lemma 2.5 (Fundamental invariance property)**.** *Assuming homogeneous Dirichlet boundary conditions for the velocity, a change of the exterior force according to* $\mathbf{f} \to \mathbf{f} + \nabla\phi$ *(with a potential* $\phi \in H^1(\mathcal{D})/\mathbb{R}$*) leads to a change of velocity and pressure by* $(\mathbf{u}, p) \to (\mathbf{u}, p + \phi)$*.*

Proof. Assuming that $(\mathbf{u}, p)$ and $(\tilde{\mathbf{u}}, \tilde{p})$ solve (7) for the exterior forces $\mathbf{f}$ and $\mathbf{f} + \nabla\phi$, one concludes that the velocity solutions $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are completely determined by testing with divergence-free velocity test functions $\mathbf{v} \in \mathbf{V}^0$, i.e., it holds

$$\nu(\nabla\tilde{\mathbf{u}}, \nabla\mathbf{v}) = (\mathbf{f} + \nabla\phi, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) = \nu(\nabla\mathbf{u}, \nabla\mathbf{v}),$$

i.e., it holds $\tilde{\mathbf{u}} = \mathbf{u}$. Further, the pressure $\tilde{p}$ is determined completely by testing with velocity functions $\mathbf{v} \in \mathbf{V}^\perp$ such that holds

$$\begin{aligned} -(\tilde{p}, \nabla\cdot\mathbf{v}) &= (\mathbf{f} + \nabla\phi, \mathbf{v}) - \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) \\ &= (\mathbf{f}, \mathbf{v}) - \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) - (\phi, \nabla\cdot\mathbf{v}) \\ &= -(p + \phi, \nabla\cdot\mathbf{v}), \end{aligned} \tag{10}$$

which proves $\tilde{p} = p + \phi$ and concludes the proof □

In this contribution, the Helmholtz projector will also be applied to functionals, as it is common in functional analysis for the incompressible Navier–Stokes equations [Soh12].

Definition 2.6. *The Helmholtz projector* $\mathbb{P}(\mathbf{f}) : \mathbf{V}^0 \to \mathbb{R}$ *is defined for all functionals* $\mathbf{f} \in \mathbf{H}^{-1}(\mathcal{D})$ *by restriction to the space* $\mathbf{V}^0$*, i.e., it holds for all* $\mathbf{v} \in \mathbf{V}^0$

$$\langle \mathbb{P}(\mathbf{f}), \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle.$$

Remark 2.7. *For all* $q \in Q$ *it holds* $\mathbb{P}(\nabla q) = \mathbf{0}$*, since testing the* $\mathbf{H}^{-1}(\mathcal{D})$*-functional* ∇q *by divergence-free velocity test functions* $\mathbf{v} \in \mathbf{V}^0$ *yields*

$$\langle \mathbb{P}(\nabla q), \mathbf{v} \rangle = -(q, \nabla\cdot\mathbf{v}) = 0,$$

and $\mathbb{P}(\nabla q)$ *is identified with the functional* $\mathbf{0} \in \mathbf{H}^{-1}(\mathcal{D})$*.*

For all $\mathbf{u} \in \mathbf{V}$, the functional $-\Delta\mathbf{u} \in \mathbf{H}^{-1}(\mathcal{D})$ is defined for all $\mathbf{v} \in \mathbf{V}$ by

$$-\langle \Delta\mathbf{u}, \mathbf{v} \rangle = (\nabla\mathbf{u}, \nabla\mathbf{v}).$$

Then, the following lemma holds.

Lemma 2.8. *For the solution of the incompressible Stokes problem* (7) *it holds*

$$\mathbb{P}(-\Delta\mathbf{u}) = \frac{1}{\nu}\mathbb{P}(\mathbf{f}),$$

where the Helmholtz projector on the left is understood in $\mathbf{H}^{-1}$*-sense, and the Helmholtz projector on the right is understood in the sense of* $\mathbf{L}^2$*.*

Proof. Indeed, for all $\mathbf{v} \in \mathbf{V}^0$ it holds

$$\begin{aligned} \nu a(\mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) && \Leftrightarrow \\ \nu a(\mathbf{u}, \mathbf{v}) &= (\mathbb{P}(\mathbf{f}), \mathbf{v}) && \Leftrightarrow \\ \langle \mathbb{P}(-\Delta \mathbf{u}), \mathbf{v} \rangle &= \frac{1}{\nu} (\mathbb{P}(\mathbf{f}), \mathbf{v}). \end{aligned}$$

This concludes the proof. □

Remark 2.9. *It should be emphasized that though it holds $\mathbb{P}(-\Delta \mathbf{u}) \in \mathbf{L}^2(\mathcal{D})$, it only holds $-\Delta \mathbf{u} \in \mathbf{H}^{-1}(\mathcal{D})$, in general.*

2.3 Mixed finite element discretizations for the incompressible (Navier–)Stokes equations

The discretization of (7) by mixed finite element methods is based on the existence of conforming finite dimensional subspaces $\mathbf{V}_h \subset \mathbf{V}$ and $Q_h \subset Q$. In the following, the operator π_{Q_h} defined by

$$\pi_{Q_h}(q) := \arg \min_{q_h \in Q_h} \|q - q_h\|_{L^2} \quad \text{for all } q \in Q,$$

denotes the best approximation operator into the subspace Q_h with respect to the norm $\|\cdot\|_{L^2}$. Similar best approximation operators for other spaces are introduced later analogously. The most important concept in the mixed discretization theory of the incompressible (Navier–)Stokes equations is the concept of the *discrete divergence* operator $\nabla_h \cdot : \mathbf{V}_h \to Q_h$ which is defined for conforming discretizations by

$$\nabla_h \cdot \mathbf{v}_h := \pi_{Q_h}(\nabla \cdot \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in \mathbf{V}_h.$$

The space of discretely divergence-free vector fields reads

$$\mathbf{V}_h^0 := \{\mathbf{v}_h \in \mathbf{V}_h : \nabla_h \cdot \mathbf{v}_h = 0\}.$$

Remark 2.10. *If the pair of spaces $(\mathbf{V}_h, Q_h)$ satisfies $\nabla \cdot \mathbf{V}_h = Q_h$, then the discrete divergence operator $\nabla_h \cdot : \mathbf{V}_h \to Q_h$ coincides with the continuous divergence operator $\nabla \cdot : \mathbf{V} \to Q$ restricted to $\mathbf{V}_h$ and the space $\mathbf{V}_h^0$ is a subspace of $\mathbf{V}^0$.*

In the following, we will assume that the pair of spaces $(\mathbf{V}_h, Q_h)$ gives rise to a so-called Fortin interpolator [GR86].

Definition 2.11 (Fortin interpolator). *A Fortin interpolator $\pi_F : \mathbf{V} \to \mathbf{V}_h$ fulfills, for all $\mathbf{v} \in \mathbf{V}$,*

1 $(\nabla \cdot \pi_F(\mathbf{v}), q_h) = (\nabla \cdot \mathbf{v}, q_h)$ *for all* $q_h \in Q_h$, *and*

2 $\|\nabla \pi_F(\mathbf{v})\|_{L_2} \leq C_S \|\nabla \mathbf{v}\|_{L_2}$ *with* $C_F \in \mathbb{R}^+$.

The following two lemmas recall classical results of the theory of mixed formulations [GR86].

Lemma 2.12 (Discrete inf-sup stability). *Assuming the existence of a Fortin interpolator π_F, it holds*

$$\inf_{q_h \in Q_h} \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{(\nabla \cdot \mathbf{v}_h, q_h)}{\|\nabla \mathbf{v}_h\|_{L_2} \|q_h\|_{L_2}} =: \beta_h > 0.$$

Indeed, it holds $\beta_h \geq \frac{\beta}{C_S}$. Consequently, the discrete divergence operator $\nabla_h \cdot : \mathbf{V}_h \to Q_h$ is surjective.

Lemma 2.13. *For all $\mathbf{v} \in \mathbf{V}^0$ it holds*

$$\inf_{\mathbf{v}_h \in \mathbf{V}_h^0} \|\nabla(\mathbf{v} - \mathbf{v}_h)\|_{L_2} \leq (1 + C_S) \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\nabla(\mathbf{v} - \mathbf{v}_h)\|_{L_2}.$$

Lemma 2.12 and 2.13 guarantee the solvability and optimal convergence behavior of the mixed discretization of (7) that seeks $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that, for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$,

$$\begin{aligned} \nu a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \\ b(\mathbf{u}_h, q_h) &= 0. \end{aligned} \tag{11}$$

3 A priori error analysis for the steady Stokes equations

This section deals with the a priori error analysis of the solution of (11). The analysis involves the *discrete Helmholtz projector*, the discrete counterpart to $\mathbb{P}$ as introduced in [LM16b]. For $\mathbf{f} \in \mathbf{L}^2(\mathcal{D})$, the discrete Helmholtz projector is defined by

$$\mathbb{P}_h(\mathbf{f}) := \arg\min_{\mathbf{v}_h \in \mathbf{V}_h^0} \|\mathbf{f} - \mathbf{v}_h\|_{L_2}.$$

3.1 Consistency error from the discrete divergence operator

The nonconformity of V_h^0 with respect to V^0 generates consistency errors that can be measured in discrete $\mathbf{H}^{-1}$ norms defined by, for $\mathbf{f} \in \mathbf{H}^{-1}(\mathcal{D})$,

$$\|\mathbf{f}\|_{\mathbf{V}_h^{0,*}} := \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h^0} \frac{\langle \mathbf{f}, \mathbf{v}_h \rangle}{\|\nabla \mathbf{v}_h\|_{L_2}} \quad \text{and} \quad \|\mathbf{f}\|_{\mathbf{V}_h^{\perp,*}} := \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h^{\perp}} \frac{\langle \mathbf{f}, \mathbf{v}_h \rangle}{\|\nabla \mathbf{v}_h\|_{L_2}}.$$

Remark 3.1. *In the following, we will sometimes use the notation ∇q for an arbitrary $q \in L^2(\mathcal{D})$, when the $\mathbf{H}^{-1}$-gradient of q is addressed. It is defined by*

$$\mathbf{v} \in \mathbf{H}^1(\mathcal{D}) \to \langle \nabla q, \mathbf{v} \rangle = -(q, \nabla \cdot \mathbf{v}).$$

For example, the expression $\|\nabla p\|_{\mathbf{V}_h^{0,*}}$ means

$$\|\nabla p\|_{\mathbf{V}_h^{0,*}} = \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h^0} \frac{-(p, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_{L_2}}.$$

Lemma 3.2. *For all $\phi \in Q$, it holds*

1. $\nabla \cdot \mathbf{V}_h = Q_h \Rightarrow \|\mathbb{P}_h(\nabla \phi)\|_{\mathbf{V}_h^{0,*}} = \|\nabla(\phi - \pi_{Q_h}(\phi))\|_{\mathbf{V}_h^{\perp,*}} = 0.$
2. $\nabla \cdot \mathbf{V}_h \neq Q_h \Rightarrow \|\mathbb{P}_h(\nabla \phi)\|_{\mathbf{V}_h^{0,*}} \leq \|\phi - \pi_{Q_h}(\phi)\|_{L^2}$, *and* $\|\nabla(\phi - \pi_{Q_h}(\phi))\|_{\mathbf{V}_h^{\perp,*}} \leq \|\phi - \pi_{Q_h}(\phi)\|_{L^2}.$

Proof. For all $\mathbf{v}_h \in \mathbf{V}_h^0$ it holds

$$\langle \mathbb{P}_h(\nabla \phi), \mathbf{v}_h \rangle = -(\phi, \nabla \cdot \mathbf{v}_h).$$

In the first case, this term is 0, since it holds $\mathbf{v}_h \in \mathbf{L}^2_\sigma(\mathcal{D})$. In the second case, one can estimate this term by

$$-(\phi, \nabla \cdot \mathbf{v}_h) = -(\phi - \pi_{Q_h}(\phi), \nabla \cdot \mathbf{v}_h)$$
$$\leq \|\phi - \pi_{Q_h}(\phi)\|_{L_2} \|\nabla \cdot \mathbf{v}_h\|_{L_2} \leq \|\phi - \pi_{Q_h}(\phi)\|_{L_2} \|\nabla \mathbf{v}_h\|_{L_2}.$$

Similarly, for all $\mathbf{v}_h \in \mathbf{V}_h^\perp$, it holds

$$(\phi - \pi_{Q_h}(\phi), \nabla \cdot \mathbf{v}_h) \leq \|\phi - \pi_{Q_h}(\phi)\|_{L_2} \|\nabla \cdot \mathbf{v}_h\|_{L_2}.$$

If $\nabla \cdot \mathbf{v}_h \in Q_h$, the left-hand side vanishes due to the best approximation properties of π_{Q_h}. □

Remark 3.3. *Lemma 3.2 constitutes the main difference between divergence-free resp. pressure-robust mixed methods and classical mixed methods that relax the divergence constraint causing a pressure-dependent consistency error. Many problems of classical mixed methods result from the fact that their discrete Helmholtz projectors do not vanish for arbitrary gradient fields [JLM+16].*

3.2 Velocity error

This section studies the distance of the discrete Stokes solution to the discrete Stokes projector of $\mathbf{u}$ defined by

$$\mathbb{S}_h(\mathbf{u}) := \underset{\mathbf{v}_h \in \mathbf{V}_h^0}{\arg\min} \|\nabla(\mathbf{u} - \mathbf{v}_h)\|_{L_2},$$

i.e. the best approximation of $\mathbf{u}$ in the space $(V_{0,h}, \|\nabla \bullet\|_{L^2})$. Note, that the approximation properties of the Stokes operator are dependent on the existence of a discrete Fortin interpolator, see Lemma 2.13.

Theorem 3.4 (Velocity error)**.** *For the continuous solution $\mathbf{u}$ of (7) and the discrete solution $\mathbf{u}_h$ of (11) it holds*

$$\mathbf{u}_h = \mathbb{S}_h(\mathbf{u}) + \mathbf{e}_h$$

where the perturbation $\mathbf{e}_h$ satisfies

$$a(\mathbf{e}_h, \mathbf{v}_h) = \frac{1}{\nu}(p, \nabla \cdot \mathbf{v}_h) \quad \textit{for all } \mathbf{v}_h \in \mathbf{V}_{0,h}$$

with the error estimate

$$\|\nabla \mathbf{e}_h\|_{L_2} \leq \frac{1}{\nu}\|\mathbb{P}_h(\nabla p)\|_{\mathbf{V}_h^{0,*}} \leq \begin{cases} 0, & \textit{for } \nabla \cdot [\mathbf{V}_h] = Q_h, \\ \frac{1}{\nu}\|p - \pi_{Q_h}(p)\|_{L^2}, & \textit{for } \nabla \cdot [\mathbf{V}_h] \neq Q_h. \end{cases}$$

Proof. The best approximation property of $\mathbb{S}_h(\mathbf{u})$ and the Galerkin orthogonality show, for $\mathbf{e}_h := \mathbf{u_h} - \mathbb{S}_h(\mathbf{u}) \in \mathbf{V}_h^0$ and any $\mathbf{v}_h \in \mathbf{V}_{0,h}$,

$$a(\mathbf{e}_h, \mathbf{v}_h) = a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) = \frac{1}{\nu}(p, \nabla \cdot \mathbf{v}_h).$$

To show the error estimate, set $\mathbf{v}_h = \mathbf{e}_h$ in the last identity to obtain

$$\|\nabla \mathbf{e}_h\|_{L_2}^2 = a(\mathbf{u}_h - \mathbf{u}, \mathbf{e}_h) \leq \frac{1}{\nu}\|\mathbb{P}_h(\nabla p)\|_{\mathbf{V}_h^{0,*}} \|\nabla \mathbf{e}_h\|_{L_2}.$$

A division by $\|\nabla \mathbf{e}_h\|$ concludes the proof. □

Remark 3.5. *For divergence-free/pressure-robust mixed finite element methods Lemma 3.2 shows* $\frac{1}{\nu}\|\mathbb{P}_h(\nabla p)\|_{\mathbf{V}_h^{0,*}} = 0$*, hence the discrete solution* $\mathbf{u}_h$ *equals the Stokes projector of* $\mathbf{u}$*. In particular, the estimate is independent of* $\frac{1}{\nu}$ *and the discrete solution* $\mathbf{u}_h$ *is just a linear function of* $\mathbf{u}$*.*

Classical mixed methods that relax the divergence constraint on the other hand show a locking phenomenon for $\nu \to 0$*. Here, the discrete solution* $\mathbf{u}_h$ *is a linear function of* $\mathbf{u}$ *and* $\frac{1}{\nu}p$*. In this sense, poor mass conservation is a consistency error of the corresponding discrete Helmholtz projector. Moreover, this shows that these methods violate the fundamental invariance property stated in Lemma 2.5 if* ∇q *with* $q \notin Q_h$ *is added to the right-hand side.*

3.3 Pressure error estimates

A similar result can be obtained for the pressure error, i.e. the distance of the discrete pressure p_h to the best approximation $\pi_{Q_h}(p)$ of the exact pressure p.

Theorem 3.6 (Pressure error)**.** *For the continuous solution* p *of* (7) *and the discrete solution* p_h *of* (11)*, it holds*

$$p_h = \pi_{Q_h}(p) + r_h$$

where the perturbation $r_h \in Q_h$ *satisfies*

$$(r_h, \nabla \cdot \mathbf{v}_h) = \nu a(\mathbf{u} - \mathbb{S}_h(\mathbf{u})), \mathbf{v}_h) + (p - \pi_{Q_h}(p), \nabla \cdot \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in \mathbf{V}_h^{\perp}$$

with the error estimate

$$\begin{aligned}\|r_h\|_{L^2} &\leq \frac{\nu}{\beta_h}\|\nabla(\mathbf{u} - \mathbb{S}_h(\mathbf{u}))\|_{L^2} + \|\nabla(p - \pi_{Q_h}(p))\|_{\mathbf{V}_h^{\perp,*}} \\ &\leq \frac{\nu}{\beta_h}\|\nabla(\mathbf{u} - \mathbb{S}_h(\mathbf{u}))\|_{L^2} + \begin{cases} 0 & \text{for } \nabla \cdot [\mathbf{V}_h] = Q_h, \\ \|\nabla(p - \pi_{Q_h}(p))\|_{\mathbf{V}_h^{\perp,*}} & \text{for } \nabla \cdot [\mathbf{V}_h] \neq Q_h. \end{cases}\end{aligned}$$

Proof. For $r_h = p_h - \pi_{Q_h}(p)$ and any $\mathbf{v}_h \in \mathbf{V}_h^{\perp}$, Galerkin orthogonality shows

$$\begin{aligned}(r_h, \nabla \cdot \mathbf{v}_h) &= (p_h - p, \nabla \cdot \mathbf{v}_h) + (p - \pi_{Q_h}(p), \nabla \cdot \mathbf{v}_h) \\ &= \nu a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + (p - \pi_{Q_h}(p), \nabla \cdot \mathbf{v}_h).\end{aligned}$$

Moreover, one can add $\nu a(\mathbf{u}_h - \mathbb{S}_h(\mathbf{u}), \mathbf{v}_h) = 0$ due to $\mathbf{v}_h \in \mathbf{V}_h^{\perp}$ and $\mathbf{u}_h - \mathbb{S}_h(\mathbf{u}) \in \mathbf{V}_h^0$. The error estimate follows from the discrete inf-sup stability, where one tests the identity with the $\mathbf{w}_h \in \mathbf{V}_h^{\perp}$ such that holds $\nabla_h \cdot \mathbf{w}_h = r_h$. □

Remark 3.7. *Note that for divergence-free/pressure-robust finite element methods, it holds* $(p - \pi_{Q_h}(p), \nabla \cdot \mathbf{v}_h) = 0$*, since* $\nabla \cdot \mathbf{v}_h \in Q_h$*. Hence, also the perturbation of the discrete pressure from the exact pressure is pressure-independent, whereas the perturbation in classical finite element methods that are not divergence-free depends on the pressure.*

4 Locking behavior of classical finite element methods

The previous section intended to show that classical mixed methods suffer from a locking phenomenon in the limit $\nu \to 0$, which does not appear in divergence-free/pressure-robust mixed methods. However, many researchers continue to claim that classical mixed methods, which are discretely inf-sup stable, would behave in an *optimal manner*. For such a statement, see e.g. [Ste90, p. 501]. This results in the belief, that classical mixed methods work well, in principle. This section argues against this claim.

4.1 Optimality (only) for $\nu = 1$

The argument for the optimality of classical mixed finite element methods seems to be based on the following observation: for the viscosity $\nu = 1$, one obtains by squaring and adding the results of Theorems 3.4 and 3.6 the following — seemingly optimal — a-priori error estimate:

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2}^2 + \|p-p_h\|_{L^2}^2 \le C(\beta_h)\left(\|\nabla(\mathbf{u}-\mathbb{S}_h(\mathbf{u}))\|_{L^2}^2 + \|p-\pi_{Q_h}(p)\|_{L^2}^2\right). \tag{12}$$

In other words, the claim is: the optimality of classical mixed methods would be revealed, if one analyzes classical mixed methods in the 'right norm'. However, the corresponding — seemingly optimal — error estimate for $\nu \neq 1$ reads:

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2}^2 + \frac{1}{\nu^2}\|p-p_h\|_{L^2}^2 \;\le\; C(\beta_h)\left(\|\nabla(\mathbf{u}-\mathbb{S}_h(\mathbf{u}))\|_{L^2}^2 + \frac{1}{\nu^2}\|p-\pi_{Q_h}(p)\|_{L^2}^2\right)$$

with the same constant $C(\beta_h)$. Obviously, in the limit $\nu \to 0$, this error estimate does not give any control over the velocity error. And indeed, numerical experience shows that the discrete pressure in classical mixed methods is always quite accurate, even if the discrete velocity is inaccurate. However, the corresponding error estimates for divergence-free/pressure-robust mixed methods in Theorems 3.4 and 3.6 deliver a reasonable control over pressure and velocity error separately, whatever small ν may be.

But maybe the reader asks herself/himself, why not just rescaling the problem from $\nu \neq 1$ to a problem with $\nu = 1$, in order to avoid this problem? In fact, such a scaling is always possible, but it only reveals the *real hidden problem* in (12), its *lack of pressure-robustness*. Since the appropriate scaling is $(\tilde{\mathbf{u}}, \tilde{p}) = (\mathbf{u}, \frac{1}{\nu}p)$, the estimate (12) only looks good, if one assumes that the physical variables $\mathbf{u}$ and p are both of comparable order, say $\mathcal{O}(1)$ simultaneously. If the pressure is magnitudes larger than the velocity, (12) still ensures only a good estimate for the pressure. However, the accuracy of divergence-free/pressure-robust mixed methods would be unaffected by large pressures p.

4.2 Optimality of divergence-free/pressure robust mixed methods

In order to further show the improved robustness of divergence-free/pressure-robust mixed methods compared to classical mixed methods, additional error estimates are presented in the following (discrete) 'right norm'

$$\left(\|\nabla(\mathbb{S}_h(\mathbf{u})-\mathbf{u}_h)\|_{L^2}^2 + \frac{1}{\nu^2}\|\pi_{Q_h}(p)-p_h\|_{L^2}^2\right)^{\frac{1}{2}} \tag{13}$$

for both kind of methods.

Lemma 4.1. *For the solutions $(\mathbf{u}, p)$ of (7) and $(\mathbf{u}_h, p_h)$ of (11), it holds*

$$\left(\|\nabla(\mathbb{S}_h(\mathbf{u})-\mathbf{u}_h)\|_{L^2}^2 + \frac{1}{\nu^2}\|\pi_{Q_h}(p)-p_h\|_{L^2}^2\right)^{\frac{1}{2}}$$
$$\le \frac{1}{\nu^2}\|\mathbb{P}_h(\nabla p)\|_{\mathbf{V}_h^{0,*}}^2 + \left(\frac{1}{\beta_h}\|\nabla(\mathbf{u}-\mathbb{S}_h(\mathbf{u}))\|_{L^2} + \frac{1}{\nu}\|\nabla(p-\pi_{Q_h}(p))\|_{\mathbf{V}_h^{\perp,*}}\right)^2$$

Note, that for divergence-free/pressure-robust methods, all pressure-dependent terms on the right-hand side disappear.

Proof. This follows directly from Theorems 3.4 and 3.6. □

Example 4.2. *To illustrate the locking behavior of classical finite element methods, consider a Stokes problem with right-hand side* $\mathbf{f} = -\nu\Delta\mathbf{u} + \nabla p$ *for* $p(x,y) = x^5 + y^5 - 1/3$ *and* $\mathbf{u} = \nabla \times (0, 0, x^2(x-1)^2y^2(y-1)^2)$ *in the domain* $\mathcal{D} = (0,1)^2$*. As depicted in Figure 1, the error of the Taylor-Hood finite element method scales with* $1/\nu$ *as predicted by Lemma 4.1, while the error of the divergence-free Scott–Vogelius finite element method is* ν*-independent. All results were obtained on the same mesh that was chosen such that both methods are guaranteed to be inf-sup stable.*

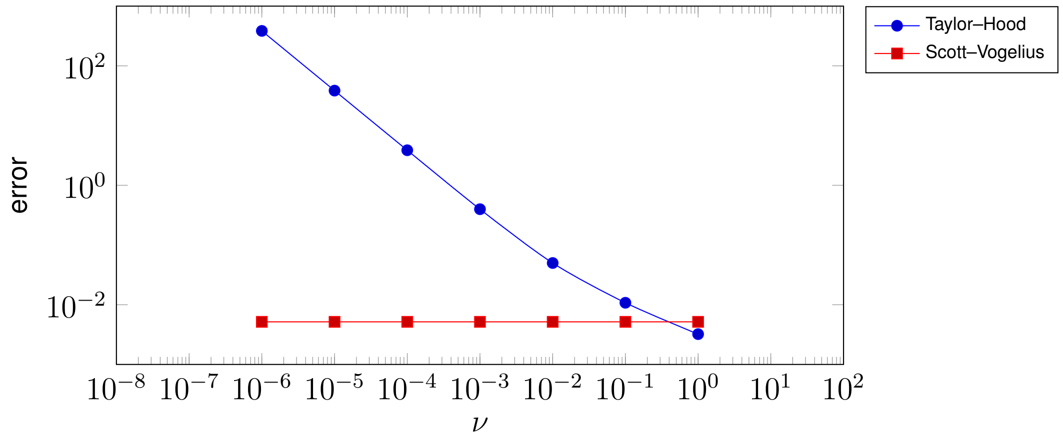

Figure 1: Best approximation error (13) of the Taylor–Hood and Scott–Vogelius finite element methods versus ν on a fixed unstructured mesh with 201 vertices in Example 4.2.

5 The transient incompressible Stokes equations

This section investigates the transient incompressible Stokes problem

$$\begin{aligned} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{f}, && (t,\mathbf{x}) \in (0,T] \times \mathcal{D}, \\ \nabla \cdot \mathbf{u} &= 0, && (t,\mathbf{x}) \in (0,T] \times \mathcal{D}, \\ \mathbf{u} &= \mathbf{0}, && (t,\mathbf{x}) \in (0,T) \times \partial\mathcal{D}, \\ \mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0(\mathbf{x}), && \mathbf{x} \in \mathcal{D}. \end{aligned} \tag{14}$$

For the numerical analysis, we assume that the solution (u, p) exists, that it is unique and that we have the following regularities in appropriate Bochner spaces

1 $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\mathcal{D})$,

2 $\mathbf{u} \in L^2(0,T;\mathbf{V})$,

3 $\mathbf{u}_t \in L^2(0,T;\mathbf{L}^2(\mathcal{D}))$,

4 $p \in L^2(0,T;Q)$.

The norm in the space $L^2(0,T;X)$ is denoted by

$$\|\bullet\|_{L^2(0,T;X)} := \left(\int_0^T \|\bullet(s)\|_X^2 \, ds\right)^{1/2}.$$

5.1 Weak formulation and discrete problem

Due to the regularity assumptions, the solution $(\mathbf{u}, p)$ fulfills the following weak formulation: we search for (u,p) such that for all $(\mathbf{v}, q) \in \mathbf{V} \times Q$ it holds for almost all $t \in (0,T]$,

$$\begin{aligned}(\mathbf{u}_t(t), \mathbf{v}) + \nu a(\mathbf{u}(t), \mathbf{v}) + b(\mathbf{v}, p(t)) &= (\mathbf{f}(t), \mathbf{v}),\\ b(\mathbf{u}(t), q) &= 0.\end{aligned} \tag{15}$$

Further, we assume $\mathbf{u}(0,\cdot) = \mathbf{u}_0$ and $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\mathcal{D}))$.

The discrete transient Stokes problem seeks $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{V}_h \times Q_h$ such that, for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$ and all $t \in (0,T]$, it holds

$$\begin{aligned}(\dot{\mathbf{u}}_h(t), \mathbf{v}_h) + \nu a(\mathbf{u}_h(t), \mathbf{v}_h) + b(\mathbf{v}_h, p_h(t)) &= (\mathbf{f}(t), \mathbf{v}_h),\\ b(\mathbf{u}_h(t), q_h) &= 0,\end{aligned} \tag{16}$$

and the discrete initial value is given by $\mathbf{u}_h(0) = \mathbb{P}_h(\mathbf{u}_0)$. The discrete solvability follows the standard theory, see, e.g., [ALM17].

5.2 A priori error analysis

For the numerical analysis, we need Gronwall's lemma [Joh16]:

Lemma 5.1 (Gronwall)**.** *For $I = (0,T]$ with $0 < T < \infty$ and $\alpha, \beta \in C(I,\mathbb{R})$ it is assumed that $\phi \in C^1(I,\mathbb{R})$ fulfills for all $t \in I$ the inequality*

$$\dot{\phi}(t) \leq \alpha(t) + \beta(t)\phi(t).$$

Then, it follows for all $t \in I$

$$\phi(t) \leq \phi(0) e^{\int_0^t \beta(\tau) d\tau} + \int_0^t \alpha(s) e^{\int_s^t \beta(\tau) d\tau} \, ds.$$

Theorem 5.2 (Velocity error)**.** *For the continuous solution $\mathbf{u}$ of (14) and the discrete solution $\mathbf{u}_h$ of (16), the following statements hold, for almost all $t \in (0,T]$,*

$$\mathbf{u}_h(t) = \mathbb{P}_h(\mathbf{u}(t)) + \mathbf{e}_h(t)$$

where $\mathbf{e}_h(t)$ satisfies the discrete evolution equation, for all $\mathbf{v}_h \in \mathbf{V}_h^0$ and $t \in [0,T]$,

$$\begin{aligned}(\dot{\mathbf{e}}_h(t), \mathbf{v}_h) + \nu(\nabla \mathbf{e}_h(t), \nabla \mathbf{v}_h) = \nu(\nabla(\mathbb{S}_h(\mathbf{u}(t)) - \mathbb{P}_h(\mathbf{u}(t))), \nabla \mathbf{v}_h)\\ - (p(t) - \pi_{Q_h}(p(t)), \nabla \cdot \mathbf{v}_h)\end{aligned}$$

with the initial state $\mathbf{e}_h(0) = 0$. Furthermore, $\mathbf{e}_h$ can be estimated either by

$$\|\mathbf{e}_h(t)\|_{L^2}^2 + \nu \|\nabla \mathbf{e}_h\|_{L^2(0,t;L^2)}^2 \leq 2\nu \|\nabla(\mathbf{u} - \mathbb{P}_h(\mathbf{u}))\|_{L^2(0,t;L^2)}^2 + \frac{2}{\nu} \|\mathbb{P}_h(\nabla p)\|_{L^2(0,t;V_h^{0,*})}^2$$

or, assuming additionally $\nabla p \in L^2(0,t;\mathbf{L}^2)$, by

$$\|\mathbf{e}_h(t)\|^2_{L^2} + \nu\|\nabla\mathbf{e}_h\|^2_{L^2(0,t;L^2)} \leq e\nu\|\nabla(\mathbf{u}-\mathbb{P}_h(\mathbf{u}))\|^2_{L^2(0,t;L^2)} + et\|\mathbb{P}_h(\nabla p)\|^2_{L^2(0,t;\mathbf{L}^2)}.$$

Again, it holds for almost all $t \in (0,T]$

$$\|\mathbb{P}_h(\nabla p(t))\|_{V_h^{0,*}} = \begin{cases} 0, & \text{for } \nabla\cdot[\mathbf{V}_h] = Q_h, \\ \|p - \pi_{Q_h}(p)\|_{L^2}, & \text{for } \nabla\cdot[\mathbf{V}_h] = Q_h. \end{cases}$$

Proof. With $\mathbf{w}_h := \mathbb{P}_h(\mathbf{u})$ and $\mathbf{e}_h := \mathbf{u}_h - \mathbf{w}_h$, the discrete and the continuous weak formulations imply

$$(\dot{\mathbf{e}}_h, \mathbf{v}_h) + \nu(\nabla\mathbf{e}_h, \nabla\mathbf{v}_h) = (\mathbf{u}_t - \dot{\mathbf{w}}_h, \mathbf{v}_h) + \nu(\nabla\mathbf{u} - \nabla\mathbf{w}_h, \nabla\mathbf{v}_h) - (p, \nabla\cdot\mathbf{v}_h).$$

Due to the special choice of $\mathbf{w}_h$ and the regularity of $\mathbf{u}_t$, it holds $(\mathbf{u}_t - \dot{\mathbf{w}}_h, \mathbf{e}_h) = 0$ which yields the claimed evolution equation for the error function.

Tested with $\mathbf{v}_h = \mathbf{e}_h$, one gets

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{ds}}\left(\|\mathbf{e}_h(s)\|^2_{L^2}\right) + \nu\|\nabla\mathbf{e}_h\|^2_{L^2} = \nu(\nabla\mathbf{u} - \nabla\mathbf{w}_h, \nabla\mathbf{e}_h) - (p, \nabla\cdot\mathbf{e}_h). \tag{17}$$

To prove the first error estimate, one can exploit $\mathbf{e}_h \in \mathbf{L}^2_\sigma$, apply Young's inequality, integrate in time, and use $\mathbf{e}_h(0) = \mathbf{0}$ to obtain

$$\frac{1}{2}\|\mathbf{e}_h(t)\|^2_{L^2} + \frac{\nu}{2}\|\nabla\mathbf{e}_h\|^2_{L^2(0,t;L^2)} \leq \nu\|\nabla(\mathbf{u}-\mathbf{w}_h)\|^2_{L^2(0,t;L^2)} + \frac{1}{\nu}\|\mathbb{P}_h(\nabla p)\|^2_{L^2(0,t;V_h^{0,*})}.$$

For the second estimate and $t > 0$, one starts again from (17) and estimates the pressure-dependent term by

$$(\nabla p(s), \mathbf{e}_h(s)) \leq \frac{t}{2}\|\mathbb{P}_h(\nabla p(s))\|^2_{L^2} + \frac{1}{2t}\|\mathbf{e}_h(s)\|^2_{L^2}$$

for all $s \in (0,t)$, yielding

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{ds}}\left(\|\mathbf{e}_h(s)\|^2_{L^2}\right) + \nu\|\nabla\mathbf{e}_h(s)\|^2_{L^2} \\ &\qquad \leq \nu\|\nabla(\mathbf{u}(s) - \mathbf{w}_h(s))\|^2_{L^2} + t\|\mathbb{P}_h(\nabla p(s))\|^2_{L^2} + \frac{1}{t}\|\mathbf{e}_h(s)\|^2_{L^2}. \end{aligned} \tag{18}$$

Finally, the differential version of Gronwall's lemma (Lemma 5.1) leads to

$$\begin{aligned} &\|\mathbf{e}_h(t)\|^2_{L^2} + \nu\|\nabla\mathbf{e}_h\|^2_{L^2(0,t;L^2)} \\ &\qquad \leq e\left(\nu\|\nabla(\mathbf{u}-\mathbf{w}_h)\|^2_{L^2(0,t;L^2)} + t\|\mathbb{P}_h(\nabla p)\|^2_{L^2(0,t;\mathbf{L}^2)}\right). \quad \square \end{aligned} \tag{19}$$

Remark 5.3. *In the divergence-free/pressure-robust first case, we get a locking-free estimate. Optimal convergence is verified on those meshes, where the $\mathbf{L}^2$ best approximation into $\mathbf{V}_h^0$ converges with optimal order in the $\mathbf{V}$ norm. In the classical case, we see a locking phenomenon for $\nu \to 0$, which is only excited by large irrotational forces in the momentum balance, which are collected by the pressure gradient. A (seemingly) better estimate without a locking phenomenon can be obtained by using the second estimate. However, this estimate is only better for small time intervals, since the error grows linearly with t.*

Theorem 5.4 (Pressure error). *For the continuous solution p of (14) and the discrete solution p_h of (16), it holds for all $t \in (0, T]$*

$$p_h(t) = \pi_{Q_h}(p(t)) + r_h(t)$$

with the error estimate

$$\|r_h(t)\|_{L^2} \leq \frac{1}{\beta_h}\Big(\|\mathbf{u}_t(t) - \dot{\mathbf{u}}_h(t)\|_{L^2} + \nu\|\nabla(\mathbf{u}(t) - \mathbf{u}_h(t))\|_{L^2} + \|\nabla(p(t) - \pi_{Q_h}(p(t)))\|_{\mathbf{V}_h^{\perp,*}}\Big). \quad (20)$$

Remark 5.5. *Note that for divergence-free/pressure-robust mixed methods, the pressure-dependent term in (20) drops out according to Lemma 3.2.*

Proof. For $r_h = p_h - \pi_{Q_h}(p)$ and any $\mathbf{v}_h \in \mathbf{V}_h^{\perp}$, the Galerkin orthogonality shows

$$\begin{aligned}(r_h, \nabla \cdot \mathbf{v}_h) &= (p_h - p, \nabla \cdot \mathbf{v}_h) + (p - \pi_{Q_h}(p), \nabla \cdot \mathbf{v}_h)\\ &= (\mathbf{u}_t - \mathbf{u}_{h,t}, \mathbf{v}_h) + \nu a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + (p - \pi_{Q_h}(p), \nabla \cdot \mathbf{v}_h).\end{aligned}$$

The rest of the proof is similar to the steady case, see Theorem 3.6. □

6 Pressure-robustness for the steady Navier–Stokes equations

The steady incompressible Navier–Stokes problem for $\mathbf{f} \in \mathbf{L}^2(\mathcal{D})$ reads

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f}, & \mathbf{x} &\in \mathcal{D},\\ \nabla\cdot\mathbf{u} &= 0, & \mathbf{x} &\in \mathcal{D},\\ \mathbf{u} &= \mathbf{u}_D, & \mathbf{x} &\in \partial\mathcal{D}. \end{aligned} \quad (21)$$

Here, $\mathbf{u}_D$ denotes the Dirichlet velocity boundary data. Denoting the nonlinear convection trilinear form by

$$c(\mathbf{a}, \mathbf{u}, \mathbf{v}) = ((\mathbf{a}\cdot\nabla)\mathbf{u}, \mathbf{v}), \quad (22)$$

a weak formulation of (21) with homogeneous Dirichlet velocity boundary conditions $\mathbf{u}_D = \mathbf{0}$ is given by: search for $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that for all $(\mathbf{v}, q) \in \mathbf{V} \times Q$ holds

$$\begin{aligned}\nu a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}),\\ b(\mathbf{u}, q) &= 0.\end{aligned} \quad (23)$$

Lemma 6.1 (Stability). *For any velocity solution $\mathbf{u} \in \mathbf{H}_0^1(\mathcal{D})$ of the steady Navier–Stokes problem (23), the following stability result holds:*

$$\|\nabla\mathbf{u}\|_{L^2} \leq \frac{C_P}{\nu}\|\mathbb{P}(\mathbf{f})\|_{L^2},$$

where C_P denotes the Poincaré-Friedrichs constant.

Proof. Testing (23) by $\mathbf{v} := \mathbf{u}$ yields

$$\nu\|\nabla\mathbf{u}\|_{L^2}^2 = (\mathbf{f}, \mathbf{u}) = (\mathbb{P}(\mathbf{f}), \mathbf{u}),$$

since c is skew-symmetric and it holds $b(\mathbf{u}, p) = 0$. Cauchy and Friedrichs inequalities conclude the proof. □

Remark 6.2. *The velocity solution* $\mathbf{u}$ does only depend *on the Helmholtz projector* $\mathbb{P}(\mathbf{f})$*, but not on the* entire *forcing* $\mathbf{f}$*. This shows that the fundamental invariance property from Lemma 2.5 for the incompressible Stokes equations holds for the steady incompressible Navier–Stokes equations, too. Further, for every irrotational forcing* $\mathbf{f} = \nabla\phi$*, one obtains the hydrostatic solution* $\mathbf{u} = \mathbf{0}$*.*

From a more mathematical viewpoint, this can be expressed equivalently as follows: for every $\mathbf{f} \in \mathbf{L}^2(\mathcal{D})$*, the expression* $\|\mathbb{P}(\mathbf{f})\|_{L^2}$ *denotes a semi-norm, since for all* $\phi \in H^1(\mathcal{D})$ *it holds* $\|\mathbb{P}(\nabla\phi)\|_{L^2} = 0$*. However, semi-norms induce naturally equivalence classes. Therefore, kinetic energy in the incompressible Navier–Stokes equations with homogeneous energy is only excited by forces from the equivalence class* $L^2_\sigma(\mathcal{D})$*, while forces from the orthogonal complement of* $L^2_\sigma(\mathcal{D})$ *in* $\mathbf{L}^2(\mathcal{D})$ *(with respect to the standard* $\mathbf{L}^2$ *scalar product) induce potential energy.*

Employing an inf-sup-stable pair of spaces $(\mathbf{V}_h, Q_h)$, the discrete steady Navier–Stokes problem seeks $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that, for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$,

$$\begin{aligned} \nu a(\mathbf{u}_h, \mathbf{v}_h) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h), \\ b(\mathbf{u}_h, q_h) &= 0. \end{aligned} \tag{24}$$

Lemma 6.3 (Discrete stability)**.** *For any velocity solution* $\mathbf{u}_h \in \mathbf{V}_h$ *of the steady Navier–Stokes problem* (24)*, the following stability result holds:*

$$\|\nabla\mathbf{u}_h\|_{L^2} \le \frac{C_P}{\nu}\|\mathbb{P}_h(\mathbf{f})\|_{L^2},$$

where C_P *denotes the Poincaré-Friedrichs constant.*

Proof. See the proof of Lemma 6.1. □

Lemma 6.4 (Nonlinear pressure-robustness for divergence-free methods)**.** *If it holds* $\nabla\cdot\mathbf{V}_h = Q_h$*, the discrete scheme is also pressure-robust in the nonlinear discrete problem* (24)*, which means: all velocity solutions* $\mathbf{u} \in \mathbf{V}$ *of* (23)*, for which it holds* $\mathbf{u} \in \mathbf{V}_h$ *are also discrete velocity solutions of* (24)*, completely independent of the pressure* p*. Indeed, in this case it further holds* $p_h = \pi_{Q_h}(p)$*.*

Proof. Testing with a divergence-free test function $\mathbf{v}_h \in \mathbf{V}_h^0$, the velocity results follows directly from the Galerkin orthogonality of (24). Indeed, for the continuous velocity solution $\mathbf{u}$ holds

$$\nu a(\mathbf{u}, \mathbf{v}_h) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h),$$

while for the discrete velocity solution $\mathbf{u}_h$ holds

$$\nu a(\mathbf{u}_h, \mathbf{v}_h) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h).$$

From $\mathbf{u} \in \mathbf{V}_h^0$ follows at once that $\mathbf{u}_h := \mathbf{u}$ is a discrete velocity solution of (24). For the pressure results, one obtains:

$$\begin{aligned} b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) - \nu a(\mathbf{u}_h, \mathbf{v}_h) - c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) \\ &= (\mathbf{f}, \mathbf{v}_h) - \nu a(\mathbf{u}, \mathbf{v}_h) - c(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) \\ &= b(\mathbf{v}_h, p) \\ &= b(\mathbf{v}_h, \pi_{Q_h} p), \end{aligned}$$

due to $\nabla\cdot\mathbf{V}_h = Q_h$. □

Remark 6.5. *It is emphasized that pressure-robustness in the sense of Lemma 6.4 for the steady Navier–Stokes equations does not hold for classical mixed finite elements like the Taylor–Hood element. For numerical examples demonstrating this statement, see [LM16c, Sections 3.3-3.4] and [LM16a, Sections 6.2-6.5].*

Remark 6.6. *An important class of benchmarks for the incompressible Navier–Stokes equations, where this concept of (nonlinear) pressure-robustness applies, are steady potential flows, see [LM16a]. For a given domain $\mathcal{D}$, we assume that $h \in H^2(\mathcal{D})$ is a harmonic potential, i.e., it holds $-\Delta h = 0$. Then,*

$$(\mathbf{u}, p) = (\nabla h, -\frac{1}{2}|\nabla h|^2)$$

fulfills the steady incompressible Navier–Stokes equations with (inhomogeneous) Dirichlet boundary data and right hand side $\mathbf{f} = \mathbf{0}$. Note that the pressure p is (in the sense of approximability) more difficult than the velocity field $\mathbf{u}$: if h is a harmonic polynomial of order $k+1$, then $\mathbf{u}$ is a polynomial of order k, and p is a polynomial of order $2k$.

The key feature of potential flows is that the nonlinear convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is irrotational, i.e. it holds $\mathbb{P}((\mathbf{u} \cdot \nabla)\mathbf{u}) = \mathbf{0}$. This requires an accurate pressure-robust space-discretization of the nonlinear convection term. Indeed, [LM16a] presents potential flow benchmarks, for which some pressure-robust discretizations are much more accurate than similar classical mixed methods.

Remark 6.7. *Also note that potential flows are not the only class of flows, for which it holds $\mathbb{P}((\mathbf{u} \cdot \nabla)\mathbf{u}) = \mathbf{0}$:*

1. *in Beltrami flows it holds $\nabla \times \mathbf{u} \neq \mathbf{0}$, but nevertheless it holds $(\nabla \times \mathbf{u}) \times \mathbf{u} = \mathbf{0}$, leading to $(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{2}\nabla(|\mathbf{u}|^2)$ as in potential flows;*
2. *in generalized Beltrami flows, $\nabla \times \mathbf{u} \neq \mathbf{0}$ and $(\nabla \times \mathbf{u}) \times \mathbf{u} \neq \mathbf{0}$ hold, but nevertheless it holds $(\nabla \times \mathbf{u}) \times \mathbf{u} = \nabla \chi$ for some potential χ. A linear, generalized Beltrami flow is given by rigid body rotations [LM16c].*

7 Pressure-robust siblings of non-divergence-free finite element methods

This section reports on a rather novel approach to obtain pressure-robust finite element methods without $\nabla \cdot \mathbf{V}_h \subseteq Q_h$.

7.1 Divergence-free reconstruction operator

The idea is to repair the orthogonality between irrotational forces ∇q and divergence-free vector fields $\mathbf{v}_h$ by replacing the scalar product

$$(\mathbf{f}, \mathbf{v}_h) \quad \text{by} \quad (\mathbf{f}, \Pi \mathbf{v}_h),$$

where Π is a linear operator with two important properties. The first one reads

$$\nabla \cdot (\Pi \mathbf{v}_h) = \pi_{Q_h}(\nabla \cdot \mathbf{v}_h) = \nabla_h \cdot \mathbf{v}_h \quad \text{for all } \mathbf{v}_h \in \mathbf{V}_h$$

and ensures that discretely divergence-free functions $\mathbf{v}_h \in \mathbf{V}_h^0$ are mapped to divergence-free functions $\Pi\mathbf{v}_h \in \mathbf{L}^2_\sigma(\mathcal{D})$. For this, Π maps into $\mathbf{H}(\mathrm{div})$-conforming finite element spaces of Raviart–Thomas or Brezzi–Douglas–Marini type [BF91]. The idea is that in the usual integration by parts formula for $(\nabla q, \mathbf{v}_h)$ not the divergence $\nabla \cdot \mathbf{v}_h$, but the discrete divergence $\nabla_h \cdot \mathbf{v}_h$ should pop up, since classical mixed finite element methods replace the continuous divergence by an appropriate discrete divergence

$$(\nabla q, \Pi\mathbf{v}_h) = -(q, \nabla_h \cdot \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in \mathbf{V}_h.$$

This also implies for all $q \in H^1(\mathcal{D})$ the desired orthogonality

$$(\nabla q, \Pi\mathbf{v}_h) = 0 \quad \text{for all } \mathbf{v}_h \in \mathbf{V}_h^0.$$

Secondly, the introduced consistency error should be small and of optimal order, i.e.

$$\|\mathbf{f} \circ (1 - \Pi)\|_{V_h^{0,\star}} := \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h^0} \frac{(\mathbf{f}, \mathbf{v}_h - \Pi\mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_{L_2}} \leq \mathcal{O}(h^k)\|\mathbf{f}\|_{H^{k-1}}.$$

Here, h denotes the mesh size, k denotes the optimal approximation order of the discrete pressure space in the L^2 norm, and $\|\bullet\|_{H^{k-1}}$ denotes the L^2-norm for $k = 1$ and the Sobolev semi norm $|\bullet|_{k-1}$ for $k \geq 2$.

Details on the design of appropriate reconstruction operators for various classical finite element methods can be found in the referenced literature, e.g. [LMT16, JLM+16, LM16b] for conforming finite element methods with discontinuous pressure spaces, [LLMS16] for the Taylor–Hood or mini finite element family or [Lin14a, BLMS15, LMW17] for the nonconforming Crouzeix–Raviart finite element.

7.2 Application to the steady Stokes equations

In case of the steady Stokes equations the modified method seeks $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that, for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$,

$$\begin{aligned} \nu a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \Pi\mathbf{v}_h) \\ b(\mathbf{u}_h, q_h) &= 0. \end{aligned} \tag{25}$$

Since only the right-hand side is modified, the inf-sup property of the pair $V_h \times Q_h$ is preserved.

Theorem 7.1 (Velocity error)**.** *For the continuous solution $\mathbf{u}$ of (7) and the discrete solution $\mathbf{u}_h$ of (25) it holds*

$$\mathbf{u}_h = \mathbb{S}_h(\mathbf{u}) + \mathbf{e}_h$$

where the perturbation $\mathbf{e}_h$ satisfies, for all $\mathbf{v}_h \in \mathbf{V}_{0,h}$,

$$a(\mathbf{e}_h, \mathbf{v}_h) = (\mathbb{P}(-\Delta\mathbf{u}), \Pi\mathbf{v}_h) - (\nabla\mathbf{u}, \nabla\mathbf{v})$$

with the error estimate

$$\|\nabla \mathbf{e}_h\|_{L_2} \leq \|\Delta\mathbf{u} \circ (1 - \Pi)\|_{\mathbf{V}_h^{0,*}}.$$

Proof. The best approximation property of $\mathbb{S}_h(\mathbf{u})$ and the Galerkin orthogonality shows, for $\mathbf{e}_h := \mathbf{u}_h - \mathbb{S}_h(\mathbf{u}) \in \mathbf{V}_h^0$ and any $\mathbf{v}_h \in \mathbf{V}_{0,h}$,

$$\begin{aligned} \nu a(\mathbf{u}_h, \mathbf{v}_h) &= (\mathbf{f}, \Pi \mathbf{v}_h) && \Leftrightarrow \\ a(\mathbf{e}_h, \mathbf{v}_h) &= \left(\frac{1}{\nu}\mathbb{P}(\mathbf{f}), \Pi \mathbf{v}_h\right) - a(\mathbf{u}, \mathbf{v}_h) && \Leftrightarrow \\ a(\mathbf{e}_h, \mathbf{v}_h) &= (-\mathbb{P}(\Delta \mathbf{u}), \Pi \mathbf{v}_h) - a(\mathbf{u}, \mathbf{v}_h). \end{aligned}$$

due to $\Pi \mathbf{v}_h \in \mathbf{L}^2_\sigma(\mathcal{D})$ and due to Lemma 2.8. Setting $\mathbf{v}_h = \mathbf{e}_h$ yields

$$\|\nabla \mathbf{e}_h\|^2_{L_2} \leq \|\Delta \mathbf{u} \circ (1 - \Pi)\|_{\mathbf{V}_h^{0,*}} \|\nabla \mathbf{e}_h\|_{L_2},$$

and a division by $\|\nabla \mathbf{e}_h\|$ concludes the proof. □

Example 7.2. *Figure 2 shows the error in the norm (13) of the classical Bernardi–Raugel finite element method and its modified sibling on the same mesh in the example specified in Section 4.2. For the modified Bernardi–Raugel element, a first order method, the operator Π is chosen as the standard interpolant of the BDM_1 element, which is elementwise defined. The classical method shows the expected locking behavior for $\nu \to 0$, while the modified method shows no error increase for smaller ν. Plenty of further examples are studied in former publications, see e.g. [JLM+16, LM16b, LM16c].*

Remark 7.3 (Application to the steady Navier–Stokes equations)**.** *In case of the steady Navier–Stokes equations [LM16b] also the nonlinear convection form c from (22) can be discretised pressure-robustly by*

$$c_h(\mathbf{a}_h, \mathbf{u}_h, \mathbf{v}_h) := ((\mathbf{a} \cdot \nabla)\mathbf{u}, \Pi \mathbf{v}_h).$$

or by a variant of the rotation form given by

$$c_h(\mathbf{a}_h, \mathbf{u}_h, \mathbf{v}_h) := (\mathrm{rot}\,\mathbf{a}_h \times \Pi \mathbf{u}_h, \Pi \mathbf{v}_h). \tag{26}$$

The latter choice has the advantage to be skew-symmetric, i.e. $c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) = -c_h(\mathbf{u}_h, \mathbf{v}_h, \mathbf{u}_h)$, which preserves the discrete kinetic energy and can be exploited in the analysis of (22).

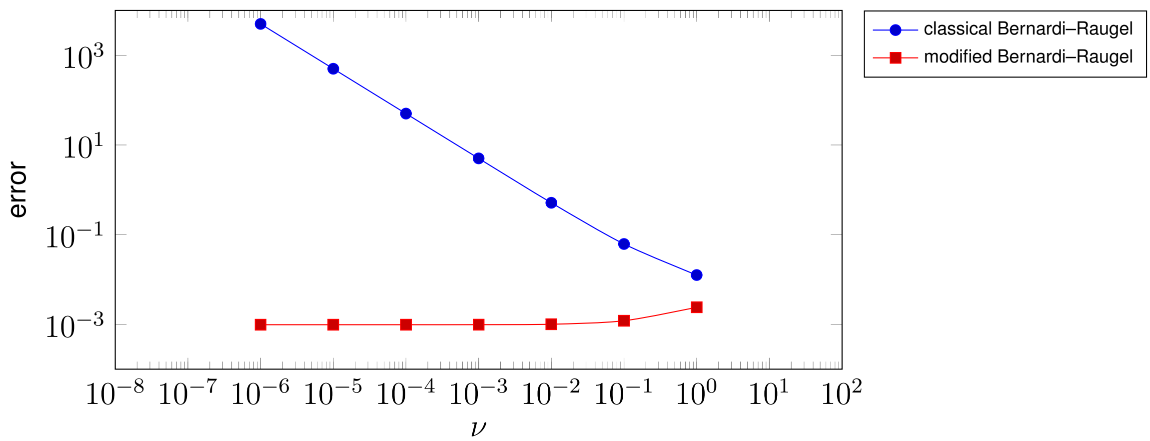

Figure 2: Best approximation error (13) of the classical and modified Bernardi–Raugel finite element method versus ν on a fixed unstructured mesh with 277 vertices in Example 7.2.

7.3 Application to the transient Stokes equations

Irrotational forces may not only appear in the right-hand side or the nonlinear convection term, but also in the time derivative. Therefore, in case of the transient Stokes equations, one can also use the reconstruction operator in the space discretization of the time derivative, i.e., replace

$$(\mathbf{u}_{t,h}, \mathbf{v}_h) \quad \text{by} \quad (\Pi \mathbf{u}_{t,h}, \Pi \mathbf{v}_h).$$

Hence, the discrete transient Stokes problem seeks $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{V}_h \times Q_h$ such that, for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$ and all $t \in (0, T]$,

$$\begin{aligned} (\Pi \dot{\mathbf{u}}_h(t), \Pi \mathbf{v}_h) + \nu a(\mathbf{u}_h(t), \mathbf{v}_h) + b(\mathbf{v}_h, p_h(t)) &= (\mathbf{f}(t), \Pi \mathbf{v}_h) \\ b(\mathbf{u}_h(t), q_h) &= 0, \end{aligned} \tag{27}$$

and the discrete initial value is given by $\mathbf{u}_h(0) = \mathbb{P}_h(\mathbf{u}_0)$. Of course, $\dot{\mathbf{u}}_h(t)$ could be replaced by any discrete approximation of the time derivative, see [ALM17] for details.

Under the additional assumption

$$C_1 \|\mathbf{v}_h\|_{L^2} \leq \|\Pi \mathbf{v}_h\|_{L^2} \leq C_2 \|\mathbf{v}_h\|_{L^2} \quad \text{for all } \mathbf{v}_h \in \ker(\Pi)^\perp, \tag{28}$$

where C_1 and C_2 denote constants, which depend on the shape-regularity of the mesh, but not on the mesh size, optimal pressure-robust error estimates can be obtained [ALM17]. The validity of this assumption is still under investigation, however, for several modified finite element methods studied in [ALM17] it is at least verified numerically.

Example 7.4. *To illustrate the benefits of pressure-robust space discretizations for the time-dependent Stokes equations, we consider a last example, which considers the potential flow* $\mathbf{u} := \nabla h$ *for the time-dependent harmonic function* $h(x,t) := \min(t,1)(x^3 - 3xy^2)$ *in the time interval* $[0,2]$ *on the domain* $\mathcal{D} := (-1,1)^2$ *and the viscosity* $\nu = 1/20$*. No exterior forces are applied, in order to concentrate on the influence of the space discretization of the time derivative which is approximated by a backward Euler scheme with equidistant time steps of length* $\tau = 0.01$*.*

Table 1 lists the best approximation errors for the unmodified and modified P_2^+ *finite element method, a second-order method that enriches the Taylor–Hood velocity ansatz space by cell bubbles to allow for piecewise linear discontinuous pressures. For the modified scheme, a second order scheme, the operator* Π *maps elementwise into* BDM_2*, and* Π *is actually the standard interpolator of* BDM_2 *[BF91]. The values show that the modified method produces the optimal best approximations (in case of the velocity it is even the exact velocity, since in this example* $\mathbb{S}_h(\mathbf{u}(t)) = \mathbf{u}(t)$*) in every time step, while the classical method shows some deviations. Also note, that the errors suddenly go down after the flow becomes stationary at* $t = 1$ *and go to to zero for* $t \to \infty$*. Hence, the error was only incited by the lack of pressure-robustness in the space discretization of the time derivative. An easy calculation shows that the exact pressure in this example reads* $p(t) = -h_t(t) = x^3 - 3xy^2$ *for* $t \in (0,1)$*.*

References

[ALM17] N. Ahmed, A. Linke, and C. Merdon, *On really locking-free mixed finite element methods for the transient incompressible Stokes equations*, WIAS Preprint 2368, 2017.

[BF91] F. Brezzi and M. Fortin, *Mixed and hybrid finite elements*, Springer Series in Computational Mathematics, vol. 15, Springer, 1991.

t	$\|\nabla(\mathbb{S}_h(\mathbf{u}) - \mathbf{u}_h)(t)\|_{L^2}$	$\|(\pi_{Q_h}(p) - p_h)(t)\|_{L^2}$
0.200	6.0985e-01/1.1158e-12	1.7151e-02/1.4948e-14
0.400	6.1211e-01/1.1218e-12	1.7135e-02/2.7488e-14
0.600	6.1233e-01/1.1777e-12	1.7130e-02/2.5178e-14
0.800	6.1235e-01/1.3101e-12	1.7128e-02/4.5991e-14
1.000	6.1234e-01/1.4053e-12	1.7127e-02/3.7512e-14
1.200	9.5237e-03/7.2701e-13	1.9888e-04/2.4882e-14
1.400	3.1821e-03/6.6881e-13	8.1772e-05/3.0147e-14
1.600	1.5822e-03/7.2887e-13	4.4317e-05/2.5122e-14
1.800	9.2117e-04/6.5868e-13	2.7249e-05/2.9482e-14
2.000	5.9139e-04/7.7912e-13	1.8137e-05/3.0043e-14
$\vdots$		
∞	3.6315e-13/4.2315e-13	2.7708e-15/8.4975e-15

Table 1: Best approximation errors for the unmodified/modified P_2^+ finite element method at particular times in Example 7.4.

[BLMS15] C. Brennecke, A. Linke, C. Merdon, and J. Schöberl, *Optimal and pressure-independent L^2 velocity error estimates for a modified Crouzeix–Raviart Stokes element with BDM reconstructions*, J. Comput. Math. **33** (2015), no. 2, 191–208. MR 3326010

[BS92a] I. Babuška and M. Suri, *On locking and robustness in the finite element method*, SIAM Journal of Numerical Analysis **29** (1992), 1261âĂŞ1293.

[BS92b] Ivo Babuška and Manil Suri, *Locking effects in the finite element approximation of elasticity problems*, Numer. Math. **62** (1992), no. 4, 439–463. MR 1174468

[Cod99] Ramon Codina, *Numerical solution of the incompressible Navier-Stokes equations with Coriolis forces based on the discretization of the total time derivative*, J. Comput. Phys. **148** (1999), no. 2, 467–496. MR 1669719 (99i:76088)

[CR73] M. Crouzeix and P.-A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge **7** (1973), no. R-3, 33–75. MR 0343661 (49 #8401)

[DGT94] O. Dorok, W. Grambow, and L. Tobiska, *Aspects of finite element discretizations for solving the Boussinesq approximation of the Navier-Stokes Equations*, Notes on Numerical Fluid Mechanics: Numerical Methods for the Navier-Stokes Equations. **47** (1994), 50–61.

[FF85] M. Fortin and A. Fortin, *Newer and newer elements for incompressible flows*, Finite Elements in Fluids **6** (1985), 171–187.

[Fro98] P. Frolkovic, *Consistent velocity approximation for density driven flow and transport*, Advanced Computational Methods in Engineering, Part 2, 1998, pp. 603–611.

[GJ05] S. Ganesan and V. John, *Pressure separation — a technique for improving the velocity error in finite element discretisations of the Navier-Stokes equations*, Appl. Math. Comp. **165** (2005), no. 2, 275–290.

[GLBB97] J.-F. Gerbeau, C. Le Bris, and M. Bercovier, *Spurious velocities in the steady flow of an incompressible fluid subjected to external forces*, International Journal for Numerical Methods in Fluids **25** (1997), no. 6, 679–695 (Anglais).

[GLCL80] P. M. Gresho, R. L. Lee, S. T. Chan, and J. M. Leone, *A new finite element for incompressible or Boussinesq fluids*, Proceedings of the Third International Conference on Finite Elements in Flow Problems, Wiley, 1980.

[GLRW12] K. Galvin, A. Linke, L. Rebholz, and N. Wilson, *Stabilizing poor mass conservation in incompressible flow problems with large irrotational forcing and application to thermal convection*, Comput. Methods Appl. Mech. Engrg. **237/240** (2012), 166–176. MR 2947652

[GMT07] S. Ganesan, G. Matthies, and L. Tobiska, *On spurious velocities in incompressible flow problems with interfaces*, Comp. Methods Appl. Mech. Engrg. **196** (2007), 1193–1202.

[GN14a] Johnny Guzmán and Michael Neilan, *Conforming and divergence-free Stokes elements in three dimensions*, IMA J. Numer. Anal. **34** (2014), no. 4, 1489–1508. MR 3269433

[GN14b] ———, *Conforming and divergence-free Stokes elements on general triangular meshes*, Math. Comp. **83** (2014), no. 285, 15–36. MR 3120580

[GR86] V. Girault and P.-A. Raviart, *Finite element methods for Navier-Stokes equations*, Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, Berlin, 1986.

[JLM+16] V. John, A. Linke, C. Merdon, M. Neilan, and L. Rebholz, *On the divergence constraint in mixed finite element methods for incompressible flows*, SIAM Review **accepted** (2016).

[Joh16] Volker John, *Finite element methods for incompressible flow problems*, 1st ed. ed., Springer Series in Computational Mathematics, vol. 51, Springer International Publishing, Cham, 2016.

[Lin09] A. Linke, *Collision in a cross-shaped domain — a steady 2d Navier–Stokes example demonstrating the importance of mass conservation in CFD*, Computer Methods in Applied Mechanics and Engineering **198** (2009), no. 41–44, 3278–3286.

[Lin14a] A. Linke, *On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime*, Comput. Methods Appl. Mech. Engrg. **268** (2014), 782–800. MR 3133522

[Lin14b] Alexander Linke, *On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime*, Comput. Methods Appl. Mech. Engrg. **268** (2014), 782–800. MR 3133522

[LLMS16] P. Lederer, A. Linke, C. Merdon, and J. Schöberl, *Divergence-free reconstruction operators for pressure-robust Stokes discretizations with continuous pressure finite elements*, SIAM Journal on Numerical Analysis **accepted** (2016).

[LM16a] A. Linke and C. Merdon, *Pressure-robustness and discrete Helmholtz projectors in mixed finite element methods for the incompressible Navier–Stokes equations*, Comput. Methods Appl. Mech. Engrg. **311** (2016), 304–326.

[LM16b] ———, *Pressure-robustness and discrete helmholtz projectors in mixed finite element methods for the incompressible navier-stokes equations*, Computer Methods in Applied Mechanics and Engineering **311** (2016), 304–326.

[LM16c] Alexander Linke and Christian Merdon, *On velocity errors due to irrotational forces in the Navier–Stokes momentum balance*, Journal of Computational Physics **313** (2016), 654–661.

[LMT16] Alexander Linke, Gunar Matthies, and Lutz Tobiska, *Robust arbitrary order mixed finite element methods for the incompressible Stokes equations with pressure independent velocity errors*, ESAIM: M2AN **50** (2016), no. 1, 289–309.

[LMW17] A. Linke, C. Merdon, and W. Wollner, *Optimal L^2 velocity error estimate for a modified pressure-robust Crouzeix–Raviart Stokes element*, IMA J. Numer. Anal. **37** (2017), no. 1, 354–374.

[LS16] Christoph Lehrenfeld and Joachim Schöberl, *High order exactly divergence-free hybrid discontinuous Galerkin methods for unsteady incompressible flows*, Comput. Methods Appl. Mech. Engrg. **307** (2016), 339–361. MR 3511719

[Nei15] Michael Neilan, *Discrete and conforming smooth de Rham complexes in three dimensions*, Math. Comp. **84** (2015), no. 295, 2059–2081. MR 3356019

[OR04] M. Olshanskii and A. Reusken, *Grad-div stabilization for Stokes equations*, Math. Comp. **73** (2004), no. 248, 1699–1718.

[PFC89] Dominique Pelletier, André Fortin, and Ricardo Camarero, *Are FEM solutions of incompressible flows really incompressible? (or how simple flows can cause headaches!)*, Internat. J. Numer. Methods Fluids **9** (1989), no. 1, 99–112. MR 977201

[Qin94] J. Qin, *On the convergence of some low order mixed finite elements for incompressible fluids*, Ph.D. thesis, Pennsylvania State University, 1994.

[QZ07] Jinshui Qin and Shangyou Zhang, *Stability and approximability of the P_1-P_0 element for Stokes equations*, Internat. J. Numer. Methods Fluids **54** (2007), no. 5, 497–515. MR 2322456 (2008b:65153)

[Sch97] Friedhelm Schieweck, *Parallele*, Habilitation, University of Magdeburg, 1997.

[Soh12] H. Sohr, *The navier-stokes equations — an elementary functional analytic approach*, Birkhäuser, 2012.

[Ste90] Rolf Stenberg, *Error analysis of some finite element methods for the Stokes problem*, Math. Comp. **54** (1990), no. 190, 495–508. MR 1010601

[TS87] R. W. Thatcher and D. Silvester, *A locally mass conserving quadratic velocity, linear pressure element*, 1987, Numerical Analysis Report No, 147, Manchester University/UMIST.

[TTK88] D. M. Tidd, R. W. Thatcher, and A. Kaye, *The free surface flow of Newtonian and non-Newtonian fluids trapped by surface tension*, Internat. J. Numer. Methods Fluids **8** (1988), no. 9, 1011–1027. MR 961777

[Zha05] S. Zhang, *A new family of stable mixed finite elements for the 3D Stokes equations*, Math. Comp. **74** (2005), no. 250, 543–554.

[Zha09] Shangyou Zhang, *A family of $Q_{k+1,k} \times Q_{k,k+1}$ divergence-free finite elements on rectangular grids*, SIAM J. Numer. Anal. **47** (2009), no. 3, 2090–2107. MR MR2519595

[Zha11a] ______, *Divergence-free finite elements on tetrahedral grids for $k \geq 6$*, Math. Comp. **80** (2011), no. 274, 669–695. MR 2772092 (2012b:65191)

[Zha11b] ______, *Quadratic divergence-free finite elements on Powell-Sabin tetrahedral grids*, Calcolo **48** (2011), no. 3, 211–244. MR 2827006