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Stochastic homogenization of rate-dependent models of monotone type in plasticity

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Abstract

In this work we deal with the stochastic homogenization of the initial boundary value problems of monotone type. The models of monotone type under consideration describe the deformation behaviour of inelastic materials with a microstructure which can be characterised by random measures. Based on the Fitzpatrick function concept we reduce the study of the asymptotic behaviour of monotone operators associated with our models to the problem of the stochastic homogenization of convex functionals within an ergodic and stationary setting. The concept of Fitzpatrick's function helps us to introduce and show the existence of the weak solutions for rate-dependent systems. The derivations of the homogenization results presented in this work are based on the stochastic two-scale convergence in Sobolev spaces. For completeness, we also present some two-scale homogenization results for convex functionals, which are related to the classical Γ-convergence theory.

1 Introduction

In this work we are concerned with the homogenization of the initial boundary value problem describing the deformation behavior of inelastic materials with a microstructure which can be characterised by random measures.

While the periodic homogenization theory for elasto/visco-plastic models is sufficiently well established (see [2, 11, 17, 18, 19, 26, 27, 30, 31] and references therein), some improvement in the development of the techniques for the stochastic homogenization of the quasi-static initial boundary value problems of monotone type has to be achieved yet. To the best knowledge of the authors, there are only two works ([13, 14]) available on the market which are concerned with the homogenization problem of rate-independent systems in plasticity within an ergodic and stationary setting. In this work we extend the results obtained in [14] for perfectly elasto-plastic models to rate-dependent plasticity. Our main ingredient in the construction of the stochastic homogenization theory for rate-dependent models of monotone type is the combination of the Fitzpatrick function concept and the two-scale convergence technique in spaces equipped with random measures due to V.V. Zhikov and A.L. Pyatnitskii (see [34]). The Fitzpatrick function is used here to reduce the study of the asymptotic behaviour of monotone operators associated with the models under consideration to the problem of the stochastic homogenization of convex functionals defined on Sobolev spaces with random measures.

Setting of the problem. Let $\boldsymbol{Q} \subset \mathbb{R}^3$ be an open bounded set, the set of material points of the solid body, with a Lipschitz boundary $\partial\boldsymbol{Q}$, the number $\eta > 0$ denote the scaling parameter of the microstructure and T_e be some positive number (time of existence). For $0 < t \leq T_e$

$$\boldsymbol{Q}_t = \boldsymbol{Q} \times (0, t).$$

Let $\mathcal{S}^3$ denote the set of symmetric 3×3-matrices, and let $u_\eta(x,t) \in \mathbb{R}^3$ be the unknown displacement of the material point x at time t, $\sigma_\eta(x,t) \in \mathcal{S}^3$ be the unknown Cauchy stress tensor and $z_\eta(x,t) \in \mathbb{R}^N$ denote the unknown vector

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of internal variables. The model equations of the problem (the microscopic problem) are

$$
\begin{aligned}
-\operatorname{div}_x \sigma_\eta(x,t) &= b(x,t), && (1)\\
\sigma_\eta(x,t) &= \mathbb{C}_\eta[x](\varepsilon(\nabla_x u_\eta(x,t)) - Bz_\eta(x,t)), && (2)\\
\partial_t z_\eta(x,t) &\in g_\eta\big(x, B^T\sigma_\eta(x,t) - L_\eta[x]z_\eta(x,t)\big), && (3)
\end{aligned}
$$

together with the homogeneous Dirichlet boundary condition

$$u_\eta(x,t) = 0, \qquad (x,t) \in \partial\boldsymbol{Q} \times (0,\infty), \tag{4}$$

and the initial condition

$$z_\eta(x,0) = z_\eta^{(0)}(x), \qquad x \in \boldsymbol{Q}. \tag{5}$$

In model equations (1) - (5)

$$\varepsilon(\nabla_x u_\eta(x,t)) = \frac{1}{2}(\nabla_x u_\eta(x,t) + (\nabla_x u_\eta(x,t))^T) \in \mathcal{S}^3$$

denotes the strain tensor (the measure of deformation), $B : \mathbb{R}^N \to \mathcal{S}^3$ is a linear mapping, which assigns to each vector of internal variables $z_\eta(x,t)$ the plastic strain tensor $\varepsilon_{p,\eta}(x,t) \in \mathcal{S}^3$, i.e. the following relation $\varepsilon_{p,\eta}(x,t) = Bz_\eta(x,t)$ holds. We recall that the space $\mathcal{S}^3$ can be isomorphically identified with the space $\mathbb{R}^6$ (see [1, p. 31]). Therefore, the linear mapping $B : \mathbb{R}^N \to \mathcal{S}^3$ is defined as a composition of a projector from $\mathbb{R}^N$ onto $\mathbb{R}^6$ and the isomorphism between $\mathbb{R}^6$ and $\mathcal{S}^3$. The transpose $B^T : \mathcal{S}^3 \to \mathbb{R}^N$ is given by

$$B^T v = (\hat{z}, 0)^T$$

for $v \in \mathcal{S}^3$ and $z = (\hat{z}, \tilde{z})^T \in \mathbb{R}^N$, $\hat{z} \in \mathbb{R}^6$, $\tilde{z} \in \mathbb{R}^{N-6}$.

For every $x \in \boldsymbol{Q}$ we denote by $\mathbb{C}_\eta[x] : \mathcal{S}^3 \to \mathcal{S}^3$ a linear symmetric mapping, the elasticity tensor. It is assumed that the mapping $x \to \mathbb{C}_\eta[x]$ is measurable. Further, we suppose that there exist two positive constants $0 < \alpha < \beta$ such that the two-sided inequality

$$\alpha|\xi|^2 \le \mathbb{C}_\eta[x]\xi\cdot\xi \le \beta|\xi|^2 \quad \text{for any } \xi \in \mathcal{S}^3.$$

is satisfied uniformly with respect to $x \in \boldsymbol{Q}$ and $\eta > 0$. The given function $b : \boldsymbol{Q} \times [0,\infty) \to \mathbb{R}^3$ is the volume force. The $(N \times N)$-matrix $L_\eta[x]$ represents hardening effects. It is assumed to be positive semi-definite, only. For all $x \in \boldsymbol{Q}$ the function $z \to g_\eta(x,z) : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is maximal monotone and satisfies the following condition

$$0 \in g_\eta(x,0), \qquad x \in \boldsymbol{Q}.$$

The mapping $x \to (L_\eta[x], g_\eta(x,\cdot))$ is measurable.

Remark 1.1. *Visco-plasticity is typically included in the former conditions by choosing the function g_η to be in Norton-Hoff form, i.e.*

$$g_\eta(x,\Sigma) = [|\Sigma| - \sigma_{\mathrm{y}}(x)]_+^{r_\eta(x)}\frac{\Sigma}{|\Sigma|}, \qquad \Sigma \in \mathcal{S}^3,\ x \in \boldsymbol{Q},$$

where $\sigma_{\mathrm{y}} : \boldsymbol{Q} \to (0,\infty)$ is the flow stress function and $r_\eta : \boldsymbol{Q} \to (0,\infty)$ is some material function together with $[x]_+ := \max(x,0)$.

In order to specify the dependence of the model coefficients in (1) - (5) on the microstructure scaling parameter $\eta > 0$, we introduce the concept of a spatial dynamical system. Throughout this paper, we follow the setting of Papanicolaou and Varadhan [22] and make the following assumptions.

Assumption 1.1. *Let $(\Omega, \mathcal{F}_\Omega, \mathcal{P})$ be a probability space with countably generated σ-algebra $\mathcal{F}_\Omega$. Further, we assume we are given a family $(\tau_x)_{x\in\mathbb{R}^n}$ of measurable bijective mappings $\tau_x : \Omega \mapsto \Omega$, having the properties of a* dynamical system *on $(\Omega, \mathcal{F}_\Omega, \mathcal{P})$, i.e. they satisfy (i)-(iii):*

(i) $\tau_x \circ \tau_y = \tau_{x+y}$, $\tau_0 = id$ *(Group property)*

(ii) $\mathcal{P}(\tau_{-x}B) = \mathcal{P}(B) \quad \forall x \in \mathbb{R}^n,\ B \in \mathcal{F}_\Omega$ *(Measure preserving)*

(iii) $A : \mathbb{R}^n \times \Omega \to \Omega$, $(x, \omega) \mapsto \tau_x\omega$ *is measurable (Measurablility of evaluation)*

We finally assume that the system $(\tau_x)_{x\in\mathbb{R}^n}$ is ergodic. This means that for every measurable function $f : \Omega \to \mathbb{R}$ there holds

$$[f(\omega) = f(\tau_x\omega)\ \forall x \in \mathbb{R}^n\,,\ a.e.\ \omega \in \Omega] \Rightarrow [f(\omega) = const\ \mathcal{P}\text{-a.e.}\,\omega \in \Omega]\ . \tag{6}$$

For reader's convenience, we recall the following well-known result (see [9, Section VI.15]).

Lemma 1.1. *Let $(A, \mathcal{F}, \mu)$ be a finite measure space with countably generated σ-algebra $\mathcal{F}$. Then, for all $1 \le p < \infty$, $L^p(A; \mu)$ contains a countable dense set of simple functions.*

The coefficients in (1) - (5) are defined as follows. First, we define the stationary random fields through the relations

$$\mathbb{C}[x, \omega] = \tilde{\mathbb{C}}[\tau_x\omega], \quad L[x, \omega] = \tilde{L}[\tau_x\omega],$$

and for every fixed $v \in \mathbb{R}^N$

$$g(x, \omega, v) = \tilde{g}(\tau_x\omega, v),$$

where $\tilde{\mathbb{C}}$, $\tilde{L}$ are measurable functions over Ω and $\omega \mapsto \tilde{g}(\omega, \cdot)$ is measurable in the sense of Definition 2.2. Then, given the specified assumptions on the random fields, the coefficients $\mathbb{C}_\eta[x]$, $L_\eta[x]$ and the mapping $x \mapsto g_\eta(x, \cdot)$ are defined as

$$\mathbb{C}_\eta[x] = \mathbb{C}\left[\frac{x}{\eta}, \omega\right], \quad L_\eta[x] = L\left[\frac{x}{\eta}, \omega\right],$$

and for each fixed $v \in \mathbb{R}^N$

$$g_\eta(x, v) = g\left(\frac{x}{\eta}, \omega, v\right).$$

Furthermore, we assume that

$$z_\eta^{(0)}(x) = \tilde{z}^{(0)}\left(x, \tau_{\frac{x}{\eta}}\omega\right), \quad x \in \boldsymbol{Q}.$$

for some ergodic function $\tilde{z}^{(0)} \in L^2(\boldsymbol{Q} \times \Omega; \mathcal{L} \otimes \mu)$.

From a modelling perspective, this construction is equivalent to the assumption that the coefficients and the given functions in (1) - (5) are statistically homogeneous (see [7], for example).

Notation. The symbols $|\cdot|$ and $(\cdot,\cdot)$ will denote a norm and a scalar product in $\mathbb{R}^k$, respectively. Let S be a measurable set in $\mathbb{R}^s$. For $m \in \mathbb{N}$, $q \in [1,\infty]$, we denote by $W^{m,q}(S,\mathbb{R}^k)$ the Banach space of Lebesgue integrable functions having q-integrable weak derivatives up to order m. This space is equipped with the norm $\|\cdot\|_{m,q,S}$. If $m = 0$, we write $\|\cdot\|_{q,S}$; and if (additionally) $q = 2$, we also write $\|\cdot\|_S$. We set $H^m(S,\mathbb{R}^k) = W^{m,2}(S,\mathbb{R}^k)$. We choose the numbers p, q satisfying $1 < p, q < \infty$ and $1/p + 1/q = 1$. For such p and q one can define the bilinear form on the product space $L^p(S,\mathbb{R}^k) \times L^q(S,\mathbb{R}^k)$ by

$$(\xi,\zeta)_S = \int_S (\xi(s),\zeta(s))ds.$$

For functions v defined on $\Omega\times[0,\infty)$ we denote by $v(t)$ the mapping $x \mapsto v(x,t)$, which is defined on Ω. The space $L^q(0,T_e;X)$ denotes the Banach space of all Bochner-measurable functions $u : [0,T_e) \to X$ such that $t \mapsto \|u(t)\|_X^q$ is integrable on $[0,T_e)$. Finally, we frequently use the spaces $W^{m,q}(0,T_e;X)$, which consist of Bochner measurable functions having q-integrable weak derivatives up to order m.

2 Preliminaries.

In this section we briefly recall some basic facts from convex analysis and nonlinear functional analysis which are needed for further discussions. For more details see [5, 15, 23, 33], for example.

Let V be a reflexive Banach space with the norm $\|\cdot\|$, V^* be its dual space with the norm $\|\cdot\|_*$. The brackets $\langle\cdot,\cdot\rangle$ denote the duality pairing between V and V^*. By V we shall always mean a reflexive Banach space throughout this section.

For a function $\phi : V \to \overline{\mathbb{R}}$ the sets

$$\mathrm{dom}(\phi) = \{v \in V \mid \phi(v) < \infty\}, \quad \mathrm{epi}(\phi) = \{(v,t) \in V \times \mathcal{R} \mid \phi(v) \le t\}$$

are called the *effective domain* and the *epigraph* of ϕ, respectively. One says that the function ϕ is *proper* if $\mathrm{dom}(\phi) \neq \emptyset$ and $\phi(v) > -\infty$ for every $v \in V$. The epigraph is a non-empty closed convex set iff ϕ is a proper lower semi-continuous convex function or, equivalently, iff ϕ is a proper weakly lower semi-continuous convex function (see [33, Theorem 2.2.1]).

The Legendre-Fenchel conjugate of a proper convex lower semi-continuous function $\phi : V \to \overline{\mathbb{R}}$ is the function ϕ^* defined for each $v^* \in V^*$ by

$$\phi^*(v^*) = \sup_{v\in V}\{\langle v^*, v\rangle - \phi(v)\}.$$

The Legendre-Fenchel conjugate ϕ^* is convex, lower semi-continuous and proper on the dual space V^*. Moreover, the *Young-Fenchel inequality* holds

$$\forall v \in V,\ \forall v^* \in V^* : \quad \phi^*(v^*) + \phi(v) \ge \langle v^*, v\rangle\,, \tag{7}$$

and the inequality $\phi \le \psi$ implies $\psi^* \le \phi^*$ for any two proper convex lower semi-continuous functions $\psi, \phi : V \to \overline{\mathbb{R}}$ (see [33, Theorem 2.3.1]).

Due to Proposition II.2.5 in [5] a proper convex lower semi-continuous function ϕ satisfies the following identity

$$\operatorname{int}\operatorname{dom}(\phi) = \operatorname{int}\operatorname{dom}(\partial\phi), \tag{8}$$

where $\partial\phi : V \to 2^{V^*}$ denotes the subdifferential of the function ϕ. We note that the equality in (7) holds iff $v^* \in \partial\phi(v)$.

Remark 2.1. *We recall that the subdifferential of a lower semi-continuous proper and convex function is maximal monotone (see [5, Theorem II.2.1]) in the sense of Definition 2.1 below.*

Convex integrands. Let the numbers p, q satisfy $1 < q \leq 2 \leq p < \infty, 1/p + 1/q = 1$. For a proper convex lower semi-continuous function $\phi : \mathbb{R}^k \to \overline{\mathbb{R}}$ we define a functional I_ϕ on $L^p(G, \mathbb{R}^k)$ by

$$I_\phi(v) = \begin{cases} \int_G \phi(v(x))dx, & \phi(v) \in L^1(G, \mathbb{R}^k) \\ +\infty, & \text{otherwise} \end{cases},$$

where G is a bounded domain in $\mathbb{R}^N$ with some $N \in \mathbb{N}$. Due to Proposition II.8.1 in [28], the functional I_ϕ is proper, convex, lower semi-continuous, and $v^* \in \partial I_\phi(v)$ iff

$$v^* \in L^q(G, \mathbb{R}^k), \quad v \in L^p(G, \mathbb{R}^k) \quad \text{and} \quad v^*(x) \in \partial\phi(v(x)), \text{ a.e.}$$

Due to the result of Rockafellar in [24, Theorem 2], the Legendre-Fenchel conjugate of I_ϕ is equal to I_{ϕ^*}, i.e.

$$\big(I_\phi\big)^* = I_{\phi^*},$$

where ϕ^* is the Legendre-Fenchel conjugate of ϕ.

Maximal monotone operators. For a multivalued mapping $A : V \to 2^{V^*}$ the sets

$$D(A) = \{v \in V \mid Av \neq \emptyset\}, \quad GrA = \{[v, v^*] \in V \times V^* \mid v \in D(A),\ v^* \in Av\}$$

are called the *effective domain* and the *graph* of A, respectively.

Definition 2.1. *A mapping $A : V \to 2^{V^*}$ is called* monotone *if and only if the following inequality holds*

$$\langle v^* - u^*, v - u\rangle \geq 0 \qquad \forall\ [v, v^*], [u, u^*] \in GrA.$$

A monotone mapping $A : V \to 2^{V^*}$ is called maximal monotone *iff the inequality*

$$\langle v^* - u^*, v - u\rangle \geq 0 \qquad \forall\ [u, u^*] \in GrA$$

implies $[v, v^*] \in GrA$.

It is well known ([23, p. 105]) that if A is a maximal monotone operator, then for any $v \in D(A)$ the image Av is a closed convex subset of V^* and the graph GrA is demi-closed[1].

[1] A set $A \in V \times V^*$ is demi-closed if v_n converges strongly to v_0 in V and v_n^* converges weakly to v_0^* in V^* (or v_n converges weakly to v_0 in V and v_n^* converges strongly to v_0^* in V^*) and $[v_n, v_n^*] \in GrA$, then $[v, v^*] \in GrA$

Canonical extensions of maximal monotone operators. In this subsection we briefly present some facts about measurable multi-valued mappings (see [4, 6, 15, 21], for example). We assume that V, and hence V^*, is separable and denote the set of maximal monotone operators from V to V^* by $\mathfrak{M}(V \times V^*)$. Further, let $(S, \Sigma(S), \mu)$ be a $\sigma-$finite $\mu-$complete measurable space.

Definition 2.2. *A mapping $A : S \to \mathfrak{M}(V \times V^*)$ is measurable iff for every open set $U \in V \times V^*$ (respectively closed set, Borel set, open ball, closed ball),*

$$\{x \in S \mid A(x) \cap U \neq \emptyset\}$$

is measurable in S.

The fact that the closed or Borel sets can be equivalently used in Definition 2.2 follows from the closedness of the values of the mapping $A : S \to \mathfrak{M}(V \times V^*)$ (see [4, Theorem 8.1.4]).

Remark 2.2. *Theorem 8.1.4 in [4] also implies that under the above conditions the measurability of a mapping $A : S \to \mathfrak{M}(V \times V^*)$ is equivalent to the existence of a countable dense subset consisting of measurable selectors, i.e. there exists a sequence of measurable functions $\{v_n\}_{n\in\mathbb{N}} : S \to V \times V^*$ such that for any $x \in S$ the image $A(x)$ can be represented as follows*

$$A(x) = \overline{\cup_{n\in\mathbb{N}} v_n(x)}.$$

The following lemma will be used in the sequel (see [32, Lemma 3.1]).

Lemma 2.1. *Let a mapping $A : S \to \mathfrak{M}(V \times V^*)$ be measurable. For any $\mathcal{L}(S)$-measurable function $v : S \to V$, the multivalued mapping $\hat{A} : x \mapsto A(x, v(x))$ is then closed-valued and measurable.*

Given a mapping $A : S \to \mathfrak{M}(V \times V^*)$, one can define a monotone graph from $L^p(S, V)$ to $L^q(S, V^*)$, where $1/p + 1/q = 1$, as follows:

Definition 2.3. *Let $A : S \to \mathfrak{M}(V \times V^*)$. The canonical extension of A from $L^p(S, V)$ to $L^q(S, V^*)$, where $1/p + 1/q = 1$, is defined by:*

$$Gr\mathcal{A}_p = \{[v, v^*] \in L^p(S, V) \times L^q(S, V^*) \mid [v(x), v^*(x)] \in GrA(x) \ for \ a.e. \ x \in S\}.$$

In the following, we will drop the index p for readability. Since we always work fix p at the beginning of a statement, there cannot occur confusion with this notation. Monotonicity of $\mathcal{A}$ defined in Definition 2.3 is obvious, while its maximality follows from the next proposition (see [8, Proposition 2.13]).

Proposition 2.1. *Let $A : S \to \mathfrak{M}(V \times V^*)$ be measurable. If $Gr\mathcal{A} \neq \emptyset$, then $\mathcal{A}$ is maximal monotone.*

Remark 2.3. *We point out that the maximality of $A(x)$ for almost every $x \in S$ does not imply the maximality of $\mathcal{A}$ as the latter can be empty (see [8]).*

Fitzpatrick's function. For a proper operator $\beta : V \to 2^{V^*}$ the Fitzpatrick function is defined as the convex and lower semicontinuous function given by

$$f_\beta(v, v^*) = \sup\{\langle v^*, v_0\rangle - \langle v_0^*, v_0 - v\rangle \mid v_0^* \in \beta(v_0)\}, \quad \forall (v, v^*) \in V \times V^*. \quad (9)$$

It is known ([10]) that, whenever β is maximal monotone,

$$f_\beta(v, v^*) \geq \langle v^*, v\rangle\,, \quad \forall (v, v^*) \in V \times V^*, \quad (10)$$
$$f_\beta(v, v^*) = \langle v^*, v\rangle \quad \Leftrightarrow \quad v^* \in \beta(v). \quad (11)$$

Any measurable maximal monotone operator $A : S \to \mathfrak{M}(V \times V^*)$ can be represented by its Fitzpatrick function $f_A : S \times V \times V^* \to \overline{\mathbb{R}}$, which is $\Sigma(S) \otimes \mathcal{B}(V \times V^*)$-measurable. Namely, the graph of a mapping $A : S \to \mathfrak{M}(V \times V^*)$ can be written in the form (see [32, Proposition 3.2])

$$\text{for } a.e.\ x \in S\ GrA(x) = \{[v, v^*] \in V \times V^* \mid f_A(x, v, v^*) = \langle v, v^*\rangle\}\,.$$

We note that the measurability of the Fitzpatrick function $f_A : S \times V \times V^* \to \overline{\mathbb{R}}$ follows directly from its definition and Remark 2.2.

The graph of the canonical extension of a measurable operator $A : S \to \mathfrak{M}(V \times V^*)$ can be equivalently represented in terms of its Fitzpatrick function $F_{\mathcal{A}_p} : L^p(S, V) \times L^q(S, V^*) \to \overline{\mathbb{R}}$, i.e.

$$Gr\mathcal{A}_p = \left\{[v, v^*] \in L^p(S, V) \times L^q(S, V^*) \mid F_{\mathcal{A}_p}(v, v^*) = \langle v, v^*\rangle\right\}\,.$$

Again, we omit p if no confusion occurs. Moreover, the following result holds (see [32, Proposition 3.3])

- the functional $F_{\mathcal{A}}$ is convex and lower semi-continuous;
- for any $[v, v^*] \in L^p(S, V) \times L^q(S, V^*)$, the integral

  $$F_{\mathcal{A}}(v, v^*) = \int_S f_A(x, v(x), v^*(x))dx$$

  exists either finite or equal to $+\infty$;
- if there exists a pair $[v, v^*] \in L^p(S, V) \times L^q(S, V^*)$ such that $F_{\mathcal{A}}(v, v^*) < +\infty$, then

  $$F_{\mathcal{A}}^*(v^*, v) = \int_S f_A^*(x, v^*(x), v(x))dx$$

  holds for all $[v^*, v] \in L^q(S, V^*) \times L^p(S, V)$.

3 Existence of solutions

In this section we introduce and show the existence of weak solutions for the initial boundary value (1) - (5). To simplify the notations, throughout the whole section we ignore the fact the coefficients and the given functions in (1) - (5) depend on $\omega \in \Omega$. The results proved below hold for a.e. $\omega \in \Omega$.

Solvability concept. We start this section with the presentation of the intuitive ideas which lead to the definition of weak solutions for the initial boundary value problem (1) - (5). To give a meaning for the solvability of problem (1) - (5) we are going to use the concept of Fitzpatrick functions defined in (9).

We assume first that a triple of functions $(u_\eta, \sigma_\eta, z_\eta)$ is given with the following properties: for every $t \in (0, T_e)$ the function $(u_\eta(t), \sigma_\eta(t))$ is a weak solution of the boundary value problem

$$
\begin{aligned}
-\operatorname{div}_x \sigma_\eta(x,t) &= b(x,t), && (12)\\
\sigma_\eta(x,t) &= \mathbb{C}_\eta[x](\varepsilon(\nabla_x u_\eta(x,t)) - Bz_\eta(x,t)), && (13)\\
u_\eta(x,t) &= 0, \qquad x \in \partial\boldsymbol{Q}. && (14)
\end{aligned}
$$

This particularly holds for $z_\eta(0) = z_\eta^{(0)}$ and the corresponding initial values $(u_\eta(0), \sigma_\eta(0)) = (u_\eta^{(0)}, \sigma_\eta^{(0)})$. The equations (3) - (5) are satisfied pointwise for almost every (x,t), and b as well as $(u_\eta, \sigma_\eta, z_\eta)$ are smooth enough. Then, based on equivalence (11), we can rewrite equation (3) as follows

$$
\begin{aligned}
&f_{g_\eta}\left(x, B^T\sigma_\eta(x,t) - L_\eta[x]z_\eta(x,t), \partial_t z_\eta(x,t)\right)\\
&= \left(B^T\sigma_\eta(x,t) - L_\eta[x]z_\eta(x,t), \partial_t z_\eta(x,t)\right),
\end{aligned}
$$

which holds for almost every $(x,t) \in \boldsymbol{Q} \times (0, T_e)$. Integrating the last equality over $\boldsymbol{Q}$ gives

$$
\int_{\boldsymbol{Q}} f_{g_\eta}\left(x, B^T\sigma_\eta - L_\eta z_\eta, \partial_t z_\eta\right) dx = \int_{\boldsymbol{Q}} \left(B^T\sigma_\eta - L_\eta z_\eta, \partial_t z_\eta\right) dx. \tag{15}
$$

Using (1), (2) and (4) the right hand side in (15) becomes ($\mathbb{A}_\eta := \mathbb{C}_\eta^{-1}$)

$$
\begin{aligned}
\int_{\boldsymbol{Q}} \left(B^T\sigma_\eta - L_\eta z_\eta, \partial_t z_\eta\right) dx &= \left(B^T\sigma_\eta, \partial_t z_\eta\right)_{\boldsymbol{Q}} - \frac{1}{2}\frac{d}{dt}\left\|L_\eta^{1/2} z_\eta\right\|_{\boldsymbol{Q}}^2\\
&= (\sigma_\eta, \varepsilon(\partial_t \nabla_x u_\eta))_{\boldsymbol{Q}} - (\mathbb{A}_\eta\sigma_\eta, \partial_t\sigma_\eta)_{\boldsymbol{Q}} - \frac{1}{2}\frac{d}{dt}\left\|L_\eta^{1/2} z_\eta\right\|_{\boldsymbol{Q}}^2\\
&= (b, \partial_t u_\eta)_{\boldsymbol{Q}} - \frac{1}{2}\frac{d}{dt}\left\{\left\|\mathbb{A}_\eta^{1/2}\sigma_\eta\right\|_{\boldsymbol{Q}}^2 + \left\|L_\eta^{1/2} z_\eta\right\|_{\boldsymbol{Q}}^2\right\}. \qquad (16)
\end{aligned}
$$

Integrating relations (15) and (16) with respect to t leads to

$$
\begin{aligned}
&\int_{\boldsymbol{Q}} (\mathbb{A}_\eta[x]\sigma_\eta(x,t), \sigma_\eta(x,t))\, dx + \int_{\boldsymbol{Q}} (L_\eta[x]z_\eta(x,t), z_\eta(x,t))\, dx\\
&+ \int_0^t \int_{\boldsymbol{Q}} f_{g_\eta}\left(x, B^T\sigma_\eta(x,\tau) - L_\eta[x]z_\eta(x,\tau), \partial_\tau z_\eta(x,\tau)\right) dx d\tau \qquad (17)\\
&= \int_{\boldsymbol{Q}} (\mathbb{A}_\eta[x]\sigma_\eta(x,0), \sigma_\eta(x,0))\, dx + \int_{\boldsymbol{Q}} \left(L_\eta[x]z_\eta^{(0)}(x), z_\eta^{(0)}(x)\right) dx + (b, \partial_\tau u_\eta)_{\boldsymbol{Q}_t}.
\end{aligned}
$$

Taking into account the inequality (10), we conclude that the triple of functions

$(u_\eta, \sigma_\eta, z_\eta)$ satisfies equality (17) if and only if the inequality

$$\begin{aligned}
&\int_{\boldsymbol{Q}} (\mathbb{A}_\eta[x]\sigma_\eta(x,t), \sigma_\eta(x,t))\, dx + \int_{\boldsymbol{Q}} (L_\eta[x] z_\eta(x,t), z_\eta(x,t))\, dx \\
&+ \int_0^t \int_{\boldsymbol{Q}} f_{g_\eta}\left(x, B^T\sigma_\eta(x,\tau) - L_\eta[x]z_\eta(x,\tau), \partial_\tau z_\eta(x,\tau)\right) dx d\tau \qquad (18)\\
&\leq \int_{\boldsymbol{Q}} \left(\mathbb{A}_\eta[x]\sigma_\eta^{(0)}(x), \sigma_\eta^{(0)}(x)\right) dx + \int_{\boldsymbol{Q}} \left(L_\eta[x] z_\eta^{(0)}(x), z_\eta^{(0)}(x)\right) dx + (b, \partial_\tau u_\eta)_{\boldsymbol{Q}_t}
\end{aligned}$$

holds for all $t \in (0, T_e)$ and some function $\sigma_\eta^{(0)} \in L^2(\boldsymbol{Q}, \mathcal{S}^3)$ solving the elliptic boundary value problem (12) - (14).

The above computations suggest the following notion of weak solutions for the initial boundary value problem (1) - (5).

Definition 3.1. *Let the numbers p, q satisfy $1 < q \leq 2 \leq p < \infty, 1/p + 1/q = 1$. A function $(u_\eta, \sigma_\eta, z_\eta)$ such that*

$$(u_\eta, \sigma_\eta) \in W^{1,q}(0, T_e; W_0^{1,q}(\boldsymbol{Q}, \mathbb{R}^3) \times L^q(\boldsymbol{Q}, \mathcal{S}^3)),$$

$$z_\eta \in W^{1,q}(0, T_e; L^q(\boldsymbol{Q}, \mathbb{R}^N)), \quad \Sigma_\eta := B^T\sigma_\eta - L_\eta z_\eta \in L^p(\boldsymbol{Q}_{T_e}, \mathbb{R}^N)$$

with

$$(\sigma_\eta, L_\eta^{1/2} z_\eta) \in L^\infty(0, T_e; L^2(\boldsymbol{Q}, \mathcal{S}^3 \times \mathbb{R}^N))$$

is called a weak solution *of the initial boundary value problem (1) - (5), if for every $t \in (0, T_e)$ the function $(u_\eta(t), \sigma_\eta(t))$ is a weak solution of the boundary value problem (1) - (2), (4) for every given $Bz_\eta(t) \in L^q(\boldsymbol{Q}, \mathcal{S}^3)$, the initial condition (5) is satisfied pointwise for almost every (x,t) and the inequality (18) holds for all $t \in (0, T_e)$ and the function $\sigma_\eta^{(0)} \in L^2(\boldsymbol{Q}, \mathcal{S}^3)$ determined by equations* (12) - (14).

Now, we show that the above definition of weak solutions for (1) - (5) is consistent. Namely, we are going to prove that if a triple of functions $(u_\eta, \sigma_\eta, z_\eta)$ is a weak solution of (1) - (5) in the sense of Definition 3.1 and possesses additional regularity, then this triple of functions is a solution of the initial boundary value problem (1) - (5), i.e. the constitutive inclusion (3) is satisfied pointwise for a.e. $(x,t) \in \boldsymbol{Q}_{T_e}$. To this end, we assume that the weak solution $(u_\eta, \sigma_\eta, z_\eta)$ has the following regularity

$$(u_\eta, \sigma_\eta) \in W^{1,1}(0, T_e; H_0^1(\boldsymbol{Q}, \mathbb{R}^3) \times L^2(\boldsymbol{Q}, \mathcal{S}^3)),$$

$$z_\eta \in W^{1,1}(0, T_e; L^2(\boldsymbol{Q}, \mathbb{R}^N)).$$

Then, it is immediately seen that the function $\sigma_\eta^{(0)} \in L^2(\boldsymbol{Q}, \mathcal{S}^3)$ as a unique solution of the problem (12) - (14) satisfies the relation $\sigma_\eta^{(0)}(x) = \sigma_\eta(x, 0)$ for a.e. $x \in \boldsymbol{Q}$ and the following identity

$$\left(\mathbb{A}_\eta\sigma_\eta(t), \sigma_\eta(t)\right)_{\boldsymbol{Q}} - \left(\mathbb{A}_\eta\sigma_\eta^{(0)}, \sigma_\eta^{(0)}\right)_{\boldsymbol{Q}} = \int_{\boldsymbol{Q}_t} \frac{\partial}{\partial\tau} \left(\mathbb{A}_\eta\sigma_\eta(x,\tau), \sigma_\eta(x,s)\right) ds dx$$

Moreover, we have that

$$\left\|L_\eta^{1/2} z_\eta(t)\right\|_{\boldsymbol{Q}}^2 - \left\|L_\eta^{1/2} z_\eta^{(0)}\right\|_{\boldsymbol{Q}}^2 = \int_0^t \frac{\partial}{\partial\tau} \|L_\eta^{1/2} z_\eta(\tau)\|_{\boldsymbol{Q}}^2 d\tau.$$

Then, the inequality (18) can be rewritten as follows

$$\int_{\boldsymbol{Q}_t} \Big((\mathbb{A}_\eta \partial_\tau \sigma_\eta, \sigma_\eta) + (L_\eta z_\eta, \partial_\tau z_\eta) + f_{g_\eta}\left(x, B^T\sigma_\eta - L_\eta z_\eta, \partial_\tau z_\eta\right) \Big) d\tau dx \le (b, \partial_\tau u_\eta)_{\boldsymbol{Q}_t}.$$

Handling the equations (1) - (2) as above we obtain that the last inequality takes the following form

$$\int_{\boldsymbol{Q}_t} \Big((L_\eta z_\eta, \partial_\tau z_\eta) + f_{g_\eta}\left(x, B^T\sigma_\eta - L_\eta z_\eta, \partial_\tau z_\eta\right) \Big) d\tau dx \le (B^T\sigma_\eta, \partial_\tau z_\eta)_{\boldsymbol{Q}_t}.$$

or, equivalently,

$$\int_{\boldsymbol{Q}_t} f_{g_\eta}\left(x, B^T\sigma_\eta - L_\eta z_\eta, \partial_\tau z_\eta\right) d\tau dx \le \int_{\boldsymbol{Q}_t} (B^T\sigma_\eta - L_\eta z_\eta, \partial_\tau z_\eta) dx d\tau.$$

Therefore, by (10) and the standard localization argument we get that

$$\begin{aligned} &f_{g_\eta}\left(x, B^T\sigma_\eta(x,t) - L_\eta[x] z_\eta(x,t), \partial_t z_\eta(x,t)\right) \\ &\quad = \left(B^T\sigma_\eta(x,t) - L_\eta[x] z_\eta(x,t), \partial_t z_\eta(x,t)\right), \end{aligned}$$

which holds for a.e. $(x,t) \in \boldsymbol{Q}_{T_e}$. Now, based on the equivalence result (11) we conclude that the inclusion (3) is satisfied pointwise from the assumed temporal regularity of $(u_\eta, \sigma_\eta, z_\eta)$. The pointwise meaning of (5) follows.

Existence result. First, we define a class of maximal monotone functions we deal with in this work.

Definition 3.2. *Let S be a measurable set in $\mathbb{R}^s$ and $m \in L^1(S,\mathbb{R})$. For $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\mathcal{M}(S, \mathbb{R}^k, \alpha_1, \alpha_2, m)$ is the set of measurable multi-valued functions $h : S \to \mathfrak{M}(\mathbb{R}^k \times \mathbb{R}^k)$ (in the sense of Definition 2.2) such that with the following inequality*

$$(v, v^*) \ge m(x) + \alpha_1 |v^*|^q + \alpha_2 |v|^p \tag{19}$$

holds for a.e. $x \in S$ and every $v^* \in h(x,v)$, where p and q satisfy the relations $2 \le p < \infty$ and $q = p/(p-1)$.

The main properties of the class $\mathcal{M}(S, \mathbb{R}^k, \alpha_1, \alpha_2, m)$ are collected in the following proposition (see [8, Corollary 2.15]).

Proposition 3.1. *Let $\mathcal{H}$ be a canonical extension of a function $h : S \to \mathfrak{M}(\mathbb{R}^k \times \mathbb{R}^k)$ in the sense of Definition 2.3, which belongs to $\mathcal{M}(S, \mathbb{R}^k, \alpha_1, \alpha_2, m)$. Then $\mathcal{H}$ is maximal monotone, surjective and $D(\mathcal{H}) = L^p(S, \mathbb{R}^k)$.*

Now, we can state the main result of this section.

Theorem 3.1. *Assume that L_η is positive semi-definite, $\mathbb{C}_\eta$ is uniformly positive definite and $\mathbb{C}_\eta \in C(\bar{\boldsymbol{Q}}, \mathcal{L}(\mathcal{S}^3, \mathcal{S}^3))$, the mappings $g_\eta \in \mathcal{M}(\boldsymbol{Q}, \mathbb{R}^N, \alpha_1, \alpha_2, m)$ with a function m from $L^1(\boldsymbol{Q}, \mathbb{R})$. Suppose that $b \in W^{1,p}(0, T_e; W^{-1,p}(\boldsymbol{Q}, \mathbb{R}^3))$ and $z_\eta^{(0)} \in L^2(\boldsymbol{Q}, \mathbb{R}^N)$.*

Then the initial boundary value problem (1) - (5) has at least one weak solution (u_η, T_η, z_η) in the sense of Definition 3.1.

Remark 3.1. *We point out that the requirement of the continuity of $\mathbb{C}_\eta$ is superfluous and is only made to simplify the proof of Theorem 3.1. The proof itself works for the measurable function $\mathbb{C}_\eta$ as well. The continuity assumption allows us to apply the L^p-regularity theory for linear elliptic systems in [12] directly to our problem. In case of $\mathbb{C}_\eta \in L^\infty(\boldsymbol{Q}, \mathcal{L}(\mathcal{S}^3, \mathcal{S}^3))$, some extra technical work has to be done before one can use the L^p-regularity theory for linear elliptic systems (this strategy is realized in [20]). To avoid the technicalities we assume the continuity of $\mathbb{C}_\eta$ here.*

Proof. To simplify the notations we drop η. The proof of the theorem is presented in [19]. Therefore, we only sketch it here. We show this by the Rothe method (a time-discretization method, see [25] for details). In order to introduce a time-discretized problem, let us fix any $m \in \mathbb{N}$ and set

$$h = h_m := \frac{T_e}{2^m},\ z_m^0 := z^{(0)},\ b_m^n := \frac{1}{h}\int_{(n-1)h}^{nh} b(s)ds \in W^{-1,p}(\boldsymbol{Q}, \mathbb{R}^3),\ \ n = 1, ..., 2^m.$$

We are looking for functions $u_m^n \in H_0^1(\boldsymbol{Q}, \mathbb{R}^3)$, $\sigma_m^n \in L^2(\boldsymbol{Q}, \mathcal{S}^3)$ and $z_m^n \in L^2(\boldsymbol{Q}, \mathbb{R}^N)$ with

$$\Sigma_{n,m} := B^T\sigma_m^n - \frac{1}{m}z_m^n - Lz_m^n \in L^p(\boldsymbol{Q}, \mathbb{R}^N)$$

solving the following problem

$$-\operatorname{div}_x \sigma_m^n(x) = b_m^n(x), \tag{20}$$

$$\sigma_m^n(x) = \mathbb{C}[x](\varepsilon(\nabla_x u_m^n(x)) - Bz_m^n(x)), \tag{21}$$

$$\frac{z_m^n(x) - z_m^{n-1}(x)}{h} \in g\big(x, \Sigma_{n,m}(x)\big), \tag{22}$$

together with the boundary conditions

$$u_m^n(x) = 0, \quad x \in \partial\boldsymbol{Q}\,. \tag{23}$$

The proof of the existence of the triple

$$(u_m^n, \sigma_m^n, z_m^n) \in H_0^1(\boldsymbol{Q}, \mathbb{R}^3) \times L^2(\boldsymbol{Q}, \mathcal{S}^3) \times L^2(\boldsymbol{Q}, \mathbb{R}^N)$$

satisfying (20) - (23) can be found in [19].

A-priori estimates. Multiplying (20) by $(u_m^n - u_m^{n-1})/h$ and then integrating over $\boldsymbol{Q}$ we get

$$\big(\sigma_m^n, \varepsilon(\nabla_x(u_m^n - u_m^{n-1}))/h\big)_{\boldsymbol{Q}} = \big(b_m^n, (u_m^n - u_m^{n-1})/h\big)_{\boldsymbol{Q}}\,. \tag{24}$$

Applying $g^{-1}(x)$ to both sides of (22), multiplying by $w_m^n := (z_m^n - z_m^{n-1})/h$ and then integrate over $\boldsymbol{Q}$ to obtain

$$\int_{\boldsymbol{Q}} \big(g^{-1}(w_m^n), w_m^n\big)dx = (\sigma_m^n, Bw_m^n)_{\boldsymbol{Q}} - \frac{1}{mh}\Big(z_m^n - z_m^{n-1}, z_m^n\Big)_{\boldsymbol{Q}} - \frac{1}{h}\Big(z_m^n - z_m^{n-1}, Lz_m^n\Big)_{\boldsymbol{Q}}.$$

With (24) we get that

$$\frac{1}{h}\Big(\mathbb{C}^{-1}\sigma_m^n, \sigma_m^n - \sigma_m^{n-1}\Big)_{\boldsymbol{Q}} + \frac{1}{h}\Big(L^{1/2}(z_m^n - z_m^{n-1}), L^{1/2}z_m^n\Big)_{\boldsymbol{Q}}$$

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$$+\frac{1}{m}\frac{1}{h}\Big(z_m^n - z_m^{n-1}, z_m^n\Big)_{\boldsymbol{Q}} + \int_{\boldsymbol{Q}} \big(g^{-1}(w_m^n), w_m^n\big)dx = \frac{1}{h}\left(b_m^n, u_m^n - u_m^{n-1}\right)_{\boldsymbol{Q}}.$$

Multiplying by h and summing the obtained relation for $n = 1, ..., l$ for any fixed $l \in [1, 2^m]$ we derive the following inequality ($\mathbb{A} = \mathbb{C}^{-1}$)

$$\frac{1}{2}\Big(\|\mathbb{A}^{1/2}\sigma_m^l\|_{\boldsymbol{Q}}^2 + \|L^{1/2}z_m^l\|_{\boldsymbol{Q}}^2 + \frac{1}{m}\|z_m^l\|_{\boldsymbol{Q}}^2\Big) + h\sum_{n=1}^{l}\int_{\boldsymbol{Q}} \big(g^{-1}(w_m^n), w_m^n\big)dx$$
$$\leq C^{(0)} + h\sum_{n=1}^{l}\left(b_m^n, \frac{u_m^n - u_m^{n-1}}{h}\right)_{\boldsymbol{Q}}, \tag{25}$$

where

$$2C^{(0)} := \|\mathbb{A}^{1/2}\sigma_m^0\|_{\boldsymbol{Q}}^2 + \|L^{1/2}z_m^0\|_{\boldsymbol{Q}}^2 + \frac{1}{m}\|z_m^0\|_{\boldsymbol{Q}}^2.$$

We estimate now the right hand side of the last inequality. Since u_m^n is a solution of the linear elliptic problem formed by the equations (20), (21) and (23), it satisfies (see [12]) the inequality

$$\|u_m^n\|_{1,q,\boldsymbol{Q}} \leq C\big(\|b_m^n\|_{q,\boldsymbol{Q}} + \|z_m^n\|_{q,\boldsymbol{Q}}\big), \tag{26}$$

where C is a positive constant independent of n and m. Therefore, using the linearity of the problem formed by (20), (21) and (23), the inequality (26) and Young's inequality with $\epsilon > 0$ we get that

$$\left(b_m^n, \frac{u_m^n - u_m^{n-1}}{h}\right)_{\boldsymbol{Q}} \leq \|b_m^n\|_{p,\boldsymbol{Q}}\|(u_m^n - u_m^{n-1})/h\|_{1,q,\boldsymbol{Q}} \leq CC_\epsilon\|b_m^n\|_{p,\boldsymbol{Q}}^p$$
$$+\epsilon C\|(b_m^n - b_m^{n-1})/h\|_{q,\boldsymbol{Q}}^q + \epsilon C\|(z_m^n - z_m^{n-1})/h\|_{q,\boldsymbol{Q}}^q, \tag{27}$$

where C_ϵ is a positive constant appearing in the Young inequality. Combining the inequalities (25) and (27), applying (10) and (19) and choosing an appropriate value for $\epsilon > 0$ we obtain the following estimate

$$\frac{1}{2}\Big(\|\mathbb{A}^{1/2}\sigma_m^l\|_{\boldsymbol{Q}}^2 + \|L^{1/2}z_m^l\|_{\boldsymbol{Q}}^2 + \frac{1}{m}\|z_m^l\|_{\boldsymbol{Q}}^2\Big) + h\hat{C}_\epsilon\sum_{n=1}^{l}\int_{\boldsymbol{Q}}\Big|\frac{z_m^n - z_m^{n-1}}{h}\Big|^q dx$$
$$\leq C^{(0)} + h\tilde{C}_\epsilon\sum_{n=1}^{l}\Big(\|b_m^n\|_{p,\boldsymbol{Q}}^p + \|(b_m^n - b_m^{n-1})/h\|_{q,\boldsymbol{Q}}^q\Big), \tag{28}$$

where $\tilde{C}, \tilde{C}_\epsilon$ and $\hat{C}_\epsilon$ are some positive constants. Now, using the definition of Rothe's approximation functions (see (68)) we rewrite (28) as follows

$$\|\mathbb{A}^{1/2}\bar{\sigma}_m(t)\|_{\boldsymbol{Q}}^2 + C_1\|L^{1/2}\bar{z}_m(t)\|_{\boldsymbol{Q}}^2 + \frac{1}{m}\|\bar{z}_m(t)\|_{\boldsymbol{Q}}^2 \tag{29}$$
$$+2\hat{C}_\epsilon\int_0^{T_e}\int_{\boldsymbol{Q}}\big|\partial_t z_m(x,t)\big|^q dxdt \leq 2C^{(0)} + 2\tilde{C}_\epsilon\|b\|_{W^{1,p}(0,T_e;L^p(\boldsymbol{Q},\mathbb{R}^3))}^p.$$

From the estimate (29) we get then that

$$\{z_m\}_m \text{ is uniformly bounded in } W^{1,q}(0,T_e;L^q(\boldsymbol{Q},\mathbb{R}^N)), \tag{30}$$

$$\{L^{1/2}\bar{z}_m\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\boldsymbol{Q},\mathbb{R}^N)), \tag{31}$$

$$\{\bar{\sigma}_m\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\boldsymbol{Q},\mathcal{S}^3)), \tag{32}$$

$$\left\{\frac{1}{\sqrt{m}}\bar{z}_m\right\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\boldsymbol{Q},\mathbb{R}^N)). \tag{33}$$

In particular, the uniform boundness of the sequences in (30) - (33) yields

$$\left\{\bar{\Sigma}_m\right\}_m \text{ is uniformly bounded in } L^p(0,T_e;L^p(\boldsymbol{Q},\mathbb{R}^N)), \tag{34}$$

$$\{u_m\}_m \text{ is uniformly bounded in } W^{1,q}(0,T_e;W_0^{1,q}(\boldsymbol{Q},\mathbb{R}^3)). \tag{35}$$

Employing (69), the estimates (31) - (34) further imply that the sequnces $\{\sigma_m\}_m$, $\{L^{1/2}z_m\}_m$, $\{z_m/\sqrt{m}\}_m$ and $\{\Sigma_m\}_m$ are also uniformly bounded in the spaces $L^\infty(0,T_e;L^2(\boldsymbol{Q},\mathcal{S}^3))$, $L^\infty(0,T_e;L^2(\boldsymbol{Q},\mathbb{R}^N))$, $L^\infty(0,T_e;L^2(\boldsymbol{Q},\mathbb{R}^N))$ and $L^p(0,T_e;L^p(\boldsymbol{Q},\mathbb{R}^N))$, respectively. Moreover, due to (30) and the following obvious relation

$$z_m^l = z_m^0 + h\sum_{n=1}^{l}\left(\frac{z_m^n - z_m^{n-1}}{h}\right)$$

we may conclude that $\{\bar{z}_m\}_m$ is uniformly bounded in $L^q(0,T_e;L^q(\boldsymbol{Q},\mathbb{R}^N))$.

In [19] it is shown that the limit functions denoted by u, T, z and Σ of the corresponding weakly convergent sequences have the following properties

$$u \in W^{1,q}(0,T_e;W_0^{1,q}(\boldsymbol{Q},\mathbb{R}^3)), \quad (\sigma, L^{1/2}z) \in L^\infty(0,T_e;L^2(\boldsymbol{Q},\mathcal{S}^3\times\mathbb{R}^N)),$$

and

$$z \in W^{1,q}(0,T_e;L^q(\boldsymbol{Q},\mathbb{R}^N)), \quad \Sigma = B^T T - Lz \in L^p(\boldsymbol{Q}_{T_e},\mathbb{R}^N).$$

To prove that the weak limit of (u_m, T_m, z_m) is a weak solution of the problem (1) - (5), we are going to employ the concept of the Fitzpatrick function again. To this end, we rewrite (25) as follows

$$\begin{aligned}&\frac{1}{2}\Big(\|\mathbb{A}^{1/2}\bar{\sigma}_m(t)\|_{\boldsymbol{Q}}^2 + \|L^{1/2}\bar{z}_m(t)\|_{\boldsymbol{Q}}^2 + \frac{1}{m}\|\bar{z}_m(t)\|_{\boldsymbol{Q}}^2\Big) \\ &+\int_0^t\int_{\boldsymbol{Q}} F_g\left(x, B^T\bar{\sigma}_m - L\bar{z}_m, \partial_\tau z_m\right)dxds \le C^{(0)} + \int_0^t \left(\bar{b}_m, \partial_\tau u_m\right)_{\boldsymbol{Q}} d\tau.\end{aligned} \tag{36}$$

Next, using the lower semi-continuity of convex functionals we get (18) after passing to the weak limit in (36). This completes the proof of Theorem 3.1. □

4 Stochastic homogenization

Throughout this section, we follow the setting for stochastic homogenization proposed in [14] for rate-independent systems.

Remark 4.1. *In the following, we introduce the concept of Palm measures. Note that we will need this concept only in the context of the results in Section 5. For the main results proved in Section 6 we will restrict to the case* $\mu_\omega^\eta = \mathcal{L}$ *which implies* $\mu_{\mathcal{P}} = \mathcal{P}$ *(this follows from the translation invariance and Fubini's theorem). In this case, we will omit* $\mu_{\mathcal{P}}$ *and every integral over* Ω *is meant with respect to* $\mathcal{P}$*. In particular, we will write* $\int_\Omega f := \int_\Omega f(\omega)d\mathcal{P}(\omega)$.

4.1 Concept of Palm measures

Let $(\Omega, \mathcal{F}_\Omega, \mathcal{P}, \tau)$ be a probability space with dynamical system satisfying Assumption 1.1 and let $\mathcal{M}(\mathbb{R}^n)$ be the set of Radon measures on $\mathbb{R}^n$ equipped with the Vague topology.

Definition 4.1. *Let* $(\Omega,\mathcal{F}_\Omega,\mathcal{P},\tau)$ *satisfy Assumtion 1.1. A* random measure *is a mapping* $\mu_\bullet:\Omega\to\mathcal{M}(\mathbb{R}^n)$, $\omega\mapsto\mu_\omega$ *such that* $\omega\mapsto\mu_\omega(A)$ *is measurable for all Borel sets* $A\subset\mathbb{R}^n$. *A random measure is called stationary, if* $\mu_{\tau_x\omega}(A)=\mu_\omega(A+x)$ *for all Borel sets* $A\subset\mathbb{R}^n$. *The* intensity $\lambda(\mu_\omega)$ *is defined by:*

$$\lambda(\mu_\omega):=\int_\Omega\int_{[0,1]^n}d\mu_\omega(x)\,d\mathcal{P}(\omega)\,. \tag{37}$$

Theorem 4.1 (Mecke [16, 7]: Existence of Palm measure)**.** *Let* $\omega\mapsto\mu_\omega$ *be a stationary random measure. Then there exists a unique measure* $\mu_\mathcal{P}$ *on* Ω *such that*

$$\int_\Omega\int_{\mathbb{R}^n}f(x,\tau_x\omega)\,d\mu_\omega(x)d\mathcal{P}(\omega)=\int_{\mathbb{R}^n}\int_\Omega f(x,\omega)\,d\mu_\mathcal{P}(\omega)dx$$

for all $\mathcal{L}\times\mu_\mathcal{P}$*-measurable non negative functions and all* $\mathcal{L}\times\mu_\mathcal{P}$*- integrable functions* f. *Furthermore for all* $A\subset\Omega$, $u\in L^1(\Omega,\mu_\mathcal{P})$ *there holds*

$$\mu_\mathcal{P}(A)=\int_\Omega\int_{\mathbb{R}^n}g(s)\chi_A(\tau_s\omega)d\mu_\omega(s)d\mathcal{P}(\omega) \tag{38}$$

$$\int_\Omega u(\omega)d\mu_\mathcal{P}=\int_\Omega\int_{\mathbb{R}^n}g(s)u(\tau_s\omega)d\mu_\omega(s)d\mathcal{P}(\omega) \tag{39}$$

for an arbitrary $g\in L^1(\mathbb{R}^n,\mathcal{L})$ *with* $\int_{\mathbb{R}^n}g(x)dx=1$ *and* $\mu_\mathcal{P}$ *is* σ*-finite.*

Remark 4.2. *a) Setting* $g(s):=\chi_{[0,1]^n}(s)$, *the Palm measure can equally be defined through (38).*

b) For the constant measure $\omega\mapsto\mathcal{L}$, *we simply find* $\mu_\mathcal{P}=\mathcal{P}$, *the original probability measure. This is a direct consequence of (38), Fubini's theorem and Assumption 1.1 (ii).*

For a random measure μ_ω, we define

$$\mu_\omega^\eta(A):=\eta^n\mu_\omega(\eta^{-1}A)\,. \tag{40}$$

Theorem 4.2 (Ergodic Theorem [7])**.** *Let Assumption 1.1 hold for* $(\Omega,\mathcal{F}_\Omega,\mathcal{P},\tau)$. *Let* μ_ω *be a stationary random measure with finite intensity and Palm measure* $\mu_\mathcal{P}$. *Then, for all* $g\in L^1(\Omega,\mu_\mathcal{P})$ *there holds* $\mathcal{P}$ *almost surely*

$$\lim_{\eta\to0}\int_A g(\tau_{\frac{x}{\eta}}\omega)d\mu_\omega^\eta(x)=|A|\int_\Omega g(\omega)d\mu_\mathcal{P}(\omega) \tag{41}$$

for all bounded Borel sets A.

The ergodic theorem only holds for functions on Ω. Nevertheless, it motivates the following generalization of the concept of ergodicity:

Definition 4.2. *Let* $f\in L^p(\boldsymbol{Q}\times\Omega;\mathcal{L}\otimes\mu_\mathcal{P})$ *for some* $1\le p<\infty$. *We say that* f *is an ergodic function if it has a* $\mathcal{B}(\boldsymbol{Q})\otimes\mathcal{F}_\Omega$*-measurable representative* $\tilde f$ *such that for* $a=\tilde f$ *and* $a=|\tilde f|^p$ *it holds*

$$\lim_{\eta\to0}\int_{\boldsymbol{Q}}a(x,\tau_{\frac{x}{\eta}}\omega)\,d\mu_\omega^\eta(x)=\int_{\boldsymbol{Q}}\int_\Omega a(x,\tilde\omega)\,d\mu_\mathcal{P}(\tilde\omega)\,dx\,. \tag{42}$$

Finally, we say $f\in L^p(\boldsymbol{Q}\times\Omega;\mathcal{L}\otimes\mu_\mathcal{P})$ *is an ergodic function if (42) holds for almost all* ω.

By Theorem 4.2, we find that every $g \in L^1(\Omega, \mu_{\mathcal{P}})$ is ergodic. In [14, Section 2.5] a larger set of ergodic functions was identified:

Lemma 4.1. *Let Assumption 1.1 hold for* $(\Omega, \mathcal{F}_\Omega, \mathcal{P}, \tau)$. *Let* $\boldsymbol{Q} \subset \mathbb{R}^n$ *be a bounded domain and let* $f \in L^\infty(\boldsymbol{Q}\times\Omega; \mathcal{L}\otimes\mu_{\mathcal{P}})$. *Then,* f *is an ergodic function.*

4.2 Potentials and Solenoidals

Let $\mathcal{BD}(\mathbb{R}^n)$ denote the set of bounded domains in $\mathbb{R}^n$. For every p with $1 < p < \infty$, we introduce the following spaces:

$$L^p_{\mathrm{pot}}(\mathbb{R}^n) := \overline{\{g \in L^p_{loc}(\mathbb{R}^n; \mathbb{R}^n) \mid \forall \boldsymbol{Q} \in \mathcal{BD}(\mathbb{R}^n)\, \exists f \in W^{1,p}(\boldsymbol{Q}) : \ g = \nabla f\}}^{\|\cdot\|_{L^p_{loc}}},$$

$$L^p_{\mathrm{sol}}(\mathbb{R}^n) := \left\{ g \in L^p_{loc}(\mathbb{R}^n; \mathbb{R}^n) \mid \forall \boldsymbol{Q} \in \mathcal{BD}(\mathbb{R}^n)\, \exists f \in W^{1,p}_0(\boldsymbol{Q}) \, : \, \int_{\boldsymbol{Q}} g \cdot \nabla f = 0 \right\}.$$

On Ω, we introduce the corresponding spaces

$$L^p_{\mathrm{pot}}(\Omega) := \left\{ f \in L^p(\Omega; \mathbb{R}^n) \mid f_\omega \in L^p_{\mathrm{pot}}(\mathbb{R}^n) \ \mathcal{P}-\text{a.s. and} \int_\Omega f = 0 \right\},$$

$$L^p_{\mathrm{sol}}(\Omega) := \left\{ f \in L^p(\Omega; \mathbb{R}^n) \mid f_\omega \in L^p_{\mathrm{sol}}(\mathbb{R}^n) \ \mathcal{P}-\text{a.s. and} \int_\Omega f = 0 \right\}.$$

Then, there holds the following orthogonal decomposition.

Lemma 4.2 ([21, Theorem 3.1.2])**.** *Let* $1 < p < \infty$ *and* $p^{-1} + q^{-1} = 1$ *and let Assumption 1.1 hold for* $(\Omega, \mathcal{F}_\Omega, \mathcal{P}, \tau)$. *Then the following relations hold in the sense of duality between the spaces* $L^p(\Omega, \mathcal{P})$ *and* $L^q(\Omega, \mathcal{P})$*:*

$$\left(L^p_{\mathrm{pot}}(\Omega)\right)^\perp = L^q_{\mathrm{sol}}(\Omega) \oplus \mathbb{R}^n, \qquad \left(L^p_{\mathrm{sol}}(\Omega)\right)^\perp = L^q_{\mathrm{pot}}(\Omega) \oplus \mathbb{R}^n.$$

Every $L^p_{\mathrm{pot}}(\Omega)$ function can be optained as the ergodic limit of a sequence of gradients with vanishing potentials. The following result can be proved like in [13, Section 2.3].

Lemma 4.3. *Let* $\upsilon \in L^p_{\mathrm{pot}}(\Omega)$, $1 < p < \infty$. *For almost every* ω *there exists* $C > 0$ *such that the following holds: For every* $\eta > 0$ *there exists a unique* V^ω_η *with* $\nabla V^\omega_\eta(x) = \upsilon(\tau_{\frac{x}{\eta}}\omega)$ *and* $\|V_\eta\|_{H^1(\boldsymbol{Q})} \le C\|\upsilon\|_{L^p_{\mathrm{pot}}(\Omega)}$ *for all* $\eta > 0$. *Furthermore,*

$$\lim_{\eta\to 0} \|V^\omega_\eta\|_{L^p(\boldsymbol{Q})} = 0.$$

Furthermore, we find the following important Korn inequality, which can be proved like in [13, Section 2.3].

Lemma 4.4. *Let Assumption 1.1 hold for* $(\Omega, \mathcal{F}_\Omega, \mathcal{P}, \tau)$. *For every* $1 < p < \infty$ *there exists* $C_p > 0$ *such that for all* $\upsilon \in L^p_{\mathrm{pot}}(\Omega; \mathbb{R}^n)$

$$\|\upsilon\|_{L^p(\Omega;\mathbb{R}^{n\times n})} \le C_p \|\upsilon^s\|_{L^p(\Omega;\mathbb{R}^{n\times n})}. \tag{43}$$

4.3 Two-scale convergence: time independent case

Let Assumption 1.1 hold for $(\Omega, \mathcal{F}_\Omega, \mathcal{P}, \tau)$ and let $\omega \mapsto \mu_\omega$ be a stationary random measure with μ_ω^η and $\mu_\mathcal{P}$ defined through (40) and (38). For the case $\mu_\omega = \mathcal{L}$ we recall Remark 4.1.

Remark 4.3. *The product σ-algebra $\mathcal{B}_Q \otimes \mathcal{F}_\Omega$ is countably generated and therefore, the space $L^p(\boldsymbol{Q} \times \Omega)$ is separable (see [9, Section VI.15, p. 92]) for every $1 \le p < \infty$. In particular, for every $1 \le p < \infty$, there exists a countable dense subset Φ_p of simple functions in $L^p(\boldsymbol{Q} \times \Omega; \mathcal{L} \otimes \mu_\mathcal{P})$. By Lemma 4.1, every $\phi \in \Phi_p$ is an ergodic function and there exists a set $\Omega_{\Phi_p} \subset \Omega$ with $\mathcal{P}(\Omega_{\Phi_p}) = 1$ such that all $\phi \in \Phi_p$ satisfy (42) (i.e. they admit ergodic realizations) for all $\omega \in \Omega_{\Phi_p}$. This corresponds to the setting of [14].*

Definition 4.3. *Let $1 < p, q < \infty$ with $1/p + 1/q = 1$. Let Φ_p be the set of Remark 4.3 and let $\omega \in \Omega_{\Phi_p}$. Let $u^\eta \in L^q(\boldsymbol{Q}; \mu_\omega^\eta)$ for all $\eta > 0$. We say that u^η converges (weakly) in two scales to $u \in L^q(\boldsymbol{Q}; L^q(\Omega, \mu_\mathcal{P}))$ and write $u^\eta \xrightharpoonup{2s} u$ if $\sup_\eta \|u^\eta\|_{L^q(\boldsymbol{Q};\mu_\omega^\eta)} < \infty$ and if for all $\phi \in \Phi_p$ there holds with $\phi_{\omega,\eta}(x) := \phi(x, \tau_{\frac{x}{\eta}}\omega)$*

$$\lim_{\eta\to 0} \int_{\boldsymbol{Q}} u^\eta \phi_{\omega,\eta} d\mu_\omega^\eta = \int_{\boldsymbol{Q}} \int_\Omega u\phi \, d\mu_\mathcal{P} \, d\mathcal{L} \,.$$

Furthermore, we say that u^η converges strongly in two scales to u, written $u^\eta \xrightarrow{2s} u$, if for all weakly two-scale converging sequences $v^\eta \in L^p(\boldsymbol{Q}; \mu_\omega^\eta)$ with $v^\eta \xrightharpoonup{2s} v \in L^p(\boldsymbol{Q}; L^p(\Omega, \mu_\mathcal{P}))$ as $\eta \to 0$ there holds

$$\lim_{\eta\to 0} \int_{\boldsymbol{Q}} u^\eta v^\eta d\mu_\omega^\eta = \int_{\boldsymbol{Q}} \int_\Omega uv \, d\mu_\mathcal{P} \, d\mathcal{L} \,.$$

Lemma 4.5 (Existence of two-scale limits [14])**.** *Let $\omega \in \Omega_{\Phi_p}$, $1 < p < \infty$ and $u^\eta \in L^p(\boldsymbol{Q}; \mu_\omega^\eta)$ be a sequence of functions such that $\|u^\eta\|_{L^p(\boldsymbol{Q};\mu_\omega^\eta)} \le C$ for some $C > 0$ independent of η. Then there exists a subsequence of $u^{\eta'}$ and $u \in L^p(\boldsymbol{Q}; L^p(\Omega, \mu_\mathcal{P}))$ such that $u^{\eta'} \xrightharpoonup{2s} u$ and*

$$\|u\|_{L^p(\boldsymbol{Q};L^p(\Omega,\mu_\mathcal{P}))} \le \liminf_{\eta\to 0} \left\| u^{\eta'} \right\|_{L^p(\boldsymbol{Q};\mu_\omega^\eta)} . \tag{44}$$

Closely connected with the definition of two-scale convergence and Lemma 4.5 is the following result.

Lemma 4.6. *Let $f \in L^1(\Omega, \mu_\mathcal{P})$. For almost all $\omega \in \Omega$ it holds: for all $\psi \in C_0(\boldsymbol{Q})$ and $u^\eta \in L^q(\boldsymbol{Q}; \mu_\omega^\eta)$, $\eta > 0$, and $u \in L^q(\boldsymbol{Q}; L^q(\Omega, \mu_\mathcal{P}))$ with $u^\eta \xrightharpoonup{2s} u$ it holds*

$$\lim_{\eta\to 0} \int_{\boldsymbol{Q}} u^\eta \psi(x) f\left(\tau_{\frac{x}{\eta}}\omega\right) d\mu_\omega^\eta = \int_{\boldsymbol{Q}} \int_\Omega u\psi f \, d\mu_\mathcal{P} \, d\mathcal{L} \,.$$

The following Lemma is well known in the periodic case ([3]) but also in the stochastic setting ([14, 34] for $p = 2$). The following version can be proofed along the same lines as Lemma 6.2 in [14].

Lemma 4.7. *Let $1 < p < \infty$. If $u^\eta \in W_0^{1,p}(\boldsymbol{Q}; \mathbb{R}^n)$ for all η with $\|\nabla u^\eta\|_{L^p(\boldsymbol{Q})} < C$ for C independent from $\eta > 0$ then there exists a subsequence $u^{\eta'}$ and functions $u \in W_0^{1,p}(\boldsymbol{Q}; \mathbb{R}^n)$ and $v \in L^p(\boldsymbol{Q}; L^p_{pot}(\Omega; \mathbb{R}^n))$ such that*

$$u^{\eta'} \xrightharpoonup{2s} u \quad \textit{and} \quad \nabla u^{\eta'} \xrightharpoonup{2s} \nabla u + v \qquad \textit{as } \varepsilon \to 0 \,.$$

We finally collect some usefull results.

Lemma 4.8. *Let $u \in L^p(\boldsymbol{Q}; L^p(\Omega; \mu_{\mathcal{P}}))$. Then, for almost every $\omega \in \Omega$, there exists a sequence $u^\eta \in L^p(\boldsymbol{Q}; \mu^\eta_\omega)$ such that $u^\eta \stackrel{2s}{\rightharpoonup} u$ as $\eta \to 0$.*

Lemma 4.9. *Let $N \in \mathbb{N}$ and let $A \in L^\infty(\boldsymbol{Q}; L^\infty(\Omega; \mathscr{L}(\mathbb{R}^N, \mathbb{R}^N)))$ be symmetric and assume A is $\mathcal{B}_Q \otimes \mathcal{F}_\Omega$ -measurable. We furthermore assume the existence of a constant $\alpha > 0$ such that*

$$\alpha\,|\xi|^2 \leq \xi A(x,\omega)\xi \leq \frac{1}{\alpha}\,|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n \text{ and for } \mathcal{L}\times\mu_{\mathcal{P}}\text{-a.e. } (x,\omega) \in \boldsymbol{Q}\times\Omega\,. \tag{45}$$

Then, for almost all $\omega \in \Omega_{\Phi_p}$ there holds: For all sequences $u^\eta \in L^2(\boldsymbol{Q}; \mu^\eta_\omega; \mathbb{R}^N)$ with weak two-scale limit $u \in L^2(\boldsymbol{Q}; L^2(\Omega; \mu_{\mathcal{P}}; \mathbb{R}^N))$ there holds with $A_{\eta,\omega}(x) := A(x, \tau_{\frac{x}{\eta}}\omega)$

$$\liminf_{\eta\to 0} \int_{\boldsymbol{Q}} u^\eta \cdot (A_{\eta,\omega} u^\varepsilon)\, d\mu^\eta_\omega \geq \int_{\boldsymbol{Q}}\int_\Omega u \cdot (Au)\, d\mu_{\mathcal{P}}\,.$$

4.4 Two-scale convergence: time dependent case

We are also interested in the convergence behavior of functions $u^\eta : [0,T] \to L^p(\boldsymbol{Q}, \mu^\eta_\omega)$.

Definition 4.4. *Let $1 < r, r', p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{r'} + \frac{1}{r} = 1$. Let Φ_q be the set of Remark 4.3 and let $\omega \in \Omega_{\Phi_q}$. Let $u^\eta \in L^r(0,T; L^p(\boldsymbol{Q}; \mu^\eta_\omega))$ for all $\eta > 0$. We say that u^η converges (weakly) in two scales to $u \in L^r(0,T; L^p(\boldsymbol{Q}; L^p(\Omega, \mu_{\mathcal{P}})))$ and write $u^\eta \stackrel{2s}{\rightharpoonup} u$ if for all continuous and piecewise affine functions $\phi : [0,T] \to \Phi_q$ there holds with $\phi_{\omega,\eta}(t,x) := \phi(t,x,\tau_{\frac{x}{\eta}}\omega)$*

$$\lim_{\eta\to 0} \int_0^T \int_{\boldsymbol{Q}} u^\eta \phi_{\omega,\eta} d\mu^\eta_\omega dt = \int_0^T \int_{\boldsymbol{Q}}\int_\Omega u\phi\, d\mu_{\mathcal{P}}\, dx\, dt$$

The following two lemmas where proved in [14].

Lemma 4.10. *Asssume that $1 < p < \infty$ and $1 < r \leq \infty$. Then, every sequence of functions $(u^\eta \in L^r(0,T; L^p(\boldsymbol{Q}; \mu^\eta_\omega)))_{\varepsilon>0}$ satisfying*

$$\|u^\eta\|_{L^r(0,T;L^p(\boldsymbol{Q};\mu^\eta_\omega))} \leq C$$

for some $C > 0$ independent from η has a weakly two-scale convergent subsequence with limit function $u \in L^r(0,T; L^p(\boldsymbol{Q}; L^p(\Omega, \mu_{\mathcal{P}})))$. Furthermore, if

$$\|\partial_t u^\eta\|_{L^r(0,T;L^p(\boldsymbol{Q};\mu^\eta_\omega))} \leq C$$

uniformly for $1 < p \leq \infty$, then also $\|\partial_t u\|_{L^r(0,T;L^p(\boldsymbol{Q};L^p(\Omega,\mu_{\mathcal{P}})))} \leq C$ and $\partial_t u^\eta \stackrel{2s}{\rightharpoonup} \partial_t u$ in the sense of Definition 4.4 as well as $u^\eta(t) \stackrel{2s}{\rightharpoonup} u(t)$ for all $t \in [0,T]$.

5 Homogenization of convex functionals

Lemma 5.1. *Let Assumption 1.1 hold for $(\Omega, \mathcal{F}_\Omega, \mathcal{P}, \tau)$ and the random measure μ_ω. Let $f : \boldsymbol{Q}\times\Omega\times\mathbb{R}^N \to \mathbb{R}$ be a convex functional in $\mathbb{R}^N$. For almost all*

$\omega \in \Omega_{\Phi_p}$ the following holds: Let $u^\eta \in L^q(\boldsymbol{Q};\mathbb{R}^N)$ be a sequence such that $\|u^\eta\|_{L^q(\boldsymbol{Q})} \leq C$ for some $0 < C < \infty$ and such that $u^\eta \xrightharpoonup{2s} u \in L^q(\boldsymbol{Q} \times \Omega; \mathcal{L} \otimes \mu_{\mathcal{P}}; \mathbb{R}^N)$. Then, it holds

$$\int_{\boldsymbol{Q}} \int_\Omega f(x,\tilde{\omega},u(x,\tilde{\omega}))\, d\mu_{\mathcal{P}}(\tilde{\omega})\, dx \leq \liminf_{\eta\to 0} \int_{\boldsymbol{Q}} f(x,\tau_{\frac{x}{\eta}}\omega,u^\eta(x))\, d\mu_\omega^\eta(x)\,.$$

The proof of Lemma 5.1 is literally the same as for Theorem 7.1 in [35]. However, we provide it here for completeness.

Proof. Let $\omega \in \Omega_{\Phi_p}$ and let f^* denote the Fenchel conjugate of f in the third variable. Without loss of generality, we may assume that

$$\lim_{\eta\to 0} \int_{\boldsymbol{Q}} f^*_{\eta,\omega}(x,\psi_\omega^\eta(x))\, d\mu_\omega^\eta(x) = \int_{\boldsymbol{Q}} \int_\Omega f^*(x,\tilde{\omega},\psi(x,\tilde{\omega}))\, d\mu_{\mathcal{P}}(\tilde{\omega})\, dx \tag{46}$$

for all $\psi \in \Phi_p^N$ and all $\omega \in \Omega_{\Phi_p}$. We first consider the case

$$f(x,\tilde{\omega},\xi) \geq |\xi|^q \tag{47}$$

for almost every $(x,\tilde{\omega}) \in \boldsymbol{Q} \times \Omega$ and all $\xi \in \mathbb{R}^N$. We then find for every $\psi \in \Phi_p^N$

$$F_\eta := \int_{\boldsymbol{Q}} f_{\eta,\omega}(x,u^\eta(x))\, d\mu_\omega^\eta(x) \geq \int_{\boldsymbol{Q}} u^\eta(x)\cdot\psi_\omega^\eta(x) d\mu_\omega^\eta(x) - \int_{\boldsymbol{Q}} f^*_{\eta,\omega}(x,\psi_\omega^\eta(x))\, d\mu_\omega^\eta(x)\,.$$

Due to $u^\eta \xrightharpoonup{2s} u$ and (46) we find

$$\liminf_{\eta\to 0} F_\eta \geq \int_{\boldsymbol{Q}} \int_\Omega \left(u(x,\tilde{\omega})\cdot\psi(x,\tilde{\omega}) - f^*(x,\tilde{\omega},\psi(x,\tilde{\omega}))\right)\, d\mu_{\mathcal{P}}(\tilde{\omega})\, dx$$

for all $\psi \in \Phi_p^N$. Since (47) holds, f^* is continuous in ξ and the last inequality implies

$$\liminf_{\eta\to 0} F_\eta \geq \int_{\boldsymbol{Q}} \int_\Omega f(x,\tilde{\omega},u(x,\tilde{\omega}))\, d\mu_{\mathcal{P}}(\tilde{\omega})\, dx\,. \tag{48}$$

In the general case, let

$$F_\eta^\delta := \int_{\boldsymbol{Q}} f_{\eta,\omega}(x,u^\eta(x))\, d\mu_\omega^\eta(x) + \delta\, \|u^\eta\|_{L^q(\boldsymbol{Q})}\,.$$

Then, $0 < F_\eta^\delta - F_\eta \leq C\delta$ and (48) implies that

$$\liminf_{\eta\to 0} F_\eta^\delta \geq \int_{\boldsymbol{Q}} \int_\Omega f(x,\tilde{\omega},u(x,\tilde{\omega}))\, d\mu_{\mathcal{P}}(\tilde{\omega})\, dx + \delta\, \|u\|_{L^q(\boldsymbol{Q}\times\Omega)}\,.$$

Hence the claim follows. □

Lemma 5.2. *Let Assumption 1.1 hold for $(\Omega,\mathcal{F}_\Omega,\mathcal{P},\tau)$ and let μ_ω be a random measure. Let $f : \boldsymbol{Q} \times \Omega \times \mathbb{R}^N \to \mathbb{R}$ be such that for a.e. $(x,\tilde{\omega})$ the function $f(x,\tilde{\omega},\cdot)$ is convex in $\mathbb{R}^N$. Then, for almost every $\omega \in \Omega_{\Phi_p}$ the following holds: If $u_\omega^\eta \in L^q(\boldsymbol{Q};\mathbb{R}^N)$ is a sequence of minimizers of the functionals*

$$F_{\eta,\omega} : u \mapsto \int_{\boldsymbol{Q}} f(x,\tau_{\frac{x}{\eta}}\omega,u(x))\, d\mu_\omega^\eta(x)$$

and if $\sup_{\eta>0}\|u^\eta_\omega\|_{L^q(\boldsymbol{Q};\mu^\eta_\omega)}<\infty$*, then there exists* $u_{0,\omega}\in L^q(\boldsymbol{Q}\times\Omega;\mathcal{L}\otimes\mu_{\mathcal{P}};\mathbb{R}^N)$ *such that* $u^\eta_\omega \xrightarrow{2s} u_{0,\omega}$ *along a subsequence and* $u_{0,\omega}$ *is a minimizer of*

$$F_0\,:\;L^q(\boldsymbol{Q}\times\Omega;\mathcal{L}\otimes\mu_{\mathcal{P}};\mathbb{R}^N)\to\mathbb{R}$$
$$u\mapsto\int_{\boldsymbol{Q}} f(x,\tilde\omega,u(x,\tilde\omega))\,d\mu_{\mathcal{P}}(\tilde\omega)\,dx\,.$$

Proof. Let $u_0\in L^q(\boldsymbol{Q}\times\Omega;\mathcal{L}\otimes\mu_{\mathcal{P}})$ be a minimizer of F_0. By [29, Theorem III-39] we can assume that $u_0(x,\tilde\omega)$ minimizes $f(x,\tilde\omega)$ for almost every $(x,\tilde\omega)$. Then, for almost all $\omega\in\Omega$ it holds $u^\eta_0(x):=u_0(x,\tau_{\frac{x}{\eta}}\omega)\in L^q(\boldsymbol{Q};\mathbb{R}^N)$ and

$$F_{\eta,\omega}(u^\eta_0)\geq F_{\eta,\omega}(u^\eta)\,.$$

We chose a subsequence $u^{\eta'}_\omega$ and $u_{0,\omega}\in L^q(\boldsymbol{Q}\times\Omega;\mathcal{L}\otimes\mu_{\mathcal{P}};\mathbb{R}^N)$ such that $u^{\eta'}_\omega\xrightarrow{2s}u_{0,\omega}$. Since $F_{\eta,\omega}(u^\eta_0)\to F_0(u_0)$, we find

$$\int_{\boldsymbol{Q}}\int_\Omega f(x,\tilde\omega,u_{0,\omega}(x,\tilde\omega))\,d\mu_{\mathcal{P}}(\tilde\omega)\,dx\leq\liminf_{\eta\to0}F_{\eta,\omega}(u^\eta_\omega)\leq F_0(u_0).$$

Hence, $u_{0,\omega}$ is a minimizer of F_0. □

Theorem 5.1. *Let Assumption 1.1 hold for* $(\Omega,\mathcal{F}_\Omega,\mathcal{P},\tau)$ *and let* μ_ω *be a random measure and let* $1<p,q<\infty$ *with* $\frac1p+\frac1q=1$*. Let* $\boldsymbol{Q}\subset\mathbb{R}^n$ *be a bounded domain and* $f:\boldsymbol{Q}\times\Omega\times\mathbb{R}^d\to\mathbb{R}$ *be measurable and for all* (x,ω)*. Let* $f(x,\omega,\cdot)$ *be continuous and convex in* $\mathbb{R}^d$ *with* $f(x,\omega,\xi)\geq|\xi|^q$*. Then, for almost every* $\omega\in\Omega_{\Phi_p}$ *it holds: If* $u^\eta\in W^{1,q}(\boldsymbol{Q})$ *is a sequence of minimizers of the functional*

$$F_\eta:\;L^q(\boldsymbol{Q})\to\mathbb{R}$$
$$u\mapsto\int_{\boldsymbol{Q}}f_{\omega,\eta}(x,\nabla u(x))\,dx$$

such that $\sup_{\eta>0}\|u^\eta_\omega\|_{L^q(\boldsymbol{Q};\mu^\eta_\omega)}<\infty$*, then there exist* $u_\omega\in W^{1,q}_0(\boldsymbol{Q})$ *and* $\upsilon_\omega\in L^q(\boldsymbol{Q};L^q_{\mathrm{pot}}(\Omega))$ *and a subsequence* $u^{\eta'}$ *such that* $u^{\eta'}\to u_\omega$ *strongly in* $L^q(\boldsymbol{Q})$ *and* $\nabla u^{\eta'}\xrightarrow{2s}\nabla u_\omega+\upsilon_\omega$ *as* $\eta\to0$ *and* $(u_\omega,\upsilon_\omega)$ *is a minimizer of the functional*

$$F_0:\;W^{1,q}_0(\boldsymbol{Q})\times L^q(\boldsymbol{Q};L^q_{\mathrm{pot}}(\Omega))\to\mathbb{R}$$
$$(u,\upsilon)\mapsto\int_{\boldsymbol{Q}}\int_\Omega f(x,\omega,\nabla u(x)+\upsilon(x,\tilde\omega))\,d\mathcal{P}(\tilde\omega)\,dx\,.$$

Proof. Let Φ_{pot} be a countable dense subset of $L^q_{\mathrm{pot}}(\Omega)$ and let $\tilde\Omega\subset\Omega_{\Phi_q}$ be a set of full measure such that Lemma lem:valid-ts-test-function holds for all $\omega\in\tilde\Omega$. By spanΦ_{pot} we denote finite linear combinations of elements of Φ_{pot}. In what follows we restrict to the case $\omega\in\tilde\Omega$.

Due to Lemma 4.7 there exist $u_\omega\in W^{1,q}(\boldsymbol{Q})$ and $\upsilon_\omega\in L^q(\boldsymbol{Q};L^q_{\mathrm{pot}}(\Omega))$ such that $\nabla u^\eta\xrightarrow{2s}\nabla u_\omega+\upsilon_\omega$ and $u^\eta\to u_\omega$ along a subsequence, which we denote u^η for simplicity. Let $u_0\in W^{1,q}_0(\boldsymbol{Q})$ and $\upsilon_0\in L^q(\boldsymbol{Q};L^q_{\mathrm{pot}}(\Omega))$ be a minimizer of the functional F_0.

Now, let $\delta > 0$. There exists $\upsilon_\delta \in L^q(\boldsymbol{Q};\mathbb{R}\Phi_{\text{pot}})$ which is simple and has compact support in $\boldsymbol{Q}$ such that $\|\upsilon_0 - \upsilon_\delta\|_{L^q(\boldsymbol{Q};L^q_{\text{pot}}(\Omega))} < \delta$. In particular, we find sets $A_i \subset \boldsymbol{Q}$, $1 \le i \le K_\delta$ and functions $\hat{\upsilon}_i \in \operatorname{span}\Phi_{\text{pot}}$ such that

$$\upsilon_\delta(x,\omega) = \sum_{i=1}^{K_\delta} \chi_{A_i}(x)\hat{\upsilon}_i(\omega)\,.$$

Let $(\varphi_\varepsilon)_{\varepsilon>0} \subset C_0^\infty(\mathrm{B}_\varepsilon)$ be a family of mollifiers. For $\varepsilon > 0$ we denote $\upsilon_{\varepsilon,\delta}(\cdot,\omega) := \varphi_\varepsilon * \upsilon(\cdot,\omega)$, where $*$ is the convolution with respect to the $\boldsymbol{Q}$-coordinate. Then $\upsilon_{\varepsilon,\delta} \in C_0^1(\overline{\boldsymbol{Q}};\operatorname{span}\Phi_{\text{pot}})$ for $\varepsilon > 0$ small enough and $\|\upsilon_{\varepsilon,\delta} - \upsilon_\delta\|_{L^q(\boldsymbol{Q};L^q_{\text{pot}}(\Omega))} \to 0$ as $\varepsilon \to 0$.

Given $x \in \boldsymbol{Q}$ we apply Lemma 4.3 and denote $V^\omega_{\eta,\varepsilon,\delta}(x,\cdot) \in H^1(\boldsymbol{Q})$ the η-potential to $\upsilon_{\varepsilon,\delta}(x)$ and η and $V^\omega_{\eta,\delta}(x,\cdot) \in H^1(\boldsymbol{Q})$ the potential to $\upsilon_\delta(x)$ and η. Further, if $\hat{V}^\omega_{i,\eta}$ is the corresponding η-potential to $\hat{\upsilon}_i$, we find

$$V^\omega_{\eta,\varepsilon,\delta}(x,z) = \sum_{i=1}^{K_\delta}(\chi_{A_i} * \varphi_\varepsilon)(x)\hat{V}^\omega_{i,\eta}(z) \quad \text{and} \quad V^\omega_{\eta,\delta}(x,z) = \sum_{i=1}^{K_\delta}\chi_{A_i}(x)\hat{V}^\omega_{i,\eta}(z)$$

Since the mapping $\hat{\upsilon}_i \mapsto \hat{V}^\omega_{i,\eta}$ is linear, we find $V^\omega_{\eta,\varepsilon,\delta} \in C_0^1(\boldsymbol{Q};H^1(\boldsymbol{Q}))$ with

$$\begin{aligned}\nabla\left(V^\omega_{\eta,\varepsilon,\delta}(x,x)\right) &= \nabla_x V^\omega_{\eta,\varepsilon,\delta}(x,x) + \nabla_z V^\omega_{\eta,\varepsilon,\delta}(x,x)\\ &= \left(V^\omega_{\eta,\delta}(\cdot,x) * \nabla\varphi_\varepsilon\right)(x) + \left(\varphi_\varepsilon * \upsilon_\delta(\cdot,\tau_{\frac{x}{\eta}})\right)(x)\end{aligned}$$

For the first term on the right hand side we obtain

$$\begin{aligned}\int_{\boldsymbol{Q}}\left|\left(V^\omega_{\eta,\delta}(\cdot,x) * \nabla\varphi_\varepsilon\right)(x)\right|^q \mathrm{d}x &\le \int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathbb{R}^d}\mathrm{d}z\,\|\nabla\varphi_\varepsilon\|^q_\infty\left|V^\omega_{\eta,\delta}(z-x,x)\right|^q\\ &\le \|\nabla\varphi_\varepsilon\|^q_\infty\int_{\boldsymbol{Q}}\mathrm{d}x\sum_{i=1}^{K_\delta}\left|\hat{V}^\omega_{i,\eta}(x)\right|^q\end{aligned}$$

Since the last expression on the right hand side converges to 0 as $\eta \to 0$ by Lemma 4.3, we find that $\nabla V_{\eta,\varepsilon,\delta}(x,x) \overset{2s}{\rightharpoonup} \upsilon_{\varepsilon,\delta}(x,\omega)$.

Hence, we find for ε small enough that $V^\omega_{\eta,\varepsilon,\delta}(x,x)$ is a valid point of evaluation for F_η and

$$\begin{aligned}F_\eta(u_0 + V^\omega_{\eta,\varepsilon,\delta}) &= \int_{\boldsymbol{Q}} f_{\omega,\eta}(x,\nabla u_0(x) + \left(V^\omega_{\eta,\delta}(\cdot,x) * \nabla\varphi_\varepsilon\right)(x) + \left(\varphi_\varepsilon * \upsilon_\delta(\cdot,\tau_{\frac{x}{\eta}}\omega)\right)(x))\,dx\\ &\to \int_{\boldsymbol{Q}}\int_\Omega f(x,\omega,\nabla u_0(x) + \upsilon_{\varepsilon,\delta}(x,\tilde{\omega}))\,d\mathcal{P}(\tilde{\omega})\,dx \qquad \text{as } \eta \to 0\,.\end{aligned}$$

On the other hand, due to Lemma 5.1, we have

$$\lim_{\eta\to 0} F_\eta(u_0 + V^\omega_{\eta,\varepsilon,\delta}) \ge \liminf_{\eta\to 0} F_\eta(u^\eta) \ge F_0(u_\omega,\upsilon_\omega) \ge F_0(u_0,\upsilon_0)\,.$$

Since f is continuous in ξ, we obtain from successively passing to the limits $\varepsilon \to 0$ and $\delta \to 0$ that

$$F_0(u_0,\upsilon_0) \ge F_0(u_\omega,\upsilon_\omega) \ge F_0(u_0,\upsilon_0)\,.$$

□

6 Homogenized system of equations

In this section, we are in the setting $\mu_\omega = \mathcal{L}$ for all ω. Hence, we frequently use the notations introduced in Remark 4.1.

The model equations of the problem (the microscopic problem) are

$$-\operatorname{div}_x \sigma_\eta(x,t) = b(x,t), \tag{49}$$

$$\sigma_\eta(x,t) = \tilde{\mathbb{C}}\left[\tau_{\frac{x}{\eta}}\tilde{\omega}\right](\varepsilon(\nabla_x u_\eta(x,t)) - Bz_\eta(x,t)), \tag{50}$$

$$\partial_t z_\eta(x,t) \in \tilde{g}\left(\tau_{\frac{x}{\eta}}\tilde{\omega}, B^T\sigma_\eta(x,t) - \tilde{L}_\eta[\tau_{\frac{x}{\eta}}\tilde{\omega}]z_\eta(x,t)\right), \tag{51}$$

together with the homogeneous Dirichlet boundary condition

$$u_\eta(x,t) = 0, \qquad (x,t) \in \partial\boldsymbol{Q} \times (0,\infty), \tag{52}$$

and the initial condition

$$z_\eta(x,0) = \tilde{z}^{(0)}(x, \tau_{\frac{x}{\eta}}\tilde{\omega}), \qquad x \in \boldsymbol{Q}. \tag{53}$$

Now, we state the main result on the stochastic homogenization of the weak solution $(u_\eta, \sigma_\eta, z_\eta)$ of problem (49) - (53).

Theorem 6.1. *Suppose that all assumptions of Theorem 3.1 are satisfied. Let $(u_\eta, \sigma_\eta, z_\eta)$ be a weak solution of the initial-boundary value problem (49) - (53). Then, there exist*

$$u_0 \in W^{1,q}(0,T_e; W_0^{1,q}(\boldsymbol{Q},\mathbb{R}^3)), \quad u_1 \in W^{1,q}(0,T_e; L^q(\boldsymbol{Q}; L^q_{\mathrm{pot}}(\Omega;\mathbb{R}^3))),$$

$$\sigma_0 \in H^1(0,T_e; L^2(\boldsymbol{Q}; L^2_{\mathrm{sol}}(\Omega;\mathcal{S}^3))), \quad z_0 \in W^{1,q}(0,T_e; L^q(\boldsymbol{Q}; L^q(\Omega;\mathbb{R}^N)))$$

such that (up to a subsequence)

$$u_\eta \overset{2s}{\rightharpoonup} u_0, \quad \nabla u_\eta \overset{2s}{\rightharpoonup} \nabla u_0 + u_1, \quad \sigma_\eta \overset{2s}{\rightharpoonup} \sigma_0 \text{ and } z_\eta \overset{2s}{\rightharpoonup} z_0. \tag{54}$$

The weak two-scale limit function $(u_0, u_1, \sigma_0, z_0)$ solves the following homogenized problem:

$$-\operatorname{div}_x\left(\int_\Omega \sigma_0(x,\omega,t)d\mathcal{P}\right) = b(x,t), \tag{55}$$

$$\begin{aligned}\sigma_0(x,\omega,t) = \tilde{\mathbb{C}}[\omega]\Big(&\varepsilon(u_1(x,\omega,t)) - Bz_0(x,\omega,t)\\ &+ \varepsilon(\nabla_x u_0(x,t))\Big),\end{aligned} \tag{56}$$

which hold for $(x,\omega,t) \in \boldsymbol{Q}\times\Omega\times[0,T_e]$, and with the boundary condition

$$u_0(x,t) = 0, \quad (x,t) \in \partial\boldsymbol{Q}\times(0,T_e). \tag{57}$$

Moreover, the following variational inequality holds ($\tilde{\mathbb{A}} := \tilde{\mathbb{C}}^{-1}$)

$$\begin{aligned}
&\int_{\boldsymbol{Q}}\int_\Omega \left\{\left(\tilde{\mathbb{A}}[\omega]\sigma(x,\omega,t),\sigma(x,\omega,t)\right) + \left(\tilde{L}[\omega]z(x,\omega,t), z(x,\omega,t)\right)\right\} d\mathcal{P}dx\\
&+ \int_0^t\int_{\boldsymbol{Q}}\int_\Omega f_{\tilde{g}}\left(\omega, B^T\sigma(x,\omega,\tau) - \tilde{L}[\omega]z(x,\omega,\tau), \partial_\tau z(x,\omega,\tau)\right) d\mathcal{P}dxd\tau\\
&\le \int_{\boldsymbol{Q}}\int_\Omega \left\{\left(\tilde{\mathbb{A}}[\omega]\sigma^{(0)}(x,\omega),\sigma^{(0)}(x,\omega)\right) + \left(\tilde{L}[\omega]\tilde{z}^{(0)}(x,\omega), \tilde{z}^{(0)}(x,\omega)\right)\right\} d\mathcal{P}dx\\
&+ \int_0^t\int_{\boldsymbol{Q}} (b(x,\tau), \partial_\tau u(x,\tau))\, dxd\tau,
\end{aligned} \tag{58}$$

where $(v^{(0)}, v_1) \in H^1(\boldsymbol{Q}, \mathbb{R}^3) \times L^2(\boldsymbol{Q}, L^2_{pot}(\Omega))$ *and* $\sigma^{(0)} \in L^2(\boldsymbol{Q}, L^2_{sol}(\Omega))$ *solve the linear elasticity problem*

$$\begin{aligned} -\operatorname{div}_x \sigma^{(0)}(x,\omega) &= b(x,0), &\quad (59)\\ \sigma^{(0)}(x,\omega) &= \tilde{\mathbb{A}}[\omega](\varepsilon(\nabla v^{(0)}(x) + v_1(x,\omega)) - B\tilde{z}^{(0)}(x,\omega)), &\quad (60)\\ v^{(0)}(x) &= 0, \qquad x \in \partial\boldsymbol{Q}. &\quad (61)\end{aligned}$$

The careful reading of the proof of Theorem 3.1 suggests the following result.

Proposition 6.1. *Suppose that all assumptions of Theorem 3.1 are satisfied. Then, the weak solution* $(u_\eta, \sigma_\eta, z_\eta)$ *of problem (49) - (53) (in the sense of Definition 3.1) fullfills the uniform estimates*

$$\{u_\eta\}_\eta \ \textit{is uniformly bounded in}\ W^{1,q}(0,T_e; L^q(\boldsymbol{Q};\mathbb{R}^3)),$$

$$\{\sigma_\eta\}_\eta \ \textit{is uniformly bounded in}\ L^\infty(0,T_e; L^2(\boldsymbol{Q};\mathcal{S}^3)),$$

$$\{z_\eta\}_\eta \ \textit{is uniformly bounded in}\ W^{1,q}(0,T_e; L^q(\boldsymbol{Q};\mathbb{R}^N)),$$

$$\{L^{1/2}z_\eta\}_\eta \ \textit{is uniformly bounded in}\ L^\infty(0,T_e; L^2(\boldsymbol{Q};\mathbb{R}^N)),$$

$$\{\Sigma_\eta\}_\eta \ \textit{is uniformly bounded in}\ L^p(0,T_e; L^p(\boldsymbol{Q};\mathbb{R}^N)).$$

The result of Proposition 6.1 plays an important role in the proof of Theorem 6.1 below.

Proof of Theorem 6.1

Proof. Proposition 6.1 provides the required uniform estimates for the solution of the microscopic problem (49) - (53). Therefore, due to Lemma 4.10 there exist functions u_0, u_1, σ_0 and z_0 with the prescribed regularities in Theorem 6.1 such that the convergence results in (54) hold. We note that (50) gives equation (56), namely

$$\sigma_0(x,\omega,t) = \tilde{\mathbb{C}}[\omega]\big(\varepsilon(\nabla_x u_0(x,t) + u_1(x,\omega,t)) - Bz_0(x,\omega,t)\big), \quad \text{a.e.} \tag{62}$$

Next, we test equation (49) with a function $\phi \in C_0^\infty(\boldsymbol{Q}, \mathbb{R}^3)$. Passing to the stochastic two-scale limit in the integral identity corresponding to (49) yields

$$\int_{\boldsymbol{Q}}\int_\Omega (\sigma_0(x,\omega,t), \varepsilon(\nabla_x\phi(x)))dx d\mathcal{P}(\omega) = \int_{\boldsymbol{Q}} (b(x,t),\phi(x))dx. \tag{63}$$

Now, we consider $\phi_{\tilde{\omega},\eta}(x,t) = \eta\phi(x,t)V_\eta^{\tilde{\omega}}(x)$, where $\phi \in C_0^\infty(\boldsymbol{Q}_{T_e}, \mathbb{R})$ and $\upsilon \in L^q_{\mathrm{pot}}(\Omega)$ with potential $V_\eta^{\tilde{\omega}}$ given by Lemma 4.3, as another test function in (49) and obtain

$$\begin{aligned} &\eta\int_0^{T_e}\int_{\boldsymbol{Q}} \big(\sigma_\eta(x,t), V_\eta^{\tilde{\omega}}(x)\otimes\nabla_x\phi(x,t)\big)dx\,dt\\ &+\int_0^{T_e}\int_{\boldsymbol{Q}} \big(\sigma_\eta(x,t), \phi(x,t)\varepsilon(\nabla_x \upsilon(\tau_{\frac{x}{\eta}}\tilde{\omega}))\big)dx\,dt \\ &= \int_0^{T_e}\int_{\boldsymbol{Q}} (b(x,t), \phi_\eta(x,t))dx\,dt. \end{aligned}\tag{64}$$

The stochastic two-scale limit in equation (64) yields

$$\int_0^{T_e}\int_{\boldsymbol{Q}}\int_{\Omega}\big(\sigma_0(x,\omega,t),\varepsilon(\nabla_\omega\upsilon(\omega))\big)\phi(x,t)d\mathcal{P}(\omega)dxdt=0. \tag{65}$$

Equation (65) implies that the integral identity

$$\int_{\Omega}\big(\sigma_0(x,\omega,t),\varepsilon(\nabla_\omega\upsilon(\omega))\big)d\mathcal{P}(\omega)=0 \tag{66}$$

holds for ever $\upsilon\in L^q_{\mathrm{pot}}(\Omega)$ for a.e. $(x,t)\in\boldsymbol{Q}\times(0,T_e)$. Integral equality (66) yields that $\sigma_0(x,\cdot,t)\in L^2_{\mathrm{sol}}(\Omega;\mathcal{S}^3)$ for a.e. $(x,t)\in\boldsymbol{Q}\times(0,T_e)$.

To pass to the stochastic two-scale limit in the inequality

$$\begin{aligned}
&\int_{\boldsymbol{Q}}\Big\{\Big(\tilde{\mathbb{A}}[\tau_{\frac{x}{\eta}}\tilde\omega]\sigma_\eta(x,t),\sigma_\eta(x,t)\Big)+\Big(\tilde L[\tau_{\frac{x}{\eta}}\tilde\omega]z_\eta(x,t),z_\eta(x,t)\Big)\Big\}\,dx\\
&+\int_0^t\int_{\boldsymbol{Q}}f_{\tilde g}\left(\tau_{\frac{x}{\eta}}\tilde\omega,B^T\sigma_\eta(x,\tau)-\tilde L[\tau_{\frac{x}{\eta}}\tilde\omega]z_\eta(x,\tau),\partial_\tau z_\eta(x,\tau)\right)dx\,d\tau\\
&\le\int_{\boldsymbol{Q}}\Big(\tilde{\mathbb{A}}[\tau_{\frac{x}{\eta}}\tilde\omega]\sigma_\eta^{(0)}(x),\sigma_\eta^{(0)}(x)\Big)\,dx+\int_0^t\int_{\boldsymbol{Q}}(b(x,\tau),\partial_\tau u_\eta(x,\tau))\,dx\,d\tau\\
&+\int_{\boldsymbol{Q}}\Big(\tilde L[\tau_{\frac{x}{\eta}}\tilde\omega]\tilde z^{(0)}(x,\tau_{\frac{x}{\eta}}\tilde\omega),\tilde z^{(0)}(x,\tau_{\frac{x}{\eta}}\tilde\omega)\Big)\,dx,
\end{aligned}$$

we use the results of Lemma 4.9 and Lemma 5.1 and obtain

$$\begin{aligned}
&\int_{\boldsymbol{Q}}\int_{\Omega}\Big\{\Big(\tilde{\mathbb{A}}[\omega]\sigma(x,\omega,t),\sigma(x,\omega,t)\Big)+\Big(\tilde L[\omega]z(x,\omega,t),z(x,\omega,t)\Big)\Big\}\,d\mathcal{P}(\omega)dx\\
&+\int_0^t\int_{\boldsymbol{Q}}\int_{\Omega}f_{\tilde g}\left(\omega,B^T\sigma(x,\omega,\tau)-\tilde L[\omega]z(x,\omega,\tau),\partial_\tau z(x,\omega,\tau)\right)d\mathcal{P}(\omega)dxd\tau\\
&\le\int_{\boldsymbol{Q}}\int_{\Omega}\Big\{\Big(\tilde{\mathbb{A}}[\omega]\sigma^{(0)}(x,\omega),\sigma^{(0)}(x,\omega)\Big)+\Big(\tilde L[\omega]\tilde z^{(0)}(x,\omega),\tilde z^{(0)}(x,\omega)\Big)\Big\}\,d\mathcal{P}(\omega)dx\\
&+\int_0^t\int_{\boldsymbol{Q}}(b(x,\tau),\partial_\tau u(x,\tau))\,dxd\tau,
\end{aligned}$$

where $\sigma^{(0)}\in L^2(\boldsymbol{Q},L^2(\Omega))$ solves the following linear elasticity problem (59)–(61), which is obtained by the passage to the stochastic two-scale limit in equations (12) - (14). Here, $v^{(0)}\in H^1(\boldsymbol{Q},\mathbb{R}^3)$ and $v_1\in L^2(\boldsymbol{Q},L^2_{\mathrm{pot}}(\Omega))$.

Therefore, we conclude that the limit function (u_0,u_1,σ_0,z_0) satisfies the homogenized problem (55) - (57) and the variational inequality (58). $\square$

Appendix: Rothe's approximation functions

Here we recall the definition of Rothe's approximation functions. For any family $\{\xi_m^n\}_{n=0,\dots,2m}$ of functions in a reflexive Banach space X and for $h=T_e/m$, we define *the piecewise affine interpolant* $\xi_m\in C([0,T_e],X)$ by

$$\xi_m(t):=\left(\frac{t}{h}-(n-1)\right)\xi_m^n+\left(n-\frac{t}{h}\right)\xi_m^{n-1}\quad\text{for }(n-1)h\le t\le nh, \tag{67}$$

and *the piecewise constant interpolant* $\bar{\xi}_m \in L^\infty(0, T_e; X)$ by

$$\bar{\xi}_m(t) := \xi_m^n \text{ for } (n-1)h < t \le nh,\ n = 1, ..., 2^m, \text{ and } \bar{\xi}_m(0) := \xi_m^0. \tag{68}$$

For the further analysis we recall the following property of $\bar{\xi}_m$ and ξ_m:

$$\|\xi_m\|_{L^p(0,T_e;X)} \le \|\bar{\xi}_m\|_{L^p(-h,T_e;X)} \le \left(h\|\xi_m^0\|_X^p + \|\bar{\xi}_m\|_{L^p(0,T_e;X)}^p\right)^{1/p}, \tag{69}$$

where $\bar{\xi}_m$ is formally extended to $t \le 0$ by ξ_m^0 and $1 \le p \le \infty$ (see [25]).

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