Global Solutions to a Coupled Parabolic–Hyperbolic System With Hysteresis in 1-D Magnetoelasticity

P. Krejčí [1,2] J. Sprekels [3]

1991 Mathematics Subject Classification. 35G25, 73R05, 82D40.
Keywords. Magnetic hysteresis, ferromagnetism, Preisach operator, magnetoelasticity, PDE's with hysteresis, parabolic-hyperbolic systems.

---

[1]Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 11567 Praha 10, Czech Republic ; partially supported by the Grant Agency of the Czech Republic under Grant No. 201/95/0568.
[2]This work has been done during the first author's visit at the Weierstrass Institute in Berlin.
[3]Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, D-10117 Berlin, Germany; partially supported by the Deutsche Forschungsgemeinschaft, DFG-SPP "Echtzeitoptimierung großer Systeme".

Abstract

In this paper the system of field equations governing the one-dimensional magnetoelastic evolution in a ferromagnet, which is immersed in an electromagnetic field and subjected to mechanical loads at a constant temperature below the Curie point, is considered. It is assumed that displacement currents are negligible and that all field quantities depend on one space variable only. The hysteretic relation between the applied magnetic field and the magnetization in the ferromagnet are modeled using the notion of hysteresis operators; in particular, hysteresis operators of Preisach type are included. It is shown that an initial-boundary value problem for the system admits global solutions for arbitrary initial data, if viscosity is present in the material, and for small initial data, if not. The considered field equations may be regarded as a model for the effect of magnetostriction in ferromagnets.

1 Introduction

In this paper, we consider the system of PDEs

$$b_t - h_{xx} + (b\,u_t)_x = f\,, \tag{1.1a}$$
$$u_{tt} - u_{xx} - \eta\,u_{xxt} + h_x\,b = g\,, \tag{1.1b}$$
$$b = h + m = h + \mathcal{P}[h]\,, \tag{1.1c}$$

which are to be satisfied in Ω_T, where $\Omega = (0,1)$ and, for $t > 0$, $\Omega_t = \Omega \times (0,t)$. In this connection, f, g are given functions, $\eta \geq 0$ is a constant, and $\mathcal{P}$ denotes a *hysteresis operator* whose properties will be specified below. We complement the equations (1.1a - c) by the initial and boundary conditions

$$u(x,0) = u^0(x), \quad u_t(x,0) = v^0(x), \quad h(x,0) = h^0(x), \quad x \in \bar{\Omega}, \tag{1.1d}$$
$$u(0,t) = u(1,t) = 0 = h(0,t) = h(1,t)\,, \quad t \in [0,T]\,. \tag{1.1e}$$

The system (1.1a - e) may be regarded as a simplified model for the one-dimensional magnetoelastic or magnetostrictive developments in ferromagnets. To confirm this, consider a sample of ferromagnetic material of unit length immersed in an electromagnetic field which is possibly subjected to mechanical loads. The temperature is maintained constant below Curie temperature so that the material is ferromagnetic. Let us make the following simplifying assumptions.

(i) The electric displacement (together with the charge density) is negligible.

(ii) The displacement vector $\mathbf{u}$ is parallel to the x-axis, i. e. $\mathbf{u} = (u,0,0)$.

(iii) All field quantities depend on the coordinate x and on the time t only.

(iv) The medium is isotropic with constant electric conductivity $\sigma_E > 0$, magnetic permeability $\mu_H > 0$, elastic (Lamé) coefficients λ_e, μ_e, and viscosity coefficient $\eta \geq 0$.

Under these assumptions the governing field equations are

$$B_{2,t} - \frac{1}{\sigma_E} H_{2,xx} + (B_2\, u_t)_x = 0\,, \tag{1.2a}$$

$$B_{3,t} - \frac{1}{\sigma_E} H_{3,xx} + (B_3\, u_t)_x = 0\,, \tag{1.2b}$$

$$\rho\, u_{tt} - (\lambda_e + 2\mu_e) u_{xx} - \eta\, u_{xxt} + B_2\, H_{2,x} + B_3\, H_{3,x} = g\,, \tag{1.2c}$$

and are to be satisfied in Ω_T.

The variables have the following physical meaning: ρ is the mass density; $\mathbf{H} = (H_1, H_2, H_3)$ is the magnetic field; $\mathbf{B} = (B_1, B_2, B_3)$ is the magnetic induction; g are the distributed volume forces in x-direction.

Let us give a brief derivation of the equations (1.2a - c). At first, we recall that under the assumption **(i)** the magnetic field $\mathbf{H}$ is described by the equations (cf. [5, p. 219])

$$\mathbf{B}_t - \operatorname{curl}(\mathbf{u}_t \times \mathbf{B}) - \frac{1}{\sigma_E} \operatorname{curl}(\operatorname{curl}\mathbf{H}) = \mathbf{0}\,, \tag{1.3a}$$

$$\operatorname{div}\mathbf{B} = 0\,, \tag{1.3b}$$

where the term $\mathbf{u}_t \times \mathbf{B}$ corresponds to the Lorentz force. Invoking **(ii), (iii)**, we obtain (1.2a, b).

Next, we consider the equations of motion which under **(iv)** have the form

$$\rho\, \mathbf{u}_{tt} - (\lambda_e + \mu_e) \operatorname{grad}\operatorname{div}\mathbf{u} - \mu_e\, \Delta\mathbf{u} - \eta\, \Delta\mathbf{u}_t - \operatorname{curl}\mathbf{H} \times \mathbf{B} = \mathbf{g}\,, \tag{1.4}$$

where $\mathbf{g}$ is the load vector. Using **(ii), (iii)**, and denoting by g the first component of $\mathbf{g}$, we arrive at (1.2c).

It remains to specify the relation between $\mathbf{B}$ and $\mathbf{H}$. We have

$$\mathbf{B} = \mu_H\, \mathbf{H} + \mathbf{M}\,, \tag{1.5}$$

where $\mathbf{M}$ denotes the magnetization. Since the sample is ferromagnetic, the relation between $\mathbf{M}$ and $\mathbf{H}$ has the form of a hysteresis, i. e. it has to be expressed as

$$\mathbf{M} = \mathcal{P}[\mathbf{H}]\,, \tag{1.6}$$

where $\mathcal{P}$ is a vector-valued *hysteresis operator*. If we assume that the hysteretic relation is *diagonal*, i. e. of the form

$$M_i = \mathcal{P}_i[H_i]\,, \quad i = 1, 2, 3\,, \tag{1.7}$$

then (1.1a - c) can be regarded as the system (1.2a - c) if (1.2b) is discarded, $b := B_2$, $h := H_2$, $m := M_2$, and if all physical constants are normalized to unity. Notice that from the mathematical viewpoint it makes no difference whether (1.2b) is included or not; the arguments generalize easily.

In our analysis, it will turn out that the cases $\eta > 0$ (with viscosity) and $\eta = 0$ (no viscosity) differ considerably: for $\eta > 0$ the system is parabolic, and the existence of global solutions for large data can be shown, while for $\eta = 0$ equation (1.1b) is hyperbolic, and global existence can only be expected for small data that guarantee that h remains within the convexity domain of the hysteresis operator $\mathcal{P}$ (see Fig. 1.). A similar phenomenon has been observed in the case of a single equation with hysteresis, where parabolic equations admit global solutions for large data (cf. Visintin [6, 7, 8]), while hyperbolic equations require the convexity of hysteresis loops (cf. Krejčí [3, 4]).

Fig. 1. Hysteresis diagram $m = P(h)$.

The paper is organized as follows. In Section 2, we give a precise statement of the problem under investigation, in particular, of the properties of the hysteresis operator $\mathcal{P}$. In addition, the main results of this paper (existence for both $\eta > 0$ and $\eta = 0$) are formulated. In Section 3, we approximate the system (1.1a - e) using a space discretization, and we prove a number of a priori estimates for the approximating solutions. In the concluding Sections 4 and 5 we use compactness arguments and a passage-to-the-limit procedure to prove the existence results for $\eta > 0$ (Section 4) and $\eta = 0$ (Section 5), respectively.

2 Statement of the Problem

Consider the initial-boundary value problem (1.1a - e). We do not prescribe any specific form of the operator $\mathcal{P}$, although the properties that are assumed to hold are typical in particular for the Preisach hysteresis model of one-dimensional ferromagnetism.

We suppose that there exists an operator $P : C[0,T] \to C[0,T]$ such that the value of the operator $\mathcal{P}$ for each input function $h \in C(\overline{\Omega_T})$ and each $x \in [0,1]$ is given by the formula

$$\mathcal{P}[h](x,t) \;=\; P[h(x,\cdot)](t)\,. \tag{2.1}$$

We generally assume:

(H1)

(i) The operator P is continuous and has the Volterra property, i. e. if $h_1, h_2 \in C[0,T]$ satisfy $h_1(t) = h_2(t)$ for all $t \in [0,t_0]$, then $P[h_1](t_0) = P[h_2](t_0)$.

(ii) There exists some Lipschitz continuous function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(0) = 0$ and $P[h](0) = \psi(h(0))$ for all $h \in C[0,T]$.

(iii) There exists a Lipschitz continuous inverse $(I+P)^{-1} : C[0,T] \to C[0,T]$, where I denotes the identity operator.

(iv) P and $(I+P)^{-1}$ map $W^{1,p}(0,T)$ into $W^{1,p}(0,T)$ for every $p \in [0,\infty]$, and there exists a constant $a > 0$ such that

$$0 \le \frac{dh}{dt}\cdot\frac{d\,P[h]}{dt} \le a\Big(\frac{dh}{dt}\Big)^2, \quad \text{a.e., for every } h \in W^{1,1}(0,T)\,. \tag{2.2}$$

(v) There exists an *internal energy density operator* $\mathcal{U} : W^{1,1}(0,T) \to W^{1,1}(0,T)$ such that

$$\mathcal{U}[h](t) \,\ge\, 0\,, \quad \forall\; h \,\in\, W^{1,1}(0,T)\,, \quad \forall\; t \,\in\, [0,T]\,, \tag{2.3}$$

$$\exists\; c > 0 : \quad \mathcal{U}[h](0) \,\le\, c\,h^2(0)\,, \quad \forall\; h \,\in\, W^{1,1}(0,T)\,, \tag{2.4}$$

$$\frac{d}{dt}\mathcal{U}[h](t) \,\le\, h(t)\,\frac{d}{dt}P[h](t) \quad \text{a.e.,} \quad \forall\; h \,\in\, W^{1,1}(0,T)\,. \tag{2.5}$$

(vi) There exists a constant $M > 0$ (*saturation*) such that

$$|P[h](t)| \,\le\, M\,, \quad \forall\; h \,\in\, C[0,T]\,, \quad \forall\; t \,\in\, [0,T]\,. \tag{2.6}$$

The properties **(i)-(vi)** of hypothesis **(H1)** are rather general and do not necessarily imply the occurence of hysteretic effects; for instance, the superposition operator $P[h](t) = \psi(h(t))$ generated by a bounded, Lipschitz continuous and non-decreasing function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(0) = 0$ satisfies **(H1)**. This is no longer the case for the following hypothesis which is typical for hysteresis operators having convex increasing and concave decreasing branches for inputs restricted to the interval $[-H_0, H_0]$ (see Fig. 1).

(H2) There exists a constant $H_0 > 0$ such that the following holds: whenever $h \in W^{1,\infty}(0,T)$ with $\max_{0\le t\le T}|h(t)| \le H_0$ is given and $b := (I+P)[h] \in W^{2,1}(0,T)$, then, for almost every $0 < t_1 < t_2 < T$,

$$\int_{t_1}^{t_2} \ddot{b}(t)\,\dot{h}(t)\,dt \,\ge\, \frac{1}{2}\big(\dot{b}(t_2)\,\dot{h}(t_2) \,-\, \dot{b}(t_1)\,\dot{h}(t_1)\big)\,. \tag{2.7}$$

(Here, and throughout this paper, the superimposed dot denotes the time derivative.)

A detailed discussion of the connections between the convexity of hysteresis loops and higher order energy-type inequalities can be found in [4]. Note also that the Preisach model of ferromagnetism generated by a regular measure satisfies **(H1)** and **(H2)**, see [2, 3, 4].

Next, we state the existence results for the system (1.1a - e) for the cases $\eta > 0$ and $\eta \geq 0$, respectively.

Theorem 2.1 *Let $\eta > 0$, let P satisfy* **(H1)**, *and suppose that*

$$h^0, v^0 \in \overset{o}{W}{}^{1,2}(\Omega), \quad u^0 \in W^{2,2}(\Omega) \cap \overset{o}{W}{}^{1,2}(\Omega), \quad f, g \in L^2(\Omega_T). \tag{2.8}$$

Then there exist functions $h, u \in C(\overline{\Omega_T})$ satisfying (1.1d, e) and

$$h_x, u_{xx}, u_{xt} \in L^\infty(0,T; L^2(\Omega)), \quad h_t, u_{tt}, u_{xxt} \in L^2(\Omega_T), \tag{2.9}$$

such that for almost every $(x,t) \in \Omega_T$ it holds

$$\begin{aligned} &\int_\Omega \Big[\big((h + \mathcal{P}[h])_t - f \big)(x,t)\, w(x) + h_x(x,t)\, w'(x) \\ &-\big((h + \mathcal{P}[h])\, u_t \big)(x,t)\, w'(x) \Big]\, dx = 0, \quad \forall\ w \in \overset{o}{W}{}^{1,2}(\Omega), \end{aligned} \tag{2.10a}$$

$$u_{tt} - u_{xx} - \eta\, u_{xxt} + h_x\, b = g, \tag{2.10b}$$

where $\mathcal{P}$ is the operator defined in (2.1).

In the case $\eta \geq 0$, we obtain a weaker result.

Theorem 2.2 *Let $\eta \geq 0$, and suppose that P satisfies* **(H1)** *and* **(H2)**. *Then there exists some $\delta > 0$ such that for every*

$$u^0, v^0, h^0 \in W^{2,2}(\Omega) \cap \overset{o}{W}{}^{1,2}(\Omega), \quad f, g \in W^{1,1}(0,T; L^2(\Omega)), \tag{2.11}$$

satisfying the inequalities

$$\|u^0\|_{W^{2,2}(\Omega)} + \|v^0\|_{W^{2,2}(\Omega)} + \|h^0\|_{W^{2,2}(\Omega)} \leq \delta, \tag{2.12a}$$

$$\int_0^T \left(\|f(t)\|^2 + \|g(t)\|^2 + \|f_t(t)\|^2 + \|g_t(t)\|^2 \right)^{1/2} dt \leq \delta, \tag{2.12b}$$

there exist functions $h, u \in C(\overline{\Omega_T})$ such that (1.1d, e), (2.10a, b) and

$$h_t, u_{tt}, u_{xt} \in L^\infty(0,T; L^2(\Omega)), \quad h_x, h_{xt}, u_{xx}, \eta\, u_{xxt} \in L^2(\Omega_T), \tag{2.13}$$

hold.

(Here, and throughout the paper, we denote by $\|\cdot\|$ the $L^2(\Omega)$-norm.)

We note that both Theorem 2.1 and Theorem 2.2 give existence results for global solutions to the system (1.1a - e); however, in Theorem 2.2 only small data are admitted. It should also be noted that even in the hyperbolic case $\eta = 0$ no shocks can develop for small data. This fact is a consequence of the hysteresis (see also [1] and [4]). In physical terms this means that the effect of magnetostriction in a one-dimensional ferromagnet is a "smooth" effect if either the fields are small or (mechanical) viscosity is present.

3 Space Discretization and A Priori Estimates

Let $n \in \mathbb{N}$ be fixed. For $k = 1, \dots, n-1$, we consider the system of ordinary differential equations

$$\dot{b}_k - n^2 (h_{k+1} - 2 h_k + h_{k-1}) + n (b_k \dot{u}_k - b_{k-1} \dot{u}_{k-1}) = f_k , \tag{3.1a}$$

$$\ddot{u}_k - n^2 (u_{k+1} - 2 u_k + u_{k-1}) - \eta n^2 (\dot{u}_{k+1} - 2 \dot{u}_k + \dot{u}_{k-1}) + n b_k (h_{k+1} - h_k) = g_k, \tag{3.1b}$$

$$u_0 = u_n = h_0 = h_n = b_0 = b_n = 0 , \tag{3.1c}$$

where

$$b_k := h_k + P[h_k] , \tag{3.1d}$$

$$g_k(t) := n \int_{k/n}^{(k+1)/n} g(x,t)\, dx , \quad f_k(t) := n \int_{k/n}^{(k+1)/n} f(x,t)\, dx , \tag{3.1e}$$

together with the initial conditions

$$h_k(0) = n \int_{k/n}^{(k+1)/n} h^0(x)\, dx , \quad b_k(0) = h_k(0) + \psi\big(h_k(0)\big) ,$$

$$u_k(0) = n \int_{k/n}^{(k+1)/n} u^0(x)\, dx , \quad \dot{u}_k(0) = n \int_{k/n}^{(k+1)/n} v^0(x)\, dx . \tag{3.1f}$$

Owing to hypothesis **(H1)**, the operator $(I+P)^{-1}$ is Lipschitz continuous on $C[0,T]$; hence there is some (maximal) $T_n \in (0,T]$ such that the initial value problem (3.1a - f) admits a unique solution $(h_k, u_k) \in W^{1,2}(0,T_n) \times W^{2,2}(0,T_n)$, $1 \le k \le n-1$.

In the sequel, we will derive some a priori estimates for (h_k, u_k), $1 \le k \le n-1$, that will ensure that $T_n = T$ and that a passage to the limit as $n \to \infty$ is possible. To this end, we will denote by $C_i, \hat{C}_i$, $i \in \mathbb{N}$, constants that may depend on the data but neither on n nor on η.

Lemma 3.1 *There is some* $\hat{C}_1 > 0$ *satisfying*

$$\frac{1}{2\,n}\sum_{k=0}^{n-1}\left(h_k^2 + \dot{u}_k^2 + n^2(u_{k+1} - u_k)^2\right)(t) + n\sum_{k=0}^{n-1}\int_0^t \left((h_{k+1} - h_k)^2\right.$$

$$\left. + \eta\,(\dot{u}_{k+1} - \dot{u}_k)^2\right)(\tau)\,d\tau \le \hat{C}_1\,, \quad \text{for all } t \in [0, T_n]\,. \tag{3.2}$$

Proof. We multiply (3.1a) by $\frac{1}{n}h_k$ and (3.1b) by $\frac{1}{n}\dot{u}_k$, add the results, and sum over k. Using the boundary conditions (3.1c), we find that

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n-1}\dot{b}_k\,h_k + n\sum_{k=0}^{n-1}(h_{k+1} - h_k)^2 + \frac{1}{n}\sum_{k=1}^{n-1}\ddot{u}_k\,\dot{u}_k \\
&+ n\sum_{k=0}^{n-1}\left[(\dot{u}_{k+1} - \dot{u}_k)(u_{k+1} - u_k) + \eta(\dot{u}_{k+1} - \dot{u}_k)^2\right] \\
= \;& \frac{1}{n}\sum_{k=1}^{n-1}(f_k\,h_k + g_k\,\dot{u}_k) \\
\le \;& \left(\frac{1}{n}\sum_{k=1}^{n-1}f_k^2\right)^{1/2}\left(\frac{1}{n}\sum_{k=1}^{n-1}h_k^2\right)^{1/2} + \left(\frac{1}{n}\sum_{k=1}^{n-1}g_k^2\right)^{1/2}\left(\frac{1}{n}\sum_{k=1}^{n-1}\dot{u}_k^2\right)^{1/2},
\end{aligned} \tag{3.3}$$

where

$$\frac{1}{n}\sum_{k=1}^{n-1}f_k^2(t) \le \int_\Omega f^2(x,t)\,dx\,, \quad \frac{1}{n}\sum_{k=1}^{n-1}g_k^2(t) \le \int_\Omega g^2(x,t)\,dx\,. \tag{3.4}$$

Using **(H1), (v)** and (2.8), we conclude (3.2) from Gronwall's lemma. □

From Lemma 3.1 it follows, in particular, that $T = T_n$. To derive further estimates, we need the following discrete Nirenberg inequality.

Lemma 3.2 *Let* $z_1, ..., z_n \in \mathbb{R}$. *Then it holds*

$$\max_{1\le j\le n}|z_j| \le \left(\frac{1}{n}\sum_{k=1}^{n}z_k^2\right)^{1/2} + \sqrt{2}\left(n\sum_{k=1}^{n-1}(z_{k+1} - z_k)^2\right)^{1/4}\left(\frac{1}{n}\sum_{k=1}^{n}z_k^2\right)^{1/4}. \tag{3.5}$$

Proof. For any $j \in \{1, ..., n\}$ it holds

$$z_j^2 \le z_k^2 + \sum_{i=1}^{j-1}|z_{i+1}^2 - z_i^2|\,, \quad \text{if } j \ge k\,, \tag{3.6a}$$

$$z_j^2 \le z_k^2 + \sum_{i=j}^{n-1}|z_{i+1}^2 - z_i^2|\,, \quad \text{if } j \le k\,. \tag{3.6b}$$

Therefore,

$$z_j^2 \;\le\; \frac{1}{n}\sum_{k=1}^{n} z_k^2 \;+\; \left(\sum_{i=1}^{n-1}(z_{i+1} \;-\; z_i)^2\right)^{1/2} \left(\sum_{i=1}^{n-1}(z_{i+1} \;+\; z_i)^2\right)^{1/2}, \tag{3.7}$$

whence the assertion easily follows. □

Lemma 3.3 *Let $\eta > 0$. Then there exists some $\hat{C}_2 > 0$ such that*

$$\begin{aligned} &\eta\, n^3 \sum_{k=1}^{n-1} \left(u_{k+1}(t) - 2\,u_k(t) + u_{k-1}(t)\right)^2 + n^3 \sum_{k=1}^{n-1} \int_0^t \left(u_{k+1} - 2\,u_k + u_{k-1}\right)^2(\tau)\,d\tau \\ \le\quad & \hat{C}_2\left(1 + \frac{1}{\eta}\right), \quad \textit{for any } t \in [0,T]\,. \end{aligned} \tag{3.8}$$

Proof. Let $t \in [0,T]$ be arbitrary. We multiply (3.1b) by $-\,n(u_{k+1} \,-\, 2\,u_k \,+\, u_{k-1})$ and sum over k in order to obtain

$$\begin{aligned} & n^3 \sum_{k=1}^{n-1}\Big[(u_{k+1} \,-\, 2\,u_k \,+\, u_{k-1})^2 \,+\, \frac{\eta}{2}\frac{d}{dt}(u_{k+1} \,-\, 2\,u_k \,+\, u_{k-1})^2\Big] \\ =\;\; & n \sum_{k=1}^{n-1} \frac{d}{dt}\left[\dot{u}_k\,(u_{k+1} \,-\, 2\,u_k \,+\, u_{k-1})\right] \,+\, n \sum_{k=0}^{n-1}(\dot{u}_{k+1} \,-\, \dot{u}_k)^2 \\ & + n^2 \sum_{k=1}^{n-1} b_k\,(h_{k+1} \,-\, h_k)\,(u_{k+1} \,-\, 2\,u_k \,+\, u_{k-1}) - \,n\, \sum_{k=1}^{n-1} g_k\,(u_{k+1} \,-\, 2\,u_k \,+\, u_{k-1})\,. \end{aligned} \tag{3.9}$$

The initial conditions (3.1f) entail that

$$\begin{aligned} n^3 \sum_{k=1}^{n-1}(u_{k+1} \,-\, 2\,u_k \,+\, u_{k-1})^2(0) \;\;=\;\; & n^5 \sum_{k=1}^{n-1}\left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} \left(u^0\left(x + \frac{1}{n}\right) \,-\, 2\,u^0(x) +\right.\right. \\ & \left.\left. u^0\left(x \,-\, \frac{1}{n}\right)\right)dx\right)^2, \end{aligned} \tag{3.10}$$

where we have put $u^0(x) = 0$ for $x \in \mathbb{R}\backslash[0,1]$. Hence,

$$\begin{aligned} & n^3 \sum_{k=1}^{n-1}(u_{k+1} \,-\, 2\,u_k \,+\, u_{k-1})^2(0) \;=\; n^5 \sum_{k=1}^{n-1}\Big(\int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{x-\frac{1}{n}}^{x} \int_{\xi}^{\xi+\frac{1}{n}} (u^0)''(s)\,ds\,d\xi\,dx\Big)^2 \\ & \le\, n^5 \sum_{k=1}^{n-1}\Big(\int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{\frac{k-1}{n}}^{\frac{k+1}{n}} \int_{\frac{k-1}{n}}^{\frac{k+2}{n}} \Big|(u^0)''(s)\Big|\,ds\,d\xi\,dx\Big)^2 \,\le\, 36 \int_0^1 |(u^0)''(x)|^2 dx. \end{aligned} \tag{3.11}$$

Integrating (3.9) over $[0,t]$, we obtain

$$
\begin{aligned}
& \frac{\eta}{2}\, n^3 \sum_{k=1}^{n-1} (u_{k+1} - 2\,u_k + u_{k-1})^2(t) + n^3 \sum_{k=1}^{n-1} \int_0^t (u_{k+1} - 2\,u_k + u_{k-1})^2(\tau)\, d\tau \\
\leq \quad & C_1 + \left(\frac{1}{n} \sum_{k=1}^{n} \dot{u}_k^2 \right)^{1/2} \left(n^3 \sum_{k=1}^{n-1} (u_{k+1} - 2\,u_k + u_{k-1})^2 \right)^{1/2} (t) \\
& + n \sum_{k=0}^{n-1} \int_0^t (\dot{u}_{k+1} - \dot{u}_k)^2(\tau)\, d\tau + \int_0^t \left(M + \max_{1 \leq j \leq n-1} |h_j| \right) . \\
& \cdot \left(n \sum_{k=0}^{n-1} (h_{k+1} - h_k)^2 \right)^{1/2} \left(n^3 \sum_{k=1}^{n-1} (u_{k+1} - 2\,u_k + u_{k-1})^2 \right)^{1/2} (\tau)\, d\tau \\
& + \int_0^t \left(\frac{1}{n} \sum_{k=1}^{n-1} g_k^2 \right)^{1/2} \left(n^3 \sum_{k=1}^{n-1} (u_{k+1} - 2\,u_k + u_{k-1})^2 \right)^{1/2} (\tau)\, d\tau \\
\leq \quad & C_2 \left[1 + \frac{1}{\eta} + \int_0^t \left(n \sum_{k=0}^{n-1} (h_{k+1} - h_k)^2 \right)^{3/4} \left(n^3 \sum_{k=1}^{n-1} (u_{k+1} - 2\,u_k + u_{k-1})^2 \right)^{1/2} (\tau)\, d\tau \right] \\
& + \frac{\eta}{8}\, n^3 \sum_{k=1}^{n-1} (u_{k+1} - 2\,u_k + u_{k-1})^2\, (t) + \frac{1}{4}\, n^3 \sum_{k=1}^{n-1} \int_0^t (u_{k+1} - 2\,u_k + u_{k-1})^2(\tau)\, d\tau ,
\end{aligned}
\tag{3.12}
$$

where we have used the inequalities (2.6), (3.2) and (3.5). In addition, using Hölder's and Young's inequalities,

$$
\begin{aligned}
& \int_0^t \left(n \sum_{k=0}^{n-1} (h_{k+1} - h_k)^2 \right)^{3/4} \left(n^3 \sum_{k=1}^{n-1} (u_{k+1} - 2\,u_k + u_{k-1})^2 \right)^{1/2} (\tau)\, d\tau \\
\leq \quad & \left(\int_0^t n \sum_{k=0}^{n-1} (h_{k+1} - h_k)^2\, (\tau)\, d\tau \right)^{3/4} \left(\int_0^t n^3 \sum_{k=1}^{n-1} (u_{k+1} - 2\,u_k + u_{k-1})^2(\tau)\, d\tau \right)^{1/4} \cdot \\
& \cdot \max_{0 \leq \tau \leq t} \left(n^3 \sum_{k=1}^{n-1} (u_{k+1} - 2\,u_k + u_{k-1})^2\, (\tau) \right)^{1/4} \\
\leq \quad & \frac{\eta}{8} \max_{0 \leq \tau \leq t} \left(n^3 \sum_{k=1}^{n-1} (u_{k+1} - 2\,u_k + u_{k-1})^2\, (\tau) \right) \\
& + \frac{1}{4}\, n^3 \sum_{k=1}^{n-1} \int_0^t (u_{k+1} - 2\,u_k + u_{k-1})^2(\tau)\, d\tau + \frac{C_3}{\eta} ,
\end{aligned}
\tag{3.13}
$$

where (3.2) has been employed. Combining (3.12) and (3.13), and taking the maximum with respect to t on both sides, we obtain (3.8). $\square$

Lemma 3.4 *Let $\eta > 0$. Then there exists some $\hat{C}_3 > 0$ such that, for any $t \in [0,T]$,*

$$n \sum_{k=0}^{n-1} \left[(h_{k+1} - h_k)^2 + \eta\, (\dot{u}_{k+1} - \dot{u}_k)^2 \right](t) + \frac{1}{n} \sum_{k=1}^{n-1} \int_0^t (\dot{h}_k^2 + \ddot{u}_k^2)(\tau)\, d\tau \le \hat{C}_3 \left(1 + \frac{1}{\eta} \right). \tag{3.14}$$

Proof. We multiply (3.1a) by $\frac{1}{n}\dot{h}_k$ and (3.1b) by $\frac{1}{n}\ddot{u}_k$, add the resulting equations, and sum over k. It follows

$$\begin{aligned}
& \frac{1}{n}\sum_{k=1}^{n-1} \dot{b}_k \dot{h}_k + n \sum_{k=0}^{n-1} (h_{k+1} - h_k)(\dot{h}_{k+1} - \dot{h}_k) + \frac{1}{n}\sum_{k=1}^{n-1} \ddot{u}_k^2 \\
& + \eta\, n \sum_{k=0}^{n-1} (\dot{u}_{k+1} - \dot{u}_k)(\ddot{u}_{k+1} - \ddot{u}_k) - n \sum_{k=1}^{n-1} \ddot{u}_k\, (u_{k+1} - 2\,u_k + u_{k-1}) \\
= \quad & \sum_{k=0}^{n-1} \left[\left(\dot{h}_{k+1} - \dot{h}_k \right) b_k\, \dot{u}_k - (h_{k+1} - h_k)\, b_k\, \ddot{u}_k \right] + \frac{1}{n}\sum_{k=1}^{n-1} (f_k\, \dot{h}_k + g_k\, \ddot{u}_k) \\
\le \quad & \frac{d}{dt}\left(\sum_{k=0}^{n-1} (h_{k+1} - h_k)\, b_k\, \dot{u}_k \right) - \sum_{k=0}^{n-1} (h_{k+1} - h_k)(\dot{b}_k\, \dot{u}_k + 2\, b_k\, \ddot{u}_k) \\
& + \left(\frac{1}{n}\sum_{k=1}^{n-1} f_k^2 \right)^{1/2} \left(\frac{1}{n}\sum_{k=1}^{n-1} \dot{h}_k \right)^{1/2} + \left(\frac{1}{n}\sum_{k=1}^{n-1} g_k^2 \right)^{1/2} \left(\frac{1}{n}\sum_{k=1}^{n-1} \ddot{u}_k^2 \right)^{1/2}.
\end{aligned} \tag{3.15}$$

Integrating (3.15) over $[0,t]$, we obtain from **(H1), (iv)**, Young's inequality, and (2.11), that

$$\begin{aligned}
& \frac{n}{2}\sum_{k=0}^{n-1} \left[(h_{k+1} - h_k)^2 + \eta\, (\dot{u}_{k+1} - \dot{u}_k)^2 \right](t) + \frac{1}{2\,n}\sum_{k=1}^{n-1} \int_0^t \left(\dot{h}_k^2 + \ddot{u}_k^2 \right)(\tau)\, d\tau \\
\le \quad & C_1 + \sum_{k=0}^{n-1} \left[(h_{k+1} - h_k)\, b_k\, \dot{u}_k \right](t) + \sum_{k=0}^{n-1} \int_0^t |h_{k-1} - h_k| \left(|\dot{b}_k\, \dot{u}_k| + 2|b_k\, \ddot{u}_k| \right)(\tau)\, d\tau \\
& + \int_0^t \left(\frac{1}{n}\sum_{k=1}^{n-1} \ddot{u}_k^2 \right)^{1/2} \left(n^3 \sum_{k=1}^{n-1} (u_{k+1} - 2\,u_k + u_{k-1})^2 \right)^{1/2} (\tau)\, d\tau \\
=: \quad & C_1 + A_1 + A_2 + A_3.
\end{aligned} \tag{3.16}$$

The expressions on the right-hand side of (3.16) are estimated individually. At first, we infer from (3.2), (3.5), and Young's inequality, that

$$
\begin{aligned}
|A_1| &\leq \left(M + \max_{1\leq j\leq n-1}\ \max_{0\leq\tau\leq t}\ |h_j(\tau)|\right)\left(\frac{1}{n}\sum_{k=1}^{n-1}\dot{u}_k^2(t)\right)^{1/2}\left(n\ \sum_{k=0}^{n-1}(h_{k+1}-h_k)^2\ (t)\right)^{1/2}\\
&\leq C_2\left(1+\max_{0\leq\tau\leq t}\left(n\ \sum_{k=0}^{n-1}(h_{k+1}-h_k)^2(\tau)\right)^{3/4}\right)\\
&\leq \frac{n}{8}\ \max_{0\leq\tau\leq t}\ \sum_{k=0}^{n-1}(h_{k+1}-h_k)^2(\tau)+C_3\ . \qquad (3.17)
\end{aligned}
$$

Next, observe that (3.8) implies that

$$
\begin{aligned}
|A_3| &\leq C_4\left(1+\frac{1}{\eta}\right)^{1/2}\ \left(\frac{1}{n}\ \sum_{k=1}^{n-1}\ \int_0^t\ \ddot{u}_k^2\ (\tau)\ d\tau\right)^{1/2}\\
&\leq \frac{1}{4n}\ \sum_{k=1}^{n-1}\ \int_0^t\ \ddot{u}_k^2\ (\tau)\ d\tau\ \ +\ \ C_5\left(1+\frac{1}{\eta}\right). \qquad (3.18)
\end{aligned}
$$

We also have, using (2.2), (3.2), (3.5), and Hölder's and Young's inequalities,

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\ \int_0^t\ |h_{k+1}-h_k|\ \ |\dot{b}_k\ \dot{u}_k|(\tau)\ d\tau\\
\leq\ &\int_0^t\ \max_{0\leq j\leq n-1}\ |\dot{u}_j|\ \left(\frac{1}{n}\sum_{k=1}^{n-1}\ \dot{b}_k^2\right)^{1/2}\left(n\ \sum_{k=1}^{n-1}\ (h_{k+1}-h_k)^2\right)^{1/2}(\tau)\ d\tau\\
\leq\ &C_6\int_0^t\left(1+\left(n\ \sum_{k=0}^{n-1}\ (\dot{u}_{k+1}-\dot{u}_k)^2\right)^{1/4}\right)\ \left(\frac{1}{n}\sum_{k=0}^{n-1}\ \dot{h}_k^2\right)^{1/2}\cdot\\
&\qquad\left(n\sum_{k=0}^{n-1}\ (h_{k+1}-h_k)^2\right)^{1/2}(\tau)\ d\tau\\
\leq\ &C_7\left(\int_0^t\left(1+n\ \sum_{k=0}^{n-1}\ (\dot{u}_{k+1}-\dot{u}_k)^2\right)(\tau)\ d\tau\right)^{1/4}\cdot\left(\int_0^t\ \frac{1}{n}\sum_{k=1}^{n-1}\ \dot{h}_k^2\ (\tau)\ d\tau\right)^{1/2}\cdot\\
&\cdot\left(\int_0^t\left(n\ \sum_{k=0}^{n-1}\ (h_{k+1}-h_k)^2\right)^2(\tau)\ d\tau\right)^{1/4}\\
\leq\ &C_8\ \eta^{-1/4}\ \max_{0\leq\tau\leq t}\left(n\sum_{k=0}^{n-1}\ (h_{k+1}-h_k)^2(\tau)\right)^{1/4}\ \left(\int_0^t\ \frac{1}{n}\sum_{k=1}^{n-1}\ \dot{h}_k^2\ (\tau)\ d\tau\right)^{1/2}\\
\leq\ &\frac{n}{8}\max_{0\leq\tau\leq t}\ \sum_{k=0}^{n-1}\ (h_{k+1}-h_k)^2(\tau)+\frac{1}{2n}\int_0^t\sum_{k=1}^{n-1}\ \dot{h}_k^2(\tau)\ d\tau+\frac{C_9}{\eta}. \qquad (3.19)
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&\sum_{k=0}^{n-1} \int_0^t |h_{k+1}-h_k|\,|b_k|\,|\ddot{u}_k|(\tau)\,d\tau \\
\leq\quad & \max_{0\le j\le n-1}\left(M+\max_{0\le\tau\le t}|h_j(\tau)|\right)\left(\int_0^t n\sum_{k=0}^{n-1}(h_{k+1}-h_k)^2(\tau)\,d\tau\right)^{1/2} \cdot \\
&\cdot\left(\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\ddot{u}_k^2(\tau)\,d\tau\right)^{1/2} =: A_4. \qquad (3.20)
\end{aligned}
$$

Using (3.2) and (3.5) again, we find that

$$
\begin{aligned}
A_4 \;&\le\; C_{10}\left(1+\max_{0\le\tau\le t} n\sum_{k=0}^{n-1}(h_{k+1}-h_k)^2(\tau)\right)^{1/4}\cdot\left(\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\ddot{u}_k^2(\tau)\,d\tau\right)^{1/2} \\
&\le\; \frac{1}{4}\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\ddot{u}_k^2(\tau)\,d\tau+\frac{n}{8}\max_{0\le\tau\le t}\sum_{k=0}^{n-1}(h_{k+1}-h_k)^2(\tau)+C_{11}. \qquad (3.21)
\end{aligned}
$$

Combining (3.16) to (3.21), and taking the maximum with respect to t on both sides, we conclude that (3.14) holds. The assertion is proved. $\square$

We now come to the final a priori estimate.

Lemma 3.5 *Let $\eta > 0$. Then there exists a constant $\hat{C}_4 > 0$ such that, for all $t \in [0,T]$,*

$$
\begin{aligned}
& n\sum_{k=0}^{n-1}(\dot{u}_{k+1}-\dot{u}_k)^2(t)+n^3\sum_{k=1}^{n-1}\Big(u_{k+1}-2u_k+u_{k-1}\Big)^2(t) \\
&+\eta\, n^3\sum_{k=1}^{n-1}\int_0^t\Big(\dot{u}_{k+1}-2\dot{u}_k+\dot{u}_{k-1}\Big)^2(\tau)\,d\tau \le \hat{C}_4\left(1+\frac{1}{\eta^2}\right). \qquad (3.22)
\end{aligned}
$$

Proof. Let $t \in [0,T]$. We multiply (3.1b) by $-n(\dot{u}_{k+1}-2\dot{u}_k+\dot{u}_{k-1})$ and integrate over

$[0,T]$. Summation over k yields

$$\frac{n}{2}\sum_{k=0}^{n-1}\Big(\dot u_{k+1}-\dot u_k\Big)^2(t)+\frac{n^3}{2}\sum_{k=1}^{n-1}\Big(u_{k+1}-2u_k+u_{k-1}\Big)^2(t)+$$

$$+\,\eta\, n^3\sum_{k=1}^{n-1}\int\limits_0^t\Big(\dot u_{k+1}-2\dot u_k+\dot u_{k-1}\Big)^2(\tau)\,d\tau$$

$$\leq\quad C_1+\frac{\eta}{2}\,n^3\sum_{k=1}^{n-1}\int\limits_0^t\Big(\dot u_{k+1}-2\dot u_k+\dot u_{k-1}\Big)^2(\tau)\,d\tau$$

$$+\,\frac{2}{\eta}\int\limits_0^t\left(\frac{1}{n}\sum_{k=1}^{n-1}\left(g_k^2+n^2\,b_k^2\Big(h_{k+1}-h_k\Big)^2\right)\right)(\tau)\,d\tau. \tag{3.23}$$

Using (3.14), (3.2) and (3.5), we easily obtain (3.22). □

4 Proof of Theorem 2.1

In this section, we will prove Theorem 2.1 using compactness arguments and a passage-to-the-limit procedure for $n\to\infty$. To this end, we define for every $n\in\mathbb{N}$ the functions (where the index n is added to the functions considered in the last section in order to stress the dependence on n)

$$h^{(n)}(x,t)\;:=\;h_k^{(n)}(t)\,+\,n\left(x-\frac{k}{n}\right)\left(h_{k+1}^{(n)}(t)\,-\,h_k^{(n)}(t)\right), \tag{4.1a}$$

$$\tilde h^{(n)}(x,t)\;:=\;h_k^{(n)}(t)\,, \tag{4.1b}$$

$$u^{(n)}(x,t)\;:=\;\frac{1}{2}\left(u_k^{(n)}\,+\,u_{k-1}^{(n)}\right)(t)\,+\,n\left(x-\frac{k}{n}\right)\left(u_k^{(n)}\,-\,u_{k-1}^{(n)}\right)(t)$$

$$+\,\frac{n^2}{2}\left(x-\frac{k}{n}\right)^2\left(u_{k+1}^{(n)}\,-\,2\,u_k^{(n)}\,+\,u_{k-1}^{(n)}\right)(t)\,, \tag{4.1c}$$

$$\tilde u^{(n)}(x,t)\;:=\;u_k^{(n)}(t)\,, \tag{4.1d}$$

for $x\in\left[\frac{k}{n},\frac{k+1}{n}\right)$, $0\le k\le n-1$, $t\in[0,T]$, where we have put

$$u_{-1}^{(n)}(t)\;:=\;-\,u_1^{(n)}(t)\,. \tag{4.1e}$$

In terms of these functions, the a priori estimates (3.2), (3.8), (3.14), and (3.22), respectively, take the form

$$\sup_{t\,\in\,(0,T)}\left(\|\tilde h^{(n)}(t)\|^2\,+\,\|h^{(n)}(t)\|^2\,+\,\|\tilde u_t^{(n)}(t)\|^2\,+\,\|u_x^{(n)}(t)\|^2\right)$$

$$+\int_0^T\left(\|h_x^{(n)}(t)\|^2\,+\,\eta\,\|u_{xt}^{(n)}(t)\|^2\right)dt\;\le\;\hat C_1\,, \tag{4.2a}$$

$$\sup_{t\,\in\,(0,T)}\left(\eta\,\|u_{xx}^{(n)}(t)\|^2\right)\,+\int_0^T\|u_{xx}^{(n)}(t)\|^2\,dt\;\le\;\hat C_2\left(1\,+\,\frac{1}{\eta}\right), \tag{4.2b}$$

$$\sup_{t\in(0,T)}\left(\|h_x^{(n)}(t)\|^2+\eta\,\|u_{xt}^{(n)}(t)\|^2\right)+\int_0^T\left(\|\tilde h_t^{(n)}(t)\|^2+\|h_t^{(n)}(t)\|^2+\|\tilde u_{tt}^{(n)}(t)\|^2\right.$$
$$\left.+\|u_{tt}^{(n)}(t)\|^2\right)dt\le\hat C_3\left(1+\frac{1}{\eta}\right), \tag{4.2c}$$

$$\sup_{t\in(0,T)}\left(\|u_{xt}^{(n)}(t)\|^2+\|u_{xx}^{(n)}(t)\|^2\right)+\eta\int_0^T\|u_{xxt}^{(n)}(t)\|^2\,dt\le\tilde C_4\left(1+\frac{1}{\eta^2}\right). \tag{4.2d}$$

Hence there exist functions $h,u\in C(\overline{\Omega_T})$ such that, possibly taking subsequences, we have

$$u_{xx}^{(n)}\to u_{xx},\quad u_{xt}^{(n)}\to u_{xt},\quad h_x^{(n)}\to h_x,\ \text{all weakly-star in } L^\infty(0,T;L^2(\Omega)), \tag{4.3a}$$

$$u_{tt}^{(n)}\to u_{tt},\quad u_{xxt}^{(n)}\to u_{xxt},\quad h_t^{(n)}\to h_t,\ \text{all weakly in } L^2(\Omega_T), \tag{4.3b}$$

and, by compact imbedding,

$$u^{(n)}\to u,\quad u_x^{(n)}\to u_x,\quad u_t^{(n)}\to u_t,\quad h^{(n)}\to h,\ \text{all strongly in } C(\overline{\Omega_T}). \tag{4.3c}$$

In addition, thanks to (3.14) we have, for all $(x,t)\in\overline{\Omega_T}$,

$$|h^{(n)}(x,t)-\tilde h^{(n)}(x,t)|\le\left(\sum_{k=0}^{n-1}(h_{k+1}-h_k)^2(t)\right)^{1/2}\le\frac{1}{\sqrt n}\left(\hat C_3\left(1+\frac{1}{\eta}\right)\right)^{1/2}, \tag{4.4a}$$

$$|u_t^{(n)}(x,t)-\tilde u_t^{(n)}(x,t)|\le\left(\sum_{k=0}^{n-1}(\dot u_{k+1}-\dot u_k)^2(t)\right)^{1/2}\le\frac{1}{\sqrt n}\left(\frac{\hat C_3}{\eta}\left(1+\frac{1}{\eta}\right)\right)^{1/2}, \tag{4.4b}$$

so that

$$\tilde u^{(n)}\to u,\quad \tilde u_t^{(n)}\to u_t,\quad \tilde h^{(n)}\to h,\ \text{all uniformly.} \tag{4.5a}$$

Therefore, in view of (4.2c), we also have

$$\tilde u_{tt}^{(n)}\to u_{tt},\quad \tilde h_t^{(n)}\to h_t,\quad \text{weakly in } L^2(\Omega_T). \tag{4.5b}$$

We will now prove that (h,u) satisfies (1.1d - e) and (2.10a, b), i.e. is a solution to (1.1a - e) in the sense of Theorem 2.1. To this end, note first that (3.1f) and (4.5a) imply that the initial conditions (1.1d) are satisfied. Moreover, we have by construction that $h^{(n)}(0,t)=h^{(n)}(1,t)=0$ for all $n\in\mathbb{N}$ and $t\in[0,T]$, so that (4.3c) yields $h(0,t)=h(1,t)=0$ for all $t\in[0,T]$. On the other hand, by (3.2),

$$|u^{(n)}(0,t)|+|u^{(n)}(1,t)|=\frac12\left(|u_1(t)|+|u_{n-1}(t)|\right)$$
$$\le\frac12\left(\sum_{k=0}^{n-1}(u_{k+1}-u_k)^2(t)\right)^{1/2}\le\frac{1}{\sqrt n}\sqrt{\hat C_1}, \tag{4.6}$$

and hence $u(0,t) = u(1,t) = 0$ for all $t \in [0,T]$.

Next, observe that (3.1b) can be rewritten in terms of the functions defined in (4.1a - e) as

$$\tilde{u}_{tt}^{(n)} - u_{xx}^{(n)} - \eta\, u_{xxt}^{(n)} + h_x^{(n)} \left(\tilde{h}^{(n)} + \mathcal{P}\left[\tilde{h}^{(n)}\right]\right) = g^{(n)}, \tag{4.7}$$

where

$$g^{(n)}(x,t) := n \int_{k/n}^{(k+1)/n} g(x,t)\,dx\,, \quad \text{for } x \in \left[\frac{k}{n}, \frac{k+1}{n}\right), \quad 0 \le k \le n-1. \tag{4.8}$$

Now observe that $g^{(n)} \to g$ strongly in $L^2(\Omega_T)$ and, owing to (4.5a) and to the continuity of $\mathcal{P}$ (cf. **(H1), (i)**), $\tilde{h}^{(n)} + \mathcal{P}[\tilde{h}^{(n)}] \to h + \mathcal{P}[h]$ strongly in $C(\overline{\Omega_T})$. Hence, passing to the limit as $n \to \infty$ in (4.7), we find that h, u satisfy (2.10b).
To conclude the proof of Theorem 2.1, it remains to confirm that h, u satisfy (2.10a). To this end, let $w \in \overset{o}{W}{}^{1,2}(\Omega)$ be arbitrary. Multiplying (3.1a) by $\frac{1}{n} w(\frac{k}{n})$, summing over k, and using summation by parts and (3.1c), we find after a straightforward computation that the functions $h^{(n)}, \tilde{h}^{(n)}, u^{(n)}, \tilde{u}^{(n)}$ satisfy for almost every $t \in (0,T)$

$$\begin{aligned}&\int_\Omega \left[\left((\tilde{h}^{(n)} + \mathcal{P}[\tilde{h}^{(n)}])_t - f\right)(x,t)w(x) + \left(h_x^{(n)} - \tilde{u}_t^{(n)}\left(\tilde{h}^{(n)} + \mathcal{P}[\tilde{h}^{(n)}]\right)\right)(x,t)w'(x)\right] dx \\ &= \sum_{k=0}^{n-1} \left[\dot{b}_k(t) \int_{k/n}^{(k+1)/n} \left(w(x) - w\left(\frac{k}{n}\right)\right) dx + \int_{k/n}^{(k+1)/n} f(x,t) \left(w\left(\frac{k}{n}\right) - w(x)\right) dx\right],\end{aligned} \tag{4.9}$$

where the right-hand side is bounded from above by

$$\frac{1}{\sqrt{n}}\, ||w'|| \left(\int_0^1 |f(x,t)|dx + \frac{1}{n}\sum_{k=1}^{n-1} |\dot{b}_k(t)|\right),$$

which, by (2.8), (2.2), (3.14) and Hölder's inequality, tends to zero as $n \to \infty$ for almost every $t \in (0,T)$. Hence it remains to show that

$$\mathcal{P}[\tilde{h}^{(n)}]_t \to (\mathcal{P}[h])_t \quad \text{, weakly in } L^2(\Omega_T). \tag{4.10}$$

By (2.2) and (4.2c), we may without loss of generality assume that $(\mathcal{P}[\tilde{h}^{(n)}])_t \to q$ weakly in $L^2(\Omega_T)$ for some $q \in L^2(\Omega_T)$. Since $\mathcal{P}[\tilde{h}^{(n)}] \to \mathcal{P}[h]$ uniformly, it follows for every $\varphi \in C_0^\infty(\Omega_T)$ that

$$0 = \lim_{n\to\infty} \int_0^T \int_\Omega \left(\left(\mathcal{P}[\tilde{h}^{(n)}]\right)_t - q\right) \varphi\, dx\, dt = -\int_0^T \int_\Omega \left(\mathcal{P}[h] - z\right)\varphi_t\, dx\, dt, \tag{4.11}$$

where $z_t = q$. Hence $q = (\mathcal{P}[h])_t$, and the assertion of Theorem 2.1 is completely proved.
□

5 Proof of Theorem 2.2

Suppose that P satisfies **(H1)** and **(H2)** and that (2.11) holds. Let us assume that (2.12a, b) hold for some fixed $\delta > 0$ (which is yet to be determined). As in the proof of Theorem 2.1, the functions h_0 , ..., h_n , u_0 , ..., u_n will denote the solutions to the system (3.1a - f) in $[0,T]$. We shall denote in the sequel by K_i , $\hat{K}_i$, $i \in \mathbb{N}$, constants that are independent of n , η and δ. From (3.3) we immediately get an estimate that is analogous to (3.2), namely

$$\frac{1}{n}\sum_{k=0}^{n-1}\left(h_k^2 + \dot{u}_k^2 + n^2\,(u_{k+1} - u_k)^2\right)(t) + n\sum_{k=0}^{n-1}\int_0^t \left((h_{k+1} - h_k)^2\right.$$
$$\left. + \eta\,(\dot{u}_{k+1} - \dot{u}_k)^2\right)(\tau)\,d\tau \le \hat{K}_1\,\delta^2, \quad \forall\ t \in [0,T]. \tag{5.1}$$

We further have, by (3.1f),

$$\max_{0\le k\le n}|h_k(0)| \le \max_{x\in\bar{\Omega}}\left|h^0(x)\right| \le \hat{K}_2\,\|h^0\|_{W^{2,2}(\Omega)} \le \hat{K}_2\,\delta. \tag{5.2}$$

Therefore, choosing

$$\delta < \frac{H_0}{\hat{K}_2}, \tag{5.3}$$

we can find a maximal $T_n^* \in (0,T]$ such that

$$\max_{0\le k\le n}|h_k(t)| < H_0, \quad \forall\ t \in [0,T_n^*]. \tag{5.4}$$

Next, we show the following result.

Lemma 5.1 *There exist constants $\hat{\delta} > 0$, $\hat{K}_3 > 0$, $\hat{K}_4 > 0$ such that for any $\delta \in (0,\hat{\delta}]$ it holds (5.3), $T_n^* = T$,*

$$\max_{0\le k\le n}|h_k(t)| < H_0, \quad \forall\ t \in [0,T], \tag{5.5}$$

and

$$\frac{1}{n}\sum_{k=0}^{n-1}\left(\dot{h}_k^2 + \ddot{u}_k^2 + n^2\,(\dot{u}_{k+1} - \dot{u}_k)^2\right)(t)$$
$$+ n\sum_{k=0}^{n-1}\int_0^t\left(\left(\dot{h}_{k+1} - \dot{h}_k\right)^2 + \eta\,(\ddot{u}_{k+1} - \ddot{u}_k)^2\right)(\tau)\,d\tau \le \hat{K}_3\,\delta^2\,e^{\hat{K}_4\,\delta^2\,t},$$
$$\forall\ t \in [0,T]. \tag{5.6}$$

Proof. The functions h_k , b_k , $\dot{u}_k$ belong to $W^{1,1}(0,T)$, and since $f_t, g_t \in L^1(0,T;L^2(\Omega))$, we even have $\dot{u}_k, b_k \in W^{2,1}(0,T)$. Differentiating (3.1a) and (3.1b) with respect to t, multiplying by $\frac{1}{n}\dot{h}_k$ and $\frac{1}{n}\ddot{u}_k$, respectively, adding the results and summing over k, we find that a.e. in $[0, T_n^*]$ it holds

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n-1}(\ddot{b}_k\,\dot{h}_k + \dddot{u}_k\,\ddot{u}_k) + n\sum_{k=0}^{n-1}\Big[\Big(\dot{h}_{k+1} - \dot{h}_k\Big)^2 + \eta\,(\ddot{u}_{k+1} - \ddot{u}_k)^2 \\
&+ (\ddot{u}_{k+1} - \ddot{u}_k)\,(\dot{u}_{k+1} - \dot{u}_k)\Big] + \sum_{k=0}^{n-1}\dot{b}_k\Big[\ddot{u}_k\,(h_{k+1} - h_k) - \dot{u}_k\,\Big(\dot{h}_{k+1} - \dot{h}_k\Big)\Big] \\
= \;\; &\frac{1}{n}\sum_{k=1}^{n-1}(\dot{f}_k\,\dot{h}_k + \dot{g}_k\,\ddot{u}_k)\,.
\end{aligned} \tag{5.7}$$

Using **(H1), (iv)** and **(H2)**, we see that for all k and almost every $0 < t_1 < t_2 < T_n^*$ it holds

$$\int_{t_1}^{t_2}\ddot{b}_k(t)\,\dot{h}_k(t)\,dt \;\geq\; \frac{1}{2}\left(\dot{h}_k^2\,(t_2) - \dot{b}_k^2\,(t_1)\right)\,. \tag{5.8}$$

Hence, owing to the continuity of $\dot{b}_k$,

$$\int_0^t \ddot{b}_k(\tau)\,\dot{h}_k(\tau)\,d\tau \;\geq\; \frac{1}{2}\left(\dot{h}_k^2\,(t) - \dot{b}_k^2\,(0)\right)\,, \quad \text{for a.e. } t \in (0, T_n^*)\,, \tag{5.9}$$

where (3.1a) entails that

$$\frac{1}{n}\sum_{k=1}^{n-1}\dot{b}_k^2(0) \;\leq\; K_1\,\delta^2\,. \tag{5.10}$$

Similarly, from (3.1b),

$$\frac{1}{n}\sum_{k=1}^{n-1}\ddot{u}_k^2(0) \;\leq\; K_2\,\delta^2\,. \tag{5.11}$$

Integrating (5.7) over time, we therefore find that for a.e. $t \in [0, T_n^*]$

$$\begin{aligned}
&\frac{1}{2\,n}\sum_{k=0}^{n-1}\left(\dot{h}_k^2 + \ddot{u}_k^2 + n\,(\dot{u}_{k+1} - \dot{u}_k)^2\right)(t) \\
&+ n\sum_{k=0}^{n-1}\int_0^t\left[\left(\dot{h}_{k+1} - \dot{h}_k\right)^2 + \eta\,(\ddot{u}_{k+1} - \ddot{u}_k)^2\right](\tau)\,d\tau \\
\leq \;\; &K_3\left(\delta^2 + \sum_{k=0}^{n-1}\int_0^t|\dot{h}_k|\left(|h_{k+1} - h_k|\,|\ddot{u}_k| + |\dot{h}_{k+1} - \dot{h}_k|\,|\dot{u}_k|\right)(\tau)\,d\tau\right) \\
&+ \int_0^t\left[\left(\frac{1}{n}\sum_{k=1}^{n-1}\dot{f}_k^2\right)^{1/2}\left(\frac{1}{n}\sum_{k=1}^{n}\dot{h}_k^2\right)^{1/2} + \left(\frac{1}{n}\sum_{k=1}^{n}\dot{g}_k^2\right)^{1/2}\left(\frac{1}{n}\sum_{k=1}^{n}\ddot{u}_k^2\right)^{1/2}\right](\tau)\,d\tau.
\end{aligned} \tag{5.12}$$

Using the discrete Nirenberg inequality (3.5), as well as (5.1), we can infer that

$$\begin{aligned}
&\sum_{k=0}^{n-1}\int_0^t \left(|\dot h_k|\,|h_{k+1}\,-\,h_k|\,|\ddot u_k|\right)(\tau)\,d\tau \\
\le\;& \int_0^t \left[\max_{0\le j\le n-1}|\dot h_j|\left(\frac{1}{n}\sum_{k=1}^{n-1}\ddot u_k^2\right)^{1/2}\left(n\sum_{k=0}^{n-1}(h_{k+1}\,-\,h_k)^2\right)^{1/2}\right](\tau)\,d\tau \\
\le\;& K_4\int_0^t \left[\left(\frac{1}{n}\sum_{k=1}^{n-1}\dot h_k^2\right)^{1/4}+\left(n\sum_{k=0}^{n-1}\left(\dot h_{k+1}\,-\,\dot h_k\right)^2\right)^{1/4}\right]\left(\frac{1}{n}\sum_{k=1}^{n-1}\dot h_k^2\right)^{1/4}\cdot \\
&\qquad\cdot\left(\frac{1}{n}\sum_{k=1}^{n-1}\ddot u_k^2\right)^{1/2}\left(n\sum_{k=0}^{n-1}(h_{k+1}\,-\,h_k)^2\right)^{1/2}(\tau)\,d\tau \\
\le\;& K_5\max_{0\le\tau\le t}\left[\left(\frac{1}{n}\sum_{k=1}^{n-1}\dot h_k^2\right)^{1/4}\left(\frac{1}{n}\sum_{k=1}^{n-1}\ddot u_k^2\right)^{1/4}(\tau)\right]\cdot \\
&\cdot\left(\int_0^t n\sum_{k=0}^{n-1}(h_{k+1}\,-\,h_k)^2\,(\tau)\,d\tau\right)^{1/2}\cdot\left(\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\ddot u_k^2(\tau)\,d\tau\right)^{1/4}\cdot \\
&\cdot\left[\left(\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\dot h_k^2(\tau)\,d\tau\right)^{1/4}+\left(\int_0^t n\sum_{k=0}^{n-1}\left(\dot h_{k+1}\,-\,\dot h_k\right)^2(\tau)\,d\tau\right)^{1/4}\right] \\
\le\;& \frac{1}{8\,n}\max_{0\le\tau\le t}\left(\sum_{k=1}^{n-1}\left(\dot h_k^2\,+\,\ddot u_k^2\right)(\tau)\right)+\frac{n}{4}\sum_{k=0}^{n-1}\int_0^t\left(\dot h_{k+1}\,-\,\dot h_k\right)^2(\tau)\,d\tau \\
&+K_6\,\delta^2\frac{1}{n}\sum_{k=1}^{n-1}\int_0^t\left(\dot h_k^2\,+\,\ddot u_k^2\right)(\tau)\,d\tau\,,\quad\text{for a.e. } t\,\in\,[0,T_n^*]\,. \qquad (5.13)
\end{aligned}$$

Similarly, we obtain for a.e. $t\in[0,T_n^*]$ the estimate

$$\begin{aligned}
&\sum_{k=0}^{n-1}\int_0^t|\dot h_k|\,|\dot h_{k+1}\,-\,\dot h_k|\,|\dot u_k|(\tau)\,d\tau \\
\le\;& \int_0^t\max_{0\le j\le n-1}|\dot h_j|\left(\frac{1}{n}\sum_{k=1}^{n-1}\dot u_k^2\right)^{1/2}\left(n\sum_{k=0}^{n-1}\left(\dot h_{k+1}\,-\,\dot h_k\right)^2\right)^{1/2}(\tau)\,d\tau \\
\le\;& K_7\,\delta\int_0^t\left[\left(\frac{1}{n}\sum_{k=1}^{n-1}\dot h_k^2\right)^{1/2}\left(n\sum_{k=0}^{n-1}\left(\dot h_{k+1}\,-\,\dot h_k\right)^2\right)^{1/2}\right. \\
&\qquad\left.+\left(\frac{1}{n}\sum_{k=1}^{n-1}\dot h_k^2\right)^{1/4}\left(n\sum_{k=0}^{n-1}\left(\dot h_{k+1}\,-\,\dot h_k\right)^2\right)^{3/4}\right](\tau)\,d\tau \\
\le\;& \frac{n}{4}\sum_{k=0}^{n-1}\int_0^t\left(\dot h_{k+1}\,-\,\dot h_k\right)^2(\tau)\,d\tau\,+\,K_8\,\delta^2\,\frac{1}{n}\sum_{k=1}^{n-1}\int_0^t\dot h_k^2(\tau)\,d\tau\,. \qquad (5.14)
\end{aligned}$$

Combining (2.12b), and (5.12) to (5.14), we obtain that for a.e. $t \in [0, T_n^*]$

$$\begin{aligned}
&\max_{0\le\tau\le t}\left(\frac{1}{n}\sum_{k=0}^{n-1}\left(\dot h_k^2 \,+\, \ddot u_k^2 \,+\, n^2\,(\dot u_{k+1} \,-\, \dot u_k)^2\right)(\tau)\right)\\
&\quad + n\sum_{k=0}^{n-1}\int_0^t\left(\left(\dot h_{k+1} \,-\, \dot h_k\right)^2 \,+\, \eta\,(\ddot u_{k+1} \,-\, \ddot u_k)^2\right)(\tau)\,d\tau\\
&\le\; K_9\,\delta^2 \,+\, K_{10}\,\delta^2\,\frac{1}{n}\sum_{k=1}^{n-1}\int_0^t\left(\dot h_k^2 \,+\, \ddot u_k^2\right)(\tau)\,d\tau\,. \qquad (5.15)
\end{aligned}$$

Gronwall's lemma, applied to (5.15), yields

$$\begin{aligned}
&\max_{0\le\tau\le t}\left(\frac{1}{n}\sum_{k=0}^{n-1}\left(\dot h_k^2 \,+\, \ddot u_k^2 \,+\, n^2\,(\dot u_{k+1} \,-\, \dot u_k)^2\right)(t)\right)\\
&\quad + n\sum_{k=0}^{n-1}\int_0^t\left(\left(\dot h_{k+1} \,-\, \dot h_k\right)^2 \,+\, \eta\,(\ddot u_{k+1} \,-\, \ddot u_k)^2\right)(\tau)\,d\tau\\
&\le\; K_9\,\delta^2\,e^{K_{10}\,\delta^2\,t}\,, \quad \text{for a.e. } t\,\in\,[0, T_n^*]\,. \qquad (5.16)
\end{aligned}$$

Next, observe that for all $j \in \{0, ..., n\}$ and $t \in [0, T_n^*]$ it holds, using (3.5) and (5.16),

$$\begin{aligned}
|h_j(t)| \;&\le\; |h_j(0)| \,+\, \int_0^t |\dot h_j(\tau)|\,d\tau\\
&\le\; K_{11}\left(\delta \,+\, \int_0^t\left[\left(\frac{1}{n}\sum_{k=1}^{n-1}\dot h_k^2\right)^{1/2} \,+\, \left(n\sum_{k=0}^{n-1}\left(\dot h_{k+1} \,-\, \dot h_k\right)^2\right)^{1/2}\right](\tau)\,d\tau\right)\\
&\le\; K_{12}\,\delta\,e^{K_{13}\,\delta^2}\,. \qquad (5.17)
\end{aligned}$$

Choosing $\hat\delta > 0$ so small that (5.3) holds and that

$$K_{12}\,\hat\delta\,e^{K_{13}\,\hat\delta^2} < H_0\,, \qquad (5.18)$$

we conclude that $T_n^* = T$ and that (5.5) and (5.6) are satisfied. This concludes the proof of the assertion. $\square$

For the conclusion of the proof of Theorem 2.2, we still need another a priori estimate.

Lemma 5.2 *Let $\hat\delta > 0$ denote the constant defined in Lemma 5.1. Then there are constants $\hat K_5 > 0$, $\hat K_6 > 0$, such that for every $\delta \in (0, \hat\delta]$ and every $t \in [0, T]$ it holds*

$$\begin{aligned}
&n^3\sum_{k=1}^{n-1}\int_0^t (u_{k+1} \,-\, 2\,u_k \,+\, u_{k-1})^2\,(\tau)\,d\tau \,+\, \eta\,n^3\sum_{k=1}^{n-1}(u_{k+1} \,-\, 2\,u_k \,+\, u_{k-1})^2\,(t)\\
&\le\; \hat K_5\,\delta^2\,e^{\hat K_6\,\delta^2\,T}\,, \qquad (5.19a)
\end{aligned}$$

$$\eta^2\,n^3\sum_{k=1}^{n-1}\int_0^t (\dot u_{k+1} \,-\, 2\,\dot u_k \,+\, \dot u_{k-1})^2\,(\tau)\,d\tau \;\le\; \hat K_5\,\delta^2\,e^{\hat K_6\,\delta^2\,T}\,. \qquad (5.19b)$$

Proof. As in the proof of Lemma 3.3, we multiply (3.1b) by $-n(u_{k+1} - 2u_k + u_{k-1})$, sum over k, and integrate over time. Using Young's inequality, (2.12a, b), (3.11), and (5.6), we arrive at the estimate

$$\frac{n^3}{2}\sum_{k=1}^{n-1}\int_0^t (u_{k+1} - 2u_k + u_{k-1})^2(\tau)\,d\tau + \frac{\eta}{2}n^3\sum_{k=1}^{n-1}(u_{k+1} - 2u_k + u_{k-1})^2(t)$$
$$\leq K_1\left(\delta^2 + \frac{1}{n}\sum_{k=0}^{n-1}\int_0^t \left[\ddot{u}_k^2 + g_k^2 + n^2 b_k^2(h_{k+1} - h_k)^2\right](\tau)\,d\tau\right). \tag{5.20}$$

Owing to (2.12b) and (5.6),

$$\frac{1}{n}\sum_{k=0}^{n-1}\int_0^t \left(\ddot{u}_k^2 + g_k^2\right)(\tau)\,d\tau \leq K_2\,\delta^2 e^{K_3\delta^2 T}, \tag{5.21}$$

and (2.6), (5.1) and (5.5) imply that

$$n\sum_{k=0}^{n-1}\int_0^t b_k^2(h_{k+1} - h_k)^2(\tau)\,d\tau \leq K_4\,\delta^2\left(M + \max_{0\leq j\leq n}\max_{0\leq\tau\leq t}|h_k(\tau)|\right)^2$$
$$\leq K_4\,\delta^2(M + H_0)^2. \tag{5.22}$$

Combining (5.20) to (5.22), we have confirmed the validity of (5.19a). Finally, using (5.19a), (5.20) and (5.21), we obtain (5.19b) directly from (3.1b). □

After these preparations, we may now conclude the proof of Theorem 2.2. To this end, consider the functions $h^{(n)}$, $\tilde{h}^{(n)}$, $u^{(n)}$, $\tilde{u}^{(n)}$ defined in (4.1a - e). As in Section 4, we can infer from the a priori estimates (5.1), (5.6), (5.19a), and (5.19b), that there exist functions $h, u \in C(\overline{\Omega_T})$ such that, possibly selecting subsequences, for $n \to \infty$ it holds

$$h_t^{(n)} \to h_t\,,\, u_{tt}^{(n)} \to u_{tt}\,,\, u_{xt}^{(n)} \to u_{xt}, \quad \text{all weakly-star in } L^\infty(0,T;L^2(\Omega))\,, \tag{5.23a}$$

$$h_x^{(n)} \to h_x\,,\, h_{xt}^{(n)} \to h_{xt}\,,\; u_{xx}^{(n)} \to u_{xx}\,,\, \eta\, u_{xxt}^{(n)} \to \eta\, u_{xxt}\;, \text{all weakly in } L^2(\Omega_T)\,. \tag{5.23b}$$

From this point, we can follow the lines of the proof of Theorem 2.1 to verify that (h,u) satisfy (2.10a, b). In this connection, we obtain an estimate analogous to (4.4b) using (5.6), while a counterpart of (4.4a) follows from (5.1), (5.6) and the elementary inequality

$$\max_{t\in[0,T]}\zeta^2(t) \leq \left(1 + \frac{1}{T}\right)\int_0^T \zeta^2(s)\,ds + \int_0^T \dot{\zeta}^2(s)\,ds\,,$$

where $\zeta := h_{k+1} - h_k$. This concludes the proof of the assertion of Theorem 2.2. □

6 Concluding remarks

We conclude our paper by adding some comments.

(i) The question whether the solutions found in the Theorems 2.1, 2.2 are unique is still open.

(ii) It seems that the above line of argumentation does not apply if the displacement current is not discarded.

(iii) The hypothesis of linear elasticity can be relaxed considerably. In fact, if the equation of motion (1.1b) is replaced by

$$u_{tt} - (F(u_x))_x - \eta\, u_{xxt} + h_x\, b = g\,, \tag{6.1}$$

then the result stated in Theorem 2.1 remains valid, provided that F, together with its inverse F^{-1}, is a Lipschitz continuous increasing function on $\mathbb{R}$ (*nonlinear elasticity*). In addition, weak solutions are obtained in the context of Theorem 2.2 if $P_F := F^{-1} - I$, where I is the identity, is a hysteresis operator satisfying **(H1)** and **(H2)**. This case corresponds to *rate independent elastoplasticity*.

References

1. Brokate, M. & Sprekels, J., *Hysteresis and Phase Transitions.* Springer, New York (to appear 1996).

2. Brokate, M. & Visintin, A., Properties of the Preisach model for hysteresis, *J. Reine Angew. Math.* **402**, 1–40 (1989).

3. Krejčí, P., On Maxwell equations with the Preisach hysteresis operator: the one–dimensional time–periodic case, *Appl. Math.* **34**, 364–374 (1989).

4. Krejčí, P., *Hysteresis, Convexity and Dissipation in Hyperbolic Equations.* Gakkotosho, Tokyo (to appear 1996).

5. Landau L. D., Lifschitz E. M. & Pitaevskij, L. P., *Electrodynamics of Continuous Media.* Pergamon Press, New York (1984).

6. Visintin, A., A model for hysteresis of distributed systems, *Ann. Mat. Pura Appl.* **81**, 203–231 (1982).

7. Visintin, A., On the Preisach model for hysteresis, *Nonlinear Anal.* **9**, 977–996 (1984).

8. Visintin, A., *Differential Models of Hysteresis.* Springer, New York (1994).