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Quasi-optimality of a pressure-robust nonconforming finite element method for the Stokes problem

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Abstract

Nearly all classical inf-sup stable mixed finite element methods for the incompressible Stokes equations are not pressure-robust, i.e., the velocity error is dependent on the pressure. However, recent results show that pressure-robustness can be recovered by a non-standard discretization of the right hand side alone. This variational crime introduces a consistency error in the method which can be estimated in a straightforward manner provided that the exact velocity solution is sufficiently smooth. The purpose of this paper is to analyze the pressure-robust scheme with low regularity. The numerical analysis applies divergence-free H^1-conforming Stokes finite element methods as a theoretical tool. As an example, pressure-robust velocity and pressure a-priori error estimates will be presented for the (first order) nonconforming Crouzeix–Raviart element. A key feature in the analysis is the dependence of the errors on the Helmholtz projector of the right hand side data, and not on the entire data term. Numerical examples illustrate the theoretical results.

1 Introduction

Nearly all inf-sup stable mixed finite elements methods for the incompressible Stokes problem on shape-regular meshes (with constant $\nu > 0$)

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f}, \qquad \text{in } \Omega, \\ -\nabla\cdot \boldsymbol{u} &= 0, \qquad \text{in } \Omega \end{aligned}$$

relax the divergence constraint and, as a result, their a priori velocity error estimates have the form

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,h} \le C\big(\inf_{\boldsymbol{w}_h\in \boldsymbol{X}_h} \|\boldsymbol{u}-\boldsymbol{w}_h\|_{1,h} + \nu^{-1} \inf_{q_h\in Q_h} \|p-q_h\|\big)$$

with a constant $C = \mathcal{O}(1) \ge 1$ independent of h, ν, and ($\boldsymbol{u}$,p) [14, 7, 24]. Here, $\boldsymbol{X}_h$ denotes the space of discrete velocity functions, Q_h denotes the space of discrete pressure functions, $\|\cdot\|_{1,h}$ denotes some (possibly discrete) H^1 norm, and $\|\cdot\|$ denotes the L^2 norm. While such discretization schemes are relatively popular, they may not be the best possible ones from a qualitative point of view. Indeed, it is possible to construct inf-sup stable, H^1-conforming schemes, which fulfill an a priori velocity error estimate of the form

$$\|\nabla \boldsymbol{u} - \nabla \boldsymbol{u}_h\| \le C \inf_{\boldsymbol{w}_h\in \tilde{\boldsymbol{X}}_h} \|\nabla \boldsymbol{u} - \nabla \boldsymbol{w}_h\|,$$

with some (possibly different) constant $C = \mathcal{O}(1) \ge 1$, and some $\boldsymbol{H}^1$-conforming discrete velocity space $\tilde{\boldsymbol{X}}_h$. Such schemes, which do not relax the divergence-free constraint, are called *divergence-free* [43, 42, 39], and require the identity $\nabla\cdot\tilde{\boldsymbol{X}}_h = Q_h$; they have become — modestly — popular only very recently [44, 17, 18, 45, 37, 35, 16, 8, 32, 10, 30, 11, 38, 3, 29].

The main advantage of divergence-free schemes is that they are *pressure-robust* [19, 27, 26, 28], i.e., their velocity error is independent of the pressure. Classical inf-sup stable schemes guarantee a small velocity error whenever the velocity $\boldsymbol{u}$ and the scaled pressure $\frac{1}{\nu}p$ can be accurately approximated on a given regular finite element mesh. Numerical errors of classical mixed methods that arise in such a case are often called *poor mass conservation* [19, 13, 2, 25, 23]. Instead, *pressure-robust* schemes guarantee a small velocity error whenever *the velocity $\boldsymbol{u}$ alone* can be accurately approximated. Even further, for many pressure-robust schemes, it was recently proven that even some *discrete* a-priori pressure estimates can be pressure-independent. In such cases one can show that the difference of the discrete pressure to the best approximation [28, 19] or some projection [21] of the continuous pressure is only velocity-dependent.

Quite recently it was realized that the pressure-dependence of the velocity error of inf-sup stable Stokes discretizations is due to a lack of L^2 orthogonality of gradient fields and discretely divergence-free *velocity test functions* [24].

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This means that nearly all inf-sup stable Stokes discretizations can be made *pressure-robust* [24, 28, 21, 20, 1] by only replacing the standard discretization of the *right hand side*

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \, dx \to \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\pi}^F(\boldsymbol{v}_h) \, dx,$$

while the *Stokes stiffness matrix* remains unchanged. Here, $\boldsymbol{\pi}^F$ is an appropriate *velocity reconstruction operator* that approximates discretely-divergence-free test functions by divergence-free ones in the sense of $\boldsymbol{H}(\mathrm{div})$.

Using a non-standard velocity test function in the discretization of the right hand side introduces a variational crime and a consistency error [24]. Classical estimates of the consistency error require a minimal regularity of $\boldsymbol{u} \in \boldsymbol{H}^{1+s}(\Omega)$ with $s > 1/2$ in order for edge integrals to be defined. For the nonconforming Crouzeix–Raviart element with the standard BDM_1 interpolator as velocity reconstruction operator $\boldsymbol{\pi}^F := \boldsymbol{\pi}^{\mathrm{BDM}}$, such classical arguments deliver optimal error estimates [4]. However, the behavior of the consistency error in the case of a low regularity $s \leq 1/2$ is not addressed. This question seems to be important, since assuming $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and assuming homogeneous Dirichlet velocity boundary conditions for a (polygonal) slit domain yields a velocity regularity $\boldsymbol{u} \in \boldsymbol{H}^{3/2-\epsilon}(\Omega)$. Further, assuming different kinds of boundary conditions such a low regularity is typical [12].

New ideas have recently been proposed to handle consistency errors in the case of low regularity [15, 31, 22]. In the paper [31] the consistency error of nonconforming finite element methods for scalar diffusion equations is represented as a Céa-lemma like term and a data oscillation term that vanishes with optimal order. This estimate is performed using some finite element interpolation operator that maps nonconforming finite element functions to $\boldsymbol{H}^1$ conforming ones. In [22] this approach is extended to classical inf-sup stable discretizations of the incompressible Stokes problem, which are not pressure-robust.

In this contribution, we will now extend the (scalar) approach of [31] to the *pressure-robust* modification of the Crouzeix–Raviart Stokes element using the velocity construction operator $\boldsymbol{\pi}^F = \boldsymbol{\pi}^{\mathrm{BDM}}$. The main challenge is to avoid any pressure-dependent terms in the estimate of the consistency error. Moreover, the data (oscillation) term should not depend on $\boldsymbol{f}$, but only on its Helmholtz projection $\mathbb{P}(\boldsymbol{f})$, i.e., its divergence-free part, since the irrotational part of $\boldsymbol{f}$ corresponds to the pressure gradient ∇p [24]. These goals will be achieved by constructing some finite element interpolation operator that maps nonconforming discretely-divergence-free Crouzeix–Raviart finite element functions to divergence-free $\boldsymbol{H}^1$ conforming vector fields. The approach exploits recent progress on the construction of divergence-free, inf-sup stable mixed methods for the Stokes equations and uses rational bubble functions [17, 18, 19]. A preliminary version of this contribution was presented in F. Neumann's master thesis [36].

2 Preliminaries

Let $\Omega \in \mathbb{R}^d$ with $d \in \{2,3\}$ be a domain with polyhedral boundary $\partial\Omega$. Slit domains are explicitly allowed. We denote by $(\cdot,\cdot)_D$ the L^2 inner product over a d-dimensional domain $D \subset \Omega$, and drop the subscript in the case $D = \Omega$. The L^2 inner product over a k-dimensional domain D, with $k < d$, is denoted by $\langle \cdot,\cdot \rangle_D$. The L^2 norm over D is denoted by $\|\cdot\|_D$, and again, we drop the subscript if $D = \Omega$. For a number $m > 0$, we denote by $\|\cdot\|_m$ the H^m norm over Ω.

We consider the steady incompressible Stokes equations with homogeneous boundary conditions to be our model problem

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f}, \text{ in } \Omega, \\ -\nabla \cdot \boldsymbol{u} &= 0, \text{ in } \Omega, \\ \boldsymbol{u} &= 0, \text{ on } \partial\Omega. \end{aligned} \tag{2.1}$$

Here, we assume $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega) := L^2(\Omega)^d$ and $\nu > 0$ denotes the kinematic viscosity. Introducing trial and test spaces $\boldsymbol{X} := \boldsymbol{H}^1_0(\Omega) := H^1_0(\Omega)^d, Q := L^2_0(\Omega)$ and bilinear forms

$$a(\boldsymbol{u},\boldsymbol{v}) = \nu \int_\Omega \nabla\boldsymbol{u} \colon \nabla\boldsymbol{v} \, \mathrm{d}x, \qquad b(\boldsymbol{u},q) = -\int_\Omega q(\nabla\cdot\boldsymbol{u}) \, \mathrm{d}x,$$

the weak formulation of (2.1) reads: *Find* $(\boldsymbol{u}, p) \in \boldsymbol{X} \times Q$ *such that for all* $(\boldsymbol{v}, q) \in \boldsymbol{X} \times Q$ *it holds*

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= (\boldsymbol{f}, \boldsymbol{v}), \\ b(\boldsymbol{u}, q) &= 0. \end{aligned} \tag{2.2}$$

Over the space of weakly divergence-free functions

$$\boldsymbol{V} := \{\boldsymbol{v} \in \boldsymbol{X} : \, b(\boldsymbol{v}, q) = 0 \quad \forall q \in Q\}, \tag{2.3}$$

we can formulate (2.2) as an elliptic equation for the velocity alone: *Seek* $\boldsymbol{u} \in \boldsymbol{V}$ *such that for all* $\boldsymbol{v} \in \boldsymbol{V}$ *it holds*

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}). \tag{2.4}$$

In the following, we recall some fundamental properties of the Helmholtz decomposition and of the corresponding Helmholtz projector [40, 19]. First, we introduce the following space of divergence-free $\boldsymbol{L}^2$ vector fields

$$\boldsymbol{L}^2_\sigma(\Omega) = \{\boldsymbol{w} \in \boldsymbol{L}^2(\Omega) : -(\boldsymbol{w}, \nabla\phi) = 0 \text{ for all } \phi \in H^1(\Omega)\}. \tag{2.5}$$

Note that for a vector field $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$, the mapping $\phi \in C_0^\infty(\Omega) \to -(\boldsymbol{w}, \nabla\phi)$ denotes the distributional divergence of $\boldsymbol{w}$. Therefore, all vector fields in $\boldsymbol{L}^2_\sigma(\Omega)$ are weakly divergence-free. Further, it holds $\boldsymbol{w} \cdot \boldsymbol{n} = 0$ at the boundary of Ω. From the definition (2.5), one recognizes that all divergence-free, smooth vector fields with compact support belong to $\boldsymbol{L}^2_\sigma(\Omega)$. Indeed, $\boldsymbol{L}^2_\sigma(\Omega)$ is the topological closure of these vector fields with respect to the $\boldsymbol{H}(\mathrm{div})$-norm.

Theorem 2.1 (Helmholtz Decomposition)**.** Let $\Omega \subset \mathbb{R}^d$ be a polyhedral domain. Then, any vector field $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ can be uniquely decomposed into a gradient of a scalar potential $\phi \in H^1(\Omega)/\mathbb{R}$ and a divergence-free vector field $\boldsymbol{\psi} \in \boldsymbol{L}^2_\sigma(\Omega)$:

$$\boldsymbol{f} = \nabla\phi + \boldsymbol{\psi}. \tag{2.6}$$

Proof. For a given vector field $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ one defines the following (well-posed) problem: Find $\phi \in H^1(\Omega)/\mathbb{R}$ such that for all $\chi \in H^1(\Omega)/\mathbb{R}$ it holds

$$(\nabla\phi, \nabla\chi) = (\boldsymbol{f}, \nabla\chi),$$

which allows us to introduce $\boldsymbol{\psi} := \boldsymbol{f} - \nabla\phi \in \boldsymbol{L}^2(\Omega)$. Obviously, it holds $\boldsymbol{\psi} \in \boldsymbol{L}^2_\sigma(\Omega)$. Further, $\nabla\phi$ and $\boldsymbol{\psi}$ are orthogonal in $\boldsymbol{L}^2(\Omega)$ by the definition of $\boldsymbol{L}^2_\sigma(\Omega)$, thus implying the uniqueness of the Helmholtz decomposition. q.e.d.

Definition 2.2. For $\boldsymbol{f} = \nabla\phi + \boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega)$ with $\boldsymbol{\psi} \in \boldsymbol{L}^2_\sigma(\Omega)$, one defines $\mathbb{P}(\boldsymbol{f}) = \boldsymbol{\psi}$, i.e., $\mathbb{P}(\boldsymbol{f})$ is the divergence-free component of $\boldsymbol{f}$.

Remark 2.3. The most important property of the Helmholtz projector in the following is that it holds for all $\chi \in H^1(\Omega)$:

$$\mathbb{P}(\nabla\chi) = \boldsymbol{0},$$

which is a consequence of the uniqueness of the Helmholtz decomposition (2.6).

3 Discrete formulations

In this section, we introduce the standard nonconforming Crouzeix–Raviart finite element method for discretising the Stokes equations. To this end, we first require some notation. We denote by $\mathcal{T}_h$ a conforming, shape-regular, simplicial triangulation of Ω, and by $\mathcal{F}_h^I$ and $\mathcal{F}_h^B$ the set of $(d-1)$-dimensional interior faces and boundary faces, respectively. For a face $f \in \mathcal{F}_h := \mathcal{F}_h^I \cup \mathcal{F}_h^B$, we denote its barycenter by m_f and its diameter by h_f. For an element $T \in \mathcal{T}_h$, we denote by $\mathcal{F}_j(T)$ and h_T, the set of j-dimensional subsimplices of T, and the diameter of T, respectively. The set of interior and boundary j-dimensional subsimplices of T are denoted by $\mathcal{F}_j^I(T)$ and $\mathcal{F}_j^B(T)$, respectively. We denote the outward unit normal of a $(d-1)$-dimensional face $f \in \mathcal{F}_h$ by $\boldsymbol{n}_f$. For $f \in \mathcal{F}_j(T)$, let $\{F_i\}_{i=1}^{d-j} \subset \mathcal{F}_{d-1}(T)$ be the $(d-1)$-dimensional faces such that $f \subset \partial F_i$. We then set $\boldsymbol{n}_f^{(i)} = \boldsymbol{n}_{F_i}$, and note that $\{\boldsymbol{n}_f^{(i)}\}_{i=1}^{d-j}$ spans the orthogonal subspace of the tangent space of f.

For an interior face $f = \partial T_+ \cap \partial T_- \in \mathcal{F}_h^I$, we define the jump of a scalar or vector-valued function v on f by

$$[v]|_f = v_+|_f - v_-|_f, \quad v_\pm := v|_{T_\pm}.$$

For a boundary face $f = \partial T_+ \cap \partial\Omega \in \mathcal{F}_h^B$, we set $[v]|_f = v_+|_f$. In addition, for a j-dimensional simplex f, we define the average of v on f by

$$\{v\}_f = \frac{1}{|\mathcal{T}_f|} \sum_{T \in \mathcal{T}_f} v_T|_f,$$

where $\mathcal{T}_f \subset \mathcal{T}_h$ denotes the set of simplices that have f as a subsimplex, $|\mathcal{T}_f|$ is the cardinality of the set, and $v_T := v|_T$. In the case $j = d-1$, we shall omit the subscript, i.e., we write $\{v\} = \{v\}_f$ when $f \in \mathcal{F}_h$

The Crouzeix–Raviart finite element spaces are given by

$$\begin{aligned}
\mathrm{CR}(\Omega) &:= \{v_h \in \mathcal{P}_1(\mathcal{T}_h),\ v_h(m_f) \text{ is single–valued},\ f \in \mathcal{F}_h\},\\
\mathrm{CR}_0(\Omega) &:= \{v_h \in \mathrm{CR}(\Omega):\ v_h(m_f) = 0, \quad \forall f \in \mathcal{F}_h^B\},
\end{aligned}$$

where $\mathcal{P}_k(\mathcal{T}_h)$ with $k \in \mathbb{N}_+$ denotes the space of piecewise kth degree polynomials with respect to the partition $\mathcal{T}_h$. We set

$$\boldsymbol{X}_h := \mathrm{CR}_0(\Omega)^d, \quad Q_h := L_0^2(\Omega) \cap \mathcal{P}_0(\mathcal{T}_h),$$

and let ∇_h and $\nabla_h\cdot$ denote the piecewise gradient and piecewise divergence operators respectively, i.e.,

$$\begin{aligned}
\nabla_h \colon \boldsymbol{X}_h &\longrightarrow L^2(\Omega)^{d\times d}, \quad (\nabla_h \boldsymbol{v}_h)\big|_T = \nabla\big(\boldsymbol{v}_h\big|_T\big), && \forall T \in \mathcal{T}_h,\\
\nabla_h\cdot : \boldsymbol{X}_h &\longrightarrow L^2(\Omega), \qquad (\nabla_h \cdot \boldsymbol{v}_h)\big|_T = \nabla \cdot \big(\boldsymbol{v}_h\big|_T\big), && \forall T \in \mathcal{T}_h.
\end{aligned}$$

For the discrete analogs of the bilinear forms $a(\cdot,\cdot),\ b(\cdot,\cdot)$ we define $a_h(\cdot,\cdot),\ b_h(\cdot,\cdot)$ piecewise over each element $T \in \mathcal{T}_h$:

$$\begin{aligned}
a_h \colon \boldsymbol{X}_h \times \boldsymbol{X}_h &\longrightarrow \mathbb{R}, \quad a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) := \nu(\nabla_h \boldsymbol{u}_h, \nabla_h \boldsymbol{v}_h),\\
b_h \colon \boldsymbol{X}_h \times Q_h &\longrightarrow \mathbb{R}, \quad b_h(\boldsymbol{u}_h, q_h) := -(q_h, \nabla_h \cdot \boldsymbol{u}_h).
\end{aligned}$$

The classical discrete formulation of (2.2) reads as follows: *Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h \times Q_h$ such that for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{X}_h \times Q_h$ it holds*

$$\begin{aligned}
a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) &= (\boldsymbol{f}, \boldsymbol{v}_h),\\
b_h(\boldsymbol{u}_h, q_h) &= 0.
\end{aligned} \tag{3.1}$$

Over the space of discretely divergence-free functions,

$$\boldsymbol{V}_h := \big\{\boldsymbol{v}_h \in \boldsymbol{X}_h \colon b_h(\boldsymbol{v}_h, q_h) = 0, \quad \text{for all } q_h \in Q_h\big\},$$

problem (3.1) can be reformulated solely in terms of the velocity unknown: *Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ it holds*

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h). \tag{3.2}$$

The next two results are standard, and can be found, e.g., in [7] and [5], respectively.

Theorem 3.1. The Crouzeix–Raviart finite-element pair $(\boldsymbol{X}_h, Q_h)$ is inf-sup stable. There exists a constant $\beta^* > 0$ independent of h with

$$\inf_{q_h \in Q_h \setminus\{0\}} \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h \setminus \{0\}} \frac{|b_h(\boldsymbol{v}_h, q_h)|}{\|q_h\| \|\nabla_h \boldsymbol{v}_h\|} \geq \beta^* > 0.$$

We note that for the discrete inf-sup constant of the Crouzeix–Raviart element holds $\beta^* \geq \beta$, where β denotes the continuous inf-sup constant.

Lemma 3.2. There holds for all $\boldsymbol{v}_h \in \boldsymbol{X}_h$,

$$\sum_{f \in \mathcal{F}_h} h_f^{-1} \big\|[\boldsymbol{v}_h]\big\|_f^2 \leq C \|\nabla_h \boldsymbol{v}_h\|^2.$$

3.1 A pressure-robust Crouzeix–Raviart element via velocity reconstructions

As argued in [24, 4], the classical Crouzeix–Raviart element is only discretely divergence-free, as it relaxes the L^2-orthogonality against arbitrary gradient fields. This leads to error estimates which are not pressure-robust, and hence depend on the inverse viscosity $\nu^{-1} > 0$ and the irrotational part of the right-hand side $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$. Here we describe a relatively simple procedure that recovers pressure-robustness by mapping discretely divergence-free test functions to $\boldsymbol{L}^2_\sigma(\Omega)$.

Set $\boldsymbol{Y}_h = \boldsymbol{\mathcal{P}}_1(\mathcal{T}_h) \cap \boldsymbol{H}_0(\mathrm{div};\Omega)$ to be the lowest order Brezzi-Douglas-Marini space [6, 7], consisting of piecewise linear vector-valued functions. Here, $\boldsymbol{H}_0(\mathrm{div};\Omega)$ denotes the space of $\boldsymbol{L}^2(\Omega)$ functions with divergence in $L^2(\Omega)$, whose normal component vanishes on $\partial\Omega$. We recall that any $\boldsymbol{w}_h \in \boldsymbol{Y}_h$ is uniquely determined by the moments

$$\int_f \boldsymbol{w}_h \cdot \boldsymbol{n}_f q \,\mathrm{d}s \quad \forall q_h \in \mathcal{P}_1(f), \quad \forall f \in \mathcal{F}_h^I.$$

We define $\boldsymbol{\pi}^{\mathrm{BDM}} \colon \boldsymbol{X} + \boldsymbol{X}_h \longrightarrow \boldsymbol{Y}_h$ as the unique operator satisfying

$$\int_f (\boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}) \cdot \boldsymbol{n}_f q_h \,\mathrm{d}s = \int_f \{\boldsymbol{v} \cdot \boldsymbol{n}_f\} q_h \,\mathrm{d}s \quad \forall q_h \in \mathcal{P}_1(f),\ \forall f \in \mathcal{F}_h^I. \tag{3.3}$$

Lemma 3.3. There holds

$$\sum_{T \in \mathcal{T}_h} h_T^{-2} \|\boldsymbol{v} - \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}\|_T^2 \le C \|\nabla_h \boldsymbol{v}\|^2, \tag{3.4}$$

$$\|\nabla_h \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}\| \le C \|\nabla_h \boldsymbol{v}\| \tag{3.5}$$

for all $\boldsymbol{v} \in \boldsymbol{X} + \boldsymbol{X}_h$, and

$$\sum_{T \in \mathcal{T}_h} h_T^{-2(1+s)} \|\boldsymbol{v} - \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}\|_T^2 \le C \|\boldsymbol{v}\|_{1+s}, \tag{3.6}$$

for all $\boldsymbol{v} \in \boldsymbol{H}^{1+s}(\Omega) \cap \boldsymbol{H}^1_0(\Omega)$. Moreover, $\nabla \cdot \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v} \equiv 0$ for all $\boldsymbol{v} \in \boldsymbol{V} + \boldsymbol{V}_h$.

Proof. The proof of (3.4)–(3.6) in the case $\boldsymbol{v} \in \boldsymbol{X}$ can be found in [7].

Let $\boldsymbol{v}_h \in \boldsymbol{X}_h$ and set $\boldsymbol{v}_T := \boldsymbol{v}_h\big|_T$ for some $T \in \mathcal{T}_h$. Since the values $\boldsymbol{w}_h \cdot \boldsymbol{n}_f\big|_f$ $(f \in \mathcal{F}_{d-1}(T))$ uniquely determine any $\boldsymbol{w}_h \in \mathcal{P}_1(T)$, and since $(\boldsymbol{v}_h - \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h)\big|_T \in \mathcal{P}_1(T)$, a scaling argument and the shape–regularity of $\mathcal{T}_h$ show that

$$h_T^{-2} \|\boldsymbol{v}_h - \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h\|_T^2 \le C \sum_{f \in \mathcal{F}_{d-1}(T)} h_f^{-1} \|(\boldsymbol{v}_T - \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h) \cdot \boldsymbol{n}_f\|_f^2.$$

Because $(\boldsymbol{v}_T - \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h) \cdot \boldsymbol{n}_f = \pm\frac{1}{2} [\boldsymbol{v}_h \cdot \boldsymbol{n}_f]$ on $f \in \mathcal{F}_h^I$ and $(\boldsymbol{v}_T - \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h) \cdot \boldsymbol{n}_f = \boldsymbol{v}_T \cdot \boldsymbol{n}_f$ on $f \in \mathcal{F}_h^B$, we have by Lemma 3.2,

$$\sum_{T \in \mathcal{T}_h} h_T^{-2} \|\boldsymbol{v}_h - \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h\|_T^2 \le C \sum_{f \in \mathcal{F}_h} h_f^{-1} \|[\boldsymbol{v}_h]\|_f^2 \le C \|\nabla_h \boldsymbol{v}_h\|^2.$$

This proves (3.4). The stability estimate (3.5) follows from (3.4) and an inverse estimate.

Finally, let $\boldsymbol{v} \in \boldsymbol{V} + \boldsymbol{V}_h$ so that $\nabla_h \cdot \boldsymbol{v} \equiv 0$ and $\int_f \{\boldsymbol{v} \cdot \boldsymbol{n}_f\}\, ds = \int_f \boldsymbol{v} \cdot \boldsymbol{n}_f \, ds$ for all $f \in \mathcal{F}_h$. Then by the divergence theorem, we have for each $T \in \mathcal{T}_h$,

$$\int_T \nabla \cdot \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v} \, dx = -\int_{\partial T} (\boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}) \cdot \boldsymbol{n} \, ds = -\int_{\partial T} (\boldsymbol{v} \cdot \boldsymbol{n}) \, ds = \int_T \nabla \cdot \boldsymbol{v} \, dx = 0.$$

Thus, $\nabla \cdot \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v} \equiv 0$. q.e.d.

Inspired by the $\boldsymbol{L}^2$-orthogonality (2.5) of $\boldsymbol{L}^2_\sigma(\Omega)$ against all gradient fields, the following variational crime improves the $\boldsymbol{L}^2$-orthogonality of discretely divergence-free vector fields $\boldsymbol{v}_h \in \boldsymbol{V}_h$ against the irrotational part of $\boldsymbol{f}$ in the sense of the Helmholtz decomposition [24, 4]:

$$\begin{aligned} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) &= (\boldsymbol{f}, \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h) \qquad && \forall \boldsymbol{v}_h \in \boldsymbol{X}_h \\ b_h(\boldsymbol{u}_h, q_h) &= 0 && \forall q_h \in Q_h. \end{aligned} \tag{3.7}$$

Restricting (3.7) to $\boldsymbol{V}_h$ and applying the Berger–Scott–Strang lemma gives us the following abstract error estimate that decomposes the error into two parts: one that measures the interpolation error and another that measures the consistency error.

Lemma 3.4 (Berger–Scott–Strang)**.** Let $\boldsymbol{u} \in \boldsymbol{X}$ be the continuous solution of (2.2) and $\boldsymbol{u}_h \in \boldsymbol{V}_h$ the discretely divergence-free solution of (3.7). Then the error satisfies

$$\|\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)\| \leq \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\nabla_h(\boldsymbol{u} - \boldsymbol{v}_h)\| + \nu^{-1} \sup_{\boldsymbol{w}_h \in \boldsymbol{V}_h \setminus \{0\}} \frac{|C_h(\boldsymbol{u}, \boldsymbol{w}_h)|}{\|\nabla_h \boldsymbol{w}_h\|},$$

where the consistency error is given by $C_h(\boldsymbol{u}, \boldsymbol{w}_h) := a_h(\boldsymbol{u}, \boldsymbol{w}_h) - (\boldsymbol{f}, \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{w}_h)$.

4 Conforming and divergence-free element

In this section we present the conforming and divergence-free finite-element introduced in [17, 18]. These elements are constructed by enhancing a family of $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming elements with rational bubble functions such that they possess tangential continuity.

For a simplex $T \in \mathcal{T}_h$, let $\{a_i\}_{i=0}^d = \mathcal{F}_0(T)$ and $\{\lambda_i\}_{i=0}^d \subset \mathcal{P}_1(T)$ denote the vertices and barycentric coordinates of T, i.e., λ_i is the unique linear polynomial satisfying $\lambda_i(a_j) = \delta_{i,j}$. In two dimensions, we label the edges $\mathcal{F}_1(T) = \{e_i\}_{i=0}^2$ such that a_i is not a vertex of e_i. Likewise, in three dimensions, we label the faces $\mathcal{F}_2(T) = \{f_i\}_{i=0}^3$ such that a_i is not a vertex of f_i.

The edge/face bubble functions and volume bubble function are given by

$$b_i := \prod_{\substack{j=0 \\ j \neq i}}^d \lambda_j \in \mathcal{P}_d(T), \qquad b_T := \prod_{j=0}^d \lambda_j \in \mathcal{P}_{d+1}(T),$$

and the rational edge/face bubble functions are given by (mod d)

$$\begin{aligned} &B_i := b_T b_i / \prod_{j=1}^d (\lambda_i + \lambda_{i+j}), && \text{for } 0 \leq \lambda_i \leq 1,\ 0 \leq \lambda_{i+j} < 1, \\ &B_i(a_{i+j}) = 0, && \text{else.} \end{aligned}$$

We note that $B_i \in W^{2,\infty}(T)$, $B_i|_{\partial T} = 0$, $\nabla B_i|_{\partial T} = -|\nabla \lambda_i| b_i \boldsymbol{n}_{f_i}$ (cf. [17, 18] for details). In particular, the rational bubble functions and its derivatives reduce to polynomial functions on the boundary of each element.

We set $\boldsymbol{N}_{m-1}(T) := \{\boldsymbol{w}_h \in \boldsymbol{\mathcal{P}}_{m-1}(T) :\ \boldsymbol{w}_h \cdot \boldsymbol{x} \in \mathcal{P}_{m-1}(T)\}$ to be the (local) $\boldsymbol{H}(\mathbf{curl};\Omega)$ Nedelec space of index $m-1$ [33], and define the local space of divergence–free polynomials $(m \geq 1)$

$$\begin{aligned} \boldsymbol{Q}_m(T) := \big\{ \boldsymbol{v}_h \in \boldsymbol{\mathcal{P}}_m(T) :\ & (\boldsymbol{v}_h, \boldsymbol{\rho}_h)_T = 0,\ \forall \boldsymbol{\rho}_h \in \boldsymbol{N}_{m-1}(T) \text{ and} \\ & \langle \boldsymbol{v}_h \cdot \boldsymbol{n}_f, \kappa_h \rangle_f = 0\ \forall \kappa_h \in \mathcal{P}_{m-1}(f),\ f \in \mathcal{F}_{d-1}(T) \big\}. \end{aligned}$$

Note that $\nabla q_h \in \boldsymbol{N}_{m-1}(T)$ for $q_h \in \mathcal{P}_{m-1}(T)$, and therefore

$$\int_T (\nabla \cdot \boldsymbol{w}_h) q_h \,\mathrm{d}x = -\int_T \boldsymbol{w}_h \cdot \nabla q_h \,\mathrm{d}x + \int_{\partial T} (\boldsymbol{w}_h \cdot \boldsymbol{n}) q_h \,\mathrm{d}s = 0 \quad \forall \boldsymbol{w}_h \in \boldsymbol{Q}_m(T),\ q_h \in \mathcal{P}_{m-1}(T).$$

Thus, functions in $\boldsymbol{Q}_m(T)$ are divergence–free as claimed. Moreover, since any $\boldsymbol{v}_h \in \boldsymbol{\mathcal{P}}_m(T)$ is uniquely determined by the moments $(\boldsymbol{v}_h, \boldsymbol{\rho}_h)_T$ and $\langle(\boldsymbol{v}_h \cdot \boldsymbol{n}_f), \kappa_h\rangle_f$ for $\boldsymbol{\rho}_h \in \boldsymbol{N}_{m-1}(T)$ and $\kappa_h \in \mathcal{P}_m(f)$ (cf. [34]), we conclude that the dimension of $\boldsymbol{Q}_m(T)$ is $\dim \boldsymbol{Q}_m(T) = (d+1)\big(\dim \mathcal{P}_m(\mathbb{R}^{d-1}) - \dim \mathcal{P}_{m-1}(\mathbb{R}^{d-1})\big) = (d+1)\binom{m+d-2}{d-2}$. This discussion also shows that $\boldsymbol{Q}_m(T) \cap \boldsymbol{\mathcal{P}}_{m-1}(T) = \{\boldsymbol{0}\}$.

We set

$$\boldsymbol{M}_k(T) := \boldsymbol{\mathcal{P}}_k(T) \oplus_{j=1}^{d-1} \boldsymbol{Q}_{k+j}(T) \subset \boldsymbol{\mathcal{P}}_{k+d-1}(T) \tag{4.1}$$

to be the local $\boldsymbol{H}(\mathrm{div};T)$-conforming finite element space with continuity at the vertices introduced in [17, 18] (also see [9, 41])

To summarize the divergence-free finite element spaces constructed in [17, 18] we discuss the two and three dimensional cases separately.

4.1 Two-dimensional construction

This section summarizes the two-dimensional family of divergence-free (yielding) finite elements constructed in [17]. As a first step, for an integer $k \geq 1$, we define the auxiliary space consisting of divergence-free rational bubble functions:

$$\boldsymbol{U}(T) := \sum_{i=0}^{2} \mathbf{curl}(B_i A_{k-1}^{(i)}(T)),$$
$$A_{k-1}^{(i)}(T) := \{q_h \in \mathcal{P}_{k-1}(T) : \ (q_h, B_i p_h)_T = 0 \ \forall p_h \in \mathcal{P}_{k-2}(T)\} \ (k \geq 2),$$

and $A_0^{(i)}(T) = \mathcal{P}_0(T)$. Here, $\mathbf{curl} = (\partial/\partial x_2, -\partial/\partial x_1)^t$ is the two-dimensional vector curl operator. Note that the dimension of $A_{k-1}^{(i)}(T)$ is k, and therefore $\dim \boldsymbol{U}(T) = 3k$. In addition, due to the properties of the rational bubble functions, there holds $\boldsymbol{z}_h|_{\partial T} \in \mathcal{P}_{k+1}(\partial T)$ for $z_h \in \boldsymbol{U}(T)$.

The local space of the divergence-free conforming element is then given by

$$\boldsymbol{W}_k(T) = \boldsymbol{M}_k(T) \oplus \boldsymbol{U}(T),$$

where $\boldsymbol{M}_k(T)$ given by (4.1) with $d = 2$. Since $\dim \boldsymbol{Q}_{k+1}(T) = 3$, we find that

$$\dim \boldsymbol{W}_k(T) = (k+2)(k+1) + 3 + 3k = (k+5)(k+1).$$

A unisolvent set of degrees of freedom is given in the next lemma (cf. [17, Lemma 5.1]). For completeness, we provide the proof in the appendix.

Lemma 4.1. The following degrees of freedom are unisolvent over $\boldsymbol{W}_k(T)$:

$$\boldsymbol{v}_h(a) \qquad \forall a \in \mathcal{F}_0(T) \tag{4.2a}$$
$$\langle \boldsymbol{v}_h, \boldsymbol{\kappa}_h\rangle_e \qquad \forall \boldsymbol{\kappa}_h \in \boldsymbol{\mathcal{P}}_{k-1}(e), \ e \in \mathcal{F}_1(T), \tag{4.2b}$$
$$(\boldsymbol{v}_h, \boldsymbol{\rho}_h)_T \qquad \forall \boldsymbol{\rho}_h \in \boldsymbol{N}_{k-1}(T). \tag{4.2c}$$

Remark 4.2. The rational bubble functions and local spaces are constructed such that $\boldsymbol{W}_k(T)|_{\partial T} \subset \boldsymbol{\mathcal{P}}_{k+1}(\partial T)$. Since the boundary degrees of freedom (4.2a)–(4.2b) are the same as the Lagrange finite element space of degree $(k+1)$, we see that the degrees of freedom (4.2) induce an $\boldsymbol{H}^1(\Omega)$–conforming finite element space.

For given $k \geq 1$ we set

$$\boldsymbol{W}_h = \boldsymbol{W}_h^k := \{\boldsymbol{v}_h \in \boldsymbol{X} : \boldsymbol{v}_h\big|_T \in \boldsymbol{W}_k(T), \ \forall T \in \mathcal{T}_h\}$$

as the two-dimensional, globally $\boldsymbol{H}^1(\Omega)$-conforming finite element space. The degrees of freedom (4.2) induce a Fortin operator which satisfies the following properties; see [17] for details.

Proposition 4.3. There exists $\boldsymbol{\pi}_h : \boldsymbol{X} \to \boldsymbol{W}_h^k$ such that, for all $\boldsymbol{v} \in \boldsymbol{X}$,

$$\int_\Omega (\nabla \cdot \boldsymbol{\pi}_h \boldsymbol{v}) q_h \,\mathrm{d}x = \int_\Omega (\nabla \cdot \boldsymbol{v}) q_h \,\mathrm{d}x \quad \forall q_h \in \mathcal{P}_{k-1}(\mathcal{T}_h) \cap Q,$$

and

$$\|\nabla \boldsymbol{\pi}_h \boldsymbol{v}\| \le C \|\nabla \boldsymbol{v}\|.$$

Furthermore, if $k \ge 2$, then

$$\int_T \boldsymbol{\pi}_h \boldsymbol{v} \,\mathrm{d}x = \int_T \boldsymbol{v} \,\mathrm{d}x \qquad \forall T \in \mathcal{T}_h.$$

The following lemma extends the results of Proposition 4.3 by constructing a Fortin-type operator on the Crouzeix–Raviart element space $\boldsymbol{X}_h$ in two dimensions.

Lemma 4.4. For each $k \ge 1$, there exists an operator $\boldsymbol{E}_h : \boldsymbol{X}_h \to \boldsymbol{W}_h^k$ such that

(i) $\int_e \boldsymbol{E}_h \boldsymbol{v}_h \,\mathrm{d}s = \int_e \boldsymbol{v}_h \,\mathrm{d}s$ for all $e \in \mathcal{F}_h$ and $\boldsymbol{v}_h \in \boldsymbol{X}_h$,

(ii) $\nabla \cdot (\boldsymbol{E}_h \boldsymbol{v}_h) = \nabla \cdot (\boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h) \quad \big(= \nabla_h \cdot \boldsymbol{v}_h \big)$ for all $\boldsymbol{v}_h \in \boldsymbol{X}_h$,

(iii) $\boldsymbol{E}_h : \boldsymbol{V}_h \to \boldsymbol{W}_h^k \cap \boldsymbol{V}$,

(iv) $\|\nabla \boldsymbol{E}_h \boldsymbol{v}_h\| \le C \|\nabla_h \boldsymbol{v}_h\|$ for all $\boldsymbol{v}_h \in \boldsymbol{X}_h$.

Proof. For $T \in \mathcal{T}_h$, we uniquely define the local operator $\boldsymbol{E}_T : \boldsymbol{X}_h \longrightarrow \boldsymbol{W}_k(T)$ such that

$$\big(\boldsymbol{E}_T \boldsymbol{v}_h\big)(a) = \{\boldsymbol{v}_h\}_a, \qquad \forall a \in \mathcal{F}_0^I(T) \tag{4.3a}$$
$$\big\langle (\boldsymbol{E}_T \boldsymbol{v}_h - \{\boldsymbol{v}_h\}), \boldsymbol{\kappa}_h \big\rangle_e = 0, \quad \forall \boldsymbol{\kappa}_h \in \boldsymbol{\mathcal{P}}_{k-1}(e),\ e \in \mathcal{F}_1^I(T), \tag{4.3b}$$
$$\big(\boldsymbol{E}_T \boldsymbol{v} - \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h, \boldsymbol{\rho}_h\big)_T = 0, \quad \forall \boldsymbol{\rho}_h \in \boldsymbol{N}_{k-1}(T), \tag{4.3c}$$

and $\boldsymbol{E}_T \boldsymbol{v}_h(a) = 0$ for $a \in \mathcal{F}_0^B(T)$, and $\langle \boldsymbol{E}_T \boldsymbol{v}_h, \boldsymbol{\kappa}_h \rangle_e = 0$ for $\boldsymbol{\kappa}_h \in \boldsymbol{\mathcal{P}}_{k-1}(e)$ and $e \in \mathcal{F}_1^B(T)$. Setting $\boldsymbol{E}_h \boldsymbol{v}_h\big|_T := \boldsymbol{E}_T \boldsymbol{v}_h$, we clearly see that property (i) is satisfied.

To show (ii), for $e \in \mathcal{F}_1(T)$, let $\mathbb{P}_e : L^2(e) \to \mathcal{P}_{\min\{1,k-1\}}(e)$ denote the L^2 projection onto $\mathcal{P}_{\min\{1,k-1\}}(e)$. For $\boldsymbol{v}_h \in \boldsymbol{X}_h$ we have $\{\boldsymbol{v}_h \cdot \boldsymbol{n}_e\}\big|_e \in \mathcal{P}_1(e)$, and therefore, since $(\boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h) \cdot \boldsymbol{n}_e\big|_e \in \mathcal{P}_1(e)$, (4.3b)–(4.3c) and integration by parts,

$$\begin{aligned}
(\nabla \cdot (\boldsymbol{E}_T \boldsymbol{v}_h), q_h)_T &= -(\boldsymbol{E}_T \boldsymbol{v}_h, \nabla q_h)_T + \sum_{e \in \mathcal{F}_1^I(T)} \langle (\boldsymbol{E}_T \boldsymbol{v}_h) \cdot \boldsymbol{n}_e, q_h \rangle_e \\
&= -(\boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h, \nabla q_h)_T + \sum_{e \in \mathcal{F}_1^I(T)} \langle \{\boldsymbol{v}_h \cdot \boldsymbol{n}_e\}, \mathbb{P}_e q_h \rangle_e \\
&= -(\boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h, \nabla q_h)_T + \sum_{e \in \mathcal{F}_1^I(T)} \langle \boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h \cdot \boldsymbol{n}_e, \mathbb{P}_e q_h \rangle_e \\
&= (\nabla \cdot (\boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h), q_h)_T
\end{aligned}$$

for all $q_h \in \mathcal{P}_{k-1}(T)$. Thus, due to $\nabla \cdot (\boldsymbol{E}_T \boldsymbol{v}_h) \in \mathcal{P}_{k-1}(T)$, the statement (ii) holds. Further, (iii) is a simple consequence of (ii), restricting $\boldsymbol{E}_h$ to $\boldsymbol{V}_h$.

To show (iv), we set $\boldsymbol{w}_T := \boldsymbol{E}_T \boldsymbol{v}_h$ and $\boldsymbol{v}_T := \boldsymbol{v}_h\big|_T$ for notational convenience. Since $\boldsymbol{W}_k(T)$ is finite dimensional, a simple scaling argument shows that

$$\begin{aligned}
\|\nabla(\boldsymbol{w}_T - \boldsymbol{v}_T)\|_T^2 \lesssim \sum_{a \in \mathcal{F}_0(T)} |(\boldsymbol{w}_T - \boldsymbol{v}_T)(a)|^2 + \sum_{e \in \mathcal{F}_1(T)} h_e^{-1} \Big| \sup_{\substack{\boldsymbol{\kappa}_h \in \boldsymbol{\mathcal{P}}_{k-1}(e) \\ \|\boldsymbol{\kappa}_h\|_e = 1}} \langle \boldsymbol{w}_T - \boldsymbol{v}_T, \boldsymbol{\kappa}_h \rangle_e \Big|^2 \\
+ \sup_{\substack{\boldsymbol{\rho}_h \in \boldsymbol{N}_{k-1}(T) \\ \|\boldsymbol{\rho}_h\|_T = 1}} h_T^{-2} \big| (\boldsymbol{w}_T - \boldsymbol{v}_T, \boldsymbol{\rho}_h)_T \big|^2.
\end{aligned} \tag{4.4}$$

Note that $\{\boldsymbol{v}_h\} - \boldsymbol{v}_T = \pm\frac{1}{2}[\boldsymbol{v}_h]$ on $e \in \mathcal{F}_1^I(T)$ and $\boldsymbol{w}_T = 0$ on $e \in \mathcal{F}_1^B$. It then follows from (4.3) and the Cauchy-Schwarz inequality that

$$\sum_{e\in\mathcal{F}_1(T)} h_e^{-1}\Big| \sup_{\substack{\boldsymbol{\kappa}_h\in\boldsymbol{\mathcal{P}}_{k-1}(e)\\ \|\boldsymbol{\kappa}_h\|_e=1}} \langle \boldsymbol{w}_T - \boldsymbol{v}_T, \boldsymbol{\kappa}_h\rangle_e\Big|^2 \le \sum_{e\in\mathcal{F}_1(T)} h_e^{-1}\big\|[\boldsymbol{v}_h]\big\|_e^2. \tag{4.5}$$

We also have by (4.3), for $a \in \mathcal{F}_0^I(T)$,

$$\begin{aligned}\big|\boldsymbol{w}_T - \boldsymbol{v}_T(a)\big|^2 = \big|\{\boldsymbol{v}_h\}_a - \boldsymbol{v}_T(a)\big|^2 &\le C\sum_{T'\in\mathcal{T}_a} |\boldsymbol{v}_{T'}(a) - \boldsymbol{v}_T(a)|^2 \\ &\le C \sum_{\substack{T',T''\in\mathcal{T}_a\\ T' \text{ and } T'' \text{ share a common edge}}} |\boldsymbol{v}_{T'}(a) - \boldsymbol{v}_{T''}(a)|^2.\end{aligned}$$

Letting $\mathcal{F}_a \subset \mathcal{F}_h$ denote the set of edges that have a as a vertex, we conclude from an inverse inequality that

$$\big|\boldsymbol{w}_T - \boldsymbol{v}_T(a)\big|^2 \le C\sum_{e\in\mathcal{F}_a} \|[\boldsymbol{v}_h]\|_{L^\infty(e)}^2 \le C\sum_{e\in\mathcal{F}_a} h_e^{-1}\|[\boldsymbol{v}_h]\|_e^2. \tag{4.6}$$

Likewise, for $a \in \mathcal{F}_0^B(T)$, we have $\boldsymbol{w}_T(a) = 0$, and therefore,

$$\big|\boldsymbol{w}_T - \boldsymbol{v}_T(a)\big|^2 = |\boldsymbol{v}_T(a)|^2 \le \sum_{e\in\mathcal{F}_1^B(T)} \|\boldsymbol{v}_T\|_{L^\infty(e)} \le C\sum_{e\in\mathcal{F}_a} h_e^{-1}\|[\boldsymbol{v}_h]\|_e^2. \tag{4.7}$$

Combining (4.4)–(4.7), summing over $T \in \mathcal{T}_h$, and applying Lemmas 3.2 and 3.3 yield

$$\|\nabla_h(\boldsymbol{E}_h\boldsymbol{v}_h - \boldsymbol{v}_h)\|^2 \lesssim \sum_{e\in\mathcal{F}_h} h_e^{-1}\|[\boldsymbol{v}_h]\|_e^2 + \sum_{T\in\mathcal{T}_h} h_T^{-2}\|\boldsymbol{v}_h - \boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{v}_h\|_T^2 \lesssim \|\nabla_h\boldsymbol{v}_h\|^2.$$

An application of the triangle inequality now gives (iii). This completes the proof. q.e.d.

4.2 Three-dimensional construction

To describe the three dimensional, divergence-free conforming finite element space, we first label the six edges of an element T as $\mathcal{F}_1(T) = \{e_{i,j}\}_{i<j}^3$ such that $e_{i,j} = \partial f_i \cap \partial f_j$. The quadratic edge bubble functions are given by

$$b_{i,j} = \prod_{\substack{k=0\\ k\neq i, k\neq j}}^{3} \lambda_k,$$

and the rational edge bubble functions are then defined as [18]

$$\boldsymbol{s}_{i,j} = \frac{b_T b_{i,j}}{2\big(\lambda_i\lambda_j + b_{i,j}(\lambda_i+\lambda_j)\big)(\lambda_i+\lambda_j)}\big(\nabla(\lambda_j^2 - \lambda_i^2) + 4(\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i)\big).$$

The (seemingly abstruse) function $\boldsymbol{s}_{i,j}$ is constructed such that [18, Lemma 2.2]

$$\mathbf{curl}\,\boldsymbol{s}_{i,j} \in \boldsymbol{C}^0(\bar{T}) \cap \boldsymbol{W}^{1,\infty}(T), \quad \mathbf{curl}\,\boldsymbol{s}_{i,j}\big|_{\partial T} = b_{i,j}(\nabla\lambda_i \times \nabla\lambda_j), \quad \boldsymbol{s}_{i,j}\big|_{\partial T} = 0.$$

Thus, similar to the rational face bubble functions, the rational edge bubble functions and its derivatives reduce to polynomials on the boundary of the element.

We define the auxiliary spaces consisting of divergence–free rational face and edge bubbles:

$$\begin{aligned}\boldsymbol{U}(T) &= \sum_{i=0}^{3} \mathbf{curl}(B_i\boldsymbol{\mathcal{P}}_0(T) \times \boldsymbol{n}_{f_i}),\\ \boldsymbol{Z}(T) &= \Big\{\sum_{\substack{i,j=0\\ i>j}}^{3} \mathbf{curl}(p\boldsymbol{s}_{i,j}) :\ p \in M^{(i,j)}(T)\Big\},\end{aligned}$$

where $M^{(i,j)}(T) = \mathrm{span}\{\lambda_k, \lambda_\ell\}$ and $k, \ell \neq i$, $k, \ell \neq j$, and $k \neq \ell$.

The local space of the divergence-free element is obtained by enriching the local $\boldsymbol{H}(\mathrm{div};)$-element (4.1) with rational edge and rational face bubble functions:

$$\boldsymbol{W}(T) = \boldsymbol{M}_1(T) \oplus \boldsymbol{U}(T) \oplus \boldsymbol{Z}(T). \tag{4.8}$$

Note that, since the last two spaces in (4.8) are divergence-free, there holds $\nabla \cdot \boldsymbol{W}(T) = \nabla \cdot \boldsymbol{M}_1(T) = \nabla \cdot \boldsymbol{\mathcal{P}}_1(T) \subset \mathcal{P}_0(T)$. Moreover, restricted to the boundary, we have $\boldsymbol{W}(T)|_{\partial T} \subset \boldsymbol{\mathcal{P}}_3(\partial T)$. A unisolvent set of degrees of freedom that induce an $\boldsymbol{H}^1$-conforming finite element space is given in the next lemma [18, Theorem 3.1].

Lemma 4.5. The dimension of $\boldsymbol{W}(T)$ is 60, and a function $\boldsymbol{v}_h \in \boldsymbol{W}(T)$ is uniquely determined by the values

$$\boldsymbol{v}_h(a) \qquad \forall a \in \mathcal{F}_0(T) \tag{4.9a}$$

$$\langle \boldsymbol{v}_h, \boldsymbol{\kappa}_h \rangle_e \qquad \forall \boldsymbol{\kappa}_h \in \boldsymbol{\mathcal{P}}_1(e),\ e \in \mathcal{F}_1(T), \tag{4.9b}$$

$$\langle \boldsymbol{v}_h, \boldsymbol{\kappa}_h \rangle_f \qquad \forall \boldsymbol{\kappa}_h \in \boldsymbol{\mathcal{P}}_0(f),\ f \in \mathcal{F}_2(T). \tag{4.9c}$$

We set

$$\boldsymbol{W}_h := \{\boldsymbol{v}_h \in \boldsymbol{X} \colon \boldsymbol{v}_h\big|_T \in \boldsymbol{W}(T),\ \forall T \in \mathcal{T}_h\}$$

Analogous to Proposition 4.3 (with $k = 1$), the degrees of freedom (4.9) induce a Fortin operator.

Proposition 4.6. There exists $\boldsymbol{\pi}_h : \boldsymbol{X} \to \boldsymbol{W}_h$ such that, for all $\boldsymbol{v} \in \boldsymbol{X}$,

$$\int_\Omega (\nabla \cdot \boldsymbol{\pi}_h \boldsymbol{v}) q_h \,\mathrm{d}x = \int_\Omega (\nabla \cdot \boldsymbol{v}) q_h \,\mathrm{d}x \quad \forall q_h \in \mathcal{P}_0(\mathcal{T}_h) \cap Q,$$

and

$$\|\nabla \boldsymbol{\pi}_h \boldsymbol{v}\| \leq C \|\nabla \boldsymbol{v}\|.$$

Similar to Lemma 4.4, we construct a Fortin-type operator on the Crouzeix–Raviart element space $\boldsymbol{X}_h$.

Lemma 4.7. In three dimensions there exists an operator $\boldsymbol{E}_h : \boldsymbol{X}_h \to \boldsymbol{W}_h$ such that

(i) $\displaystyle\int_f \boldsymbol{E}_h \boldsymbol{v}_h \,\mathrm{d}s = \int_f \boldsymbol{v}_h \,\mathrm{d}s$ for all $f \in \mathcal{F}_h$ and $\boldsymbol{v}_h \in \boldsymbol{X}_h$,

(ii) $\nabla \cdot (\boldsymbol{E}_h \boldsymbol{v}_h) = \nabla \cdot (\boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h) \quad \big(= \nabla_h \cdot \boldsymbol{v}_h\big)$ for all $\boldsymbol{v}_h \in \boldsymbol{X}_h$,

(iii) $\boldsymbol{E}_h : \boldsymbol{V}_h \to \boldsymbol{W}_h \cap \boldsymbol{V}$,

(iv) $\|\nabla \boldsymbol{E}_h \boldsymbol{v}_h\| \leq C \|\nabla_h \boldsymbol{v}_h\|$ for all $\boldsymbol{v}_h \in \boldsymbol{X}_h$.

Proof. The proof closely follows the proof of Lemma 4.4, so we only sketch the argument.

For $T \in \mathcal{T}_h$, define $\boldsymbol{E}_T : \boldsymbol{X}_h \longrightarrow \boldsymbol{W}(T)$ such that

$$\big(\boldsymbol{E}_T \boldsymbol{v}_h\big)(a) = \{\boldsymbol{v}_h\}_a, \qquad \forall a \in \mathcal{F}_0^I(T) \tag{4.10a}$$

$$\big\langle (\boldsymbol{E}_T \boldsymbol{v}_h - \{\boldsymbol{v}_h\}_e), \boldsymbol{\kappa}_h \big\rangle_e = 0, \quad \forall \boldsymbol{\kappa}_h \in \boldsymbol{\mathcal{P}}_1(e),\ e \in \mathcal{F}_1^I(T), \tag{4.10b}$$

$$\big\langle (\boldsymbol{E}_T \boldsymbol{v}_h - \boldsymbol{v}_h), \boldsymbol{\kappa}_h \big\rangle_f = 0, \qquad \forall \boldsymbol{\kappa}_h \in \boldsymbol{\mathcal{P}}_0(f),\ f \in \mathcal{F}_2(T), \tag{4.10c}$$

and $\boldsymbol{E}_T \boldsymbol{v}_h(a) = 0$ for $a \in \mathcal{F}_0^B(T)$, and $\langle \boldsymbol{E}_T \boldsymbol{v}_h, \boldsymbol{\kappa}_h \rangle_e = 0$ for $\boldsymbol{\kappa}_h \in \boldsymbol{\mathcal{P}}_1(e)$, for $e \in \mathcal{F}_1^B(T)$. Setting $\boldsymbol{E}_h \boldsymbol{v}_h\big|_T := \boldsymbol{E}_T \boldsymbol{v}_h$, we clearly see that property (i) is satisfied. Moreover, since $(\nabla \cdot \boldsymbol{E}_T \boldsymbol{v}_h) \in \mathcal{P}_0(T)$ and $(\nabla \cdot \boldsymbol{v}_h)|_T \in \mathcal{P}_0(T)$, condition (4.10c) and integration by parts shows that $\nabla \cdot \boldsymbol{E}_h \boldsymbol{v}_h = \nabla_h \cdot \boldsymbol{v}_h = \nabla \cdot (\boldsymbol{\pi}^{\mathrm{BDM}} \boldsymbol{v}_h)$. Thus, (ii)-(iii) holds.

Setting $\boldsymbol{w}_T = \boldsymbol{E}_T \boldsymbol{v}_h$ and $\boldsymbol{v}_T = \boldsymbol{v}_h\big|_T$, a scaling argument yields

$$\begin{aligned}\|\nabla(\boldsymbol{w}_T - \boldsymbol{v}_T)\|_T^2 \lesssim \sum_{a\in\mathcal{F}_0(T)} h_T|(\boldsymbol{w}_T-\boldsymbol{v}_T)(a)|^2 + \sum_{e\in\mathcal{F}_1(T)} \Big|\sup_{\substack{\boldsymbol{\kappa}_h\in\boldsymbol{\mathcal{P}}_1(e)\\ \|\boldsymbol{\kappa}_h\|_e=1}} \langle \boldsymbol{w}_T-\boldsymbol{v}_T,\boldsymbol{\kappa}_h\rangle_e\Big|^2 \\ + \sum_{f\in\mathcal{F}_2(T)} \Big|\sup_{\substack{\boldsymbol{\kappa}_h\in\boldsymbol{\mathcal{P}}_0(f)\\ \|\boldsymbol{\kappa}_h\|_f=1}} h_F^{-1}\langle \boldsymbol{w}_T-\boldsymbol{v}_T,\boldsymbol{\kappa}_h\rangle_f\Big|^2 \\ \le \sum_{a\in\mathcal{F}_0(T)} h_T|\{\boldsymbol{v}_h\}_a - \boldsymbol{v}_T(a)|^2 + \sum_{e\in\mathcal{F}_1(T)} \|\{\boldsymbol{v}_h\}_e - \boldsymbol{v}_T\|_e^2.\end{aligned} \tag{4.11}$$

Applying similar arguments found in the proof of Lemma 4.4, we have (cf. (4.6))

$$\sum_{e\in\mathcal{F}_1(T)} \|\{\boldsymbol{v}_h\}_e - \boldsymbol{v}_T\|_e^2 \le C \sum_{e\in\mathcal{F}_1(T)}\sum_{f\in\mathcal{F}_e} h_F^{-1}\big\|[\boldsymbol{v}_h]\|_f^2, \tag{4.12}$$

where $\mathcal{F}_e$ denotes the set of faces that have e as an edge. Likewise, we have for $a\in\mathcal{F}_0^I(T)$,

$$\begin{aligned}|\{\boldsymbol{v}_h\}_a - \boldsymbol{v}_T(a)|^2 &\le C\sum_{T'\in\mathcal{T}_a} |\boldsymbol{v}_{T'}(a) - \boldsymbol{v}_T(a)|^2 \\ &\le C \sum_{\substack{T',T''\in\mathcal{T}_a\\ T' \text{ and } T'' \text{ share a common face}}} |\boldsymbol{v}_{T'}(a) - \boldsymbol{v}_{T''}(a)|^2 \\ &\le C\sum_{f\in\mathcal{F}_a} \big\|[\boldsymbol{v}_h]\|_{L^\infty(f)}^2 \le C\sum_{f\in\mathcal{F}_a} h_f^{-2}\big\|[\boldsymbol{v}_h]\big\|_f^2,\end{aligned} \tag{4.13}$$

where $\mathcal{F}_a$ denotes the set of faces that have a as a vertex. For $a\in\mathcal{F}_0^B(T)$ we have

$$|\{\boldsymbol{v}_h\}_a - \boldsymbol{v}_T(a)|^2 = |\boldsymbol{v}_T(a)|^2 \le C\sum_{f\in\mathcal{F}_a} h_f^{-2}\big\|[\boldsymbol{v}_h]\|_f^2. \tag{4.14}$$

Combining the estimates (4.12)–(4.14) to (4.11) and summing over $T\in\mathcal{T}_h$ yields

$$\|\nabla(\boldsymbol{E}\boldsymbol{v}_h - \boldsymbol{v}_h)\|^2 \le C\sum_{f\in\mathcal{F}} h_f^{-1}\|[\boldsymbol{v}_h]\|_f^2.$$

Applying Lemma 3.2 and the triangle inequality, we obtain (iv). This completes the proof. q.e.d.

5 Pressure-robust error estimates

Following the Berger-Scott-Strang-Lemma 3.4, estimates of the energy error contain a consistency error $C_h(\boldsymbol{u},\boldsymbol{w}_h)$. Classical estimates of the consistency error require a minimal regularity of $\boldsymbol{u}\in\boldsymbol{H}^{1+s}(\Omega)$ with $s>1/2$ in order for edge-integrals to be defined. Together with the preceding section, we are now in position to estimate the energy and $\boldsymbol{L}^2$ errors of the modified Crouzeix–Raviart element method (3.7) for arbitrary regularities

$$\boldsymbol{u}\in\boldsymbol{X}\cap\boldsymbol{H}^{1+s}(\Omega),\quad s\ge 0.$$

We will use the Fortin-type operator defined in Theorem 4.4 to estimate the consistency error by the velocity-best approximations and additional higher-order oscillations.

Theorem 5.1. Let $k\ge 1$ if $d=2$, and $k=1$ if $d=3$. Let $\boldsymbol{u}\in\boldsymbol{V}$ be the continuous solution of (2.4) and $\boldsymbol{u}_h\in\boldsymbol{V}_h$ be the discrete solution to the reconstructed scheme (3.7). There holds

$$\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\| \le C\Big(\inf_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\|\nabla_h(\boldsymbol{u}-\boldsymbol{v}_h)\| + \nu^{-1}\mathrm{R}_{k-2}\big(\mathbb{P}(\boldsymbol{f})\big)\Big),$$

with

$$\mathrm{R}_{k-2}(\boldsymbol{g})^2 = \begin{cases} \sum_{T\in\mathcal{T}_h} h_T^2 \|\boldsymbol{g}\|_T^2 & \text{for } k=1,\\ \sum_{T\in\mathcal{T}_h} h_T^2 \inf_{\boldsymbol{q}_h\in\boldsymbol{\mathcal{P}}_{k-2}(T)} \|\boldsymbol{g}-\boldsymbol{q}_h\|_T^2 & \text{for } k\geq 2,\end{cases} \tag{5.1}$$

and $C>0$ is independent of h, ν and $(\boldsymbol{u},p)$.

Proof. Let $\boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{V}_h$ be arbitrary. Using the BDM reconstruction leads to the modified consistency error $C_h(\boldsymbol{u},\boldsymbol{w}_h)$ defined in Lemma 3.4. For $\boldsymbol{E}_h\boldsymbol{w}_h \in \boldsymbol{V}$ being conforming and divergence-free, it holds $a_h(\boldsymbol{u},\boldsymbol{E}_h\boldsymbol{w}_h)$ $=(\boldsymbol{f},\boldsymbol{E}_h\boldsymbol{w}_h)$, and therefore,

$$\begin{aligned} C_h(\boldsymbol{u},\boldsymbol{w}_h) &= a_h(\boldsymbol{u},\boldsymbol{w}_h) - (\boldsymbol{f},\boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{w}_h)\\ &= a_h(\boldsymbol{u},\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h) - (\boldsymbol{f},\boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h)\\ &= \underbrace{a_h(\boldsymbol{u}-\boldsymbol{v}_h,\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h)}_{=:I_1} + \underbrace{a_h(\boldsymbol{v}_h,\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h)}_{=:I_2} - \underbrace{(\boldsymbol{f},\boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h)}_{=:I_3}\end{aligned}$$

for arbitrary $\boldsymbol{v}_h\in\boldsymbol{X}_h$.

To bound the first term I_1 we apply Lemmas 4.4 and 4.7, and the Cauchy–Schwarz inequality:

$$I_1 \leq \nu\|\nabla_h(\boldsymbol{u}-\boldsymbol{v}_h)\|\|\nabla_h(\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h)\| \leq \nu C\|\nabla_h(\boldsymbol{u}-\boldsymbol{v}_h)\|\|\nabla_h\boldsymbol{w}_h\|.$$

Since $\boldsymbol{v}_h$ is piecewise linear, an integration by parts for the second term I_2 yields

$$\begin{aligned} I_2 &= \nu\sum_{T\in\mathcal{T}_h}\int_T \nabla\boldsymbol{v}_h : \nabla\big(\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h\big)\,\mathrm{d}x\\ &= -\nu\sum_{T\in\mathcal{T}_h}\int_T \underbrace{\Delta\boldsymbol{v}_h}_{\equiv\boldsymbol{0}}\cdot\big(\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h\big)\,\mathrm{d}x + \nu\sum_{T\in\mathcal{T}_h}\int_{\partial T}\underbrace{\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}}_{\equiv\text{ const}}\cdot\big(\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h\big)\,\mathrm{d}s\\ &= 0. \end{aligned}\tag{5.2}$$

Concerning the last term I_3, it follows for $k\geq 2$ from (4.3c) that for any $\boldsymbol{q}_h\in\boldsymbol{\mathcal{P}}_{k-2}(\mathcal{T}_h)$,

$$\begin{aligned} I_3 &= \big(\boldsymbol{f},\boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h\big) = \big(\mathbb{P}(\boldsymbol{f}),\boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h\big)\\ &= \big(\mathbb{P}(\boldsymbol{f})-\boldsymbol{q}_h,\boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h\big). \end{aligned}\tag{5.3}$$

In the case $k=1$, a similar argument follows with $\boldsymbol{q}_h=\boldsymbol{0}$.

Next, it follows from Lemmas 4.4 and 4.7 that the integral of $\boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h$ vanishes on each edge/face. Applications of the Poincaré and Cauchy–Schwarz inequalities then lead to

$$\begin{aligned} I_3 &\leq \Big(\sum_{T\in\mathcal{T}_h} h_T^2\|\mathbb{P}(\boldsymbol{f})-\boldsymbol{q}_h\|_T^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-2}\big\|\boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h\big\|_T^2\Big)^{1/2}\\ &\leq C\Big(\sum_{T\in\mathcal{T}_h} h_T^2\|\mathbb{P}(\boldsymbol{f})-\boldsymbol{q}_h\|_T^2\Big)^{1/2}\|\nabla_h(\boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{w}_h-\boldsymbol{E}_h\boldsymbol{w}_h)\|. \end{aligned}$$

Using the $\boldsymbol{H}^1$-stability results of $\boldsymbol{E}_h$ and $\boldsymbol{\pi}^{\mathrm{BDM}}$ then yield

$$I_3 \leq C\mathrm{R}_{k-2}\big(\mathbb{P}(\boldsymbol{f})\big)\|\nabla_h\boldsymbol{w}_h\|.$$

A combination of all preceding estimates yields

$$E_h(\boldsymbol{u},\boldsymbol{w}_h) \leq C\Big(\nu\|\nabla_h(\boldsymbol{u}-\boldsymbol{v}_h)\| + \mathrm{R}_{k-2}\big(\mathbb{P}(\boldsymbol{f})\big)\Big)\|\nabla_h\boldsymbol{w}_h\|.$$

Finally, inf-sup stability implies [5]

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\|\nabla_h(\boldsymbol{u}-\boldsymbol{v}_h)\| \leq C\inf_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\|\nabla_h(\boldsymbol{u}-\boldsymbol{v}_h)\|.$$

Combining these results with Lemma 3.4 then gives the desired result (5.1). q.e.d.

Remark 5.2. The dependence of the error estimate on the term $\nu^{-1}\mathbb{P}(\boldsymbol{f})$ is briefly discussed in a special case here. For a more detailed discussion, see Subsection 5.1. Assume that it holds $\Delta \boldsymbol{u}, \nabla p \in \boldsymbol{L}^2(\Omega)$. Then, one obtains

$$\frac{1}{\nu}\mathbb{P}(\boldsymbol{f}) = \frac{1}{\nu}\mathbb{P}(-\nu\Delta\boldsymbol{u} + \nabla p) = \mathbb{P}(\Delta\boldsymbol{u}),$$

due to Remark 2.3. Hence, $\nu^{-1}\mathbb{P}(\boldsymbol{f})$ is ν-independent. Note that $\nu^{-1}\boldsymbol{f} = \Delta\boldsymbol{u} + \nu^{-1}\nabla p$ is not ν-independent, instead, and hence, any error estimate that depends on this term is not pressure-robust. Indeed, a dependence on $\nu^{-1}\boldsymbol{f}$ indicates a locking-phenomenon, see the discussion in [1].

In order to estimate the L^2 error, we follow the lines of Aubin-Nitsche [5]. First we define $(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \in \boldsymbol{V} \times \boldsymbol{V}_h$ as the solutions to the following dual problems:

$$\big(\nabla\boldsymbol{\phi}, \nabla\boldsymbol{v}\big) = (\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{5.4a}$$

$$\big(\nabla_h\boldsymbol{\phi}_h, \nabla_h\boldsymbol{v}_h\big) = (\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{5.4b}$$

We assume that the continuous dual problem (5.4a) satisfies the following regularity

$$\|\boldsymbol{\phi}\|_{1+s_0} \le C\|\boldsymbol{u} - \boldsymbol{u}_h\|, \tag{5.5}$$

with $s_0 \in [0, 1]$ and for some constant $C > 0$.

Lemma 5.3 (Dual energy error)**.** Let $\boldsymbol{\phi} \in \boldsymbol{V}$ be the continuous solution of (5.4a) and $\boldsymbol{\phi}_h \in \boldsymbol{V}_h$ be the discrete dual solution of (5.4b). Then it follows that the dual energy error satisfies

$$\|\nabla_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\| \le C h^{s_0}\|\boldsymbol{u} - \boldsymbol{u}_h\|.$$

Proof. The dual energy error can be estimated by Theorem 5.1 for $\nu = 1$ and $k = 1$:

$$\|\nabla_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\| \le C\Big(\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \|\nabla_h(\boldsymbol{\phi} - \boldsymbol{v}_h)\| + \sqrt{\sum_{T\in\mathcal{T}_h} h_T^2\|\boldsymbol{u} - \boldsymbol{u}_h\|_T^2}\Big).$$

By standard approximation results and (5.5) we have

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \|\nabla_h(\boldsymbol{\phi} - \boldsymbol{v}_h)\| \le C h^{s_0}\|\boldsymbol{\phi}\|_{1+s_0} \le C h^{s_0}\|\boldsymbol{u} - \boldsymbol{u}_h\|. \tag{5.6}$$

and hence, by the definition of R_{k-2} in (5.1), for meshes satisfying $h_T \le 1$

$$\|\nabla_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\| \le h^{s_0}\|\boldsymbol{u} - \boldsymbol{u}_h\|.$$

q.e.d.

Theorem 5.4. Let $\boldsymbol{u} \in \boldsymbol{V}$ be the solution of (2.4) and $\boldsymbol{u}_h \in \boldsymbol{V}_h$ be the discrete solution of the reconstructed scheme (3.7). Then there holds

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| \le C h^{s_0}\left(\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \|\nabla_h(\boldsymbol{u} - \boldsymbol{v}_h)\| + h\nu^{-1}\|\mathbb{P}(\boldsymbol{f})\|\right), \tag{5.7}$$

with $s_0 \in [0, 1]$ being the dual regularity (5.5).

Proof. Let us define the following terms

$$\begin{aligned}
I_1 &:= \big(\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h), \nabla_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\big),\\
I_2 &:= \big(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{u} - \boldsymbol{u}_h)\big) - \big(\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h), \nabla\boldsymbol{\phi}\big),\\
I_3 &:= \nu^{-1}\big(\mathbb{P}(\boldsymbol{f}), \boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\big) - \big(\nabla\boldsymbol{u}, \nabla_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\big),\\
I_4 &:= \big(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h - \boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{u} - \boldsymbol{u}_h)\big),\\
I_5 &:= \nu^{-1}\big(\mathbb{P}(\boldsymbol{f}), \boldsymbol{\phi} - \boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{\phi}\big),
\end{aligned}$$

such that the $\boldsymbol{L}^2$ error splits up as follows

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|^2 &= (\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\boldsymbol{u}_h) - \big(\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h), \nabla_h \boldsymbol{\phi}_h\big) - \big(\nabla \boldsymbol{u}, \nabla_h(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\big) \\ &\quad + \nu^{-1}\big(\boldsymbol{f}, \boldsymbol{\phi} - \boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{\phi}_h\big) \\ &= \big(\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h), \nabla_h(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\big) - \big(\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h), \nabla\boldsymbol{\phi}\big) - \big(\nabla \boldsymbol{u}, \nabla_h(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\big) \\ &\quad + \big(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\boldsymbol{u}_h - \boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{u}-\boldsymbol{u}_h)\big) + \big(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{u}-\boldsymbol{u}_h)\big) \\ &\qquad + \nu^{-1}\big(\boldsymbol{f}, \boldsymbol{\phi} - \boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{\phi}_h\big) \\ &= I_1 + I_2 + I_4 + \nu^{-1}\big(\mathbb{P}(\boldsymbol{f}), \boldsymbol{\phi} - \boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{\phi}\big) + \nu^{-1}\big(\mathbb{P}(\boldsymbol{f}), \boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\big) \\ &\quad - \big(\nabla \boldsymbol{u}, \nabla_h(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\big) \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{5.8}$$

The transition to the Helmholtz-projection $\mathbb{P}(\boldsymbol{f})$ is admissible since $\boldsymbol{\phi} \in \boldsymbol{V}$ and $\boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{\phi}-\boldsymbol{\phi}_h)$ is divergence-free.

For the first term we use the Cauchy-Schwarz inequality and apply the preceding Lemma 5.3 to estimate the dual energy error $\|\nabla_h(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|$. It follows for mesh sizes $h \leq 1$

$$I_1 \leq \|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|\|\nabla_h(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\| \leq C h^{s_0}\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|\|\boldsymbol{u}-\boldsymbol{u}_h\|. \tag{5.9}$$

In order to estimate the second term, we make use of the Fortin operators $\boldsymbol{\pi}_h$ and $\boldsymbol{E}_h$ given in Proposition 4.3 and Lemma 4.4, respectively (with $k \geq 2$). For $\boldsymbol{v}_h \in \boldsymbol{V}_h$ arbitrary, it follows that

$$\begin{aligned} I_2 &= \big(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{u}-\boldsymbol{u}_h)\big) - \big(\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h), \nabla\boldsymbol{\phi}\big) \\ &\overset{\text{(5.4a)}}{=} \big(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{u}-\boldsymbol{u}_h) - (\boldsymbol{\pi}_h\boldsymbol{u} - \boldsymbol{E}_h\boldsymbol{u}_h)\big) + \big(\nabla\boldsymbol{\phi}, \nabla_h(\boldsymbol{\pi}_h\boldsymbol{u} - \boldsymbol{E}_h\boldsymbol{u}_h - (\boldsymbol{u}-\boldsymbol{u}_h))\big) \\ &= \big(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{u}-\boldsymbol{u}_h) - (\boldsymbol{\pi}_h\boldsymbol{u} - \boldsymbol{E}_h\boldsymbol{u}_h)\big) + \big(\nabla_h(\boldsymbol{\phi}-\boldsymbol{v}_h), \nabla_h(\boldsymbol{\pi}_h\boldsymbol{u} - \boldsymbol{E}_h\boldsymbol{u}_h - (\boldsymbol{u}-\boldsymbol{u}_h))\big) \\ &\qquad + \underbrace{\big(\nabla_h\boldsymbol{v}_h, \nabla_h(\boldsymbol{\pi}_h\boldsymbol{u} - \boldsymbol{E}_h\boldsymbol{u}_h - (\boldsymbol{u}-\boldsymbol{u}_h))\big)}_{=0} \\ &=: J_1 + J_2. \end{aligned}$$

Concerning the first contribution, we apply the Poincaré inequality, Lemma 3.3, Lemma 4.4, and Proposition 4.3 to obtain

$$\begin{aligned} J_1 &\leq \|\boldsymbol{u}-\boldsymbol{u}_h\|\|\boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{u}-\boldsymbol{u}_h) - (\boldsymbol{\pi}_h\boldsymbol{u} - \boldsymbol{E}_h\boldsymbol{u}_h)\| \\ &\leq C h\|\boldsymbol{u}-\boldsymbol{u}_h\|\big\|\nabla_h\big(\boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{u}-\boldsymbol{u}_h) - (\boldsymbol{\pi}_h\boldsymbol{u} - \boldsymbol{E}_h\boldsymbol{u}_h)\big)\big\| \\ &\leq C h\|\boldsymbol{u}-\boldsymbol{u}_h\|\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|. \end{aligned}$$

Likewise, for the second contribution, we apply Lemmas 3.3 and 4.4 and Proposition 4.3:

$$\begin{aligned} J_2 &\leq \|\nabla_h(\boldsymbol{\phi}-\boldsymbol{v}_h)\|\big\|\nabla_h\big(\boldsymbol{\pi}_h\boldsymbol{u} - \boldsymbol{E}_h\boldsymbol{u}_h - (\boldsymbol{u}-\boldsymbol{u}_h)\big)\big\| \\ &\leq C\|\nabla_h(\boldsymbol{\phi}-\boldsymbol{v}_h)\|\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|. \end{aligned}$$

Altogether it holds for the second term I_2

$$\begin{aligned} I_2 &\leq C\big(\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\nabla_h(\boldsymbol{\phi}-\boldsymbol{v}_h)\| + h\|\boldsymbol{u}-\boldsymbol{u}_h\|\big)\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\| \\ &\leq C\big(\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \|\nabla_h(\boldsymbol{\phi}-\boldsymbol{v}_h)\| + h\|\boldsymbol{u}-\boldsymbol{u}_h\|\big)\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\| \\ &\leq C h^{s_0}\|\boldsymbol{u}-\boldsymbol{u}_h\|\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|. \end{aligned} \tag{5.10}$$

The estimate of I_3 follows from the same arguments as I_2 by interchanging $\boldsymbol{\phi}$ with $\boldsymbol{u}$, $\boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{u}-\boldsymbol{u}_h)$ with $\boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{\phi}-\boldsymbol{\phi}_h)$ and $\boldsymbol{u}-\boldsymbol{u}_h$ with $\nu^{-1}\mathbb{P}(\boldsymbol{f})$; thus,

$$\begin{aligned} I_3 &\leq C\big(\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \|\nabla_h(\boldsymbol{u}-\boldsymbol{v}_h)\| + h\nu^{-1}\|\mathbb{P}(\boldsymbol{f})\big)\|\nabla_h(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\| \\ &\leq C h^{s_0}\big(\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \|\nabla_h(\boldsymbol{u}-\boldsymbol{v}_h)\| + h\nu^{-1}\|\mathbb{P}(\boldsymbol{f})\|\big)\|\boldsymbol{u}-\boldsymbol{u}_h\|. \end{aligned} \tag{5.11}$$

Next, applying Lemma 3.3 we obtain

$$I_4 = (\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h - \boldsymbol{\pi}^{\mathrm{BDM}}(\boldsymbol{u} - \boldsymbol{u}_h)) \leq Ch\|\boldsymbol{u} - \boldsymbol{u}_h\|\|\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)\|.$$

In order to bound the last contribution I_5 we again employ Lemma 3.3:

$$\begin{aligned} I_5 &= \nu^{-1}\big(\mathbb{P}(\boldsymbol{f}), \boldsymbol{\phi} - \boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{\phi}\big) \leq \nu^{-1}\|\mathbb{P}(\boldsymbol{f})\|\|\boldsymbol{\phi} - \boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{\phi}\| \\ &\leq C\nu^{-1}h^{1+s_0}\|\mathbb{P}(\boldsymbol{f})\|\|\boldsymbol{u} - \boldsymbol{u}_h\|. \end{aligned} \tag{5.12}$$

Finally we combine the estimates (5.9)–(5.12) to (5.8) to obtain (5.7). The proof is complete. q.e.d.

Theorem 5.5. Let $k \geq 1$ if $d = 2$ and $k = 1$ if $d = 3$. Let $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ be the solution of (2.2) and $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ be the discrete solution of the reconstructed scheme (3.7). Then there holds

$$\|\pi_h p - p_h\| \leq C\frac{\nu}{\beta^*}\|\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)\| + \frac{C}{\beta^*}\mathrm{R}_{k-2}\big(\mathbb{P}(\boldsymbol{f})\big), \tag{5.13}$$

where $\pi_h p$ denotes the L^2 best approximation of p in Q_h.

Proof. For an arbitrary $\boldsymbol{w}_h \in \boldsymbol{X}_h$ one obtains

$$\begin{aligned} (\pi_h p - p_h, \nabla_h \cdot \boldsymbol{w}_h) &= (\pi_h p, \nabla \cdot (\boldsymbol{E}_h \boldsymbol{w}_h)) - (p_h, \nabla_h \cdot \boldsymbol{w}_h) \\ &= (p, \nabla \cdot (\boldsymbol{E}_h \boldsymbol{w}_h)) - (p_h, \nabla_h \cdot \boldsymbol{w}_h), \end{aligned}$$

since it holds for all elements T in the mesh, $(\nabla_h \cdot \boldsymbol{w}_h)_{|T} = (\nabla \cdot (\boldsymbol{E}_h \boldsymbol{w}_h))_{|T}$ (see Lemmas 4.4 and 4.7 (ii)). Using the definitions of the continuous and the discrete Stokes problems (2.2) and (3.7), one obtains

$$(\pi_h p - p_h, \nabla_h \cdot \boldsymbol{w}_h) = \underbrace{a(\boldsymbol{u}, \boldsymbol{E}_h \boldsymbol{w}_h) - a_h(\boldsymbol{u}_h, \boldsymbol{w}_h)}_{I_1} + \underbrace{\big(\boldsymbol{f}, \boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{w}_h - \boldsymbol{E}_h \boldsymbol{w}_h\big)}_{I_2}.$$

The first term can be estimated using the arguments for (5.2)

$$I_1 = a(\boldsymbol{u}, \boldsymbol{E}_h \boldsymbol{w}_h) - a_h(\boldsymbol{u}_h, \boldsymbol{w}_h) = a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{E}_h \boldsymbol{w}_h) \leq C\nu\|\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)\|\,\|\nabla_h \boldsymbol{w}_h\|.$$

For the second term, the right hand side data $\boldsymbol{f}$ is represented via the Helmholtz decomposition as $\boldsymbol{f} = \mathbb{P}(\boldsymbol{f}) + \nabla\phi$ with some $\phi \in H^1(\Omega)$, see Theorem 2.1. Hence, one obtains

$$I_2 = (\big(\boldsymbol{f}, \boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{w}_h - \boldsymbol{E}_h \boldsymbol{w}_h\big) = (\big(\mathbb{P}(\boldsymbol{f}), \boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{w}_h - \boldsymbol{E}_h \boldsymbol{w}_h\big) - (\phi, \nabla \cdot (\boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{w}_h - \boldsymbol{E}_h \boldsymbol{w}_h)),$$

and the last term is zero, since it holds $\nabla \cdot (\boldsymbol{\pi}^{\mathrm{BDM}}\boldsymbol{w}_h) = \nabla \cdot (\boldsymbol{E}_h \boldsymbol{w}_h)$ due to Lemma 4.4 ii). Now we remark that I_2 is the same term as I_3 in (5.3). The discrete inf-sup stability concludes the proof. q.e.d.

Remark 5.6 (Pressure-robustness of the discrete pressure error)**.** Assuming again that $\Delta\boldsymbol{u}, \nabla p \in \boldsymbol{L}^2(\Omega)$, we see that the discrete pressure p_h equals the best approximation $\pi_h p$ up to an error, which is only velocity-dependent, since it holds in this special case

$$\|\pi_h p - p_h\| \leq C\frac{\nu}{\beta^*}\|\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)\| + C\frac{\nu}{\beta^*}\mathrm{R}_{k-2}\big(\mathbb{P}(\Delta\boldsymbol{u})\big),$$

In this sense, the discrete pressure error $\|\pi_h p - p_h\|$ is pressure-robust.

Remark 5.7 (Hydrostatics)**.** Classical mixed methods and pressure-robust mixed methods differ most dramatically for hydrostatic problems with complicated pressures $p \in Q$. Assume that $\boldsymbol{f} = \nabla\phi$ for some $\phi \in H^1(\Omega) \cap Q$. Then, the continuous solution of (2.2) is given by $(\boldsymbol{u}, p) = (\boldsymbol{0}, \phi)$. Due to $\mathbb{P}(\boldsymbol{f}) = \boldsymbol{0}$ it holds, according to Theorems 5.1 and 5.5, for the discrete solution $(\boldsymbol{u}_h, p_h) = (\boldsymbol{0}, \pi_h\phi)$. Therefore, the pressure-robust discrete solution is the best possible on the given grid. On the contrary, the classical Crouzeix–Raviart element will show (at least on unstructured grids) for $\nu \ll 1$ extremely large errors, if ϕ is complicated, i.e., if it holds $\nu^{-1}\|\phi - \pi_h\phi\| \gg 1$.

Remark 5.8 (Pressure error)**.** The full pressure error $\|p - p_h\|$ can be obtained by

$$\|p - p_h\|^2 = \|p - \pi_h p\|^2 + \|\pi_h p - p_h\|^2.$$

The convergence order of $\|p - p_h\|$ is given by the minimum of the convergence order of the velocity error and the order of the pressure best approximation error.

ndof	$\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|$ (classical CR)	$\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|$ (modified CR)	quotient
2431	7.7508e-03	1.2310e-02	0.6296
9855	3.9152e-03	6.2873e-03	0.6227
39679	1.9652e-03	3.1713e-03	0.6197

Table 6.1: Comparison of the gradient errors of the classical and the modified Crouzeix–Raviart method for zero pressure p_1 and $\nu = 1$ in the first example.

5.1 Impact of the velocity-reconstruction

In this section we study the advantages of the velocity reconstruction on the error estimates in Theorem 5.1. In [22], it was shown that the classical Crouzeix–Raviart energy error satisfies

$$\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\| \leq C\big(\inf_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\|\nabla_h(\boldsymbol{u}-\boldsymbol{v}_h)\| + \nu^{-1}\inf_{q_h\in Q_h}\|p-q_h\| + \nu^{-1}\mathrm{R}_{k-2}(\boldsymbol{f})\big). \tag{5.14}$$

On the contrary, let $\boldsymbol{u}_h$ be the discrete solution to the reconstructed scheme (3.7). Then it follows from Theorem 5.1 that

$$\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\| \leq C\big(\inf_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\|\nabla_h(\boldsymbol{u}-\boldsymbol{v}_h)\| + \nu^{-1}\mathrm{R}_{k-2}\big(\mathbb{P}(\boldsymbol{f})\big)\big). \tag{5.15}$$

Remark 5.9. In Remark 5.2 it is argued that the term $\nu^{-1}\mathbb{P}(\boldsymbol{f})$ indicates a pressure-robust and locking-free error estimate for $\nu \ll 1$, if one assumes that $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\Delta\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ hold, simultaneously.

Avoiding the assumption $\Delta\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ requires first to extend the domain of the Helmholtz projector $\mathbb{P}$ from $\boldsymbol{L}^2(\Omega)$ to $\boldsymbol{H}^{-1}(\Omega)$ by simply restricting the application of $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ to the divergence-free test space $C^\infty_{0,\sigma}(\Omega)$, see [40]. Again, an important property of the Helmholtz projector in the $\boldsymbol{H}^{-1}$-sense is that all gradients in distributional sense vanish for divergence-free vector fields from $C^\infty_{0,\sigma}(\Omega)$ [40]. Exploiting the weak formulation (2.4) for $\boldsymbol{u}$, one obtains for the Helmholtz projector in the $\boldsymbol{H}^{-1}$-sense

$$\mathbb{P}(-\Delta\boldsymbol{u}) = \frac{1}{\nu}\mathbb{P}(\boldsymbol{f}) \in \boldsymbol{L}^2(\Omega),$$

which shows that the expression $\|\mathbb{P}(\Delta\boldsymbol{u})\|$ has a precise meaning, even if the assumption $\Delta\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ does not hold. Therefore, the error estimate in Theorem 5.1 is pressure-robust and does not suffer from any kind of locking phenomenon for $\nu \ll 1$.

Remark 5.10. The operator $\boldsymbol{E}_h : \boldsymbol{X}_h \to \boldsymbol{W}_h$ for $k = 1$ is also a useful tool for the numerical analysis of the classical Crouzeix–Raviart element, i.e., where one uses the classical right hand side discretization $\boldsymbol{w}_h \to (\boldsymbol{f}, \boldsymbol{w}_h)$. Then, a similar reasoning as in Theorem 5.1 will deliver the a-priori error estimate

$$\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\| \leq C\Big(\inf_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\|\nabla_h(\boldsymbol{u}-\boldsymbol{v}_h)\| + \nu^{-1}\mathrm{R}_{-1}\big(\boldsymbol{f}\big)\Big). \tag{5.16}$$

From a qualitative point of view, this is a better estimate than the estimate (5.14) presented in [22], since the new estimate does not contain any terms depending *explicitly* on the pressure regularity. But note that also this estimate is not pressure-robust, since $\nu^{-1}\mathrm{R}_{-1}\big(\boldsymbol{f}\big)$ depends *implicitly* on the pressure via the data term $\boldsymbol{f} = -\nu\Delta\boldsymbol{u} + \nabla p$.

Please, note also that using the operator $\boldsymbol{E}_h$ for $k \geq 2$ in a similar way for the analysis of the classical Crouzeix–Raviart element (with the classical right hand side discretization $\boldsymbol{w}_h \to (\boldsymbol{f}, \boldsymbol{w}_h)$), does not deliver further qualitative improvements of the estimate (5.16), since the inclusion of the BDM-operator in the definition of $\boldsymbol{E}_h$ (in order to get H^1-conforming divergence-free velocities $\boldsymbol{E}_h(\boldsymbol{w}_h)$ for all $\boldsymbol{w}_h \in \boldsymbol{V}_h$) is in contradiction to the necessary property for the volume moments (4.3c), in order to get an oscillation term in the error estimate.

6 Numerical Experiments

6.1 Illustration of pressure-robustness

The first example studies the velocity field

$$\boldsymbol{u}(x,y) := (\partial/\partial y, -\partial/\partial x)\, x^2(x-1)^2y^2(y-1)^2$$

ndof	$\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|$ (classical CR)	$\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|$ (modified CR)	quotient
2431	2.7103e-02	1.2310e-02	2.2017
9855	1.4029e-02	6.2873e-03	2.2313
39679	7.1242e-03	3.1713e-03	2.2465

Table 6.2: Comparison of the gradient errors of the classical and the modified Crouzeix–Raviart method for pressure p_2 and $\nu = 1$ in the first example.

ndof	$\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|$ (classical CR)	$\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|$ (modified CR)	quotient
2431	2.6109e+02	1.2310e-02	2121.0
9855	1.3540e+02	6.2873e-03	2153.5
39679	6.8819e+01	3.1713e-03	2170.1

Table 6.3: Comparison of the gradient errors of the classical and the modified Crouzeix–Raviart method for pressure p_2 and $\nu = 10^{-4}$ in the first example.

and two different pressure fields

$$p_1 := 0 \quad \text{and} \quad p_2 := x^3 + y^3 - 1/2$$

on the unit cube $\Omega := (0,1)^2$ and the matching right hand sides $\boldsymbol{f}_j := -\nu\Delta\boldsymbol{u} + \nabla p_j$ for different values of ν and $j = 1, 2$. The choice $j = 1$ yields a worst-case for the modified Crouzeix–Raviart method, since the pressure is then in the pressure ansatz space and so the pressure-dependent term in the classical estimate vanishes. However, the modified method makes a consistency error and by comparing the errors of both methods one can estimate the size of this consistency error. Table 6.1 shows that the error of the modified method in this worst-case scenario is about 60 percent larger than the error of the classical method.

In presence of a nonzero pressure that is not in the pressure ansatz space, like p_2, the situation changes. Table 6.2 shows that the error of the classical method is more than 120 percent larger than the error of the modified pressure-robust method, even for $\nu = 1$. For smaller ν the quotient increases proportional to $1/\nu$, see Table 6.3 for $\nu = 10^{-4}$ which results in factors of more than 2100. Note, that the error of the modified Crouzeix–Raviart method is the same in all three tables since its discrete velocity is pressure-independent.

6.2 The impact of quadrature rules

The second example employs the exact velocity $\boldsymbol{u} \equiv 0$ on the square domain $\Omega := (-1,1)^2$, where the pressure is given (up to a constant) by

$$p(x, y) := 1/(0.01 + x^2 + y^2).$$

Since the pressure is non-polynomial, the right-hand side $\boldsymbol{f} = \nabla p$ cannot be integrated exactly by simple quadrature rules. This leads to some quadrature error that pollutes the pressure-robustness. The reason is that the application of a quadrature rule in the right-hand side is similar to a projection of $\boldsymbol{f}$ onto some polynomial space.

	$\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|$ (classical CR)			$\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|$ (modified CR)		
ν	k=2	k=7	k=15	k=2	k=7	k=15
1	8.0921e-01	8.0951e-01	8.0951e-01	3.7434e-04	7.5452e-09	8.7045e-15
1e-1	8.0921e+00	8.0951e+00	8.0951e+00	3.7434e-03	7.5452e-08	2.3337e-14
1e-2	8.0921e+01	8.0951e+01	8.0951e+01	3.7434e-02	7.5452e-07	2.2706e-13
1e-3	8.0921e+02	8.0951e+02	8.0951e+02	3.7434e-01	7.5452e-06	2.3470e-12
1e-4	8.0921e+03	8.0951e+03	8.0951e+03	3.7434e+00	7.5452e-05	2.3913e-11
1e-5	8.0921e+04	8.0951e+04	8.0951e+04	3.7434e+01	7.5452e-04	2.4806e-10
1e-6	8.0921e+05	8.0951e+05	8.0951e+05	3.7434e+02	7.5452e-03	2.2993e-09
1e-7	8.0921e+06	8.0951e+06	8.0951e+06	3.7434e+03	7.5452e-02	2.4221e-08

Table 6.4: Comparison of the gradient errors of the classical and the modified Crouzeix–Raviart method on a fixed mesh with 16173 degrees of freedom and different ν and three different quadrature orders $k \in \{2, 7, 15\}$ in the second example.

Even if $\boldsymbol{f}$ is irrotational, its projection needs not to be exactly irrotational. Therefore, the error, though theoretically pressure-independent, shows some pressure-dependence that can be reduced by better quadrature rules. For a fixed mesh and different choices of ν, Table 6.4 compares the gradient errors of the classical and the modified methods for three different quadrature rules of degrees 2, 7 and 15.

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A Proof of Lemma 4.1

Recall that the dimension of $\boldsymbol{W}_k(T)$ is $(k+5)(k+1)$. On the other hand, the number of conditions given in (4.2) is equal to

$$2(3) + 3\dim \boldsymbol{\mathcal{P}}_{k-1}(\mathbb{R}) + \dim \boldsymbol{N}_{k-1}(T) = 6 + 6k + (k-1)(k+1) = (k+5)(k+1).$$

We show that $\boldsymbol{v}_h \in \boldsymbol{W}_k(T)$ vanishes on (4.2) if and only if $\boldsymbol{v}_h \equiv 0$.

First, since $\boldsymbol{v}_h|_{\partial T} \in \boldsymbol{\mathcal{P}}_{k+1}(\partial T)$, we have $\boldsymbol{v}_h|_{\partial T} = 0$. Now write $\boldsymbol{v}_h = \boldsymbol{w}_h + \boldsymbol{z}_h$ with $\boldsymbol{w}_h \in \boldsymbol{\mathcal{P}}_k(T) \oplus \boldsymbol{Q}_{k+1}(T)$, $\boldsymbol{z}_h = \sum_{i=0}^2 \mathbf{curl}(B_i z_h^{(i)})$ and $z_h^{(i)} \in A_{k-1}^{(i)}(T)$. Since $B_i|_{\partial T} = 0$ and $\nabla B_i|_{\partial T} = -|\nabla \lambda_i| b_i \boldsymbol{n}_{e_i}$, we find that $\boldsymbol{z}_h|_{e_i} = -|\nabla\lambda_i| b_i z_h^{(i)} \boldsymbol{t}_{e_i}$, where $\boldsymbol{t}_{e_i}$ the unit tangent of e_i, obtained by rotating $\boldsymbol{n}_{e_i}$ counter-clockwise 90 degrees. Thus, $\boldsymbol{z}_h \cdot \boldsymbol{n}|_{\partial T} = 0$, and therefore $0 = \boldsymbol{v}_h \cdot \boldsymbol{n}|_{\partial T} = \boldsymbol{w}_h \cdot \boldsymbol{n}|_{\partial T}$. In addition, by the definition of $A_{k-1}^{(i)}(T)$ and (4.2b),

$$0 = (\boldsymbol{v}_h, \boldsymbol{\rho}_h)_T = (\boldsymbol{w}_h + \boldsymbol{z}_h, \boldsymbol{\rho}_h)_T = (\boldsymbol{w}_h, \boldsymbol{\rho}_h)_T + \sum_{i=0}^{2} (B_i z_h^{(i)}, \mathrm{curl}(\boldsymbol{\rho}_h))_T = (\boldsymbol{w}_h, \boldsymbol{\rho}_h)_T$$

for all $\boldsymbol{\rho}_h \in \boldsymbol{N}_{k-1}(T)$. In summary, we have $\boldsymbol{w}_h \cdot \boldsymbol{n}|_{\partial T} = 0$ and $(\boldsymbol{w}_h, \boldsymbol{\rho}_h)_T = 0$ for all $\boldsymbol{\rho}_h \in \boldsymbol{N}_{k-1}(T)$. We now show that these conditions imply that $\boldsymbol{w}_h \equiv 0$.

Write $\boldsymbol{w}_h = \boldsymbol{p}_h + \boldsymbol{q}_h$ with $\boldsymbol{p}_h \in \boldsymbol{\mathcal{P}}_k(T)$ and $\boldsymbol{q}_h \in \boldsymbol{Q}_{k+1}(T)$. From the definition of $\boldsymbol{Q}_{k+1}(T)$ we see that

$$0 = (\boldsymbol{w}_h, \boldsymbol{\rho}_h)_T = (\boldsymbol{p}_h, \boldsymbol{\rho}_h)_T \qquad \forall \boldsymbol{\rho}_h \in \boldsymbol{N}_{k-1}(T),$$

and

$$0 = \langle \boldsymbol{w}_h \cdot \boldsymbol{n}_e, \boldsymbol{p}_h \cdot \boldsymbol{n}_e \rangle_e = \langle \boldsymbol{p}_h \cdot \boldsymbol{n}_e, \boldsymbol{p}_h \cdot \boldsymbol{n}_e \rangle_e \quad \forall e \in \mathcal{F}_1(T).$$

These two conditions imply that $\boldsymbol{p}_h \equiv 0$. Therefore $\boldsymbol{q}_h \cdot \boldsymbol{n}|_{\partial T}$, and by applying the definition of $\boldsymbol{Q}_{k+1}(T)$ once again, we get $\boldsymbol{q}_h \equiv 0$ and $\boldsymbol{w}_h \equiv 0$.

Finally, we have

$$0 = \langle \boldsymbol{v}_h \cdot \boldsymbol{t}_{e_i}, \kappa \rangle_{e_i} = \langle \boldsymbol{z}_h \cdot \boldsymbol{t}_{e_i}, \kappa \rangle_{e_i} = -|\nabla\lambda_i| \langle b_i z_h^{(i)}, \kappa \rangle_{e_i} \qquad \forall \kappa \in \mathcal{P}_{k-1}(e_i),$$

which implies $z_h^{(i)}|_{e_i} = 0$. Thus, $z_h^{(i)} = \lambda_i p_h^{(i)}$ for some $p^{(i)} \in \mathcal{P}_{k-2}(T)$. Applying the definition of $A_{k-1}^{(i)}(T)$ we conclude that

$$0 = (B_i z_h^{(i)}, p_h^{(i)})_T = (B_i \lambda_i p_h^{(i)}, p_h^{(i)})_T.$$

Since $B_i \lambda_i > 0$ on T, we conclude that $p_h^{(i)} \equiv 0$ and therefore $\boldsymbol{v}_h \equiv 0$.