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# On convergences of the squareroot approximation scheme to the Fokker–Planck operator


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# On convergences of the squareroot approximation scheme to the Fokker–Planck operator

Martin Heida

**Abstract**

We study the qualitative convergence behavior of a novel FV-discretization scheme of the Fokker-Planck equation, the squareroot approximation scheme (SQRA), that recently was proposed by Lie, Fackeldey and Weber [31] in the context of conformation dynamics. We show that SQRA has a natural gradient structure and that solutions to the SQRA equation converge to solutions of the Fokker-Planck equation using a discrete notion of G-convergence for the underlying discrete elliptic operator. The SQRA does not need to account for the volumes of cells and interfaces and is taylored for high dimensional spaces. However, based on FV-discretizations of the Laplacian it can also be used in lower dimensions taking into account the volumes of the cells. As an example, in the special case of stationary Voronoi tessellations we use stochastic two-scale convergence to prove that this setting satisfies the G-convergence property.

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## 1 Introduction

In a recent work [31], the so-called squareroot approximation (SQRA) operator has been introduced, based on earlier related works [17, 30]. The SQRA-scheme was introduced as a finite volume scheme on a random Voronoi discretization designed for numerical simulation of large molecules in the framework of conformation dynamics. Hence it is interesting to know whether the SQRA-operator converges in some sense to a physically reasonable continuous operator as the discretization becomes finer and finer. A major contribution of this work is a positive answer to that question, i.e. that the SQRA-operator converges to the (physically expected) Fokker-Planck operator, which is also known as the Smoluchowski operator in conformation dynamics. Furthermore, we will see below that the SQRA scheme possesses a gradient structure which is a natural discretization of the gradient structure behind the Fokker-Planck equation discovered in [29]. The convergence behavior will be considered for Dirichlet and for periodic boundary conditions.

In order to introduce the SQRA operator, let $\boldsymbol{Q} \subset \mathbb{R}^n$ be a bounded domain representing the state space of a given molecule with a family of points $(P_i)_{i=1,\dots m} \subset \boldsymbol{Q}$. From these points we construct a Voronoi tessellation of cells $G_i$ that correspond to $P_i$ for every $i$.

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We write $i \sim j$ if the cells $G_i$ and $G_j$ are neighbored. Thus, the finite volume space for the discretization $(G_i)_{i=1,\dots m}$ is isomorphic to $\mathbb{R}^m$. Let $u_i(t)$ be the probability that the state of the molecule lies in $G_i$ at time $t > 0$ and consider the vector $u(t) \in \mathbb{R}^m$. The corresponding master equation reads

$$\dot{u}(t) = u\mathfrak{Q} \tag{1}$$

for a matrix $\mathfrak{Q} \in \mathbb{R}^{m\times m}$, the *matrix of kinetic rates* or simply *rate-matrix.* If the evolution of the molecule is Markovian, we obtain the master equation of the form (see Section 5.2 in [24])

$$\dot{u}_i = \sum_{j\sim i} \left(\mathfrak{Q}_{ji}u_j - \mathfrak{Q}_{ij}u_i\right) , \tag{2}$$

also known as the forward Kolmogorov equation of the molecule in the discrete phase space $\mathbb{R}^m$. The corresponding backward equation $\dot{n} = \mathfrak{Q}n$ describes the evolution of an ensemble of $\int_{\boldsymbol{Q}} n$ particles in the state space and has been analyzed in [31]. It turns out that for $\pi := \exp(-\beta V)$ the rate $\mathfrak{Q}_{ij}$ is given by

$$\mathfrak{Q}_{ij} = \frac{1}{\pi_i}\int_{\partial G_i\cap\partial G_j} \Phi\pi\,, \qquad \pi_i := \pi(P_i) \approx \int_{G_i} \pi\,.$$

Here, $\Phi$ represents the flux in case $V \equiv 0$. A crucial assumption for numerical efficiency in [31] is that the mass of $G_i$ and of $\partial G_i \cap \partial G_j$ are approximately constant among $i$ and $(i,j)$. Since the problem in case $V = 0$ is isotropic, it is assumed that $⨍_{\partial G_i\cap\partial G_j} \Phi \approx \hat{\Phi}_m$ for some constant $\hat{\Phi}_m$, which only depends on the fineness of the discretization. Finally, it is assumed that $\pi \approx \sqrt{\pi_i\pi_j}$ on $\partial G_i \cap \partial G_j$, implying $\mathfrak{Q}_{ij} = \hat{\Phi}_m\sqrt{\pi_j/\pi_i}$. Hence we may reformulate (2) as

$$\dot{u}_i = \hat{\Phi}_m \sum_{j\sim i} \sqrt{\pi_i\pi_j}\left(\frac{1}{\pi_j}u_j - \frac{1}{\pi_i}u_i\right) ,$$

with $u_i = \pi_i$ being the only stationary solution. The latter, in turn, is the classical statement of Boltzmann statistics [24]. Since we may write the operator on the right hand side as

$$\left(\mathcal{F}_m u\right)_i := \hat{\Phi}_m \sum_{i\sim j}\left(u_j\frac{\sqrt{\pi_i}}{\sqrt{\pi_j}} - u_i\frac{\sqrt{\pi_j}}{\sqrt{\pi_i}}\right) , \qquad \mathcal{F}_m\pi = 0,$$

this operator is called the Squareroot Approximation Operator. In this work, it will be written as

$$\left(\mathcal{F}_m u\right)_i := \hat{\Phi}_m \sum_{i\sim j}\left(u_j\frac{v_i}{v_j} - u_i\frac{v_j}{v_i}\right) , \tag{3}$$

where $v_i = \sqrt{\pi_i} = \exp\left(-\frac{\beta}{2}V(P_i)\right)$. As we will see below in Theorem 1.13, the natural scaling of $\hat{\Phi}_m$ is $\hat{\Phi}_m \approx \Phi_0\varepsilon^{-2}$, where $\varepsilon$ is the characteristic length scale of the diameter of the Voronoi cells. It turns out that $\Phi_0$ can in principle be estimated from the case $V \equiv 0$, i.e. from the discrete Laplace operator $\mathcal{L}_m$ which is given as

$$\left(\mathcal{L}_m u\right)_i := \hat{\Phi}_m \sum_{j\sim i}\left(u_j - u_i\right) . \tag{4}$$

More precisely, Theorem 1.6 states that the convergence behavior of $\mathcal{F}_m$ is mostly characterized by the convergence behavior of $\mathcal{L}_m$: If $\mathcal{L}_m$ is G-convergent (in the discrete sense) to $\mathcal{L}u = \nabla \cdot (A_{\text{hom}} \nabla u)$, the solutions $u_m$ of the equation $\mathcal{F}_m u_m = f_m$ converge to solutions $\mathcal{F}u := \nabla \cdot (A_{\text{hom}} \nabla u) + \operatorname{div}(u A_{\text{hom}} \nabla V) = f$, provided $f_m \to f$ in a weak sense. Note that the opposite direction is trivial: If the SQRA converges for all $V \in C^2(\overline{\boldsymbol{Q}})$ then $\mathcal{L}_m \to \nabla \cdot A_{\text{hom}} \nabla$.

The derivation of (3) in [31] was done within the setting of conformation dynamics which is an alternative approach to molecular dynamics. In molecular dynamics, the behavior of a molecule is simulated using its Newtonian equations. Conformation dynamics uses a "dual" approach: Instead of one single molecule, an ensemble of molecules is studied within the energy landscape of the state space. Each region in the state space, corresponding to one of the Voronoi cells $G_i$, is then occupied by a certain number $|G_i| n_i$ of molecules and $\mathfrak{Q}_{ij} n_i$ is the flux of molecules from $G_i$ to $G_j$. The adjoint $\mathfrak{Q}^T$ operates on the probabilities $u_i$ introduced above. A conformation is a subregion of the state space such that it is very unlikely the molecule will leave this region. Hence a change of conformation occurs on very large time-scales compared to thermal oscillations. On the other hand, these long-time changes of conformation are crucial for the understanding of many biochemical processes. In the discrete setting, the conformations can be identified from $\mathfrak{Q}$ using Perron-Cluster analysis [9]. The result is a set of conformations $(C_i)_{i=1,\dots I}$, a reduced matrix $\tilde{\mathfrak{Q}} \in \mathbb{R}^{I \times I}$ and a linear equation $\dot{C} = C\tilde{\mathfrak{Q}}$, where usually $I \ll m$. With the reduced matrix $\tilde{\mathfrak{Q}}$, a large time scale simulation is then feasible, which ignores thermal short term oscillations.

Let us note that in the 1-dimensional setting, the SQRA was already mentioned in [2] and also more recently derived in [12, 11] from a completely different point of view using entropy minimization methods. The resulting formula shows some similarities to the Boltzmann collision integral (see e.g. [21]), but was not derived in this setting. Also there is a relation to the Butler-Volmer kinetics, which we will not carry out in detail at this point, but refer to [40]. We only note that the electrical current between two states can be recast into an expression that has the form $\boldsymbol{j}_0 \left( \frac{u_j}{\pi_j} - \frac{u_i}{\pi_i} \right)$.

The SQRA scheme was originally introduced under the restrictions that the underlying grid is given by a Voronoi discretization and that the volume of the cells and the interfaces is neglected. This was done in order to break the curse of dimensionality. Remark that the phase spaces of molecules are of very high dimension (order $10^3$) and even after significant dimension reduction - already for dimension 6 - it is computationally not feasible to estimate the respective volumes in reasonable time. However, the main theorems 1.6 and 1.8 only need that the underlying linear FV-operator G-converges and we could also apply the convergence result to any finite element discretization of $-\Delta$ where the volumes are incorporated into the weights $a_{ij}^{\varepsilon}$ in (9)–(10), which is interesting for low-dimensional simulations (up to dimension 3). For the case of Voronoi-cells, there exists huge literature [13, 15, 16, 38, 43] on the discretization of second order elliptic operators and the proofs of convergence usually imply G-convergence.

## 1.1 The gradient structure of the SQRA equation

Interestingly, the SQRA possesses a gradient structure, which is asymptotically compatible with the gradient structure of the Fokker-Planck equation studied by Jordan, Kinderlehrer and Otto [29]. In a recent work by Mielke, Peletier and Renger [37], it was shown that (1) possesses a gradient structure for the energy potential $\mathcal{E}_{\pi}^{\varepsilon}$ and dissipation potential $\Psi_\varepsilon^*$ given through

$$\mathcal{E}_\pi^\varepsilon(u) = \frac{1}{2}\sum_i u_i \ln\frac{u_i}{\pi_i}\,, \tag{5}$$

$$\Psi_\varepsilon^*(u,\xi) = \frac{1}{2}\sum_i\sum_{i\sim j}\sqrt{u_iu_j\mathfrak{Q}_{ij}\mathfrak{Q}_{ji}}\left(\cosh\left(\xi_j-\xi_i\right)-1\right)\,, \tag{6}$$

iff the process is weakly reversible (meaning $\mathfrak{Q}_{ij}>0$ iff $\mathfrak{Q}_{ji}>0$). In particular, this means that

$$\dot u(t) = u\mathfrak{Q} = \partial_\xi\Psi_\varepsilon^*\left(u,\,-D\mathcal{E}_\pi^\varepsilon(u)\right)\,.$$

In the setting of this work, the above gradient structure can be verified from $D_{u_i}\mathcal{E}_\pi(u) = \frac{1}{2}\ln\frac{u_i}{\pi_i}+1$ and

$$\partial_{\xi_i}\Psi_\varepsilon^*(u,\xi) = \varepsilon^{-2}\Phi_0\frac{1}{2}\sum_{i\sim j}\sqrt{u_iu_j}\left(-\exp\left(\xi_j-\xi_i\right)+\exp\left(\xi_i-\xi_j\right)\right)\,,$$

where we used that $\hat\Phi_m\approx\Phi_0\varepsilon^{-2}$ and hence $\mathfrak{Q}_{ij}\mathfrak{Q}_{ji}\approx\Phi_0^2\varepsilon^{-4}$. This yields

$$\begin{aligned}\partial_{\xi_i}\Psi_\varepsilon^*\left(u,\,-D\mathcal{E}_\pi^\varepsilon(u)\right) &= \varepsilon^{-2}\Phi_0\sum_{i\sim j}\sqrt{u_iu_j}\left(-\frac{\exp\left(-\frac{1}{2}\ln\frac{u_j}{\pi_j}\right)}{\exp\left(-\frac{1}{2}\ln\frac{u_i}{\pi_i}\right)}+\frac{\exp\left(-\frac{1}{2}\ln\frac{u_i}{\pi_i}\right)}{\exp\left(-\frac{1}{2}\ln\frac{u_j}{\pi_j}\right)}\right)\\ &= \varepsilon^{-2}\Phi_0\sum_{i\sim j}\sqrt{u_iu_j}\left(-\sqrt{\frac{u_i}{\pi_i}}\sqrt{\frac{u_j}{\pi_j}}^{-1}+\sqrt{\frac{u_j}{\pi_j}}\sqrt{\frac{u_i}{\pi_i}}^{-1}\right)\\ &= \left(\mathfrak{Q}^Tu\right)_i\,.\end{aligned}$$

The formal limits of $\mathcal{E}_\pi^\varepsilon(u)$ and $\Psi_\varepsilon^*$ as $\varepsilon\to0$ are given by

$$\mathcal{E}(u) = \int_{\boldsymbol{Q}}\frac{1}{2}u\ln\frac{u}{\pi}\,,\qquad \Psi_{\mathscr{L}}^*(u,\xi) = \frac{1}{2}\int_{\boldsymbol{Q}}u\left|\nabla\xi\right|^2\,, \tag{7}$$

where we used that (formally) $\sqrt{u_iu_j}\to u$ and $\varepsilon^{-2}\cosh\left(\xi_j-\xi_i\right)\approx\varepsilon^{-2}\left(\xi_j-\xi_i\right)^2\to\left|\nabla\xi\right|^2$ for ever finer discretizations of size $\varepsilon$. This is the setting in [29] but the chemical potential $\ln u$ has already been used before by DiPerna and Lions [10] for analysis of the Boltzmann equation. Hence, Theorem 1.8 can be interpreted in a sense that solutions to the gradient flows with respect to (5)–(6) converge to a solution of the gradient flow with respect to (7) provided the underlying linear operator G-converges. However, we do not prove (evolutionary) $\Gamma$-convergence of (5)–(6) to (7).

A related ansatz for a discrete Fokker-Planck equation by Chow et.al. [6] and Maas[32] uses the same energy but a different dissipation potential. In the resulting discrete equation appear nonlinear terms which are not well suited in numerical calculations. A further related approach by Mielke [35] in one space dimension comes up with the same discretized equation as the SQRA.

## 1.2 The linear finite-volume operator

Voronoi finite volume schemes for Fokker–Planck equations are used widely in literature. A first breakthrough for those methods was the Scharfetter-Gummel scheme [42], which has been used extensively in the simulation of semiconductor devices, though the idea even goes back to a work by Macneal [33]. These approximation schemes use the knowledge on the volumes of the cells and the interfaces, as their aim is a simulation in spaces of low dimension (up to 3) while SQRA has been developed for high dimensions. Hence, we cannot use the convergence behavior of the discrete linear operators derived for such low dimension. In contrast, we will use methods from homogenization theory such as G-convergence in the sense of Dal Maso [7]. However, as mentioned above, our result can also be applied to the classical Voronoi discretization in the above sense, since these discretizations also G-converge.

As an example for a non-classical discretization scheme without volumetric terms, we will study the exemplary case of a stationary ergodic partition with Voronoi cells in Theorem 1.13 and show that this operator is $G$-convergent. The scheme in [14] suggests that $P_m$ sometimes is *"reasonably close"* to such a stationary ergodic process and the results in [14] show that the discretization (3) has good properties in application. In the general Theorem 1.6 we prove convergence of (3) under the assumption that the discrete Laplace operator G-converges. Moreover, Theorem 1.6 tells us that the normalizing constants in (4) and (3) should be the same. Hence, in numerical application, one can determine $C_m$ by comparing the first eigenvalue of $\mathcal{L}_m$ with the first eigenvalue of $\Delta$. On the other hand, this ansatz provides us with a practical criteria to qualitatively validate the convergence of $\mathcal{F}_m$ apriori. More precisely, we can expect that the numerical approximation is good if $-\mathcal{L}_m u \approx -C\Delta u$ for the first $k$ eigenvectors of $-\Delta$ on $\boldsymbol{Q}$.

The stochastic homogenization of the discrete Laplace operator (also known as homogenization in the random conductance model) has been studied very well in recent years, as it is of great interest for physicists (see [4]) and mathematicians (see [3]). The motivation originally comes from random walk theory, where the elliptic operator is the generator of the semigroup generated by the random walk.

In view of the vast literature on stochastic homogenization of elliptic problems, Theorem 1.13 is not a surprising result. However, we are not aware of a suitable proof in literature that applies to this particular setting. The method used in order to proof Theorem 1.13 is a weak$^*$ convergence method called two-scale convergence. It is based on the two-scale convergence introduced by Zhikov and Piatnitsky in [44] and generalized and applied in the context of random walk theory in the works [18, 19]. In a slightly different way, two-scale convergence has also been applied in [34].

A novelty of the theory presented below is the application of two-scale convergence to a grid that differs from $\mathbb{Z}^n$, which made it necessary to modify certain notions and concepts. In this context, note that our spaces $L^2_{\mathrm{pot}}$ and $L^2_{\mathrm{sol}}$ indeed differ from the standard definition in [3], as we drop for example the covariance condition. Another approach to unstructured grids has recently been followed by Alicandro, Cicalese and Gloria [1]. They study homogenization of nonlinear elasticity problems and in the quadratic case their result could also be applied to the elliptic operator $\mathcal{L}_m$, yielding somehow a different concept of notation (i.e. $\Gamma$-convergence) and a formally different formulation of the limiting matrix $A_{\mathrm{hom}}$.

For further reference to the random conductance model, we refer to the aforementioned review by Biskup [3].

Let us finally comment on the convergence rate. We will only prove qualitative convergence and the question of quantitative convergence is completely open. However, we know from literature on stochastic homogenization of the continuous and the discrete Laplace operator that the best convergence rate we can expect is $\varepsilon^{\frac{1}{2}}$ in presence of Dirichlet boundary conditions, see the above mentioned literature for Voronoi FV-methods, and $\varepsilon$ for unbounded domains or periodic boundary conditions, see the recent work [20] and references therein. Since the Fokker-Planck equation is a linearly perturbed Laplace equation, we expect the same convergence rate for the SQRA-operator as for the underlying discrete Laplace operator. However, for the discretization presented in this work, the author is not aware of results for the convergence rate of the discrete Laplace operator.

## 1.3 Main results

We now formulate the major results of this article in a rigorous way. For a definition of the notions stationarity and ergodicity, we refer to Section 2.

For every $\varepsilon > 0$ let $P^\varepsilon = \bigcup_{i\in\mathbb{N}} P_i^\varepsilon$ be a countable family of points in $\mathbb{R}^n$ with corresponding Voronoi cells $G^\varepsilon := \bigcup_i G_i^\varepsilon$. We denote by $E^\varepsilon$ the set of all natural pairs $(i,j) \in \mathbb{N}^2$ such that $G_i^\varepsilon$ and $G_j^\varepsilon$ are neighbored where we identify $(i,j)$ with $(j,i)$ and write $i \sim j$. For $(i,j) \in E^\varepsilon$ we define $\Gamma_{ij}^\varepsilon := \frac{1}{2}\left(P_i^\varepsilon + P_j^\varepsilon\right)$.

*Notation* 1.1. We denote by $\mathcal{S}^\varepsilon$ the set of all functions $(P_i^\varepsilon)_{i\in\mathbb{N}} \to \mathbb{R}$ which is a Hilbert space with the scalar product

$$\langle \varphi, \psi \rangle_{\mathcal{S}^\varepsilon} := \sum_i \varphi_i \psi_i \,.$$

For every $u \in \mathcal{S}^\varepsilon$ we write $u_i^\varepsilon := u(P_i^\varepsilon)$ and for every $f : \Gamma^\varepsilon \to \mathbb{R}$ we write $f_{ij}^\varepsilon := f(\Gamma_{ij}^\varepsilon)$. Furthermore, we write $\bar{u}_{ij}^\varepsilon := \frac{1}{2}\left(u_i^\varepsilon + u_j^\varepsilon\right)$ such that $\bar{u}_{ij}^\varepsilon : \Gamma^\varepsilon \to \mathbb{R}$.

We define $\mathcal{R}_\varepsilon : L^2_{loc}(\mathbb{R}^n) \to \mathcal{S}^\varepsilon$ and its adjoint $\mathcal{R}_\varepsilon^* : \mathcal{S}^\varepsilon \to L^2_{loc}(\mathbb{R}^n)$ through

$$\left(\mathcal{R}_\varepsilon \phi\right)_i = |G_i^\varepsilon|^{-1} \int_{G_i^\varepsilon} \phi \,, \qquad \text{and} \qquad \left(\mathcal{R}_\varepsilon^* u\right)[x] = u(P_i^\varepsilon) \qquad \text{if} \quad x \in G_i^\varepsilon \,,$$

If $(i,j) \in E^\varepsilon$, we denote $\partial G_{ij}^\varepsilon$ the interface between $G_i^\varepsilon$ and $G_j^\varepsilon$ and $\nu_{ij}$ the unit vector pointing from $P_i^\varepsilon$ to $P_j^\varepsilon$. Hence, we find $\nu_{ij} = -\nu_{ji}$. Furthermore, we define

$$\Gamma^\varepsilon := \bigcup_{(i,j)\in E^\varepsilon} \Gamma_{ij}^\varepsilon \qquad \text{and} \qquad \partial G^\varepsilon := \bigcup_{(i,j)\in E^\varepsilon} \partial G_{ij}^\varepsilon \,.$$

The *jump operator* on $\partial G_{ij}^\varepsilon$ for a function $u \in \mathcal{S}^\varepsilon$ is given through $[u]_{ij} := (u_j - u_i)$. Then, for every $\phi \in \mathcal{S}^\varepsilon$ and $\psi \in C_c^1(\mathbb{R}^n)^n$ it holds:

$$\int_{G^\varepsilon} \left(\mathcal{R}_\varepsilon^* \phi\right) \nabla \cdot \psi \, d\mathcal{L} = \sum_i \sum_{i\sim j} \int_{\partial G_{ij}^\varepsilon} \phi_i \nu_{ij} \cdot \psi d\mathcal{H}^{n-1} = -\sum_{(i,j)\in E^\varepsilon} \int_{\partial G_{ij}^\varepsilon} [\![\phi]\!]_{ij} \cdot \psi d\mathcal{H}^{n-1} \,, \tag{8}$$

where we introduced $[\![\phi]\!]_{ij} = [\phi]_{ij}\nu_{ij} = [\phi]_{ji}\nu_{ji}$, which is invariant under the transformation $(i,j) \to (j,i)$. Hence, the operator $[\![\phi]\!] d\mathcal{H}^{n-1}$ is a distributional gradient of $\mathcal{R}_\varepsilon^* \phi$. Moreover, for $\phi \in \mathcal{S}^\varepsilon$ the quantity $[\![\phi]\!]_{ij}$ can be equally interpreted as a function on $\Gamma_{ij}^\varepsilon$.

**The general case**

On a given bounded Lipschitz domain $\boldsymbol{Q}$ and for a given family of points $P_\varepsilon$, strictly positive numbers $a_{ij}^\varepsilon$ and a bounded continuously differentiable function $v \in C^1(\overline{\boldsymbol{Q}})$ with $v \neq 0$ on $\overline{\boldsymbol{Q}}$, we consider the following two operators on $u \in \mathcal{S}^\varepsilon$:

$$(\mathcal{L}^\varepsilon u)_i := \frac{1}{\varepsilon^2} \sum_{(i,j)\in E^\varepsilon} a_{ij}^\varepsilon \left(u_j^\varepsilon - u_i^\varepsilon\right) , \tag{9}$$

$$(\mathcal{F}_v^\varepsilon u)_i := \frac{1}{\varepsilon^2} \sum_{(i,j)\in E^\varepsilon} a_{ij}^\varepsilon \left(u_j^\varepsilon \frac{v_i^\varepsilon}{v_j^\varepsilon} - u_i^\varepsilon \frac{v_j^\varepsilon}{v_i^\varepsilon}\right) , \tag{10}$$

where we use the Notation 1.1.

**Condition 1.2.** For a bounded Lipschitz domain $\boldsymbol{Q}$ and every $\varepsilon > 0$ let $(P_i^\varepsilon)_{i\in\mathbb{N}}$ be a family of points in $\mathbb{R}^n$ and let $(G_i^\varepsilon)_{i\in\mathbb{N}}$ be all Voronoi cells that intersect with $\boldsymbol{Q}$. We say that $(P_i^\varepsilon)_{i\in\mathbb{N}}$ is admissible if there exists $\alpha > 0$ such that

$$\forall \varepsilon > 0 : \quad \alpha\varepsilon \le \inf_{i\in\mathbb{N}} \underline{\operatorname{diam}}\, G_i^\varepsilon \le \sup_{i\in\mathbb{N}} \overline{\operatorname{diam}}\, G_i^\varepsilon \le \varepsilon , \tag{11}$$

where $\underline{\operatorname{diam}}\, G_i^\varepsilon$ and $\overline{\operatorname{diam}}\, G_i^\varepsilon$ denote the minimal and the maximal diameter of the cell $G_i^\varepsilon$, respectively.

**Corollary 1.3.** *Let $\boldsymbol{Q}$ be a bounded domain and let $\sup_i \overline{\operatorname{diam}}\, G_i^\varepsilon < \varepsilon$, then for every $u \in L^2(\boldsymbol{Q})$ holds $(\mathcal{R}_\varepsilon^* \mathcal{R}_\varepsilon u) \to u$ in $L^2(\boldsymbol{Q})$ as $\varepsilon \to 0$.*

In fact, Condition 1.2 is already sufficient to proof unique existence of solutions to the SQRA scheme, as we will see in the proof of Theorem 1.6.

**Definition 1.4** (G-convergence)**.** Let $\boldsymbol{Q}$ be a bounded Lipschitz domain. For every $\varepsilon > 0$, let $P^\varepsilon$ be a family of points with strictly positive numbers $a_{ij}^\varepsilon$. We call $(P^\varepsilon)_{\varepsilon>0}$ and $a_{ij}^\varepsilon$ G-convergent if there exists a symmetric positive definite matrix $A_{\mathrm{hom}}$ such that for every $f \in L^2(\boldsymbol{Q})$ the sequence of unique solutions $u^\varepsilon \in \mathcal{S}_0^\varepsilon(\boldsymbol{Q})$ to the problem

$$\mathcal{L}^\varepsilon u^\varepsilon = \mathcal{R}_\varepsilon f$$

satisfies $\mathcal{R}_\varepsilon^* u^\varepsilon \to u$ strongly in $L^2(\boldsymbol{Q})$ where $u \in H^2(\boldsymbol{Q}) \cap H_0^1(\boldsymbol{Q})$ solves

$$\nabla \cdot (A_{\mathrm{hom}} \nabla u) = f . \tag{12}$$

**Definition 1.5.** A sequence of functionals $F^\varepsilon : L^2(\boldsymbol{Q}) \to \mathbb{R}$ is weakly (strongly) $\Gamma$-convergent to a functional $F : L^2(\boldsymbol{Q}) \to \mathbb{R}$ if

1. $u^\varepsilon \rightharpoonup u$ weakly ($u^\varepsilon \to u$ strongly) in $L^2(\boldsymbol{Q})$ implies
$$F(u) \le \liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) ,$$

2. For every $u \in L^2(\boldsymbol{Q})$ there exists a weakly (strongly) convergent sequence $u^\varepsilon \rightharpoonup u$ ($u^\varepsilon \to u$) such that
$$F(u) = \limsup_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) .$$

The notion of G-convergence comes from homogenization theory, see [7, 28, 39]. Our definition coincides with the general definition of Dal Maso [7] applied to the particular setting of this work. Note that the Dirichlet-version of Theorem 1.13 below guaranties that the class of G-convergent point processes is not empty. As Dal Maso shows in Theorem 13.5 of his book [7], the G-convergence of $P^\varepsilon$ implies weak $\Gamma$-convergence of the functional $F^\varepsilon$ to the functional $F$, where

$$F^\varepsilon(u) := \langle \mathcal{R}_\varepsilon u,\, \mathcal{L}^\varepsilon \mathcal{R}_\varepsilon u\rangle_{\mathcal{S}_\varepsilon}\,, \quad F(u) := \int_{\boldsymbol{Q}} \nabla u\cdot A_{\mathrm{hom}}\nabla u\,.$$

Furthermore, $F^\varepsilon$ strongly $\Gamma$-converges to $F$ iff $\lambda + \mathcal{L}^\varepsilon$ is G-convergent for some $\mu > 0$ (Theorem 13.6).

**Theorem 1.6.** *Let $\boldsymbol{Q}\subset\mathbb{R}^n$ be a bounded Lipschitz domain and for every $\varepsilon>0$ let $P^\varepsilon$ be a distribution of points on $\mathbb{R}^n$ with strictly positive numbers $a_{ij}^\varepsilon$ such that $(P^\varepsilon)_{\varepsilon>0}$ satisfies Condition 1.2 and $a^{-1}\le a_{ij}^\varepsilon \le a$ for some $a>0$ and for every $\varepsilon>0$, $(i,j)\in E^\varepsilon$. Let $v(x) = \exp\left(-\frac{1}{2}\beta V(x)\right)$ for some bounded and twice continuously differentiable function $V\in C^2(\overline{\boldsymbol{Q}})$. Then, for every $\varepsilon>0$ and $f^\varepsilon\in S^\varepsilon$ there exists a unique solution $u^\varepsilon\in S_0^\varepsilon(\boldsymbol{Q})$ to*

$$-\left(\mathcal{F}_v^\varepsilon u^\varepsilon\right)_i = f_i^\varepsilon \quad \forall P_i^\varepsilon\in\boldsymbol{Q}\,. \tag{13}$$

*satisfying the estimate*

$$\|\mathcal{R}_\varepsilon^* u^\varepsilon\|_{L^2(\boldsymbol{Q})}^2 + \|\mathcal{R}_\varepsilon^*\left(\mathcal{L}^\varepsilon u^\varepsilon\right)\|_{L^2(\boldsymbol{Q})}^2 \le C\left(\|f^\varepsilon\|_{P^\varepsilon}^2\,,\ \|v\|_{C^2(\overline{\boldsymbol{Q}})}^2\right)\,. \tag{14}$$

*If $(P^\varepsilon)_{\varepsilon>0}$ additionally is G-convergent and $\mathcal{R}_\varepsilon^* f^\varepsilon \rightharpoonup f$ weakly in $L^2(\boldsymbol{Q})$, then there exists a function $u\in H_0^1(\boldsymbol{Q})$ such that $\mathcal{R}_\varepsilon^* u^\varepsilon\to u$ strongly in $L^2(\boldsymbol{Q})$ and $\frac{1}{\varepsilon}[\![u^\varepsilon]\!]d\mathcal{H}^{n-1}\to\nabla u$ in the sense of distribution as $\varepsilon\to 0$. Furthermore, $u$ is a solution to the problem*

$$-\nabla\cdot\left(A_{\mathrm{hom}}\nabla u\right) - \nabla\cdot\left(A_{\mathrm{hom}} u\beta\nabla V\right) = f\,. \tag{15}$$

*Remark* 1.7. A critical point in our studies is the assumption of a bound on the potential $V$, which might not exist in application. Note in this context, that the proof of Theorem 1.6 formally remains the same for $V$ bounded from below, but unbounded from above. This holds since Step 3 of the proof holds for arbitrary $v$, as long as $v$ is bounded continuous. However, the transformed operator

$$\frac{1}{\varepsilon^2}\sum_{(i,j)\in E^\varepsilon} a_{ij}^\varepsilon v_i^\varepsilon v_j^\varepsilon\left(U_j^\varepsilon - U_i^\varepsilon\right) = f_i^\varepsilon$$

becomes degenerate and one needs to spend more work in the derivation of proper apriori estimates for $U^\varepsilon$ and $u^\varepsilon = (v^\varepsilon)^2 U^\varepsilon$. Note in particular, that this involves weighted discrete Sobolev inequalities and, at least in the current version of the proof, also different notion of G-convergence, due to the lower regularity of $U^\varepsilon$. This lies beyond the scope of the present work, aiming at the presentation of the SQRA to a broader audience. On the other hand, let us note that one can often reduce the problems of large molecules to a problem in a few angles neglecting variations in the distances of neighbored atoms. In these reduced spaces, $V$ usually is bounded both from above and from below.

What we have found so far is that the solutions of

$$\partial_\xi\Psi_\varepsilon^*\left(u^\varepsilon, -D\mathcal{E}_\pi^\varepsilon(u^\varepsilon)\right) = f^\varepsilon$$

converge to solutions of

$$\partial_\xi \Psi^* \left(u, -D\mathcal{E}(u)\right) = f\,.$$

However, we are interested in the homogenization limit of solutions of the full gradient system (5)–(6) to solutions (7). This is given by the following result. However, note that we will not discuss the evolutionary Γ-convergence of the gradient systems in the sense of Sandier-Serfaty [41] or Mielke [36]. This is left to a future studies.

**Theorem 1.8.** *Let $\boldsymbol{Q} \subset \mathbb{R}^n$ be a bounded Lipschitz domain and for every $\varepsilon > 0$ let $P^\varepsilon$ be a distribution of points on $\mathbb{R}^n$ with strictly positive numbers $a^\varepsilon_{ij}$ such that $(P^\varepsilon)_{\varepsilon>0}$ satisfies Condition 1.2 and is G-convergent and $a^{-1} \le a^\varepsilon_{ij} \le a$ for some $a > 0$ and for every $\varepsilon > 0$, $(i,j) \in E^\varepsilon$. Let $v(x) = \exp\left(-\frac{1}{2}\beta V(x)\right)$ for some bounded and twice continuously differentiable function $V \in C^2(\overline{\boldsymbol{Q}})$ and for every $\varepsilon > 0$ let $f^\varepsilon \in L^2(0,T;S^\varepsilon)$ and $u_0^\varepsilon \in \mathcal{S}^\varepsilon$ with*

$$\sup_\varepsilon \left( \|\mathcal{R}^*_\varepsilon u_0^\varepsilon\|^2_{L^2(\boldsymbol{Q})} + \varepsilon^{n-2} \sum_{i\sim j} (u^\varepsilon_{0,j} - u^\varepsilon_{0,i})^2 \right) < \infty\,.$$

*Then, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ there exists a unique solution $u^\varepsilon$ to*

$$\partial_t u_i^\varepsilon - \left(\mathcal{F}^\varepsilon_v u^\varepsilon\right)_i = f_i^\varepsilon\,. \tag{16}$$

*If $\mathcal{R}^*_\varepsilon f^\varepsilon \rightharpoonup f$ weakly in $L^2(0,T;L^2(\boldsymbol{Q}))$, then*

$$\sup_\varepsilon \left( \|\partial_t \mathcal{R}^*_\varepsilon u^\varepsilon\|^2_{L^2(0,T;L^2(\boldsymbol{Q}))} + \|\mathcal{R}^*_\varepsilon \left(\mathcal{L}^\varepsilon u^\varepsilon\right)\|^2_{L^2(0,T;L^2(\boldsymbol{Q}))} + \int_0^T \varepsilon^{n-2} \sum_{i\sim j} (u^\varepsilon_{0,j} - u^\varepsilon_{0,i})^2 \right) < \infty$$

*and there exists a function $u \in L^2(0,T;H^1_0(\boldsymbol{Q}))$ with $\partial_t u \in L^2(0,T;L^2(\boldsymbol{Q}))$ such that $\mathcal{R}^*_\varepsilon u^\varepsilon \to u$ strongly in $L^2(0,T;L^2(\boldsymbol{Q}))$, $\partial_t \mathcal{R}^*_\varepsilon u^\varepsilon \rightharpoonup \partial_t u$ weakly in $L^2(0,T;L^2(\boldsymbol{Q}))$ and $\frac{1}{\varepsilon}[\![u^\varepsilon]\!] d\mathcal{H}^{n-1} \to \nabla u$ in the sense of distribution as $\varepsilon \to 0$ and $u$ is the unique solution to the problem*

$$\partial_t u - \nabla\cdot\left(A_{\mathrm{hom}}\nabla u\right) - \nabla\cdot\left(A_{\mathrm{hom}} u \beta \nabla V\right) = f\,. \tag{17}$$

We will prove the Theorems 1.6 and 1.8 in Section 4.

*Remark* 1.9. Theorem 1.6 and 1.8 can also be formulated an proved with periodic boundary conditions on a rectangular domain. The modification of the proofs are minor and straight forward.

### The stationary ergodic case

Let $(\Omega, \mathscr{F}, \mathcal{P})$ be a probability space and let $\omega \mapsto P(\omega) = (P_i(\omega))_{i\in\mathbb{N}}$ be a stationary random point process on $\mathbb{R}^n$. We then define $P^\varepsilon(\omega) := \varepsilon P(\omega)$ and construct from $P^\varepsilon(\omega)$ the sets $G^\varepsilon_{ij}(\omega)$, $\Gamma^\varepsilon(\omega)$ and $E^\varepsilon(\omega)$ according to the beginning of Section 1.3.

**Condition 1.10.** Using the notation of Condition 1.2, a Voronoi-tessellation $(G_i)_{i\in\mathbb{N}}$, based on a point process $(P_i)_{i\in\mathbb{N}}$ is admissible if there exists $\alpha > 0$ such that

$$\alpha \le \inf_i \underline{\mathrm{diam}}\, G_i \le \sup_i \overline{\mathrm{diam}}\, G_i \le 1\,. \tag{18}$$

A similar condition has been imposed in [1]. Note that if $P(\omega)$ satisfies 1.10, this implies that $P^\varepsilon(\omega)$ satisfies the admissibility Condition 1.2.

If $\boldsymbol{Q}$ is a cuboid, we denote $P^\varepsilon_{\text{per}}(\boldsymbol{Q},\omega)$ the periodization of $\boldsymbol{Q}\cap P^\varepsilon(\omega)$. From the $\boldsymbol{Q}$-periodic point process $P^\varepsilon_{\text{per}}(\boldsymbol{Q},\omega)$, we construct $G^\varepsilon_{\text{per}}(\boldsymbol{Q},\omega)$, $\Gamma^\varepsilon_{\text{per}}(\boldsymbol{Q},\omega)$ and $E^\varepsilon_{\text{per}}(\boldsymbol{Q},\omega)$ according to the beginning of Section 1.3. Furthermore, we set $\mathcal{S}^\varepsilon_{\text{per}}(\boldsymbol{Q},\omega)$ the set of all functions $P^\varepsilon_{\text{per}}(\boldsymbol{Q},\omega)\to\mathbb{R}$ that are $\boldsymbol{Q}$-periodic. The operators $\mathcal{R}_\varepsilon$ and $\mathcal{R}^*_\varepsilon$ are defined on $\mathcal{S}^\varepsilon_{\text{per}}(\boldsymbol{Q},\omega)$ in an obvious way. Furthermore, we denote $H^1_{\text{per}}(\boldsymbol{Q})$ the set of all $H^1(\boldsymbol{Q})$-functions with periodic boundary conditions.

*Remark* 1.11. Note that for the periodized point process and the corresponding Voronoi tessellation the Condition 1.10 is still satisfied with $\alpha$ in inequality (18) being replaced by $\frac{\alpha}{2}$.

For the stochastic results, we will need the following Assumption.

**Assumption 1.12.** *The random positive numbers $a_{ij}(\omega)$ are such that the measure*

$$\mu_{a\Gamma(\omega)} := \sum_{(i,j)\in E(\omega)} a_{ij}(\omega)\delta_{\Gamma_{ij}(\omega)} \tag{19}$$

*is a stationary and ergodic random measure.*

If $a_{ij}(\omega)\equiv 1$ for all $(i,j)$ and almost every $\omega$, this implies that the point process $(P_i(\omega))_{i\in\mathbb{N}}$ has to be stationary and ergodic. If we work on the periodized lattice, we set $a_{ij}=1$ for every $(i,j)\in E^\varepsilon_{\text{per}}(\omega)\backslash E^\varepsilon(\omega)$. Then, we define the following discrete elliptic operator:

$$(\mathcal{L}^\varepsilon_\omega u)_i := \sum_{(i,j)\in E^\varepsilon_{\text{per}}(\omega)} \frac{1}{\varepsilon^2} a_{ij}(\omega)\left(u_j-u_i\right). \tag{20}$$

Since we work on periodic boundary conditions, we will restrict ourselves to the following function space

$$\mathcal{S}^\varepsilon_{\text{per},0}(\boldsymbol{Q},\omega) := \left\{ u\in\mathcal{S}^\varepsilon_{\text{per}}(\boldsymbol{Q},\omega)\ :\ \sum_{P^\varepsilon_i\in P^\varepsilon_{\text{per}}(\omega)} u(P^\varepsilon_i)=0 \right\}.$$

The operator $\mathcal{L}^\varepsilon_\omega$ admits the following asymptotic behavior on $\mathcal{S}^\varepsilon_{\text{per},0}(\boldsymbol{Q},\omega)$.

**Theorem 1.13.** *Let the point process $P^\varepsilon(\omega)$ almost surely satisfy Condition 1.10 and let the random numbers $a_{ij}(\omega)$ be such that $0<c^{-1}\le a_{ij}(\omega)\le c<\infty$ almost surely for some positive constant c and such that Assumption 1.12 holds. For such $\omega$ let $f^\varepsilon\in\mathcal{S}^\varepsilon_{\text{per},0}(\boldsymbol{Q},\omega)$ be a sequence of functions such that $\mathcal{R}^*_\varepsilon f^\varepsilon \rightharpoonup f$ weakly in $L^2(\boldsymbol{Q})$ for some $f\in L^2(\boldsymbol{Q})$. Then for almost every $\omega$ the sequence $u^\varepsilon\in\mathcal{S}^\varepsilon_{\text{per},0}(\boldsymbol{Q},\omega)$ of solutions to the problems*

$$-\mathcal{L}^\varepsilon_\omega u^\varepsilon = f^\varepsilon \tag{21}$$

*has the following properties: There exists a function $u\in H^1_{\text{per}}(\boldsymbol{Q})$ such that $\mathcal{R}^*_\varepsilon u^\varepsilon\to u$ strongly in $L^2(\boldsymbol{Q})$ and $\frac{1}{\varepsilon}[\![u^\varepsilon]\!]d\mathcal{H}^{n-1}\to\nabla u$ in the sense of distribution and as $\varepsilon\to 0$. Furthermore, $u\in H^1_{\text{per}}(\boldsymbol{Q})\cap H^2(\boldsymbol{Q})$ is the unique solution to the problem*

$$-\nabla\cdot\left(A_{\text{hom}}\nabla u\right)=f\,, \qquad \int_{\boldsymbol{Q}} u = 0\,, \tag{22}$$

*where $A_{\text{hom}}$ is defined below in (39).*

Theorem 1.13 evidently implies G-convergence according to Definition 1.4. Note that it can also be formulated and proved for Dirichlet boundary conditions. In the latter case, the proof turns out to be simpler which is why the Theorem was formulated for the periodic case.

# 2 Ergodic Theorems for Voronoi-tessellations

In this work, we rely on the following assumptions.

**Assumption 2.1.** *Let $(\Omega, \mathscr{F}, \mathcal{P})$ be a probability space. We assume we are given a family $(\tau_x)_{x\in\mathbb{R}^n}$ of measurable bijective mappings $\tau_x : \Omega \mapsto \Omega$, having the properties of a* dynamical system *on $(\Omega, \mathscr{F}, \mathcal{P})$, i.e. they satisfy (i)-(iii):*

*(i)* $\tau_x \circ \tau_y = \tau_{x+y}$ , $\tau_0 = id$ *(Group property)*

*(ii)* $\mathcal{P}(\tau_{-x}B) = \mathcal{P}(B) \quad \forall x \in \mathbb{R}^n,\ B \in \mathscr{F}$ *(Measure preserving)*

*(iii)* $A : \mathbb{R}^n \times \Omega \to \Omega \qquad (x,\omega) \mapsto \tau_x\omega$ *is measurable (Measurability of evaluation)*

*We finally assume that the system $(\tau_x)_{x\in\mathbb{R}^n}$ is ergodic. This means that for every measurable function $f : \Omega \to \mathbb{R}$ there holds*

$$[f(\omega) = f(\tau_x\omega)\ \forall x \in \mathbb{R}^n,\ a.e.\ \omega \in \Omega] \Rightarrow [f(\omega) = const \text{ for } \mathcal{P} - a.e.\ \omega \in \Omega] \,. \tag{23}$$

In what follows, we recapitulate parts of the theory from [8]. Given a stationary point process $(P_i)_{i\in\mathbb{N}}$, we define $\Gamma^\varepsilon_{ij}(\omega) := \frac{1}{2}\left(P^\varepsilon_i + P^\varepsilon_j\right)$ the midpoint of the straight line connecting $P^\varepsilon_i$ and $P^\varepsilon_j$ and

$$\Gamma^\varepsilon(\omega) := \bigcup_{(i,j)\in E^\varepsilon(\Omega)} \Gamma^\varepsilon_{ij}(\omega)\,.$$

The measure $\mu_P := \sum_i \delta_{P_i}$ is stationary and the mapping $\omega \mapsto \mu_{P(\omega)}(B)$ is measurable for every open set $B \subset \mathbb{R}^n$. Similarly, we can define $\mu_{\Gamma(\omega)} := \sum_{(i,j)\in E(\omega)} \delta_{\Gamma_{ij}(\omega)}$ having the same properties as $\mu_P$. Hence, $\mu_{P(\omega)}$, $\mu_{\Gamma(\omega)}$ and $\mu_{a\Gamma(\omega)}$ from (19) are random measures, i.e. measurable mappings $\Omega \to \mathcal{M}$, where $\mathcal{M}$ is the set of all Radon measures on $\mathbb{R}^n$ equipped with the vague topology and corresponding $\sigma$-algebra.

Hence, for fixed $\omega$, the mapping $\omega \mapsto \boldsymbol{\mu}_\omega := \mu_{a\Gamma(\omega)} + \mu_{P(\omega)}$ is a random measure and therefore $(\boldsymbol{\mu}(\Omega), \boldsymbol{\mu}(\mathscr{F}), \boldsymbol{\mu}\#\mathcal{P})$ is a probability space with respect to the vague topology. Due to this observation, we may assume that $\Omega \subset \mathcal{M}$ and $\mathcal{P}$ is a probability measure on $\mathcal{M}$. This has the advantage that $\mathcal{M}$ with the vague topology is a complete separable metric space. Hence the $\sigma$-Algebra $\mathscr{F}$ becomes separable and the set $C_b(\Omega)$ of bounded continuous functions is dense in $L^p(\Omega, \mu)$ for any $1 \le p < \infty$ and any finite measure $\mu$ on $\mathcal{M}$. Finally, we observe that the mapping $\mathbb{R}^n \times \mathcal{M} \to \mathcal{M}$, $(x,\omega) \mapsto \tau_x\omega$ is even continuous (see [22]).

**Theorem 2.2** (Existence of Palm measure [8])**.** *Let $\omega \mapsto \mu_\omega$ be a stationary random measure. Then there exists a unique measure $\mu_{\mathcal{P}}$, called* Palm measure *of $\mu$, on $\Omega$ such that*

$$\int_\Omega \int_{\mathbb{R}^n} f(x, \tau_x\omega)\, d\mu_\omega(x) d\mathcal{P}(\omega) = \int_{\mathbb{R}^n} \int_\Omega f(x,\omega)\, d\mu_{\mathcal{P}}(\omega) dx$$

*for all $\mathcal{L} \otimes \mu_{\mathcal{P}}$-measurable non negative functions and all $\mathcal{L} \otimes \mu_{\mathcal{P}}$- integrable functions $f$. Furthermore for all $A \subset \Omega$, $u \in L^1(\Omega, \mu_{\mathcal{P}})$ there holds*

$$\mu_{\mathcal{P}}(A) = \int_\Omega \int_{\mathbb{R}^n} g(x)\chi_A(\tau_x\omega) d\mu_\omega(x) d\mathcal{P} \tag{24}$$

$$\int_\Omega u(\omega) d\mu_{\mathcal{P}} = \int_\Omega \int_{\mathbb{R}^n} g(x) u(\tau_x\omega) d\mu_\omega(x) d\mathcal{P}$$

*for an arbitrary $g \in L^1(\mathbb{R}^n, \mathcal{L})$ with $\int_{\mathbb{R}^n} g(x)dx = 1$ and $\mu_{\mathcal{P}}$ is $\sigma$-finite.*

**Definition.** We denote $\mu_{P,\mathcal{P}}$ and $\mu_{\Gamma,\mathcal{P}}$ the *Palm measure* of $\mu_P$ and $\mu_\Gamma$ respectively.

An application of the classical Radon-Nikodym theorem yields the following result. For a proof, we refer to [22, Lemma 2.14].

**Lemma 2.3.** *There exists a measurable set $\tilde{P} \subset \Omega$ with $\mathbb{I}_{P(\omega)}(x) = \mathbb{I}_{\tilde{P}}(\tau_x\omega)$ for $\mathcal{L} + \mu_{P(\omega)}$-almost every $x$ for $\mathcal{P}$-almost every $\omega$. Furthermore $\mathcal{P}(\tilde{P}) = 0$ and $\mu_{P,\mathcal{P}}(\Omega\backslash\tilde{P}) = 0$. The same applies to $\Gamma(\omega)$.*

Lemma 2.3 will not be used below, but it highlights the strong interaction between a point process and its Palm measure. However, the same proof also yields the following result, which we will use frequently.

**Lemma 2.4.** *Let $\omega \to \mu_{1,\omega}$ and $\omega \to \mu_{2,\omega}$ be two stationary random measures such that for a.e. $\omega$ it holds $\mu_{1,\omega} \ll \mu_{2,\omega}$. Then the corresponding Palm measure $\mu_{1,\mathcal{P}}$ and $\mu_{2,\mathcal{P}}$ satisfy $\mu_{1,\mathcal{P}} \ll \mu_{2,\mathcal{P}}$ and there exists a measurable function $f_{1,2} : \Omega \to \mathbb{R}$ such that $\mu_{1,\mathcal{P}} = f_{1,2}\mu_{2,\mathcal{P}}$.*

Hence, if $\boldsymbol{\mu}_{\mathcal{P}}$ denotes the Palm measure for $\boldsymbol{\mu}_\omega$, we find $\mu_{P,\mathcal{P}} = \tilde{P}\boldsymbol{\mu}_{\mathcal{P}}$. Furthermore, we find existence of measurable functions $a : \Omega \to \mathbb{R}$ such that $a_{ij}(\omega) = a(\tau_{\Gamma_{ij}(\omega)}\omega)$ and $\mu_{a\Gamma,\mathcal{P}} = a\mu_{\Gamma,\mathcal{P}}$. Finally, the following theorem is essential for all following calculations.

**Theorem 2.5** (Ergodic Theorem [8])**.** *Let the dynamical System $\tau_x$ be ergodic and assume that the Palm measure $\mu_{\mathcal{P}}$ of the stationary random measure $\mu_\omega$ has finite intensity. Then, with $\mu_\omega^\varepsilon(B) := \varepsilon^n\mu_\omega(\varepsilon^{-1}B)$, for all $g \in L^1(\Omega, \mu_{\mathcal{P}})$ it holds*

$$\lim_{t\to\infty} \int_A g(\tau_{\frac{x}{\varepsilon}}\omega)d\mu_\omega^\varepsilon(x) = |A| \int_\Omega g(\omega)d\mu_{\mathcal{P}}(\omega) \tag{25}$$

*for $\mathcal{P}$ almost every $\omega$ and for all bounded Borel sets $A$ that contain an open ball around 0.*

From the last result, one can derive the following generalization.

**Theorem 2.6** ([23], Section 2)**.** *Let the dynamical System $\tau_x$ be ergodic and assume that the stationary random measure $\mu_\omega$ has finite intensity. Then, defining $\mu_\omega^\varepsilon(B) := \varepsilon^n\mu_\omega(\varepsilon^{-1}B)$, it holds: for all $g \in L^1(\Omega, \mu_{\mathcal{P}})$ we find for $\mathcal{P}$-almost every $\omega$, and all $\varphi \in C_c(\mathbb{R}^n)$ that*

$$\lim_{t\to\infty} \int_{\mathbb{R}^n} g(\tau_{\frac{x}{\varepsilon}}\omega)\varphi(x)d\mu_\omega^\varepsilon(x) = \int_{\mathbb{R}^n}\int_\Omega g(\omega)\varphi(x)d\mu_{\mathcal{P}}(\omega)\,dx\,. \tag{26}$$

**Lemma 2.7.** *Let the point process $P^\varepsilon(\omega)$ be stationary and such that Condition 1.10 holds almost surely and let $\boldsymbol{Q}$ be an open cuboid that contains 0. Then for $\mathcal{P}$-almost every $\omega$ it holds for all $\varphi \in C_{\mathrm{per}}(\boldsymbol{Q})$ that*

$$\lim_{\varepsilon\to 0} \varepsilon^n \sum_{(i,j)\in E^\varepsilon_{\mathrm{per}}(\omega)} \varphi(\Gamma^\varepsilon_{\mathrm{per},ij}) = \int_{\boldsymbol{Q}}\int_\Omega \varphi(x)d\mu_{\Gamma,\mathcal{P}}\,dx\,. \tag{27}$$

*Proof.* Let $\eta > 0$ and $\phi_\eta \in C_c(\boldsymbol{Q})$ such that $1 \geq \phi_\eta \geq 0$, $\phi_\eta = 0$ on $\boldsymbol{Q}\backslash(1-\eta)\boldsymbol{Q}$ and $\phi_\eta = 1$ on $\boldsymbol{Q}_\eta := (1-2\eta)\boldsymbol{Q}$. Define $\mu_\omega^\varepsilon(B) := \varepsilon^n\mu_{\Gamma^\varepsilon(\omega)}(B)$ and $\mu_{\mathrm{per},\omega}^\varepsilon(B) := \varepsilon^n\mu_{\Gamma^\varepsilon_{\mathrm{per}}(\omega)}(B)$ .

Since $\sup_i \operatorname{diam} G_i < \infty$, we can find $\varepsilon_\eta > 0$ such that for all $\varepsilon < \varepsilon_\eta$ it holds $\mu_\omega^\varepsilon = \mu_{\mathrm{per},\omega}^\varepsilon$ on $\operatorname{supp}\phi_\eta$. Due to Condition 1.10, the integral $\mu_{\mathrm{per},\omega}^\varepsilon\big((1+2\eta)\boldsymbol{Q}\backslash\boldsymbol{Q}_\eta\big)$ is bounded from above by $C\eta$ for some constant $C$ that does not depend on $\eta$. Hence from Theorem 2.6 one obtains

$$\begin{aligned}
&\left|\lim_{\varepsilon\to 0}\int_{\boldsymbol{Q}}\varphi(x)d\mu_{\mathrm{per},\omega}^\varepsilon(x) - \int_{\boldsymbol{Q}}\int_\Omega \varphi(x)d\mu_{\mathcal{P}}\,dx\right| \\
&\qquad \le \left|\lim_{\varepsilon\to 0}\int_{\boldsymbol{Q}}\varphi\phi_\eta d\mu_{\mathrm{per},\omega}^\varepsilon(x) - \int_{\boldsymbol{Q}}\int_\Omega \varphi\phi_\eta d\mu_{\mathcal{P}}\,dx\right| \\
&\qquad\quad + \|\varphi\|_\infty \limsup_{\varepsilon\to 0}\mu_{\mathrm{per},\omega}^\varepsilon\big((1+2\eta)\boldsymbol{Q}\backslash\boldsymbol{Q}_\eta\big) + \|\varphi\|_\infty\left|(1+2\eta)\boldsymbol{Q}\backslash\boldsymbol{Q}_\eta\right| \\
&\qquad \le \|\varphi\|_\infty C\eta\,,
\end{aligned}$$

where $C$ does not depend on $\eta$. As $\eta > 0$ was arbitrary, statement follows. □

# 3 Function spaces and the effective matrix $A_{\mathrm{hom}}$

## 3.1 The jump operator

Let $u \in H^1_{loc}(\mathbb{R}^n)$ and $\phi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$. Then, $\phi$ and $\nabla\cdot\phi$ are uniformly continuous on the support of $\phi$ and for $\varepsilon \to 0$ we find in view of Corollary 1.3

$$\begin{aligned}
-\int_{\partial G^\varepsilon(\omega)} [\![\mathcal{R}_\varepsilon u]\!]\cdot\phi d\mathcal{H}^{n-1} &= \int_{G^\varepsilon(\omega)}(\mathcal{R}_\varepsilon^*\mathcal{R}_\varepsilon u)\,\nabla\cdot\phi \\
&\to \int_{\mathbb{R}^n} u\nabla\cdot\phi = -\int_{\mathbb{R}^n}\nabla u\cdot\phi\,. 
\end{aligned}\tag{28}$$

This implies that $[\![\mathcal{R}_\varepsilon u]\!]d\mathcal{H}^{n-1} \to \nabla u$ in the sense of distributions as $\varepsilon \to 0$.

The convergence (28) requires more attention, as this is the convergence behavior we expect for the solutions of equations (13) or (21). We start denoting $\gamma_{ij}^\varepsilon := \varepsilon^{1-n}|\partial G_{ij}^\varepsilon|$ and quoting a Poincaré inequality due to Hummel.

**Lemma 3.1** (Compactness property, see also [26])**.** *Let $\boldsymbol{Q}$ be a bounded Lipschitz domain in $\mathbb{R}^n$ with Lipschitz boundary and let the families of points $(P_i^\varepsilon)_{i\in\mathbb{N}}$ satisfy Condition 1.2. Then, for every $s \in ]0,\frac12[$ there exists a constant $C_s$ independent from $\varepsilon$ such that for every $\varepsilon > 0$ and every $u^\varepsilon \in \mathcal{S}_0^\varepsilon(\boldsymbol{Q})$:*

$$\|\mathcal{R}_\varepsilon^* u^\varepsilon\|^2_{H_0^s(\boldsymbol{Q})} \le C_s\left(\varepsilon^{n-2}\sum_{(i,j)\in E^\varepsilon}[\![u^\varepsilon]\!]_{ij}^2\gamma_{ij}\right). \tag{29}$$

*If $\boldsymbol{Q}$ is a cube and $u^\varepsilon \in \mathcal{S}_{\mathrm{per}}^\varepsilon(\boldsymbol{Q},\omega)$, the following relation holds:*

$$\|\mathcal{R}_\varepsilon^* u^\varepsilon\|^2_{H^s_{\mathrm{per}}(\boldsymbol{Q})} \le C_s\left(\varepsilon^{n-2}\sum_{(i,j)\in E^\varepsilon_{\mathrm{per}}(\omega)}[\![u^\varepsilon]\!]_{ij}^2\gamma_{ij}(\omega) + \left(\int_{\boldsymbol{Q}}\mathcal{R}_\varepsilon^* u^\varepsilon\,d\mathcal{L}\right)^2\right). \tag{30}$$

*The constant $C_s$ only depends on the constant $\alpha$ in (11) resp. (18) and the dimension.*

*Sketch of proof.* Inequality (29) is a direct consequence of [26, Proposition 3.16] (a periodic version is given in [25]), noting that for functions $u^\varepsilon \in \mathcal{S}_0(\boldsymbol{Q})$ it holds

$$\varepsilon^{n-2} \sum_{(i,j)\in E^\varepsilon} [\![u^\varepsilon]\!]_{ij}^2 \gamma_{ij} = \varepsilon^{-1} \int_{\partial G^\varepsilon} [\![u^\varepsilon]\!]^2 \, d\mathcal{H}^{n-1} .$$

Inequality (30) now follows from [26, Proposition 3.16] and Remark 1.11, since $\Gamma^\varepsilon_{\mathrm{per}}(\omega)$ and $G^\varepsilon_{\mathrm{per}}(\omega)$ satisfy Condition 1.10 with a $\alpha$ replaced by $\frac{\alpha}{2}$. □

Using Lemma 3.1, we obtain the following result.

**Lemma 3.2.** *Let $\boldsymbol{Q}$ be a cube, $(G_i^\varepsilon(\omega))_{i\in\mathbb{N}}$ a random Voronoi-tessellation satisfying Condition 1.10 and $u^\varepsilon \in \mathcal{S}^\varepsilon_{\mathrm{per}}(\boldsymbol{Q},\omega)$ a sequence such that*

$$\left( \varepsilon^{n-2} \sum_{(i,j)\in E^\varepsilon_{\mathrm{per}}(\boldsymbol{Q},\omega)} [\![u^\varepsilon]\!]_{ij}^2 \gamma^\varepsilon_{\mathrm{per},ij}(\omega) + \left( \int_Q \mathcal{R}^*_\varepsilon u^\varepsilon d\mathcal{L} \right)^2 \right) \le C \tag{31}$$

*for some $C$ independent from $\varepsilon$. Then there exists a subsequence, not relabeled, and $u \in H^1_{\mathrm{per}}(\boldsymbol{Q})$ such that $\mathcal{R}^*_\varepsilon u^\varepsilon \to u$ strongly in $L^2(\boldsymbol{Q})$ and $[\![u^\varepsilon]\!] d\mathcal{H}^{n-1} \to \nabla u$ in the sense of distributions as $\varepsilon \to 0$. Furthermore, it holds*

$$\|\nabla u^\varepsilon\|^2_{L^2(\boldsymbol{Q})} \le C \liminf_{\varepsilon\to 0} \varepsilon^{n-2} \sum_{(i,j)\in E^\varepsilon_{\mathrm{per}}(\omega)} [\![u^\varepsilon]\!]_{ij}^2 \gamma^\varepsilon_{\mathrm{per},ij}(\omega) \tag{32}$$

*for $C = \mu_{\Gamma,\mathcal{P}}(\Omega)^{\frac12} \sup_{ij} |\gamma_{ij}|^{\frac12}$.*

*Proof.* Due to Lemma 3.1, we find $u \in L^2(\boldsymbol{Q})$ such that $\mathcal{R}^*_\varepsilon u^\varepsilon \to u$ strongly in $L^2(\boldsymbol{Q})$ along a subsequence. Furthermore, for every $\phi \in C^1_{\mathrm{per}}(\boldsymbol{Q};\mathbb{R}^n)$ we find

$$-\int_{\boldsymbol{Q}\cap\partial G^\varepsilon_{\mathrm{per}}(\omega)} [\![u^\varepsilon]\!] \cdot \phi d\mathcal{H}^{n-1} = \int_{\boldsymbol{Q}\cap G^\varepsilon_{\mathrm{per}}(\omega)} (\mathcal{R}^*_\varepsilon u^\varepsilon)\, \nabla\cdot\phi \to \int_{\boldsymbol{Q}} u \nabla\cdot\phi . \tag{33}$$

Using first the Cauchy-Schwarz inequality with (31) and then the boundedness of $\gamma_{ij}$ and equicontinuity of $\phi$ we have

$$\begin{aligned} \left| \int_{\boldsymbol{Q}\cap\partial G^\varepsilon_{\mathrm{per}}(\omega)} [\![u^\varepsilon]\!] \cdot \phi d\mathcal{H}^{n-1} \right| &\le C^{\frac12} \left( \varepsilon \int_{\partial G^\varepsilon_{\mathrm{per}}(\omega)} \phi^2 d\mathcal{H}^{n-1} \right)^{\frac12} \\ &\le C^{\frac12} \sup_{ij} |\gamma_{ij}|^{\frac12} \left( \varepsilon^n \sum_{(i,j)\in E^\varepsilon_{\mathrm{per}}} \phi^2_{ij} \right)^{\frac12} + C\eta , \end{aligned}$$

where $\eta$ as a modulus of continuity of $\phi$ is arbitrary small if $\varepsilon$ is small enough. In the limit $\varepsilon \to 0$, Lemma 2.7 and (33) applied to the last inequality becomes

$$\left| \int_{\boldsymbol{Q}} u \nabla\cdot\phi \right| \le C^{\frac12} \sup_{ij} |\gamma_{ij}|^{\frac12} \left( \mu_{\Gamma,\mathcal{P}}(\Omega) \int_{\boldsymbol{Q}} \phi^2 \right)^{\frac12} . \tag{34}$$

Since $C^1_{\mathrm{per}}(\boldsymbol{Q})$ is dense in $H^1_{\mathrm{per}}(\boldsymbol{Q})$ this implies $\nabla u \in L^2(\boldsymbol{Q})$ and (32) (see Brezis [5, Proposition 9.3]).

Equation (33) together with $\int_{\boldsymbol{Q}} u \nabla\cdot\phi = -\int_{\boldsymbol{Q}} \nabla u \cdot \phi$ proves $[\![u]\!] d\mathcal{H}^{n-1} \to \nabla u$ in the sense of distributions as $\varepsilon \to 0$. □

**Lemma 3.3.** *Let $\boldsymbol{Q}$ be a cube, $(G_i^\varepsilon)_{i\in\mathbb{N}}$ be a family of Voronoi-tessellations satisfying Condition 1.2 and $u^\varepsilon \in \mathcal{S}_0^\varepsilon(\boldsymbol{Q})$ a sequence such that*

$$\left(\varepsilon^{n-2}\sum_{(i,j)\in E^\varepsilon}[\![u^\varepsilon]\!]_{ij}^2\gamma_{ij}^\varepsilon\right)\le C$$

*for some $C$ independent from $\varepsilon$. Then there exists a subsequence, not relabeled, and $u \in H_0^1(\boldsymbol{Q})$ such that $\mathcal{R}_\varepsilon^* u^\varepsilon \to u$ strongly in $L^2(\boldsymbol{Q})$ and $[\![u^\varepsilon]\!]d\mathcal{H}^{n-1} \to \nabla u$ in the sense of distributions as $\varepsilon \to 0$. Furthermore, it holds*

$$\|\nabla u^\varepsilon\|_{L^2(\boldsymbol{Q})}^2 \le C\liminf_{\varepsilon\to 0}\varepsilon^{n-2}\sum_{(i,j)\in E^\varepsilon}[\![u^\varepsilon]\!]_{ij}^2\gamma_{ij}^\varepsilon\,.$$

*Proof.* The proof follows the lines of the proof of Lemma 3.2, except for equation (34), where $\mu_{\Gamma,\mathcal{P}}(\Omega)$ is replaced by $n^n\alpha^{-n}$. □

The distributional gradients $[\![\cdot]\!]d\mathcal{H}^{n-1}$ are vector-valued. However, at every edge $(i,j)\in E^\varepsilon$, the jump $[\![u]\!]$ of a function $u\in\mathcal{S}^\varepsilon$ is oriented only along the direction $\nu_{ij} = -\nu_{ji}$. Hence, for every $(i,j)\in E^\varepsilon$ the set $\{[\![u]\!]_{ij} \,:\, u\in\mathcal{S}^\varepsilon\}$ spans a 1-dimensional space, which suggests to work with the scalar quantities $[\![u]\!]_{ij}\cdot\nu_{ij}$ instead of $[\![u]\!]_{ij}$. However, the quantity $[\![u]\!]_{ij}\cdot\nu_{ij}$ is not invariant under the permutation of $i$ and $j$. Thus, we introduce the following definition.

**Definition 3.4** (Normal Field)**.** Let $e_0 = 0$ and $(e_i)_{i=1,\dots,n}$ be the canonical basis of $\mathbb{R}^n$. Define:

$$D^{n-1} := \{\nu\in S^{n-1} \mid \exists m\in\{1,\cdots,n\} : \nu\cdot e_i = 0\ \forall\ i\in\{0,1,\cdots,m-1\}\ \text{and}\ \nu\cdot e_m > 0\}$$

Thus, for every $\nu\in S^{n-1}$ it holds $\nu\in D^{n-1}$ if and only if $-\nu\notin D^{n-1}$.

For each $(i,j)\in E^\varepsilon$ let $\tilde\nu_{ij} = \nu_{ij}$ if $\nu_{ij}\in D^{n-1}$ and $\tilde\nu_{ij} = \nu_{ji} = -\nu_{ij}$ if $\nu_{ji}\in D^{n-1}$. Hence, $\tilde\nu_{ij} = \tilde\nu_{ji}$ is stationary and invariant under the transformation $(i,j)\to(j,i)$. Note that $\nu_{ij}$ and $\tilde\nu_{ij}$ do not have an index $\varepsilon$ for simplicity of notation as they will only be used in context with other quantities having an index $\varepsilon$. In case $E^\varepsilon(\omega)$ and $E^\varepsilon_{\mathrm{per}}(\omega)$ the normal field is defined accordingly.

Using $\tilde\nu$ we define the invariant field $[\![u]\!]_{ij}^\sim := [\![u]\!]_{ij}\cdot\tilde\nu_{ij}$. The operator $[\![\cdot]\!]^\sim$ then defines a linear operator

$$\begin{aligned}\mathcal{S}^\varepsilon &\to L^2_{loc}(\Gamma^\varepsilon,\mu_\Gamma^\varepsilon)\\ \text{or}\quad \mathcal{S}^\varepsilon_{\mathrm{per}}(\boldsymbol{Q},\omega) &\to L^2_{loc}(\Gamma^\varepsilon_{\mathrm{per}}(\boldsymbol{Q},\omega);\mu^\varepsilon_{\Gamma(\omega),\mathrm{per}})\end{aligned}$$

with $\mu_\Gamma^\varepsilon$ defined in (36) below. We are interested in the adjoint operator (with respect to the topological structure in Section 3.2), which we denote $-\mathrm{div}_P := ([\![\cdot]\!]^\sim)^*$ and which can be calculated as follows:

Given $u\in\mathcal{S}^\varepsilon$ and $\phi:\Gamma^\varepsilon\to\mathbb{R}$ having compact support in $\boldsymbol{Q}$, we use $[\![u]\!]_{ij} = u_j\nu_{ij} + u_i\nu_{ji} = [\![u]\!]_{ij}^\sim\tilde\nu_{ij}$ to get

$$\begin{aligned}\sum_{(i,j)\in E}[\![u]\!]_{ij}^\sim\phi_{ij} &= \sum_{(i,j)\in E}[\![u]\!]_{ij}^\sim\tilde\nu_{ij}\cdot\tilde\nu_{ij}\phi_{ij}\\ &= \sum_{(i,j)\in E}(u_j\nu_{ij}+u_i\nu_{ji})\cdot\tilde\nu_{ij}\phi_{ij}\\ &= \sum_i u_i\sum_{j\sim i}\nu_{ji}\cdot\tilde\nu_{ij}\phi_{ij} = -\sum_i u_i\sum_{j\sim i}\nu_{ij}\cdot\tilde\nu_{ij}\phi_{ij}\,.\end{aligned}$$

Hence, we obtain

$$(\mathrm{div}_P\phi)_i = \sum_{j\sim i} \nu_{ij}\cdot\tilde{\nu}_{ij}\phi_{ij}\,. \tag{35}$$

The calculations for the case of periodic functions $\mathcal{S}^\varepsilon_{\mathrm{per}}(\boldsymbol{Q},\omega)$ and $\phi:\Gamma^\varepsilon_{\mathrm{per}}(\omega)\to\mathbb{R}$ are similar.

*Remark* 3.5. The definitions of the operators $[\![\cdot]\!]^\sim$ and $\mathrm{div}_P$ are coupled to the choice of the point process $P^\varepsilon$ and also vary with scaling $\varepsilon$. However, they do not scale with the parameter $\varepsilon$. More precisely, for $u\in\mathcal{S}^1(\omega)$ and $u^\varepsilon := u(\frac{x}{\varepsilon})$ we have $u^\varepsilon\in\mathcal{S}^\varepsilon(\omega)$ and

$$[\![u]\!]^\sim(\frac{x}{\varepsilon}) = [\![u^\varepsilon]\!]^\sim(x)\,,$$

while for functions $\phi\in C^1(\mathbb{R}^n)$ and the usual gradient we have $\nabla\phi(\frac{x}{\varepsilon}) = \varepsilon\nabla\phi^\varepsilon(x)$.

## 3.2 Function spaces

In the rest of this paper, we will frequently use the following measures

$$\mu^\varepsilon_P := \mu_{P^\varepsilon} := \varepsilon^n\sum_{i\in\mathbb{N}}\delta_{P^\varepsilon_i}\,,\quad \mu^\varepsilon_\Gamma := \mu_{\Gamma^\varepsilon} := \varepsilon^n\sum_{(i,j)\in E^\varepsilon}\delta_{\Gamma^\varepsilon_{ij}}\,, \tag{36}$$

and use them to introduce the following scalar products:

$$\begin{aligned}\langle u,v\rangle_{P^\varepsilon,\boldsymbol{Q}} = \langle u,v\rangle_{\mathcal{S}^\varepsilon,\boldsymbol{Q}} &:= \varepsilon^n\sum_{P^\varepsilon_i\in\boldsymbol{Q}} u(P^\varepsilon_i)\,v(P^\varepsilon_i) = \int_{\boldsymbol{Q}} u(x)v(x)\,d\mu_{P^\varepsilon}(x)\\ \langle u,v\rangle_{\Gamma^\varepsilon,\boldsymbol{Q}} &:= \varepsilon^n\sum_{\Gamma^\varepsilon_{ij}\in\boldsymbol{Q}} u(\Gamma^\varepsilon_{ij})\,v(\Gamma^\varepsilon_{ij}) = \int_{\boldsymbol{Q}} u(x)v(x)\,d\mu_{\Gamma^\varepsilon}(x)\end{aligned}$$

with the corresponding norms $\|\cdot\|_{P^\varepsilon,\boldsymbol{Q}}$ and $\|\cdot\|_{\Gamma^\varepsilon,\boldsymbol{Q}}$ on $\mathcal{S}^\varepsilon(\boldsymbol{Q}) := L^2(\boldsymbol{Q};\mu_{P^\varepsilon})$ and $L^2(\boldsymbol{Q};\mu_{\Gamma^\varepsilon})$. By an abuse of notation, we also write $\langle u,v\rangle_{\mathcal{S}^\varepsilon,\boldsymbol{Q}}$ resp. $\langle u,v\rangle_{\Gamma^\varepsilon,\boldsymbol{Q}}$ for the pairing of $L^1$- and $L^\infty$ functions. We emphasize that due to the discrete character of the measures $\mu_{P^\varepsilon}$ and $\mu_{\Gamma^\varepsilon}$ every integral with respect to one of these measures over a bounded domain corresponds to a finite sum and we will frequently make use of this duality. In particular, we emphasize that for $u\in C(\overline{\boldsymbol{Q}})$:

$$\varepsilon^n\sum_{P^\varepsilon_i\in\boldsymbol{Q}} u(P^\varepsilon_i) = \int_{\boldsymbol{Q}} u(x)\,d\mu_{P^\varepsilon}(x)\,,\qquad \varepsilon^n\sum_{\Gamma^\varepsilon_{ij}\in\boldsymbol{Q}} u(\Gamma^\varepsilon_{ij}) = \int_{\boldsymbol{Q}} u(x)\,d\mu_{\Gamma^\varepsilon}(x)\,,$$

and we choose the notation depending on what aspect seems suitable for presentation.

If the point process $P(\omega)$ is stationary, so is the measure $\omega\mapsto\mu_{\partial G(\omega)} := \mathcal{H}^{n-1}(\cdot\cap\partial G(\omega))$ and the measure $\mu_{\gamma\Gamma(\omega)} := \sum_{(i,j)\in E(\omega)}\gamma_{ij}(\omega)\delta_{\Gamma_{ij}(\omega)}$, where $\gamma_{ij}(\omega) = |\partial G_{ij}(\omega)|$. Then, by Lemma 2.4 there exists a measurable function $\gamma:\Omega\to\mathbb{R}$ such that $\gamma_{ij}(\omega) = \gamma(\tau_{\Gamma_{ij}(\omega)}\omega)$. Furthermore, by Condition 1.10, $0<\gamma_{ij}(\omega)\le C<\infty$ for some constant $C$ independent from $\omega$. By Lemma 2.4 we find

$$\tilde{\nu}:\Omega\to\mathbb{R}^n\quad\text{such that}\quad \tilde{\nu}_{ij}(\omega) = \tilde{\nu}(\tau_{\Gamma_{ij}(\omega)}\omega)\,. \tag{37}$$

In the periodic case we similarly construct $\Gamma^\varepsilon_{\mathrm{per}}(\boldsymbol{Q},\omega)$ $\partial G^\varepsilon_{\mathrm{per}}$ and $\gamma^\varepsilon_{\mathrm{per},ij}(\omega)$, where $\gamma^\varepsilon_{\mathrm{per},ij}(\omega)$ are the interface volumes of $\varepsilon^{-1}\partial G^\varepsilon_{\mathrm{per}}$ on the torus $\boldsymbol{Q}/\varepsilon$.

For every $f \in C_b(\Omega)$ and fixed $\omega \in \Omega$ for the functions $f_\omega(x) := f(\tau_x\omega)$ and $f_{\omega,\varepsilon}(x) := f(\tau_{\frac{x}{\varepsilon}}\omega)$ it holds $f_\omega \in C_b(\mathbb{R}^n)$. Furthermore, by the Ergodic Theorem, for every $f \in L^p(\Omega, \mu_{P,\mathcal{P}})$ it holds $f_{\omega,\varepsilon} \in L^p_{loc}(\mathbb{R}^n; \mu_{P^\varepsilon(\omega)})$ for almost every $\omega \in \Omega$ and every $\varepsilon$. The same holds for $f \in L^p(\Omega, \mu_{\Gamma,\mathcal{P}})$ where $f_{\omega,\varepsilon} \in L^p_{loc}(\mathbb{R}^n; \mu_{\Gamma^\varepsilon(\omega)})$ for almost every $\omega \in \Omega$ and every $\varepsilon$.

Hence, for every $f \in C_b(\Omega)$ and fixed $\omega \in \Omega$ and the expression $[\![f]\!]^\sim_{\mathrm{Om}}(\omega) = [\![f_\omega]\!]^\sim(0)$ is well defined provided $0 \in \Gamma(\omega)$. Therefore, $[\![f]\!]^\sim_{\mathrm{Om}}(\omega)$ is $\mu_{\Gamma,\mathcal{P}}$-almost everywhere well defined. In a similar manner, we may define $\mathrm{div}_{\mathrm{Om}}$ as an operator on $C_b(\Omega; \mathbb{R}^n)$ via the realizations and equation (35). We observe that $[\![\cdot]\!]^\sim_{\mathrm{Om}}$ is a linear operator from $C_b(\Omega)$ to $L^2(\Omega; \mu_{\Gamma,\mathcal{P}})$ and like for the operator $[\![\cdot]\!]^\sim$ on $\mathbb{R}^n$ we claim that $-\mathrm{div}_{\mathrm{Om}} = ([\![\cdot]\!]^\sim_{\mathrm{Om}})^*$ also holds on $\Omega$.

Similar to the above scalar products for function spaces on $\mathbb{R}^n$, we define the following scalar products for function spaces on $\Omega$:

$$\langle u, v\rangle_{P,\mathcal{P}} := \int_\Omega u\, v\, d\mu_{P,\mathcal{P}}\,, \qquad\qquad \langle u, v\rangle_{\Gamma,\mathcal{P}} := \int_\Omega u\, v\, d\mu_{\Gamma,\mathcal{P}}\,.$$

**Lemma 3.6.** *For every $u \in C_b(\Omega)$, $f \in L^1(\Omega, \mu_{\Gamma,\mathcal{P}}; \mathbb{R}^n)$ with $\mathrm{div}_{\mathrm{Om}} f \in L^1(\Omega, \mu_{P,\mathcal{P}})$ and every $\varphi \in C_c(\mathbb{R}^n)$ it holds for almost every $\omega \in \Omega$*

$$\lim_{\varepsilon\to 0} \varepsilon^n \left\langle \mathrm{div}_P\left(f_{\omega,\varepsilon}\varphi\right), u_{\omega,\varepsilon}\right\rangle_{P^\varepsilon(\omega)} = \int_{\mathbb{R}^n} \varphi(x) \left\langle \mathrm{div}_{\mathrm{Om}} f,\, u\right\rangle_{P,\mathcal{P}} dx\,.$$

*The same holds if $u \in L^1(\Omega, \mu_{P,\mathcal{P}})$, $f \in C_b(\Omega; \mathbb{R}^n)$.*

Once Lemma 3.6 is proved, one easily obtains the following corollary.

**Corollary.** *The operator $-\mathrm{div}_{\mathrm{Om}} : L^2(\Omega, \mu_{\Gamma,\mathcal{P}}; \mathbb{R}^n) \to L^2(\Omega, \mu_{P,\mathcal{P}})$ is the adjoint of $[\![\cdot]\!]^\sim_{\mathrm{Om}}$.*

*Proof of Lemma 3.6.* We define $f_{ij}(\omega) := f(\tau_{\Gamma_{ij}(\omega)}\omega) = f(\tau_{\Gamma^\varepsilon_{ij}(\omega)/\varepsilon}\omega)$ and $u_i(\omega) := u(\tau_{P_i}\omega)$ as well as $\varphi^\varepsilon_i(\omega) := \varphi(P^\varepsilon_i(\omega))$ and $\varphi^\varepsilon_{ij}(\omega) := \varphi(\Gamma^\varepsilon_{ij}(\omega))$. For readability, we omit $\omega$ whenever possible and observe that

$$\begin{aligned}\left\langle \mathrm{div}_P\left(f_{\omega,\varepsilon}\varphi\right), u_{\omega,\varepsilon}\right\rangle_{P^\varepsilon(\omega)} &= \varepsilon^n \sum_{P^\varepsilon_i \in \boldsymbol{Q}} u_i \sum_{i\sim j} f_{ij}\nu_{ij}\cdot\tilde{\nu}_{ij}\varphi^\varepsilon_{ij} \\ &= \varepsilon^n \sum_{P^\varepsilon_i \in \boldsymbol{Q}} u_i \sum_{i\sim j} f_{ij}\nu_{ij}\cdot\tilde{\nu}_{ij}\varphi^\varepsilon_i + \varepsilon^n \sum_{P^\varepsilon_i \in \boldsymbol{Q}} u_i \sum_{i\sim j} f_{ij}\nu_{ij}\cdot\tilde{\nu}_{ij}\left[\varphi^\varepsilon_{ij} - \varphi^\varepsilon_i\right]. \end{aligned} \tag{38}$$

For the first sum on the right hand side of (38) we obtain

$$\begin{aligned}\varepsilon^n \sum_{P^\varepsilon_i \in \boldsymbol{Q}} u_i(\omega) \sum_{i\sim j} f_{ij}(\omega)\nu_{ij}\cdot\tilde{\nu}_{ij}\varphi(P^\varepsilon_i) = \left\langle \mathrm{div}_P\left(f_{\omega,\varepsilon}\right), \varphi u_{\omega,\varepsilon}\right\rangle_{P^\varepsilon(\omega)} = \left\langle \left(\mathrm{div}_{\mathrm{Om}} f\right)_{\omega,\varepsilon}, \varphi u_{\omega,\varepsilon}\right\rangle_{P^\varepsilon(\omega)} \\ \to \int_{\mathbb{R}^n} \varphi(x) \left\langle \mathrm{div}_{\mathrm{Om}} f,\, u\right\rangle_{P,\mathcal{P}}\,.\end{aligned}$$

Thus it only remains to estimate the second term on the right hand side of (38).

Due to Condition 1.10 and the uniform continuity of $\varphi$, for every $\eta > 0$ there exists $\varepsilon_0$ such that for all $\varepsilon < \varepsilon_0$ and all $i, j$ it holds $\left|\varphi(\Gamma^\varepsilon_{ij}) - \varphi(P^\varepsilon_i)\right| \le \eta$. We distinguish two cases.

**Case 1:** Let $u\in C_b(\Omega)$, $f\in L^1(\Omega,\mu_{\Gamma,\mathcal{P}};\mathbb{R}^n)$. We write $\bar u_{ij}:=\frac12(u_i+u_j)$ and $\tilde f_{ij}:=f_{ij}\tilde\nu_{ij}$ and obtain (omitting the $\omega$)

$$\begin{aligned}
&\varepsilon^n\sum_i u_i\sum_{i\sim j}\tilde f_{ij}\cdot\nu_{ij}\left[\varphi^\varepsilon_{ij}-\varphi^\varepsilon_i\right]\\
&\quad=-\varepsilon^n\sum_{(i,j)\in E^\varepsilon(\omega)}\left[\bar u_{ij}\tilde f_{ij}\cdot\nu_{ij}\left(\varphi^\varepsilon_j-\varphi^\varepsilon_i\right)+\frac12\left(u_i-u_j\right)\tilde f_{ij}\cdot\nu_{ij}\left(2\varphi^\varepsilon_{ij}-\varphi^\varepsilon_i-\varphi^\varepsilon_j\right)\right]\\
&\quad=-\varepsilon^n\sum_{(i,j)\in E^\varepsilon(\omega)}\left[\bar u_{ij}f_{ij}[\![\varphi^\varepsilon]\!]^\sim_{ij}-\frac12[\![u_\omega]\!]^\sim_{ij}f_{ij}\left(2\varphi^\varepsilon_{ij}-\varphi^\varepsilon_i-\varphi^\varepsilon_j\right)\right].
\end{aligned}$$

The first term on the right hand side becomes arbitrarily small since $[\![\varphi^\varepsilon]\!]^\sim_{ij}<\eta$ and $\|u\|_\infty<\infty$. The second term on the right hand side becomes small since $\left|2\varphi^\varepsilon_{ij}-\varphi^\varepsilon_i-\varphi^\varepsilon_j\right|<2\eta$ and $\left\|[\![u]\!]^\sim_{ij}\right\|_\infty<2\|u\|_\infty$.

**Case 2:** Let $u\in L^1(\Omega,\mu_{P,\mathcal{P}})$, $f\in C(\Omega;\mathbb{R}^n)$. For the limit of the second sum, we define $P^\varepsilon_\varphi(\omega)=P^\varepsilon(\omega)\cap\operatorname{supp}\varphi$ and obtain for $\varepsilon>\varepsilon_0$ that

$$\begin{aligned}
\left|\varepsilon^n\sum_i u_i(\omega)\sum_{i\sim j}f_{ij}(\omega)\left[\varphi(\Gamma^\varepsilon_{ij})-\varphi(P^\varepsilon_i)\right]\right|&\le\eta C\|f\|_\infty\int_{P^\varepsilon_\varphi(\omega)}\left|u(\tau_{\frac{x}{\varepsilon}}\omega)\right|d\mu^\varepsilon_{P(\omega)}(x)\\
&\to\eta\left|\operatorname{supp}\varphi\right|C\|f\|_\infty\int_\Omega|u|\,d\mu_{P,\mathcal{P}}.
\end{aligned}$$

Again, since $\eta$ is arbitrarily small, the statement follows. □

We use the definition of $[\![u]\!]^\sim$ to define the following subspace of $L^2(\Omega,\mu_{a\Gamma,\mathcal{P}})$, where $\mathrm{d}\mu_{a\Gamma,\mathcal{P}}(\omega)=a(\omega)\mathrm{d}\mu_{\Gamma,\mathcal{P}}(\omega)$:

$$L^2_{\mathrm{pot}}(\Gamma)=\operatorname{closure}_{L^2(\Omega,\mu_{a\Gamma,\mathcal{P}})}\left\{[\![f]\!]^\sim\,:\,f\in C_b(\Omega)\right\}\qquad L^2_{\mathrm{sol}}(\Gamma)=L^2_{\mathrm{pot}}(\Gamma)^\perp$$

and make the following observation:

**Lemma 3.7.** *For every $f\in L^2_{\mathrm{sol}}(\Gamma)$ it holds $\operatorname{div}_{\mathrm{Om}}(fa)=0$ $\mu_{\Gamma,\mathcal{P}}$-almost surely. Hence, for almost every realization $f_\omega$ holds $\operatorname{div}_P(a_\omega f_\omega)=0$ locally on $P(\omega)$.*

*Proof.* Let $f\in L^2_{\mathrm{sol}}(\Gamma)$ and let $\varphi\in C_c(\mathbb{R}^n)$. Then, for every $u\in C_b(\Omega)$ we obtain from Theorem 2.6 and Lemma 3.6 for some $\omega\in\Omega$ that

$$\begin{aligned}
0&=\int_{\mathbb{R}^n}\varphi\left\langle[\![u]\!]^\sim_{\mathrm{Om}},af\right\rangle_{\Gamma,\mathcal{P}}=\lim_{\varepsilon\to0}\left\langle\varphi f_{\omega,\varepsilon}a_{\omega,\varepsilon},\left([\![u]\!]^\sim_{\mathrm{Om}}\right)_{\omega,\varepsilon}\right\rangle_{\Gamma^\varepsilon(\omega)}\\
&=-\lim_{\varepsilon\to0}\left\langle\operatorname{div}_P\left(f_{\omega,\varepsilon}a_{\omega,\varepsilon}\varphi\right),u_{\omega,\varepsilon}\right\rangle_{P^\varepsilon(\omega)}=-\int_{\mathbb{R}^n}\varphi(x)\left\langle\operatorname{div}_{\mathrm{Om}}(fa),u\right\rangle_{P,\mathcal{P}}.
\end{aligned}$$

Since this holds true for every $\varphi\in C_c(\mathbb{R}^n)$ and every $u\in C_b(\Omega)$, the claim follows. □

## 3.3 The homogenized matrix in the stationary ergodic setting

Let $(e_i)_{i=1,\dots n}$ be an orthonormal basis of $\mathbb{R}^n$, $\tilde\nu$ from (37) and let $\chi_i\in L^2_{\mathrm{pot}}(\Gamma)$ be the unique minimizers of the functional

$$\begin{aligned}
E_i:\ &L^2_{\mathrm{pot}}(\Gamma)\to\mathbb{R}\\
&\chi\mapsto\int_\Omega a\left|e_i-\chi\tilde\nu\right|^2d\mu_{\Gamma,\mathcal{P}}.
\end{aligned}$$

We define the matrix $A_{\mathrm{hom}}$ through

$$\begin{aligned} A_{\mathrm{hom}} &= (A_{i,j})_{i,j=1,\dots,n} \\ \text{with} \qquad A_{i,j} &= \int_\Omega a\,(e_i - \chi_i\tilde\nu)\cdot(e_j - \chi_j\tilde\nu)\; d\mu_{\Gamma,\mathcal{P}}. \end{aligned} \tag{39}$$

As usual in random conductance theory, the matrix $A_{\mathrm{hom}}$ and the space $L^2_{\mathrm{sol}}(\Gamma)$ satisfy the following properties.

**Lemma 3.8.** *The matrix $A_{\mathrm{hom}}$ is positive definite.*

*Proof.* The proof is standard (see [19, Lemma 5.5]) and we provide it here for completeness.

**Step 1:**Recall the definition of $\gamma_{ij}$ at the beginning of Section 3.1. We first prove that every $v \in L^2_{\mathrm{pot}}(\Gamma)$ satisfies

$$\forall \xi \in \mathbb{R}^n : \quad \int_\Omega v\tilde\nu\cdot\xi\gamma d\mu_{\Gamma,\mathcal{P}} = 0\,. \tag{40}$$

In order to prove (40) let $u \in C_b(\Omega)$ and choose a bounded open ball $B$ around 0 with normal vector $\nu_B$. Let $\tilde u^\varepsilon_\omega(P^\varepsilon_i) := u(\tau_{P^\varepsilon_i}\omega)$ such that $\tilde u^\varepsilon \in \mathcal{S}^\varepsilon(\omega)$. We obtain

$$\begin{aligned} |B|\left|\int_\Omega [\![u]\!]^\sim_{\mathrm{Om}}\tilde\nu\cdot\xi\gamma d\mu_{\Gamma,\mathcal{P}}\right| &= \lim_{\varepsilon\to0}\left|\int_{B\cap\Gamma^\varepsilon(\omega)} \left([\![u]\!]^\sim_{\mathrm{Om}}\gamma\tilde\nu\right)(\tau_{\frac{x}{\varepsilon}}\omega)\cdot\xi d\mu^\varepsilon_{\Gamma(\omega)}\right| \\ &= \lim_{\varepsilon\to0}\left|\varepsilon\int_{B\cap\partial G^\varepsilon(\omega)} [\![\mathcal{R}^*_\varepsilon\tilde u^\varepsilon_\omega]\!]\cdot\xi d\mathcal{H}^{n-1}\right| \\ &= \lim_{\varepsilon\to0}\left|\varepsilon\int_{\partial B} \mathcal{R}^*_\varepsilon\tilde u^\varepsilon_\omega\xi\cdot\nu_B\, d\mathcal{H}^{n-1}\right| \\ &\le \lim_{\varepsilon\to0}\varepsilon\,\|u\|_\infty\,|\xi|\,|\partial B| = 0\,. \end{aligned}$$

Hence (40) follows from the density of $[\![u]\!]^\sim$ in $L^2_{\mathrm{pot}}(\Gamma)$.

**Step 2:** Let $\xi \in \mathbb{R}^n\backslash 0$. Using (40) and the Cauchy-Schwarz inequality we find with $C_\gamma = \int_\Omega \gamma d\mu_{\Gamma,\mathcal{P}} > 0$

$$\begin{aligned} \xi_k^2 &= \xi_k C_\gamma^{-1}\int_\Omega \xi\cdot e_k\gamma d\mu_{\Gamma,\mathcal{P}} \\ &= \xi_k C_\gamma^{-1}\int_\Omega e_k\cdot\sum_{i=1}^n\left(\xi_i e_i - \xi_i\chi_i\tilde\nu\right)\gamma d\mu_{\Gamma,\mathcal{P}} \\ &\le |\xi_k|\,C_\gamma^{-1}\left(\int_\Omega \gamma^2 a^{-1}\, d\mu_{\Gamma,\mathcal{P}}\right)^{\frac12}\left(\sum_{i,j=1}^n \xi_i\xi_j A_{ij}\right)^{\frac12}\,. \end{aligned}$$

Summing up the last inequality over $k = 1,\dots,n$ yields

$$\frac{|\xi|_2^2}{|\xi|_1} \le C\sqrt{\xi\cdot A_{\mathrm{hom}}\xi}\,.$$

The Lemma now follows from the equivalence of norms in $\mathbb{R}^n$. □

**Lemma 3.9.** *It holds $\mathbb{R}^n = \mathrm{span}\left\{\int_\Omega f\tilde\nu d\mu_{\Gamma,\mathcal{P}} \,:\, f \in L^2_{\mathrm{sol}}(\Gamma)\right\}$.*

*Proof.* We follow the proof of [18, Lemma 4.5]. Due to the minimizing properties of $\chi_i$ in $L^2_{\mathrm{pot}}(\Gamma)$ we have $(e_i - \chi_i\tilde{\nu})\cdot\tilde{\nu} \in L^2_{\mathrm{sol}}(\Gamma)$, i.e.

$$\forall i,j: \int_\Omega ((e_i - \chi_i\tilde{\nu})\cdot\tilde{\nu})\,\chi_j a\,d\mu_{\Gamma,\mathcal{P}} = 0\,. \tag{41}$$

Defining $V := \operatorname{span}\left\{\int_\Omega f\tilde{\nu}d\mu_{\Gamma,\mathcal{P}} \,:\, f \in L^2_{\mathrm{sol}}(\Gamma)\right\}$ we choose $\xi \in V^\perp\backslash\{0\}$. Then, for all $i = 1,\dots,n$ it holds

$$\int_\Omega \xi\cdot\tilde{\nu}\,((e_i - \chi_i\tilde{\nu})\cdot\tilde{\nu})\,a\,d\mu_{\Gamma,\mathcal{P}} = 0\,. \tag{42}$$

Combining (41) and (42) implies that

$$\int_\Omega \sum_{j=1}^n \xi_j\,(e_j\cdot\tilde{\nu} + \chi_j)\,((e_i - \chi_i\tilde{\nu})\cdot\tilde{\nu})\,a\,d\mu_{\Gamma,\mathcal{P}} = 0\,.$$

Multiplying the last equality by $\xi_i$ and summing over $i$ yields

$$\xi A_{\mathrm{hom}}\xi = 0\,.$$

Due to Lemma 3.8 this implies $\xi = 0$, a contradiction. □

# 4 Proof of Theorems 1.6 and 1.8

We first observe the following behavior.

**Lemma 4.1** ($L^2$ - G-convergence)**.** *Let the family $(P_\varepsilon)_{\varepsilon>0}$ be G-convergent in sense of Definition 1.4, $a^{-1} \le a^\varepsilon_{ij} \le a$ uniformly in $\varepsilon, i, j$ for some $a > 0$ and let Condition 1.2 be satisfied. Let $f_\varepsilon \in \mathcal{S}^\varepsilon(\omega)$ and $f \in L^2(\boldsymbol{Q})$ such that $\mathcal{R}^*_\varepsilon f^\varepsilon \rightharpoonup f$ weakly in $L^2(\boldsymbol{Q})$ and let the sequence $u^\varepsilon \in \mathcal{S}^\varepsilon_0(\boldsymbol{Q})$ be solutions of the problems*

$$\varepsilon^{-2}\sum_{i\sim j} a^\varepsilon_{ij}\left(u^\varepsilon_i - u^\varepsilon_j\right) = f^\varepsilon_i \tag{43}$$

*and let $u \in H^1_0(\boldsymbol{Q}) \cap H^2(\boldsymbol{Q})$ be the unique solution to*

$$-\nabla\cdot(A_{\mathrm{hom}}\nabla u) = f\,. \tag{44}$$

*Then $\mathcal{R}^*_\varepsilon u^\varepsilon \to u$ strongly in $L^2(\boldsymbol{Q})$ and $\frac{1}{\varepsilon}[\![u^\varepsilon]\!]d\mathcal{H}^{n-1} \to \nabla u$ in the sense of distribution as $\varepsilon \to 0$.*

*Proof.* The operator $-\mathcal{L}^\varepsilon$ is strictly positive definite and symmetric as follows from

$$-\langle\mathcal{L}^\varepsilon u^\varepsilon, u^\varepsilon\rangle = \varepsilon^{n-2}\sum_{i\sim j} a^\varepsilon_{ij}\left(u^\varepsilon_j - u^\varepsilon_i\right)^2\,.$$

Due to Lemma 3.1, the family $\mathcal{L}^\varepsilon$ is uniformly elliptic in $\varepsilon$ and we obtain the apriori estimate

$$\|u^\varepsilon\|^2_{P^\varepsilon} = \varepsilon^n\sum_i (u^\varepsilon_i)^2 \le C\varepsilon^{n-2}\sum_{i\sim j}\left(u^\varepsilon_j - u^\varepsilon_i\right)^2 \le Ca\,\|f^\varepsilon\|_{P^\varepsilon}\,\|u^\varepsilon\|_{P^\varepsilon}\,. \tag{45}$$

By the Lax-Milgram Lemma, the solution to (43) exists and is unique. Let $\hat{u}^\varepsilon$ be the unique solution of $-\mathcal{L}^\varepsilon \hat{u}^\varepsilon = \mathcal{R}_\varepsilon^* f$. Then $\hat{u}^\varepsilon$ satisfies (45) with $f^\varepsilon$ replaced by $\mathcal{R}_\varepsilon^* f$ and we find

$$\left\|\mathcal{R}_\varepsilon^*\left(u^\varepsilon - \hat{u}^\varepsilon\right)\right\|_{L^2(\boldsymbol{Q})}^2 \leq \varepsilon^n C \sum_{(i,j)\in E^\varepsilon(\omega)} \frac{1}{\varepsilon^2} a_{ij}^\varepsilon [\![u^\varepsilon - \hat{u}^\varepsilon]\!]_{ij}^2 = \varepsilon^n C \sum_i \left(f_i^\varepsilon - \left(\mathcal{R}_\varepsilon^* f\right)_i\right)\left(u^\varepsilon - \hat{u}^\varepsilon\right) .$$

Due to Lemma 3.1, $\mathcal{R}_\varepsilon^* u^\varepsilon$ and $\mathcal{R}_\varepsilon^* \hat{u}^\varepsilon$ are both precompact sequences in $L^2(\boldsymbol{Q})$. Since $P^\varepsilon$ is G-convergent and

$$\mathcal{R}_\varepsilon^*\left(\mathcal{R}_\varepsilon f\right) - \mathcal{R}_\varepsilon^* f_i^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(\boldsymbol{Q}) , \tag{46}$$

we obtain from Lemma 3.3 and the above estimates that

$$\lim_{\varepsilon\to 0}\left( \left\|\mathcal{R}_\varepsilon^*\left(u^\varepsilon - \hat{u}^\varepsilon\right)\right\|_{L^2(\boldsymbol{Q})}^2 + \varepsilon^n \sum_{(i,j)\in E^\varepsilon(\omega)} \frac{1}{\varepsilon^2} [\![u^\varepsilon - \hat{u}^\varepsilon]\!]_{ij}^2 \right) = 0$$

and hence $\mathcal{R}_\varepsilon^* u^\varepsilon \to u$ strongly in $L^2(\boldsymbol{Q})$ and $\frac{1}{\varepsilon}[\![u^\varepsilon]\!] d\mathcal{H}^{n-1} \to \nabla u$ in the sense of distribution. Since $P^\varepsilon$ is G-convergent, we obtain that $u$ solves (44). □

**Lemma 4.2.** *Let the family $\left(P_\varepsilon\right)_{\varepsilon>0}$ be G-convergent in sense of Definition 1.4 and let Condition 1.2 be satisfied. Then, there exists a constant $C>0$ such that for every $\varepsilon>0$ and every $\phi, u \in \mathcal{S}^\varepsilon$ it holds*

$$\varepsilon^n \sum_i |v_i| \sum_{j\sim i} |u_j - u_i| \leq C \left\|v\right\|_{P^\varepsilon} \left\|[\![u]\!]\right\|_{\Gamma^\varepsilon} .$$

*Proof.* We obtain

$$\varepsilon^n \sum_i |v_i| \sum_{j\sim i} |u_j - u_i| \leq \left\|v\right\|_{P^\varepsilon} \left( \varepsilon^n \sum_i 2C \sum_{i\sim j} |u_j - u_i|^2 \right)^{\frac{1}{2}} \leq 4C \left\|v\right\|_{P^\varepsilon} \left( \varepsilon^n \sum_{i\sim j} |u_j - u_i|^2 \right)^{\frac{1}{2}} ,$$

where $C$ denotes the maximum number of neighbors of a cell, which is bounded due to Condition 1.2. □

**Lemma 4.3.** *Let let the sequence $u^\varepsilon \in \mathcal{S}_0^\varepsilon(\boldsymbol{Q})$ satisfy $\mathcal{R}_\varepsilon^* u^\varepsilon \to u$ strongly in $L^2(\boldsymbol{Q})$ and*

$$\sup_{\varepsilon>0} \varepsilon^n \sum_{(i,j)\in E^\varepsilon(\omega)} \frac{1}{\varepsilon^2} [\![u^\varepsilon]\!]_{ij}^2 + \varepsilon^n \sum_i \left( \varepsilon^{-2} \sum_{i\sim j} \left(u_j^\varepsilon - u_i^\varepsilon\right) \right)^2 < \infty . \tag{47}$$

*Then $u \in H_0^1(\boldsymbol{Q})$ and for every $\phi \in C_c^1(\overline{\boldsymbol{Q}})$ with $\phi_i^\varepsilon = \phi(P_i^\varepsilon)$ it holds*

$$\varepsilon^n \sum_{(i,j)\in E^\varepsilon(\omega)} \frac{1}{\varepsilon^2} a_{ij}^\varepsilon [\![u^\varepsilon]\!]_{ij} [\![\phi^\varepsilon]\!]_{ij} \to \int_{\boldsymbol{Q}} \nabla u \cdot \left(A_{\mathrm{hom}} \nabla\phi\right) .$$

*Proof.* The regularity $u \in H_0^1(\boldsymbol{Q})$ follows from (47) and Lemma 3.1. Writing $f^\varepsilon := \varepsilon^{-2} \sum_{i\sim j} a_{ij}^\varepsilon \left(u_j^\varepsilon - u_i^\varepsilon\right)$ we find $\sup_{\varepsilon>0} \left\|f^\varepsilon\right\|_{P^\varepsilon} < \infty$ and hence along a subsequence we find $\mathcal{R}_\varepsilon^* f^\varepsilon \rightharpoonup f \in L^2(\boldsymbol{Q})$ and G-convergence implies $f = \nabla\cdot\left(A_{\mathrm{hom}}\nabla u\right)$. Therefore

$$\varepsilon^n \sum_{(i,j)\in E^\varepsilon(\omega)} \frac{1}{\varepsilon^2} a_{ij}^\varepsilon [\![u^\varepsilon]\!]_{ij} [\![\phi^\varepsilon]\!]_{ij} = -\varepsilon^n \sum_i \frac{1}{\varepsilon^2} f_i^\varepsilon \phi_i^\varepsilon \to -\int \nabla\cdot\left(A_{\mathrm{hom}}\nabla u\right)\phi = \int_{\boldsymbol{Q}} \nabla u \cdot \left(A_{\mathrm{hom}}\nabla\phi\right) .$$

□

## Proof of Theorem 1.6

We define $U_i^\varepsilon := u_i^\varepsilon / (v_i^\varepsilon)^2$ satisfying

$$-\frac{1}{\varepsilon^2}\sum_{j\sim i} v_i^\varepsilon v_j^\varepsilon a_{ij}^\varepsilon \left(U_j^\varepsilon - U_i^\varepsilon\right) = f_i^\varepsilon \,. \tag{48}$$

For simplicity and readability of the proof, we assume $a_{ij}^\varepsilon = 1$.

**Step 1: Apriori estimates on $U^\varepsilon$.** Testing (48) with $U_i^\varepsilon$ and using boundedness of $v \geq C > 0$ from below we obtain similar to the proof of Lemma 4.1 that

$$\varepsilon^n \sum_i \left(U_i^\varepsilon\right)^2 \leq C\varepsilon^{n-2}\sum_{i\sim j}\left(U_j^\varepsilon - U_i^\varepsilon\right)^2 \leq C\left\|f^\varepsilon\right\|_{P^\varepsilon}^2 \,,$$

and hence the sequence $\mathcal{R}_\varepsilon^* U^\varepsilon$ is precompact and

$$\frac{1}{\varepsilon^2}\left\|[\![U^\varepsilon]\!]^\sim\right\|_{\Gamma^\varepsilon}^2 + \left\|U^\varepsilon\right\|_{P^\varepsilon}^2 \leq C\left\|f\right\|_{P^\varepsilon}^2 \,. \tag{49}$$

Next, we recall the definition of $\mathcal{L}^\varepsilon$ and test (48) with $\phi_i^\varepsilon := \left(\mathcal{L}^\varepsilon U_i^\varepsilon\right) = \varepsilon^{-2}\sum_{j\sim i}\left(U_j^\varepsilon - U_i^\varepsilon\right)$ and use

$$\sum_{i\sim j} v_i^\varepsilon v_j^\varepsilon\left(U_j^\varepsilon - U_i^\varepsilon\right) = \left(v_i^\varepsilon\right)^2\sum_{i\sim j}\left(U_j^\varepsilon - U_i^\varepsilon\right) + v_i^\varepsilon\sum_{i\sim j}\left(v_j^\varepsilon - v_i^\varepsilon\right)\left(U_j^\varepsilon - U_i^\varepsilon\right)$$

and Lemma 4.2 to obtain

$$\begin{aligned}
\varepsilon^n\sum_i\left(v_i^\varepsilon\right)^2\left(\mathcal{L}^\varepsilon U_i^\varepsilon\right)^2 &\leq -\varepsilon^n\sum_i f_i^\varepsilon\left(\mathcal{L}^\varepsilon U_i^\varepsilon\right) + \left\|\nabla v\right\|_\infty\frac{1}{\varepsilon}\left\|[\![U^\varepsilon]\!]^\sim\right\|_{\Gamma^\varepsilon}\left\|\mathcal{L}^\varepsilon U^\varepsilon\right\|_{P^\varepsilon}\\
&\leq \left\|f\right\|_{P^\varepsilon}\left\|\mathcal{L}^\varepsilon U^\varepsilon\right\|_{P^\varepsilon} + \left\|\nabla v\right\|_\infty\frac{1}{\varepsilon}\left\|[\![U^\varepsilon]\!]^\sim\right\|_{\Gamma^\varepsilon}\left\|\mathcal{L}^\varepsilon U^\varepsilon\right\|_{P^\varepsilon}\,.
\end{aligned}$$

Using (49) we obtain that

$$\frac{1}{\varepsilon^2}\left\|[\![U^\varepsilon]\!]^\sim\right\|_{\Gamma^\varepsilon}^2 + \left\|U^\varepsilon\right\|_{P^\varepsilon}^2 + \left\|\mathcal{L}^\varepsilon U^\varepsilon\right\|_{P^\varepsilon}^2 \leq C\left(\left\|f\right\|_{P^\varepsilon}^2, \left\|v\right\|_{C^2(\overline{\boldsymbol{Q}})}^2\right)\,. \tag{50}$$

**Step 2: Apriori Estimates on $u^\varepsilon$.** In what follows, we write $\tilde{v}_i^\varepsilon := \left(v_i^\varepsilon\right)^2$. From $u_i^\varepsilon = \tilde{v}_i^\varepsilon U_i^\varepsilon$ we obtain

$$u_j^\varepsilon - u_i^\varepsilon = \frac{1}{2}\left(\tilde{v}_j^\varepsilon - \tilde{v}_i^\varepsilon\right)\left(U_j^\varepsilon + U_i^\varepsilon\right) + \frac{1}{2}\left(\tilde{v}_j^\varepsilon + \tilde{v}_i^\varepsilon\right)\left(U_j^\varepsilon - U_i^\varepsilon\right)$$

which gives an estimate on $\frac{1}{\varepsilon^2}\left\|[\![u^\varepsilon]\!]^\sim\right\|_{\Gamma^\varepsilon}^2$. In order to proof the estimate on $\mathcal{L}^\varepsilon u^\varepsilon$, we multiply $\mathcal{L}^\varepsilon u^\varepsilon$ with an arbitrary test function $\phi \in C_c^\infty(\boldsymbol{Q})$ and obtain

$$\begin{aligned}
-\left\langle\mathcal{L}^\varepsilon u^\varepsilon, \phi\right\rangle_{P^\varepsilon} &= \varepsilon^{n-2}\sum_{i\sim j}\left(\tilde{v}_j^\varepsilon U_j^\varepsilon - \tilde{v}_i^\varepsilon U_i^\varepsilon\right)\left(\phi_j^\varepsilon - \phi_i^\varepsilon\right)\\
&= \varepsilon^{n-2}\sum_{i\sim j}\left(\frac{1}{2}\left(\tilde{v}_j^\varepsilon + \tilde{v}_i^\varepsilon\right)\left(U_j^\varepsilon - U_i^\varepsilon\right)\left(\phi_j^\varepsilon - \phi_i^\varepsilon\right) + \frac{1}{2}\left(\tilde{v}_j^\varepsilon - \tilde{v}_i^\varepsilon\right)\left(U_j^\varepsilon + U_i^\varepsilon\right)\left(\phi_j^\varepsilon - \phi_i^\varepsilon\right)\right)\\
&= \varepsilon^{n-2}\sum_{i\sim j}\frac{1}{2}\left(\left(U_j^\varepsilon - U_i^\varepsilon\right)\left(\tilde{v}_j^\varepsilon\phi_j^\varepsilon - \tilde{v}_i^\varepsilon\phi_i^\varepsilon\right) + \left(\tilde{v}_j^\varepsilon - \tilde{v}_i^\varepsilon\right)\left(\phi_j^\varepsilon U_j^\varepsilon - \phi_i^\varepsilon U_i^\varepsilon\right)\right)\\
&\quad + \varepsilon^{n-2}\sum_{i\sim j}\left(\tilde{v}_j^\varepsilon - \tilde{v}_i^\varepsilon\right)\left(U_j^\varepsilon - U_i^\varepsilon\right)\left(\phi_j^\varepsilon + \phi_i^\varepsilon\right)\\
&= \varepsilon^n\sum_i\phi_i^\varepsilon U_i^\varepsilon\frac{1}{2\varepsilon^2}\sum_{j\sim i}\left(\tilde{v}_j^\varepsilon - \tilde{v}_i^\varepsilon\right) + \varepsilon^n\sum_i\phi_i^\varepsilon\tilde{v}_i^\varepsilon\frac{1}{2\varepsilon^2}\sum_{j\sim i}\left(U_j^\varepsilon - U_i^\varepsilon\right)\\
&\quad + \varepsilon^{n-2}\sum_{i\sim j}\left(\tilde{v}_j^\varepsilon - \tilde{v}_i^\varepsilon\right)\left(U_j^\varepsilon - U_i^\varepsilon\right)\left(\phi_j^\varepsilon + \phi_i^\varepsilon\right)
\end{aligned}$$

Hence, we obtain with help of Lemma 4.2

$$|\langle \mathcal{L}^\varepsilon u^\varepsilon, \phi\rangle_{P^\varepsilon}| \le \|\phi\|_{P^\varepsilon} C\left(\|U^\varepsilon\|_{P^\varepsilon} \|v\|_{C^2(\boldsymbol{Q})} + \|\mathcal{L}^\varepsilon U^\varepsilon\|_{P^\varepsilon} \|v\|_\infty + \left\|\nabla v^2\right\|_\infty \frac{1}{\varepsilon} \|[\![U^\varepsilon]\!]^\sim\|_{\Gamma^\varepsilon}\right),$$

where $C$ does not depend on $\varepsilon$. Together with (50), it follows

$$\limsup_{\varepsilon\to 0} \|\mathcal{R}_\varepsilon^* \mathcal{L}^\varepsilon u^\varepsilon\|_{L^2(\boldsymbol{Q})} \le \limsup_{\varepsilon\to 0} C\left(\|f\|_{P^\varepsilon} + \left\|v^2\right\|_{C^2(\overline{\boldsymbol{Q}})}\right)\left\|v^2\right\|_{C^2(\overline{\boldsymbol{Q}})} \le \infty .$$

This concludes the proof.

**Step 3: Convergence.** We use the above estimates in order to pass to the limit in (48). We choose a countable dense family $\Phi := (\phi^k)_{k\in\mathbb{N}} \subset H_0^1(\boldsymbol{Q})$ of functions $\phi^k \in C_c^\infty(\boldsymbol{Q})$ for every $k \in \mathbb{N}$ and use these as test functions in (48).

We write $v_{ij}^\varepsilon := v(\Gamma_{ij}^\varepsilon)$, recall (36) and define

$$I_1^\varepsilon := \varepsilon^n \sum_{(i,j)\in E^\varepsilon} v_i^\varepsilon v_j^\varepsilon \frac{1}{\varepsilon}[\![U^\varepsilon]\!]_{ij}^\sim \frac{1}{\varepsilon}[\![\phi]\!]_{ij}^\sim = \varepsilon^n \sum_{(i,j)\in E^\varepsilon} \left(v_{ij}^\varepsilon\right)^2 \frac{1}{\varepsilon}[\![U^\varepsilon]\!]_{ij}^\sim \frac{1}{\varepsilon}[\![\phi]\!]_{ij}^\sim + I_2^\varepsilon .$$

Since $v$ is uniformly continuous, for every $\eta > 0$ there exists $\varepsilon_0$ such that for $\varepsilon < \varepsilon_0$ it holds $\left|\left(v_{ij}^\varepsilon\right)^2 - v_i^\varepsilon v_j^\varepsilon\right| < \eta$ for every $(i,j) \in E^\varepsilon$. Hence with

$$|I_2^\varepsilon| \le \eta \|\nabla\phi\|_\infty \|[\![U^\varepsilon]\!]^\sim\|_{\Gamma^\varepsilon} \le \eta \|\nabla\phi\|_\infty \sup_{\varepsilon>0} \|[\![U^\varepsilon]\!]^\sim\|_{\Gamma^\varepsilon} \le \eta C \|\nabla\phi\|_\infty .$$

Due to Lemma 4.3, we obtain for every $\phi \in \Phi$ that

$$\varepsilon^n \sum_{(i,j)\in E^\varepsilon} \frac{1}{\varepsilon}[\![U^\varepsilon]\!]_{ij}^\sim \frac{1}{\varepsilon}[\![\phi]\!]_{ij}^\sim \to \int_{\boldsymbol{Q}} \nabla\phi \cdot (A_{\mathrm{hom}}\nabla U) . \tag{51}$$

Furthermore, we note that

$$\sup_{\varepsilon>0} \varepsilon^n \sum_{(i,j)\in E^\varepsilon} \frac{1}{\varepsilon^2}[\![U^\varepsilon]\!]_{ij}^2 \frac{1}{\varepsilon^2}[\![\phi]\!]_{ij}^2 \le \|\nabla\phi\|_\infty^2 \sup_{\varepsilon>0} \varepsilon^n \sum_{(i,j)\in E^\varepsilon} \frac{1}{\varepsilon^2}[\![U^\varepsilon]\!]_{ij}^2 < \infty .$$

Hence, for every $\phi \in \Phi$, the pair $\left(\frac{1}{\varepsilon}[\![U^\varepsilon]\!]_{ij}^\sim \frac{1}{\varepsilon}[\![\phi]\!]_{ij}^\sim, \mu_{\Gamma^\varepsilon}\right)$ is a measure-function pair w.r.t. the quadratic function in the sense of Hutchinson and we can apply [27, Theorem 4.4.2]. In particular, since $\Phi$ is countable, we obtain from [27, Theorem 4.4.2] that

$$\forall \phi \in \Phi : \qquad \varepsilon^{n-2} \sum_{i\sim j} \left(v_{ij}^\varepsilon\right)^2 \frac{1}{\varepsilon}[\![U^\varepsilon]\!]_{ij}^\sim \frac{1}{\varepsilon}[\![\phi]\!]_{ij}^\sim \to \int_{\boldsymbol{Q}} v^2 \nabla\phi \cdot (A_{\mathrm{hom}}\nabla U) .$$

Furthermore, we obtain from the above apriori estimates and Lemma 3.2 that $U^\varepsilon \to U$ and $u^\varepsilon \to u$ strongly in $L^2(\boldsymbol{Q})$ and due to $U_i^\varepsilon := u_i^\varepsilon / \left(v_i^\varepsilon\right)^2$ we find $u = v^2 U$. From the weak convergence of $f^\varepsilon$ we finally obtain that $u$ solves

$$\int_{\boldsymbol{Q}} v^2 \nabla\phi \cdot \left(A_{\mathrm{hom}} \nabla\left(\frac{u}{v^2}\right)\right) = \int_{\boldsymbol{Q}} f\phi ,$$

or equivalently

$$\int_{\boldsymbol{Q}} \nabla\phi \cdot (A_{\mathrm{hom}}\nabla u) - 2\int_{\boldsymbol{Q}} \frac{u}{v}\nabla\phi \cdot (A_{\mathrm{hom}}\nabla v) = \int_{\boldsymbol{Q}} f\phi .$$

Using that $v = \exp\left(-\frac{\beta}{2}V\right)$, we obtain that $u$ solves (15).

### Proof of Theorem 1.8

Like in the previous proof, we assume $a^\varepsilon_{ij} = 1$ for simplicity and readability.

Due to the first part of Theorem 1.6 the operator $\mathcal{F}^\varepsilon_v$ is invertible for $\varepsilon$ small enough and unique existence of solutions to (16) follows. Let us first note that writing $V^\varepsilon_i := V(x^\varepsilon_i)$ and using the Taylor formula we obtain

$$\frac{v^\varepsilon_i}{v^\varepsilon_j} = \exp\left(-\frac{\beta}{2}\left(V^\varepsilon_i - V^\varepsilon_j\right)\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{\beta^k}{2^k}\left(V^\varepsilon_j - V^\varepsilon_i\right)^k$$

and hence

$$\begin{aligned}\left(u^\varepsilon_j \frac{v^\varepsilon_i}{v^\varepsilon_j} - u^\varepsilon_i \frac{v^\varepsilon_j}{v^\varepsilon_i}\right) &= \sum_{k=0}^{\infty} \frac{1}{k!}\frac{\beta^k}{2^k}\left(u^\varepsilon_j - (-1)^k u^\varepsilon_i\right)\left(V^\varepsilon_j - V^\varepsilon_i\right)^k \\ &= \left(u^\varepsilon_j - u^\varepsilon_i\right) + \frac{1}{2}\left(u^\varepsilon_j + u^\varepsilon_i\right)\beta\left(V^\varepsilon_j - V^\varepsilon_i\right) \\ &\quad + \sum_{k=2}^{\infty} \frac{1}{k!}\frac{1}{2^k}\left((-1)^k u^\varepsilon_j - u^\varepsilon_i\right)\beta^k\left(V^\varepsilon_i - V^\varepsilon_j\right)^k . \end{aligned} \tag{52}$$

Testing (16) with $u^\varepsilon$, using (52) and Lemma 4.2 yields

$$\begin{aligned}\frac{1}{2}\left\|u^\varepsilon\right\|^2_{P^\varepsilon}\Big|_0^T + \frac{1}{\varepsilon^2}\int_0^T \left\|[\![u^\varepsilon]\!]^\sim\right\|^2_{\Gamma^\varepsilon} &\le \int_0^T \left\|f^\varepsilon\right\|^2_{P^\varepsilon}\left\|u^\varepsilon\right\|^2_{P^\varepsilon} + C\frac{1}{\varepsilon}\left\|\nabla v\right\|_\infty \int_0^T \left\|[\![u^\varepsilon]\!]^\sim\right\|_{\Gamma^\varepsilon}\left\|u^\varepsilon\right\|_{P^\varepsilon} \\ &\quad + C\varepsilon\left\|\nabla v\right\|_\infty \int_0^T \frac{1}{\varepsilon}\left\|[\![u^\varepsilon]\!]^\sim\right\|_{\Gamma^\varepsilon}\left(\frac{1}{\varepsilon}\left\|[\![u^\varepsilon]\!]^\sim\right\|_{\Gamma^\varepsilon} + \left\|u^\varepsilon\right\|_{P^\varepsilon}\right) .\end{aligned}$$

From this inequality, the apriori estimate on $\left\|u^\varepsilon\right\|^2_{P^\varepsilon}$ and $\frac{1}{\varepsilon^2}\int_0^T \left\|[\![u^\varepsilon]\!]^\sim\right\|^2_{\Gamma^\varepsilon}$ follows using the Gronwall inequality, provided $\varepsilon$ is small enough. Furthermore, the last inequality yields $u^\varepsilon = 0$ if $f^\varepsilon = 0$ and $u^\varepsilon_0 = 0$. Next, we test (16) with $\partial_t u^\varepsilon$ and use once more (52) and Lemma 4.2 to obtain

$$\begin{aligned}\int_0^T \left\|\partial_t u^\varepsilon\right\|^2_{P^\varepsilon} + \frac{1}{\varepsilon^2}\frac{1}{2}\left\|[\![u^\varepsilon]\!]^\sim\right\|^2_{\Gamma^\varepsilon}\Big|_0^T &\le \int_0^T \left\|f^\varepsilon\right\|^2_{P^\varepsilon}\left\|\partial_t u^\varepsilon\right\|^2_{P^\varepsilon} + C\frac{1}{\varepsilon}\left\|\nabla v\right\|_\infty \int_0^T \left\|[\![u^\varepsilon]\!]^\sim\right\|_{\Gamma^\varepsilon}\left\|\partial_t u^\varepsilon\right\|_{P^\varepsilon} \\ &\quad + C\varepsilon\left\|\nabla v\right\|_\infty \int_0^T \frac{1}{\varepsilon}\left\|[\![u^\varepsilon]\!]^\sim\right\|_{\Gamma^\varepsilon}\left(\frac{1}{\varepsilon}\left\|\partial_t[\![u^\varepsilon]\!]^\sim\right\|_{\Gamma^\varepsilon} + \left\|\partial_t u^\varepsilon\right\|_{P^\varepsilon}\right) .\end{aligned}$$

Hence the the apriori estimate on $\left\|\partial_t u^\varepsilon\right\|^2_{P^\varepsilon}$ follows from the Gronwall inequality. From the apriori estimates, Lemma 3.1 and the Aubin-Lions Theorem, we obtain strong convergence $\mathcal{R}^*_\varepsilon u^\varepsilon \to u$ in $L^2(0,T;L^2(\boldsymbol{Q}))$ for some $u \in L^2(0,T;L^2(\boldsymbol{Q}))$. From Lemma 3.2 we infer that $u \in L^2(0,T;H^1_0(\boldsymbol{Q}))$ and $\frac{1}{\varepsilon}[\![u^\varepsilon]\!]d\mathcal{H}^{n-1} \to \nabla u$ in the sense of distribution. The weak convergence $\partial_t\mathcal{R}^*_\varepsilon u^\varepsilon \to \partial_t u$ in $L^2(0,T;L^2(\boldsymbol{Q}))$ as well as $\partial_t u \in L^2(0,T;L^2(\boldsymbol{Q}))$ is straight forward.

Integrating the right hand side $f^\varepsilon$ and the solutions $u^\varepsilon$ of (16) over time intervals $(s,t) \subset (0,T)$ and applying Theorem 1.6 it follows that $u$ solves (17).

## 5 Two-scale Convergence

We recall the notation (36). Since $C_b(\Omega)$ lies densely in the separable space $L^2(\Omega;\mu_{\Gamma,\mathcal{P}})$, we can chose a countable dense family $\Phi_\Omega = (\phi_i)_{i\in\mathbb{N}} \subset L^2(\Omega;\mu_{\Gamma,\mathcal{P}})$ of $C_b(\Omega)$-functions and a countable dense family of functions $\Phi_{\boldsymbol{Q}} = (\psi_i)_{i\in\mathbb{N}} \subset C_0(\boldsymbol{Q})$ of functions $\psi_i \in C_c(\boldsymbol{Q})$. We furthermore assume that $\Phi_\Omega = \Phi_{\mathrm{pot}} \oplus \Phi_{\mathrm{sol}}$ for dense subsets $\Phi_{\mathrm{sol}} \subset L^2_{\mathrm{sol}}(\Gamma)$ and $\Phi_{\mathrm{pot}} \subset L^2_{\mathrm{pot}}(\Gamma)$ where $\Phi_{\mathrm{pot}}$ is such that $v \in \Phi_{\mathrm{pot}}$ if and only if $v = [\![u]\!]^{\sim}_{\mathrm{Om}}$ for some $u \in C_b(\Omega)$. Finally, let $\Omega_\Phi \subset \Omega$ be the set of all $\omega$ such that the Ergodic Theorems 2.7–2.5 hold for all $v \in \Phi_\Omega$ and $\phi \in \Phi_{\boldsymbol{Q}}$.

**Definition 5.1** (Two-scale convergence)**.** Let $\boldsymbol{Q}$ be a bounded open domain, $\omega \in \Omega_\Phi$ and let $v^\varepsilon \in L^2(\boldsymbol{Q};\mu^\varepsilon_{\Gamma(\omega)})$ be a sequence such that

$$\sup_{\varepsilon>0} \|v^\varepsilon\|_{\Gamma^\varepsilon(\omega)} < \infty$$

and let $v \in L^2(\boldsymbol{Q};L^2(\Omega;\mu_{\Gamma,\mathcal{P}}))$. We say that $v^\varepsilon$ converges in two scales to $v$, written $v^\varepsilon \xrightarrow{2s}_\omega v$ if for every $\phi \in \Phi_\Omega$ and every $\psi \in \Phi_{\boldsymbol{Q}}$ it holds

$$\lim_{\varepsilon\to0} \langle v^\varepsilon\,,\, \phi_{\omega,\varepsilon}\psi a_{\omega,\varepsilon}\rangle_{\Gamma^\varepsilon(\omega)} = \int_{\boldsymbol{Q}} \langle v(x,\cdot),\, \phi a\rangle_{\Gamma,\mathcal{P}}\, \psi(x)\,dx\,.$$

This definition makes sense in view of the following result.

**Lemma 5.2** (Existence of two-scale limits)**.** *For every $\omega \in \Omega_\Phi$ it holds: Let $v^\varepsilon \in L^2(\boldsymbol{Q})$ be a sequence of functions such that $\sup_{\varepsilon>0}\|v^\varepsilon\|_{\Gamma^\varepsilon(\omega),\boldsymbol{Q}} \le C$ for some $C > 0$ independent from $\varepsilon$. Then there exists a subsequence of $(v^{\varepsilon'})_{\varepsilon'\to0}$ and $v \in L^2(\boldsymbol{Q};L^2(\Omega;\mu_{\Gamma,\mathcal{P}}))$ such that $v^{\varepsilon'} \xrightarrow{2s}_\omega v$ and*

$$\|v\|_{L^2(\boldsymbol{Q};L^2(\Omega;\mu_{\Gamma,\mathcal{P}}))} \le \liminf_{\varepsilon'\to0} \left\|v^{\varepsilon'}\right\|_{\Gamma^{\varepsilon'}(\omega),\boldsymbol{Q}}\,. \tag{53}$$

The proof of Lemma 5.2 is standard. However, we provide it here for completeness.

*Proof.* Let $\omega \in \Omega_\Phi$, let $(\phi_k)_{k\in\mathbb{N}}$ be an enumeration of $\Phi_\Omega$ and $(\psi_j)_{j\in\mathbb{N}}$ an enumeration of $\Phi_{\boldsymbol{Q}}$ and for $\varepsilon > 0$ we write $\phi_{k,\omega,\varepsilon}(x) := \phi_k(\tau_{\frac{x}{\varepsilon}}\omega)$. For fixed $j,k \in \mathbb{N}$, we obtain from Theorem 2.6 that

$$\begin{aligned}\limsup_{\varepsilon\to0}\left|\langle v^\varepsilon,\, \psi_j\phi_{k,\omega,\varepsilon}a_{\omega,\varepsilon}\rangle_{\Gamma^\varepsilon(\omega)}\right| &= \limsup_{\varepsilon\to0}\left|\int_{\boldsymbol{Q}} v^\varepsilon(x)\psi_j(x)\phi_{k,\omega,\varepsilon}(x)a(\tau_{\frac{x}{\varepsilon}}\omega)\,d\mu^\varepsilon_{\Gamma(\omega)}(x)\right|\\ &\le C\limsup_{\varepsilon\to0}\left(\int_Q \psi_j(x)^2\left(\phi_k(\tau_{\frac{x}{\varepsilon}}\omega)\right)^2 a(\tau_{\frac{x}{\varepsilon}}\omega)d\mu^\varepsilon_{\Gamma(\omega)}\right)^{\frac12}\\ &= C\,\|\psi_j\|_{L^2(\boldsymbol{Q})}\,\|\phi_k\|_{L^2(\Omega;\mu_{\Gamma,\mathcal{P}})}\,.\end{aligned}$$

Therefore, we can use Cantor's diagonalization argument to construct a subsequence of $v^\varepsilon$, not relabeled in the following, such that

$$\forall j,k \in \mathbb{N} \qquad \langle v^\varepsilon,\, \psi_j\phi_{k,\omega,\varepsilon}a_{\omega,\varepsilon}\rangle_{\Gamma^\varepsilon(\omega)} \to L_{j,k} \quad \text{as } \varepsilon\to0$$

and $L_{j,k}$ is linear in $\psi_j\phi_k \in L^2(\boldsymbol{Q};L^2(\Omega;\mu_{\Gamma,\mathcal{P}}))$. Therefore, there exists $v \in L^2(\boldsymbol{Q};L^2(\Omega;\mu_{\Gamma,\mathcal{P}}))$ such that

$$L_{j,k} = \int_{\boldsymbol{Q}} \langle v(x,\cdot)\,,\, \psi_j(x)\phi_k a\rangle_{\Gamma,\mathcal{P}}\,dx \qquad \forall k \in \mathbb{N}\,.$$

Since the span of the $\psi_j\phi_k$ is dense in $L^2(\boldsymbol{Q};L^2(\Omega;\mu_{\Gamma,\mathcal{P}}))$, the function $u$ is unique. □

The next result provides a kind of generalization of Theorem 2.6. It is needed in order to proof the main result of this section.

**Lemma 5.3.** *For a random tessellation $(G(\omega),\Gamma(\omega))$ that fulfills the compactness property 3.1 in $\mathbb{R}^n$ with $\boldsymbol{Q}\subset\mathbb{R}^n$ bounded Lipschitz domain and fixed $\omega\in\Omega$ let $u^\varepsilon\in\mathcal{S}^\varepsilon(\omega)$ and $u\in H^1(\boldsymbol{Q})$ such that $\mathcal{R}_\varepsilon^* u^\varepsilon\to u$ strongly in $L^2(\boldsymbol{Q})$. Then for every $b\in L^2(\Omega;\mu_{\Gamma,\mathcal{P}})$ such that the Ergodic Theorems 2.7–2.5 are valid for $b$ it holds that for every $\phi\in C_c^1(\boldsymbol{Q})$ and $\psi\in C(\overline{\boldsymbol{Q}})$*

$$\lim_{\varepsilon\to0}\varepsilon^n\sum_{(i,j)\in E^\varepsilon(\omega)}\bar u_{ij}^\varepsilon b_{ij}(\omega)\psi_{ij}\frac{1}{\varepsilon}[\![\phi]\!]_{ij}^{\sim}=\int_{\boldsymbol{Q}}u\psi\nabla\phi\cdot\int_\Omega b\tilde\nu d\mu_{\Gamma,\mathcal{P}}dx\,, \tag{54}$$

*where $\bar u_{ij}^\varepsilon:=\frac12\left(u_i^\varepsilon+u_j^\varepsilon\right)$.*

*Remark* 5.4. Lemma 5.3 is also valid for the space $\mathcal{S}_{\mathrm{per}}^\varepsilon(\omega,\boldsymbol{Q})$ and $H_{\mathrm{per}}^1(\boldsymbol{Q})$ if $\boldsymbol{Q}$ is a cuboid.

*Proof.* The proof follows closely the lines of Step 2a in the proof of Theorem 1.6. However, we provide the full proof for completeness. For $\delta>0$ let $\varphi_\delta$ be a smooth mollifier with support in $B_\delta(0)$ and let $u_\delta^\varepsilon:=(\mathcal{R}_\varepsilon^*u^\varepsilon)*\varphi_\delta$ and $u_\delta^0=u*\varphi_\delta$. Since $(\mathcal{R}_\varepsilon^*u^\varepsilon)\to u$ strongly in $L^2(\boldsymbol{Q})$ we obtain that for every fixed $\delta>0$ the family $\left(u_\delta^\varepsilon\right)_{\varepsilon>0}$ together with $u_\delta^0$ is uniformly equicontinuous and $u_\delta^\varepsilon\to u_\delta^0$ in $C(\overline{\boldsymbol{Q}})$. This follows from the fact that $u_\delta^\varepsilon\in C_c^\infty(2\boldsymbol{Q})$ and

$$\|\nabla u_\delta^\varepsilon\|_\infty\le C\left\|\nabla^{n+1}u_\delta^\varepsilon\right\|_{L^2}\le C\left\|\nabla^{n+1}\varphi_\delta\right\|_{L^1}\|\mathcal{R}_\varepsilon^*u_\delta^\varepsilon\|_{L^2(\boldsymbol{Q})}\,,$$

due to the Sobolev inequality and the convolution inequality.

For shortness of notation, we write $\|\cdot\|_{L^2_\varepsilon}:=\|\cdot\|_{\Gamma^\varepsilon(\omega),\boldsymbol{Q}}$ and define

$$I_1^\varepsilon=\varepsilon^n\sum_{(i,j)\in E^\varepsilon(\omega)}\bar u_{ij}^\varepsilon\psi_{ij}b_{ij}\frac{1}{\varepsilon}[\![\phi]\!]_{ij}^{\sim}\,.$$

For $(i,j)\in E^\varepsilon(\omega)$ we introduce $u_{\delta,ij}=u_\delta(\Gamma_{ij}^\varepsilon(\omega))$ and $\bar u_{\delta,ij,\varepsilon}=\frac12\left(u_\delta^\varepsilon(P_i^\varepsilon(\omega))+u_\delta^\varepsilon(P_j^\varepsilon(\omega))\right)$. Then, we write

$$I_1^\varepsilon=\varepsilon^n\sum_{(i,j)\in E^\varepsilon(\omega)}u_{\delta,ij,\varepsilon}b_{ij}\psi_{ij}\frac{1}{\varepsilon}[\![\phi]\!]_{ij}^{\sim}+I_2^\varepsilon\,, \tag{55}$$

with

$$\begin{aligned}|I_2^\varepsilon|&\le C\|\nabla\phi\|_\infty\|\psi\|_\infty\|u_\delta-\bar u^\varepsilon\|_{L^2_\varepsilon}\|b\|_{L^2_\varepsilon}\\&\le C\|\nabla\phi\|_\infty\|\psi\|_\infty\|b\|_{L^2_\varepsilon}\left(\|u_\delta-u_\delta^\varepsilon\|_{L^2_\varepsilon}+\|u_\delta^\varepsilon-\bar u_\delta^\varepsilon\|_{L^2_\varepsilon}+\|\bar u_\delta^\varepsilon-\bar u^\varepsilon\|_{L^2_\varepsilon}\right)\end{aligned}$$

Since $\left\|u_\delta-u_\delta^\varepsilon\right\|_{C(\overline{\boldsymbol{Q}})}\to0$ as $\varepsilon\to0$, we obtain from the Ergodic Theorem 2.5 that $\left\|u_\delta-u_\delta^\varepsilon\right\|_{L^2_\varepsilon}\to0$ as $\varepsilon\to0$. Furthermore, uniform equicontinuity of $\left(u_\delta^\varepsilon\right)_{\varepsilon>0}$ and the existence of a maximal cell diameter from Condition 1.10 imply that for every $\eta>0$ there exists $\varepsilon_0>0$ such that for all $\varepsilon<\varepsilon_0$ we find $\left\|u_\delta^\varepsilon-\bar u_\delta^\varepsilon\right\|_{L^2_\varepsilon}\le\eta\|1\|_{L^2_\varepsilon}\to\eta|\boldsymbol{Q}|\mu_{\Gamma,\mathcal{P}}(\Omega)$. Furthermore, Condition 1.10 implies that the number of neighbors of a cell is bounded from above by $n^n\alpha^{-n}$. Hence, we remain with

$$\begin{aligned}
\lim_{\varepsilon\to 0}|I_2^\varepsilon| &\le \lim_{\varepsilon\to 0} C\,\|\nabla\phi\|_\infty\,\|\psi\|_\infty\,\|b\|_{L^2_\varepsilon}\,\|\bar u_\delta^\varepsilon-\bar u^\varepsilon\|_{L^2_\varepsilon}\\
&\le \lim_{\varepsilon\to 0} C\,\|\nabla\phi\|_\infty\,\|\psi\|_\infty\,\|b\|_{L^2_\varepsilon}\left(\varepsilon^n\sum_{\Gamma_{ij}^\varepsilon\in\operatorname{supp}\phi}\left[\left(u_{\delta,i}^\varepsilon-u_i^\varepsilon\right)^2+\left(u_{\delta,j}^\varepsilon-u_j^\varepsilon\right)^2\right]\right)^{\frac12}\\
&\le \lim_{\varepsilon\to 0} C\,\|\nabla\phi\|_\infty\,\|\psi\|_\infty\,\|b\|_{L^2_\varepsilon}\left(\varepsilon^n\sum_{i:\Gamma_{ij}^\varepsilon\in\operatorname{supp}\phi}\left(u_{\delta,i}^\varepsilon-u_i^\varepsilon\right)^2\sum_{j\sim i}1\right)^{\frac12}\\
&= C\,\|\nabla\phi\|_\infty\,\|\psi\|_\infty\,\|b\|_{L^2(\Omega;\mu_{\Gamma,\mathcal P})}\left\|u_\delta^0-u\right\|_{L^2(\boldsymbol Q)}\,.
\end{aligned}$$

For the first term on the right hand side of (55) we find by Theorem 2.6 that

$$\begin{aligned}
-\lim_{\varepsilon\to 0}\varepsilon^n\sum_{(i,j)\in E^\varepsilon(\omega)}u_{\delta,ij}\psi_{ij}b_{ij}\frac{1}{\varepsilon}[\![\phi]\!]_{ij}^{\sim} &= -\lim_{\varepsilon\to 0}\varepsilon^n\sum_{(i,j)\in E^\varepsilon(\omega)}u_{\delta,ij}\psi_{ij}b_{ij}\tilde\nu_{ij}\cdot\nabla\phi(\Gamma_{ij}^\varepsilon(\omega))\\
&= -\int_{\boldsymbol Q}u_\delta\psi\nabla\phi\cdot\int_\Omega b\tilde\nu\, d\mu_{\Gamma,\mathcal P}dx\,.
\end{aligned}$$

Hence we obtain

$$\left|\lim_{\varepsilon\to 0}I_1^\varepsilon+\int_Q u_\delta\psi\nabla\phi\cdot\int_\Omega b\tilde\nu\, d\mu_{\Gamma,\mathcal P}dx\right|\le C\,\|\nabla\phi\|_\infty\,\|\psi\|_\infty\,\|b\|_{L^2(\Omega;\mu_{\Gamma,\mathcal P})}\left\|u_\delta^0-u\right\|_{L^2(\boldsymbol Q)}\,,$$

which finally yields (54). □

The following proposition is our main two-scale convergence result and is at the heart of the proof of Theorem 1.13.

**Proposition 5.5.** *For a random tessellation $(G(\omega),\Gamma(\omega))$ that fulfills Condition 1.10 with $\boldsymbol Q\subset\mathbb R^n$ bounded and open cuboid and fixed $\omega\in\Omega_\Phi$ let $u^\varepsilon\in\mathcal S_{\mathrm{per}}^\varepsilon(\omega,\boldsymbol Q)$ with*

$$\frac{1}{\varepsilon^2}\|[\![u^\varepsilon]\!]^\sim\|^2_{L^2(Q;\mu^\varepsilon_{\Gamma_{\mathrm{per}}(\omega)})}\le C$$

*Then there are $u\in H^1_{\mathrm{per}}(\boldsymbol Q)$ and $\mathrm v\in L^2(\boldsymbol Q;L^2_{pot}(\Gamma))$ such that:*

$$\begin{aligned}
&\mathcal R_\varepsilon^* u^\varepsilon\to u \qquad \text{in } L^2(\boldsymbol Q)\\
&[\![u^\varepsilon]\!]^\sim \overset{2s}{\rightharpoonup}_\omega \nabla u\cdot\tilde\nu+\mathrm v
\end{aligned}\tag{56}$$

*Proof.* By Lemma 3.2 there exists $u\in H^1_{\mathrm{per}}(\boldsymbol Q)$ such that $\mathcal R_\varepsilon^* u^\varepsilon\to u$ strongly in $L^2(\boldsymbol Q)$ and $[\![u^\varepsilon]\!]d\mathcal H^{n-1}\to\nabla u$ in the sense of distributions along a subsequence as $\varepsilon\to 0$. From Lemma 5.2 it follows that there exists $\mathrm w\in L^2(\boldsymbol Q;L^2(\Gamma,\mu_{\Gamma,\mathcal P}))$ such that along a further subsequence

$$\frac{1}{\varepsilon}[\![u^\varepsilon]\!]^\sim \;\overset{2s}{\rightharpoonup}_\omega\; \mathrm w\,.$$

Now take $\psi := \phi v$ with $\phi \in C_c^\infty(\boldsymbol{Q})$ and $v \in L^2_{sol}(\Gamma)$. Introducing the notation $b_{ij}(\omega) = v_{ij}(\omega)a_{ij}(\omega)$ we obtain

$$\begin{aligned}\frac{1}{\varepsilon}\left\langle [\![u^\varepsilon]\!]^\sim, \phi v_{\omega,\varepsilon}a_{\omega,\varepsilon}\right\rangle_{\Gamma^\varepsilon(\omega),\boldsymbol{Q}} &= -\frac{1}{\varepsilon}\left\langle u^\varepsilon\,,\ (\mathrm{div}_{\mathcal{P}}\phi v_{\omega,\varepsilon}a_{\omega,\varepsilon})\right\rangle_{\mathcal{P}^\varepsilon(\omega),\boldsymbol{Q}}\\
&= -\varepsilon^{n-1}\sum_{P_i^\varepsilon\in\boldsymbol{Q}} u^\varepsilon(P_i^\varepsilon)\sum_{i\sim j} b_{ij}(\omega)\phi(\Gamma_{ij}^\varepsilon)\\
&= -\varepsilon^{n-1}\sum_{P_i^\varepsilon\in\boldsymbol{Q}} u^\varepsilon(P_i^\varepsilon)\phi(P_i^\varepsilon)\sum_{i\sim j} b_{ij}(\omega)\tilde\nu_{ij}\cdot\nu_{ij}\\
&\quad - \varepsilon^{n-1}\sum_{P_i^\varepsilon\in\boldsymbol{Q}} u^\varepsilon(P_i^\varepsilon)\sum_{i\sim j} b_{ij}(\omega)\tilde\nu_{ij}\cdot\nu_{ij}\left(\phi(\Gamma_{ij}^\varepsilon)-\phi(P_i^\varepsilon)\right)\,.\end{aligned}$$

Since $\mathrm{div}_{\mathcal{P}} b = 0$ by Lemma 3.7, the first term on the right hand side vanishes. We denote the second term as $I_1^\varepsilon$ and obtain

$$I^\varepsilon = -\varepsilon^{n-1}\sum_{P_i^\varepsilon\in\boldsymbol{Q}} u^\varepsilon(P_i^\varepsilon)\sum_{i\sim j} b_{ij}(\omega)\tilde\nu_{ij}\cdot\nu_{ij}\left(\phi(\Gamma_{ij}^\varepsilon)-\phi(P_i^\varepsilon)\right)\,.$$

In what follows, we simplify notations. We write $u_i^\varepsilon := u^\varepsilon(P_i^\varepsilon)$, $\bar u_{ij}^\varepsilon := \frac{1}{2}(u_i^\varepsilon + u_j^\varepsilon)$, $\tilde b_{ij} := b_{ij}\tilde\nu_{ij}$, $\phi_{ij} = \phi(\Gamma_{ij}^\varepsilon)$ and $\phi_i = \phi(P_i^\varepsilon)$ and obtain

$$\begin{aligned}\varepsilon^{1-n}I^\varepsilon &= -\sum_{P_i^\varepsilon\in\boldsymbol{Q}} u_i^\varepsilon\sum_{i\sim j}\tilde b_{ij}\cdot\nu_{ij}\left(\phi(\Gamma_{ij}^\varepsilon)-\phi(P_i^\varepsilon)\right)\\
&= -\sum_{(i,j)\in E^\varepsilon(\omega)}\left[\bar u_{ij}^\varepsilon\tilde b_{ij}\cdot\nu_{ij}\left(\phi_j-\phi_i\right)+\frac{1}{2}\left(u_i^\varepsilon-u_j^\varepsilon\right)\tilde b_{ij}\cdot\nu_{ij}\left(2\phi_{ij}-\phi_i-\phi_j\right)\right]\\
&= -\sum_{(i,j)\in E^\varepsilon(\omega)}\left[\bar u_{ij}^\varepsilon b_{ij}[\![\phi]\!]_{ij}^\sim - \frac{1}{2}[\![u^\varepsilon]\!]_{ij}^\sim b_{ij}\left(2\phi_{ij}-\phi_i-\phi_j\right)\right]\,. \qquad (57)\end{aligned}$$

Due to the uniform size of the Voronoi-cells, we obtain that for every $\delta > 0$

$$\lim_{\varepsilon\to 0}\left|\varepsilon^n\sum_{(i,j)\in E^\varepsilon(\omega)}\frac{1}{2\varepsilon}[\![u^\varepsilon]\!]_{ij}^\sim b_{ij}\left(2\phi_{ij}-\phi_i-\phi_j\right)\right|\le\delta\,.$$

Using the last estimate and (57), Lemma 5.3 yields

$$\lim_{\varepsilon\to 0} I^\varepsilon = -\int_{\boldsymbol{Q}} u\nabla\phi\cdot\int_\Omega b\tilde\nu\, d\mu_{\Gamma,\mathcal{P}}dx\,.$$

Hence, we obtain in the limit:

$$\int_{\boldsymbol{Q}}\int_\Gamma \mathrm{w}\psi\, d\mu_{\Gamma,\mathcal{P}}dx = -\int_{\boldsymbol{Q}} u\nabla\phi\cdot\int_\Omega b\tilde\nu\, d\mu_{\Gamma,\mathcal{P}}dx\,.$$

Since $u \in H^1_{\mathrm{per}}(\boldsymbol{Q})$ we can apply integration by parts to obtain

$$\int_{\boldsymbol{Q}}\int_\Gamma \left(\mathrm{w}\tilde\nu - \nabla u\right)\phi\cdot b\tilde\nu\, a\, d\mu_{\Gamma,\mathcal{P}}\,dx = 0\,.$$

This implies that for almost every $x$ the function $(\mathrm{w} - \nabla u\cdot\tilde\nu)(x,\cdot)$ lies in $L^2_{\mathrm{pot}}(\Gamma)$, i.e. $\mathrm{w} - \nabla u\cdot\tilde\nu \in L^2(\boldsymbol{Q}; L^2_{\mathrm{pot}}(\Gamma))$. □

# 6 Proof of Theorem 1.13

Multiplying (20) by a function $\phi \in \mathcal{S}^{\varepsilon}_{\mathrm{per}}(\omega,\boldsymbol{Q})$, and summing up over $P^{\varepsilon}_{\mathrm{per},i}(\omega,\boldsymbol{Q})$, we arrive at

$$\begin{aligned}-\sum_{P^{\varepsilon}_{i,\mathrm{per}}(\omega,\boldsymbol{Q})}\left(\mathcal{L}^{\varepsilon}_{\omega}u\right)_i\phi_i &= -\sum_{P^{\varepsilon}_{i,\mathrm{per}}(\omega,\boldsymbol{Q})}\sum_{(i,j)\in E^{\varepsilon}_{\mathrm{per}}(\omega)}\frac{1}{\varepsilon^2}a_{ij}(\omega)\left(u_j-u_i\right)\phi_i\\ &= \sum_{(i,j)\in E^{\varepsilon}_{\mathrm{per}}(\omega)}\frac{1}{\varepsilon^2}a_{ij}(\omega)\left(u_j-u_i\right)\left(\phi_j-\phi_i\right)\,.\end{aligned}$$

Hence, the equation (21) is equivalent with the discrete weak formulation

$$\forall\phi\in\mathcal{S}^{\varepsilon}_{\mathrm{per}}(\omega,\boldsymbol{Q})\qquad \sum_{(i,j)\in E^{\varepsilon}_{\mathrm{per}}(\omega)}\frac{1}{\varepsilon^2}a_{ij}(\omega)\left(u^{\varepsilon}_j-u^{\varepsilon}_i\right)\left(\phi_j-\phi_i\right)=\sum_{P^{\varepsilon}_{i,\mathrm{per}}(\omega,\boldsymbol{Q})}f^{\varepsilon}_i\phi_i \tag{58}$$

Let $\omega\in\Omega_{\Phi}$ be fixed. Due to the Poincaré inequality (30) we find that

$$\|u\|^2_{\mathcal{S}^{\varepsilon}_{\mathrm{per},0}}:=\varepsilon^{n-2}\sum_{(i,j)\in E^{\varepsilon}_{\mathrm{per}}(\omega)}[\![u^{\varepsilon}]\!]^2_{ij}$$

is a norm on the subspace $\mathcal{S}^{\varepsilon}_{\mathrm{per},0}$. Since $\gamma$ is bounded from above, this norm has the property that

$$\|u^{\varepsilon}\|^2_{P^{\varepsilon}(\omega),\boldsymbol{Q}}\le\|\mathcal{R}^{*}_{\varepsilon}u^{\varepsilon}\|^2_{H^s_{\mathrm{per}}(Q)}\le C\varepsilon^{n-2}\sum_{(i,j)\in E^{\varepsilon}_{\mathrm{per}}(\omega)}[\![u^{\varepsilon}]\!]^2_{ij}\gamma^{\varepsilon}_{\mathrm{per},ij}(\omega)\le C\,\|u^{\varepsilon}\|^2_{\mathcal{S}^{\varepsilon}_{\mathrm{per},0}}\,. \tag{59}$$

The Lax-Milgram Lemma hence yields a unique solution $u^{\varepsilon}\in\mathcal{S}^{\varepsilon}_{\mathrm{per},0}$ to problem (58). Testing (58) with $\phi=u^{\varepsilon}$ and using (59) and the lower bound on $a$ yields the estimate

$$\|u^{\varepsilon}\|^2_{P^{\varepsilon}(\omega),\boldsymbol{Q}}\le C\,\|u^{\varepsilon}\|^2_{\mathcal{S}^{\varepsilon}_{\mathrm{per},0}}\le C\int_{P^{\varepsilon}_{\mathrm{per}}(\boldsymbol{Q})}f^{\varepsilon}u^{\varepsilon}\,d\mu^{\varepsilon}_{P}$$

and hence

$$\|u^{\varepsilon}\|^2_{P^{\varepsilon}(\omega),\boldsymbol{Q}}+\varepsilon^{n-2}\sum_{(i,j)\in E^{\varepsilon}_{\mathrm{per}}(\omega)}[\![u^{\varepsilon}]\!]^2_{ij}\le C\,\|f^{\varepsilon}\|^2_{\mathcal{S}^{\varepsilon}_{\mathrm{per}}(\boldsymbol{Q})}\,.$$

By Proposition 5.5 there exists a subsequence, not relabeled, and $u\in H^1_{\mathrm{per}}(\boldsymbol{Q})$, $\mathrm{v}\in L^2(\boldsymbol{Q};L^2_{\mathrm{pot}}(\Omega))$ such that

$$u^{\varepsilon}\to u\text{ strongly in }L^2(\boldsymbol{Q})\quad\text{and}\quad\frac{1}{\varepsilon}[\![u^{\varepsilon}]\!]^{\sim}\xrightarrow{2s}_{\omega}\nabla u\cdot\tilde{\nu}+\mathrm{v}\,.$$

We choose $\varphi\in\Phi_{\boldsymbol{Q}}$ and $\mathrm{w}\in\Phi_{\mathrm{pot}}$ with $\psi_{\mathrm{w}}\in C_b(\Omega)$ such that $\mathrm{w}=[\![\psi_{\mathrm{w}}]\!]^{\sim}_{\mathrm{Om}}$ and define $\phi_{\varepsilon,\omega}(x):=\varepsilon\varphi(x)\psi_{\mathrm{w}}(\tau_{\frac{x}{\varepsilon}}\omega)$. We use $\phi_{\varepsilon,\omega}$ as a test-function in (58) recall that $\varphi\in C_c(\boldsymbol{Q})$ and obtain for $\varepsilon$ small enough that

$$\sum_{(i,j)\in E^{\varepsilon}_{\mathrm{per}}(\omega)}a_{ij}\frac{1}{\varepsilon}[\![u^{\varepsilon}]\!]^{\sim}_{ij}\left(\varepsilon\,\frac{1}{\varepsilon}[\![\varphi]\!]^{\sim}_{ij}\psi_{\mathrm{w}}(\tau_{P_j}\omega)+\varphi(P^{\varepsilon}_i)\mathrm{w}(\tau_{\Gamma_{ij}}\omega)\right)=\varepsilon\sum_{P^{\varepsilon}_{i,\mathrm{per}}(\omega,\boldsymbol{Q})}f^{\varepsilon}_i\varphi(P^{\varepsilon}_i)\psi_{\mathrm{w}}(\tau_{P_i}\omega)\,.$$

As $\varepsilon\to0$, we find that $\varepsilon^{-1}[\![\varphi]\!]^{\sim}_{ij}$ is uniformly bounded by $\|\nabla\varphi\|_{\infty}$. Hence, the first term on the left hand side vanishes as $\varepsilon\to0$ and using two-scale convergence of $\frac{1}{\varepsilon}[\![u^{\varepsilon}]\!]^{\sim}$, we obtain the following limit equation:

$$\forall\varphi\in\Phi_{\boldsymbol{Q}},\,\mathrm{w}\in\Phi_{\mathrm{pot}}:\qquad\int_{\boldsymbol{Q}}\left\langle\nabla u(x)\cdot\tilde{\nu}+\mathrm{v}(x,\cdot)\,,\;a\,\mathrm{w}\varphi(x)\right\rangle_{\Gamma,\mathcal{P}}dx=0\,. \tag{60}$$

Given $u \in H^1(\boldsymbol{Q})$, equation (60) admits the solution

$$\mathrm{v} = \sum_{i=1}^{n} \partial_i u \chi_i \,, \tag{61}$$

where $\chi_i$ are the same as in (39). Since $\Phi_{\boldsymbol{Q}}$ is dense in $L^2(\boldsymbol{Q})$ and $\Phi_{\mathrm{pot}}$ is dense in $L^2_{\mathrm{pot}}(\Gamma)$, equation (60) also has to hold for all $\varphi \in L^2(\boldsymbol{Q})$ and $\mathrm{w} \in L^2_{\mathrm{pot}}(\boldsymbol{Q})$. The Lax-Milgram Lemma then yields that the solution v is unique for given $u \in H^1(\boldsymbol{Q})$.

Next, we use a test-function $\phi \in \Phi_{\boldsymbol{Q}}$ in (58) and obtain the limit equation

$$\forall \phi \in \Phi_{\boldsymbol{Q}} : \qquad \int_{\boldsymbol{Q}} \langle \nabla u(x) \cdot \tilde{\nu} + \mathrm{v}(x,\cdot) \,,\; a\nabla\phi(x) \cdot \tilde{\nu} \rangle_{\Gamma,\mathcal{P}} \, dx = \mu_{P,\mathcal{P}}(\Omega) \int_{\boldsymbol{Q}} f\phi \,. \tag{62}$$

We can use $\partial_i \phi \chi_i$ as a testfunction in (60) and add the resulting equation to (62). Using (61) and (39), this yields

$$\int_{\boldsymbol{Q}} \int_{\Omega} \nabla u \cdot A_{\mathrm{hom}} \nabla \phi d\mu_{\Gamma,\mathcal{P}} dx = \mu_{P,\mathcal{P}}(\Omega) \int_{\boldsymbol{Q}} f\phi \,,$$

and hence $u \in H^2(\boldsymbol{Q})$ and $u$ is a strong solution of (22).

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