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Optimal boundary control of a nonstandard Cahn–Hilliard system with dynamic boundary condition and double obstacle inclusions

Dedicated to our friend Prof. Dr. Gianni Gilardi on the occasion of his 70th birthday

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Abstract

In this paper, we study an optimal boundary control problem for a model for phase separation taking place in a spatial domain that was introduced by P. Podio-Guidugli in Ric. Mat. **55** (2006), pp. 105–118. The model consists of a strongly coupled system of nonlinear parabolic differential inclusions, in which products between the unknown functions and their time derivatives occur that are difficult to handle analytically; the system is complemented by initial and boundary conditions. For the order parameter of the phase separation process, a dynamic boundary condition involving the Laplace–Beltrami operator is assumed, which models an additional nonconserving phase transition occurring on the surface of the domain. We complement in this paper results that were established in the recent contribution appeared in Evol. Equ. Control Theory **6** (2017), pp. 35–58, by the two authors and Gianni Gilardi. In contrast to that paper, in which differentiable potentials of logarithmic type were considered, we investigate here the (more difficult) case of nondifferentiable potentials of double obstacle type. For such nonlinearities, the standard techniques of optimal control theory to establish the existence of Lagrange multipliers for the state constraints are known to fail. To overcome these difficulties, we employ the following line of approach: we use the results contained in the preprint arXiv:1609.07046 [math.AP] (2016), pp. 1–30, for the case of (differentiable) logarithmic potentials and perform a so-called “deep quench limit”. Using compactness and monotonicity arguments, it is shown that this strategy leads to the desired first-order necessary optimality conditions for the case of (nondifferentiable) double obstacle potentials.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ denote some open, connected and bounded domain with smooth boundary Γ (we should at least have $\Gamma \in C^2$), and let $T > 0$ be a fixed final time and $Q := \Omega \times (0,T)$, $\Sigma := \Gamma \times (0,T)$. We denote by $\partial_{\mathbf{n}}$, ∇_Γ, Δ_Γ, the outward normal derivative, the tangential gradient, and the Laplace–Beltrami operator on Γ, in this order. We study in this paper the following optimal boundary control problem:

$(\mathscr{P}_0)$ Minimize the cost functional

$$\begin{aligned} \mathscr{J}((\mu,\rho,\rho_\Gamma),u_\Gamma) := {} & \frac{\beta_1}{2}\|\mu-\hat{\mu}_Q\|^2_{L^2(Q)} + \frac{\beta_2}{2}\|\rho-\hat{\rho}_Q\|^2_{L^2(Q)} \\ & + \frac{\beta_3}{2}\|\rho_\Gamma-\hat{\rho}_\Sigma\|^2_{L^2(\Sigma)} + \frac{\beta_4}{2}\|\rho(T)-\hat{\rho}_\Omega\|^2_{L^2(\Omega)} \\ & + \frac{\beta_5}{2}\|\rho_\Gamma(T)-\hat{\rho}_\Gamma\|^2_{L^2(\Gamma)} + \frac{\beta_6}{2}\|u_\Gamma\|^2_{L^2(\Sigma)} \end{aligned} \tag{1.1}$$

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over a suitable set $\mathscr{U}_{\rm ad} \subset (H^1(0,T;L^2(\Gamma)) \cap L^\infty(\Sigma))$ of admissible controls u_Γ (to be specified later), subject to the state system

$$(1+2g(\rho))\,\mu_t + \mu\, g'(\rho)\,\rho_t - \Delta\mu = 0 \quad \text{a. e. in } Q, \tag{1.2}$$

$$\partial_{\mathbf{n}}\mu = 0 \quad \text{a. e. on } \Sigma, \quad \mu(0) = \mu_0 \quad \text{a. e. in } \Omega, \tag{1.3}$$

$$\rho_t - \Delta\rho + \xi + \pi(\rho) = \mu\, g'(\rho) \quad \text{a. e. in } Q, \tag{1.4}$$

$$\xi \in \partial I_{[-1,1]}(\rho) \quad \text{a. e. in } Q, \tag{1.5}$$

$$\partial_{\mathbf{n}}\rho + \partial_t\rho_\Gamma - \Delta_\Gamma\rho_\Gamma + \xi_\Gamma + \pi_\Gamma(\rho_\Gamma) = u_\Gamma, \quad \rho_\Gamma = \rho_{|\Sigma}, \quad \text{a. e. on } \Sigma, \tag{1.6}$$

$$\xi_\Gamma \in \partial I_{[-1,1]}(\rho_\Gamma) \quad \text{a. e. on } \Sigma, \tag{1.7}$$

$$\rho(0) = \rho_0 \quad \text{a. e. in } \Omega, \quad \rho_\Gamma(0) = \rho_{0_\Gamma} \quad \text{a. e. on } \Gamma. \tag{1.8}$$

Here, $\beta_i,\ 1 \le i \le 6$, are nonnegative weights, and $\hat\mu_Q, \hat\rho_Q \in L^2(Q),\ \hat\rho_\Sigma \in L^2(\Sigma),\ \hat\rho_\Omega \in L^2(\Omega)$, and $\hat\rho_\Gamma \in L^2(\Gamma)$ are prescribed target functions.

The physical background behind the control problem ($\mathscr{P}_0$) is the following: the state system (1.2)–(1.8) constitutes a model for phase separation taking place in the container Ω and originally introduced in [32]. In this connection, the unknowns μ and ρ denote the associated *chemical potential*, which in this particular model has to be nonnegative, and the *order parameter* of the phase separation process, which is usually the volumetric density of one of the involved phases. We assume that ρ is normalized in such a way as to attain its values in the interval $[-1,1]$. The nonlinearities π, π_Γ, g are assumed to be smooth in $[-1,1]$, and $\partial I_{[-1,1]}$ denotes the subdifferential of the indicator function of the interval $[-1,1]$. As is well known, we have that

$$I_{[-1,1]}(\rho) = \begin{cases} 0 & \text{if } \rho \in [-1,1] \\ +\infty & \text{otherwise} \end{cases}, \quad \partial I_{[-1,1]}(\rho) = \begin{cases} (-\infty,0] & \text{if } \rho = -1 \\ \{0\} & \text{if } -1 < \rho < 1 \\ [0,+\infty) & \text{if } \rho = 1 \end{cases}. \tag{1.9}$$

The state system (1.2)–(1.8) is singular, with highly nonlinear and nonstandard couplings. It has been the subject of intensive study over the past years for the case that (1.6) is replaced by a zero Neumann condition. In this conncetion, we refer the reader to [8–11, 13–16]. In [12], an associated control problem with a distributed control in (1.2) was investigated for the special case $g(\rho) = \rho$, and in [18], the corresponding case of a boundary control for μ was studied. A nonlocal version, in which the Laplacian $-\Delta\rho$ in (1.4) was replaced by a nonlocal operator, was discussed in the contributions [21–23].

In all of the works cited above a zero Neumann condition was assumed for the order parameter ρ. In contrast to this, we study in this paper the case of the dynamic boundary condition (1.6). It models a nonconserving phase transition taking place on the boundary, which could be induced by, e. g., an interaction between bulk and wall. The associated total free energy of the phase separation process is the sum of a bulk and a surface contribution and has the form

$$\begin{aligned}
&\mathscr{F}_{\text{tot}}[\mu(t),\rho(t),\rho_\Gamma(t)] \\
&:= \int_\Omega \Big(I_{[-1,1]}(\rho(x,t)) + \hat{\pi}(\rho(x,t)) - \mu(x,t)\, g(\rho(x,t)) + \frac{1}{2}|\nabla\rho(x,t)|^2 \Big)\, \mathrm{d}x \\
&+ \int_\Gamma \Big(I_{[-1,1]}(\rho_\Gamma(x,t)) + \hat{\pi}_\Gamma(\rho_\Gamma(x,t)) - u_\Gamma(x,t)\, \rho_\Gamma(x,t) + \frac{1}{2}|\nabla_\Gamma\rho_\Gamma(x,t)|^2 \Big)\, \mathrm{d}\Gamma, \qquad (1.10)
\end{aligned}$$

for $t \in [0,T]$, where $\hat{\pi}(r) = \int_0^r \pi(\xi)\mathrm{d}\xi$ and $\hat{\pi}_\Gamma(r) = \int_0^r \pi_\Gamma(\xi)\mathrm{d}\xi$. In the recent contribution [24], the state system (1.2)–(1.8) was studied systematically concerning existence, uniqueness, and regularity. A boundary control problem resembling ($\mathscr{P}_0$) was solved in [25] for the case of potentials of logarithmic type.

The mathematical literature on control problems for phase field systems involving equations of viscous or nonviscous Cahn–Hilliard type is still scarce and quite recent. We refer in this connection to the works [5,6,19,20,29,35]. Control problems for convective Cahn–Hilliard systems were studied in [33,36,37], and a few analytical contributions were made to the coupled Cahn–Hilliard/Navier–Stokes system (cf. [27,28,30,31]). The contribution [17] dealt with the optimal control of a Cahn–Hilliard type system arising in the modeling of solid tumor growth. For the optimal control of Allen–Cahn equations with dynamic boundary condition, we refer to [7,26].

In this paper, we aim to employ the results established in [25] to treat the nondifferentiable double obstacle case when ξ, ξ_Γ satisfy the inclusions (1.5), (1.7). Our approach is guided by a strategy that was introduced in [7] by the present authors and M.H. Farshbaf-Shaker: in fact, we aim to derive first-order necessary optimality conditions for the double obstacle case by performing a so-called "deep quench limit" in a family of optimal control problems with differentiable logarithmic nonlinearities that was treated in [25], and for which the corresponding state systems were analyzed in [24]. The general idea is briefly explained as follows: we replace the inclusions (1.5) and (1.7) by the identities

$$\xi = \varphi(\alpha)\, h'(\rho), \quad \xi_\Gamma = \varphi(\alpha)\, h'(\rho_\Gamma), \qquad (1.11)$$

where h is defined by

$$h(\rho) := \begin{cases} (1-\rho)\,\ln(1-\rho) + (1+\rho)\,\ln(1+\rho) & \text{if } \rho \in (-1,1) \\ 2\,\ln(2) & \text{if } \rho \in \{-1,1\} \end{cases}, \qquad (1.12)$$

and where φ is continuous and positive on $(0,1]$ and satisfies

$$\lim_{\alpha \searrow 0} \varphi(\alpha) = 0. \qquad (1.13)$$

We remark that we can simply choose $\varphi(\alpha) = \alpha^p$ for some $p > 0$. Now observe that $h'(y) = \ln\left(\frac{1+y}{1-y}\right)$ and $h''(y) = \frac{2}{1-y^2} > 0$ for $y \in (-1,1)$. Hence, in particular, we have

$$\begin{aligned}
&\lim_{\alpha \searrow 0} \varphi(\alpha)\, h'(y) = 0 \quad \text{for } -1 < y < 1, \\
&\lim_{\alpha \searrow 0} \Big(\varphi(\alpha) \lim_{y \searrow -1} h'(y) \Big) = -\infty, \quad \lim_{\alpha \searrow 0} \Big(\varphi(\alpha) \lim_{y \nearrow +1} h'(y) \Big) = +\infty. \qquad (1.14)
\end{aligned}$$

We thus may regard the graph $\varphi(\alpha)\,h'$ as an approximation to the graph of the subdifferential $\partial I_{[-1,1]}$.

Now, for any $\alpha > 0$ the optimal control problem (later to be denoted by $(\mathscr{P}_\alpha)$), which results if in $(\mathscr{P}_0)$ the relations (1.5), (1.7) are replaced by (1.11), is of the type for which in [25] the existence of optimal controls $u_\Gamma^\alpha \in \mathscr{U}_{\rm ad}$ as well as first-order necessary optimality conditions have been derived. Proving a priori estimates (uniform in $\alpha > 0$), and employing compactness and monotonicity arguments, we will be able to show the following existence and approximation result: whenever $\{u_\Gamma^{\alpha_n}\} \subset \mathscr{U}_{\rm ad}$ is a sequence of optimal controls for $(\mathscr{P}_{\alpha_n})$, where $\alpha_n \searrow 0$ as $n \to \infty$, then there exist a subsequence of $\{\alpha_n\}$, which is again indexed by n, and an optimal control $\bar{u}_\Gamma \in \mathscr{U}_{\rm ad}$ of $(\mathscr{P}_0)$ such that

$$u_\Gamma^{\alpha_n} \to \bar{u}_\Gamma \quad \text{weakly-star in } \mathscr{X} \text{ as } n \to \infty, \tag{1.15}$$

where, here and in the following,

$$\mathscr{X} := H^1(0,T;H_\Gamma) \cap L^\infty(\Sigma) \tag{1.16}$$

will always denote the control space. In other words, optimal controls for $(\mathscr{P}_\alpha)$ are for small $\alpha > 0$ likely to be 'close' to optimal controls for $(\mathscr{P}_0)$. It is natural to ask if the reverse holds, i. e., whether every optimal control for $(\mathscr{P}_0)$ can be approximated by a sequence $\{u_\Gamma^{\alpha_n}\}$ of optimal controls for $(\mathscr{P}_{\alpha_n})$, for some sequence $\alpha_n \searrow 0$.

Unfortunately, we will not be able to prove such a 'global' result that applies to all optimal controls for $(\mathscr{P}_0)$. However, a 'local' result can be established. To this end, let $\bar{u}_\Gamma \in \mathscr{U}_{\rm ad}$ be any optimal control for $(\mathscr{P}_0)$. We introduce the 'adapted' cost functional

$$\widetilde{\mathscr{J}}((\mu,\rho,\rho_\Gamma),u_\Gamma) := \mathscr{J}((\mu,\rho,\rho_\Gamma),u_\Gamma) + \frac{1}{2}\,\|u_\Gamma - \bar{u}_\Gamma\|^2_{L^2(\Sigma)} \tag{1.17}$$

and consider for every $\alpha \in (0,1]$ the *adapted control problem* of minimizing $\widetilde{\mathscr{J}}$ subject to $u_\Gamma \in \mathscr{U}_{\rm ad}$ and to the constraint that (μ,ρ,ρ_Γ) solves the approximating system (1.2)–(1.4), (1.6), (1.8), (1.11). It will then turn out that the following is true:

(i) There are some sequence $\alpha_n \searrow 0$ and minimizers $\bar{u}_\Gamma^{\alpha_n} \in \mathscr{U}_{\rm ad}$ of the adapted control problem associated with α_n, $n \in \mathbb{N}$, such that

$$\bar{u}_\Gamma^{\alpha_n} \to \bar{u}_\Gamma \quad \text{strongly in } L^2(\Sigma) \text{ as } n \to \infty. \tag{1.18}$$

(ii) It is possible to pass to the limit as $\alpha \searrow 0$ in the first-order necessary optimality conditions corresponding to the adapted control problems associated with $\alpha \in (0,1]$ in order to derive first-order necessary optimality conditions for problem $(\mathscr{P}_0)$.

The paper is organized as follows: in Section 2, we give a precise statement of the problem under investigation, and we derive some results concerning the state system (1.2)–(1.8) and its α – approximation which is obtained if in $(\mathscr{P}_0)$ the relations (1.5) and (1.7) are replaced by the relations (1.11). In Section 3, we then prove the existence of optimal controls and the approximation result formulated above in (i). The final Section 4 is devoted to the derivation of the first-order necessary optimality conditions, where the strategy outlined in (ii) is employed.

During the course of this analysis, we will make repeated use of Hölder's inequality, of the elementary Young's inequality

$$ab \le \gamma|a|^2 + \frac{1}{4\gamma}|b|^2 \quad \forall\, a,b \in \mathrm{I\!R} \quad \forall\, \gamma > 0, \tag{1.19}$$

and of the continuity of the embeddings $H^1(\Omega) \subset L^p(\Omega)$ for $1 \le p \le 6$. We will also use the denotations

$$Q_t := \Omega \times (0,t), \quad \Sigma_t := \Gamma \times (0,t), \quad \text{for } 0 < t \le T. \tag{1.20}$$

Throughout the paper, for a Banach space X we denote by $\|\cdot\|_X$ its norm and by X^* its dual space. The only exemption from this rule are the norms of the L^p spaces and of their powers, which we often denote by $\|\cdot\|_p$, for $1 \le p \le +\infty$. By $\langle v, w\rangle_X$ we will denote the dual pairing between elements $v \in X^*$ and $w \in X$. About the time derivative of a time-dependent function v, we warn the reader that we may use both the notation $\partial_t v$ and the shorter one v_t.

2 General assumptions and state equations

In this section, we formulate the general assumptions of the paper, and we state some preparatory results for the state system (1.2)–(1.8) and its $\alpha-$approximations. To begin with, we introduce some denotations. We set

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{w \in H^2(\Omega) : \partial_{\mathbf{n}} w = 0 \text{ on } \Gamma\},$$
$$H_\Gamma := L^2(\Gamma), \quad V_\Gamma := H^1(\Gamma), \quad \mathscr{V} := \{(v, v_\Gamma) \in V \times V_\Gamma : v_\Gamma = v_{|\Gamma}\},$$

and endow these spaces with their standard norms. Notice that we have $V \subset H \subset V'$ and $V_\Gamma \subset H_\Gamma \subset V_\Gamma'$, with dense, continuous and compact embeddings.

We make the following general assumptions:

(A1) $\mu_0 \in W$, $\mu_0 \ge 0$ in $\overline{\Omega}$, $\rho_0 \in H^2(\Omega)$, $\rho_{0_\Gamma} := \rho_{0_{|\Gamma}} \in H^2(\Gamma)$, and

$$-1 < \min_{x\in\overline{\Omega}} \rho_0(x), \quad \max_{x\in\overline{\Omega}} \rho_0(x) < +1. \tag{2.1}$$

(A2) $\pi, \pi_\Gamma \in C^2[-1,1]$; $g \in C^3[-1,1]$ is nonnegative and concave on $[-1,1]$.

(A3) $\mathscr{U}_{\rm ad} = \{u_\Gamma \in \mathscr{X} : u_* \le u_\Gamma \le u^* \text{ a. e. on } \Sigma \text{ and } \|u_\Gamma\|_{\mathscr{X}} \le R_0\}$, where $u_*, u^* \in L^\infty(\Sigma)$ and $R_0 > 0$ are such that $\mathscr{U}_{\rm ad} \ne \emptyset$.

Now observe that the set $\mathscr{U}_{\rm ad}$ is a bounded subset of $\mathscr{X}$. Hence, there exists a bounded open ball in $\mathscr{X}$ that contains $\mathscr{U}_{\rm ad}$. For later use it is convenient to fix such a ball once and for all, noting that any other such ball could be used instead. In this sense, the following assumption is rather a denotation:

(A4) Let $R>0$ be such that $\mathscr{U}_{\rm ad}\subset\mathscr{U}_R:=\{u_\Gamma\in\mathscr{X}:\,\|u_\Gamma\|_{\mathscr{X}}<R\}$.

For the quantities entering the cost functional $\mathscr{J}$ (see (1.1)), we assume:

(A5) The constants β_i, $1\le i\le 6$, are nonnegative but not all equal to zero, and we have that $\hat\mu_Q,\hat\rho_Q\in L^2(Q)$, $\hat\rho_\Sigma\in L^2(\Sigma)$, $\hat\rho_\Omega\in L^2(\Omega)$, $\hat\rho_\Gamma\in L^2(\Gamma)$.

We observe at this point that if (A1), (A2) and $u_\Gamma\in\mathscr{U}_R$ hold true, then all of the general assumptions made in [24] are satisfied provided we put, in the notation used there, $\hat\beta=\hat\beta_\Gamma=I_{[-1,1]}$. We thus may conclude from [24, Thm. 2.1 and Rem. 3.1] the following well-posedness result:

THEOREM 2.1: *Suppose that the assumptions* (A1)–(A4) *are fulfilled. Then the state system* (1.2)–(1.8) *has for every* $u_\Gamma\in\mathscr{U}_R$ *a unique solution* (μ,ρ,ρ_Γ) *with* $\mu\ge 0$ *a. e. in* Q*, which satisfies*

$$\mu\in C^0([0,T];V)\cap L^p(0,T;W)\cap L^2(0,T;W^{2,6}(\Omega))\cap L^\infty(Q)\quad\forall\,p\in[1,+\infty),\tag{2.2}$$

$$\mu_t\in L^p(0,T;H)\cap L^2(0,T;L^6(\Omega))\quad\forall\,p\in[1,+\infty),\tag{2.3}$$

$$\rho\in W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;H^2(\Omega)),\tag{2.4}$$

$$\rho_\Gamma\in W^{1,\infty}(0,T;H_\Gamma)\cap H^1(0,T;V_\Gamma)\cap L^\infty(0,T;H^2(\Gamma)),\tag{2.5}$$

$$\rho\in[-1,1]\quad\text{a. e. in } Q,\quad \rho_\Gamma\in[-1,1]\quad\text{a. e. on }\Sigma,\tag{2.6}$$

$$\xi\in L^\infty(0,T;H),\quad \xi_\Gamma\in L^\infty(0,T;H_\Gamma).\tag{2.7}$$

Moreover, there is a constant $K_1^*>0$*, which depends only on the data of the state system and on* R*, such that*

$$\|\mu\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W)\cap L^\infty(Q)}+\|\rho\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;H^2(\Omega))}$$
$$+\|\rho_\Gamma\|_{W^{1,\infty}(0,T;H_\Gamma)\cap H^1(0,T;V_\Gamma)\cap L^\infty(0,T;H^2(\Gamma))}+\|\xi\|_{L^\infty(Q)}+\|\xi_\Gamma\|_{L^\infty(\Sigma)}\le K_1^*,\tag{2.8}$$

whenever (μ,ρ,ρ_Γ) *is a solution to* (1.2)–(1.8) *which corresponds to some* $u_\Gamma\in\mathscr{U}_R$ *and satisfies* (2.2)–(2.7).

REMARK 2.2: Thanks to Theorem 2.1, the control-to-state operator $\mathscr{S}_0:u_\Gamma\mapsto(\mu,\rho,\rho_\Gamma)$ is well defined as a mapping from $\mathscr{U}_R$ into the space specified by the regularity properties (2.2)–(2.5). Moreover, in view of (2.4), it follows from well-known embedding results (see, e. g., [34, Sect. 8, Cor. 4]) that $\rho\in C^0([0,T];H^s(\Omega))$ for $0<s<2$. In particular, we have $\rho\in C^0(\overline{Q})$, so that $\rho_\Gamma=\rho_{|\Gamma}\in C^0(\overline{\Sigma})$.

We now turn our interest to the $\alpha-$approximating system that results if we replace (1.5) and

(1.7) by (1.11), with h given by (1.12) and φ satisfying (1.13). We then obtain the following system of equations:

$$(1+2g(\rho^\alpha))\,\mu_t^\alpha + \mu^\alpha g'(\rho^\alpha)\,\rho_t^\alpha - \Delta\mu^\alpha = 0 \quad \text{a.e. in } Q, \tag{2.9}$$

$$\partial_{\mathbf{n}}\mu^\alpha = 0 \quad \text{a.e. on } \Sigma, \quad \mu^\alpha(0) = \mu_0 \quad \text{a.e. in } \Omega, \tag{2.10}$$

$$\rho_t^\alpha - \Delta\rho^\alpha + \varphi(\alpha)\,h'(\rho^\alpha) + \pi(\rho^\alpha) \,=\, \mu^\alpha g'(\rho^\alpha) \quad \text{a.e. in } Q, \tag{2.11}$$

$$\partial_{\mathbf{n}}\rho^\alpha + \partial_t\rho_\Gamma^\alpha - \Delta_\Gamma\rho_\Gamma^\alpha + \varphi(\alpha)\,h'(\rho_\Gamma^\alpha) + \pi_\Gamma(\rho_\Gamma^\alpha) = u_\Gamma^\alpha, \quad \rho_\Gamma^\alpha = \rho^\alpha_{|\Sigma} \quad \text{a.e. on } \Sigma, \tag{2.12}$$

$$\rho^\alpha(0) = \rho_0 \quad \text{a.e. in } \Omega, \quad \rho_\Gamma^\alpha(0) = \rho_{0_\Gamma} \quad \text{a.e. on } \Gamma. \tag{2.13}$$

By virtue of [25, Thm. 2.4], the system (2.9)–(2.13) has for every $u_\Gamma^\alpha \in \mathscr{U}_R$ a unique solution $(\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha)$ satisfying $\mu^\alpha \geq 0$ in Q and (2.2)–(2.5). Moreover, there are constants $r_*(\alpha), r^*(\alpha) \in (-1,1)$, which depend only on R, α, and the data of the system, such that, for all $(x,t) \in \overline{Q}$,

$$-1 < r_*(\alpha) \leq \rho^\alpha(x,t) \leq r^*(\alpha) < 1, \quad -1 < r_*(\alpha) \leq \rho_\Gamma^\alpha(x,t) \leq r^*(\alpha) < 1. \tag{2.14}$$

Again it follows (recall Remark 2.2) that $\rho^\alpha \in C^0(\overline{Q})$ and $\rho_\Gamma^\alpha \in C^0(\overline{\Sigma})$. Therefore, we may infer from (A2) that there is a constant $K_2^* > 0$, which depends only on R and the data of the system, such that

$$\max_{0\leq i\leq 3}\left\|g^{(i)}(\rho^\alpha)\right\|_{C^0(\overline{Q})} + \max_{0\leq i\leq 2}\left(\left\|\pi^{(i)}(\rho^\alpha)\right\|_{C^0(\overline{Q})} + \left\|\pi_\Gamma^{(i)}(\rho_\Gamma^\alpha)\right\|_{C^0(\overline{\Sigma})}\right) \leq K_2^*, \tag{2.15}$$

for every solution triple $(\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha)$ corresponding to some $u_\Gamma \in \mathscr{U}_R$ and any $\alpha \in (0,1]$. Observe that a corresponding estimate cannot be concluded for the derivatives of $\varphi(\alpha)h$, since it may well happen that $r_*(\alpha) \searrow -1$ and/or $r^*(\alpha) \nearrow +1$, as $\alpha \searrow 0$.

According to the above considerations, for every $\alpha \in (0,1]$ the solution operator $\mathscr{S}_\alpha : u_\Gamma^\alpha \in \mathscr{U}_R \mapsto (\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha)$ is well defined as a mapping into the space that is specified by the regularity properties (2.2)–(2.5). We now aim to derive some a priori estimates for $(\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha)$ that are independent of $\alpha \in (0,1]$. We have the following result.

PROPOSITION 2.3: *Suppose that* (A1)–(A4) *are satisfied. Then there is some constant* $K_3^* > 0$*, which depends only on* R *and on the data of the system, such that we have: whenever* $(\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha) = \mathscr{S}_\alpha(u_\Gamma^\alpha)$ *for some* $u_\Gamma^\alpha \in \mathscr{U}_R$ *and some* $\alpha \in (0,1]$*, then it holds that*

$$\begin{aligned}
&\|\mu^\alpha\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W)\cap L^\infty(Q)} \\
&+ \|\rho^\alpha\|_{W^{1,\infty}(0,T;H)\cap H^1([0,T];V)\cap L^\infty(0,T;H^2(\Gamma))} \\
&+ \|\rho_\Gamma^\alpha\|_{W^{1,\infty}(0,T;H_\Gamma)\cap H^1([0,T];V_\Gamma)\cap L^\infty(0,T;H^2(\Gamma))} \\
&+ \|\varphi(\alpha)\,h'(\rho^\alpha)\|_{L^\infty(0,T;H)} + \|\varphi(\alpha)\,h'(\rho_\Gamma^\alpha)\|_{L^\infty(0,T;H_\Gamma)} \leq K_3^*.
\end{aligned} \tag{2.16}$$

PROOF: Let $u_\Gamma^\alpha \in \mathscr{U}_R$ and $\alpha \in (0,1]$ be arbitrary and $(\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha) = \mathscr{S}_\alpha(u_\Gamma^\alpha)$. The result will be established in a series of a priori estimates. To this end, we will in the following denote by $C>0$ constants that may depend on the quantities mentioned in the statement, but not on $\alpha\in(0,1]$. For the sake of a better readability, we will omit the superscript α of $(\mu^\alpha,\rho^\alpha,\rho_\Gamma^\alpha)$ during the estimations, writing it only at the end of each estimate. We will also make repeated use of the general bounds (2.15) without further reference.

<u>FIRST ESTIMATE:</u>

First, note that $\partial_t((\frac12+g(\rho))\,\mu^2) = (1+2g(\rho))\,\mu_t\,\mu + g'(\rho)\,\rho_t\,\mu^2$. Thus, multiplying (2.9) by μ and integrating over Q_t, where $t\in(0,T]$, we find the estimate

$$\int_\Omega \big(\tfrac12+g(\rho(t))\big|\,\mu(t)|^2\,\mathrm{d}x + \int_0^t\!\!\int_\Omega |\nabla\mu|^2\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega \big(\tfrac12+g(\rho_0))\big|\,\mu_0|^2\,\mathrm{d}x. \tag{2.17}$$

Hence, as $g(\rho)\ge 0$ by (A2), it follows that

$$\|\mu^\alpha\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le C \quad \forall\,\alpha\in(0,1]. \tag{2.18}$$

<u>SECOND ESTIMATE:</u>

Next, we multiply (2.11) by $\varphi(\alpha)\,h'(\rho^\alpha)$ and integrate over Q_t and by parts, where $t\in(0,T]$. We obtain the identity

$$\begin{aligned}
&\varphi(\alpha)\int_\Omega h(\rho(t))\,\mathrm{d}x + \varphi(\alpha)\int_\Gamma h(\rho_\Gamma(t))\,\mathrm{d}\Gamma + \int_0^t\!\!\int_\Omega |\varphi(\alpha)\,h'(\rho)|^2\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_0^t\!\!\int_\Gamma |\varphi(\alpha)\,h'(\rho_\Gamma)|^2\,\mathrm{d}\Gamma\,\mathrm{d}s + \varphi(\alpha)\int_0^t\!\!\int_\Omega h''(\rho)\,|\nabla\rho|^2\,\mathrm{d}x\,\mathrm{d}s\\
&+\varphi(\alpha)\int_0^t\!\!\int_\Gamma h''(\rho_\Gamma)\,|\nabla_\Gamma\rho_\Gamma|^2\,\mathrm{d}\Gamma\,\mathrm{d}s\\
&=\varphi(\alpha)\int_\Omega h(\rho_0)\,\mathrm{d}x + \varphi(\alpha)\int_\Gamma h(\rho_{0_\Gamma})\,\mathrm{d}\Gamma\\
&\quad+\int_0^t\!\!\int_\Omega (\mu\,g'(\rho)-\pi(\rho))\,\varphi(\alpha)\,h'(\rho)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_0^t\!\!\int_\Gamma (u_\Gamma^\alpha-\pi_\Gamma(\rho_\Gamma))\,\varphi(\alpha)\,h'(\rho_\Gamma)\,\mathrm{d}\Gamma\,\mathrm{d}s.
\end{aligned}\tag{2.19}$$

Obviously, all of the terms on the left-hand side are nonnegative, while the first two summands on the right-hand side are bounded independently of $\alpha\in(0,1]$. Thus, applying Hölder's and Young's inequalities to the last two integrals in (2.19), and invoking (2.15) and (2.18), we readily find that

$$\big\|\varphi(\alpha)\,h'(\rho^\alpha)\big\|_{L^2(Q)} + \big\|\varphi(\alpha)\,h'(\rho_\Gamma^\alpha)\big\|_{L^2(\Sigma)} \le C \quad \forall\,\alpha\in(0,1]. \tag{2.20}$$

<u>THIRD ESTIMATE:</u>

We now add ρ on both sides of (2.11) and ρ_Γ on both sides of (2.12). Then we multiply the first resulting equation by ρ_t and integrate over Q_t, where $t\in(0,T]$. Employing (2.15), we then obtain an inequality of the form

$$\int_0^t\int_\Omega |\rho_t|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Gamma |\partial_t\rho_\Gamma|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\left(\|\rho(t)\|_V^2 + \|\rho_\Gamma(t)\|_{V_\Gamma}^2\right)$$
$$\le \frac{1}{2}\left(\|\rho_0\|_V^2 + \|\rho_{0_\Gamma}\|_{V_\Gamma}^2\right) + \int_0^t\int_\Omega |\rho_t|\,(|\rho| + |\varphi(\alpha)\,h'(\rho)| + C(1+|\mu|))\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t\int_\Gamma |\partial_t\rho_\Gamma|\,(|\rho_\Gamma| + |\varphi(\alpha)\,h'(\rho_\Gamma)| + |u_\Gamma^\alpha|)\,\mathrm{d}\Gamma\,\mathrm{d}s\,. \tag{2.21}$$

Using (A1), (2.18), and (2.20), and employing Young's inequality and Gronwall's lemma, we thus conclude that

$$\|\rho^\alpha\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\rho_\Gamma^\alpha\|_{H^1(0,T;H_\Gamma)\cap L^\infty(0,T;V_\Gamma)} \le C \quad \forall\,\alpha\in(0,1]\,. \tag{2.22}$$

FOURTH ESTIMATE:

We now take advantage of the estimates (2.15), (2.18), (2.20) and (2.22). Indeed, comparison in (2.11) yields that

$$\|\Delta\rho\|_{L^2(Q)} \le C\,. \tag{2.23}$$

Now observe that, owing to [3, Thm. 3.2, p. 1.79], we have the estimate

$$\int_0^T \|\rho(t)\|_{H^{3/2}(\Omega)}^2\,\mathrm{d}t \le C\int_0^T \left(\|\Delta\rho(t)\|_H^2 + \|\rho_\Gamma(t)\|_{V_\Gamma}^2\right)\mathrm{d}t,$$

so that

$$\|\rho\|_{L^2(0,T;H^{3/2}(\Omega))} \le C. \tag{2.24}$$

Hence, by the trace theorem (cf. [3, Thm. 2.27, p. 1.64]), we infer that

$$\|\partial_{\mathbf{n}}\rho\|_{L^2(0,T;H_\Gamma)} \le C, \tag{2.25}$$

whence, by comparison in (2.12),

$$\|\Delta_\Gamma\rho_\Gamma\|_{L^2(0,T;L^2(\Gamma))} \le C. \tag{2.26}$$

Thus, by the boundary version of elliptic estimates, we deduce that

$$\|\rho_\Gamma\|_{L^2(0,T;H^2(\Gamma))} \le C, \tag{2.27}$$

whence, by virtue of standard elliptic theory, it turns out that

$$\|\rho\|_{L^2(0,T;H^2(\Omega))} \le C. \tag{2.28}$$

Since the embeddings

$$(H^1(0,T;H)\cap L^2(0,T;H^2(\Omega))) \subset C^0([0,T];V)$$

and

$$(H^1(0,T;H_\Gamma)\cap L^2(0,T;H^2(\Gamma)))\subset C^0([0,T];V_\Gamma)$$

are continuous, we have thus shown the estimate

$$\|\rho^\alpha\|_{C^0([0,T];V)\cap L^2(0,T;H^2(\Omega))}+\|\rho_\Gamma^\alpha\|_{C^0([0,T];V_\Gamma)\cap L^2(0,T;H^2(\Gamma))}\le C \quad \forall\,\alpha\in(0,1]. \tag{2.29}$$

FIFTH ESTIMATE:

In this step of the proof, we adopt a formal argument that can be made rigorous by using finite differences in time. Namely, we differentiate (2.11) formally with respect to time, multiply the resulting identity by ρ_t, and integrate over Q_t, where $0<t\le T$, and (formally) by parts. We then arrive at an inequality of the form

$$\begin{aligned}
&\frac{1}{2}\left(\|\rho_t(t)\|_H^2+\|\partial_t\rho_\Gamma(t)\|_{H_\Gamma}^2\right)+\int_0^t\!\!\int_\Omega|\nabla\partial_t\rho|^2\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\!\!\int_\Gamma|\nabla_\Gamma\partial_t\rho_\Gamma|^2\,\mathrm{d}\Gamma\,\mathrm{d}s\\
&+\varphi(\alpha)\int_0^t\!\!\int_\Omega h''(\rho)|\rho_t|^2\,\mathrm{d}x\,\mathrm{d}s+\varphi(\alpha)\int_0^t\!\!\int_\Gamma h''(\rho_\Gamma)|\partial_t\rho_\Gamma|^2\,\mathrm{d}\Gamma\,\mathrm{d}s\\
&\le\frac{1}{2}\left(\|\rho_t(0)\|_H^2+\|\partial_t\rho_\Gamma(0)\|_{H_\Gamma}^2\right)+\sum_{j=1}^4 I_j,
\end{aligned}\tag{2.30}$$

where the expressions I_j, $1\le j\le 4$, will be specified and estimated below. Notice that all of the terms on the left-hand side are nonnegative. At first, using (A1), (A2), the trace theorem, and the fact that $u_\Gamma^\alpha\in\mathscr{U}_{\mathrm{ad}}$, we find that

$$\begin{aligned}
&\|\rho_t(0)\|_H\le\|\Delta\rho_0-\varphi(\alpha)\,h'(\rho_0)-\pi(\rho_0)+\mu_0\,g'(\rho_0)\|_H\le C,\\
&\|\partial_t\rho_\Gamma(0)\|_H\le\|\partial_{\mathbf{n}}\rho_0\|_H+\|\Delta_\Gamma\rho_{0_\Gamma}-\varphi(\alpha)\,h'(\rho_{0_\Gamma})-\pi_\Gamma(\rho_{0_\Gamma})+u_\Gamma^\alpha(0)\|_H\le C.
\end{aligned}\tag{2.31}$$

Next, recalling (2.15) and (2.22), we have that

$$I_1:=-\int_0^t\!\!\int_\Omega\pi'(\rho)\,|\rho_t|^2\,\mathrm{d}x\,\mathrm{d}s\le C,\tag{2.32}$$

as well as, by also using Young's inequality,

$$I_4:=\int_0^t\!\!\int_\Gamma(\partial_t u_\Gamma^\alpha-\pi_\Gamma'(\rho_\Gamma)\,\partial_t\rho_\Gamma)\,\partial_t\rho_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}s\le C.\tag{2.33}$$

In addition, since $\mu\,g''(\rho)\le 0$, it turns out that

$$I_2:=\int_0^t\!\!\int_\Omega\mu\,g''(\rho)\,|\rho_t|^2\,\mathrm{d}x\,\mathrm{d}s\le 0.\tag{2.34}$$

The estimation of the remaining term

$$I_3:=\int_0^t\!\!\int_\Omega\mu_t\,g'(\rho)\,\rho_t\,\mathrm{d}x\,\mathrm{d}s$$

is more delicate. To this end, we use the identity (cf. (2.9))

$$\mu_t \,=\, (1+2g(\rho))^{-1}(\Delta\mu - \mu\, g'(\rho)\,\rho_t),$$

where, obviously, $1/(1+2g(\rho)) \le 1$. Substitution of this identity and integration by parts yield that

$$\begin{aligned} I_3 &= \int_0^t\!\!\int_\Omega \frac{1}{1+2g(\rho)}\left[\Delta\mu - \mu\, g'(\rho)\,\rho_t\right] g'(\rho)\,\rho_t\,\mathrm{d}x\,\mathrm{d}s \\ &= -\int_0^t\!\!\int_\Omega \nabla\mu(s)\cdot\nabla\Big(\frac{g'(\rho)\,\rho_t}{1+2g(\rho)}\Big)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\!\!\int_\Omega \frac{(g'(\rho))^2}{1+2g(\rho)}\,\mu\,|\rho_t|^2\,\mathrm{d}x\,\mathrm{d}s\,, \end{aligned} \tag{2.35}$$

where the second summand on the right is obviously nonpositive. We thus obtain the inequality

$$I_3 \,\le\, C\int_0^t\!\!\int_\Omega |\nabla\mu|\,|\nabla\rho_t|\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\!\!\int_\Omega |\nabla\mu|\,|\nabla\rho|\,|\rho_t|\,\mathrm{d}x\,\mathrm{d}s := J_1 + J_2\,. \tag{2.36}$$

Obviously, owing to Young's inequality and (2.18), we infer that

$$J_1 \,\le\, \frac{1}{4}\int_0^t\!\!\int_\Omega |\nabla\rho_t|^2\,\mathrm{d}x\,\mathrm{d}s + C. \tag{2.37}$$

On the other hand, thanks to Hölder's and Young's inequalities, we also have that

$$\begin{aligned} J_2 &\le C\int_0^t \|\nabla\mu(s)\|_2\,\|\nabla\rho(s)\|_4\,\|\rho_t(s)\|_4\,\mathrm{d}x\,\mathrm{d}s \\ &\le \frac{1}{4}\int_0^t \|\rho_t(s)\|_V^2\,\mathrm{d}s + C\int_0^t \|\nabla\mu(s)\|_H^2\,\|\nabla\rho(s)\|_V^2\,\mathrm{d}s \\ &\le C + \frac{1}{4}\int_0^t\!\!\int_\Omega |\nabla\rho_t|^2\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t \|\nabla\mu(s)\|_H^2\|\nabla\rho(s)\|_V^2\,\mathrm{d}s\,. \end{aligned} \tag{2.38}$$

The last integral cannot be controlled in this form. We thus try to estimate the expression $\|\nabla\rho(s)\|_V^2$ in terms of the expressions $\|\partial_t\rho(s)\|_H^2$ and $\|\partial_t\rho_\Gamma(s)\|_{H_\Gamma}^2$ which can be handled using the first summand on the left-hand side of (2.30). To this end, we use the regularity theory for linear elliptic equations and (2.29) to deduce that

$$\|\nabla\rho(s)\|_V^2 \,\le\, C\left(\|\rho(s)\|_V^2 + \|\Delta\rho(s)\|_H^2\right) \,\le\, C\left(1 + \|\Delta\rho(s)\|_H^2\right). \tag{2.39}$$

We now multiply, just as in the second estimate above, (2.11) by $\varphi(\alpha)\,h'(\rho(s))$, but this time we only integrate over Ω. We then obtain, for almost every $s\in(0,t)$,

$$\begin{aligned} &\|\varphi(\alpha)\,h'(\rho(s))\|_H^2 + \|\varphi(\alpha)\,h'(\rho_\Gamma(s))\|_{H_\Gamma}^2 + \varphi(\alpha)\int_\Omega h''(\rho(s))\,|\nabla\rho(s)|^2\,\mathrm{d}x \\ &+ \varphi(\alpha)\int_\Gamma h''(\rho_\Gamma(s))\,|\nabla_\Gamma\rho_\Gamma(s)|^2\,\mathrm{d}\Gamma \\ &= \int_\Omega \varphi(\alpha)\,h'(\rho(s))\left(-\rho_t(s) - \pi(\rho(s)) + \mu(s)\,g'(\rho(s))\right)\mathrm{d}x \\ &\quad + \int_\Gamma \varphi(\alpha)\,h'(\rho_\Gamma(s))\,(-\partial_t\rho_\Gamma(s) - \pi_\Gamma(\rho_\Gamma(s)) + \partial_t u_\Gamma^\alpha(s))\,\mathrm{d}\Gamma, \end{aligned} \tag{2.40}$$

whence, thanks to the already proven estimates and to Young's inequality,

$$\|\varphi(\alpha)\,h'(\rho(s))\|_H^2 + \|\varphi(\alpha)\,h'(\rho_\Gamma(s))\|_{H_\Gamma}^2 \le C\left(1+\|\partial_t\rho(s)\|_H^2+\|\partial_t\rho_\Gamma(s)\|_{H_\Gamma}^2\right)$$
$$\text{for a. e. } s\in(0,t). \tag{2.41}$$

Comparison in (2.11) then yields that

$$\|\Delta\rho(s)\|_H^2 \le C\left(1+\|\partial_t\rho(s)\|_H^2+\|\partial_t\rho_\Gamma(s)\|_{H_\Gamma}^2\right) \quad \text{for a. e. } s\in(0,t). \tag{2.42}$$

Combining the estimates (2.36)–(2.42), we have thus shown that

$$I_3 \le C + \frac{1}{2}\int_0^t\!\!\int_\Omega |\nabla\rho_t|\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t \|\nabla\mu(s)\|_H^2\left(\|\rho_t(s)\|_H^2+\|\partial_t\rho_\Gamma(s)\|_H^2\right)\mathrm{d}x\,\mathrm{d}s, \tag{2.43}$$

where the mapping $s\mapsto\|\nabla\mu(s)\|_H^2$ is known to be bounded in $L^1(0,T)$, uniformly with respect to $\alpha\in(0,1]$. We thus may combine (2.30)–(2.34) with (2.43) to infer from Gronwall's lemma that

$$\|\rho^\alpha\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} + \|\rho_\Gamma^\alpha\|_{W^{1,\infty}(0,T;H_\Gamma)\cap H^1(0,T;V_\Gamma)} \le C \quad \forall\,\alpha\in(0,1]. \tag{2.44}$$

Therefore, we can conclude from (2.41) and (2.42) that also, for all $\alpha\in(0,1]$,

$$\|\varphi(\alpha)\,h'(\rho^\alpha)\|_{L^\infty(0,T;H)} + \|\varphi(\alpha)\,h'(\rho_\Gamma^\alpha)\|_{L^\infty(0,T;H_\Gamma)} + \|\Delta\rho^\alpha\|_{L^\infty(0,T;H)} \le C. \tag{2.45}$$

Since we already know from (2.29) the bound for $\|\rho_\Gamma^\alpha\|_{C^0([0,T];V_\Gamma)}$, we can follow the same chain of estimates as in the fourth a priori estimate above, eventually obtaining that

$$\|\rho^\alpha\|_{L^\infty(0,T;H^2(\Omega))} + \|\rho_\Gamma^\alpha\|_{L^\infty(0,T;H^2(\Gamma))} \le C \quad \forall\,\alpha\in(0,1]. \tag{2.46}$$

SIXTH ESTIMATE:

Next, we multiply (2.9) by μ_t and integrate over Q_t, where $t\in(0,T]$. Recalling that $g(\rho)$ is nonnegative, and using Hölder's and Young's inequalities, we obtain from (A1) that

$$\begin{aligned}
&\int_0^t\!\!\int_\Omega |\mu_t|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\,\|\nabla\mu(t)\|_H^2 \le \frac{1}{2}\,\|\nabla\mu_0\|_H^2 + C\int_0^t\!\!\int_\Omega |\mu_t|\,|\mu|\,|\rho_t|\,\mathrm{d}x\,\mathrm{d}s\\
&\le C + C\int_0^t \|\mu_t(s)\|_2\,\|\mu(s)\|_4\,\|\rho_t(s)\|_4\,\mathrm{d}s\\
&\le C + \frac{1}{2}\int_0^t\!\!\int_\Omega |\mu_t|^2\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t \|\rho_t(s)\|_V^2\,\|\mu(s)\|_V^2\,\mathrm{d}s,
\end{aligned} \tag{2.47}$$

where, owing to (2.44), the mapping $s\mapsto\|\rho_t(s)\|_V^2$ is bounded in $L^1(0,T)$, uniformly in $\alpha\in(0,1]$. We thus can infer from Gronwall's lemma that

$$\|\mu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le C. \tag{2.48}$$

Comparison in (2.9) then shows that also

$$\|\Delta\mu\|_{L^2(0,T;H)} \le C, \tag{2.49}$$

whence, by virtue of standard elliptic estimates,

$$\|\mu\|_{L^2(0,T;W)} \leq C. \tag{2.50}$$

Since the embedding $(H^1(0,T;H) \cap L^2(0,T;H^2(\Omega))) \subset C^0([0,T];V)$ is continuous, we have thus shown the estimate

$$\|\mu^\alpha\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W)} \leq C \quad \forall\, \alpha \in (0,1]. \tag{2.51}$$

Next, we use the continuity of the embedding

$$(L^\infty(0,T;H) \cap L^2(0,T;V)) \subset L^{7/3}(0,T;L^{14/3}(\Omega)),$$

which, in view of (2.44), implies that

$$\|\rho_t^\alpha\|_{L^{7/3}(0,T;L^{14/3}(\Omega))} \leq C \quad \forall \alpha \in (0,1]. \tag{2.52}$$

With this estimate shown, we may argue as in the proof of [11, Thm. 2.3] to conclude that

$$\|\mu^\alpha\|_{L^\infty(Q)} \leq C \quad \forall\, \alpha \in (0,1]. \tag{2.53}$$

Hence, the assertion is completely proved. ∎

3 Existence and approximation of optimal controls

In this section, we aim to approximate optimal pairs of $(\mathscr{P}_0)$. To this end, we consider for $\alpha \in (0,1]$ the optimal control problem

$(\mathscr{P}_\alpha)$ Minimize the cost functional $\mathscr{J}((\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha), u_\Gamma^\alpha)$ for $u_\Gamma^\alpha \in \mathscr{U}_{\rm ad}$, subject to the state system (2.9)–(2.13).

According to [25, Thm. 4.1], this optimal control problem has an optimal pair $((\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha), u_\Gamma^\alpha)$, for every $\alpha \in (0,1]$. Our first aim in this section is to prove the following approximation result:

THEOREM 3.1: *Suppose that the assumptions* (A1)–(A5) *are satisfied, and let the sequences* $\{\alpha_n\} \subset (0,1]$ *and* $\{u_\Gamma^{\alpha_n}\} \subset \mathscr{U}_{\rm ad}$ *be given such that* $\alpha_n \searrow 0$ *and* $u_\Gamma^{\alpha_n} \to u_\Gamma$ *weakly-star in* $\mathscr{X}$ *for some* $u_\Gamma \in \mathscr{U}_{\rm ad}$. *Then it holds, for* $(\mu^{\alpha_n}, \rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}) = \mathscr{S}_{\alpha_n}(u_\Gamma^{\alpha_n})$, $n \in \mathbb{N}$,

$$\mu^{\alpha_n} \to \mu \quad \textit{weakly-star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q), \tag{3.1}$$

$$\rho^{\alpha_n} \to \rho \quad \textit{weakly-star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)), \tag{3.2}$$

$$\rho_\Gamma^{\alpha_n} \to \rho_\Gamma \quad \textit{weakly-star in } W^{1,\infty}(0,T;H_\Gamma) \cap H^1(0,T;V_\Gamma) \cap L^\infty(0,T;H^2(\Gamma)), \tag{3.3}$$

as well as

$$\varphi(\alpha_n)\,h'(\rho^{\alpha_n}) \to \xi \quad \textit{weakly-star in } L^\infty(0,T;H), \tag{3.4}$$

$$\varphi(\alpha_n)\,h'(\rho_\Gamma^{\alpha_n}) \to \xi_\Gamma \quad \textit{weakly-star in } L^\infty(0,T;H_\Gamma), \tag{3.5}$$

where $(\mu,\rho,\rho_\Gamma,\xi,\xi_\Gamma)$ *is the unique solution to the state system* (1.2)–(1.8) *associated with* u_Γ*. Moreover, with* $\mathscr{S}_0(u_\Gamma)=(\mu,\rho,\rho_\Gamma)$ *it holds that*

$$\mathscr{J}(\mathscr{S}_0(u_\Gamma),u_\Gamma) \le \liminf_{n\to\infty} \mathscr{J}(\mathscr{S}_{\alpha_n}(u_\Gamma^{\alpha_n}),u_\Gamma^{\alpha_n}), \tag{3.6}$$

$$\mathscr{J}(\mathscr{S}_0(v_\Gamma),v_\Gamma) = \lim_{n\to\infty} \mathscr{J}(\mathscr{S}_{\alpha_n}(v_\Gamma),v_\Gamma) \quad \forall\, v_\Gamma \in \mathscr{U}_{\rm ad}. \tag{3.7}$$

PROOF: Let $\{\alpha_n\}\subset(0,1]$ be any sequence such that $\alpha_n \searrow 0$ as $n\to\infty$, and suppose that $u_\Gamma^{\alpha_n} \to u_\Gamma$ weakly-star in $\mathscr{X}$ for some $u_\Gamma \in \mathscr{U}_{\rm ad}$. By virtue of Proposition 2.3, there are a subsequence of $\{\alpha_n\}$, which is again indexed by n, and some quintuple $(\mu,\rho,\rho_\Gamma,\xi,\xi_\Gamma)$ such that the convergence results (3.1)–(3.5) hold true. In particular, we have $\mu(0)=\mu_0$ and $\rho(0)=\rho_0$. Moreover, from standard compact embedding results (cf. [34, Sect. 8, Cor. 4]) we can infer that

$$\mu^{\alpha_n} \to \mu \quad \text{strongly in } C^0(0,T;H)\cap L^2(0,T;V), \tag{3.8}$$

$$\rho^{\alpha_n} \to \rho \quad \text{strongly in } C^0(\overline{Q}), \tag{3.9}$$

also including

$$\rho_\Gamma^{\alpha_n} \to \rho_\Gamma \quad \text{strongly in } C^0(\overline{\Sigma}), \tag{3.10}$$

whence we infer that $\rho_\Gamma = \rho_{|\Sigma}$. Therefore, we obviously have that

$$\Psi(\rho^{\alpha_n}) \to \Psi(\rho) \quad \text{strongly in } C^0(\overline{Q}), \text{ for } \Psi\in\{g,g',\pi\}, \tag{3.11}$$

$$\pi_\Gamma(\rho_\Gamma^{\alpha_n}) \to \pi_\Gamma(\rho_\Gamma) \quad \text{strongly in } C^0(\overline{\Sigma}), \tag{3.12}$$

and (3.2) implies that $\partial_{\mathbf{n}}\rho^{\alpha_n} \to \partial_{\mathbf{n}}\rho$ weakly in $L^2(\Sigma)$. Further, we easily verify that, at least weakly in $L^1(Q)$,

$$g(\rho^{\alpha_n})\,\mu_t^{\alpha_n} \to g(\rho)\,\mu_t, \quad \mu^{\alpha_n} g'(\rho^{\alpha_n})\,\rho_t^{\alpha_n} \to \mu\, g'(\rho)\,\rho_t, \quad \mu^{\alpha_n} g'(\rho^{\alpha_n}) \to \mu\, g'(\rho). \tag{3.13}$$

Combining the above convergence results, we may pass to the limit as $n\to\infty$ in the equations (2.9)–(2.13) (written for $\alpha=\alpha_n$) to find that the quintuple $(\mu,\rho,\rho_\Gamma,\xi,\xi_\Gamma)$ satisfies the equations (1.2)–(1.4), (1.6), and (1.8). In addition, we have $\mu\ge 0$ in Q, and the properties in (2.6) are fulfilled. We also notice that the regularities in (2.2)–(2.3) follow from $\mu_0\in W$ (cf. (A1)) and the regularity theory for solutions to linear uniformly parabolic equations with continuous coefficients and right-hand side in $L^\infty(0,T;H)\cap L^2(0,T;L^6(\Omega))$ (comments are given in [24, Section 3, Step 4 and Remark 3.1]). Then, in order to show that the quintuple $(\mu,\rho,\rho_\Gamma,\xi,\xi_\Gamma)$ is in fact the unique solution to problem (1.2)–(1.8) corresponding to u_Γ, it remains to show that $\xi\in\partial I_{[-1,1]}(\rho)$ a.e. in Q and $\xi_\Gamma\in\partial I_{[-1,1]}(\rho_\Gamma)$ a.e. in Σ.

Now, recall that h is convex in $[-1,1]$ and both h and φ are nonnegative. We thus have, for every $n \in \mathbb{N}$,

$$0 \le \int_0^T \int_\Omega \varphi(\alpha_n)\, h(\rho^{\alpha_n})\, \mathrm{d}x\, \mathrm{d}t$$
$$\le \int_0^T \int_\Omega \varphi(\alpha_n)\, h(z)\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_\Omega \varphi(\alpha_n)\, h'(\rho^{\alpha_n})\, (\rho^{\alpha_n} - z)\, \mathrm{d}x\, \mathrm{d}t$$
$$\text{for all } z \in \mathscr{K} := \{v \in L^2(Q) : |v| \le 1 \text{ a.e. in } Q\}. \tag{3.14}$$

Thanks to (1.13), the first integral on the central line of (3.14) tends to zero as $n \to \infty$. Hence, invoking (3.4) and (3.9), the passage to the limit as $n \to \infty$ yields

$$\int_0^T \int_\Omega \xi\, (\rho - z)\, \mathrm{d}x\, \mathrm{d}t \ge 0 \quad \forall z \in \mathscr{K}. \tag{3.15}$$

Inequality (3.15) entails that ξ is an element of the subdifferential of the extension $\mathscr{I}$ of $I_{[-1,1]}$ to $L^2(Q)$, which means that $\xi \in \partial \mathscr{I}(\rho)$ or, equivalently (cf. [2, Ex. 2.3.3., p. 25]), $\xi \in \partial I_{[-1,1]}(\rho)$ a. e. in Q. Similarly, we can prove that $\xi_\Gamma \in \partial I_{[-1,1]}(\rho_\Gamma)$ a. e. in Σ.

We have thus shown that, for a suitable subsequence of $\{\alpha_n\}$, we have the convergence properties (3.1)–(3.5), where $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma)$ is a solution to the state system (1.2)–(1.8). But this solution is known to be unique, which entails that the above convergence properties are valid for the entire sequence. This finishes the proof of the first claim of the theorem.

It remains to show the validity of (3.6) and (3.7). In view of (3.1)–(3.3), the inequality (3.6) is an immediate consequence of the weak sequential semicontinuity properties of the cost functional $\mathscr{J}$. To establish the identity (3.7), let $v_\Gamma \in \mathscr{U}_{\mathrm{ad}}$ be arbitrary and put $(\mu^{\alpha^n}, \rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}) = \mathscr{S}_{\alpha_n}(v_\Gamma)$, for $n \in \mathbb{N}$. Taking Proposition 2.3 into account, and arguing as in the first part of this proof, we can conclude that $\mathscr{S}_{\alpha_n}(v_\Gamma)$ converges to $(\mu, \rho, \rho_\Gamma) = \mathscr{S}_0(v_\Gamma)$ in the sense of (3.1)–(3.3) and (3.8)–(3.10). In particular, we have

$$\mathscr{S}_{\alpha_n}(v_\Gamma) \to \mathscr{S}_0(v_\Gamma) \quad \text{strongly in } C^0([0,T];H) \times C^0([0,T];H) \times C^0([0,T];H_\Gamma).$$

As the cost functional $\mathscr{J}$ is obviously continuous in the variables (μ, ρ, ρ_Γ) with respect to the strong topology of $C^0([0,T];H) \times C^0([0,T];H) \times C^0([0,T];H_\Gamma)$, we may thus infer that (3.7) is valid. ∎

COROLLARY 3.2: *The optimal control problem* ($\mathscr{P}_0$) *has a least one solution.*

PROOF: Pick an arbitrary sequence $\{\alpha_n\}$ such that $\alpha_n \searrow 0$ as $n \to \infty$. Then, by virtue of [25, Thm. 4.1], the optimal control problem ($\mathscr{P}_{\alpha_n}$) has for every $n \in \mathbb{N}$ an optimal pair $((\mu^{\alpha_n}, \rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}), u_\Gamma^{\alpha_n})$, where $u_\Gamma^{\alpha_n} \in \mathscr{U}_{\mathrm{ad}}$ and $(\mu^{\alpha_n}, \rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}) = \mathscr{S}_{\alpha_n}(u_\Gamma^{\alpha_n})$. Since $\mathscr{U}_{\mathrm{ad}}$ is a bounded subset of $\mathscr{X}$, we may without loss of generality assume that $u_\Gamma^{\alpha_n} \to u_\Gamma$ weakly-star in $\mathscr{X}$ for some $u_\Gamma \in \mathscr{U}_{\mathrm{ad}}$. Then, for the unique solution $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma)$ to (1.2)–(1.8) associated with u_Γ, we conclude from Theorem 3.1 the convergence properties (3.1)–(3.7). Invoking the optimality of $((\mu^{\alpha_n},\rho^{\alpha_n},\rho_\Gamma^{\alpha_n}),u_\Gamma^{\alpha_n})$ for $(\mathscr{P}_{\alpha_n})$, we then find, for every $v_\Gamma \in \mathscr{U}_{\rm ad}$, that

$$\begin{aligned}\mathscr{J}((\mu,\rho,\rho_\Gamma),u_\Gamma) &= \mathscr{J}(\mathscr{S}_0(u_\Gamma),u_\Gamma) \le \liminf_{n\to\infty} \mathscr{J}(\mathscr{S}_{\alpha_n}(u_\Gamma^{\alpha_n}),u_\Gamma^{\alpha_n}) \\ &\le \liminf_{n\to\infty} \mathscr{J}(\mathscr{S}_{\alpha_n}(v_\Gamma),v_\Gamma) = \lim_{n\to\infty} \mathscr{J}(\mathscr{S}_{\alpha_n}(v_\Gamma),v_\Gamma) = \mathscr{J}(\mathscr{S}_0(v_\Gamma),v_\Gamma), \qquad (3.16)\end{aligned}$$

which yields that u_Γ is an optimal control for $(\mathscr{P}_0)$ with the associate state $(\mu,\rho,\rho_\Gamma,\xi,\xi_\Gamma)$. The assertion is thus proved. ∎

Corollary 3.2 does not yield any information on whether every solution to the optimal control problem $(\mathscr{P}_0)$ can be approximated by a sequence of solutions to the problems $(\mathscr{P}_\alpha)$. As already announced in the Introduction, we are not able to prove such a general 'global' result. Instead, we can only give a 'local' answer for every individual optimizer of $(\mathscr{P}_0)$. For this purpose, we employ a trick due to Barbu [1]. To this end, let $\bar u_\Gamma \in \mathscr{U}_{\rm ad}$ be an arbitrary optimal control for $(\mathscr{P}_0)$, and let $(\bar\mu,\bar\rho,\bar\rho_\Gamma,\bar\xi,\bar\xi_\Gamma)$ be the associated solution quintuple to the state system (1.2)–(1.8) in the sense of Theorem 2.1. In particular, $(\bar\mu,\bar\rho,\bar\rho_\Gamma) = \mathscr{S}_0(\bar u_\Gamma)$. We associate with this optimal control the *adapted cost functional*

$$\widetilde{\mathscr{J}}((\mu,\rho,\rho_\Gamma),u_\Gamma) := \mathscr{J}((\mu,\rho,\rho_\Gamma),u_\Gamma) + \frac{1}{2}\,\|u_\Gamma - \bar u_\Gamma\|^2_{L^2(\Sigma)} \qquad (3.17)$$

and a corresponding *adapted optimal control problem*,

$(\widetilde{\mathscr{P}}_\alpha)$ Minimize $\widetilde{\mathscr{J}}((\mu,\rho,\rho_\Gamma),u_\Gamma)$ for $u_\Gamma \in \mathscr{U}_{\rm ad}$, subject to the condition that (2.9)–(2.13) be satisfied.

With a standard direct argument that needs no repetition here, we can show the following result.

LEMMA 3.3: *Suppose that the assumptions* (A1)–(A5), (1.12)–(1.13) *are satisfied, and let* $\alpha \in (0,1]$. *Then the optimal control problem* $(\widetilde{\mathscr{P}}_\alpha)$ *admits a solution.*

We are now in the position to give a partial answer to the question raised above. We have the following result.

THEOREM 3.4: *Let the assumptions* (A1)–(A5), (1.12)–(1.13) *be fulfilled, suppose that* $\bar u_\Gamma \in \mathscr{U}_{\rm ad}$ *is an arbitrary optimal control of* $(\mathscr{P}_0)$ *with associated state quintuple* $(\bar\mu,\bar\rho,\bar\rho_\Gamma,\bar\xi,\bar\xi_\Gamma)$, *and let* $\{\alpha_n\} \subset (0,1]$ *be any sequence such that* $\alpha_n \searrow 0$ *as* $n\to\infty$. *Then there exist a subsequence* $\{\alpha_{n_k}\}$ *of* $\{\alpha_n\}$, *and, for every* $k \in \mathbb{N}$, *an optimal control* $u_\Gamma^{\alpha_{n_k}} \in \mathscr{U}_{\rm ad}$ *of the* *adapted problem* $(\widetilde{\mathcal{P}}_{\alpha_{n_k}})$ *with associated state triple* $(\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}})$ *such that, as* $k \to \infty$,

$$u_\Gamma^{\alpha_{n_k}} \to \bar{u}_\Gamma \quad \text{strongly in } L^2(\Sigma), \tag{3.18}$$

the properties (3.1)–(3.5) *are satisfied, where* $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma)$
is replaced by $(\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma, \bar{\xi}, \bar{\xi}_\Gamma)$ *and the index* n *is replaced by* n_k, (3.19)

$$\widetilde{\mathcal{J}}((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}) \to \mathcal{J}((\mu, \rho, \rho_\Gamma), u_\Gamma). \tag{3.20}$$

PROOF: Let $\alpha_n \searrow 0$ as $n \to \infty$. For any $n \in \mathbb{N}$, we pick an optimal control $u_\Gamma^{\alpha_n} \in \mathcal{U}_{\rm ad}$ for the adapted problem $(\widetilde{\mathcal{P}}_{\alpha_n})$ and denote by $(\mu^{\alpha_n}, \rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}) = \mathcal{S}_{\alpha_n}(u_\Gamma^{\alpha_n})$ the associated solution triple of problem (2.9)–(2.13) for $\alpha = \alpha_n$. By the boundedness of $\mathcal{U}_{\rm ad}$, there is some subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that

$$u_\Gamma^{\alpha_{n_k}} \to u_\Gamma \quad \text{weakly-star in } \mathcal{X} \quad \text{as } k \to \infty, \tag{3.21}$$

with some $u_\Gamma \in \mathcal{U}_{\rm ad}$. Thanks to Theorem 3.1, the convergence properties (3.1)–(3.5) hold true, where $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma)$ is the unique solution to the state system (1.2)–(1.8). In particular, the pair $(\mathcal{S}_0(u_\Gamma), u_\Gamma) = ((\mu, \rho, \rho_\Gamma), u_\Gamma)$ is admissible for $(\mathcal{P}_0)$.

We now aim to prove that $u_\Gamma = \bar{u}_\Gamma$. Once this is shown, then the uniqueness result of Theorem 2.1 yields that also $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma) = (\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma, \bar{\xi}, \bar{\xi}_\Gamma)$, which implies that (3.19) holds true.

Now observe that, owing to the weak sequential lower semicontinuity of $\widetilde{\mathcal{J}}$, and in view of the optimality property of $((\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma), \bar{u}_\Gamma)$ for problem $(\mathcal{P}_0)$,

$$\begin{aligned}
&\liminf_{k\to\infty} \widetilde{\mathcal{J}}((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}) \geq \mathcal{J}((\mu, \rho, \rho_\Gamma), u_\Gamma) + \frac{1}{2}\|u_\Gamma - \bar{u}_\Gamma\|^2_{L^2(\Sigma)} \\
&\geq \mathcal{J}((\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma), \bar{u}_\Gamma) + \frac{1}{2}\|u_\Gamma - \bar{u}_\Gamma\|^2_{L^2(\Sigma)}.
\end{aligned} \tag{3.22}$$

On the other hand, the optimality property of $((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}})$ for problem $(\widetilde{\mathcal{P}}_{\alpha_{n_k}})$ yields that for any $k \in \mathbb{N}$ we have

$$\widetilde{\mathcal{J}}((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}) = \widetilde{\mathcal{J}}(\mathcal{S}_{\alpha_{n_k}}(u_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}) \leq \widetilde{\mathcal{J}}(\mathcal{S}_{\alpha_{n_k}}(\bar{u}_\Gamma), \bar{u}_\Gamma), \tag{3.23}$$

whence, taking the limit superior as $k \to \infty$ on both sides and invoking (3.7) in Theorem 3.1,

$$\begin{aligned}
&\limsup_{k\to\infty} \widetilde{\mathcal{J}}((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}) \\
&\leq \widetilde{\mathcal{J}}(\mathcal{S}_0(\bar{u}_\Gamma), \bar{u}_\Gamma) = \widetilde{\mathcal{J}}((\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma), \bar{u}_\Gamma) = \mathcal{J}((\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma), \bar{u}_\Gamma).
\end{aligned} \tag{3.24}$$

Combining (3.22) with (3.24), we have thus shown that $\frac{1}{2}\|u_\Gamma - \bar{u}_\Gamma\|^2_{L^2(\Sigma)} = 0$, so that $u_\Gamma = \bar{u}_\Gamma$ and thus also $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma) = (\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma, \bar{\xi}, \bar{\xi}_\Gamma)$. Moreover, (3.22) and (3.24) also imply that

$$\begin{aligned}
&\mathcal{J}((\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma), \bar{u}_\Gamma) = \widetilde{\mathcal{J}}((\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma), \bar{u}_\Gamma) = \liminf_{k\to\infty} \widetilde{\mathcal{J}}((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}) \\
&= \limsup_{k\to\infty} \widetilde{\mathcal{J}}((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}) = \lim_{k\to\infty} \widetilde{\mathcal{J}}((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}),
\end{aligned} \tag{3.25}$$

which proves (3.20) and, at the same time, also (3.18). This concludes the proof of the assertion. ∎

4 The optimality system

In this section, we aim to establish first-order necessary optimality conditions for the optimal control problem $(\mathscr{P}_0)$. This will be achieved by a passage to the limit as $\alpha \searrow 0$ in the first-order necessary optimality conditions for the adapted optimal control problems $(\widetilde{\mathscr{P}_\alpha})$ that can by derived as in [25] with only minor and obvious changes. This procedure will yield certain generalized first-order necessary optimality conditions in the limit. In this entire section, we assume that h is given by (1.12) and that (1.13) and the general assumptions (A1)–(A5) are satisfied. We also assume that a fixed optimal control $\bar{u}_\Gamma \in \mathscr{U}_{\rm ad}$ for $(\mathscr{P}_0)$ is given, along with the corresponding solution quintuple $(\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma, \bar{\xi}, \bar{\xi}_\Gamma)$ of the state system (1.2)–(1.8) established in Theorem 2.1. That is, we have $(\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma) = \mathscr{S}_0(\bar{u}_\Gamma)$ as well as $\bar{\xi} \in \partial I_{[-1,1]}(\bar{\rho})$ a. e. in Q and $\bar{\xi}_\Gamma \in \partial I_{[-1,1]}(\bar{\rho}_\Gamma)$ a. e. on Σ.

In order to be able to take advantage of the analysis performed in [25, Sect. 4], we impose the following additional compatibility condition:

(A6) It holds that $(\beta_4(\bar{\rho}(T) - \hat{\rho}_\Omega), \beta_5(\bar{\rho}_\Gamma(T) - \hat{\rho}_\Gamma)) \in \mathscr{V}$.

Obviously, (A6) is fulfilled if $\beta_4 = \beta_5$ (especially if $\beta_4 = \beta_5 = 0$) and $(\hat{\rho}_\Omega, \hat{\rho}_\Gamma) \in \mathscr{V}$. In view of the fact that always $(\bar{\rho}(T), \bar{\rho}_\Gamma(T)) \in \mathscr{V}$, these conditions for the target functions $\hat{\rho}_\Omega$ and $\hat{\rho}_\Gamma$ seem to be quite reasonable.

We begin our analysis by formulating the adjoint state system for the adapted control problem $(\widetilde{\mathscr{P}_\alpha})$. To this end, let us assume that $u_\Gamma^\alpha \in \mathscr{U}_{\rm ad}$ is an arbitrary optimal control for $(\widetilde{\mathscr{P}_\alpha})$ and that $(\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha)$ is the solution triple to the associated state system (2.9)–(2.13). In particular, $(\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha) = \mathscr{S}_\alpha(u_\Gamma^\alpha)$, the solution has the regularity properties (2.2)–(2.5), and it satisfies the global bounds (2.15), (2.16), as well as the separation property (2.14). Moreover, it follows from [25, Thm. 4.2] that the associated adjoint system

$$-\,(1+2g(\rho^\alpha))\,p^\alpha_t - g'(\rho^\alpha)\,\rho^\alpha_t\,p^\alpha - \Delta p^\alpha \,=\, g'(\rho^\alpha)\,q^\alpha + \beta_1(\mu^\alpha - \hat\mu_Q) \quad \text{a.e. in } Q, \tag{4.1}$$

$$\partial_{\mathbf n} p^\alpha = 0 \quad \text{a.e. on } \Sigma, \quad p^\alpha(T) = 0 \quad \text{a.e. in } \Omega, \tag{4.2}$$

$$-\,q^\alpha_t - \Delta q^\alpha + (\varphi(\alpha)\,h''(\rho^\alpha) + \pi'(\rho^\alpha) - \mu^\alpha\,g''(\rho^\alpha))\,q^\alpha = g'(\rho^\alpha)(\mu^\alpha\,p^\alpha_t - \mu^\alpha_t\,p^\alpha) + \beta_2(\rho^\alpha - \hat\rho_Q) \quad \text{a.e. in } Q, \tag{4.3}$$

$$\partial_{\mathbf n} q^\alpha - \partial_t q^\alpha - \Delta_\Gamma q^\alpha_\Gamma + (\varphi(\alpha)\,h''(\rho^\alpha_\Gamma) + \pi'_\Gamma(\rho^\alpha_\Gamma))\,q^\alpha_\Gamma \,=\, \beta_3(\rho^\alpha_\Gamma - \hat\rho_\Sigma), \quad \text{and } q^\alpha_\Gamma = q^\alpha_{|\Sigma}, \quad \text{a.e. on } \Sigma, \tag{4.4}$$

$$q^\alpha(T) = \beta_4(\rho^\alpha(T) - \hat\rho_\Omega) \quad \text{a.e. in } \Omega, \quad q^\alpha_\Gamma(T) = \beta_5(\rho^\alpha_\Gamma(T) - \hat\rho_\Gamma) \quad \text{a.e. on } \Gamma \tag{4.5}$$

has a unique solution $(p^\alpha, q^\alpha, q^\alpha_\Gamma)$ such that

$$p^\alpha \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W), \tag{4.6}$$

$$q^\alpha \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;H^2(\Omega)), \tag{4.7}$$

$$q^\alpha_\Gamma \in H^1(0,T;H_\Gamma) \cap C^0([0,T];V_\Gamma) \cap L^2(0,T;H^2(\Gamma)). \tag{4.8}$$

In addition, as in the proof of [25, Cor. 4.3], it follows the validity of the variational inequality

$$\int_0^T\!\!\int_\Gamma \big(q^\alpha_\Gamma + \beta_6\,u^\alpha_\Gamma + (u^\alpha_\Gamma - \bar u_\Gamma)\big)(v_\Gamma - u^\alpha_\Gamma)\,\mathrm d\Gamma\,\mathrm dt \,\ge\, 0 \quad \forall\, v_\Gamma \in \mathscr U_{\mathrm{ad}}. \tag{4.9}$$

We now prove an a priori estimate that will be fundamental for the derivation of the optimality conditions for $(\mathscr P_0)$. To this end, we introduce some further function spaces. At first, we put

$$Y := H^1(0,T;V^*) \cap L^2(0,T;V), \quad Y_\Gamma := H^1(0,T;V^*_\Gamma) \cap L^2(0,T;V_\Gamma), \tag{4.10}$$

$$\mathscr W \,:= \big(H^1(0,T;V^*) \times H^1(0,T;V^*_\Gamma)\big) \cap L^2(0,T;\mathscr V), \tag{4.11}$$

$$\mathscr W_0 \,:= \{(\eta,\eta_\Gamma) \in \mathscr W : (\eta(0),\eta_\Gamma(0)) = (0,0)\}, \tag{4.12}$$

which are Banach spaces when equipped with the natural norm of $Y \times Y_\Gamma$. Moreover, we have the dense and continuous injections $Y \subset L^2(0,T;V) \subset L^2(Q) \subset L^2(0,T;V^*) \subset Y^*$ and $Y_\Gamma \subset L^2(0,T;V_\Gamma) \subset L^2(\Sigma) \subset L^2(0,T;V^*_\Gamma) \subset Y^*_\Gamma$, where it is understood that

$$\langle z, v\rangle_Y \,=\, \int_0^T \langle z(t), v(t)\rangle_V\,\mathrm dt \quad \text{for all } z \in L^2(0,T;V^*) \text{ and } v \in L^2(0,T;V), \tag{4.13}$$

$$\langle z_\Gamma, v_\Gamma\rangle_{Y_\Gamma} \,=\, \int_0^T \langle z_\Gamma(t), v_\Gamma(t)\rangle_{V_\Gamma}\,\mathrm dt \quad \text{for all } z_\Gamma \in L^2(0,T;V^*_\Gamma) \text{ and } v_\Gamma \in L^2(0,T;V_\Gamma). \tag{4.14}$$

We also note that the embeddings $Y \subset C^0([0,T];H)$ and $Y_\Gamma \subset C^0([0,T];H_\Gamma)$ are continuous. Likewise, we have the dense and continuous embeddings $\mathscr{W} \subset L^2(0,T;\mathscr{V}) \subset L^2(0,T;H \times H_\Gamma) \subset L^2(0,T;\mathscr{V}^*) \subset \mathscr{W}^*$, as well as the continuous injection $\mathscr{W} \subset C^0([0,T];H \times H_\Gamma)$, which gives the initial condition encoded in (4.12) a proper meaning. Furthermore, since $\mathscr{W}_0$ is a closed subspace of $Y \times Y_\Gamma$, we deduce that the elements $F = (z, z_\Gamma) \in \mathscr{W}_0^*$ are exactly those that are of the form

$$\langle F, (\eta, \eta_\Gamma) \rangle_{\mathscr{W}_0} = \langle z, \eta \rangle_Y + \langle z_\Gamma, \eta_\Gamma \rangle_{Y_\Gamma} \quad \text{for all } (\eta, \eta_\Gamma) \in \mathscr{W}_0, \tag{4.15}$$

where $z \in Y^*$ and $z_\Gamma \in Y_\Gamma^*$. In particular, for $z \in L^2(0,T;V^*)$ and $z_\Gamma \in L^2(0,T;V_\Gamma^*)$ the formulas (4.13) and (4.14) apply. Observe that these representation formulas allow us to give a proper meaning to statements like

$$(z^\alpha, z_\Gamma^\alpha) \to (z, z_\Gamma) \quad \text{weakly in } \mathscr{W}_0^*.$$

In addition to the spaces introduced in (4.10)–(4.12), we also define

$$\mathscr{X} := (L^\infty(0,T;H) \times L^\infty(0,T;H_\Gamma)) \cap L^2(0,T;\mathscr{V}), \tag{4.16}$$

which is a Banach spaced when endowed with its natural norm.

We have the following result.

PROPOSITION 4.1: *Let the general assumptions* (A1)–(A6), (1.12)–(1.13) *be satisfied, and let*

$$(\lambda^\alpha, \lambda_\Gamma^\alpha) := \big(\varphi(\alpha) h''(\rho^\alpha) q^\alpha, \, \varphi(\alpha) h''(\rho_\Gamma^\alpha) q_\Gamma^\alpha\big) \quad \forall \alpha \in (0,1]. \tag{4.17}$$

Then there exists a constant $K_3^* > 0$, which depends only on the data of the system and on R, such that for all $\alpha \in (0,1]$ it holds

$$\|p^\alpha\|_{H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W)} + \max_{0 \le t \le T} (\|q^\alpha(t)\|_H + \|q_\Gamma^\alpha(t)\|_{H_\Gamma})$$
$$+ \|(q^\alpha, q_\Gamma^\alpha)\|_{L^2(0,T;\mathscr{V})} + \|(\lambda^\alpha, \lambda_\Gamma^\alpha)\|_{\mathscr{W}_0^*} + \|(\partial_t q^\alpha, \partial_t q_\Gamma^\alpha)\|_{\mathscr{W}_0^*} \le K_3^*. \tag{4.18}$$

PROOF: In the following, $C > 0$ denote positive constants that may depend on the data of the system but not on $\alpha \in (0,1]$. We make repeated use of the global estimates (2.15) and (2.16) without further reference.

First, we add p^α on both sides of (4.1), multiply the result by $-p_t^\alpha$, and integrate over $\Omega \times (t,T]$, where $t \in [0,T)$. Using the fact that $p^\alpha(T) = 0$, we obtain the inequality

$$\int_t^T \int_\Omega |p_t^\alpha|^2 \, dx \, ds + \frac{1}{2} \|p^\alpha(t)\|_V^2 \le I_1 + I_2 + I_3, \tag{4.19}$$

where the quantities I_j, $1 \le j \le 3$, are specified and estimated below. At first, Young's inequality yields that

$$\begin{aligned} I_1 :&= -\int_t^T \int_\Omega (p^\alpha + \beta_1(\mu^\alpha - \hat{\mu}_Q)) \, p_t^\alpha \, dx \, ds \\ &\le \frac{1}{5} \int_t^T \int_\Omega |p_t^\alpha|^2 \, dx \, ds + C + C \int_t^T \int_\Omega |p^\alpha|^2 \, dx \, ds. \end{aligned} \tag{4.20}$$

Likewise, we have that

$$I_2 := -\int_t^T\int_\Omega g'(\rho^\alpha)\,q^\alpha\,\rho_t^\alpha\,\mathrm{d}x\,\mathrm{d}s \le \frac{1}{5}\int_t^T\int_\Omega |\rho_t^\alpha|^2\,\mathrm{d}x\,\mathrm{d}s + C\int_t^T\int_\Omega |q^\alpha|^2\,\mathrm{d}x\,\mathrm{d}s. \tag{4.21}$$

Moreover, by also invoking Hölder's inequality and the continuity of the embedding $V \subset L^4(\Omega)$, we deduce that

$$\begin{aligned} I_3 := -\int_t^T\int_\Omega g'(\rho^\alpha)\,\rho_t^\alpha\,p^\alpha\,p_t^\alpha\,\mathrm{d}x\,\mathrm{d}s &\le C\int_t^T \|\rho_t^\alpha(s)\|_4\,\|p^\alpha(s)\|_4\,\|p_t^\alpha(s)\|_2\,\mathrm{d}s \\ &\le \frac{1}{5}\int_t^T\int_\Omega |p_t^\alpha|^2\,\mathrm{d}x\,\mathrm{d}s + C\int_t^T \|\rho_t^\alpha(s)\|_V^2\,\|p^\alpha(s)\|_V^2\,\mathrm{d}s, \end{aligned} \tag{4.22}$$

where the mapping $s \mapsto \|\rho_t^\alpha(s)\|_V^2$ is bounded in $L^1(0,T)$ uniformly with respect to $\alpha \in (0,1]$.

Next, we multiply (4.3) by q^α and integrate over $\Omega \times (t,T]$, where $t \in [0,T)$. Taking (4.4) into account, we obtain the identity

$$\begin{aligned} &\frac{1}{2}\left(\|q^\alpha(t)\|_H^2 + \|q_\Gamma^\alpha(t)\|_{H_\Gamma}^2\right) + \int_t^T\int_\Omega |\nabla q^\alpha|^2\,\mathrm{d}x\,\mathrm{d}s + \int_t^T\int_\Gamma |\nabla_\Gamma q_\Gamma^\alpha|^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\ &+ \int_t^T\int_\Omega \varphi(\alpha)\,h''(\rho^\alpha)\,|q^\alpha|^2\,\mathrm{d}x\,\mathrm{d}s + \int_t^T\int_\Gamma \varphi(\alpha)\,h''(\rho_\Gamma^\alpha)\,|q_\Gamma^\alpha|^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\ &= \frac{1}{2}\left(\|q^\alpha(T)\|_H^2 + \|q_\Gamma^\alpha(T)\|_{H_\Gamma}^2\right) \\ &\quad + \int_t^T\int_\Omega \left(\mu^\alpha g''(\rho^\alpha) - \pi'(\rho^\alpha)\right)|q^\alpha|^2\,\mathrm{d}x\,\mathrm{d}s + \int_t^T\int_\Omega \beta_2(\rho^\alpha - \hat\rho_Q)\,q^\alpha\,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \int_t^T\int_\Gamma \pi_\Gamma'(\rho_\Gamma^\alpha)\,|q_\Gamma^\alpha|^2\,\mathrm{d}\Gamma\,\mathrm{d}s + \int_t^T\int_\Gamma \beta_3(\rho_\Gamma^\alpha - \hat\rho_\Sigma)\,q_\Gamma^\alpha\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_t^T\int_\Omega g'(\rho^\alpha)\,\mu^\alpha\,p_t^\alpha\,q^\alpha\,\mathrm{d}x\,\mathrm{d}s - \int_t^T\int_\Omega g'(\rho^\alpha)\,\mu_t^\alpha\,p^\alpha\,q^\alpha\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{4.23}$$

Since $\varphi(\alpha)\,h'' \ge 0$, all summands on the left-hand side are nonnegative. Moreover, invoking (4.5) and Young's inequality, it is readily seen that the first five summands on the right-hand side are bounded by an expression of the form

$$C\left(1 + \int_t^T\int_\Omega |q^\alpha|^2\,\mathrm{d}x\,\mathrm{d}s + \int_t^T\int_\Gamma |q_\Gamma^\alpha|^2\,\mathrm{d}\Gamma\,\mathrm{d}s\right). \tag{4.24}$$

It thus remains to estimate the last two summands on the right-hand side, which we denote by J_1 and J_2, respectively. By virtue of Hölder's and Young's inequality, we first have that

$$\begin{aligned} J_1 &\le C\int_t^T \|\mu^\alpha(s)\|_\infty\,\|p_t^\alpha(s)\|_2\,\|q^\alpha(s)\|_2\,\mathrm{d}s \\ &\le \frac{1}{5}\int_t^T\int_\Omega |p_t^\alpha|^2\,\mathrm{d}x\,\mathrm{d}s + C\int_t^T\int_\Omega |q^\alpha|^2\,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{4.25}$$

while, also using the continuity of the embedding $V \subset L^4(\Omega)$,

$$
\begin{aligned}
J_2 &\le C \int_t^T \|\mu_t^\alpha(s)\|_2 \, \|p^\alpha(s)\|_4 \, \|q^\alpha(s)\|_4 \, \mathrm{d}s \\
&\le \frac{1}{2} \int_t^T \|q^\alpha(s)\|_V^2 \, \mathrm{d}s + C \int_t^T \|\mu_t^\alpha(s)\|_H^2 \, \|p^\alpha(s)\|_V^2 \, \mathrm{d}s \,, \qquad (4.26)
\end{aligned}
$$

where the mapping $s \mapsto \|\mu_t^\alpha(s)\|_H^2$ is known to be bounded in $L^1(0,T)$, uniformly in $\alpha \in (0,1]$. Therefore, combining the estimates (4.19)–(4.26), we obtain from Gronwall's lemma, taken backward in time, the estimate

$$
\begin{aligned}
\|p^\alpha\|_{H^1(0,T;H)} + \max_{0 \le t \le T} \left(\|p^\alpha(t)\|_V + \|q^\alpha(t)\|_H + \|q_\Gamma^\alpha(t)\|_{H_\Gamma} \right) \\
+ \|(q^\alpha, q_\Gamma^\alpha)\|_{L^2(0,T;\mathscr{V})} \le C. \qquad (4.27)
\end{aligned}
$$

Now observe that

$$
\begin{aligned}
\left\| g'(\rho^\alpha) \, \rho_t^\alpha \, p^\alpha \right\|_{L^2(Q)}^2 &\le C \int_0^t \int_\Omega |\rho_t^\alpha|^2 \, |p^\alpha|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\le C \int_0^T \|\rho_t^\alpha(s)\|_4^2 \, \|p^\alpha(s)\|_4^2 \, \mathrm{d}s \le C.
\end{aligned}
$$

Thus, by comparison in (4.1), we find out that $\|\Delta p^\alpha\|_{L^2(Q)} \le C$, whence, by virtue of (4.2) and standard elliptic estimates,

$$
\|p^\alpha\|_{L^2(0,T;W)} \le C. \qquad (4.28)
$$

Next, we derive the bound for the time derivatives. To this end, let $(\eta, \eta_\Gamma) \in \mathscr{W}_0$ be arbitrary. Using the continuity of the embeddings $Y \subset C^0([0,T];H)$ and $Y_\Gamma \subset C^0([0,T];H_\Gamma)$, and invoking the estimate (4.27), we obtain from integration by parts that

$$
\begin{aligned}
\langle (\partial_t q^\alpha, \partial_t q_\Gamma^\alpha), (\eta, \eta_\Gamma) \rangle_{\mathscr{W}} &= \int_0^T \int_\Omega q_t^\alpha \, \eta \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Gamma \partial_t q_\Gamma^\alpha \, \eta_\Gamma \, \mathrm{d}\Gamma \, \mathrm{d}t \\
&= \int_\Omega q^\alpha(T) \, \eta(T) \, \mathrm{d}x + \int_\Gamma q_\Gamma^\alpha(T) \, \eta_\Gamma(T) \, \mathrm{d}\Gamma \\
&\quad - \int_0^T \langle \eta_t(t), q^\alpha(t) \rangle_V \, \mathrm{d}t - \int_0^T \langle \partial_t \eta_\Gamma(t), q_\Gamma^\alpha(t) \rangle_{V_\Gamma} \, \mathrm{d}t \\
&\le \|q^\alpha(T)\|_H \, \|\eta(T)\|_H + \|q_\Gamma^\alpha(T)\|_{H_\Gamma} \, \|\eta_\Gamma(T)\|_{H_\Gamma} \\
&\quad + \int_0^T \|\eta_t(t)\|_{V^*} \, \|q^\alpha(t)\|_V \, \mathrm{d}t + \int_0^T \|\partial_t \eta_\Gamma(t)\|_{V_\Gamma^*} \, \|q_\Gamma^\alpha(t)\|_V \, \mathrm{d}t,
\end{aligned}
$$

whence

$$
\begin{aligned}
&\langle (\partial_t q^\alpha, \partial_t q_\Gamma^\alpha), (\eta, \eta_\Gamma) \rangle_{\mathscr{W}} \\
&\le C \max_{0 \le t \le T} \left(\|\eta(t)\|_H + \|\eta_\Gamma(t)\|_{H_\Gamma} \right) \\
&\quad + C \, \|(q^\alpha, q_\Gamma^\alpha)\|_{L^2(0,T;\mathscr{V})} \left(\|\eta_t\|_{L^2(0,T;V^*)} + \|\partial_t \eta_\Gamma\|_{L^2(0,T;V_\Gamma{}^*)} \right) \le C \, \|(\eta, \eta_\Gamma)\|_{\mathscr{W}_0} \,.
\end{aligned}
$$

We thus have shown that

$$\|(\partial_t q^\alpha, \partial_t q_\Gamma^\alpha)\|_{\mathscr{W}_0^*} \le C. \tag{4.29}$$

Now, let $(\eta,\eta_\Gamma) \in \mathscr{W}_0$ be arbitrary. We define the functions

$$\begin{aligned} &v_1^\alpha := \big(\mu^\alpha g''(\rho^\alpha) - \pi'(\rho^\alpha)\big)\, q^\alpha + g'(\rho^\alpha)\mu^\alpha p_t^\alpha, \quad v_2^\alpha := -g'(\rho^\alpha)\mu_t^\alpha p^\alpha, \\ &w^\alpha := -\pi_\Gamma'(\rho_\Gamma^\alpha) q_\Gamma^\alpha \,. \end{aligned} \tag{4.30}$$

Multiplying (4.3) by η, and invoking (4.4), we then easily infer the identity

$$\begin{aligned} &\langle(\lambda^\alpha,\lambda_\Gamma^\alpha),(\eta,\eta_\Gamma)\rangle_{\mathscr{W}_0} = \int_0^T\!\!\int_\Omega \lambda^\alpha\, \eta \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Gamma \lambda_\Gamma^\alpha\, \eta_\Gamma \,\mathrm{d}\Gamma\,\mathrm{d}t \\ &= \int_0^T\!\!\int_\Omega \eta\, q_t^\alpha \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Gamma \eta_\Gamma\, \partial_t q_\Gamma^\alpha \\ &\quad - \int_0^T\!\!\int_\Omega \nabla q^\alpha \cdot \nabla\eta \,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_\Gamma \nabla_\Gamma q_\Gamma^\alpha \cdot \nabla_\Gamma \eta_\Gamma \,\mathrm{d}\Gamma\,\mathrm{d}t \\ &\quad + \int_0^T\!\!\int_\Omega v_1^\alpha\, \eta \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega v_2^\alpha\, \eta \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Gamma w^\alpha\, \eta_\Gamma \,\mathrm{d}\Gamma\,\mathrm{d}t \\ &\quad + \int_0^T\!\!\int_\Omega \beta_2(\rho^\alpha - \hat\rho_Q) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Gamma \beta_3(\rho_\Gamma^\alpha - \hat\rho_\Sigma) \,\mathrm{d}\Gamma\,\mathrm{d}t. \end{aligned} \tag{4.31}$$

Now observe that v_1^α and w^α are known to bounded in $L^2(Q)$ and in $L^2(\Sigma)$, respectively, uniformly in $\alpha \in (0,1]$. Also, using the continuity of the embedding $H^2(\Omega) \subset L^\infty(\Omega)$, we have that

$$\begin{aligned} &\int_0^T\!\!\int_\Omega v_2^\alpha\, \eta \,\mathrm{d}x\,\mathrm{d}t \le C \int_0^T \|\mu_t^\alpha(t)\|_2\, \|\eta(t)\|_2\, \|p^\alpha(t)\|_\infty \,\mathrm{d}t \\ &\le C \max_{0\le t\le T} \|\eta(t)\|_H\, \|\mu_t^\alpha\|_{L^2(Q)}\, \|p^\alpha\|_{L^2(0,T;H^2(\Omega))} \le C\, \|\eta\|_Y \,. \end{aligned} \tag{4.32}$$

Therefore, taking (4.27) and (4.29) into account, we have shown that

$$\|(\lambda^\alpha,\lambda_\Gamma^\alpha)\|_{\mathscr{W}_0^*} \le C\,. \tag{4.33}$$

This conclude the proof of the assertion. ∎

After these preliminaries, we are now in a position to establish first-order necessary optimality conditions for $(\mathscr{P}_0)$ by performing a limit as $\alpha \searrow 0$ in the approximating problems. To this end, recall that a fixed optimal control $\bar u_\Gamma \in \mathscr{U}_{\rm ad}$ for $(\mathscr{P}_0)$, along with a solution quintuple $(\bar\mu,\bar\rho,\bar\rho_\Gamma,\bar\xi,\bar\xi_\Gamma)$ of the associated state system (1.2)–(1.8) is given.

Now, we choose an arbitrary sequence $\{\alpha_n\}$ such that $\alpha_n \searrow 0$ as $n \to \infty$. By virtue of Theorem 3.4, we can find a subsequence, which is again indexed by n, such that, for any $n \in \mathbb{N}$, we can find an optimal control $u_\Gamma^{\alpha_n} \in \mathscr{U}_{\rm ad}$ for $(\widetilde{\mathscr{P}}_{\alpha_n})$ with associated state triple $(\mu^{\alpha_n},\rho^{\alpha_n},\rho_\Gamma^{\alpha_n})$

that satisfies the convergence properties (3.18)–(3.20). From [34, Sect. 8, Cor. 4], without loss of generality we may assume that

$$\mu^{\alpha_n} \to \bar{\mu} \quad \text{strongly in } C^0([0,T];L^p(\Omega)) \quad \text{for } 1 \le p < 6, \tag{4.34}$$

$$\rho^{\alpha_n} \to \bar{\rho} \quad \text{strongly in } C^0(\overline{Q}), \quad \rho_\Gamma^{\alpha_n} \to \bar{\rho}_\Gamma \quad \text{strongly in } C^0(\overline{\Sigma}), \tag{4.35}$$

which entail that

$$\Psi(\rho^{\alpha_n}) \to \Psi(\bar{\rho}) \quad \text{strongly in } C^0(\overline{Q}) \quad \text{for } \Psi \in \{g, g', g'', \pi, \pi'\} \tag{4.36}$$

$$\Psi_\Gamma(\rho_\Gamma^{\alpha_n}) \to \Psi_\Gamma(\bar{\rho}) \quad \text{strongly in } C^0(\overline{\Sigma}) \quad \text{for } \Psi \in \{\pi_\Gamma, \pi_\Gamma'\}. \tag{4.37}$$

Moreover, thanks to Proposition 4.1 and to [34, Sect. 8, Cor. 4], we may assume that the associated adjoint variables $(p^{\alpha_n}, q^{\alpha_n}, q_\Gamma^{\alpha_n})$ satisfy

$$p^{\alpha_n} \to p \quad \text{weakly-star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$$
$$\text{and strongly in } C^0([0,T];L^p(\Omega)) \quad \text{for } 1 \le p < 6, \tag{4.38}$$

$$(q^{\alpha_n}, q_\Gamma^{\alpha_n}) \to (q, q_\Gamma) \quad \text{weakly-star in } \mathscr{X}, \tag{4.39}$$

$$(\partial_t q^{\alpha_n}, \partial_t q_\Gamma^{\alpha_n}) \to (\partial_t q, \partial_t q_\Gamma) \quad \text{weakly in } \mathscr{W}_0^*, \tag{4.40}$$

$$(\lambda^{\alpha_n}, \lambda_\Gamma^{\alpha_n}) \to (\lambda, \lambda_\Gamma) \quad \text{weakly in } \mathscr{W}_0^*, \tag{4.41}$$

for suitable limits (p, q, q_Γ) and $(\lambda, \lambda_\Gamma)$, where $\lambda \in Y^*$ and $\lambda_\Gamma \in Y_\Gamma^*$, as explained around (4.15). Obviously, (4.38) implies that $\partial_{\mathbf{n}} p = 0$ almost everywhere on Σ and $p(T) = 0$ almost everywhere in Ω. Therefore, passing to the limit as $n \to \infty$ in the variational inequality (4.9), written for α_n, $n \in \mathbb{N}$, we obtain that (p, q, q_Γ) satisfies

$$\int_0^T \int_\Gamma (q_\Gamma + \beta_6 \bar{u}_\Gamma)(v_\Gamma - \bar{u}_\Gamma) \, d\Gamma \, dt \ge 0 \quad \forall v_\Gamma \in \mathscr{U}_{\text{ad}}. \tag{4.42}$$

Next, we aim to show that in the limit as $n \to \infty$ a limiting adjoint system for $(\mathscr{P}_0)$ is satisfied. At first, it easily follows from the convergence properties stated above that

$$g(\rho^{\alpha_n}) p_t^{\alpha_n} \to g(\bar{\rho}) p_t, \quad g'(\rho^{\alpha_n}) \rho_t^{\alpha_n} p^{\alpha_n} \to g'(\bar{\rho}) \bar{\rho}_t p, \quad g'(\rho^{\alpha_n}) q^{\alpha_n} \to g'(\bar{\rho}) q, \tag{4.43}$$

all weakly in $L^1(Q)$. It thus follows, by taking the limit as $n \to \infty$ in (4.1) and (4.2), that the limits p, q satisfy

$$-(1 + 2g(\bar{\rho})) p_t - g'(\bar{\rho}) \bar{\rho}_t p - \Delta p = g'(\bar{\rho}) q + \beta_1 (\bar{\mu} - \hat{\mu}_Q) \quad \text{a.e. in } Q, \tag{4.44}$$

$$\partial_{\mathbf{n}} p = 0 \quad \text{a.e. on } \Sigma, \quad p(T) = 0 \quad \text{a.e. in } \Omega. \tag{4.45}$$

The limiting equation corresponding to (4.3)–(4.5) has to be formulated in a weak form. To this end, we multiply (4.3), written for α_n, $n \in \mathbb{N}$, by an arbitrary $(\eta, \eta_\Gamma) \in \mathscr{W}_0$ and integrate the resulting equation over Q. Integrating by parts with respect to time and space, and invoking the endpoint conditions for q and q_Γ, as well as the zero initial conditions for (η,η_Γ), we arrive at the identity

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega \lambda^{\alpha_n}\,\eta\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Gamma \lambda_\Gamma^{\alpha_n}\,\eta_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}t + \int_0^T \langle\partial_t\eta(t),q^{\alpha_n}(t)\rangle_V\,\mathrm{d}t\\
&+\int_0^T \langle\partial_t\eta_\Gamma(t),q_\Gamma^{\alpha_n}(t)\rangle_{V_\Gamma}\,\mathrm{d}t + \int_0^T\!\!\int_\Omega \nabla q^{\alpha_n}\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Gamma \nabla_\Gamma q_\Gamma^{\alpha_n}\cdot\nabla_\Gamma\eta_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}t\\
&-\int_0^T\!\!\int_\Omega v_1^{\alpha_n}\,\eta\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_\Omega v_2^{\alpha_n}\eta\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_\Gamma w^{\alpha_n}\,\mathrm{d}\Gamma\,\mathrm{d}t\\
&=\beta_2\int_0^T\!\!\int_\Omega(\rho^{\alpha_n}-\hat\rho_Q)\,\eta\,\mathrm{d}x\,\mathrm{d}t + \beta_3\int_0^T\!\!\int_\Gamma(\rho_\Gamma^{\alpha_n}-\hat\rho_\Sigma)\,\eta_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}t\\
&\quad+\beta_4\int_\Omega(\rho^{\alpha_n}(T)-\hat\rho_\Omega)\,\eta(T)\,\mathrm{d}x + \beta_5\int_\Gamma(\rho_\Gamma^{\alpha_n}(T)-\hat\rho_\Gamma)\,\eta_\Gamma(T)\,\mathrm{d}\Gamma\,. \qquad (4.46)
\end{aligned}$$

Now, owing to (4.13)–(4.15), the sum of the first two integrals on the left-hand side of (4.46) is equal to $\langle(\lambda^{\alpha_n},\lambda_\Gamma^{\alpha_n}),(\eta,\eta_\Gamma)\rangle_{\mathscr{W}_0}$, which, by (4.41), converges to $\langle(\lambda,\lambda_\Gamma),(\eta,\eta_\Gamma)\rangle_{\mathscr{W}_0}$. Moreover, it is straightforward to verify (and this may be left to the reader) that also the remaining integrals in (4.46) converge. We therefore obtain, for every $(\eta,\eta_\Gamma)\in\mathscr{W}_0$,

$$\begin{aligned}
&\langle(\lambda,\lambda_\Gamma)(\eta,\eta_\Gamma)\rangle_{\mathscr{W}_0} + \int_0^T \langle\partial_t\eta(t),q(t)\rangle_V\,\mathrm{d}t + \int_0^T \langle\partial_t\eta_\Gamma(t),q_\Gamma(t)\rangle_{V_\Gamma}\,\mathrm{d}t\\
&+\int_0^T\!\!\int_\Omega \nabla q\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Gamma \nabla_\Gamma q_\Gamma\cdot\nabla_\Gamma\eta_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}t + \int_0^T\!\!\int_\Gamma \pi_\Gamma'(\bar\rho_\Gamma)\,q_\Gamma\,\eta_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}t\\
&+\int_0^T\!\!\int_\Omega \big[(\pi'(\bar\rho)-\bar\mu\,g''(\bar\rho))\,q + g'(\bar\rho)\,(\bar\mu_t\,p-\bar\mu\,p_t)\big]\,\eta\,\mathrm{d}x\,\mathrm{d}t\\
&=\beta_2\int_0^T\!\!\int_\Omega(\bar\rho-\hat\rho_Q)\,\eta\,\mathrm{d}x\,\mathrm{d}t + \beta_3\int_0^T\!\!\int_\Gamma(\bar\rho_\Gamma-\hat\rho_\Sigma)\,\eta_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}t\\
&\quad+\beta_4\int_\Omega(\bar\rho(T)-\hat\rho_\Omega)\,\eta(T)\,\mathrm{d}x + \beta_5\int_\Gamma(\bar\rho_\Gamma(T)-\hat\rho_\Gamma)\,\eta_\Gamma(T)\,\mathrm{d}\Gamma\,. \qquad (4.47)
\end{aligned}$$

Next, we show that the limit pair $((\lambda,\lambda_\Gamma),(q,q_\Gamma))$ satisfies some sort of a complementarity slackness condition. To this end, observe that (cf. (4.17)) for all $n\in\mathbb{N}$ we obviously have

$$\int_0^T\!\!\int_\Omega \lambda^{\alpha_n}\,q^{\alpha_n}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_\Omega \varphi(\alpha_n)\,h''(\rho^{\alpha_n})\,|q^{\alpha_n}|^2\,\mathrm{d}x\,\mathrm{d}t \ge 0\,.$$

An analogous inequality holds for the corresponding boundary terms. Hence, it is found that

$$\liminf_{n\to\infty}\int_0^T\!\!\int_\Omega \lambda^{\alpha_n}\,q^{\alpha_n}\,\mathrm{d}x\,\mathrm{d}t \ge 0, \quad \liminf_{n\to\infty}\int_0^T\!\!\int_\Gamma \lambda_\Gamma^{\alpha_n}\,q_\Gamma^{\alpha_n}\,\mathrm{d}\Gamma\,\mathrm{d}t \ge 0\,. \qquad (4.48)$$

Finally, we derive a relation which gives some indication that the limit (λ,λ_Γ) should somehow be concentrated on the set where $|\bar\rho|=1$ and $|\bar\rho_\Gamma|=1$ (which, however, we cannot prove rigorously). To this end, we test the pair $(\lambda^{\alpha_n}, \lambda_\Gamma^{\alpha_n})$ by the function

$$\big((1-(\rho^{\alpha_n})^2)\,\phi, (1-(\rho_\Gamma^{\alpha_n})^2)\,\phi_\Gamma\big)$$

that belongs to $\mathscr{V}$, since (ϕ, ϕ_Γ) is any smooth test function satisfying

$$(\phi(0), \phi_\Gamma(0)) = (0,0), \quad \int_\Omega (1-(\rho^{\alpha_n})^2)\,\phi(t)\,\mathrm{d}x = 0 \quad \forall t \in [0,T]. \tag{4.49}$$

As $h''(r) = 2/\left(1-r^2\right)$ for every $r \in (-1,1)$, we obtain that

$$\lim_{n\to\infty}\left(\int_0^T\!\!\int_\Omega \lambda^{\alpha_n}(1-(\rho^{\alpha_n})^2)\,\phi\,\mathrm{d}x\,\mathrm{d}t\,, \int_0^T\!\!\int_\Gamma \lambda_\Gamma^{\alpha_n}(1-(\rho_\Gamma^{\alpha_n})^2)\,\phi_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}t\right)$$
$$= \lim_{n\to\infty}\left(2\int_0^T\!\!\int_\Omega \varphi(\alpha_n)\,q^{\alpha_n}\,\phi\,\mathrm{d}x\,\mathrm{d}t\,, 2\int_0^T\!\!\int_\Gamma \varphi(\alpha_n)\,q_\Gamma^{\alpha_n}\,\phi_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}t\right) = (0,0). \tag{4.50}$$

We now collect the results established above. We have the following statement.

THEOREM 4.2: *Let the assumptions* (A1)–(A6) *and* (1.12)–(1.13) *be satisfied. Moreover, let* $\bar{u}_\Gamma \in \mathscr{U}_{\mathrm{ad}}$ *be an optimal control for* $(\mathscr{P}_0)$ *with the associated quintuple* $(\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma, \bar{\xi}, \bar{\xi}_\Gamma)$ *solving the corresponding state system* (1.2)–(1.8) *in the sense of Theorem 2.1. Moreover, let* $\{\alpha_n\} \subset (0,1]$ *be a sequence with* $\alpha_n \searrow 0$ *as* $n \to \infty$ *such that there are optimal pairs* $((\mu^{\alpha_n}, \rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}), u_\Gamma^{\alpha_n})$ *for the adapted control problem* $(\widetilde{\mathscr{P}}_{\alpha_n})$ *satisfying* (3.18)–(3.20) *(such sequences exist by Theorem 3.4) and having the associated adjoint variables* $(p^{\alpha_n}, q^{\alpha_n}, q_\Gamma^{\alpha_n})$. *Then, for any subsequence* $\{n_k\}_{k\in\mathbb{N}}$ *of* $\mathbb{N}$, *there are a subsequence* $\{n_{k_\ell}\}_{\ell\in\mathbb{N}}$ *and some quintuple* $(p, q, q_\Gamma, \lambda, \lambda_\Gamma)$ *such that*

$$p \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;H^2(\Omega)),$$
$$(q, q_\Gamma) \in \mathscr{X}, \quad (\partial_t q, \partial_t q_\Gamma) \in \mathscr{W}_0^*, \quad (\lambda, \lambda_\Gamma) \in \mathscr{W}_0^*, \tag{4.51}$$

and such that the relations (4.38)–(4.41) *are valid (where the sequences are indexed by* n_{k_ℓ} *and the limits are taken as* $\ell \to \infty$*). Moreover, the variational inequality* (4.42) *and the adjoint state equations* (4.44), (4.45), *and* (4.47) *are satisfied.*

REMARK 4.3: Unfortunately, we cannot show that the limit quintuple

$$(p, q, q_\Gamma, \lambda, \lambda_\Gamma)$$

solving the adjoint problem associated with the optimal pair

$$((\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma, \bar{\xi}, \bar{\xi}_\Gamma), \bar{u}_\Gamma)$$

is unique. Therefore, it may well happen that the limits differ for different subsequences. However, it follows from the variational inequality (4.42) that for any such limit $(p, q, q_\Gamma, \lambda, \lambda_\Gamma)$ it holds, with the orthogonal projection $\mathbb{P}_{\mathscr{U}_{\mathrm{ad}}}$ onto $\mathscr{U}_{\mathrm{ad}}$ with respect to the standard inner product in $L^2(\Sigma)$, that for $\beta_6 > 0$ we have that $\bar{u}_\Gamma = \mathbb{P}_{\mathscr{U}_{\mathrm{ad}}}\left(-\beta_6^{-1} q_\Gamma\right)$.

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References

[1] V. Barbu, *Necessary conditions for nonconvex distributed control problems governed by elliptic variational inequalities*, J. Math. Anal. Appl. **80** (1981) 566–597.

[2] H. Brezis, *“Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert”*, North-Holland Math. Stud. **5**, North-Holland, Amsterdam, 1973.

[3] F. Brezzi, G. Gilardi, Chapters 1-3 in *“Finite Element Handbook”*, H. Kardestuncer and D.H. Norrie (eds.), McGraw-Hill Book Co., New York, 1987.

[4] L. Calatroni, P. Colli, *Global solution to the Allen–Cahn equation with singular potentials and dynamic boundary conditions*, Nonlinear Anal. **79** (2013), 12–27.

[5] P. Colli, M.H. Farshbaf-Shaker, G. Gilardi, J. Sprekels, *Optimal boundary control of a viscous Cahn–Hilliard system with dynamic boundary condition and double obstacle potentials*, SIAM J. Control Optim. **53** (2015), 2696–2721.

[6] P. Colli, M.H. Farshbaf-Shaker, G. Gilardi, J. Sprekels, *Second-order analysis of a boundary control problem for the viscous Cahn–Hilliard equation with dynamic boundary conditions*, Ann. Acad. Rom. Sci. Math. Appl. **7** (2015), 41–66.

[7] P. Colli, M. H. Farshbaf-Shaker, J. Sprekels, *A deep quench approach to the optimal control of an Allen–Cahn equation with dynamic boundary conditions and double obstacles*, Appl. Math. Optim. **71** (2015), 1–24.

[8] P. Colli, G. Gilardi, P. Krejčí, P. Podio-Guidugli, J. Sprekels, *Analysis of a time discretization scheme for a nonstandard viscous Cahn–Hilliard system*, ESAIM Math. Model. Numer. Anal. **48** (2014), 1061–1087.

[9] P. Colli, G. Gilardi, P. Krejčí, J. Sprekels, *A continuous dependence result for a nonstandard system of phase field equations*, Math. Methods Appl. Sci. **37** (2014), 1318–1324.

[10] P. Colli, G. Gilardi, P. Krejčí, J. Sprekels, *A vanishing diffusion limit in a nonstandard system of phase field equations*, Evol. Equ. Control Theory **3** (2014), 257–275.

[11] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, *Well-posedness and long-time behavior for a nonstandard viscous Cahn–Hilliard system*, SIAM J. Appl. Math. **71** (2011), 1849–1870.

[12] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, *Distributed optimal control of a non-standard system of phase field equations*, Contin. Mech. Thermodyn. **24** (2012), 437–459.

[13] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, *Continuous dependence for a nonstandard Cahn–Hilliard system with nonlinear atom mobility*, Rend. Sem. Mat. Univ. Politec. Torino **70** (2012), 27–52.

[14] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, *Global existence for a strongly coupled Cahn–Hilliard system with viscosity*, Boll. Unione Mat. Ital. (9) **5** (2012), 495–513.

[15] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, *An asymptotic analysis for a nonstandard Cahn–Hilliard system with viscosity*, Discrete Contin. Dyn. Syst. Ser. S **6** (2013), 353–368.

[16] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, *Global existence and uniqueness for a singular/degenerate Cahn–Hilliard system with viscosity*, J. Differential Equations **254** (2013), 4217–4244.

[17] P. Colli, G. Gilardi, E. Rocca, J. Sprekels, *Optimal distributed control of a diffuse interface model of tumor growth*, preprint arXiv:1601.04567 [math.AP] (2016), 1–32.

[18] P. Colli, G. Gilardi, J. Sprekels, *Analysis and optimal boundary control of a nonstandard system of phase field equations*, Milan J. Math. **80** (2012), 119–149.

[19] P. Colli, G. Gilardi, J. Sprekels, *A boundary control problem for the pure Cahn–Hilliard equation with dynamic boundary conditions*, Adv. Nonlinear Anal. **4** (2015), 311–325.

[20] P. Colli, G. Gilardi, J. Sprekels, *A boundary control problem for the viscous Cahn–Hilliard equation with dynamic boundary conditions*, Appl. Math. Optim. **73** (2016), 195–225.

[21] P. Colli, G. Gilardi, J. Sprekels, *On an application of Tikhonov's fixed point theorem to a nonlocal Cahn-Hilliard type system modeling phase separation*, J. Differential Equations **260** (2016), 7940–7964.

[22] P. Colli, G. Gilardi, J. Sprekels, *Distributed optimal control of a nonstandard nonlocal phase field system*, AIMS Mathematics **1** (2016), 225–260.

[23] P. Colli, G. Gilardi, J. Sprekels, *Distributed optimal control of a nonstandard nonlocal phase field system with double obstacle potential*, Evol. Equ. Control Theory **6** (2017), 35–58.

[24] P. Colli, G. Gilardi, J. Sprekels, *Global existence for a nonstandard viscous Cahn–Hilliard system with dynamic boundary condition*, preprint arXiv:1608.00854 [math.AP] (2016), 1–29.

[25] P. Colli, G. Gilardi, J. Sprekels, *Optimal boundary control of a nonstandard viscous Cahn–Hilliard system with dynamic boundary condition*, preprint arXiv:1609.07046 [math.AP] (2016), 1–30.

[26] P. Colli, J. Sprekels, *Optimal control of an Allen–Cahn equation with singular potentials and dynamic boundary condition*, SIAM J. Control Optim. **53** (2015), 213–234.

[27] S. Frigeri, E. Rocca, J. Sprekels, *Optimal distributed control of a nonlocal Cahn–Hilliard/Navier–Stokes system in two dimensions*, SIAM J. Control Optim. **54** (2016), 221–250.

[28] M. Hintermüller, T. Keil, D. Wegner, *Optimal control of a semidiscrete Cahn–Hilliard–Navier–Stokes system with non-matched fluid densities*, preprint arXiv:1506.03591 [math.AP] (2015), 1–35.

[29] M. Hintermüller, D. Wegner, *Distributed optimal control of the Cahn–Hilliard system including the case of a double-obstacle homogeneous free energy density*, SIAM J. Control Optim. **50** (2012), 388–418.

[30] M. Hintermüller, D. Wegner, *Optimal control of a semidiscrete Cahn–Hilliard–Navier–Stokes system*, SIAM J. Control Optim. **52** (2014), 747–772.

[31] M. Hintermüller, D. Wegner, *Distributed and boundary control problems for the semidiscrete Cahn–Hilliard/Navier–Stokes system with nonsmooth Ginzburg–Landau energies*, Isaac Newton Institute Preprint Series No. NI14042-FRB (2014), 1–29.

[32] P. Podio-Guidugli, *Models of phase segregation and diffusion of atomic species on a lattice*, Ric. Mat. **55** (2006), 105–118.

[33] E. Rocca, J. Sprekels, *Optimal distributed control of a nonlocal convective Cahn–Hilliard equation by the velocity in three dimensions*, SIAM J. Control Optim. **53** (2015), 1654–1680.

[34] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4) **146** (1987), 65–96.

[35] Q.-F. Wang, S.-i. Nakagiri, *Weak solutions of Cahn–Hilliard equations having forcing terms and optimal control problems*, Mathematical models in functional equations (Japanese) (Kyoto, 1999), Sūrikaisekikenkyūsho Kōkyūroku No. 1128 (2000), 172–180.

[36] X. Zhao, C. Liu, *Optimal control of the convective Cahn–Hilliard equation*, Appl. Anal. **92** (2013), 1028–1045.

[37] X. Zhao, C. Liu, *Optimal control for the convective Cahn–Hilliard equation in 2D case*, Appl. Math. Optim. **70** (2014), 61–82.