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Large deviations of specific empirical fluxes of independent Markov chains, with implications for Macroscopic Fluctuation Theory

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Abstract

We consider a system of independent particles on a finite state space, and prove a dynamic large-deviation principle for the empirical measure-empirical flux pair, taking the specific fluxes rather than net fluxes into account. We prove the large deviations under deterministic initial conditions, and under random initial conditions satisfying a large-deviation principle. We then show how to use this result to generalise a number of principles from Macroscopic Fluctuation Theory to the finite-space setting.

1 Introduction

A well-known strengthening of the Second Law states that for thermodynamically closed systems not only is the free energy non-increasing in time, but the system is *driven* by its free energy. Exactly how the free energy drives the system can often be made precise by linear response theory, that is, a constitutive relation between thermodynamic driving forces, defined as the derivative of the free energy, and the thermodynamic velocities, that is, $\dot\rho(t) = -K\big(\rho(t)\big)D\mathcal{F}\big(\rho(t)\big)$. If the state-dependent operator $K(\rho)$ is symmetric and positive definite, then such equations are actually gradient flows on the manifold with inverse metric tensor $K(\rho)$, and the Second Law follows as a consequence. Moreover, one then also has the free energy-dissipation balance, that is, the dissipation of free energy equals the free energy production. Since the work of Onsager it is well-known that the symmetry of the operator $K(\rho)$ is closely connected to detailed balance of an underlying microscopic system [Ons31] via large-deviation principles. In a general setting, one can always derive free energy-dissipation balances from microscopic systems satisfying detailed balance, although this may lead to non-linear response theories, and non-quadratic dissipations, in particular on discrete spaces [MPR14].

For microscopic systems that are not in detailed balance with macroscopic systems that are not thermodynamically closed, this is no longer true. A detailed balance condition can be violated either due to bulk forces, or due to boundary effects. Typical examples of the latter are the boundary-driven systems studied in for example [BDSG$^+$02, BDSG$^+$03, DLE03]. We will focus our attention on bulk forces. The typical example for such systems have a uniform stationary state and an external non-conservative force field that causes mass to flow around in cycles, leaving the stationary state intact. Hence the work done by the external force field may have no effect on the state, and an energy-dissipation balance as described above is not to be expected. Naturally, the work done is not simply lost, but the resulting effect cannot be observed unless both state and the fluxes are taken into account. This idea is the core of Macroscopic Fluctuation Theory (MFT), which still allows to derive many thermodynamic properties for systems that are not in detailed balance, see the recent overview paper [BDSG$^+$15] and the references therein. The aim of the current paper is to provide a few steps towards a more general Macroscopic Fluctuation Theory, in two ways.

Firstly, most known results in MFT are stated for systems on a continuous space (at least macroscopically). Therefore the stochastic noise is usually driven by some white noise (at least approximately), leading to large deviations that can be written as some squared Hilbert norm. This Hilbert structure allows one to split all terms into orthogonal components: gradient parts and solenoidal parts. However,

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large deviations for jump processes on discrete spaces are often not quadratic. It is an open question how to generalise orthogonal decompositions to structures with more general convex functionals. Nevertheless, as a first step into generalising MFT, we will focus our attention to a simple system of jump processes on a discrete space. In order not to blur the main ideas we study a system of independent particles (more general jump processes on more general discrete spaces will be studied in the subsequent paper [PR17]). We then prove a dynamic large-deviation principle for the state-flux pair, and use these large deviations to generalise known principles from MFT to the discrete setting. Some of these generalisations can also be found in the more formal, physics-oriented work [BMN09].

For the second generalisation, it should be noted that most known MFT studies net fluxes only. In the current paper we take the *specific* fluxes into account. This has a few advantages. Firstly, there could be forces that produce strong fluxes without changing the net flux; such forces speed up the mixing behaviour of the system and can therefore be important quantities to study. Secondly, it turns out that on discrete spaces the large deviation rate for the specific fluxes takes on a nice explicit form, whereas the large deviations for the net fluxes is defined either via a convex dual, or via a minimisation problem. Indeed, this minimisation comes from a simple contraction principle, which can always be applied whenever only the net fluxes are of interest.

As a by-product, the large deviations of the empirical measure follows immediately from the large deviations of the empirical fluxes by a contraction principle, and this might even be easier to prove via a contraction than to prove directly. In particular, a number of assumptions like weak detailed balance [PR16] and log-bounded jump rates [SW95, Ch. 5] are no longer needed.

1.1 Microscopic model

We consider a Markov chain with generator $Q \in \mathbb{R}^{I\times I}$ on a finite state space I. Let $X_1(t), X_2(t), \ldots$ be independent copies of the Markov chain with generator Q. We first assume the following initial conditions:

Fix a $0 < \mu \in \mathcal{P}(I)$, and fix the initial positions $X_1(0) = x_1, X_2(0) = x_2, \ldots$ deterministically such that $\rho^{(n)}(0) := \frac{1}{n}\sum_{k=1}^n \mathbb{1}_{x_k} \xrightarrow[n\to\infty]{} \mu$. (1)

Later we will consider random initial conditions as well. Throughout this paper, all processes will be of bounded variation, where we implicitly assume càdlàg representatives. Therefore the (random) set $\mathrm{jump}_t(X_k) := \{\hat t \in (0,t); X_k(\hat t^-) \neq X_k(\hat t)\}$ of any Markov chain X_k will be at most countable (actually finite a.s.). Define the empirical measure and the empirical (specific) flux by

$$\rho^{(n)}(t) := \frac{1}{n}\sum_{k=1}^n \mathbb{1}_{X_k(t)}, \qquad \text{and} \qquad \mathsf{W}^{(n)}(t) := \frac{1}{n}\sum_{k=1}^n \sum_{\hat t\in \mathrm{jump}_t(X_k)} \mathbb{1}_{\left(X_k(\hat t^-),X_k(\hat t)\right)}.$$

Then the pair $\left(\rho^{(n)}(t), \mathsf{W}^{(n)}(t)\right)$ is a Markov process in $\mathcal{P}(I)\times \mathfrak{l}^1_+(I\times I)$ with generator

$$(\mathcal{Q}^{(n)}\phi)(\rho,\mathsf{w}) := n\sum\sum_{i\neq j} \rho_i Q_{ij}\left[\phi(\rho^{i\to j},\mathsf{w}^{ij+}) - \phi(\rho,\mathsf{w})\right],$$

and deterministic initial condition $\left(\rho^{(n)}(0),\mathsf{W}^{(n)}(0)\right) = \left(n^{-1}\sum_{k=1}^n \mathbb{1}_{x_k}, 0\right)$, where $\rho^{i\to j} := \rho - \frac{1}{n}\mathbb{1}_i + \frac{1}{n}\mathbb{1}_j$ and $\mathsf{w}^{ij+} := \mathsf{w} + \frac{1}{n}\mathbb{1}_{ij}$. This notation will only be used for fixed n, and so we can repress the dependence on that parameter. Here, $\mathfrak{l}^1_+(I\times I)$ is the space of non-negative matrices $\mathbb{R}^{I\times I}_+$ equipped with the norm $|\mathsf{w}|_1 = \sum\sum_{i\neq j}\mathsf{w}_{ij}$, which is conveniently compatible with the natural norm on $\mathcal{P}(I)\subset\mathbb{R}^I_+$. Note that by a slight abuse of notation we ignore diagonal elements in this space.

1.2 Macroscopic model and large deviations

In the many-particle limit, the process converges to the solution of

$$\dot\rho(t) = -\operatorname{div} \dot{\mathsf{w}}(t), \tag{2a}$$

$$\dot{\mathsf{w}}(t) = \rho(t) \otimes Q, \tag{2b}$$

with initial condition $\big(\rho(0), \mathsf{w}(0)\big) = \big(\mu, 0\big)$, using the notation $(\rho \otimes Q)_{ij} := \rho_i Q_{ij}$ and $(\operatorname{div} \mathsf{w})_i := \sum_{j\in I} \mathsf{w}_{ij} - \mathsf{w}_{ji}$. We will make this convergence result rigorous in Proposition 3.3. Observe that the net flux through an edge (i,j) is just the anti-symetric matrix $\mathsf{w}_{ij} - \mathsf{w}_{ji}$, so that the specific fluxes really encode more information. Equation (2a) is the *continuity equation*, or conservation law, and it plays an important role in MFT. Naturally, the microscopic equation also satisfies the continuity equation, hence microscopic fluctuations can only occur around (2b).

In Theorems 4.1 and 4.2 we prove large-deviation principles that characterise the dynamic microscopic fluctuations around the macroscopic limit:

$$\operatorname{Prob}\Big(\big(\rho^{(n)}(\cdot), \mathsf{W}^{(n)}(\cdot)\big) \approx (\rho, \mathsf{w})\Big) \sim e^{-n\mathcal{I}_0(\rho(0)) - n\mathcal{I}_{(0,T)}(\rho,\mathsf{w})} \qquad \text{as } n \to \infty,$$

$$\mathcal{I}_{(0,T)}(\rho,\mathsf{w}) := \begin{cases} \int_0^T \mathcal{S}\big(\dot{\mathsf{w}}(t) | \rho(t) \otimes Q\big)\,dt, & \text{if } (\rho,\mathsf{w}) \in W^{1,1}\big(0,T; \mathcal{P}(I) \times \ell^1(I\times I)\big) \\ & \text{and } \dot\rho + \operatorname{div}\dot{\mathsf{w}} = 0, \\ \infty, & \text{otherwise.} \end{cases} \tag{3}$$

Here $\mathcal{S}(\hat Q | Q) := \sum\sum_{i\neq j} \lambda_B(\hat Q_{ij}/Q_{ij}) Q_{ij}$ is the usual relative entropy and $\lambda_B(z) := z\log z - z + 1$ for $z > 0$ and $\lambda_B(0) := 0$ is the Boltzmann function. We first prove the large deviations under initial condition (1.1), in which case

$$\mathcal{I}_0\big(\rho(0)\big) = \begin{cases} 0, & \rho(0) = \mu, \\ \infty, & \text{otherwise.} \end{cases} \tag{4}$$

After that we will prove the large-deviation principle for more general initial conditions, where $\rho^{(n)}(0)$ is assumed to be random, and satisfying a large-deviation principle with some given rate functional $\mathcal{I}_0$.

1.3 Overview

In Section 2 we introduce the space, the topology and the sigma algebra that will be used for the large deviations. Section 3 we discuss well-posedness for the microscopic and macroscopic models, and show that the microscopic process converges to the deterministic macroscopic equation. In Section 4 we rigorously prove the large-deviation principle (3), first under deterministic initial conditions and then under random initial conditions. As common in dynamical large deviation theory, the difficulty lies in showing that the rate functional can be approximated by a set of sufficiently regular paths in order to prove the lower bound. It is also common to either regularise the perturbation factor [PR16], or replace the probabilities by exponentially equivalent ones [SW95, Ch. 5]. However, in the current work we directly regularise the paths (ρ, w), and exploit the explicit formulation (3) of the rate functional. Finally, in Section 5 we discuss some implications of the large-deviation principles for MFT on discrete spaces with specific fluxes.

2 Preliminaries

2.1 Paths of bounded variation

Throughout this paper we consider paths of bounded variation on an arbitrary time interval $(0,T)$. For any Banach space valued function $f\in L^1(0,T;X^*)$, the essential pointwise variation is

$$\operatorname{epvar}(f):=\inf_{g=f\text{ a.e.}}\sup_{0<t_1<\ldots<t_K<T}\sum_{k=1}^K\|g(t_k)-g(t_{k-1})\|_{X^*},$$

where the supremum runs over all finite partitions of the interval $(0,T)$. The space of paths of bounded variation is defined as

$$\mathrm{BV}(0,T;X^*):=\Big\{f\in L^1(0,T;X^*):\operatorname{epvar}(f)<\infty\Big\}.$$

A function of bounded variation f always has a weak derivative $\dot f\in\mathcal{M}(0,T;X^*)$ which is a X^*-valued bounded measure on $(0,T)$ with total variation norm $\|\dot f\|_{\mathrm{TV}}=\operatorname{epvar}(f)$ [AFP06, Prop. 3.6 & Th. 3.27]. In addition, if a function of bounded variation is absolutely continuous, then f lies in $W^{1,1}(0,T;X^*)$ [AFP06, p. 139]. Moreover, any path of bounded variation $f\in\mathrm{BV}(0,T;X^*)$ has a càdlàgrepresentative, that satisfies $f(t)=f(0)+\dot f\big((0,t]\big)$ [AFP06, Th. 3.28].

We will be particarly concerned with the set

$$\mathrm{BV}_{\text{flux}}:=\mathrm{BV}_{\text{flux}}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big):=\Big\{(\rho,\mathsf{w})\in\mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big):\dot\rho=-\operatorname{div}\dot{\mathsf{w}}\Big\}.$$

Observe that the continuity equation in this definition is in general a measure-valued equation in time, i.e. $\dot\rho(dt)=-\operatorname{div}\dot{\mathsf{w}}(dt)$. In this sense both the microscopic model and the macroscopic model satisfies the continuity equation.

Throughout the paper we will need the following three simple estimates for any $(\rho,\mathsf{w})\in\mathrm{BV}_{\text{flux}}$:

$$\operatorname{epvar}(\rho)=\|\dot\rho\|_{TV}=\|\operatorname{div}\dot{\mathsf{w}}\|_{TV}\le 2\,\|\dot{\mathsf{w}}\|_{TV}=2\operatorname{epvar}(\mathsf{w}),\tag{5}$$

$$\|\rho\|_{L^1(0,T;\mathcal{P}(I))}=\int_0^T|\rho(t)|_1\,dt=T,\tag{6}$$

$$\begin{aligned}\|\mathsf{w}\|_{L^1(0,T;\mathfrak{l}^1(I\times I))}&=\int_0^T|\mathsf{w}(t)|_1\,dt=\int_0^T|\dot{\mathsf{w}}((0,t])|_1\,dt\\&\le\int_0^T|\dot{\mathsf{w}}|_1((0,T])\,dt=T\,\|\dot{\mathsf{w}}\|_{TV}=T\operatorname{epvar}(\mathsf{w}).\end{aligned}\tag{7}$$

Remark 2.1. Inequality (5) actually becomes an equality if w does not have simultaneous jumps. For microscopic paths $(\rho^{(n)},\mathsf{w}^{(n)})$ with finite n this is indeed almost surely the case. □

2.2 The hybrid topology and Borel sigma-algebra

In this paper we always equip the space of paths of bounded variation with the *hybrid topology*. This topology is defined via the convergent nets as follows. We say that a sequence (or net) $(\rho^n,\mathsf{w}^n)_n$ in $\mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ *converges hybridly* to a path of bounded variation (ρ,w) whenever

$(\rho^n,\mathsf{w}^n)\to(\rho,\mathsf{w})$ strongly in $L^1\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ and $(\dot\rho^n,\dot{\mathsf{w}}^n)\to(\dot\rho,\dot{\mathsf{w}})$ vaguely, that is, against all test functions in $C_0\big(0,T;\mathbb{R}^{I\times I\times I}\big)$. It should be noted that this topology is mostly known in the literature as the weak-* topology [AFP06, Def. 3.11]. The term hybrid topology was recently introduced in [HPR16] to distinguish it from the functional analytical weak-* topology; strictly speaking, the two topologies only coincide in finite dimensions and restricted to bounded variation balls.

The space $\mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ equipped with the hybrid topology is not a Polish space - it is not metrisable. This could make it difficult to do probability theory on this space. It is, however, completely regular and perfectly normal [HPR16, Th. 3.15]. Therefore, the corresponding Borel sigma-algebra 'behaves nicely', and the standard tools from probability theory that we will need are still valid [HPR16, Sec. 4].

We choose to work with the space of paths of bounded variation rather than the usual Skorohod space since the compactness criteria are very easy. This simplifies the proof of the exponential tightness, Proposition 4.6, considerably. In particular, because of the estimates (5),(6) and (7) above,

Proposition 2.2 ([AFP06, Th. 3.23] and [HPR16, Th. 3.18]). *Let $\mathcal{F}\subset\mathrm{BV}_{\mathrm{flux}}$ be a subset of finite variation: $\sup_{(\rho,\mathsf{w})\in\mathcal{F}}\mathrm{epvar}(\mathsf{w})<\infty$. Then $\mathcal{F}$ is hybrid-compact.*

A similar argument is used in [PR16] to prove large deviations for infinite chemical reaction networks, and related approaches can be found in [Jak97] and [BFG15].

3 Well-posedness and many-particle limit

We now discuss the well-posedness of both the microscopic and macroscopic model, and provide a short proof of the convergence from the micro to the macro model. The proofs of the results in this section are fairly standard, but we include them for completeness. For the proof of the large deviations in Section 4, we will in fact need these results for a slightly more general process, perturbed by an abitrary $u\in L^\infty\big(0,T;\mathfrak{l}^\infty(I\times I)\big)$, with generator:

$$(\mathcal{Q}^{(n)}_{u(t)}\phi)(\rho,\mathsf{w}):=n\sum\sum_{i\neq j}\rho_iQ_{ij}u_{ij}(t)\big[\phi(\rho^{i\to j},\mathsf{w}^{ij+})-\phi(\rho,\mathsf{w})\big]. \tag{8}$$

3.1 Well-posedness

Since the jump rates $n\rho_iQ_{ij}u_{ij}(t)$ are uniformly bounded in t and the hybrid sigma-algebra is uniquely characterised by the finite-dimensional distributions [HPR16, Th. 4.5], the following result becomes trivial:

Proposition 3.1 (Existence for the microscopic system). *Let $n\in\mathbb{N}$, $\rho^{(n)}(0)\in\mathcal{P}(I)\cap\frac{1}{n}\mathbb{N}^I$, and $\mathsf{W}^{(n)}(0)=0$, and let the process $(\rho^{(n)},\mathsf{W}^{(n)})$ be defined by the perturbed generator $\mathcal{Q}^{(n)}_{u(t)}$. Then there exists a unique probability measure for $(\rho^{(n)},\mathsf{W}^{(n)})$ on the space $\mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$, i.e. for any hybridly-measurable set $\mathcal{U}\subset\mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$,*

$$\mathbb{P}^{(n)}_u(\mathcal{U}):=\mathrm{Prob}\left((\rho^{(n)},\mathsf{W}^{(n)})\in\mathcal{U}\right) \tag{9}$$

and similarly we define the measure $\mathbb{P}^{(n)}$ whenever $u\equiv 0$. Moreover, the process $(\rho^{(n)},\mathsf{W}^{(n)})$ lies almost surely in $\mathrm{BV}_{\mathrm{flux}}$.

The well-posedness of the macroscopic system is a bit more involved:

Proposition 3.2 (Well-posedness of the macroscopic system). *(i) Let $u \in L^\infty\big(0,T;\mathfrak{l}^\infty(I\times I)\big)$, and let $(\mu,v) \in \mathcal{P}(I)\times \mathfrak{l}^1(I\times I)$ be given. Then there exists a unique solution $(\rho,\mathsf{w}) \in W^{1,1}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ to the perturbed system of equations*

$$\begin{cases} \dot\rho(t) = -\operatorname{div}\dot{\mathsf{w}}(t), & t\in(0,T)\\ \dot{\mathsf{w}}(t) = \rho(t)\otimes Q\otimes u(t), & t\in(0,T)\\ \lim_{t\searrow 0}\rho(t) = \mu, \\ \lim_{t\searrow 0}\mathsf{w}(t) = v. \end{cases} \tag{10}$$

Moreover, the solution (ρ,w) also lies in $W^{1,\infty}\big(0,T;\mathfrak{l}^1(I\times I)\big)$.

(ii) Let $u \in L^\infty_{\geq 0}\big(0,T;\mathfrak{l}^\infty(I\times I)\big)$. For any $t\in(0,T)$ the solution operator $\psi(t) : \big(\mu,v\big) \mapsto \big(\rho(t),\mathsf{w}(t)\big)$ for the initial value problem (10) *is a linear and bounded operator mapping $\mathfrak{l}^1(I\times I\times I)$ into itself.*

Proof. (i) The proof is by a standard Picard-Lindelöf fixed point argument.

We first prove existence and uniqueness of solutions for the slowed-down system

$$\begin{cases} \dot{\mathsf{w}}(t) = \alpha\big(\mu - \operatorname{div}\mathsf{w}(t)\big)\otimes Q\otimes u(t), & t\in(0,T),\\ \lim_{t\searrow 0}\mathsf{w}(t) = v, \end{cases} \tag{11}$$

where $0<\alpha\leq 1$ will be determined later. Observe that for $\alpha=1$ the problem (11) coincides with (10) once we retrieve the variable ρ through the continuity equation (2a). Denote

$$W^{1,1}_v := \Big\{\mathsf{w}\in W^{1,1}\big(0,T;\mathfrak{l}^1(I\times I)\big) : \lim_{t\searrow 0}\mathsf{w}(t) = v\Big\},$$

and define the operator $A : W^{1,1}_v \to W^{1,1}_v$ by

$$A[\mathsf{w}](t) := v + \alpha\int_0^t \big(\mu - \operatorname{div}\mathsf{w}(s)\big)\otimes Q\otimes u(s)\,ds.$$

Naturally, solutions of (11) are fixed points of A. We can estimate for any $\mathsf{w},\hat{\mathsf{w}}\in W^{1,1}_v$:

$$\|A[\mathsf{w}] - A[\hat{\mathsf{w}}]\|_{\mathrm{BV}} \leq \alpha(T+1)\|\mathsf{w}-\hat{\mathsf{w}}\|_{L^1}|Q|_1\|u\|_{L^\infty}.$$

Hence for α small enough, the Banach Fixed Point Theorem gives the existence and uniqueness of a solution to the slowed-down system (11). By rescaling time we find a unique solution of the original system (10) up to time αT. Repeating this process a finite number of times gives existence and uniqueness in $W^{1,1}$ of the solution up to time T.

For the regularity in $W^{1,\infty}$, observe that $\dot{\mathsf{w}}(t) = \rho(t)\otimes Q\otimes u(t)$ is uniformly bounded, and hence so are $\dot\rho(t) = -\operatorname{div}\dot{\mathsf{w}}(t)$ and $\rho(t)$ and $\mathsf{w}(t)$.

(ii) The linearity of ψ is immediate from the linearity of the system (10). The boundedness in the first variable is also trivial since $|\rho(t)|_1 = |\mu|_1$. For the boundedness in w, note that $u\geq 0$ implies that $\mathsf{w}(t)$ is non-negative and non-decreasing. Therefore $\frac{d}{dt}|\mathsf{w}(t)|_1 = |\dot{\mathsf{w}}(t)|_1 \leq |\mu|_1|Q|_1\|u\|_\infty$, which proves that $|\mathsf{w}(t)|_1 \leq t|\mu|_1|Q|_1\|u\|_\infty$. □

3.2 Many-particle limit

We now state that the microscopic system converges to the macroscopic system in the many-particle limit. Of course, this limit is an immediate consequence of the law of the large numbers, see for example [Dud89, Th. 11.4.1]. However, the proof below has a bit more structure that allows for generalisations.

Theorem 3.3 (Many-particle limit). *Fix (deterministic) $(\rho^{(n)}(0))_n$ and μ in $\mathcal{P}(I)$ such that $\rho^{(n)}(0) \to \mu$, and $\mathsf{W}^{(n)}(0) \equiv 0$. Let $0 \leq u \in L^\infty\big(0,T;\mathfrak{l}^\infty(I\times I)\big)$ be non-negative and bounded, and let $\big(\rho^{(n)},\mathsf{w}^{(n)}\big)$ be the perturbed process with generator* (8) *starting from $\big(\rho^{(n)}(0),0\big)$. Then $\big(\rho^{(n)},\mathsf{W}^{(n)}\big)$ converges in probability in $\mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ with the hybrid topology to the (deterministic) solution (ρ,w) of the perturbed problem* (10).

Proof. We first prove convergence of the finite-dimensional distributions by operator convergence. Let $Q_{u(t)}: C_b^1\big(\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big) \to C_b\big(\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ be the deterministic generator corresponding to the system (10), i.e.:

$$(\mathcal{Q}_{u(t)}\phi)(\rho,\mathsf{w}) = \sum\sum_{i\neq j} \rho_i Q_{ij} u_{ij}(t)\, \nabla\phi(\rho,\mathsf{w})\cdot(\mathbb{1}_j-\mathbb{1}_i,\mathbb{1}_{ij}). \tag{12}$$

Naturally, the corresponding semigroup is simply $\big(S(t)\phi\big)(\rho,\mathsf{w}) = \phi\big(\psi(t)(\rho,\mathsf{w})\big)$, where $\psi(t)$ is the solution operator. Because of Proposition 3.2(ii), the semigroup $S(t)$ maps $C_b^1\big(\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ into itself, and hence $C_b^1\big(\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ is a core for the limit generator $Q_{u(t)}$ [EK05, Ch. 1, Prop. 3.3]. Moreover, for any test function ϕ in this core we clearly have $\|Q^{(n)}_{u(t)}\phi - Q_{u(t)}\phi\|_{C_b(\mathcal{P}(I)\times\mathfrak{l}^\infty(I\times I))} \to 0$. Therefore, due to the Trotter-Kurtz Theorem [Kal97, Th. 17.25] all finite-dimensional distributions convergence.

By the tightness of the process $(\rho^{(n)},\mathsf{W}^{(n)})$ in $\mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$, which will be proven in Proposition 4.6, we also have narrow convergence of the paths in $\mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$, see [HPR16, Prop. 4.8]. Finally, narrow convergence to a deterministic limit implies convergence in probability. $\square$

4 Large deviations

In this section we rigorously prove the dynamic large-deviation principle. First we prove the dynamic large deviations under deterministic initial conditions:

Theorem 4.1 (Large-deviation principle I). *Fix a $\mu\in\mathcal{P}(I), \mu>0$ (coordinate-wise) satisfying initial condition* (1.1). *Then the pair $(\rho^{(n)},\mathsf{W}^{(n)})$ satisfies a large-deviation principle in $\mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ with the hybrid topology, with good rate functional $\mathcal{I}_{(0,T)}$ from* (3)*, where we implicitly set $\mathcal{I}_{(0,T)}(\rho,\mathsf{w})=\infty$ whenever $\lim_{t\searrow 0}\big(\rho(t),\mathsf{w}(t)\big) = \big(\mu,0\big)$ is violated.*

Next we derive the coupled large-deviation principle if the initial conditions satisfy a large-deviation principle themselves:

Theorem 4.2 (Large-deviation principle II). *Let the random variables $\rho^{(n)}(0)$ satisfy a large-deviation principle in $\mathcal{P}(I)$ with rate functional $\mathcal{I}_0$, and assume that $\mathcal{I}_0$ is left-continuous at the lower boundary $\{\rho\in\mathcal{P}(I):\rho_i=0 \text{ for some } i\in I\}$. Then the pair $(\rho^{(n)},\mathsf{W}^{(n)})$ satisfies a large-deviation principle in $\mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ with the hybrid topology, with good rate functional $(\rho,\mathsf{w})\mapsto\mathcal{I}_0\big(\rho(0)\big)+\mathcal{I}_{(0,T)}(\rho,\mathsf{W})$, where we implicitly set $\mathcal{I}_{(0,T)}(\rho,\mathsf{w})=\infty$ whenever $\lim_{t\searrow0}\mathsf{w}=0$ is violated.*

We largely follow the ideas from [SW95, Ch. 5] and [PR16]. The assumption that $\mu > 0$ is believed to be technical, and has to do with the construction of the approximation sequence in the lower bound. A similar assumption is used in for example [BDSG$^+$07, Th. 2.1], although the approximation argument is very different. In our Theorem 4.2 above, this assumption is no longer needed.

In Subsection 4.2 we prove that the sequence is exponentially tight, in Subsection 4.3 we prove the large-deviation lower bound, and in Subsection 4.4 we prove the large deviation upper bound on compact sets. Together, these results imply the large-deviation principle Theorem 4.1, with a good rate functional [DZ87, Lem. 1.2.18]. In order to prove both bounds, we first need some results about the rate functional, which are proven in Subsection 4.1 below. In Subsection 4.5 we prove large-deviation Theorem 4.2 for the coupled system with random initial conditions.

4.1 Analysis of the rate functional

As briefly mentioned above, we follow the ideas of [PR16], which studies the large deviations of the empirical measure for a very general class of reactions. That general setting applies partially to the setting of the current paper. To apply the general results to the current setting we need to consider each jump as a chemical reaction in the space $I \times I \times I$, where each reaction, parametrised by $i, j \in I \times I$, causes a state change of $\gamma^{(ij)} := (\mathbb{1}_j - \mathbb{1}_i, \mathbb{1}_{ij})$, and occurs with intensity $\rho_i Q_{ij}$. Some conditions from the paper [PR16] are violated, but the ones that we will need in this section are easily verified. We can then use the following result:

Proposition 4.3. *Let $(\rho, \mathsf{w}) \in \mathrm{BV}_{\mathrm{flux}}$ such that $\mathcal{I}_{(0,T)}(\rho, \mathsf{w}) < \infty$. Then (ρ, w) is absolutely continuous, that is, we can identify the path (ρ, w) with a function in $W^{1,1}\big(0, T; \mathcal{P}(I) \times \mathfrak{l}^1(I \times I)\big)$.*

Proof. For this theorem we can immediately apply [PR16, Prop. 4.1] after checking the following conditions:

- $1 \leq \inf_{i,j \in I: i \neq j} |\gamma^{(ij)}|_1 = \sup_{i,j \in I: i \neq j} |\gamma^{(ij)}|_1 = 3 < \infty$,
- $\sup_{(\rho, \mathsf{w}) \in \mathcal{P}(I) \times \mathfrak{l}^1(I \times I)} \sum\sum_{i \neq j} \rho_i Q_{ij} = \sum\sum_{i \neq j} Q_{ij} < \infty$.

□

As often in large deviation theory it will be beneficial to have a dual formulation of the rate functional at hand. For any $\rho \in L^1\big(0, T; \mathcal{P}(I)\big)$, $\mathsf{j} \in L^1\big(0, T; \mathfrak{l}^1(I \times I)\big)$ and $\zeta \in L^\infty\big(0, T; \mathfrak{l}^\infty(I \times I)\big)$, let

$$G(\rho, \mathsf{j}, \zeta) := {}_{L^\infty}\langle \zeta, \mathsf{j} \rangle_{L^1} - \int_0^T \mathcal{H}\big(\rho(t), \zeta(t)\big)\, dt \quad \text{and} \quad \mathcal{H}(\rho, \zeta) := \sum_{i \neq j}\sum \rho_i Q_{ij}\big(e^{\zeta_{ij}} - 1\big),$$

denoting ${}_{L^\infty}\langle \zeta, \mathsf{j} \rangle_{L^1} := \sum\sum_{i \neq j} \int_0^T \zeta_{ij}(t)\, \mathsf{j}_{ij}(t)\, dt$, again excluding the diagonal. We then have the following:

Proposition 4.4. *For any $(\rho, \mathsf{w}) \in W^{1,1}\big(0, T; \mathcal{P}(I) \times \mathfrak{l}^1(I \times I)\big) \cap \mathrm{BV}_{\mathrm{flux}}$ with initial condition $\lim_{t \searrow 0}(\rho(t), \mathsf{w}(t)) = (\mu, 0)$,*

$$\mathcal{I}_{(0,T)}(\rho, \mathsf{w}) = \sup_{\zeta \in C_b^2(0, T; \mathfrak{l}^\infty(I \times I))} G(\rho, \dot{\mathsf{w}}, \zeta).$$

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Moreover, if $\mathcal{I}_{(0,T)}(\rho,\mathsf{w})<\infty$ *then*

$$\mathcal{I}_{(0,T)}(\rho,\mathsf{w})=\sup_{\zeta\in C_c^2([0,T);\mathfrak{l}^\infty(I\times I))} G(\rho,\dot{\mathsf{w}},\zeta). \tag{13}$$

Proof. Fix a $\rho\in\mathrm{BV}\big(0,T;\mathcal{P}(I)\big)$. For any $\zeta\in L^\infty\big(0,T;\mathfrak{l}^\infty(I\times I)\big)$ there holds

$$\begin{aligned}\sup_{\mathsf{j}\in L^1(0,T;\mathfrak{l}^1(I\times I))} {}_{L^\infty}\langle\zeta,\mathsf{j}\rangle_{L^1}-\int_0^T\mathcal{S}\big(\mathsf{j}(t)|\rho(t)\otimes Q\big)\,dt\\ \leq\int_0^T\sup_{\mathsf{j}(t)\in\mathfrak{l}^1(I\times I)}\Big[\zeta(t)\cdot\mathsf{j}(t)-\mathcal{S}\big(\mathsf{j}(t)|\rho(t)\otimes Q\big)\Big]\,dt=\int_0^T\mathcal{H}\big(\rho(t),\zeta(t)\big)\,dt.\end{aligned} \tag{14}$$

The point-wise maximiser on the right-hand side is $\mathsf{j}:(i,j,t)\mapsto\rho_i(t)Q_{ij}\big(e^{\zeta_{ij}(t)}-1\big)$, which lies in $L^1\big(0,T;\mathfrak{l}^1(I\times I)\big)$. Hence the point-wise maximiser is also the global maximiser of the left-hand side, and inequality (14) is in fact an equality. From the Moreau-Fenchel Theorem [Bré83, Th. I.10] it follows that

$$\int_0^T\mathcal{S}\big(\dot{\mathsf{w}}(t)|\rho(t)\otimes Q\big)\,dt=\sup_{\zeta\in L^\infty(0,T;\mathfrak{l}^\infty(I\times I))}G(\rho,\dot{\mathsf{w}},\zeta).$$

We can now show that we can replace the supremum over L^∞-functionals by a supremum over C_b^2-functions. To prove this, take any $\zeta\in L^\infty\big(0,T;\mathfrak{l}^\infty(I\times I)\big)$ and approximate it by $\zeta^{(\epsilon)}:=\zeta*\theta^{(\epsilon)}$ where

$$\theta^{(\epsilon)}(t)=\frac{1}{4\pi\epsilon}e^{-t^2/4\epsilon}, \tag{15}$$

such that $\zeta^{(\epsilon)}\in C_b^2\big(0,T;\mathfrak{l}^\infty(I\times I)\big)$ and $\zeta^{(\epsilon)}\to\zeta$ weakly in $L^\infty\big(0,T;\mathfrak{l}^\infty(I\times I)\big)$. Note that by equality (14), the functional $\zeta\mapsto\int_0^T\mathcal{H}\big(\rho(t),\zeta(t)\big)\,dt$ is the supremum of weakly continuous functionals, and hence it is itself weakly lower semicontinuous in L^∞. Therefore we get

$$\limsup_{\epsilon\to0}G(\rho,\dot{\mathsf{w}},\zeta^{(\epsilon)})\leq\limsup_{\epsilon\to0}{}_{L^\infty}\langle\zeta^{(\epsilon)},\dot{\mathsf{w}}\rangle_{L^1}-\liminf_{\epsilon\to0}\int_0^T\mathcal{H}\big(\rho(t),\zeta^{(\epsilon)}(t)\big)\,dt\leq G(\rho,\dot{\mathsf{w}},\zeta),$$

which proves the claim.

Finally, we show (13), assuming $\mathcal{I}_{(0,T)}<\infty$. Take a $\zeta\in C_b^2\big(0,T;\mathfrak{l}^\infty(I\times I)\big)$, and approximate it by $\zeta\eta^{(\epsilon)}$, where $\eta^{(\epsilon)}:(0,T)\to[0,1]$ is smooth such that $\eta^{(\epsilon)}|_{(0,T-\epsilon]}=1$ and $\eta^{(\epsilon)}|_{(T-\frac12\epsilon,T]}=0$. Let $g_{ij}(t,\zeta_{ij}):=\zeta_{ij}\dot{\mathsf{w}}_{ij}(t)-\rho_i(t)Q_{ij}\big(e^{\zeta_{ij}}-1\big)$ and $s_{ij}(t)=\lambda_B\big(\dot{\mathsf{w}}_{ij}(t)/(\rho_i(t)Q_{ij}\big)\dot{\mathsf{w}}_{ij}(t)$. For almost every t and any coordinate pair $i\neq j$, we can distinguish three different cases (see Figure 1):

(i) if $\zeta_{ij}(t)<0$, then $g_{ij}\big(t,\zeta_{ij}(t)\big)\leq g_{ij}\big(t,\zeta_{ij}(t)\eta^{(\epsilon)}(t)\big)\leq0\leq s\big(\dot{\mathsf{w}}_{ij}(t)|\rho_i(t)Q_{ij}\big)$,

(ii) if $\zeta_{ij}(t)\geq0$ and $g_{ij}\big(t,\zeta_{ij}(t)\big)\geq0$, then $0\leq g_{ij}\big(t,\zeta_{ij}(t)\eta^{(\epsilon)}(t)\big)\leq s\big(\dot{\mathsf{w}}_{ij}(t)|\rho_i(t)Q_{ij}\big)$,

(iii) if $\zeta_{ij}(t)\geq0$ and $g_{ij}\big(t,\zeta_{ij}(t)\big)<0$, then $g_{ij}\big(t,\zeta_{ij}(t)\big)\leq g_{ij}\big(t,\zeta_{ij}(t)\eta^{(\epsilon)}(t)\big)\leq s\big(\dot{\mathsf{w}}_{ij}(t)|\rho_i(t)Q_{ij}\big)$.

In all cases we have the t-almost everywhere bounds

$$g^-(t):=\sum\sum_{i\neq j}g_{ij}\big(t,\zeta_{ij}(t)\big)\wedge0\leq\sum\sum_{i\neq j}g_{ij}\big(t,\zeta_{ij}(t)\eta^{(\epsilon)}(t)\big)\leq\sum\sum_{i\neq j}s_{ij}(t).$$

The right-hand side lies in $L^1(0,T)$ by the assumption that $\mathcal{I}_{(0,T)}(\rho,\mathsf{w})<\infty$, and the left-hand as well since $\|g^-(t)\|_{L^1(0,T)}\leq\mathcal{I}_{(0,T)}-G(\rho,\dot{\mathsf{w}},\zeta)<\infty$ where $G(\rho,\dot{\mathsf{w}},\zeta)>-\infty$ as ζ is bounded. Clearly $\zeta\eta^{(\epsilon)}$ converges pointwise to ζ; by dominated convergence it follows that $G(\rho,\dot{\mathsf{w}},\zeta\eta^{(\epsilon)})\to G(\rho,\dot{\mathsf{w}},\zeta)$, which was to be proven. □

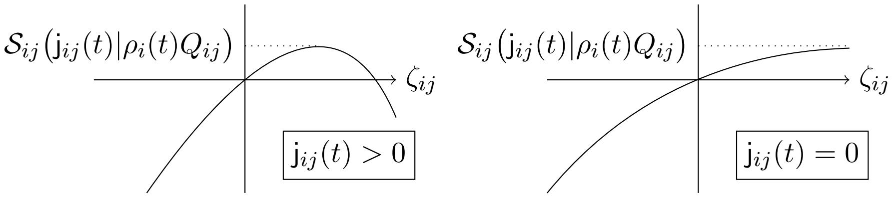

Figure 1: The function $g_{ij}(t, \zeta_{ij})$.

The functional $G(\rho, \dot{\mathsf{w}}, \zeta)$ is generally not hybrid-continuous in (ρ, w), but all we will need is the following:

Lemma 4.5. *For any $\zeta \in C_c^2\big([0,T); \mathfrak{l}^\infty(I \times I)\big)$, the functional $(\rho, \mathsf{w}) \mapsto G(\rho, \dot{\mathsf{w}}, \zeta)$ is hybrid-continuous on $\{\mathcal{I}_{(0,T)} < \infty\}$.*

Proof. Paths for which $\mathcal{I}_{(0,T)}$ is finite satisfy the initial condition $\big(\rho(0), \mathsf{w}(0)\big) = (\mu, 0)$. As ζ is differentiable and 0 at the right boundary we can write:

$$G(\rho, \dot{\mathsf{w}}, \zeta) = -{}_{L^\infty}\langle \dot{\zeta}, \mathsf{w}\rangle_{L^1} - \sum\sum_{i \neq j} \int_0^T \rho_i(t) Q_{ij}(e^{\zeta_{ij}(t)} - 1)\, dt,$$

which is strongly continuous in $L^1\big(0,T; \mathcal{P}(I) \times \mathfrak{l}^1(I \times I)\big)$ and hence also hybrid-continuous. □

4.2 Exponential tightness

As mentioned in the Preliminaries Section, with the choice of the hybrid topology, the proof for the exponential tightness becomes very simple. Define the total-variation balls:

$$B_r := \big\{ \big(\rho, \mathsf{w}\big) \in \mathrm{BV}_{\mathrm{flux}} : \mathrm{epvar}(\mathsf{w}) \leq r \big\}. \tag{16}$$

Note that by (5) the variation $\mathrm{epvar}(\rho)$ is automatically uniformly bounded within such balls.

Proposition 4.6 (Exponential tightness)**.** *Let $u \in L^\infty\big(0,T; \mathfrak{l}^\infty(I \times I)\big)$. For any $\eta > 0$, and for $r := \eta + T\, e \|u\|_{L^\infty} |Q|_1$,*

$$\mathbb{P}^{(n)}_u(B_r^{\mathsf{c}}) \leq e^{-n\eta},$$

and the balls B_r are hybrid-compact in $\mathrm{BV}(0,T; \mathcal{P}(I) \times \mathfrak{l}^1(I \times I)\big)$.

Proof. Observe that the perturbed Markov jump process $\sum\sum_{i \neq j} \mathsf{W}^{(n)}_{i,j}(t)$ is bounded by a Poisson process $\frac{1}{n} N_\lambda$ with intensity $\lambda := n\|u\|_{L^\infty} |Q|_1 \geq n \sum\sum_{i \neq j} \rho_i(t) Q_{ij} u_{ij}(t)$. A standard Chernoff bound therefore yields

$$\mathbb{P}^{(n)}(B_r^{\mathsf{c}}) = \mathbb{P}^{(n)}\big(\big\{\tfrac{1}{n} \# \mathrm{jump}(\mathsf{W}^{(n)}) > r\big\}\big) \leq \mathrm{Prob}\,\big(N_\lambda > nr\big) \leq e^{\lambda T e - nr} = e^{-n\eta}.$$

Moreover, by Proposition 2.2 the balls B_r are automatically hybrid-compact. □

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4.3 Lower bound

We now prove the large-deviation lower bound. As usual in dynamic large deviations, the proof is based on a Girsanov transformation together with an approximation argument.

Lemma 4.7 (Girsanov transformation)**.** *Fix an $n \in \mathbb{N}$, $\rho^{(n)}(0) \in \mathcal{P}(I) \cap \frac{1}{n}\mathbb{N}^I$ and a function $\zeta \in C_b^2\big(0,T;\mathfrak{l}^\infty(I\times I)\big)$. Let $\mathbb{P}^{(n)}_{e^\zeta,r}$ be the path measure for the process with initial conditions $(\rho^{(n)}(0),0)$ and generator*

$$(\mathcal{Q}^{(n)}_{e^{\zeta(t)},r}\phi)(\rho,\mathsf{w}) := n\sum\sum_{i\neq j} \rho_i Q_{ij} e^{\zeta_{ij}(t)\mathbb{1}_{\{\mathsf{w}_{ij}(t)\leq r-\frac{1}{n}\}}}\big[\phi(\rho^{(i\to j)},\mathsf{w}^{ij+}) - \phi(\rho,\mathsf{w})\big].$$

Then

$$\frac{1}{n}\log\frac{d\mathbb{P}^{(n)}_{e^\zeta,r}}{d\mathbb{P}^{(n)}}(\rho,\mathsf{w}) = -G_r(\rho,\mathsf{w},\zeta), \tag{17}$$

where

$$G_r(\rho,\mathsf{w},\zeta) := \sum\sum_{i\neq j}\int_0^{\inf\{\hat t\in(0,T):\mathsf{w}_{ij}(\hat t)\geq r\}} \Big[\zeta_{ij}(t)\dot{\mathsf{w}}_{ij}(dt) - \rho_i(t)Q_{ij}\big(e^{\zeta_{ij}}-1\big)\,dt\Big].$$

Proof. We can apply the Girsanov Theorem for jump processes [KL99, Ch. A.1, Th. 7.3] to the functional $F_t(\mathsf{w}) := n\,\zeta(t)\cdot(\mathsf{w}\wedge r)$ for $\mathsf{w}\in\mathfrak{l}^1(I\times I)$, which lies in $C_b^2\big(0,T;L^\infty(\mathfrak{l}^1(I\times I))\big)$. □

Remark 4.8. The right-hand side of (17) does not involve the particle number n due to the independence of the particles. □

The next lemma shows that the following set constitutes of sufficiently regular paths that can be retrieved via the Girsanov transformation:

$$\mathcal{A} := \Big\{(\rho,\mathsf{w})\in W^{1,1}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big) : \mathcal{I}_{(0,T)}(\rho,\mathsf{w})<\infty \quad\text{and}\quad t\mapsto\log\frac{\dot{\mathsf{w}}(t)}{\rho(t)\otimes Q}\in C_c^2\big([0,T);\mathfrak{l}^\infty(I\times I)\big)\Big\}. \tag{18}$$

Lemma 4.9. *For any hybrid-open set $\mathcal{U}\subset\mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ and any $(\rho,\mathsf{w})\in\mathcal{U}\cap\mathcal{A}$,*

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}^{(n)}(\mathcal{U}) \geq -\mathcal{I}_{(0,T)}(\rho,\mathsf{w}).$$

Proof. Let a hybrid-open $\mathcal{U}\subset\mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ and a $(\rho,\mathsf{w})\in\mathcal{U}\cap\mathcal{A}$ be given, and define $\zeta(t) := t\mapsto\log\frac{\dot{\mathsf{w}}(t)}{\rho(t)\otimes Q}\in C_c^2\big([0,T);\mathfrak{l}^\infty(I\times I)\big)$. For an arbitrary $\epsilon>0$, let $\mathcal{U}_\epsilon(\rho,\mathsf{w}) := \{(\hat\rho,\hat{\mathsf{w}})\in\mathrm{BV}\left(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\right) : G(\hat\rho,\hat{\mathsf{w}},\zeta) < G(\rho,\dot{\mathsf{w}},\zeta)+\epsilon\}$. By applying the transformation Lemma 4.7,

$$\begin{aligned}\frac{1}{n}\log\mathbb{P}^{(n)}(\mathcal{U}) &\geq \frac{1}{n}\log\mathbb{P}^{(n)}\big(\mathcal{U}\cap\mathcal{U}_\epsilon(\rho,\mathsf{w})\cap B_n\big)\\ &\geq \frac{1}{n}\log\mathbb{P}^{(n)}_{e^\zeta,n}\big(\mathcal{U}\cap\mathcal{U}_\epsilon(\rho,\mathsf{w})\cap B_n\big)+\frac{1}{n}\log \mathbb{P}^{(n)}\text{-}\operatorname*{ess\,inf}_{(\hat\rho,\hat{\mathsf{w}})\in\mathcal{U}\cap\mathcal{U}_\epsilon(\rho,\mathsf{w})\cap B_n}\frac{d\mathbb{P}^{(n)}}{d\mathbb{P}^{(n)}_{e^\zeta,n}}(\hat\rho,\hat{\mathsf{w}})\\ &\geq \frac{1}{n}\log\mathbb{P}^{(n)}_{e^\zeta,n}\big(\mathcal{U}\cap\mathcal{U}_\epsilon(\rho,\mathsf{w})\cap B_n\big)-\sup_{(\hat\rho,\hat{\mathsf{w}})\in\mathcal{U}\cap\mathcal{U}_\epsilon(\rho,\mathsf{w})\cap B_n}G_n(\hat\rho,\hat{\mathsf{w}},\zeta)\\ &\geq \frac{1}{n}\log\mathbb{P}^{(n)}_{e^\zeta}\big(\mathcal{U}\cap\mathcal{U}_\epsilon(\rho,\mathsf{w})\cap B_n\big)-\sup_{(\hat\rho,\hat{\mathsf{w}})\in\mathcal{U}\cap\mathcal{U}_\epsilon(\rho,\mathsf{w})\cap B_n}G(\hat\rho,\hat{\mathsf{w}},\zeta). \qquad (19)\end{aligned}$$

To bound the first term, observe that the perturbed limit equation (10) with $u = e^\zeta = \dot{\mathsf{w}}/(\rho\otimes Q)$ yields the given path (ρ,w). Hence by the many-particle limit (Theorem 3.3), $\mathbb{P}^{(n)}_{e^\zeta}\rightharpoonup\delta_{(\rho,\mathsf{w})}$. Furthermore, because of Lemma 4.5 the sets $\mathcal{U}_\epsilon(\rho,\mathsf{w})$ are hybrid-open, so that the Portemanteau Theorem gives $\liminf_n\mathbb{P}^{(n)}_{e^\zeta}\big(\mathcal{U}\cap\mathcal{U}_\epsilon(\rho,\mathsf{w})\big)\geq 1$. Then we also have, exploiting the exponential tightness (Proposition 4.6),

$$\liminf_{n\to\infty}\mathbb{P}^{(n)}_{e^\zeta}\big(\mathcal{U}\cap\mathcal{U}_\epsilon(\rho,\mathsf{w})\cap B_n\big)\geq\liminf_{n\to\infty}\Big[\mathbb{P}^{(n)}_{e^\zeta}\big(\mathcal{U}\cap\mathcal{U}_\epsilon(\rho,\mathsf{w})\big)-\mathbb{P}^{(n)}_{e^\zeta}\big(B_n^{\mathsf{c}}\big)\Big]\geq 1,$$

which shows that the first term in (19) vanishes in the limes inferior.

For the second term, we use the definition of the set $\mathcal{U}_\epsilon(\rho,\mathsf{w})$ together with the fact that ζ maximises $G(\rho,\dot{\mathsf{w}},\cdot)$. Putting everything together we find:

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}^{(n)}(\mathcal{U}) &\geq \liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}^{(n)}_{e^\zeta}\big(\mathcal{U}\cap\mathcal{U}_\epsilon(\rho,\mathsf{w})\cap B_n\big)-G(\rho,\dot{\mathsf{w}},\zeta)-\epsilon\\ &\geq -G(\rho,\dot{\mathsf{w}},\zeta)-\epsilon=\mathcal{I}_{(0,T)}(\rho,\mathsf{w})-\epsilon.\end{aligned}$$

Since ϵ was arbitrary the claim follows. □

The next three approximation lemmas show that the rate fuctional can be approximated by taking paths in $\mathcal{A}$.

Lemma 4.10 (Lower bound approximation I)**.** *Let $(\rho,\mathsf{w})\in W^{1,1}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ with $\mathcal{I}_{(0,T)}(\rho,\mathsf{w})<\infty$. Then there exists a sequence $(\rho^{(\epsilon)},\mathsf{w}^{(\epsilon)})_{0<\epsilon<\hat\epsilon}\subset W^{1,1}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ such that $\mathcal{I}_{(0,T)}(\rho^{(\epsilon)},\mathsf{w}^{(\epsilon)})\xrightarrow[\epsilon\to0]{}\mathcal{I}_{(0,T)}(\rho,\mathsf{w})$ and $\|\dot{\mathsf{w}}^{(\epsilon)}_{ij}\|_{L^1(0,T)}=0\implies\|\rho^{(\epsilon)}_iQ_{ij}\|_{L^1(0,T)}=0$ for all $i\neq j\in I\times I$ and for all $0<\epsilon<\hat\epsilon$.*

Proof. If $\|\dot{\mathsf{w}}_{ij}\|_{L^1(0,T)}=0\implies\|\rho_iQ_{ij}\|_{L^1(0,T)}=0$ for all $i,j\in I$ then the construction is trivial. Now assume that there is one and only one pair $i\neq j\in I\times I$ for which $\|\rho_i\|_{L^1}|Q_{ij}|=\|\rho_iQ_{ij}\|_{L^1}>0$ but $\|\dot{\mathsf{w}}_{ij}\|_{L^1}=0$. If there would be more pairs with this property then we can simply repeat the construction below for each such pair separately.

Since $\mathcal{I}_{(0,T)}(\rho,\mathsf{w})<\infty$, by Proposition 4.3 the path ρ must be continuous, and so there exists an $\hat\epsilon>0$ and a $\hat t\in(0,T)$ such that, see Figure 2:

$$0\leq(t-\hat t)\mathbb{1}_{(\hat t,\hat t+\epsilon)}(t)\leq(t-\hat t)\mathbb{1}_{(\hat t,\hat t+\hat\epsilon)}(t)\leq\rho_i(t)-\hat\epsilon\qquad\text{for all }0<\epsilon<\hat\epsilon.$$

Define the sequence:

$$\begin{aligned}&\mathsf{w}^{(\epsilon)}(t):=\mathsf{w}(t)+(t-\hat t)\mathbb{1}_{(\hat t,\hat t+\epsilon)}(t)\mathbb{1}_{ij},\quad\text{and}\\ &\rho^{(\epsilon)}(t):=\mu-\operatorname{div}\mathsf{w}^{(\epsilon)}(t)=\rho(t)-(t-\hat t)\mathbb{1}_{(\hat t,\hat t+\epsilon)}(t)\mathbb{1}_i,\end{aligned}$$

and note that by construction $\rho^{(\epsilon)}(0) = \mu$, the continuity equation is satisfied, and we have that both $\dot{\mathsf{w}}^{(\epsilon)}_{ij}(t) > 0$ and $\rho^{(\epsilon)}_i(t)Q_{ij} > 0$, as required. To prove the convergence of the rate functional, we will use formulation (3) which is equal to $\mathcal{I}_{(0,T)}$ by Proposition 4.4. Recall that $\dot{\mathsf{w}}_{ij} \equiv 0$ and note that the continuous function $\rho_i(t)$ is uniformly bounded by some $C > 0$ on the compact interval $[\hat t, \hat t + \hat\epsilon]$, so that $0 < \hat\epsilon \le \rho^{(\epsilon)}_i(t) = \rho_i(t) - t + \hat t \le C$ on $[\hat t, \hat t + \hat\epsilon]$. It follows that

$$\begin{aligned}\left|\mathcal{I}_{(0,T)}(\rho^{(\epsilon)}, \mathsf{w}^{(\epsilon)}) - \mathcal{I}_{(0,T)}(\rho^{(\epsilon)}, \mathsf{w}^{(\epsilon)})\right| &\le \int_{\hat t}^{\hat t+\epsilon} \left|1 \log \frac{1}{\rho^{(\epsilon)}_i(t)Q_{ij}} - 1 + \rho^{(\epsilon)}_i(t)Q_{ij} - \rho_i(t)Q_{ij}\right| dt \\ &\le \int_{\hat t}^{\hat t+\epsilon} \Big(|\log(\rho^{(\epsilon)}_i(t)Q_{ij})| + 1 + \hat t - t\Big)\, dt \to 0.\end{aligned}$$

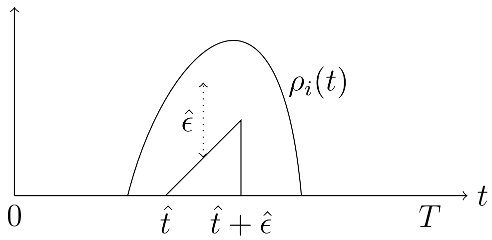

Figure 2: There exists a small triangle below the graph of $\rho_i(t)$.

□

Lemma 4.11 (Lower bound approximation II)**.** *Let $(\rho, \mathsf{w}) \in W^{1,1}\big(0, T; \mathcal{P}(I) \times \ell^1(I \times I)\big)$ with $\mu > 0$, $\mathcal{I}_{(0,T)}(\rho, \mathsf{w}) < \infty$, and $\|\dot{\mathsf{w}}_{ij}\|_{L^1(0,T)} = 0 \implies \|\rho_i Q_{ij}\|_{L^1(0,T)} = 0$ for all $i \ne j \in I \times I$. Then there exists a sequence $(\rho^{(\epsilon)}, \mathsf{w}^{(\epsilon)})_{\epsilon>0} \subset W^{1,1}\big(0, T; \mathcal{P}(I) \times \ell^1(I \times I)\big)$ such that $\mathcal{I}_{(0,T)}(\rho^{(\epsilon)}, \mathsf{w}^{(\epsilon)}) \xrightarrow[\epsilon\to0]{} \mathcal{I}_{(0,T)}(\rho, \mathsf{w})$ and $\log \frac{\dot{\mathsf{w}}^{(\epsilon)}}{\rho^{(\epsilon)} \otimes Q} \in C^2_b\big(0, T; \ell^\infty(I \times I)\big)$ for all $\epsilon > 0$ (with the convention that $\log \frac{0}{0} \equiv 0$).*

Proof. Let $\theta^{(\epsilon)}$ be the heat kernel (15) and define the approximating sequence (coordinate-wise):

$$\begin{aligned}\mathsf{w}^{(\epsilon)}(t) &:= \frac{\mu}{(\rho * \theta^{(\epsilon)})(0)}\big[(\mathsf{w} * \theta^{(\epsilon)})(t) - (\mathsf{w} * \theta^{(\epsilon)})(0)\big], \qquad \text{and} \\ \rho^{(\epsilon)}(t) &:= \mu - \operatorname{div} \mathsf{w}^{(\epsilon)}(t) = \frac{\mu}{(\rho * \theta^{(\epsilon)})(0)}(\rho * \theta^{(\epsilon)})(t). \qquad (20)\end{aligned}$$

Here the convolutions run over the whole real line, where we extended $(\rho(t), \mathsf{w}(t)) = (\mu, 0)$ for $t < 0$ and $\big(\rho(t), \mathsf{w}(t)\big) = \big(\rho(T), \mathsf{w}(T)\big)$ for $t > T$; these values are well-defined since functions of bounded variation have left and right limits. Observe that by construction, the initial condition $\big(\rho^{(\epsilon)}(0), \mathsf{w}^{(\epsilon)}(0)\big) = (\mu, 0)$ and the continuity equation are satisfied, $\rho^{(\epsilon)}(t)$ and $\dot{\mathsf{w}}^{(\epsilon)}(t)$ are non-negative, and so is $\mathsf{w}^{(\epsilon)}(t)$.

We first prove that $\log \frac{\dot{\mathsf{w}}^{(\epsilon)}_{ij}}{\rho^{(\epsilon)}_i \otimes Q_{ij}} \in C^2_b(0, T)$ for all $\epsilon > 0$ and $i, j \in I$. We distinguish between two cases. If the path $\rho_i(t)Q_{ij} = 0$ for almost every $t \in (0, T)$, then also $\dot{\mathsf{w}}_{ij}(t) = 0$ for almost every t, since $\mathcal{I}_{(0,T)} < \infty$ implies $\dot{\mathsf{w}}(t) \ll \rho(t) \otimes Q$. Then we also have $\rho^{(\epsilon)}_i(t)Q_{ij} = 0$ and $\mathsf{w}^{(\epsilon)}_{ij}(t) = 0$ for almost every $t \in (0, T)$, and hence $\log \frac{\dot{\mathsf{w}}^{(\epsilon)}_{ij}}{\rho^{(\epsilon)}_i Q_{ij}} = \log \frac{0}{0} := 0 \in C^2_b(0, T)$. For the second case we can assume that $\|\rho_i\|_{L^1(0,T)} > 0$ and $Q_{ij} > 0$, and so by the main assumption of the lemma also $\mathsf{w}_{ij}(T) = \|\dot{\mathsf{w}}_{ij}\|_{L^1(0,T)} > 0$. Clearly $(\rho^{(\epsilon)}, \mathsf{w}^{(\epsilon)}) \in C_b^\infty\big(0,T;\mathcal{P}(I)\times \mathfrak{l}^\infty(I\times I)\big)$, and in particular, the function and all its derivatives are uniformly bounded. More precisely, we have the following bounds from below, uniformly in t:

$$\rho_i^{(\epsilon)}(t) \geq \frac{\mu}{(\rho*\theta^{(\epsilon)})(0)}\theta^{(\epsilon)}(T)\|\rho_i\|_{L^1(0,T)} > 0 \quad \text{and} \quad \dot{\mathsf{w}}_{ij}^{(\epsilon)}(t) \geq \frac{\mu}{(\rho*\theta^{(\epsilon)})(0)}\theta^{(\epsilon)}(T)\mathsf{w}_{ij}(T) > 0. \tag{21}$$

Therefore the three functions

$$\begin{aligned}
&\log \frac{\dot{\mathsf{w}}_{ij}^{(\epsilon)}(t)}{\rho_i^{(\epsilon)}(t)Q_{ij}}\\
&\frac{d}{dt}\log \frac{\dot{\mathsf{w}}_{ij}^{(\epsilon)}(t)}{\rho_i^{(\epsilon)}(t)Q_{ij}} = \frac{\ddot{\mathsf{w}}_{ij}^{(\epsilon)}(t)}{\dot{\mathsf{w}}_{ij}^{(\epsilon)}(t)} - \frac{\dot{\rho}_i^{(\epsilon)}(t)}{\rho_i^{(\epsilon)}(t)}\\
&\frac{d^2}{dt^2}\log \frac{\dot{\mathsf{w}}_{ij}^{(\epsilon)}(t)}{\rho_i^{(\epsilon)}(t)Q_{ij}} = \frac{\dot{\mathsf{w}}_{ij}^{(\epsilon)}(t)\dddot{\mathsf{w}}_{ij}^{(\epsilon)}(t) - \ddot{\mathsf{w}}_{ij}^{(\epsilon)}(t)^2}{\dot{\mathsf{w}}_{ij}^{(\epsilon)}(t)^2} - \frac{\rho_i^{(\epsilon)}(t)\ddot{\rho}_i^{(\epsilon)}(t) - \dot{\rho}_i^{(\epsilon)}(t)^2}{\rho_i^{(\epsilon)}(t)^2}
\end{aligned}$$

are all bounded and continuous which was to be shown.

We now show that $\mathcal{I}_{(0,T)}(\rho^{(\epsilon)},\mathsf{w}^{(\epsilon)}) \xrightarrow[\epsilon\to0]{} \mathcal{I}_{(0,T)}(\rho,\mathsf{w})$. Because $(\rho,\mathsf{w}) \in W^{1,1}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ we surely have $(\rho^{(\epsilon)},\mathsf{w}^{(\epsilon)}) \xrightarrow[\epsilon\to0]{W^{1,1}} (\rho,\mathsf{w})$. Since $\mathcal{I}_{(0,T)}$ is the supremum over $W^{1,1}$-continuous functionals, $\mathcal{I}_{(0,T)}$ is lower semi-continuous, and so

$$\liminf_{\epsilon\to0} \mathcal{I}_{(0,T)}(\rho^{(\epsilon)},\mathsf{w}^{(\epsilon)}) \geq \mathcal{I}_{(0,T)}(\rho,\mathsf{w}). \tag{22}$$

We use the relative entropy formulation of $\mathcal{I}_{(0,T)}$ to prove the inequality in the other direction. Since $\mathcal{S}(\cdot|\cdot)$ is jointly convex one finds by a two-dimensional Jensen inequality that

$$\begin{aligned}
&\int_0^T \mathcal{S}\big(\dot{\mathsf{w}}^{(\epsilon)}(t)|\rho^{(\epsilon)}\otimes Q(t)\big)\,dt\\
&\quad= \frac{\mu}{(\rho*\theta^{(\epsilon)})(0)}\int_0^T \mathcal{S}\Big(\int_{-\infty}^\infty \dot{\mathsf{w}}(t-s)\,\theta^{(\epsilon)}(s)\,ds\,\Big|\,\int_{-\infty}^\infty \rho(t-s)\theta^{(\epsilon)}(s)\,ds\otimes Q\Big)\,dt\\
&\quad\leq \frac{\mu}{(\rho*\theta^{(\epsilon)})(0)}\int_0^T\int_{-\infty}^\infty \mathcal{S}\big(\dot{\mathsf{w}}(t-s)|\rho(t-s)\otimes Q\big)\theta^{(\epsilon)}(s)\,ds\,dt\\
&\quad\xrightarrow[\epsilon\to0]{} \int_0^T \mathcal{S}\big(\dot{\mathsf{w}}(t)|\rho(t)\otimes Q\big)\,dt = \mathcal{I}_{(0,T)}(\rho,\mathsf{w}),
\end{aligned}$$

where the convergence follows from $(\rho*\theta^{(\epsilon)})(0)\to\mu$ together with the fact that the non-negative mapping $t\mapsto\mathcal{S}(\dot{\mathsf{w}}(t)|\rho(t)\otimes Q)$ lies in $L^1(0,T)$ since $\mathcal{I}_{(0,T)}(\rho,\mathsf{w})<\infty$ (see for example [Eva98, App. C.4, Th. 6]). □

Lemma 4.12 (Lower bound approximation III)**.** *Let* $(\rho,\mathsf{w}) \in W^{1,1}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ *with* $\mathcal{I}_{(0,T)}(\rho,\mathsf{w})<\infty$ *and* $\log\frac{\dot{\mathsf{w}}}{\rho\otimes Q} \in C_b^2\big(0,T;\mathfrak{l}^\infty(I\times I)\big)$ *(with the convention that* $\log\frac{0}{0}\equiv0$*). Then there exists a sequence* $(\rho^{(\epsilon)},\mathsf{w}^{(\epsilon)})_{\epsilon>0} \subset W^{1,1}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ *such that* $\mathcal{I}_{(0,T)}(\rho^{(\epsilon)},\mathsf{w}^{(\epsilon)}) \to \mathcal{I}_{(0,T)}(\rho,\mathsf{w})$ *and* $\log\frac{\dot{\mathsf{w}}^{(\epsilon)}}{\rho^{(\epsilon)}\otimes Q} \in C_c^2\big([0,T);\mathfrak{l}^\infty(I\times I)\big)$ *for all* $\epsilon>0$*.*

Proof. Let $\psi(t)$ be the solution map from Proposition 3.2(ii), (where the perturbation factor $u \equiv 0$), and define

$$(\hat\rho^{(\epsilon)}(t), \hat{\mathsf{w}}^{(\epsilon)}(t)) := \begin{cases} (\rho(t), \mathsf{w}(t)), & t \in (0, T-\epsilon), \\ \psi(t-T+\epsilon)\big[\rho(T-\epsilon), \mathsf{w}(T-\epsilon)\big], & t \in (T-\epsilon, T). \end{cases}$$

Then clearly $\log \frac{\dot{\hat{\mathsf{w}}}^{(\epsilon)}}{\hat\rho^{(\epsilon)}\otimes Q}$ has C_b^2-regularity on $(0, T-\epsilon]$, and it is constant 0 on $(T-\epsilon, T)$. To deal with the lack of regularity at time $t = T-\epsilon$, we again mollify like in (20), but now with smooth, compactly supported bump functions $\eta^{(\epsilon)} \in C_c^\infty(-\frac12\epsilon, \frac12\epsilon)$, $\int \eta^{(\epsilon)}(t)\,dt = 1$, i.e.

$$\mathsf{w}^{(\epsilon)}(t) := \frac{\mu}{(\hat\rho^{(\epsilon)} * \eta^{(\epsilon)})(0)}\big[(\hat{\mathsf{w}}^{(\epsilon)} * \eta^{(\epsilon)})(t) - (\hat{\mathsf{w}}^{(\epsilon)} * \eta^{(\epsilon)})(0)\big], \quad \text{and} \quad \rho^{(\epsilon)}(t) \quad := \mu - \operatorname{div}\mathsf{w}^{(\epsilon)}(t).$$

Due to the small compact support, we still have that $(\rho^{(\epsilon)}, \mathsf{w}^{(\epsilon)})$ follows the macroscopic flow on $(T - \frac12\epsilon, T)$, and so $\log \frac{\dot{\mathsf{w}}^{(\epsilon)}}{\rho^{(\epsilon)}\otimes Q}$ has the desired regularity.

We now prove that the rate functional converges. The convolution with the smooth kernel can be dealt with in exactly the same manner as in the proof of Lemma 4.11, so we only need to prove the convergence of $\mathcal{I}_{(0,T)}(\hat\rho^{(\epsilon)}, \hat{\mathsf{w}}^{(\epsilon)})$. This follows immediately by monotone convergence:

$$\mathcal{I}_{(0,T)}(\hat\rho^{(\epsilon)}, \hat{\mathsf{w}}^{(\epsilon)}) = \int_0^{T-\epsilon} \mathcal{S}\big(\dot{\hat{\mathsf{w}}}^{(\epsilon)}(t)|\hat\rho^{(\epsilon)}(t)\otimes Q\big)\,dt \to \int_0^T \mathcal{S}\big(\dot{\hat{\mathsf{w}}}^{(\epsilon)}(t)|\hat\rho^{(\epsilon)}(t)\otimes Q\big)\,dt.$$

□

Proposition 4.13 (Large-deviation lower bound)**.** *Assume $\mu > 0$. For any hybrid-open set $\mathcal{U} \subset \mathrm{BV}\big(0,T;\mathcal{P}(I)\times \mathfrak{l}^1(I\times I)\big)$,*

$$\liminf_{n\to\infty} \frac1n \log \mathbb{P}^{(n)}(\mathcal{U}) \ge -\inf_{(\rho,\mathsf{w})\in\mathcal{U}} \mathcal{I}_{(0,T)}(\rho,\mathsf{w}).$$

Proof. Take an arbitrary hybrid-open $\mathcal{U} \subset \mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$. As a consequence of Lemmas 4.10, 4.11 and 4.12, any path with $\mathcal{I}_{(0,T)}(\rho,\mathsf{w}) < \infty$ can be approximated by paths in the set $\mathcal{A}$ from (18) such that the rate functional also converges. In particular, due to this denseness, the set $\mathcal{U}\cap\mathcal{A}$ is never empty. Combining this approximation with Lemma 4.9 yields

$$\frac1n \log \mathbb{P}^{(n)}(\mathcal{U}) \ge -\inf_{(\rho,\mathsf{w})\in\mathcal{U}\cap\mathcal{A}} \mathcal{I}_{(0,T)}(\rho,\mathsf{w}) = -\inf_{(\rho,\mathsf{w})\in\mathcal{U}} \mathcal{I}_{(0,T)}(\rho,\mathsf{w}).$$

□

4.4 Upper bound

We now prove the large-deviation weak upper bound via a standard covering technique.

Proposition 4.14. *For any compact set $\mathcal{K} \subset \mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ and any $r > 0$,*

$$\liminf_{n\to\infty} \frac1n \log \mathbb{P}^{(n)}\big(\mathcal{K}\cap B_r\big) \le -\inf_{(\rho,\mathsf{w})\in\mathcal{K}\cap B_r} \mathcal{I}_{(0,T)}(\rho,\mathsf{w}).$$

where B_r is the bounded-variation ball (16)*.*

Proof. Fix an $\epsilon > 0$, and observe that because of (13) one can find for any $(\rho, \mathsf{w}) \in \mathrm{BV}\big(0,T;\mathcal{P}(I)\times \ell^1(I\times I)\big)$ a $\zeta[\rho,\mathsf{w}] \in C_c^2\big([0,T);\ell^\infty(I\times I)\big)$ such that $G(\rho,\dot{\mathsf{w}},\zeta[\rho,\mathsf{w}]) \geq \mathcal{I}_{(0,T)}(\rho,\mathsf{w}) - \epsilon$. Now define for each $(\rho,\mathsf{w}) \in \mathrm{BV}\big(0,T;\mathcal{P}(I)\times\ell^1(I\times I)\big)$ the sets $\mathcal{V}_\epsilon(\rho,\mathsf{w}) := \Big\{(\tilde\rho,\tilde{\mathsf{w}}) \in \mathrm{BV}_{\mathrm{flux}} : G(\tilde\rho,\tilde{\mathsf{w}},\zeta[\rho,\mathsf{w}]) > G(\rho,\dot{\mathsf{w}},\zeta[\rho,\mathsf{w}]) - \epsilon\Big\}$, which are open by Lemma 4.5. Surely $\bigcup_{(\rho,\mathsf{w})\in\mathcal{K}} \mathcal{V}_\epsilon(\rho,\mathsf{w}) \supset \mathcal{K}$ and so by compactness there exists a finite subcovering $\cup_{k=1}^K \mathcal{V}_\epsilon(\rho^{(k)},\mathsf{w}^{(k)}) \supset \mathcal{K}$. We can then use Lemma 4.7 together with the fact that $G_r = G$ on B_r, to estimate for every $k = 1,\dots,K$,

$$
\begin{aligned}
\limsup_{n\to\infty}\frac1n \log \mathbb{P}^{(n)}\big(\mathcal{V}_\epsilon(\rho^{(k)},\mathsf{w}^{(k)})\cap B_r\big) &\leq \limsup_{n\to\infty}\frac1n \underbrace{\log \mathbb{P}^{(n)}_{e^{\zeta[\rho^{(k)},\mathsf{w}^{(k)}]},r}\big(\mathcal{V}_\epsilon(\rho^{(k)},\mathsf{w}^{(k)})\cap B_r\big)}_{\leq 0} \\
&\qquad + \frac1n \log \mathop{\mathbb{P}^{(n)}\text{-}\operatorname{ess\,sup}}_{(\tilde\rho,\tilde{\mathsf{w}})\in\mathcal{V}_\epsilon(\rho^{(k)},\mathsf{w}^{(k)})\cap B_r} \frac{d\mathbb{P}^{(n)}}{d\mathbb{P}^{(n)}_{e^{\zeta[\rho^{(k)},\mathsf{w}^{(k)}]},r}}(\tilde\rho,\tilde{\mathsf{w}}) \\
&\leq -\inf_{(\tilde\rho,\tilde{\mathsf{w}})\in\mathcal{V}_\epsilon(\rho^{(k)},\mathsf{w}^{(k)})\cap B_r} G_r(\tilde\rho,\tilde{\mathsf{w}},\zeta[\rho^{(k)},\mathsf{w}^{(k)}]) \\
&= -\inf_{(\tilde\rho,\tilde{\mathsf{w}})\in\mathcal{V}_\epsilon(\rho^{(k)},\mathsf{w}^{(k)})\cap B_r} G(\tilde\rho,\tilde{\mathsf{w}},\zeta[\rho^{(k)},\mathsf{w}^{(k)}]) \\
&\leq -G(\rho^{(k)},\dot{\mathsf{w}}^{(k)},\zeta[\rho^{(k)},\mathsf{w}^{(k)}])\chi_{(\rho^{(k)},\mathsf{w}^{(k)})}(B_r) + \epsilon \\
&\leq -\mathcal{I}_{(0,T)}(\rho^{(k)},\mathsf{w}^{(k)})\chi_{(\rho^{(k)},\mathsf{w}^{(k)})}(B_r) + 2\epsilon,
\end{aligned}
$$

with the usual characteristic function $\chi_{(\rho^{(k)},\mathsf{w}^{(k)})}(B_r) = 0$ whenever $(\rho^{(k)},\mathsf{w}^{(k)}) \in B_r$ and ∞ otherwise. Due to the finiteness of the covering one can use the Laplace Principle to get:

$$
\begin{aligned}
\limsup_{n\to\infty}\frac1n \log \mathbb{P}^{(n)}(\mathcal{K}\cap B_r) &\leq \max_{k=1,\dots,K}\limsup_{n\to\infty}\frac1n\log\mathbb{P}^{(n)}\big(\mathcal{V}_\epsilon(\rho^{(k)},\mathsf{w}^{(k)})\cap B_r\big) \\
&\leq \max_{k=1,\dots,K} -\mathcal{I}_{(0,T)}(\rho^{(k)},\mathsf{w}^{(k)})\chi_{(\rho^{(k)},\mathsf{w}^{(k)})}(B_r) + 2\epsilon \\
&\leq -\inf_{(\rho,\mathsf{w})\in\mathcal{K}\cap B_r} \mathcal{I}_{(0,T)}(\rho^{(k)},\mathsf{w}^{(k)}) + 2\epsilon.
\end{aligned}
$$

Since ϵ was arbitrary, this proves the claim. □

Corollary 4.15 (Large-deviation weak upper bound)**.** *For any compact set* $\mathcal{K} \subset \mathrm{BV}\big(0,T;\mathcal{P}(I)\times\ell^1(I\times I)\big)$,

$$
\liminf_{n\to\infty}\frac1n\log\mathbb{P}^{(n)}(\mathcal{K}) \leq -\inf_{(\rho,\mathsf{w})\in\mathcal{K}} \mathcal{I}_{(0,T)}(\rho,\mathsf{w}).
$$

Proof. This is a consequence of the exponential tightness Proposition 4.6 and Proposition 4.14, as follows. Note that the balls B_r are defined as subsets of $\mathrm{BV}_{\mathrm{flux}}$, and that $\mathcal{I}_{(0,T)}|_{\mathrm{BV}^{\mathsf{c}}_{\mathrm{flux}}} = \infty$. For any $\eta > 0$ and $r := \eta + T\,e|Q|_1$ we can apply Laplace's principle:

$$
\begin{aligned}
\limsup_{n\to\infty}\frac1n\log\mathbb{P}^{(n)}(\mathcal{K}) &\leq \limsup_{n\to\infty}\frac1n\log\big(\mathbb{P}^{(n)}(\mathcal{K}\cap B_r) + \mathbb{P}^{(n)}(B_r^{\mathsf{c}})\big) \\
&\leq -\big(\inf_{\mathcal{K}\cap B_r}\mathcal{I}_{(0,T)}\wedge\eta\big) \leq -\big(\inf_{\mathcal{K}}\mathcal{I}_{(0,T)}\wedge\eta\big).
\end{aligned}
$$

Since η was arbitrary the claim follows. □

4.5 Coupled large deviations

We can now use the conditional large deviations to prove the coupled large deviations.

Proof of Theorem 4.2. The coupled probabilities can be disintegrated as

$$\begin{aligned}&\mathrm{Prob}\big(\rho^{(n)}, \mathsf{W}^{(n)} \in d\rho\, d\mathsf{w}\big)\\ &\qquad = \int \mathrm{Prob}\big(\rho^{(n)}, \mathsf{W}^{(n)} \in d\rho\, d\mathsf{w} \mid \rho^{(n)}(0) = \rho(0)\big)\, \mathrm{Prob}\big(\rho^{(n)}(0) \in d\rho(0)\big),\end{aligned}$$

where initial probabilities satisfy a large-deviation principle with rate $\mathcal{I}_0$, and the conditional probabilities satisfy a large-deviation principle with rate $\mathcal{I}_{(0,T)}$ whenever $\rho(0) > 0$. Therefore, apart from the condition $\rho(0) > 0$, we can immediately apply [Big04] to get the large deviations for the coupled system. Observe that the condition $\rho(0) > 0$ is not needed in the conditional upper bound, Corollary 4.15, and hence by [Big04] the coupled large-deviation upper bound holds. However, for the lower bound we only get for any hybrid-open set $\mathcal{U} \subset \mathrm{BV}\big(0, T; \mathcal{P}(I) \times \mathfrak{l}^1(I \times I)\big)$ that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}^{(n)}(\mathcal{U}) \geq - \inf_{\substack{(\rho,\mathsf{w})\in\mathcal{U}:\\ \rho(0)>0}} \mathcal{I}_0\big(\rho(0)\big) + \mathcal{I}_{(0,T)}(\rho, \mathsf{w}).$$

In order to replace the infimum above by $-\inf_{(\rho,\mathsf{w})\in\mathcal{U}} \mathcal{I}_0\big(\rho(0)\big) + \mathcal{I}_{(0,T)}(\rho,\mathsf{w})$, we need to show that any $(\rho, \mathsf{w}) \in \mathrm{BV}\big(0, T; \mathcal{P}(I) \times \mathfrak{l}^1(I \times I)\big)$ can be approximated by trajectories $(\rho^{(\epsilon)}, \mathsf{w}^{(\epsilon)})$ with $\rho^{(\epsilon)}(0) > 0$ such that $\mathcal{I}_{(0,T)}(\rho^{(\epsilon)}, \mathsf{w}^{(\epsilon)}) \to \mathcal{I}_{(0,T)}(\rho, \mathsf{w})$.

Take an arbitrary $(\rho, \mathsf{w}) \in \mathrm{BV}\big(0, T; \mathcal{P}(I) \times \mathfrak{l}^1(I \times I)\big)$ where $\rho(0)$ lies on the boundary $\{\rho \in \mathcal{P}(I) : \rho_i = 0 \text{ for some } i \in I\}$. We apply two approximations. The first approximation is similar to (20) but slightly different since $\rho(0)$ might be zero:

$$\begin{aligned}\mathsf{w}^{(\epsilon)}(t) &:= (\mathsf{w} * \theta^{(\epsilon)})(t) - (\mathsf{w} * \theta^{(\epsilon)})(0),\\ \hat\rho^{(\epsilon)}(t) &:= (\rho * \theta^{(\epsilon)})(0) - \mathrm{div}\, \mathsf{w}^{(\epsilon)}(t) = (\rho * \theta^{(\epsilon)})(t),\end{aligned}$$

where as before we extend the functions constantly outside $(0, T)$. By the same argument as in Lemma 4.11 we have $\mathcal{I}_{(0,T)}(\hat\rho^{(\epsilon)}, \mathsf{w}^{(\epsilon)}) \to \mathcal{I}_{(0,T)}(\rho, \mathsf{w})$; we will use this below. Moreover by the assumed continuity on the boundary we also have $\mathcal{I}_0\big(\hat\rho^{(\epsilon)}(0)\big) = \mathcal{I}_0\big((\rho * \theta^{(\epsilon)})(0)\big) \to \mathcal{I}_0\big(\rho(0)\big)$.

For the second approximation, let $\hat i \in I$ be such that $\rho_{\hat i}(0) > 0$. For this coordinate, we have, similarly to (21), the lower bound $\hat\rho^{(\epsilon)}_{\hat i}(t) \geq \theta^{(\epsilon)}(T)\|\rho_{\hat i}\|_{L^1(0,T)} > 0$. We can then define

$$\begin{aligned}\rho^{(\epsilon)}(t) &:= \rho^{(\epsilon)}(0) - \mathrm{div}\, \mathsf{w}^{(\epsilon)}(t), \qquad \text{where}\\ \rho^{(\epsilon)}_i(0) &:= \begin{cases} \hat\rho^{(\epsilon)}_{\hat i}(0) - \epsilon\theta^{(\epsilon)}(T)\|\rho_{\hat i}\|_{L^1(0,T)}, & i = \hat i,\\ \hat\rho^{(\epsilon)}_i(0) + \frac{\epsilon}{|I|-1}\theta^{(\epsilon)}(T)\|\rho_{\hat i}\|_{L^1(0,T)}, & i \neq \hat i.\end{cases}\end{aligned}$$

Observe that by construction $\rho^{(\epsilon)}(0) > 0$ and $\rho^{(\epsilon)}(t)$ remains positive. Moreover $\rho^{(\epsilon)}(0) \to \rho(0)$ and hence $\mathcal{I}_0\big(\rho^{(\epsilon)}(0)\big) \to \mathcal{I}_0\big(\rho(0)\big)$. To prove convergence of $\mathcal{I}_{(0,T)}(\rho^{(\epsilon)})$, we exploit the lower semicontinuity as in (22); hence we only need to prove an upper bound. For any pair $i \neq \hat i, j$ we get

$$\begin{aligned}&\int_0^T \Big[\dot{\mathsf{w}}^{(\epsilon)}_{ij}(t) \log \frac{\dot{\mathsf{w}}^{(\epsilon)}_{ij}(t)}{\underbrace{\rho^{(\epsilon)}_i(t)}_{\geq \hat\rho^{(\epsilon)}_i(t)}\, Q_{ij}} - \dot{\mathsf{w}}^{(\epsilon)}_{ij}(t) + \rho^{(\epsilon)}_i(t) Q_{ij}\Big]\, dt\\ &\quad \leq \int_0^T \Big[\dot{\mathsf{w}}^{(\epsilon)}_{ij}(t) \log \frac{\dot{\mathsf{w}}^{(\epsilon)}_{ij}(t)}{\hat\rho^{(\epsilon)}_i(t) Q_{ij}} - \dot{\mathsf{w}}^{(\epsilon)}_{ij}(t) + \hat\rho^{(\epsilon)}_i(t) Q_{ij}\Big]\, dt + \underbrace{\frac{T\epsilon}{(|I|-1)}\theta^{(\epsilon)}(T)\|\rho_{\hat i}\|_{L^1(0,T)} Q_{ij}}_{\to 0}.\end{aligned}$$

On the other hand, for any pair $\hat{i} \neq j$,

$$\begin{aligned}
&\int_0^T \Big[\dot{\mathsf{w}}^{(\epsilon)}_{\hat{i}j}(t)\log\frac{\dot{\mathsf{w}}^{(\epsilon)}_{\hat{i}j}(t)}{\rho^{(\epsilon)}_{\hat{i}}(t)Q_{\hat{i}j}} - \dot{\mathsf{w}}^{(\epsilon)}_{\hat{i}j}(t) + \rho^{(\epsilon)}_{\hat{i}}(t)Q_{\hat{i}j}\Big]\,dt\\
&\quad= \int_0^T \Big[\dot{\mathsf{w}}^{(\epsilon)}_{\hat{i}j}(t)\log\frac{\dot{\mathsf{w}}^{(\epsilon)}_{\hat{i}j}(t)}{\hat{\rho}^{(\epsilon)}_{\hat{i}}(t)Q_{\hat{i}j}} - \dot{\mathsf{w}}^{(\epsilon)}_{\hat{i}j}(t) + \hat{\rho}^{(\epsilon)}_{\hat{i}}(t)Q_{\hat{i}j}\\
&\qquad\qquad - \epsilon\theta^{(\epsilon)}(T)\|\rho_{\hat{i}}\|_{L^1(0,T)}Q_{\hat{i}j} - \dot{\mathsf{w}}^{(\epsilon)}_{\hat{i}j}(t)\log\Big(1-\frac{\epsilon\theta^{(\epsilon)}(T)\|\rho_{\hat{i}}\|_{L^1(0,T)}}{\underbrace{\hat{\rho}^{(\epsilon)}_{\hat{i}}(t)}_{\geq\theta^{(\epsilon)}(T)\|\rho_{\hat{i}}\|_{L^1(0,T)}}}\Big)\Big]\,dt.
\end{aligned}$$

Putting both bounds together yields

$$\limsup_{\epsilon\to0}\mathcal{I}_{(0,T)}(\rho^{(\epsilon)},\mathsf{w}^{(\epsilon)}) \leq \limsup_{\epsilon\to0}\mathcal{I}_{(0,T)}(\hat{\rho}^{(\epsilon)},\mathsf{w}^{(\epsilon)}) = \mathcal{I}_{(0,T)}(\rho,\mathsf{w}),$$

which was to be proven. □

In particular, as a consequence of Theorem 4.2 and Sanov's Theorem, we have:

Corollary 4.16. *Let $\rho^{(n)}(0)=\frac{1}{n}\sum_{k=1}^n \mathbb{1}_{X_k}$ where $X_1,\dots,X_n$ are i.i.d. according to $\pi\in\mathcal{P}(I)$, the unique invariant measure for the Markov chain with generator Q. Then $(\rho^{(n)},\mathsf{W}^{(n)})$ satisfies a large-deviation principle in $\mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$ with the hybrid topology, with good rate functional $\mathcal{S}\big(\rho(0)|\pi\big)+\mathcal{I}_{(0,T)}(\rho,\mathsf{w})$.*

Remark 4.17. The dynamic large deviations could also be proven by first proving the large deviations of the empirical measure on the path space, and then contracting to the path of the empirical measure-flux pair. This is the strategy used in [Fen94, Léo95] to prove large deviations for the empirical measure. □

5 Implications for Macroscopic Fluctuation Theory

Using the large-deviation principle that we proved in the previous section, we now explore which parts of MFT can be applied to the setting of this paper.

5.1 Time-reversal symmetry for the flux

Usually in MFT, time is reversed simply by replacing time t by $T-t$. However this would turn non-decreasing fluxes into non-increasing ones, which are non-feasible as we consider specific rather than net fluxes. The solution is evident: whenever a particle jumps from state i to j at some time t, a particle should jump from j to i at time $T-t$ for the reversed path. More precisely, we define, for any $(\rho,\mathsf{w})\in\mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$,

$$(\theta\rho)(t) := \rho(T-t) \qquad \text{and} \qquad (\theta\mathsf{w})_{ij}(t) := \mathsf{w}_{ij}(T) - \mathsf{w}_{ji}(T-t) \quad \text{for all } i,j\in I.$$

Observe that the continuity equation is indeed invariant under this operation:

$$\partial_t(\theta\rho)(t) + \operatorname{div}\partial_t(\theta\mathsf{w})(t) = -\dot{\rho}(T-t) + \operatorname{div}\dot{\mathsf{w}}^{\mathsf{T}}(T-t) = -\dot{\rho}(T-t) - \operatorname{div}\dot{\mathsf{w}}(T-t) = 0.$$

Let us from now on assume that the chain is irreducible, with a unique, (coordinate-wise) positive invariant measure $\pi > 0$. The time-reversed process $(\theta\rho^{(n)}, \theta\mathsf{W}^{(n)})$ is then a Markov process with generator

$$(\overleftarrow{\mathcal{Q}}^{(n)}\phi)(\rho,\mathsf{w}) := n\sum\sum_{i\neq j}\frac{\rho_i}{\pi_i}Q_{ji}\pi_j\big[\phi(\rho^{i\to j},\mathsf{w}^{ij+})-\phi(\rho,\mathsf{w})\big], \tag{23}$$

and we denote by $\overleftarrow{\mathcal{I}}_{[0,T]}$ the corresponding large-deviation rate functional, defined as in (3) where Q_{ij} is replaced by $(\frac{1}{\pi}\otimes Q^\mathsf{T}\otimes\pi)_{ij} := \frac{1}{\pi_i}Q_{ji}\pi_j$. Let us assume that $\rho^{(n)}(0) = \frac{1}{n}\sum_{k=1}^n \mathbb{1}_{X_k}$ where $X_1,\dots,X_n$ are i.i.d. according to the stationary measure $\pi\in\mathcal{P}(I)$. Then $\theta\rho^{(n)}(0)$ has the same distribution, and so for any hybrid-measurable $\mathcal{A}\subset \mathrm{BV}\big(0,T;\mathcal{P}(I)\times\mathfrak{l}^1(I\times I)\big)$,

$$\mathrm{Prob}\big((\rho^{(n)},\mathsf{W}^{(n)})\in\mathcal{A}\big) = \mathrm{Prob}\big((\theta\rho^{(n)},\theta\mathsf{W}^{(n)})\in\theta\mathcal{A}\big).$$

By Corollary 4.16 this implies the time-reversal symmetry:

$$\mathcal{S}\big(\rho(0)|\pi\big) + \mathcal{I}_{(0,T)}(\rho,\mathsf{w}) = \mathcal{S}\big(\rho(T)|\pi\big) + \overleftarrow{\mathcal{I}}_{[0,T]}(\theta\rho,\theta\mathsf{w}). \tag{24}$$

Symmetries of this type are fundamental to MFT, and we see that with the right choice of the time-reveral operator θ the symmetry is also valid when using specific rather than net fluxes. One should however be aware that we only assumed an invariant measure for $\rho^{(n)}$, and not for the flux $\mathsf{W}^{(n)}$. Indeed if the density is in equilibrium, net fluxes are always zero, whereas specific fluxes will continue to increase.

Since $\overleftarrow{Q}$ is known, relation (24) can also be calculated directly:

$$\begin{aligned}\mathcal{I}_{(0,T)}(\rho,\mathsf{w}) - \overleftarrow{\mathcal{I}}_{[0,T]}(\theta\rho,\theta\mathsf{w}) &= \int_0^T \mathcal{S}\big(\dot{\mathsf{w}}(t)|\rho(t)\otimes Q\big)\,dt - \int_0^T \mathcal{S}\big(\dot{\mathsf{w}}^\mathsf{T}(t)\big|\frac{\rho(t)}{\pi}\otimes Q^\mathsf{T}\otimes\pi\big)\,dt\\
&= \sum_i\int_0^T\Big[\underbrace{\sum_{j\neq i}(\dot{\mathsf{w}}_{ij}(t)-\dot{\mathsf{w}}_{ji}(t))}_{=-\operatorname{div}_i\mathsf{w}(t)=\dot\rho_i(t)}\log\frac{\rho_i(t)}{\pi_i} + \rho_i(t)\underbrace{\sum_{j\neq i}(Q_{ij}-\frac{1}{\pi_i}Q_{ji}\pi_j)}_{=0}\Big]\,dt\\
&= \mathcal{S}(\rho_T|\pi) - \mathcal{S}(\rho_0|\pi).\end{aligned} \tag{25}$$

In the case of more general particle systems, relation (24) still holds, and can be exploited to characterise the reversed hydrodynamics. Consider for example a system with mean-field interaction, where a particle jumps from state i to j with rate $\mathsf{k}_{ij}(\rho^{(n)})$ that depends on the empirical measure/mean field. The large-deviation rate for the pair $(\rho^{(n)}, W^{(n)})$ is then [PR17]

$$\mathcal{I}_{(0,T)}(\rho,\mathsf{w}) = \int_0^T \mathcal{S}\big(\dot{\mathsf{w}}(t)|\mathsf{k}(\rho(t))\big)\,dt,$$

whenever $(\rho,\mathsf{w})\in W^{1,1}(0,T;\mathfrak{l}^1(I\times I))$ and (2a) holds. Now assume that there exists a unique invariant measure for $\rho^{(n)}$, with corresponding large-deviation rate $\mathcal{I}_0$ as in Corollary 4.16. Let us further assume that the time reversed process $(\theta\rho^{(n)},\theta\mathsf{W}^{(n)})$ is a Markov process with corresponding dynamical large-deviation rate

$$\overleftarrow{\mathcal{I}}_{[0,T]}(\rho,\mathsf{w}) = \int_0^T \mathcal{S}\big(\dot{\mathsf{w}}(t)|\overleftarrow{\mathsf{k}}(\rho(t))\big)\,dt,$$

for some unknown $\overleftarrow{k}$. Following [BDSG+15, Sec. II.C], we replace the time-interval by $(t-\epsilon,t+\epsilon)$ for some $t\in(0,T)$, divide (24) by ϵ and then let $\epsilon\to 0$, yielding

$$\begin{aligned}\nabla D\mathcal{I}_0\big(\rho(t)\big)\cdot\dot{\mathsf{w}}(t) = D\mathcal{I}_0\big(\rho(t)\big)\cdot\dot\rho(t) &= \mathcal{S}\big(\dot{\mathsf{w}}(t)|\mathsf{k}(\rho(t))\big) - \mathcal{S}\big(\dot{\mathsf{w}}^\mathsf{T}(t)|\overleftarrow{\mathsf{k}}(\rho(t))\big)\\
&= -\log\frac{\mathsf{k}(\rho(t))}{\overleftarrow{k}(\rho(t))}\cdot\dot{\mathsf{w}}(t) - \big|\mathsf{k}(\rho(t))\big|_1 + \big|\overleftarrow{\mathsf{k}}(\rho(t))\big|_1,\end{aligned} \tag{26}$$

using the discrete partial integration $\nabla\xi \cdot \mathsf{j} := -\xi \cdot \operatorname{div} \mathsf{j} = \xi \cdot \dot\rho$. Since (26) has to hold for all $\big(\rho(t), \dot{\mathsf{w}}(t)\big)$, the reversed rates have to satisfy the following two conditions for all ρ:

$$\nabla D\mathcal{I}_0(\rho) = \log \frac{\mathsf{k}(\rho)}{\overleftarrow{\mathsf{k}}(\rho)},$$
$$\big|\mathsf{k}(\rho)\big|_1 = \big|\overleftarrow{\mathsf{k}}(\rho)\big|_1.$$

The first condition is sometimes interpreted as a fluctuation-dissipation equation, see [BDSG$^+$15, Sec. II.C].

Remark 5.1. At least formally, the average flux $\bar{\mathsf{J}}^{(n)} := T^{-1}\int_0^T \dot{\mathsf{W}}^{(n)}(dt) = T^{-1}\mathsf{W}^{(n)}(T)$ satisfies a large-deviation principle as $n \to \infty$ and subsequently $T \to \infty$ with rate functional (or the lower semicontinuous regularisation thereof):

$$\bar{\mathsf{j}} \mapsto \lim_{T\to\infty} \inf_{\substack{\mathsf{w}\in W^{1,1}(0,T;\mathfrak{l}^1(I\times I)):\\ \mathsf{w}(T)/T=\bar{\mathsf{j}}}} \frac{1}{T}\mathcal{I}_{[0,T]}(\rho,\mathsf{w}), \qquad \rho(t) := \rho(0) + \operatorname{div}\mathsf{w},$$

and so-called a Galavotti-Cohen time-reversal symmetry also holds for this functional. This is beyond the scope of the current paper. □

5.2 Time-reversal symmetry for the measure

As briefly mentioned in the introduction, by a simple contraction principle we retrieve the large deviations of the empirical measure $\rho^{(n)}$ with rate functional

$$\mathcal{J}_{(0,T)}(\rho) = \inf_{\substack{\mathsf{w}\in \mathrm{BV}(0,T;\mathfrak{l}^1_+(I\times I)):\\ \dot\rho + \operatorname{div}\dot{\mathsf{w}}=0}} \mathcal{I}_{(0,T)}(\rho,\mathsf{w}),$$

see for example [Fen94, Léo95, SW95, PR16]. Note in particular that there are no additional assumptions needed on the jump rates $\rho\otimes Q$, like boundedness away from zero, reversibility or weak reversibility. Therefore this method has the potential to generalise some earlier large-deviation results, which we shall pursue in [PR17].

Following [BDSG$^+$15, Sec. IV.A], we find the time-reversal symmetry for the empirical measures by taking the infimum on both sides of (24),

$$\mathcal{S}\big(\rho(0)|\pi\big) + \mathcal{J}_{(0,T)}(\rho) = \mathcal{S}\big(\rho(T)|\pi\big) + \overleftarrow{\mathcal{J}}_{[0,T]}(\theta\rho).$$

Moreover, from this relation the quasipotential can be retrieved via (again, see [BDSG$^+$15, Sec. IV.A])

$$\mathcal{S}(\rho|\pi) = \inf_{\substack{\nu\in\mathrm{BV}(-\infty,0:\mathcal{P}(I))\\ \nu(-\infty)=\pi,\nu(0)=\rho}} \mathcal{J}_{(-\infty,0]}(\nu).$$

5.3 Flux decompositions

The advantage of working with specific fluxes is that the rate functional $\mathcal{I}_{(0,T)}$ has a nice explicit formula, from which we can deduce additional structure as in (25). Of course the specific fluxes encode more information, and by a straight-forward contraction one finds the large-deviation rate for the empirical measure-net flux pair $(\rho^{(n)}, \mathsf{W}^{(n)} - \mathsf{W}^{(n)\mathsf{T}})$:

$$\hat{\mathsf{w}} \mapsto \inf_{\substack{\mathsf{w}\in W^{1,1}(0,T;\mathfrak{l}^1(I\times I)):\\ \mathsf{w}-\mathsf{w}^{\mathsf{T}}=\hat{\mathsf{w}}}} \mathcal{I}_{(0,T)}(\rho,\mathsf{w}).$$

For more information, analysis and thermodynamic interpretation of this functional we refer to [BMN09].

Alternatively, if the non-negativity of the net fluxes is of importance, one could also split $\mathsf{W}^{(n)}_{ij} = [\mathsf{W}^{(n)}_{ij} \wedge \mathsf{W}^{(n)}_{ji}] + [\mathsf{W}^{(n)}_{ij} - \mathsf{W}^{(n)}_{ij} \wedge \mathsf{W}^{(n)}_{ji}]$, and apply contractions to find the large deviations for the cancelling fluxes and the net fluxes respectively.

The net fluxes can be decomposed even further. It is however unclear how to do this in a meaningful way. As common in MFT, we could decompose any flux into a gradient and a solenoidal part $\mathsf{w} = \mathsf{w}_\nabla + \mathsf{w}_{\mathrm{sol}}$, where $\mathsf{w}_\nabla = \nabla\xi$ for some $\xi \in \mathbb{R}^I$, and $\operatorname{div}\mathsf{w}_{\mathrm{sol}} = 0$. However, one usually exploits the quadratic structure of the rate functional to devise an *orthogonal* decomposition such that the rate functional split into two parts. In the discrete setting of this paper however, the rate functional $\mathcal{I}_{(0,T)}$ is entropic rather than quadratic. Therefore it remains an open question how a meaningful decomposition of the fluxes would look like.

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