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Linearized elasticity as Mosco-limit of finite elasticity
in the presence of cracks

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Abstract

The small-deformation limit of finite elasticity is considered in presence of a given crack. The rescaled finite energies with the constraint of global injectivity are shown to Γ -converge to the linearized elastic energy with a local constraint of non-interpenetration along the crack.

1 Introduction

In [DNP02] Dal Maso, Negri and Percivale showed that finite-strain elasticity Γ -converges to small-strain linearized elasticity under the assumptions of small loadings. Later, this result was extended to different settings, e.g. to multi-well energies by Schmidt [Sch08], to materials with residual stresses by Paroni and Tomassetti [PaT09, PaT11] or to evolutionary problems like plasticity by Mielke and Stefanelli [MiS13] or crack propagation by Negri and Zanini [NeZ14]. In this contribution we discuss an extension of the results in [DNP02] to a setting where the reference domain has a crack Γ_{cr} of a certain class including cracks with kinks, see Section 2 for details.

The presence of the crack destroys the Lipschitz property of the cracked domain $\Omega_{\text{cr}} := \Omega \setminus \Gamma_{\text{cr}}$ and therefore crucial tools, such as the well-known rigidity estimate from [FJM02], have to be adapted to the setting of cracked domains, see Proposition 3.2. More importantly, the setting of domains with cracks requires to introduce an additional constraint of global injectivity of the deformations $y : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$. A crucial step for the small-deformation Γ -limit is to show that this particular *global injectivity condition* leads to a *local non-interpenetration condition* along the crack $\Gamma_{\text{cr}} \subset \Omega$.

In [CiN87] Ciarlet and Nečas proposed the condition $\int_{\Omega} \det \nabla y(x) \, dx \leq \text{vol}(y(\Omega))$, where $\text{vol}(A)$ denotes the d -dimensional volume. This conditions had been used in various applications, e.g. by Giacomini and Ponsiglione [GiP08] in the SBV-theory for brittle materials or by Mariano and Modica [MaM09] in the theory of weak diffeomorphisms to describe deformations in “complex bodies”. In [GMS98, Prop. 3.2.1], Giaquinta, Modica, and Souček showed that the above condition is equivalent to the condition

$$\int_{\Omega} \varphi(y(x)) |\det \nabla y(x)| \, dx \leq \int_{\mathbb{R}^d} \varphi(z) \, dz \quad \text{for all } \varphi \in C_c(\mathbb{R}^d, \mathbb{R}) \text{ with } \varphi \geq 0. \quad (1.1)$$

which we will simply call GMS condition.

This latter condition turns out to be an appropriate formulation for our purpose. In particular, assuming that $y_{\varepsilon} : \Omega \rightarrow \mathbb{R}^d$ satisfy (1.1) we will deduce that a weak limit $u : \Omega \rightarrow \mathbb{R}^d$ for $\varepsilon \rightarrow 0$ of the rescaled displacements

$$u_{\varepsilon} : x \mapsto \frac{1}{\varepsilon} (y_{\varepsilon}(x) - x)$$

satisfies the following local jump condition on the crack:

$$0 \leq \llbracket u(x) \rrbracket_{\Gamma_{\text{cr}}} := (u^+(x) - u^-(x)) \cdot \nu(x), \quad (1.2)$$

where u^+ and u^- are the traces of u on Γ_{cr} from the upper and the lower side, respectively, see Theorem 4.1.

Our analysis is based on elastic energies of integral type, i.e. $\mathcal{E}(y) = \int W(x, \nabla y(x)) \, dx$. Apart from the classical assumptions in elasticity of W being orientation preserving and satisfying the lower bound stated below, the most crucial assumption of the following is

that about the quadratic behavior of W near the identity matrix $F = I$. With $\text{GL}_+(d) := \{A \in \mathbb{R}^{d \times d} \mid \det A > 0\}$ and $\text{SO}(d) := \{R \in \mathbb{R}^{d \times d} \mid R^\top R = I, \det(R) = 1\}$ pose the following conditions on the stored-energy density $W : \mathbb{R}^{d \times d} \rightarrow [0, \infty]$:

$$\forall F \in \mathbb{R}^{d \times d} \setminus \text{GL}_+(d): W(F) = \infty; \quad (1.3a)$$

$$\forall F \in \mathbb{R}^{d \times d}, R \in \text{SO}(d): W(RF) = W(F); \quad (1.3b)$$

$$\exists p > d, c_W, C_W > 0 \forall F \in \mathbb{R}^{d \times d}: W(F) \geq c_W \max \{ \text{dist}(F, \text{SO}(d))^2, |F|^p - C_W \}; \quad (1.3c)$$

$$\left. \begin{aligned} &\exists \mathbb{C} \geq 0 \text{ with } \mathbb{C}^\top = \mathbb{C} \forall \delta > 0 \exists r_\delta > 0 \forall A \in B_{r_\delta}(0) \subset \mathbb{R}^{d \times d}: \\ &\left| W(I+A) - \frac{1}{2} \langle A, \mathbb{C}A \rangle \right| \leq \delta \langle A, \mathbb{C}A \rangle. \end{aligned} \right\} \quad (1.3d)$$

In particular, condition (1.3d) states that $A \mapsto \frac{1}{2} \langle A, \mathbb{C}A \rangle$ is the second order Taylor expansion of W around id . It implies $W(id) = 0$, $\partial_F W(id) = 0$ and $\partial_F^2 W(id) = \mathbb{C}$, where the second part yields that the material is stress free and, if W would be C^2 in a neighborhood of id , from the third part the assumed symmetry of \mathbb{C} could be deduced. Moreover the semi norm given by $|A|_{\mathbb{C}}^2 := \frac{1}{2} \langle A, \mathbb{C}A \rangle$ is equivalent to the norm $A \mapsto |A^{\text{sym}}|$ as on the one hand the frame indifference (1.3b) implies $\mathbb{C}A = \mathbb{C}A^{\text{sym}}$ for every $A \in \mathbb{R}^{d \times d}$ and on the other hand the first part of assumption (1.3c) being $W(F) \geq c_W \text{dist}^2(F, \text{SO}(d))$ and assumption (1.3c) imply $c_W |A^{\text{sym}}| \leq \frac{1}{2} |A|_{\mathbb{C}}^2$ (see [MiS13] for the details).

To take the small-deformation limit one considers *small deformations* of the form $y_\varepsilon = \text{id} + \varepsilon u_\varepsilon$ for small parameters $\varepsilon > 0$, where u_ε remains bounded in a suitable function space. As the above discussed quadratic behavior of W around I suggests, the scaling of $W(\nabla y_\varepsilon) = W(I + \varepsilon \nabla u)$ by $\frac{1}{\varepsilon^2}$ will be appropriate to obtain linearized elasticity in the bulk, namely

$$\overline{W}_\varepsilon(A) := \frac{1}{\varepsilon^2} W(I + \varepsilon \cdot) \xrightarrow{M} \frac{1}{2} |\cdot|_{\mathbb{C}}^2. \quad (1.4)$$

The correspondingly rescaled elastic energies (cf. [DNP02]) without GMS condition reads

$$\tilde{\mathcal{F}}_\varepsilon(u) := \int_\Omega \frac{1}{\varepsilon} W(x, I + \varepsilon \nabla u(x)) \, dx$$

while we are interested in the elastic energy with the GMS condition (1.1), namely

$$\mathcal{F}_\varepsilon : \mathbf{H}_{g, \text{Dir}}^1 \rightarrow \mathbb{R} \cup \{\infty\}, \quad u \mapsto \begin{cases} \tilde{\mathcal{F}}_\varepsilon(u) & \text{if } \text{id} + \varepsilon u \text{ satisfies (1.1),} \\ \infty & \text{else,} \end{cases} \quad (1.5)$$

where Γ_{Dir} and $\mathbf{H}_{g, \text{Dir}}^1$ are specified in (2.7) such that $u \in \mathbf{H}_{g, \text{Dir}}^1$ implies $(u-g)|_{\Gamma_{\text{Dir}}} = 0$. The functional $\tilde{\mathcal{F}}_\varepsilon$ is the one considered in [DNP02], and it is shown to Γ -converge to

$$\tilde{\mathcal{F}}(u) = \int_{\Omega_{\text{cr}}} \frac{1}{2} \langle e(u), \mathbb{C}e(u) \rangle \, dx, \quad \text{where } e(u) := (\nabla u)^{\text{sym}} := \frac{1}{2} (\nabla u + (\nabla u)^\top).$$

The main result of this work is the Mosco convergence (i.e. Γ -convergence with respect to both weak and strong \mathbf{H}^1 -topology) of \mathcal{F}_ε to the functional \mathcal{F} , which is obtained from $\tilde{\mathcal{F}}$ by adding the local non-interpenetration condition (1.2), namely

$$\mathcal{F} : \mathbf{H}_{g, \text{Dir}}^1 \rightarrow \mathbb{R} \cup \{\infty\}, \quad u \mapsto \begin{cases} \tilde{\mathcal{F}}(u) & \text{if } u \text{ satisfies (1.2),} \\ \infty & \text{else.} \end{cases} \quad (1.6)$$

The equi-coercivity of the functionals \mathcal{F}_ε is directly implied by the equi-coercivity of $\widetilde{\mathcal{F}}_\varepsilon$, once the rigidity result of [FJM02] has been generalized to our class of crack domains $\Omega_{\text{cr}} := \Omega \setminus \Gamma_{\text{cr}}$ as specified in Section 2. Thus, the coercivity (1.3c) and the energy bound $\widetilde{\mathcal{F}}_\varepsilon(u_\varepsilon) \leq C < \infty$ imply $\|u_\varepsilon\|_{H^1} \leq C$ and $\|\varepsilon u_\varepsilon\|_{L^p} \leq C$, which gives $\|\varepsilon u_\varepsilon\|_{L^\infty} \leq C\varepsilon^r$ for some $r > 0$, see Proposition 3.6. Our main Theorem 2.1 states the following Γ -convergence:

$$\begin{aligned} \mathcal{F}_\varepsilon &\xrightarrow{\text{M}} \mathcal{F} \text{ in } H_{g,\text{Dir}}^1, \text{ i.e.} \\ \forall u_\varepsilon \rightharpoonup u \text{ in } H_{g,\text{Dir}}^1 : &\quad \mathcal{F}(u) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon), \\ \forall \tilde{u} \in H_{g,\text{Dir}}^1 \exists \tilde{u}_\varepsilon \rightarrow \tilde{u} : &\quad \mathcal{F}(\tilde{u}) \geq \limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(\tilde{u}_\varepsilon). \end{aligned} \tag{1.7}$$

In Section 4 we provide the liminf estimate (in the weak topology of $H^1(\Omega; \mathbb{R}^d)$), where because of the result in [DNP02] it remains to establish the local non-interpenetration condition (1.2) as a limit of the global condition (1.1), which is not too difficult, see Theorem 4.1. The construction of recovery sequences for the limsup estimate (now in the strong topology of H^1) is more delicate, as in general (even for very smooth) displacements $u \in H^1(\Omega_{\text{cr}}; \mathbb{R}^d)$ satisfying the local non-interpenetration condition (1.2) the associated close-to-identity deformation $y_\varepsilon = \text{id} + \varepsilon u$ does not satisfy the GMS condition (1.1) for global injectivity, see Example 5.1. On the one hand, our construction of recovery sequences invokes an approximation of functions in $H^1(\Omega_{\text{cr}}; \mathbb{R}^d)$ satisfying (1.2) by functions in $W^{1,\infty}(\Omega_{\text{cr}}; \mathbb{R}^d)$ still satisfying (1.2), which is reminiscent to the density results in Proposition 5.4 for under convex constraints derived in [HiR15, HRR16]. On the other hand, we have to use an artificial forcing apart of the two crack sides to be able to guarantee (1.1), see Proposition 5.2.

In the present work, we are only able to treat the static situation as in [DNP02], which is in contrast to [MiS13, NeZ14] where the passage from finite-strain to linearized elasticity is handled in the rate-independent setting. However, the treatment of the contact problem in finite-strain seems still to be too difficult. In [LaT11] the quasistatic evolution of fracture in linearized elasticity is developed, where cracks may occur along arbitrary paths that have $C^{1,\text{Lip}}$ regularity, which is the same regularity needed for our analysis.

2 Transformation and main result

Throughout this paper we consider a reference configuration with a Lipschitz domain Ω and a given crack Γ_{cr} on which the displacements $u \in H^1(\Omega_{\text{cr}}; \mathbb{R}^d)$ may have jumps. We expect that our theory works for general domains Ω and cracks Γ_{cr} that are piecewise $C^{1,\text{Lip}}$, if all the edges and corners are non-degenerate. However, to avoid an overload of technicalities we concentrate on the essential difficulties that arise by (i) smooth pieces of the crack, (ii) by the edge of the crack, (iii) by kinks inside a crack, and (iv) through the intersection of the crack with the boundary $\partial\Omega$.

Thus, we define a model domain $\widehat{\Omega}$ with a model crack $\widehat{\Gamma}_{\text{cr}}$ that displays all these difficulties and then consider all domains Ω with cracks Γ_{cr} that are obtained by a bi-Lipschitz mapping $T : \Omega \rightarrow \widehat{\Omega}$ such that $\widehat{\Gamma}_{\text{cr}} = T(\Gamma_{\text{cr}})$.

Conditions on the model pair $(\widehat{\Omega}, \widehat{\Gamma}_{\text{cr}})$. Our conditions essentially say that $\widehat{\Omega}_{\text{cr}} = \widehat{\Omega} \setminus \widehat{\Gamma}_{\text{cr}}$ can be written as the union of two Lipschitz domains A_+ and A_- that have a nontrivial intersection $A_+ \cap A_-$, which is a Lipschitz set again, and that define $\widehat{\Gamma}_{\text{cr}}$ as the

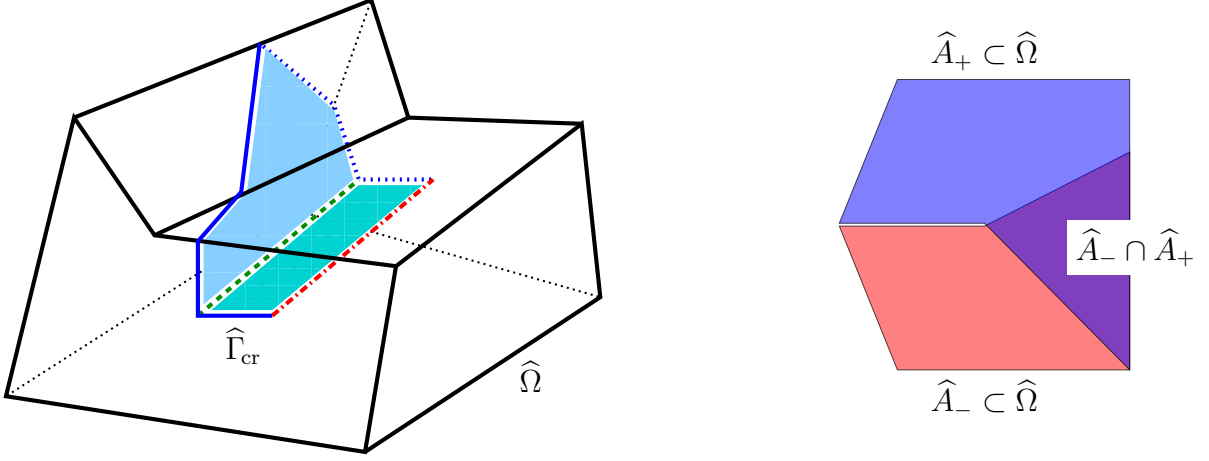


Figure 1: Left: Crack $\widehat{\Gamma}_{\text{cr}}$ (areas shaded in light blue) inside the domain $\widehat{\Omega}$, the crack edge $\widehat{\Gamma}_{\text{edge}}$ is red, the crack kink $\widehat{\Gamma}_{\text{kink}}$ is green lying between the two shaded areas, and $\partial\Omega \cap \widehat{\Gamma}_{\text{cr}}$ is blue. Right: Decomposition of a planar $\widehat{\Omega}$ into \widehat{A}_+ and \widehat{A}_- according to (2.1c).

intersection of the boundaries ∂A_+ and ∂A_- . Using the normal vector $\widehat{\nu} \in \mathbb{S}^{d-1}$ of the crack $\widehat{\Gamma}_{\text{cr}}$ and the outward normal vector $\widehat{n} \in \mathbb{S}^{d-1}$ on $\partial\Omega$, our precise assumptions are the following.

$$\widehat{\Omega} \subset \mathbb{R}^d \text{ is a bounded Lipschitz domain;} \quad (2.1a)$$

$$\widehat{\Gamma}_{\text{cr}} := \left(([0, 1] \times \{0\} \times \mathbb{R}^{d-2}) \cup (\{0\} \times [0, \infty] \times \mathbb{R}^{d-2}) \right) \cap \widehat{\Omega}; \quad (2.1b)$$

$$\left. \begin{array}{l} \text{there exist open, Lipschitz sets } \widehat{A}_+, \widehat{A}_- \subset \widehat{\Omega} \text{ such that } \widehat{\Omega}_{\text{cr}} \subset \widehat{A}_+ \cup \widehat{A}_- \\ \text{and both } \widehat{A}_+ \cap \widehat{A}_- \text{ and } \widehat{A}_+ \setminus \widehat{A}_- \text{ have Lipschitz boundary;} \end{array} \right\} \quad (2.1c)$$

$$\left. \begin{array}{l} \text{Transversality of } \widehat{\Gamma}_{\text{cr}}: \partial\Omega \text{ and } \widehat{\Gamma}_{\text{cr}} \text{ intersect transversally, i.e.} \\ \exists \delta > 0: \quad |\widehat{n}(\widehat{x}) \cdot \widehat{\nu}(\widehat{x})| \leq 1 - \delta \text{ for } \mathcal{H}^{d-2}\text{-a.e. } \widehat{x} \in \partial\Omega \cap \widehat{\Gamma}_{\text{cr}}. \end{array} \right\} \quad (2.1d)$$

$$\left. \begin{array}{l} \text{Transversality of } \widehat{\Gamma}_{\text{edge}} := \{(0, 1)\} \times \mathbb{R}^{d-2} \cap \widehat{\Omega}: \\ \partial\Omega \text{ and } \widehat{\Gamma}_{\text{edge}} \text{ intersect transversally, i.e.} \\ \exists \delta > 0: \quad |\widehat{n}(\widehat{x}) \cdot e_2| \leq 1 - \delta \text{ for } \mathcal{H}^{d-3}\text{-a.e. } \widehat{x} \in \partial\Omega \cap \widehat{\Gamma}_{\text{edge}}. \end{array} \right\} \quad (2.1e)$$

The conditions on $(\widehat{\Omega}, \widehat{\Gamma}_{\text{cr}})$ are illustrated in Figure 1. The model crack $\widehat{\Gamma}_{\text{cr}}$ defined in (2.1b) contains two special subsets, (i) the crack edge $\widehat{\Gamma}_{\text{edge}} := (\{(1, 0)\} \times \mathbb{R}^{d-2}) \cap \widehat{\Omega}$ and (ii) the crack kink $\widehat{\Gamma}_{\text{kink}} := (\{(0, 0)\} \times \mathbb{R}^{d-2}) \cap \widehat{\Omega}$. For all other points we have the well-defined crack normal $\nu(\widehat{x}) = (1, 0, \dots, 0)^\top \in \mathbb{R}^d$ or $(0, 1, 0, \dots, 0)^\top$, respectively. Condition (2.1d) asked that the crack $\widehat{\Gamma}_{\text{cr}}$ does not meet the boundary $\partial\widehat{\Omega}$ tangentially.

The decomposition in (2.1c) will be used for two purposes, namely (i) for the derivation of a rigidity result for the cracked domain and (ii) to construct enough good test functions for deriving the jump condition in Theorem 4.1.

The domains Ω and the cracks $\Gamma \subset \Omega$ for which we will formulate our theory are now obtained by a bi-Lipschitzian mapping $T : \Omega \rightarrow \widehat{\Omega}$ that are additionally $C^{1, \text{Lip}} = W^{2, \infty}$. Thus, the conditions on the pair (Ω, Γ) or the cracked domain $\Omega_{\text{cr}} := \Omega \setminus \Gamma$ are the

following:

Assumptions on (Ω, Γ) :

$$\begin{aligned} (\widehat{\Omega}, \widehat{\Gamma}_{\text{cr}}) \text{ satisfy (2.1) and there exists a bi-Lipschitz map } T : \Omega \rightarrow \widehat{\Omega} \quad (2.2) \\ \text{such that } \widehat{\Omega} = T(\Omega), \quad \widehat{\Gamma}_{\text{cr}} = T(\Gamma_{\text{cr}}), \quad \text{and } T \in C^{1,\text{Lip}}(\Omega; \mathbb{R}^d). \end{aligned}$$

Note that the true crack Γ_{cr} will be piecewise $C^{1,\text{Lip}}$, since we allowed for a kink in $\widehat{\Gamma}_{\text{cr}}$.

As a first consequence of this assumption we see that Ω_{cr} can also be decomposed similarly to $\widehat{\Omega}_{\text{cr}}$ in (2.1c). Defining $A_{\pm} := T^{-1}(A_{\pm})$ with \widehat{A}_{\pm} from (2.1c) we have that

$$\left. \begin{aligned} A_+, A_- \subset \Omega \text{ are Lipschitz domains with } A_+ \cup A_- = \Omega_{\text{cr}} \\ \text{such that } A_+ \cap A_- \text{ and } A_- \setminus A_+ \text{ are also Lipschitz domains.} \end{aligned} \right\} \quad (2.3)$$

This overlapping cover of Ω_{cr} in assumption (2.3) is used for two different purposes. First, it allows us to extend the rigidity result from Lipschitz domains to our crack domains Ω_{cr} , see Corollary 3.3. Second, it allows us to derive the jump condition (1.2) in Theorem 4.1 by applying the divergence theorem on a disjoint cover given by A_+ and $A_- \setminus A_+$.

The assumption that T is a bi-Lipschitz mapping means that it is bijective and that $T : \Omega \rightarrow \widehat{\Omega}$ and $T^{-1} : \widehat{\Omega} \rightarrow \Omega$ are Lipschitz. The additional condition $T \in C^{1,\text{Lip}}(\Omega; \mathbb{R}^d)$ then implies $T^{-1} \in C^{1,\text{Lip}}(\widehat{\Omega}; \mathbb{R}^d)$. Note that we may assume that T and T^{-1} are also $C^{1,\text{Lip}}$ on open neighborhoods of Ω and $\widehat{\Omega}$, respectively. Thus, a near-identity diffeomorphism $y : \Omega \rightarrow \mathbb{R}^d$ (i.e. $y - \text{id}$ is small in L^∞) can be transformed to a near-identity mapping on $\widehat{\Omega}$ via the transform

$$\widehat{y}(\widehat{x}) = T(y(T^{-1}(\widehat{x}))) \quad \text{or} \quad y(x) = T^{-1}(\widehat{y}(T(x))).$$

In particular, for $\widehat{y}_{\varepsilon, \widehat{u}} := \text{id} + \varepsilon \widehat{u} : \widehat{\Omega} \rightarrow \mathbb{R}^d$ we find the expansion

$$y_\varepsilon(x) = T^{-1}(\widehat{y}_{\varepsilon, \widehat{u}}(T(x))) = x + \varepsilon \nabla T(x)^{-1} \widehat{u}(T(x)) + O(\varepsilon^2),$$

The mapping from \widehat{u} to the corresponding term in y_ε is called the *Piola transform* P_T for *vector fields*, cf. also [KMZ08, KnS12]. Under the assumption (2.2) the mapping

$$P_T : \begin{cases} H^1(\widehat{\Omega}) & \rightarrow & H^1(\Omega) \\ \widehat{u} & \mapsto & u : x \mapsto \nabla T(x)^{-1} \widehat{u}(T(x)) \end{cases} \quad (2.4)$$

is a bijective bounded linear mapping as well as its inverse $P_{T^{-1}} : H^1(\Omega) \rightarrow H^1(\widehat{\Omega})$.

The Piola transform is especially useful for us, as it also transforms the local non-interpenetration condition in the correct way, see e.g. [KMZ08, KnS12]. If $\widehat{\nu}(\widehat{x})$ is the normal vector at $\widehat{x} \in \widehat{\Gamma}_{\text{cr}}$, then it is related to the normal vector $\nu(x)$ at $x = T^{-1}(\widehat{x}) \in \Gamma$ via

$$\nu(x) = \frac{1}{|\nabla T(x)^\top \widehat{\nu}(T(x))|} \nabla T(x)^\top \widehat{\nu}(T(x)) \quad \text{or} \quad \widehat{\nu}(T(x)) = \frac{1}{|\nabla T(x)^{-\top} \nu(x)|} \nabla T(x)^{-\top} \nu(x).$$

Thus, for the jump over the crack we obtain the relation

$$\begin{aligned} \llbracket u \rrbracket_{\Gamma_{\text{cr}}}(x) &= (u^+(x) - u^-(x)) \cdot \nu(x) \\ &= (\nabla T(x)^{-1} \widehat{u}^+(T(x)) - \nabla T(x)^{-1} \widehat{u}^-(T(x))) \cdot \nu(x) \\ &= (\widehat{u}^+(T(x)) - \widehat{u}^-(T(x))) \cdot \nabla T(x)^{-\top} \nu(x) \\ &= |\nabla T(x)^{-\top} \nu(x)| \llbracket \widehat{u} \rrbracket_{\widehat{\Gamma}_{\text{cr}}}(T(x)). \end{aligned} \quad (2.5)$$

Thus, the jumps translate correctly if we take into account the prefactor that associates with the stretching of surface elements.

For future use of the above assumptions on $(\Omega, \Gamma_{\text{cr}})$ we derive the following well-known consequences, which will be employed below in our theory of Γ -convergence:

$$\left. \begin{array}{l} \Omega \text{ Lipschitz domain, and for all } x_0 \in \partial\Omega \text{ there exists an open} \\ \text{neighborhood } U \subset \mathbb{R}^d \text{ of } x_0 \text{ and a bi-Lipschitz } \Psi_{x_0}: U \rightarrow V \subset \mathbb{R}^d \\ \text{such that } U \cap \Omega \subset \Psi_{x_0}^{-1}(\{v_d > 0\}) \text{ and } U \cap \partial\Omega \subset \Psi_{x_0}^{-1}(\{v_d = 0\}); \end{array} \right\} \quad (2.6a)$$

$$\left. \begin{array}{l} \textbf{transversality of } \Gamma_{\text{cr}} \textbf{ and } \partial\Omega: \text{ for all } x_0 \in \Gamma_{\text{cr}} \cap \partial\Omega \text{ there exist} \\ \widehat{\eta}_{x_0} \in \mathbb{S}^{d-1}, \kappa > 0, \text{ and } U \text{ and } \Psi_{x_0} \text{ as in (2.6a), such that} \\ \text{(i) } \nabla \Psi_{x_0}(x) e_d \cdot \nabla T(x)^{-1} \widehat{\eta}_{x_0} \geq \kappa \quad \text{for a.a. } x \in U \cap \Omega, \\ \text{(ii) } \widehat{\eta}_{x_0} \cdot \nabla T(x)^{-\top} \nu(x) = 0 \quad \text{for a.a. } x \in U \cap \Gamma_{\text{cr}}, \\ \text{(iii) } \widehat{\eta}_{x_0} \in \{(0, 0)\} \times \mathbb{R}^{d-2} \quad \text{if } x_0 \in \partial\Omega \cap \Gamma_{\text{edge}}, \end{array} \right\} \quad (2.6b)$$

where $\Gamma_{\text{edge}} := T^{-1}(\widehat{\Gamma}_{\text{edge}})$ with $\widehat{\Gamma}_{\text{edge}} := \{(1, 0)\} \times \mathbb{R}^{d-2}$.

Note that condition (ii) in (2.6b) simply means $\widehat{\eta}_{x_0} \cdot \widehat{\nu}(T(x)) = 0$, where $\widehat{\nu}$ takes one of the values $e_1, e_2 \in \mathbb{R}^d$, or even both values if $T(x_0) \in \widehat{\Gamma}_{\text{kink}}$. Hence, this condition follows directly from (2.1d), but we will use the form as given in (2.6b) for a full neighborhood. Similarly, condition (iii) in (2.6b) is a direct consequence of (2.1e).

Note that the angle of $\frac{\pi}{2}$ at the kink of $\widehat{\Gamma}_{\text{cr}}$ is not essential and will be varied by the mapping $\nabla T^{-1}(y)$ for $y \in \widehat{\Gamma}_{\text{cr}} \cap \widehat{\Omega}_{\text{cr}}$. Furthermore the choice of $\widehat{\Gamma}_{\text{cr}} = T(\Gamma_{\text{cr}}) \subset \widehat{\Omega}$ in (2.2) is just an example as easy as possible while still showing the crucial difficulties. We expect that the theory works for any $C^{1,\text{Lip}}$ -surface which is piecewise C^2 . The proofs and constructions are made with the intention to be adaptable to other special situations.

The transversality condition (2.6b) requires the crack Γ_{cr} and the boundary $\partial\Omega$ to intersect transversally. Technically it enables us to use the following implicit function theorem for Lipschitz maps to conclude $\partial\widehat{\Omega}$ being a graph in the direction η , which is parallel to $\widehat{\Gamma}_{\text{cr}}$ in a whole open neighborhood of $T(x_0)$. You can interpret this graphically when having in mind the fact, that normal vectors transform by the cofactor of the gradient. Then the first equation of (2.6b) can be read as the vector field η_{x_0} , which is constant on the flat configuration $\widehat{\Omega} \setminus \widehat{\Gamma}_{\text{cr}}$ having an angle bounded away from $\frac{\pi}{2}$ to the normal on the boundary, which is given by $e_d := (0, \dots, 0, 1)^\top$ transferred by the cofactor of $\Psi_{x_0}^{-1}$ and T . The last requirement specifies that for $x_0 \in \Gamma_{\text{cr}}$ the vector η_{x_0} is tangential to $\widehat{\Gamma}_{\text{cr}}$.

To collect all the assumptions we now specify the boundary conditions in terms of the part $\Gamma_{\text{Dir}} \subset \partial\Omega$, where the Dirichlet boundary conditions $(u-g)|_{\Gamma_{\text{Dir}}} = 0$ are imposed.

$$\begin{aligned} \overline{\Gamma}_{\text{Dir}} \cap \overline{\Gamma}_{\text{cr}} &= \emptyset, \quad \int_{\Gamma_{\text{Dir}}} 1 \, d\mathcal{H}^{d-1} > 0, \quad g \in W^{1,\infty}(\Omega; \mathbb{R}^d) \\ H_{g,\text{Dir}}^1 &:= \text{clos}_{H^1(\Omega_{\text{cr}})} \left(\{ u \in W^{1,\infty}(\Omega_{\text{cr}}; \mathbb{R}^d) \mid (u-g)|_{\Gamma_{\text{Dir}}} = 0 \} \right). \end{aligned} \quad (2.7)$$

Theorem 2.1 (Mosco convergence $\mathcal{F}_\varepsilon \xrightarrow{M} \mathcal{F}$) *Let the assumptions (1.3), (2.2), and (2.7) be satisfied and \mathcal{F}_ε defined in (1.5) Mosco-converges to \mathcal{F} defined in (1.6) in $H^1(\Omega_{\text{cr}}; \mathbb{R}^d)$.*

The proof of this result is the content of the following sections. In particular, the liminf estimate is established in Proposition 4.3, and the limsup estimate in Theorem 5.5.

The following result is a weak version of the implicit function theorem that will be needed to represent the boundary $\partial\Omega$ near a point $x_0 \in \partial\Omega \cap \Gamma_{\text{cr}}$, see Corollary 2.3.

Theorem 2.2 (Special version of Implicit Function Theorem) *Let $U_m \subset \mathbb{R}^m$, $U_n \subset \mathbb{R}^n$ be open sets, $a \in U_m$, $b \in U_n$ and $F : U_m \times U_n \rightarrow \mathbb{R}^n$ be a Lipschitz map with $F(a, b) = 0$. Suppose there exists a constant $K > 0$ such that for all $x \in U_m$ and $y_1, y_2 \in U_n$ it holds*

$$|F(x, y_1) - F(x, y_2)| \geq K|y_1 - y_2|. \quad (2.8)$$

Then there exists an open neighborhood V_m of a , $V_m \subset U_m$ and a Lipschitz map $\varphi : V_m \rightarrow \mathbb{R}^n$ such that $\varphi(a) = b$ and

$$F^{-1}(0) = \{(x, \varphi(x)) | x \in V_m\}.$$

Proof. We will sketch the proof briefly.

By (2.8), which is a Lipschitz analog of the invertibility of $\nabla_y F$ in the differentiable version of the inverse function theorem, the map $f : U_m \times U_n \supset \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n}$, $(x, y) \mapsto (x, \delta F(x, y))$ is bi-Lipschitz for $0 < \delta < \|\nabla F\|_{L^\infty}^{-1}$. In particular f is continuous, injective and maps an open subset of \mathbb{R}^{m+n} to \mathbb{R}^{m+n} , thus by Brouwer's invariance of domain theorem f is an open map, i.e. $f(U_m \times U_n)$ is open in \mathbb{R}^{m+n} and f^{-1} is continuous. Consider the embedding $e_m : \mathbb{R}^m \rightarrow \mathbb{R}^{m+n}$, $x \mapsto (x, 0)$ and the projection $p_n : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $(x, y) \mapsto y$. Both e_m and p_n are Lipschitz continuous, thus $\varphi := p_n \circ f^{-1} \circ e_m$ defines a Lipschitz map on $V_m := e_m^{-1}(f(U_m \times U_n))$, which is open by continuity of e_m and f^{-1} . Because of the assumption $F(a, b) = 0$ we have $a \in V_m$ and $\varphi(a) = b$. Regarding the claimed equality $F^{-1}(0) = \{(x, \varphi(x)) | x \in V_m\}$ we get on the one hand the inclusion " \supset " from $F(x, \varphi(x)) = 0$, which follows by construction of φ . On the other hand for every $(x, y) \in U_m \times U_n$ with $F(x, y) = 0$ we have $f(x, y) = (x, 0)$ such that x lies in the domain V_m of φ by construction of V_m , which gives the other inclusion " \subset ". \blacksquare

We are now able to write the boundary $\partial\widehat{\Omega}$ near $\widehat{x}_0 \in \partial\widehat{\Omega} \cap \widehat{\Gamma}_{\text{cr}}$ as a Lipschitz graph over the plane $\widehat{\eta}_{\widehat{x}_0}^\perp$ through $\widehat{x}_0 = T(x_0)$ that is normal to $\widehat{\eta}_{\widehat{x}_0}$. This construction will be needed in the proof of Proposition 5.4.

Corollary 2.3 *Let $\widehat{x}_0 = T(x_0) \in \widehat{\Gamma}_{\text{cr}} \cap \partial\widehat{\Omega}$ and U and $\widehat{\eta}_{x_0}$ as in the transversality condition (2.6b). Set $\widehat{\eta}_{\widehat{x}_0}^\perp := \{\widehat{x} \in \mathbb{R}^d \mid (\widehat{x} - T(x_0)) \cdot \widehat{\eta}_{x_0} = 0\}$. Then, there is an open neighborhood \widehat{V} of $T(x_0)$ and a Lipschitz continuous function $\varphi_{x_0} : \widehat{V} \cap \widehat{\eta}_{\widehat{x}_0}^\perp \rightarrow \mathbb{R}$ such that the function*

$$\widehat{g} : \widehat{V} \rightarrow \mathbb{R}; \quad \widehat{g}(\widehat{x}) := \varphi_{x_0}(\widehat{x} - ((\widehat{x} - T(x_0)) \cdot \widehat{\eta}_{x_0}) \widehat{\eta}_{x_0}) - (\widehat{x} - T(x_0)) \cdot \widehat{\eta}_{x_0}$$

characterizes $\partial\widehat{\Omega}$ locally via $\widehat{g}(\widehat{x}) > 0$ for $\widehat{x} \in \widehat{\Omega}$, $\widehat{g}(\widehat{x}) = 0$ for $\widehat{x} \in \partial\widehat{\Omega}$, and $\widehat{g}(\widehat{x}) < 0$ for $\widehat{x} \in \mathbb{R}^d \setminus \text{clos}(\widehat{\Omega})$.

Similarly, the boundary $\partial\Omega$ near a point $x_0 \in \Gamma_{\text{cr}} \cap \partial\Omega$ can be characterized by a function $g = \widehat{g} \circ T^{-1}$, where \widehat{g} is obtained as above for $\widehat{x}_0 = T(x_0)$.

Proof. Take Ψ_{x_0} as in the transversality condition (2.6b) and consider the map

$$F : U \cap \widehat{\eta}_{x_0}^\perp \times \mathbb{R} \rightarrow \mathbb{R}; \quad F(\tilde{x}, y) := e_d \cdot \Psi_{x_0}(T^{-1}(\tilde{x} + y\widehat{\eta}_{x_0})),$$

where $(\tilde{x}, y) \in \widehat{\eta}_{x_0}^\perp \times \mathbb{R}$ provides a unique representation of $\widehat{x} \in \mathbb{R}^d$ via $\widehat{x} = \tilde{x} + y\widehat{\eta}_{x_0}$. By construction F is Lipschitz and $F^{-1}(0) \subset \partial\widehat{\Omega}$. Up to an isomorphism it is $F : \mathbb{R}^{d-1} \times \mathbb{R} \rightarrow \mathbb{R}$, and by the transversality condition we have $\frac{\partial}{\partial y} F(\tilde{x}, y) \geq \kappa$. Thus, Theorem 2.2 gives $y = \varphi_{x_0}(\tilde{x})$, and the remaining assertions follow by simple computations. \blacksquare

3 Coercivity of \mathcal{F}_ε via rigidity

The equi-coercivity of the \mathcal{F}_ε is directly implied by the equi-coercivity of the $\tilde{\mathcal{F}}_\varepsilon$, since $\mathcal{F}_\varepsilon \geq \tilde{\mathcal{F}}_\varepsilon$ holds. For extending the proof of the equi-coercivity of $\tilde{\mathcal{F}}_\varepsilon$ from [DNP02] we have to generalize the rigidity estimate from [FJM02] from Lipschitz domains to domains with cracks. For this we will use the overlapping decomposition $\Omega_{\text{cr}} = A_+ \cup A_-$ from (2.3).

Definition 3.1 (Rigidity domains) *A domain $\tilde{\Omega} \subset \mathbb{R}^d$ is called a rigidity domain, if*

$$\exists C > 0 \forall v \in H^1(\tilde{\Omega}, \mathbb{R}^d) : \quad \inf_{R \in \text{SO}(d)} \|\nabla v - R\|_{L^2(\tilde{\Omega})}^2 \leq C \|\text{dist}(\nabla v, \text{SO}(d))\|_{L^2(\tilde{\Omega})}^2. \quad (3.1)$$

The smallest such constant we call rigidity constant $\mathcal{R}(\tilde{\Omega})$.

In [FJM02] it is proved, that every bounded Lipschitz domain is a rigidity domain. Furthermore a doubling argument can be found therein similar to the one used in the following proof.

Proposition 3.2 *Let $A, B \subset \mathbb{R}^d$ be bounded rigidity domains such that $A \cap B$ is a rigidity domain with positive volume. Then $A \cup B$ is a rigidity domain, and we have*

$$\mathcal{R}(A \cup B) \leq (2+4\mu_A)\mathcal{R}(A) + (2+\mu_B)\mathcal{R}(B) + 4(\mu_A+\mu_B)\mathcal{R}(A \cap B),$$

where $\mu_A = \text{vol}(A)/\text{vol}(A \cap B) \geq 1$ and $\mu_B = \text{vol}(B)/\text{vol}(A \cap B) \geq 1$.

Proof. We fix $v \in H^1(A \cup B, \mathbb{R}^d)$ and denote by $R_A, R_B, R_{A \cap B} \in \text{SO}(d)$ the minimizers $R \in \text{SO}(d)$ in (3.1) on the corresponding domains. Hence on $A \cup B$ we obtain the estimate

$$\int_{A \cup B} |\nabla v(x) - R_{A \cap B}|^2 dx \leq I_A + I_B, \quad \text{where } I_D := \int_D |\nabla v(x) - R_{A \cap B}|^2 dx.$$

Writing shortly $\delta(F) := \text{dist}(F, \text{SO}(d))^2$ we can estimate:

$$\begin{aligned} I_A &\leq 2 \int_A |\nabla v(x) - R_A|^2 dx + 2 \int_A |R_A - R_{A \cap B}|^2 dx \\ &\leq 2\mathcal{R}(A) \int_A \delta(\nabla v(x)) dx + 2\mu_A \int_{A \cap B} |R_A - R_{A \cap B}|^2 dx, \end{aligned}$$

where we used that R_A is the minimizer for the set A , that $|R_A - R_{A \cap B}|$ is constant and the definition of μ_A . For the second term of I_A we have

$$\begin{aligned} \int_{A \cap B} |R_A - R_{A \cap B}|^2 dx &\leq 2 \int_{A \cap B} |R_A - \nabla v(x)|^2 dx + 2 \int_{A \cap B} |\nabla v(x) - R_{A \cap B}|^2 dx \\ &\leq 2\mathcal{R}(A) \int_A \delta(\nabla v(x)) dx + 2\mathcal{R}(A \cap B) \int_{A \cap B} \delta(\nabla v(x)) dx. \end{aligned}$$

Together we find $I_A \leq ((2+4\mu_A)\mathcal{R}(A) + 4\mu_A\mathcal{R}(A \cap B)) \int_{A \cup B} \delta(\nabla v(x)) dx$.

Interchanging A and B we find the analogous estimate for I_B , and the result follows. \blacksquare

In this form, the rigidity estimate applies to our situation by our assumption (2.1c) on the decomposition of Ω in two overlapping Lipschitz domains. We simply apply the above proposition to $\Omega_{\text{cr}} = A \cup B$ with $A = A_+$ and $B = A_-$, see (2.3).

Corollary 3.3 (Ω_{cr} is a rigidity domain) *Let $(\Omega, \Gamma_{\text{cr}})$ satisfy (2.2). Then, $\Omega_{\text{cr}} = \Omega \setminus \Gamma_{\text{cr}}$ is a rigidity domain, i.e. there is a constant $C > 0$ such that*

$$\forall v \in H^1(\Omega_{\text{cr}}; \mathbb{R}^d) \exists R \in SO(d) : \quad \|\nabla v - R\|_{L^2(\Omega_{\text{cr}})} \leq C \|\text{dist}(\nabla v, SO(d))\|_{L^2(\Omega_{\text{cr}})}.$$

Before proving coercivity, let us note the following quantitative statement on the rotations showing up when applying the rigidity estimate to small deformations $y_\varepsilon = \text{id} + \varepsilon u$. In [DNP02] as well as for us, it is a main step in the proof of the equi-coercivity. Moreover, we will need it for proving Theorem 4.1 on the local non-interpenetration in the next chapter. The main point is to show that for mappings $y_\varepsilon = \text{id} + \varepsilon u$ the corresponding rotation matrices R_ε that are minimizers in the rigidity estimate are also close to the identity matrix $I \in \mathbb{R}^{d \times d}$. For this we use the boundary conditions $u|_{\Gamma_{\text{Dir}}} = g$.

Lemma 3.4 *Let $\Omega, \Gamma_{\text{cr}}$, and W satisfy the assumption (2.2) and (1.3) and fix $g \in W^{1,\infty}(\Omega)$. Then, there exist constants $C_{\mathcal{F}}, C_{\mathcal{R}} > 0$ such that for all $\varepsilon \in]0, 1[$ and all $u \in H_{g, \text{Dir}}^1$ the following holds:*

$$\int_{\Omega_{\text{cr}}} |I + \varepsilon \nabla u(x) - R_{\text{id} + \varepsilon u}|^2 dx \leq C_F \varepsilon^2 \tilde{\mathcal{F}}_\varepsilon(u), \quad (3.2a)$$

$$|I - R_{\text{id} + \varepsilon u}|^2 \leq C_R \varepsilon^2 \left(\tilde{\mathcal{F}}_\varepsilon(u) + \int_{\Gamma_{\text{Dir}}} |g|^2 d\mathcal{H}^{d-1} \right), \quad (3.2b)$$

where R_v denotes the minimizer $R \in SO(d)$ in (3.1) for fixed $v \in H^1(\Omega_{\text{cr}}; \mathbb{R}^d)$.

Proof. Combining the coercivity of W in (1.3c) with the rigidity constant from Corollary 3.3 we obtain (3.2a) with $C_F = \mathcal{R}(\Omega_{\text{cr}})/c_W$.

To derive the second estimate we set $R_\varepsilon = R_{\text{id} + \varepsilon u}$ and $\zeta_\varepsilon := \int_{\Omega_{\text{cr}}} (x + \varepsilon u(x) - R_\varepsilon x) dx$. By continuity of the traces and Poincaré's inequality we find

$$\begin{aligned} \int_{\Gamma_{\text{Dir}}} |(x + \varepsilon u(x)) - R_\varepsilon x - \zeta_\varepsilon|^2 d\mathcal{H}^{d-1} &\leq C_2 \|(x + \varepsilon u(x)) - R_\varepsilon x - \zeta_\varepsilon\|_{H^1(\Omega_{\text{cr}}, \mathbb{R}^d)} \\ &\leq C_3 \int_{\Omega_{\text{cr}}} |(I + \varepsilon \nabla u(x)) - R_\varepsilon|^2 dx \leq C_4 \varepsilon^2 \tilde{\mathcal{F}}_\varepsilon(u) \end{aligned}$$

with $C_4 = C_F C_3$. Exploiting $u|_{\Gamma_{\text{Dir}}} = g$ and the prefactor ε we obtain

$$\int_{\Gamma_{\text{Dir}}} |(I - R_\varepsilon)x - \zeta_\varepsilon|^2 d\mathcal{H}^{d-1} \leq C_5 \varepsilon^2 \left(\tilde{\mathcal{F}}_\varepsilon(u) + \int_{\Gamma_{\text{Dir}}} |g|^2 d\mathcal{H}^{d-1} \right).$$

Note that $R_\varepsilon - I$ is an element of the closed cone K generated by $SO(d) - I$, on which Lemma 3.3 from [DNP02] applies (see the derivation of (3.14) therein). Thus

$$|I - R_\varepsilon|^2 \leq C_6 \min_{\zeta \in \mathbb{R}^d} \int_{\Gamma_{\text{Dir}}} |(I - R_\varepsilon)x - \zeta|^2 d\mathcal{H}^{d-1},$$

and the estimate (3.2b) follows with $C_R = C_5 C_4$. ■

Now we can proof the equi-coercivity of $\tilde{\mathcal{F}}_\varepsilon$ on $H_{g, \text{Dir}}^1$.

Proposition 3.5 (First a priori bound) *Assume that $\Omega, \Gamma_{\text{cr}}$, and W satisfy (2.2) and (1.3). Then, there exists $c_{\mathcal{F}}, C_{\mathcal{F}} > 0$ such that*

$$\forall \varepsilon \in]0, 1[\forall u \in H_{g, \text{Dir}}^1 : \quad \tilde{\mathcal{F}}_\varepsilon(u) \geq c_{\mathcal{F}} \|u\|_{H^1}^2 - C_{\mathcal{F}}.$$

Proof. By the first part of assumption (1.3c) on W and Corollary 3.3 we have

$$\begin{aligned} \|(I+\varepsilon\nabla u) - R_\varepsilon\|_{L^2}^2 &\leq C_1 \int_{\Omega_{\text{cr}}} \text{dist}^2(I+\varepsilon\nabla u(x), \text{SO}(d)) \, dx \\ &\leq C_2 \int_{\Omega_{\text{cr}}} W(I+\varepsilon\nabla u(x)) \, dx \leq C_2 \varepsilon^2 \mathcal{F}_\varepsilon(u). \end{aligned}$$

Using both estimates from Lemma 3.4 we proceed to obtain

$$\varepsilon^2 \|\nabla u\|_{L^2}^2 \leq 2(\|I-R_\varepsilon\|_{L^2}^2 + \|I+\varepsilon\nabla u - R_\varepsilon\|_{L^2}^2) \leq \varepsilon^2 C_3 \left(\mathcal{F}_\varepsilon(u) + \int_{\Gamma_{\text{Dir}}} |g|^2 \, d\mathcal{H}^{d-1} \right)$$

with $C_3 = 2C_F + 2C_R$. Dividing by ε^2 and exploiting the boundary conditions as well as Poincaré's inequality we arrive at the desired result. \blacksquare

The above result shows that sequences $(u_\varepsilon)_\varepsilon$ with bounded energy $\tilde{\mathcal{F}}_\varepsilon(u_\varepsilon) \leq C < \infty$ are bounded in $H^1(\Omega_{\text{cr}}; \mathbb{R}^d)$. The next results provides a weaker, but still useful a priori bound, which implies that $\varepsilon u_\varepsilon$ converges to 0 in $L^\infty(\Omega; \mathbb{R}^d)$ for energy bounded sequences.

Proposition 3.6 (Second a priori bound) *Let W satisfy (1.3). Consider a sequence $(u_\varepsilon)_{\varepsilon>0}$ with $\sup_{\varepsilon>0} \tilde{\mathcal{F}}_\varepsilon(u_\varepsilon) < \infty$. Then, there exists a constant $C > 0$ such that*

$$\|\varepsilon u_\varepsilon\|_{W^{1,p}} \leq C \quad \text{and} \quad \|\varepsilon u_\varepsilon\|_{L^\infty} \leq C \varepsilon^r \tag{3.3}$$

with $r \in]0, 1[$ arbitrary for $d = 2$, and $r = \frac{2(p-d)}{d(p-2)} \in]0, 1[$ for $d \geq 3$.

Proof. The first estimate in (3.3) follows directly from the coercivity (1.3c) for W :

$$\varepsilon^2 C \geq \varepsilon^2 \tilde{\mathcal{F}}_\varepsilon(u_\varepsilon) \geq \int_{\Omega_{\text{cr}}} c_W (|I+\varepsilon\nabla u_\varepsilon(x)|^p - C_W) \, dx \geq \frac{c_W}{2} \|\varepsilon\nabla u_\varepsilon\|_{L^p}^p - \bar{C}.$$

Together with Proposition 3.5 we obtain a uniform bound in $W^{1,p}(\Omega; \mathbb{R}^d)$.

For the second estimate in (3.3) we use the Gagliardo-Nirenberg interpolation estimate for $f = \varepsilon u_\varepsilon$, where we crucially $p > d$ as provided in (1.3c):

$$\|f\|_{L^\infty} \leq C \|f\|_{W^{1,p}}^\theta \|f\|_{H^1}^{1-\theta}.$$

For $d = 1$ we can take $\theta = 0$ because $H^1 \subset L^\infty$, and for $d = 2$ any $\theta \in]0, 1[$ is sufficient. For $d \geq 3$ we can choose $\theta = \frac{d-2}{d} \frac{p-2}{p} \in]0, 1[$, and the result follows by using Proposition 3.5, which gives $\|f\|_{H^1}^{1-\theta} \leq \varepsilon^{1-\theta} C$. \blacksquare

4 The lim inf estimate

In contrast to the equi-coercivity the Γ -lim inf estimate for \mathcal{F}_ε does not follow directly from the Γ -lim inf estimate for $\tilde{\mathcal{F}}_\varepsilon$, since we have to consider the case $\mathcal{F}(u) = \infty$ carefully, i.e. we have to show that the global injectivity condition (1.1) generates the local non-interpenetration condition (1.2) in the limit $\varepsilon \rightarrow 0$. This is the content of the following result.

Theorem 4.1 (Local non-interpenetration) *Consider $u_\varepsilon, u \in H^1(\Omega_{\text{cr}}, \mathbb{R}^d)$ such that $u_\varepsilon \xrightarrow{H^1} u$ and $\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty$; then $\llbracket u \rrbracket_{\Gamma_{\text{cr}}} \geq 0$ holds.*

To prove this theorem we will first prove the following linearization result concerning the determinant of $I + \varepsilon \nabla u$:

Lemma 4.2 *There exists $C_{\det} > 0$ depending on Ω , Γ_{Dir} , Γ_{cr} and the exponent $p > d$ and constants from assumption (1.3c) such that*

$$\forall \varepsilon \in]0, 1[\quad \forall u \in H_{g, \text{Dir}}^1 : \quad (4.1)$$

$$\int_{\Omega_{\text{cr}}} \left| \det(I + \varepsilon \nabla u(x)) - 1 - \varepsilon \operatorname{div} u(x) \right| dx \leq \varepsilon^2 C_{\det} (\tilde{\mathcal{F}}_\varepsilon(u) + C_{\det}).$$

Proof. For matrices $A \in \mathbb{R}^{d \times d}$ we have $|\det(I+A) - (1 + \operatorname{tr} A)| \leq C_d(|A|^2 + |A|^d)$, where we will insert $A = \varepsilon \nabla u(x)$. To control the term $|A|^d$ we will use $W(I+A)$ and $|I+A|^p \geq \frac{1}{2}|A|^p - C_1$, which yields

$$W(I+A) \geq c_w(|I+A|^p - C_w) \geq \frac{c_w}{2}(|A|^p - C_2).$$

Using $W(F) \geq 0$ we even have $W(I+A) \geq \frac{c_w}{2}[|A|^p - C_2]_+$, where $[a]_+ := \max\{a, 0\}$. Because of $p > d \geq 2$ there exists $C_* > 0$ such that $t^d \leq C_*(t^2 + (t^p - C_2)_+)$ for all $t \geq 0$. Thus, inserting $t = |A| = |\varepsilon \nabla u(x)|$, setting $C_3 = C_d(C_* + 1)$, and integrating over Ω_{cr} results in

$$\begin{aligned} \int_{\Omega_{\text{cr}}} \left| \det(I + \varepsilon \nabla u(x)) - (1 + \varepsilon \operatorname{div} u(x)) \right| dx &\leq \int_{\Omega_{\text{cr}}} C_d(|\varepsilon \nabla u|^2 + |\varepsilon \nabla u|^d) dx \\ &\leq \int_{\Omega_{\text{cr}}} C_3 \left(|\varepsilon \nabla u|^2 + [|\varepsilon \nabla u|^p - C_2]_+ \right) dx \leq \varepsilon^2 C_3 \|\nabla u\|_{L^2}^2 + \varepsilon^2 \frac{2C_3}{c_w} \tilde{\mathcal{F}}_\varepsilon(u). \end{aligned}$$

Together with Proposition 3.5 we see that the assertion holds with C_{\det} chosen as the maximum of $C_3/c_{\mathcal{F}} + 2C_3/c_w$ and $C_3C_{\mathcal{F}}/c_{\mathcal{F}}$. \blacksquare

With this lemma at hand, we are now able to complete the proof of the main theorem of this section. The idea is to consider the GMS condition (1.1) for global injectivity for $y_\varepsilon = \operatorname{id} + \varepsilon u_\varepsilon$ with non-negative test functions $\varphi \in C^\infty(\Omega)$. Dividing by ε and passing to the limit with the help of the above lemma one can derive the relation $\int_{\Omega_{\text{cr}}} \nabla \varphi \cdot u dx \geq 0$, which provides the local non-interpenetration condition (1.2).

Proof of Theorem 4.1: As $\alpha := \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty$ there is a subsequence (ε_j, u_j) such that $\operatorname{id} + \varepsilon_j u_j$ fulfills the GMS-condition (1.1) and $\det(I + \varepsilon_j \nabla u_j) > 0$ a.e. on Ω . Hence, by rearranging (1.1), for every $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$ we have:

$$\begin{aligned} 0 &\geq \frac{1}{\varepsilon_j} \int_{\Omega_{\text{cr}}} \left(\varphi(x + \varepsilon_j u_j(x)) \det(I + \varepsilon_j \nabla u_j(x)) - \varphi(x) \right) dx \\ &= \frac{1}{\varepsilon_j} \int_{\Omega_{\text{cr}}} \varphi(x + \varepsilon_j u_j(x)) \left(\det(I + \varepsilon_j \nabla u_j(x)) - (1 + \varepsilon_j \operatorname{div} u_j(x)) \right) dx \\ &\quad + \int_{\Omega_{\text{cr}}} \varphi(x + \varepsilon_j u_j(x)) \operatorname{div} u_j(x) dx + \int_{\Omega_{\text{cr}}} \frac{1}{\varepsilon_j} \left(\varphi(x + \varepsilon_j u_j(x)) - \varphi(x) \right) dx. \end{aligned}$$

By Lemma 4.2 the first summand on the right-hand side converges to 0 for $j \rightarrow \infty$. The second summand converges to $\int_{\Omega_{\text{cr}}} \varphi(x) \operatorname{div} u(x) dx$, as $\operatorname{div} u_j \xrightarrow{L^2} \operatorname{div} u$ weakly and $\varphi \circ (\operatorname{id} + \varepsilon_j u_j) \rightarrow \varphi$ strongly in $L^2(\Omega)$, where we use Proposition 3.6, namely $\|\varepsilon_j u_j\|_{L^\infty} \rightarrow 0$.

Finally, the third term can be treated by using the relation

$$\frac{1}{\varepsilon_j}(\varphi(x+\varepsilon_j u_j(x)) - \varphi(x)) = \int_{s=0}^1 \nabla \varphi(x+s\varepsilon_j u_j(x)) \cdot u_j(x) ds,$$

such that, again using $\|\varepsilon_j u_j\|_{L^\infty} \rightarrow 0$, and weak convergence $u_j \rightharpoonup u$ shows convergence to $\int_{\Omega_{\text{cr}}} \nabla \varphi(x) \cdot u(x) dx$. Altogether the limit $\varepsilon \rightarrow 0$ provides the estimate get

$$\begin{aligned} 0 &\geq 0 + \int_{\Omega_{\text{cr}}} \varphi(x) \operatorname{div} u(x) dx + \int_{\Omega_{\text{cr}}} \nabla \varphi(x) \cdot u(x) dx \\ &= \int_{\Omega_{\text{cr}}} \operatorname{div}(\varphi u)(x) dx = - \int_{\Gamma_{\text{cr}}} \varphi(x) \llbracket u \rrbracket_{\Gamma_{\text{cr}}}(x) da(x). \end{aligned}$$

For the last identity we have to recall that u lies in $H_{g,\text{Dir}}^1 \subset H^1(\Omega_{\text{cr}}; \mathbb{R}^d)$ such that the upper and lower traces at the crack Γ_{cr} may be different. Applying the divergence theorem on the Lipschitz sets A_+ and $A_- \setminus A_+$ (see (2.3)) separately, all terms cancel except for the jump along Γ_{cr} . As $\varphi \geq 0$ was arbitrary, we conclude $\llbracket u \rrbracket_{\Gamma_{\text{cr}}} \geq 0$ a.e. on Γ_{cr} . \square

We are now ready for deriving the lim inf part for our Mosco convergence $\mathcal{F}_\varepsilon \xrightarrow{\text{M}} \mathcal{F}$.

Proposition 4.3 (Liminf estimate) *For every sequence $\varepsilon_j \rightarrow 0$ and $u_j, u \in H_{g,\text{Dir}}^1$ with $u_j \rightharpoonup u$ in $H^1(\Omega_{\text{cr}}; \mathbb{R}^d)$ we have*

$$\mathcal{F}(u) \leq \liminf_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_j).$$

Proof. We can assume that $\alpha := \liminf_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_j) < \infty$, since otherwise the inequality holds trivially. Thus, there is a subsequence (ε_j, u_j) such that $\mathcal{F}_{\varepsilon_j}(u_j) = \tilde{\mathcal{F}}_{\varepsilon_j}(u_j) \rightarrow \alpha$ and that $\text{id} + \varepsilon_j u_j$ is globally injective. By Theorem 4.1 we conclude $\llbracket u \rrbracket_{\Gamma_{\text{cr}}} \geq 0$. Consequently the liminf estimate above reduces to the liminf estimate for $\tilde{\mathcal{F}}_\varepsilon$:

$$\mathcal{F}(u) = \tilde{\mathcal{F}}(u) \leq \alpha = \lim_{j \rightarrow \infty} \tilde{\mathcal{F}}_{\varepsilon_j}(u_j) = \lim_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_j).$$

Because $\tilde{\mathcal{F}}$ is convex, by [MiS13, Lem. 4.2] it suffices to show the pointwise lim inf estimate of the respective densities. From (1.3d) we even obtain pointwise equality using

$$\left| \frac{1}{\varepsilon^2} W(I + \varepsilon G) - \frac{1}{2} \langle G, \mathbb{C}G \rangle \right| \leq \frac{\delta}{2} \langle G, \mathbb{C}G \rangle \leq \delta \frac{|\mathbb{C}|}{2} |G|^2 \quad \text{for } G \in B_{r_\delta/\varepsilon}(0).$$

Since $\delta > 0$ is arbitrary, for each fixed G we have $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} W(I + \varepsilon G) = \frac{1}{2} \langle G, \mathbb{C}G \rangle$. \blacksquare

5 The lim sup estimate

Showing the lim sup estimate in (1.7) amounts in the construction of a recovery sequence $u_\varepsilon \rightarrow u$ converging strongly in $H_{g,\text{Dir}}^1 \subset H^1(\Omega_{\text{cr}}; \mathbb{R}^d)$. In the case without constraints (1.1) or (1.2) the limsup estimate for the Γ -convergence $\tilde{\mathcal{F}}_\varepsilon \xrightarrow{\Gamma} \tilde{\mathcal{F}}$ is much simpler since for $u \in W^{1,\infty}(\Omega_{\text{cr}}; \mathbb{R}^d)$ we can take the constant recovery sequence $u_j = u$. Then, the extension to general $u \in H_{g,\text{Dir}}^1$ follows by density and the strong continuity of $\tilde{\mathcal{F}}$, see [DNP02, Prop. 4.1].

Due to the constraints (1.1) and (1.2) in the functionals \mathcal{F}_ε and \mathcal{F} , respectively, we have to do some extra work. First, setting

$$\mathbf{C}_g := \{ u \in H_{g, \text{Dir}}^1 \mid \llbracket u \rrbracket_{\Gamma_{\text{cr}}} \geq 0 \}$$

we have to show that $W^{1,\infty} \cap \mathbf{C}_g$ is dense in \mathbf{C}_g with respect to the H^1 norm. Second we would like to use that $u \in W^{1,\infty} \cap \mathbf{C}_g$ implies that the close-to-identity deformations $\text{id} + \varepsilon u$ is globally injective for sufficiently small $\varepsilon > 0$. The following example shows that this cannot be expected in general.

Example 5.1 (Non-injectivity) Consider the domain $\tilde{\Omega} =]-1, 1[^2 \subset \mathbb{R}^2$, the crack $\tilde{\Gamma}_{\text{cr}} = \{0\} \times]0, \infty[$, the cracked domain $\tilde{\Omega}_{\text{cr}} := \tilde{\Omega} \setminus \tilde{\Gamma}_{\text{cr}}$ and the displacement

$$u : \tilde{\Omega}_{\text{cr}} \rightarrow \mathbb{R}^2; \quad u(x_1, x_2) = \begin{cases} (0, 0)^\top & \text{for } x_2 < 0, \\ (x_2 + (x_2)^2, x_2)^\top & \text{for } x_2 \geq 0 \text{ and } x_1 > 0, \\ (x_2, 0)^\top & \text{for } x_2 \geq 0 \text{ and } x_1 < 0. \end{cases}$$

Then, $u \in W^{1,\infty}(\tilde{\Omega}_{\text{cr}}; \mathbb{R}^2)$, and along the crack we have $\tilde{\nu}(0, x_2) = e_1 = (1, 0)^\top$ and the jump $\llbracket u \rrbracket_{\tilde{\Gamma}_{\text{cr}}}(0, x_2) = (x_2)^2 > 0$, except on the crack tip $\tilde{\Gamma}_{\text{edge}} = (0, 0)^\top$.

However, $y_\varepsilon := \text{id} + \varepsilon u$ is not injective for any $\varepsilon > 0$ near the crack tip. To see this, we set $x_\varepsilon^+ = ((\frac{\varepsilon}{2})^3, \frac{\varepsilon}{2})^\top$ and $x_\varepsilon^- = (-(\frac{\varepsilon}{2})^3, \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2})^\top$ which lie in the first and second quadrant, respectively. We have $y_\varepsilon(x_\varepsilon^+) = (\frac{\varepsilon^2}{2} + 3(\frac{\varepsilon}{2})^3, \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2})^\top = y_\varepsilon(x_\varepsilon^-)$, which violates injectivity. Even more, we see that the second quadrant is mapped to the set $\{ y \in \mathbb{R}^2 \mid y_2 \geq 0, y_1 < \varepsilon y_2 \}$ while the first quadrant is mapped to $\{ y \in \mathbb{R}^2 \mid y_2 \geq 0, y_1 > h_\varepsilon(y_1) \}$ with $h_\varepsilon(z) = \varepsilon z(1 + \varepsilon + z)/(1 + \varepsilon)^2$. Thus, each point in the area

$$\{ (y_1, y_2) \mid 0 < y_2 < \varepsilon(1 + \varepsilon), \varepsilon y_2 < y_1 < h_\varepsilon(y_2) \}$$

has two preimages.

The main problem in handling domains with cracks is that the missing Lipschitz property. For Lipschitz domains Ω we have $C^{\text{Lip}}(\Omega) = W^{1,\infty}(\Omega)$ with an estimate

$$\text{Lip}_\Omega(u) \leq C_\Omega \|\nabla u\|_{L^\infty(\Omega)}. \quad (5.1)$$

For convex domains one has $C_\Omega = 1$ but for general domains the constant depends on the relation between Euclidian distance and the inner distance

$$d_\Omega : \Omega \times \Omega \rightarrow \mathbb{R}; \quad d_\Omega(x, \tilde{x}) = \inf \{ \text{Length}(\gamma) \mid \gamma \text{ connects } x \text{ with } \tilde{x} \text{ inside } \Omega \}.$$

Then, the chain rule guarantees $|u(x) - u(\tilde{x})| \leq \|\nabla u\|_\infty d_\Omega(x, \tilde{x})$. Thus, we can choose $C_\Omega = \sup \{ d_\Omega(x, \tilde{x}) / |x - \tilde{x}| \mid x, \tilde{x} \in \Omega, x \neq \tilde{x} \}$ in (5.1).

In a domain Ω_{cr} with a crack, we obviously have $C_{\Omega_{\text{cr}}} = \infty$, since points x^+ and x^- on two opposite sides may have arbitrary small Euclidian distance $|x^+ - x^-|$ but large inner distance $d_{\Omega_{\text{cr}}}(x^+, x^-)$. This explains the difficulty in proving global injectivity, since for a close-to-identity mapping $y_\varepsilon = \text{id} + \varepsilon u$ we have

$$|y_\varepsilon(x^+) - y_\varepsilon(x^-)| \geq |x^+ - x^-| - \varepsilon |u(x^+) - u(x^-)| \geq |x^+ - x^-| - \varepsilon \|\nabla u\|_{L^\infty(\Omega_{\text{cr}})} d_{\Omega_{\text{cr}}}(x^+, x^-).$$

Thus, for Lipschitz domains Ω with $C_\Omega < \infty$ the global injectivity follows easily if $\varepsilon \|\|\nabla u\|_{L^\infty(\Omega)} C_\Omega \leq 1/2$, but for cracked domains Ω_{cr} we have to be much more careful. Indeed, we have to require that our functions $u \in \mathbf{C}_g \cap W^{1,\infty}(\Omega_{\text{cr}}; \mathbb{R}^d)$ also have a crack opening that is bounded from below linearly by the distance of the points on the crack from the edge Γ_{edge} . In the next result, we will show that we can achieve this by a suitable forcing apart.

Proposition 5.2 *Let $u \in W^{1,\infty}(\Omega_{\text{cr}}, \mathbb{R}^d) \cap \mathbf{C}_g$, then there is a sequence $u_k \in W^{1,\infty}(\Omega_{\text{cr}}, \mathbb{R}^d) \cap H_{g,\text{Dir}}^1$ satisfying $u_k \xrightarrow{H^1} u$ and*

$$\forall k \in \mathbb{N} \exists \varepsilon_k > 0 \forall \varepsilon \in]0, \varepsilon_k[: \quad \text{id} + \varepsilon u_k \text{ satisfies (1.1)}. \quad (5.2)$$

Proof. Motivated by the above example we will use the displacement $\widehat{\varphi}_{\delta,\eta} : \widehat{\Omega}_{\text{cr}} \rightarrow \mathbb{R}^d$, which forces to two sides of the crack $\widehat{\Gamma}_{\text{cr}}$ apart. For two small parameters $\delta, \eta > 0$ we set $\widehat{\varphi}_{\delta,\eta}(\widehat{x}) = \delta \lambda_\eta(\widehat{x}) \widehat{n} \in H^1(\widehat{\Omega}_{\text{cr}}, \mathbb{R}^d)$ with $\widehat{n} = (1, 1, 0, \dots, 0)^\top \in \mathbb{R}^d$. The scalar function $\lambda_\eta \in W^{1,\infty}(\widehat{\Omega}_{\text{cr}})$ with $\gamma \in]0, 1[$ is given by

$$\lambda_\eta(x_1, x_2, \dots, x_d) = \begin{cases} 0 & \text{if } x_1 > 1, \\ \min \left\{ 1, \frac{1}{\eta}(1-x_1) \right\} & \text{for } x_1 \in]0, 1] \text{ and } x_2 > 0, \\ -\min \left\{ 1, \frac{1}{\eta}(1-x_1) \right\} & \text{for } x_1 \in]0, 1] \text{ and } x_2 < 0, \\ -1 & \text{for } x_2 \leq 0. \end{cases}$$

Hence the jump of λ_η grows linearly with slope $1/\eta$ with the distance from $\widehat{\Gamma}_{\text{edge}}$ and then saturates at the values ± 1 .

We now choose an exponent $\alpha \in]1, 2[$ and a positive sequence $\delta_k \rightarrow 0$ and set $\eta_k = \delta_k^\alpha$. With this we define $\widehat{\varphi}_k := \widehat{\varphi}_{\delta_k, \eta_k}$ on $\widehat{\Omega}_{\text{cr}}$. Using the pullback of $\widehat{\varphi}_k$ to the reference configuration Ω via the Piola transform $\varphi_k(x) = \nabla T(x)^{-1} \widehat{\varphi}_k(T(x))$, see (2.2). Moreover, using (2.7) we can choose a cut-off function $\gamma \in W^{1,\infty}(\Omega; [0, 1])$ that is 1 on a neighborhood of Γ_{cr} and vanishes on Γ_{Dir} . With this we define the required sequence

$$u_k \in W^{1,\infty}(\Omega_{\text{cr}}, \mathbb{R}^d); \quad x \mapsto u_k(x) = u(x) + \gamma(x) \varphi_k(x).$$

Note that the boundary value on Γ_{Dir} is not changed, i.e. $u_k \in H_{g,\text{Dir}}^1$.

To show the convergence $u_k = u + \gamma \varphi_k \xrightarrow{H^1} u$ we need the smallness of $\gamma \varphi_k$. Using

$$\|\gamma \varphi_k\|_{H^1(\Omega_{\text{cr}})} \leq \|\gamma\|_{W^{1,\infty}(\Omega)} \|\nabla T^{-1}\|_{W^{1,\infty}(\widehat{\Omega})} \|\widehat{\varphi}_k\|_{H^1(\widehat{\Omega}_{\text{cr}})}.$$

will give the first condition for α :

$$\begin{aligned} \|\widehat{\varphi}_k\|_{L^2(\widehat{\Omega})}^2 &\leq \text{vol}(\widehat{\Omega}) |\widehat{n}|^2 \delta_k^2 \\ \|\nabla \widehat{\varphi}_k\|_{L^2(\widehat{\Omega}_{\text{cr}})}^2 &\leq \int_{\widehat{\Omega} \cap \{1-\eta_k \leq x_1 \leq 1\}} \left(\delta_k / \eta_k \right)^2 dx \leq \text{diam}(\widehat{\Omega})^{d-1} \delta_k^{2-\alpha}, \end{aligned}$$

where we used $\eta_k = \delta_k^\alpha$. Because of $\alpha < 2$ we have $\|u_k - u\|_{H^1} \rightarrow 0$ as desired.

Let us now come to the global invertibility. We establish the existence of $\varepsilon_k > 0$ by a contradiction argument. For this, we fix k for the moment and assume there is a sequence $\varepsilon_j \rightarrow 0$ such that $\text{id} + \varepsilon_j u_k$ is not globally invertible for all $j \in \mathbb{N}$. Thus, there exist $x_j, y_j \in \Omega_{\text{cr}}$ with $(\text{id} + \varepsilon_j u_k)(x_j) = (\text{id} + \varepsilon_j u_k)(y_j)$, i. e.

$$0 \neq x_j - y_j = \varepsilon_j (u_k(y_j) - u_k(x_j)). \quad (5.3)$$

By boundedness of Ω there is a (not relabeled) subsequence, such that x_j and y_j both converge. Since (5.3) gives $|x_j - y_j| \leq \varepsilon_j \|u_k\|_{L^\infty(\Omega_{\text{cr}})} \leq \varepsilon_j (\|u\|_{L^\infty(\Omega_{\text{cr}})} + 3\delta_k)$, these two limits are the same, from now denoted by z_∞ . We next establish the following claim:

Claim: The point z_∞ lies in the crack edge $\Gamma_{\text{edge}} = T^{-1}(\{(1, 0)\} \times \mathbb{R}^{d-2})$, and the convergence gives a very specific picture, i.e. $T(x_j) \cdot e_2 > 0$, $T(y_j) \cdot e_2 < 0$, $T(x_j) \cdot e_1 < 1$, $T(y_j) \cdot e_1 < 1$, and

$$\frac{|x_j - y_j|}{(1 - T(x_j) \cdot e_1) + (1 - T(y_j) \cdot e_1)} \rightarrow 0 \text{ as } j \rightarrow \infty. \quad (5.4)$$

That means that x_j and y_j converge to z_∞ by approaching the crack asymptotically from above and below, respectively.

A major part of the proof of the claim is due to Lipschitz continuity. If $z_\infty \notin \Gamma_{\text{cr}}$, there would be a Lipschitz neighborhood $U \subset \Omega_{\text{cr}}$ of z_∞ , which contains subsequences of x_j and y_j . With $L_k = \|\nabla u_k\|_{L^\infty(U)} \leq \|\nabla u_k\|_{L^\infty(\Omega_{\text{cr}})} < \infty$ we obtain

$$|x_j - y_j| = \varepsilon_j |u_k(y_j) - u_k(x_j)| \leq \varepsilon_j L_k d_U(x_j, y_j) \leq \varepsilon_j L_k C_U |x_j - y_j|.$$

For $\varepsilon_j L_k C_U < 1$ this implies $x_j = y_j$, which contradicts (5.3). Thus, $z_\infty \in \Gamma_{\text{cr}}$ is established.

To conclude $z_\infty \in \Gamma_{\text{edge}}$ we define the overlapping decomposition $\Omega_{\text{cr}} = \Omega^+ \cup \Omega^-$ via

$$\begin{aligned} \Omega^+ &:= T^{-1}(\{\hat{x} \in \hat{\Omega} \mid (x_1 > 0 \wedge x_2 > 0) \vee x_1 > 1\}) \text{ and} \\ \Omega^- &:= T^{-1}(\{\hat{x} \in \hat{\Omega} \mid x_1 < 0 \vee x_1 > 1 \vee x_2 < 0\}). \end{aligned}$$

Obviously there cannot be subsequences with $x_j, y_j \in \Omega^+$ or $x_j, y_j \in \Omega^-$ because both Ω^+ and Ω^- have Lipschitz boundary and the Lipschitz continuity of u_k would lead to a contradiction as in the step above. So without loss of generality we can assume $x_j \in \Omega^+ \setminus \Omega^-$ and $y_j \in \Omega^- \setminus \Omega^+$.

If $\hat{z}_\infty := T(z_\infty) \in \hat{\Gamma}_{\text{cr}} \setminus \hat{\Gamma}_{\text{kink}}$ we have a normal vector to $\hat{\Gamma}_{\text{cr}}$ given by

$$\hat{\nu} = \begin{cases} e_1 := (1, 0, \dots, 0) & \text{for } e_1 \cdot z_\infty = 0, \\ e_2 := (0, 1, 0, \dots, 0) & \text{for } e_2 \cdot z_\infty = 0. \end{cases}$$

By the above choice $x_j \in \Omega^+ \setminus \Omega^-$ and $y_j \in \Omega^- \setminus \Omega^+$ we obtain

$$(T(x_j) - T(y_j)) \cdot \hat{\nu} > 0 \quad (5.5)$$

for sufficiently big $j \in \mathbb{N}$. Thus, exploiting the smoothness of T across the crack and the relation (5.3) again we obtain

$$\begin{aligned} 0 < \frac{1}{\varepsilon_j} (T(x_j) - T(y_j)) \cdot \hat{\nu} &= \int_0^1 \nabla T(x_j + t(y_j - x_j)) dt \frac{1}{\varepsilon_j} (x_j - y_j) \cdot \hat{\nu} \\ &\stackrel{(5.3)}{=} \int_0^1 \nabla T(x_j + t(y_j - x_j)) dt (u_k(y_j) - u_k(x_j)) \cdot \hat{\nu}. \end{aligned}$$

Passing to the limit $j \rightarrow \infty$ we find the jump condition

$$0 \leq \nabla T(z_\infty) (u_k^-(z_\infty) - u_k^+(z_\infty)) \cdot \hat{\nu} = (u_k^-(z_\infty) - u_k^+(z_\infty)) \cdot T(z_\infty)^\top \hat{\nu}.$$

However, because of the non-interpenetration condition $\llbracket u_k \rrbracket_{\Gamma_{\text{cr}}} = \llbracket u \rrbracket_{\Gamma_{\text{cr}}} + \llbracket \varphi_k \rrbracket_{\Gamma_{\text{cr}}} \geq 0$, where $\llbracket \varphi_k \rrbracket_{\Gamma_{\text{cr}}} > 0$ except on the crack edge, we have

$$\begin{aligned} (u_k^+(z_\infty) - u_k^-(z_\infty)) \cdot \nabla T(z_\infty)^\top \hat{\nu} &\geq 0, \\ \text{where equality holds if and only if } z_\infty &\in \Gamma_{\text{edge}}. \end{aligned} \quad (5.6)$$

Thus, we conclude that z_∞ cannot lie in $\Gamma_{\text{cr}} \setminus (\Gamma_{\text{kink}} \cup \Gamma_{\text{edge}})$.

It remains to exclude $z_\infty \in \Gamma_{\text{kink}}$. If this is the case, then both (5.5) and (5.6) still hold for some $\widehat{\nu}$ but for different reasons: (5.5) holds for $\widehat{\nu} = e_1$ or $\widehat{\nu} = e_2$ for at least a subsequence by the Pigeonhole principle because $x_j \in \Omega^+$ and $y_j \in \Omega^-$ and (5.6) holds for both $\widehat{\nu} = e_1$ and $\widehat{\nu} = e_2$ by continuity of u_k . Thus, we similarly conclude $z_\infty \notin \Gamma_{\text{kink}}$, and $z_\infty \in \Gamma_{\text{edge}}$, which is the first part of the above claim.

From here on let $\widehat{U} := B_\varrho(T(z_\infty)) \subset \widehat{\Omega}$ with $\varrho < 1$, such that \widehat{U} does not touch Γ_{kink} . Then, $x_j, y_j \in \widehat{U}$ for j big enough, and $x_j \in \Omega^+ \setminus \Omega^-$ and $y_j \in \Omega^- \setminus \Omega^+$ gives

$$T(x_j) \cdot e_1 < 1, \quad T(x_j) \cdot e_2 > 0, \quad T(y_j) \cdot e_1 < 1, \quad T(y_j) \cdot e_2 < 0,$$

which is the second part of the above claim.

To see the last part of the claim note that we have either (5.4) as claimed or

$$(1 - T(x_j) \cdot e_1) + (1 - T(y_j) \cdot e_1) \leq C |x_j - y_j| \tag{5.7}$$

with some positive constant C independent of j . We assume now (5.7) in order to generate a contradiction. Indeed, the smallness of the quantities on the left-hand side allow us to use the Lipschitz continuity of u_k on $T^{-1}(\{\widehat{x} \in \widehat{U} \mid \widehat{x}_1 \geq 1\})$. Introducing the projections

$$x'_j := T^{-1}\left(T(x_j) - (1 - T(x_j) \cdot e_1) e_1\right) \quad \text{and} \quad y'_j := T^{-1}\left(T(y_j) - (1 - T(y_j) \cdot e_1) e_1\right),$$

we can compare them with x_j and y_j , respectively, as well as x'_j and y'_j to each other:

$$\begin{aligned} \frac{1}{\varepsilon_j} |x_j - y_j| &= |u_k(x_j) - u_k(y_j)| \\ &\leq |u_k(x_j) - u_k(x'_j)| + |u_k(x'_j) - u_k(y'_j)| + |u_k(y'_j) - u_k(y_j)| \\ &\leq L_k \left(|x_j - x'_j| + |x'_j - y'_j| + |y'_j - y_j| \right) \\ &\leq L_k \left(2|x_j - x'_j| + |x_j - y_j| + 2|y'_j - y_j| \right) \\ &\leq L_k \left(|x_j - y_j| + 2\|\nabla T^{-1}\|_{L^\infty} (|T(x_j) - T(x'_j)| + |T(y'_j) - T(y_j)|) \right) \\ &\leq L \left(|x_j - y_j| + 2\|\nabla T^{-1}\|_{L^\infty} ((1 - T(x_j) \cdot e_1) + (1 - T(y_j) \cdot e_1)) \right) \\ &\stackrel{(5.7)}{\leq} L |x_j - y_j| (1 + 2\|\nabla T^{-1}\|_{L^\infty} C). \end{aligned}$$

After dividing by $|x_j - y_j| \neq 0$, we see that this contradicts $\varepsilon_j \rightarrow 0$, such that (5.7) must be false, and hence (5.4) and the whole above claim is established.

We still have to produce a contradiction to show that (5.3) is false. But now we can use the relations in the above claim, in particular the convergence (5.4). To this we will use the assumption $\alpha > 1$ in the definition $\eta_k = \delta_k^\alpha$.

In the following calculation we use the abbreviation $A_j := \int_0^1 \nabla T(x_j + t(y_j - x_j)) dt$ and

insert the relation (5.3):

$$\begin{aligned}
0 &\leq \frac{1}{\varepsilon_j} (T(x_j) - T(y_j)) \cdot e_2 = \frac{1}{\varepsilon_j} A_j(x_j - y_j) \cdot e_2 = A_j(u_k(y_j) - u_k(x_j)) \cdot e_2 \quad (5.8) \\
&= A_j \left((u(y_j) - u(y'_j)) + (u(y'_j) - u(x'_j)) + (u(x'_j) - u(x_j)) + (\varphi_k(y_j) - \varphi_k(x_j)) \right) \cdot e_2 \\
&\leq \|\nabla T\|_{L^\infty} \|\nabla u\|_{L^\infty} (|y_j - y'_j| + |y'_j - x'_j| + |x'_j - x_j|) + A_j(\varphi_k(y_j) - \varphi_k(x_j)) \cdot e_2 \\
&\leq \|\nabla T\|_{L^\infty} \|\nabla u\|_{L^\infty} \left(|x_j - y_j| + 2(|x_j - x'_j| + |y_j - y'_j|) \right) + A_j(\varphi_k(y_j) - \varphi_k(x_j)) \cdot e_2 \\
&\leq \|\nabla T\|_{L^\infty} \|\nabla u\|_{L^\infty} \left(|x_j - y_j| + 2\|\nabla T^{-1}\|_{L^\infty} ((1 - T(x_j) \cdot e_1) + (1 - T(y_j) \cdot e_1)) \right) \\
&\quad + A_j(\varphi_k(y_j) - \varphi_k(x_j)) \cdot e_2
\end{aligned}$$

Dividing by $(1 - T(x_j) \cdot e_1) + (1 - T(y_j) \cdot e_1)$ and taking the limit $j \rightarrow \infty$, the assumed convergence (5.4) implies that the first summand of the right-hand side converges to the constant $C_u := 2\|\nabla T\|_{L^\infty} \|\nabla u\|_{L^\infty} \|\nabla T^{-1}\|_{L^\infty}$, which is independent of k . The idea is now to show that for our choice of $\alpha > 1$ the second summand makes the right-hand side negative for sufficiently small δ_k , which then produces a contradiction.

For this, we exploit the definition of φ_k via the function λ_{η_k} and the choices $x_j \in \Omega^+ \setminus \Omega^-$ and $y_j \in \Omega^- \setminus \Omega^+$. Since x_j and y_j are near Γ_{edge} we obtain

$$\lambda_\eta(T(x_j)) = \frac{1}{\eta} (1 - T(x_j) \cdot e_1) \quad \text{and} \quad \lambda_\eta(T(y_j)) = -\frac{1}{\eta} (1 - T(y_j) \cdot e_1).$$

Inserting this with $\eta = \delta_k^\alpha$ we find

$$\begin{aligned}
A_j(\varphi_k(y_j) - \varphi_k(x_j)) \cdot e_2 &= \delta_k A_j \left(\nabla T(y_j)^{-1} \lambda_{\delta_k^\alpha}(T(y_j)) \widehat{n} - \nabla T(x_j)^{-1} \lambda_{\delta_k^\alpha}(T(x_j)) \widehat{n} \right) \cdot e_2 \\
&= \delta_k A_j \left(-\frac{1}{\delta_k^\alpha} \nabla T(y_j)^{-1} (1 - T(y_j) \cdot e_1) \widehat{n} - \frac{1}{\delta_k^\alpha} \nabla T(x_j)^{-1} (1 - T(x_j) \cdot e_1) \widehat{n} \right) \cdot e_2 \\
&= -\delta_k^{1-\alpha} \left((1 - T(y_j) \cdot e_1) e_2 \cdot A_j \nabla T(y_j)^{-1} \widehat{n} + (1 - T(x_j) \cdot e_1) e_2 \cdot A_j \nabla T(x_j)^{-1} \widehat{n} \right).
\end{aligned}$$

The matrices $A_j \nabla T(y_j)^{-1}$ and $A_j \nabla T(x_j)^{-1}$ converge to $I \in \mathbb{R}^{d \times d}$ by dominated convergence and continuity of ∇T , thus we have $e_2 \cdot A_j \nabla T(x_j)^{-1} \widehat{n} \rightarrow e_2 \cdot \widehat{n} = 1$ and similarly for y_j . Because both $(1 - T(x_j) \cdot e_1)$ and $(1 - T(y_j) \cdot e_1)$ are positive, this implies the convergence

$$\delta_k^{\alpha-1} \frac{A_j(\varphi_k(y_j) - \varphi_k(x_j)) \cdot e_2}{(1 - T(x_j) \cdot e_1) + (1 - T(y_j) \cdot e_1)} \rightarrow -1 \quad \text{for } j \rightarrow \infty.$$

Inserting this into (5.8) divided by $(1 - T(x_j) \cdot e_1) + (1 - T(y_j) \cdot e_1)$ we obtain $0 \leq 2C_u - \frac{1}{2} \delta_k^{1-\alpha}$ for fixed k and sufficiently large j . Thus, making δ_k smaller if necessary, we arrive at a contraction, because $\delta_k \rightarrow 0$ and $\alpha > 1$.

This shows that (5.3) cannot hold for $\varepsilon_j \rightarrow 0$. Thus, the existence of $\varepsilon_k > 0$ is established, and Proposition 5.2 is proved. \blacksquare

To extend the achieved knowledge from the dense set $W^{1,\infty}(\Omega_{\text{cr}}; \mathbb{R}^d) \cap \mathbf{C}_g$ to the general case $u \in \mathbf{C}_g$, we have to show that all functions $u \in \mathbf{C}_g$ can be approximated by $u_k \in W^{1,\infty}(\Omega_{\text{cr}}; \mathbb{R}^d) \cap \mathbf{C}_g$, i.e. we have to approximate under the convex constraint of local non-interpenetration. Similar approximation results for more classical state constraints are contained in [HiR15, HRR16].

To handle our conditions of non-negativity of jumps over the crack we can use a reflection and decomposition into odd and even parts. To simplify the reading of the following proof, we illustrate this idea by a simple two-dimensional problem.

Example 5.3 (Straight crack in \mathbb{R}^2) We consider $\Omega = \mathbb{R}^2$, $\Gamma_{\text{cr}} = \mathbb{R} \times \{0\}$, and a function $u \in H^1(\Omega \setminus \Gamma_{\text{cr}})$ with $\llbracket u \rrbracket_{\Gamma_{\text{cr}}} \geq 0$. To find a smooth approximation we define

$$u^{\text{even}}(x) = \frac{1}{2}(u(x_1, x_2) + u(x_1, -x_2)) \quad \text{and} \quad u^{\text{odd}}(x) = \frac{1}{2}(u(x_1, x_2) - u(x_1, -x_2)),$$

such that $u = u^{\text{even}} + u^{\text{odd}}$, $\llbracket u^{\text{even}} \rrbracket_{\Gamma_{\text{cr}}} = 0$, and $\llbracket u^{\text{odd}} \rrbracket_{\Gamma_{\text{cr}}} = \llbracket u \rrbracket_{\Gamma_{\text{cr}}}$.

We can easily approximate u^{even} by $v_k \in C_c^\infty(\mathbb{R}^2)$, since it lies in $H^1(\mathbb{R}^2)$. For u^{odd} we don't want to smoothen the jump along Γ . Hence, we define

$$\tilde{u}(x_1, x_2) = \begin{cases} u^{\text{odd}}(x_1, x_2) & \text{for } x_2 > 0, \\ \max\{0, u^{\text{odd}}(x_1, -x_2)\} & \text{for } x_2 < 0. \end{cases}$$

Because of $\tilde{u}(\cdot, 0^+) = u(\cdot, 0^+) = \frac{1}{2}\llbracket u^{\text{odd}} \rrbracket_{\Gamma_{\text{cr}}} \geq 0$ we conclude that $\llbracket \tilde{u} \rrbracket_{\Gamma_{\text{cr}}} = 0$, which implies $\tilde{u} \in H^1(\mathbb{R}^2)$. Defining convolution kernels $\psi_k \in C_c^\infty(\mathbb{R}^2)$ with $\psi_k \geq 0$, $\int_{\mathbb{R}^2} \psi_k \, dy = 1$, and $\text{supp}(\psi_k) \subset B_{1/k}((0, -1/k)) \subset \mathbb{R} \times]-\infty, 0[$ we can define $\tilde{v}_k = \psi_k * \tilde{u} \in C^\infty(\mathbb{R}^2)$ and check that $\tilde{v}_k \rightarrow \tilde{u}$ in $H^1(\mathbb{R}^2)$ and that $\tilde{v}_k(x_1, 0) \geq 0$, because $\tilde{u}(x_1, x_2) \geq 0$ for $x_2 \leq 0$. Thus, setting

$$u_k(x_1, x_2) = v_k(x_1, x_2) + \text{sign}(x_2) \tilde{v}_k(x_1, |x_2|)$$

we obtain $u_k \in C^\infty(\Omega \setminus \Gamma_{\text{cr}})$ with $u_k \rightarrow u$ in $H^1(\Omega \setminus \Gamma_{\text{cr}})$ and $\llbracket u_k \rrbracket_{\Gamma_{\text{cr}}} \geq 0$.

The analogous construction for our general $\Gamma_{\text{cr}} \subset \Omega$ works similarly by mapping the displacements $u : \Omega_{\text{cr}} \rightarrow \mathbb{R}^d$ via the Piola transform onto displacements $\hat{u} : \hat{\Omega}_{\text{cr}} \rightarrow \mathbb{R}^d$, where the positivity of the jumps are preserved, see (2.5). Of course, we can take full advantage that the crack $\hat{\Gamma}_{\text{cr}}$ is piecewise flat. The only point that is more delicate arises for points in the intersection of Γ_{cr} and $\partial\Omega$.

Proposition 5.4 Let $u \in H_{g, \text{Dir}}^1$ with $\llbracket u \rrbracket_{\Gamma_{\text{cr}}} \geq 0$, then there is a sequence $u_k \in H_{g, \text{Dir}}^1 \cap W^{1, \infty}(\Omega_{\text{cr}}, \mathbb{R}^d)$ with $\llbracket u_k \rrbracket_{\Gamma_{\text{cr}}} \geq 0$ such that $u_k \rightarrow u$ in $H^1(\Omega_{\text{cr}}; \mathbb{R}^d)$.

Proof. It suffices to show the assertion locally in a neighborhood U of each point x^* from $\bar{\Omega}$ because by compactness we have a finite cover of such neighborhoods and recombination by partition of unity gives the result.

Bulk points in Ω_{cr} : For $x^* \in \Omega_{\text{cr}}$ and an open neighborhood U with $\bar{U} \subset \Omega_{\text{cr}}$ the convolution $v_k = \varphi_k * u$ is smooth and converges in $H^1(\bar{U}, \mathbb{R}^d)$ to u , where $\varphi_k(z) = k^d \varphi(kz)$ with any mollifier $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\varphi \geq 0$, $\text{supp} \varphi \subset B_1(0)$, $\int_{\mathbb{R}^d} \varphi \, dz = 1$, as u is of class H^1 in an open neighborhood of \bar{U} .

Free boundary: The case $x^* \in \partial\Omega \setminus (\Gamma_{\text{cr}} \cup \Gamma_{\text{Dir}})$ is more difficult. To extend u to the outside of Ω , one takes an open neighborhood V with $\bar{V} \cap \Gamma_{\text{cr}} = \emptyset$ where we have by Lipschitz property of $\partial\Omega$ a bi-Lipschitz chart $\Psi : \bar{V} \rightarrow \mathbb{R}^d$ such that $\Omega \cap \bar{V} \subset \Psi^{-1}(\{x_d > 0\})$, $\partial\Omega \cap \bar{V} \subset \Psi^{-1}(\{x_d = 0\})$ and $\bar{V} \setminus \bar{\Omega} \subset \Psi^{-1}(\{x_d < 0\})$ hold. An $H^1(\bar{V}, \mathbb{R}^d)$ -extension of u can then be given via a reflection by $u \circ \Psi^{-1} \circ R \circ \Psi$ where $R : (x_1, \dots, x_d) \mapsto (x_1, \dots, x_{d-1}, |x_d|)$. On every open neighborhood U with $\bar{U} \subset V$ again mollification converges to u in $H^1(\bar{U}, \mathbb{R}^d)$.

Dirichlet part of the boundary: For $x^* \in \Gamma_{\text{Dir}}$ there is an open neighborhood U disjoint from the crack Γ_{cr} and by definition of $H_{g, \text{Dir}}^1$ there is a $W^{1, \infty}$ -sequence coinciding with the Dirichlet data g on Γ_{Dir} .

Flat parts of the crack: For $x^* \in \Gamma_{\text{cr}} \setminus (\Gamma_{\text{edge}} \cup \Gamma_{\text{kink}} \cup \partial\Omega)$ we proceed similarly as in Example 5.3 after using the transformation $T : \Omega \rightarrow \hat{\Omega}$ to the simpler geometry $(\hat{\Omega}, \hat{\Gamma}_{\text{cr}})$ via the

Piola transform $\widehat{u}(T(x)) = \nabla T(x)u(x)$, which preserves the local non-interpenetration, see (2.5).

Since x^* is neither a point in $\partial\Omega$ nor in the crack kink Γ_{kink} or the crack edge Γ_{edge} tip nor the kink, we can assume without loss of generality that $T(x^*) \in \{0\} \times (0, \infty) \times \mathbb{R}^{d-2}$, the case $T(x^*) \in (0, 1) \times \{0\} \times \mathbb{R}^{d-2}$ is analogous. Take a neighborhood $V = T^{-1}\left(B_\delta(T(x^*))\right) \subset \Omega$ that touches neither of the critical parts. On this neighborhood we define the odd and even parts of the Piola transform $\widehat{u}^{(1)}$ and $\widehat{u}^{(0)}$ by

$$\widehat{u}^{(i)}(x_1, \dots, x_d) = \frac{1}{2} \left(\widehat{u}(x_1, \dots, x_d) + (-1)^i \widehat{u}(-x_1, x_2, \dots, x_d) \right).$$

These two parts need to be approximated on $B_\gamma^+(T(x^*)) := \{\widehat{x} \in B_\gamma(T(x^*)) \mid \widehat{x}_1 > 0\}$ for some $\gamma < \delta$, as if we have such $\widehat{u}_k^{(0)}, \widehat{u}_k^{(1)} \in W^{1,\infty}(B_\gamma^+(T(x^*)))$ approximated $\widehat{u}^{(0)}$ and $\widehat{u}^{(1)}$ on $B_\gamma^+(T(x^*))$, respectively. An approximation $u_k \in W^{1,\infty}(U)$ with $U := T^{-1}(B_\gamma(T(x^*)))$ is given by $u_k(x') = \nabla T(x')^{-1} \widehat{u}_k(T(x'))$, where

$$\widehat{u}_k(x_1, \dots, x_d) = \widehat{u}_k^{(0)}(|x_1|, x_2, \dots, x_d) + \text{sign}(x_1) \widehat{u}_k^{(1)}(|x_1|, x_2, \dots, x_d).$$

This way, the requirement $\llbracket u_k \rrbracket_{\Gamma_{\text{cr}}} \geq 0$ translates into asking $\llbracket \widehat{u}_k \rrbracket_{\widehat{\Gamma}_{\text{cr}}}(0, x_2, \dots, x_d) = 2u_k^{(1)}(0^+, x_2, \dots, x_d) \cdot e_1 \geq 0$ in the sense of traces. So $\widehat{u}^{(0)}$ and the every component except the first of $\widehat{u}^{(1)}$ can be approximated on $B_\delta^+(T(x^*))$ by standard means for Lipschitz domains. For the construction of $\widehat{u}_{1,k}^{(1)}$ we exploit $\widehat{u}_1^{(1)} \geq 0$ on $B_\gamma(T(x^*)) \cap \widehat{\Gamma}_{\text{cr}}$ by extending it from $B_\gamma^+(T(x^*))$ to $B_\gamma(T(x))$ by $\max\{0, \widehat{u}_1^{(1)}(-x_1, x_2, \dots, x_d)\}$ for $x_1 \leq 0$. As in Example 5.3 mollifying with φ satisfying $\text{supp } \varphi \subset \{z \in B_1(0) \mid z_1 < 0\}$ gives an approximation of $\widehat{u}_1^{(1)}$ with nonnegative trace on the crack.

Crack edge: For a point $x^* \in \Gamma_{\text{edge}} = T^{-1}(\{(1, 0)\} \times \mathbb{R}^d)$ and $V = T^{-1}(B_\delta(T(x^*))) \subset \Omega$ with $\delta < 1$ we proceed similar. The odd and even parts of the Piola transform in this case read

$$\widehat{u}^{(i)}(x_1, \dots, x_d) = \frac{1}{2} \left(\widehat{u}(x_1, \dots, x_d) + (-1)^i \widehat{u}(x_1, -x_2, x_3, \dots, x_d) \right)$$

and as before we approximate these separately and put $u_k(x') = \nabla T(x')^{-1} \widehat{u}_k(T(x'))$ where

$$\widehat{u}_k(x_1, \dots, x_d) = \widehat{u}_k^{(0)}(|x_1|, x_2, \dots, x_d) + \text{sign}(x_1) \widehat{u}_k^{(1)}(|x_1|, x_2, \dots, x_d).$$

Now, in addition to $0 \leq \widehat{u}_{2,k}^{(1)} = \frac{1}{2} \llbracket \widehat{u} \rrbracket_{\widehat{\Gamma}_{\text{cr}}}$ we have to guarantee $\widehat{u}_{2,k}^{(1)} = 0$ on $\{0\} \times [1, \delta] \times \mathbb{R}^{d-2}$ to obtain $u_k \in H^1(\Omega_{\text{cr}})$. Therefore we we have to refine the extension slightly. Consider a bi-Lipschitz map $S : \mathbb{R}^d \rightarrow \mathbb{R}^d$ that maps $\mathbb{R} \times [0, \infty[\times \mathbb{R}^{d-2}$ to $\{x \in \mathbb{R}^d \mid x_1 \leq 0, x_2 \leq 1\}$ in such a way, that it is the identity on $] -\infty, 1[\times \{0\} \times \mathbb{R}^{d-2}$, while it maps $] 1, \infty[\times \{0\} \times \mathbb{R}^{d-2}$ to $\{1\} \times] -\infty, 0[\times \mathbb{R}^{d-2}$. (Think of taking (x_1, x_2) in polar coordinates around $(1, 0)$ and then bisecting and translating the angle $\mathbb{R}_+ \times] 0, 2\pi[\ni (r, \varphi) \mapsto (r, \frac{1}{2}\varphi + \pi)$.) Now $\widehat{u}_2^{(1)}$ is extended from $B_\delta^+(T(x^*))$ to $B_\delta(T(x^*))$ by $\max\{0, \widehat{u}_2^{(1)}(S^{-1}(x_1, \dots, x_d))\}$ for $x_2 \leq 0$ and $x_1 \leq 1$ and by 0 for $x_2 \leq 0$ and $x_1 \geq 1$. Mollifying with φ satisfying $\text{supp } \varphi \subset \{z \in B_1(0) \mid z_1 > 0, z_2 < 0\}$ gives an approximation of $\widehat{u}_2^{(1)}$ with the desired property of the trace.

Crack kink: Let us come to $x^* \in \Gamma_{\text{kink}} = T^{-1}(\{(0, 0)\} \times \mathbb{R}^{d-2}) \cap \Omega$ and a neighborhood $V = T^{-1}(B_\delta(T(x^*))) \subset \Omega$ with $\delta < 1$ such that V does not touch the crack edge. We

again consider the Piola transform \widehat{u} and want to decompose in odd and even parts, except we now have two hyperplanes to consider and consequently four parts:

$$\begin{aligned} \widehat{u}^{(i,j)}(x) = \frac{1}{4} & \left(\widehat{u}(x_1, x_2, x_3, \dots, x_d) + (-1)^i \widehat{u}(-x_1, x_2, x_3, \dots, x_d) \right. \\ & \left. + (-1)^j \widehat{u}(x_1, -x_2, x_3, \dots, x_d) + (-1)^{i+j} \widehat{u}(-x_1, -x_2, x_3, \dots, x_d) \right) \end{aligned}$$

with $i, j \in \{0, 1\}$. After approximating each of the four functions on $B_\delta^+(T(x^*)) := \{x \in B_\delta(T(x^*)) \mid x_1 > 0 \wedge x_2 > 0\}$ separately, the plan would be again to put $u_k(x') = \nabla T(x')^{-1} \widehat{u}_k(T(x'))$ where

$$\begin{aligned} \widehat{u}_k^{(i,j)}(x) = \widehat{u}_k^{(0,0)}(|x_1|, |x_2|, x_3, \dots, x_d) + \text{sign}(x_1) \widehat{u}_k^{(1,0)}(|x_1|, |x_2|, x_3, \dots, x_d) \\ + \text{sign}(x_2) \widehat{u}_k^{(0,1)}(|x_1|, |x_2|, x_3, \dots, x_d) + \text{sign}(x_1 x_2) \widehat{u}_k^{(1,1)}(|x_1|, |x_2|, x_3, \dots, x_d). \end{aligned}$$

There are now two scalar functions, the trace of which we have to take care of. On $\{0\} \times [0, \delta] \times \mathbb{R}^{d-2}$ we have $[\widehat{u}]_{\widehat{\Gamma}_{\text{cr}}} = 2(\widehat{u}_1^{(1,0)} + \widehat{u}_1^{(1,1)}) \geq 0$ and on $[0, \delta] \times \{0\} \times \mathbb{R}^{d-2}$ we have $[\widehat{u}]_{\widehat{\Gamma}_{\text{cr}}} = 2(\widehat{u}_2^{(0,1)} + \widehat{u}_2^{(1,1)}) \geq 0$. Consider first $s := \widehat{u}_1^{(1,0)} + \widehat{u}_1^{(1,1)}$, the other one can be treated analogously. We can define an H^1 -extension of s to the full ball $B_\delta(T(x^*))$ by taking $s(|x_1|, |x_2|, x_3, \dots, x_d)$ where $x_1 \geq 0$ and $x_2 \leq 0$ and by $\max\{0, s(|x_1|, |x_2|, x_3, \dots, x_d)\}$ where $x_2 \leq 0$. Mollifying with φ satisfying $\text{supp } \varphi \subset \{z \in B_1(0) \mid z_1 < 0, z_2 < 0\}$ gives approximations s_k of s with non-negative trace on $\{0\} \times [0, \delta] \times \mathbb{R}^{d-2}$. For any approximating sequence $\widehat{u}_{1,k}^{(1,0)}$ of $\widehat{u}_1^{(1,0)}$ we can now approximate $\widehat{u}_1^{(1,1)}$ by $\widehat{u}_{1,k}^{(1,1)} := s_k - \widehat{u}_{1,k}^{(1,0)}$ to get the non-negative trace of the sum.

Crack and boundary: When we now come to $x^* \in \partial\Omega \cap \Gamma_{\text{cr}}$ we again use reflection to extend \widehat{u} from $\widehat{\Omega} \cap T(U)$ to the outside but this time specialized by using Corollary 2.3. With U , φ_{x^*} , and η_{x^*} from there, we define the map $R : B_\delta(T(x^*)) \rightarrow \widehat{\Omega}$ with

$$R(x) = x - 2 \max\{0, (x - T(x^*)) \cdot \eta_{x^*} - \varphi_{x^*}(x - \eta_{x^*} \cdot (x - T(x^*)) \eta_{x^*})\} \eta_{x^*},$$

which is Lipschitz continuous and satisfies the property $R^{-1}(\widehat{\Omega} \cap \widehat{\Gamma}_{\text{cr}}) \subset \widehat{\Gamma}_{\text{cr}}$. Thus, we can extend \widehat{u} by $\widehat{u} \circ R \in H^1(V \setminus \widehat{\Gamma}_{\text{cr}}, \mathbb{R}^d)$ where $V = R^{-1}(\widehat{\Omega} \cap U)$ is an open neighborhood of x . Now one can proceed as in the case $x^* \in \Omega \cap \Gamma_{\text{cr}}$ above.

Thus Proposition 5.4 is established. \blacksquare

We are now ready to proof the desired limsup estimate by constructing a recovery sequence $(u_\varepsilon)_\varepsilon$ that converges strongly in $H^1(\Omega_{\text{cr}}; \mathbb{R}^d)$. This result also provides the final part of the proof of the main Theorem 2.1 on the Mosco convergence $\mathcal{F}_\varepsilon \xrightarrow{\text{M}} \mathcal{F}$.

Theorem 5.5 (Limsup estimate) *For every $u \in H_{g,\text{Dir}}^1$ there exists a sequence (ε_j, u_j) with*

$$\varepsilon_j \rightarrow 0, \quad u_j \rightarrow u \text{ in } H_{g,\text{Dir}}^1 \subset H^1(\Omega_{\text{cr}}; \mathbb{R}^d), \quad \text{and} \quad \limsup_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_j) \leq \mathcal{F}(u).$$

Proof. For $\mathcal{F}(u) = \infty$ there is nothing to show, so we restrict to the case $\mathcal{F}(u) < \infty$ which implies $[\![u]\!]_{\Gamma_{\text{cr}}} \geq 0$.

Case $u \in W^{1,\infty}(\Omega_{\text{cr}}; \mathbb{R}^d)$: We apply Proposition 5.2 and obtain a sequence (ε_k, u_k) with $u_k \rightarrow u$ such that $y_k = \text{id} + \varepsilon_k u_k$ satisfies the GMS condition (1.1), which implies

$$\mathcal{F}_{\varepsilon_k}(u_k) = \widetilde{\mathcal{F}}_{\varepsilon_k}(u_k) = \int_{\Omega_{\text{cr}}} \frac{1}{\varepsilon_k^2} W(I + \varepsilon_k \nabla u_k(x)) \, dx = \int_{\Omega_{\text{cr}}} \overline{W}_\varepsilon(\nabla u_k(x)) \, dx.$$

Since all u_k lie in $W^{1,\infty}$ we may assume that $\varepsilon_k \|\nabla u_k\|_{L^\infty} \leq r_{1/2}$ with $\delta > 0$ and $r_\delta > 0$ from (1.3d). Thus, we have

$$W_\varepsilon(\nabla u_k(x)) = \frac{1}{\varepsilon_k^2} W(I + \varepsilon_k \nabla u_k(x)) \leq \left(\frac{1}{2} + \frac{1}{2}\right) |\nabla u_k(x)|_{\mathbb{C}}^2 \leq |\mathbb{C}| |\nabla u_k(x)|^2.$$

Moreover, we may choose a subsequence (not relabeled) and $h \in L^1(\Omega)$ such that

$$\nabla u_k(x) \xrightarrow{k \rightarrow \infty} \nabla u(x) \quad \text{and} \quad |\nabla u_k(x)|^2 \leq h(x) \quad \text{for a.a. } x \in \Omega.$$

Using the convergence (1.4), i.e. $\overline{W}_\varepsilon(\cdot) \xrightarrow{M} \frac{1}{2} |\cdot|_{\mathbb{C}}^2$ we conclude $\overline{W}_{\varepsilon_k}(\nabla u_k(x)) \rightarrow \frac{1}{2} |\nabla u(x)|_{\mathbb{C}}^2$ a.e. in Ω . Now Lebesgue's dominated convergence theorem provides the desired limsup estimate

$$\lim_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k}(u_k) = \lim_{k \rightarrow \infty} \int_{\Omega_{\text{cr}}} \overline{W}_{\varepsilon_k}(\nabla u_k(x)) \, dx = \int_{\Omega_{\text{cr}}} \frac{1}{2} \langle \nabla u(x), \mathbb{C} \nabla u(x) \rangle \, dx = \mathcal{F}(u).$$

General $u \in \mathbf{C}_g$: For a general $u \in \mathbf{C}_g$ Proposition 5.4 guarantees the existence of an approximating sequence $u_j \in \mathbf{C}_g \cap W^{1,\infty}(\Omega_{\text{cr}}; \mathbb{R}^d)$. By the first case there are for each j sequences $(\varepsilon_{j,k}, u_{j,k})_{k \in \mathbb{N}}$ with $u_{j,k} \in \mathbf{C}_g \cap W^{1,\infty}(\Omega_{\text{cr}}; \mathbb{R}^d)$, $\varepsilon_{j,k} \rightarrow 0$, $u_{j,k} \rightarrow u_j$, and $\mathcal{F}_{\varepsilon_{j,k}}(u_{j,k}) \rightarrow \mathcal{F}(u_j)$ as $k \rightarrow \infty$.

To construct a diagonal sequence we use the strong continuity of \mathcal{F} restricted to the convex set \mathbf{C}_g , namely

$$\exists C_F > 0 \quad \forall v \in \mathbf{C}_g \quad \text{with} \quad \|v - u\|_{H^1} \leq 1 : \quad |\mathcal{F}(v) - \mathcal{F}(u)| \leq C_F \|v - u\|_{H^1}.$$

With this we can construct a diagonal sequence as follows. For $n \in \mathbb{N}$ we choose $j_n \geq n$ with $\|u - u_{j_n}\|_{H^1} < 1/n$. Next we choose $k_n \geq n$ with

$$\varepsilon_{j_n, k_n} < 1/n, \quad \|u_{j_n, k_n} - u_{j_n}\|_{H^1} < 1/n, \quad \text{and} \quad |\mathcal{F}_{\varepsilon_{j_n, k_n}}(u_{j_n, k_n}) - \mathcal{F}(u_{j_n})| < 1/n.$$

Setting $\tilde{\varepsilon}_n = \varepsilon_{j_n, k_n}$ and $\tilde{u}_n = u_{j_n, k_n}$ we obtain $\tilde{\varepsilon}_n < 1/n$, $\|\tilde{u}_n - u\|_{H^1} < 2/n$, and

$$|\mathcal{F}_{\tilde{\varepsilon}_n}(\tilde{u}_n) - \mathcal{F}(u)| \leq |\mathcal{F}_{\varepsilon_{j_n, k_n}}(u_{j_n, k_n}) - \mathcal{F}(u_{j_n})| + |\mathcal{F}(u_{j_n}) - \mathcal{F}(u)| \leq 1/n + C_F/n \rightarrow 0.$$

Thus, $(\tilde{\varepsilon}_n, \tilde{u}_n)_{n \in \mathbb{N}}$ is a strongly converging recovery sequence for $u \in \mathbf{C}_g$. ■

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